ANISOTROPIC FUNCTION ESTIMATION USING MULTI-BANDWIDTH GAUSSIAN PROCESSES[1]

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In nonparametric regression problems involving multiple predictors, there is typically interest in estimating an anisotropic multivariate regression surface in the important predictors while discarding the unimportant ones. Our focus is on defining a Bayesian procedure that leads to the minimax optimal rate of posterior contraction (up to a log factor) adapting to the unknown dimension and anisotropic smoothness of the true surface. We propose such an approach based on a Gaussian process prior with dimension-specific scalings, which are assigned carefully-chosen hyperpriors. We additionally show that using a homogenous Gaussian process with a single bandwidth leads to a sub-optimal rate in anisotropic cases.

1. Introduction. Gaussian processes [Rasmussen (2004), van der Vaart and van Zanten (2008b)] are widely used as priors on functions due to tractable posterior computation and attractive theoretical properties. The law of a mean zero Gaussian process (GP) W_t is entirely characterized by its covariance kernel $c(s, t) = E(W_s W_t)$. A squared exponential covariance kernel given by $c(s, t) = \exp(-a\|s - t\|^2)$ is commonly used in the literature.

Given n independent observations, the optimal rate of estimation of a d-variable function that is only known to be α-smooth is $n^{-\alpha/(2\alpha+d)}$ [Stone (1982)]. The quality of estimation thus improves with increasing smoothness of the "true" function while it deteriorates with increase in dimensionality. In practice, the smoothness α is typically unknown and one would like an estimation procedure that adapts to any possible $\alpha > 0$. Accordingly, a lot of effort has been employed to develop adaptive estimation methods that are rate-optimal for every regularity level of the unknown function.

The literature on adaptive estimation in a minimax setting was initiated by Lepskiĭ in a series of papers [Lepskiĭ (1990, 1991, 1992)]; see also Birgé (2001) for a discussion on this topic. We also refer the reader to Hoffmann and Lepski (2002), which contains an extensive list of developments in the frequentist literature on adaptive estimation. There is a growing literature on Bayesian adaptation over the

last decade. Previous works include Belitser and Ghosal (2003), Ghosal, Lember and Van Der Vaart (2003, 2008), Huang (2004), Scricciolo (2006), Rousseau (2010), Kruijer, Rousseau and van der Vaart (2010), de Jonge and van Zanten (2010), Shen, Tokdar and Ghosal (2013).

A key idea in frequentist adaptive estimation is to narrow down the search for an "optimal" estimator within a class of estimators indexed by a smoothness or bandwidth parameter, and make a data-driven choice to select the proper bandwidth. In a Bayesian context, one would place a prior on the bandwidth parameter and model-average across different values of the bandwidth through the posterior distribution. The parameter a in the squared-exponential covariance kernel c plays the role of a scaling or inverse bandwidth. van der Vaart and van Zanten (2009) showed that with a gamma prior on a^d, one obtains the minimax rate of posterior contraction $n^{-\alpha/(2\alpha+d)}$ up to a logarithmic factor for α-smooth functions adaptively over all $\alpha > 0$.

In most multivariate applications, the isotropic smoothness assumption seems too restrictive. Potentially, one can incorporate a separate scaling variable a_j for each dimension using the covariance kernel $c(s, t) = \exp(-\sum_{j=1}^{d} a_j |s_j - t_j|^2)$, intuitively enabling better approximation of anisotropic functions. Such kernels, going by the name *automatic relevance determination* (ARD), have been heavily used in the machine learning community; see, for example, Rasmussen (2004) and references therein. Zou et al. (2010) and Savitsky, Vannucci and Sha (2011) recently considered such a model, with point mass mixture priors on the a_j's. Although this is an attractive approach with encouraging empirical performance, there has not been any theoretical studies of asymptotic properties of related models in a Bayesian framework.

In the frequentist literature, minimax rates of convergence in anisotropic Sobolev, Besov and Hölder spaces have been studied in Ibragimov and Hasminskiĭ (1981), Nussbaum (1985), Birgé (1986), with adaptive estimation procedures developed in Barron, Birgé and Massart (1999), Kerkyacharian, Lepski and Picard (2001), Hoffmann and Lepski (2002), Klutchnikoff (2005) among others. The traditional way of dealing with anisotropy is to employ a separate bandwidth or scaling parameter for the different dimensions, and choose an optimal combination of scales in a data-driven way. However, the multidimensional nature of the problem makes the optimal bandwidth selection difficult compared to the isotropic case, as there is no natural ordering among the estimators with multiple bandwidths [Lepski and Levit (1999)].

It is known [Hoffmann and Lepski (2002)] that the minimax rate of convergence for a function with smoothness α_i along the ith dimension is given by $n^{-\alpha_0/(2\alpha_0+1)}$, where $\alpha_0^{-1} = \sum_{i=1}^{d} \alpha_i^{-1}$ is an *exponent of global smoothness* [Birgé (1986)]. When $\alpha_i = \alpha$ for all $i = 1, \ldots, d$, one reduces back to the optimal rate for isotropic classes. On the contrary, if the true function belongs to an anisotropic class, the assumption of isotropy would lead to loss of efficiency which would be more and

more accentuated in higher dimensions. In addition, if the true function depends on a subset of coordinates $I = \{i_1, \ldots, i_{d_0}\} \subset \{1, \ldots, d\}$ for some $1 \leq d_0 \leq d$, the minimax rate would further improve to $n^{-\alpha_{0I}/(2\alpha_{0I}+1)}$, with $\alpha_{0I}^{-1} = \sum_{j \in I} \alpha_j^{-1}$.

The objective of this article is to study whether one can fully adapt to this larger class of functions in a Bayesian framework using dimension-specific rescalings of a homogenous Gaussian process, referred to as a multi-bandwidth Gaussian process from now on. We answer the question in the affirmative to establish rate adaptiveness of the posterior distribution in a variety of settings involving a multi-bandwidth Gaussian process through a novel prior specification on the vector of bandwidths. For simplicity of exposition, we initially study the problem in two parts: (i) adaptive estimation over anisotropic Hölder functions of d arguments, and (ii) adaptive estimation over functions that can possibly depend on fewer coordinates and have isotropic Hölder smoothness over the remaining coordinates. The proposed prior specification for the two cases above are intuitively interpretable and can be easily connected to prescribe a unified prior leading to adaptivity over (i) and (ii) combined.

Although our prior specification involving dimension-specific bandwidth parameters leads to adaptivity, a stronger result is required to conclude that a single bandwidth would be inadequate for the above classes of functions. We prove that the optimal prior choice in the isotropic case leads to a sub-optimal convergence rate if the true function has anisotropic smoothness by obtaining a lower bound on the posterior contraction rate. Previous results on posterior lower bounds in nonparametric problems include Castillo (2008), van der Vaart and van Zanten (2011).

The remaining paper is organized as follows. In Section 2, we introduce relevant notations and conventions used throughout the paper. The multi-bandwidth Gaussian process is introduced in Section 3. Sections 3.1 and 3.2 discuss the main developments with applications to anisotropic Gaussian process mean regression and logistic Gaussian process density estimation described in Section 3.4. Section 3.5 establishes the necessity of the multi-bandwidth Gaussian process by showing a lower-bound result. In Sections 4.1 and 4.2, we study various properties of rescaled Gaussian processes which are crucially used in the proofs of the main theorems in Section 5.

2. Preliminaries. To keep the notation clean, we shall only use boldface for $\mathbf{a}, \mathbf{b}$ and $\boldsymbol{\alpha}$ to denote vectors. We shall make frequent use of the following multi-index notations. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, let $\mathbf{a}. = \sum_{j=1}^d a_j, \mathbf{a}^* = \prod_{j=1}^d a_j, \mathbf{a}! = \prod_{j=1}^d a_j!, \bar{\mathbf{a}} = \max_j a_j, \underline{\mathbf{a}} = \min_j a_j, \mathbf{a}./\mathbf{b} = (a_1/b_1, \ldots, a_d/b_d)^{\mathrm{T}}, \mathbf{a} \cdot \mathbf{b} = (a_1 b_1, \ldots, a_d b_d)^{\mathrm{T}}, \mathbf{a}^{\mathbf{b}} = \prod_{j=1}^d a_j^{b_j}$. Denote $\mathbf{a} \leq \mathbf{b}$ if $a_j \leq b_j$ for all $j = 1, \ldots, d$. For $n = (n_1, \ldots, n_d)$, let $D^n f$ denote the mixed partial derivatives of order $(n_1, \ldots, n_d)$ of f.

Let $C[0, 1]^d$ and $C^\beta[0, 1]^d$ denote the space of all continuous functions and the Hölder space of β-smooth functions $f : [0, 1]^d \to \mathbb{R}$, respectively, endowed

with the supremum norm $\|f\|_\infty = \sup_{t \in [0,1]^d} |f(t)|$. For $\beta > 0$, the Hölder space $C^\beta[0,1]^d$ consists of functions $f \in C[0,1]^d$ that have bounded mixed partial derivatives up to order $\lfloor \beta \rfloor$, with the partial derivatives of order $\lfloor \beta \rfloor$ being Lipschitz continuous of order $\beta - \lfloor \beta \rfloor$. Also, denote by $H^\beta[0,1]^d$ the Sobolev space of functions $f : [0,1]^d \to \mathbb{R}$ that are restrictions of a function $f : \mathbb{R}^d \to \mathbb{R}$ with Fourier transform $\hat{f}(\lambda) = (2\pi)^{-d} \int e^{i(\lambda,t)} f(t)\, dt$ such that

$$\int (1 + \|\lambda\|^2)^\beta |\hat{f}(\lambda)|^2\, d\lambda < \infty.$$

Note that using the above convention, the inverse Fourier transform $f(t) = \int e^{-i(\lambda,t)} \hat{f}(\lambda)\, d\lambda$. Next, we define an anisotropic Hölder class of functions previously used in Barron, Birgé and Massart (1999), Klutchnikoff (2005). For a function $f \in C[0,1]^d$, $x \in [0,1]^d$, and $1 \le i \le d$, let $f_i(\cdot \mid x)$ denote the univariate function $y \mapsto f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d)$. For a vector of positive numbers $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, the anisotropic Hölder space $C^{\boldsymbol{\alpha}}[0,1]^d$ consists of functions f which satisfy, for some $L > 0$,

$$(2.1) \qquad \max_{1 \le i \le n} \sup_{x \in [0,1]^d} \sum_{j=0}^{\lfloor \alpha_i \rfloor} \|D^j f_i(\cdot \mid x)\|_\infty \le L$$

and, for any $y \in [0,1]$, h small such that $y + h \in [0,1]$ and for all $1 \le i \le d$,

$$(2.2) \qquad \sup_{x \in [0,1]^d} \|D^{\lfloor \alpha_i \rfloor} f_i(y+h \mid x) - D^{\lfloor \alpha_i \rfloor} f_i(y \mid x)\|_\infty \le L |h|^{\alpha_i - \lfloor \alpha_i \rfloor}.$$

For $t \in \mathbb{R}^d$ and a subset $I \subset \{1, \ldots, d\}$ of size $|I| = \tilde{d}$ with $1 \le \tilde{d} \le d$, let t_I denote the vector of size $\tilde{d}$ consisting of the coordinates $(t_j : j \in I)$. Let $C[0,1]^I$ denote the subset of $C[0,1]^d$ consisting of functions f such that $f(t) = g(t_I)$ for some function $g \in C[0,1]^{\tilde{d}}$. Also, let $C^{\boldsymbol{\alpha}}[0,1]^I$ denote the subset of $C^{\boldsymbol{\alpha}}[0,1]^d$ consisting of functions f such that $f(t) = g(t_I)$ for some function $g \in C^{\boldsymbol{\alpha}_I}[0,1]^{\tilde{d}}$.

The ε-covering number $N(\varepsilon, S, d)$ of a semimetric space S relative to the semimetric d is the minimal number of balls of radius ε needed to cover S. The logarithm of the covering number is referred to as the entropy.

We write "$\precsim$" for inequality up to a constant multiple. Let $\phi(x) = (2\pi)^{-1/2} \times \exp(-x^2/2)$ denote the standard normal density, and let $\phi_\sigma(x) = (1/\sigma)\phi(x/\sigma)$. Let an asterisk denote a convolution, for example, $(\phi_\sigma * f)(y) = \int \phi_\sigma(y - x) f(x)\, dx$.

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers and let $\mathbb{R}_*$ denote positive reals. Denote by $\mathcal{S}_{d-1}$ the $(d-1)$-dimensional simplex $\{x \in \mathbb{R}^d : x_i \ge 0, 1 \le i \le d, \sum_{i=1}^d x_i = 1\}$.

Unless otherwise stated, $c, C, C', C_1, C_2, \ldots, K, K_1, K_2, \ldots$ shall denote global constants irrelevant to our purpose.

3. Main results. Let $W = \{W_t : t \in [0, 1]^d\}$ be a centered homogeneous Gaussian process with covariance function $\mathrm{E}(W_s W_t) = c(s - t)$. A detailed review of the facts on Gaussian processes relevant to the present application can be found in van der Vaart and van Zanten (2008b). If $c : \mathbb{R}^d \to \mathbb{R}$ is continuous, by Bochner's theorem, there exists a finite positive measure ν on $\mathbb{R}^d$, called the spectral measure of W, such that

$$c(t) = \int_{\mathbb{R}^d} e^{-i(\lambda, t)} \nu(d\lambda),$$

where for $u, v \in \mathbb{C}^d$, (u, v) denotes the complex inner product. As in van der Vaart and van Zanten (2009), we shall restrict ourselves to processes with spectral measure ν having subexponential tails, that is, for some $\delta > 0$,

$$(3.1) \qquad\qquad \int e^{\delta \|\lambda\|} \nu(d\lambda) < \infty.$$

The spectral measure ν of a squared exponential covariance kernel with $c(t) = \exp(-\|t\|^2)$ has a density w.r.t. the Lebesgue measure given by $f(\lambda) = 1/(2^d \pi^{d/2}) \exp(-\|\lambda\|^2/4)$ which clearly satisfies (3.1).

Let $\mathbb{H}$ be the RKHS of W; see van der Vaart and van Zanten (2008b) for a review of relevant facts. van der Vaart and van Zanten (2008a) showed that the rate of posterior contraction with a Gaussian process prior W, using a metric where appropriate testing is possible, is determined by the behavior of the concentration function $\phi_{w_0}(\varepsilon)$ for ε close to zero, where

$$(3.2) \qquad \phi_{w_0}(\varepsilon) = \inf_{h \in \mathbb{H} : \|h - w_0\|_\infty \le \varepsilon} \frac{\|h\|_{\mathbb{H}}^2}{2} - \log P(\|W\|_\infty \le \varepsilon).$$

We tacitly assume that there is a given statistical problem where the true parameter g_0 is a known function of w_0. van der Vaart and van Zanten (2009) studied rescaled Gaussian processes $W^A = \{W_{At} : t \in [0, 1]^d\}$ for a real positive random variable A stochastically independent of W, and showed that with a Gamma prior on A^d, one obtains the minimax-optimal rate of convergence $n^{-\alpha/(2\alpha+d)}$ (up to a logarithmic factor) for α-smooth functions. Since their prior specification does not involve the unknown smoothness α, the procedure is fully adaptive.

The key result of van der Vaart and van Zanten (2009) was to construct sets $B_n \subset C[0, 1]^d$ so that given $\alpha > 0$, a function $w_0 \in C^\alpha[0, 1]^d$, and a constant $C > 1$, there exists a constant $D > 0$ such that, for every sufficiently large n,

$$(3.3) \qquad\qquad \log N(\bar{\varepsilon}_n, B_n, \|\cdot\|_\infty) \le Dn\bar{\varepsilon}_n^2,$$

$$(3.4) \qquad\qquad P(W^A \notin B_n) \le e^{-Cn\varepsilon_n^2},$$

$$(3.5) \qquad\qquad P(\|W^A - w_0\|_\infty \le \varepsilon_n) \ge e^{-n\varepsilon_n^2}$$

with $\varepsilon_n = n^{-\alpha/(2\alpha+1)}(\log n)^{\kappa_1}$, $\bar{\varepsilon}_n = n^{-\alpha/(2\alpha+1)}(\log n)^{\kappa_2}$ for constants $\kappa_1, \kappa_2 > 0$.

In this article, we shall study multi-bandwidth Gaussian processes of the form $\{W_t^{\mathbf{a}} = W_{\mathbf{a} \cdot t} : t \in [0,1]^d\}$ for a vector of rescalings (or inverse-bandwidths) $\mathbf{a} = (a_1, \ldots, a_d)^{\mathrm{T}}$ with $a_j > 0$ for all $j = 1, \ldots, d$. We consider two function classes defined in Section 2:

(i) Hölder class of functions $C^{\boldsymbol{\alpha}}[0,1]^d$ with anisotropic smoothness ($\boldsymbol{\alpha} \in \mathbb{R}_*^d$).

(ii) Hölder class of functions $C^{\alpha}[0,1]^I$ with isotropic smoothness that can possibly depend on fewer dimensions ($\alpha > 0$ and $I \subset \{1, \ldots, d\}$).

For a continuous function in the support of a Gaussian process, the probability assigned to a sup-norm neighborhood of the function is controlled by centered small ball probability and how well the function can be approximated from the RKHS of the process [Section 5 of van der Vaart and van Zanten (2008b)]. With the target class of functions as in (i) or (ii), a single scaling seems inadequate and it is intuitively appealing to introduce multiple bandwidth parameters to enlarge the RKHS and facilitate improved approximation from the RKHS.

The main technical challenge for adaptation in our setting is to find a joint prior on $\mathbf{a}$ and devise sets B_n (henceforth called *sieves*) so that (3.3)–(3.5) are satisfied with w_0 in the above function classes (i)–(ii) and ε_n being the optimal rate of convergence for the same. To that end, we propose a novel class of joint priors on the rescaling vector $\mathbf{a}$ that leads to adaptation over function classes (i) and (ii) in Sections 3.1 and 3.2, respectively. Connections between the two prior choices are discussed and a unified framework is prescribed for the function class $\{C^{\boldsymbol{\alpha}}[0,1]^I : \boldsymbol{\alpha} \in \mathbb{R}_*^d, I \subset \{1, \ldots, d\}\}$ combining (i) and (ii).

The construction of the sieves B_n are laid out in Section 5. With such B_n, one can use standard results to establish adaptive minimax rate of convergence in various statistical settings; refer to the discussion following Theorem 3.1 in van der Vaart and van Zanten (2009). Some such specific applications are described in Section 3.4.

3.1. *Adaptive estimation of anisotropic functions.* Let $\mathbf{A} = (A_1, \ldots, A_d)^{\mathrm{T}}$ be a random vector in $\mathbb{R}^d$ with each A_j a nonnegative random variable stochastically independent of W. We can then define a scaled process $W^{\mathbf{A}} = \{W_{\mathbf{A} \cdot t} : t \in [0,1]^d\}$, to be interpreted as a Borel measurable map in $C[0,1]^d$ equipped with the sup-norm $\|\cdot\|_{\infty}$. The basic idea here is to scale the different dimensions by different amounts so that the resulting process becomes suitable for approximating functions having different smoothness along the different coordinate axes.

We shall define a joint distribution on $\mathbf{A}$ induced through the following hierarchical specification. Let $\Theta = (\Theta_1, \ldots, \Theta_d)$ denote a random vector with a density supported on the simplex $\mathcal{S}_{d-1}$.

(PA1) Draw $\Theta \sim \mathrm{Dir}(\beta_1, \ldots, \beta_d)$ for some $\beta = (\beta_1, \ldots, \beta_d)$.

(PA2) Given $\Theta = \theta$, draw $A_j^{1/\theta_j} \sim g$ independently, where g is a density on the positive real line satisfying

$$(3.6) \qquad C_1 x^p \exp(-D_1 x \log^q x) \le g(x) \le C_2 x^p \exp(-D_2 x \log^q x)$$

for positive constants C_1, C_2, D_1, D_2 and nonnegative constants p, q and every sufficiently large $x > 0$.

In particular, g corresponds to a gamma(b_1, b_2) distribution if $b_1 = p + 1$, $D_1 = D_2 = b_2$, $q = 0$, $C_1 = C_2$ in (3.6). For notational simplicity, we shall assume g to be gamma(b_1, b_2) from now on, noting that the main results would all hold for the general form of g above.

Let $\pi_{\mathbf{A}}$ denote the joint prior on $\mathbf{A}$ induced through (PA1)–(PA2), so that $\pi_{\mathbf{A}}(\mathbf{a}) = \int \prod_{j=1}^{d} \pi(a_j \mid \theta_j) \, d\pi(\theta)$. We now state our main theorem for the anisotropic smoothness class in (i), with a detailed proof provided in Section 5.

THEOREM 3.1. *Let W be a centered homogeneous Gaussian random field on $\mathbb{R}^d$ with spectral measure ν that satisfies* (3.1) *and let $W^{\mathbf{A}}$ denote the multi-bandwidth process with $\mathbf{A} \sim \pi_{\mathbf{A}}$ as in* (PA1)–(PA2). *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be a vector of positive numbers and $\alpha_0 = (\sum_{i=1}^{d} \alpha_i^{-1})^{-1}$. Suppose w_0 belongs to the anisotropic Hölder space $C^{\boldsymbol{\alpha}}[0, 1]^d$. Then for every constant $C > 1$, there exist Borel measurable subsets B_n of $C[0, 1]^d$ and a constant $D > 0$ such that, for every sufficiently large n, the conditions* (3.3)–(3.5) *are satisfied by $W^{\mathbf{A}}$ with $\varepsilon_n = n^{-\alpha_0/(2\alpha_0+1)}(\log n)^{\kappa_1}, \bar{\varepsilon}_n = n^{-\alpha_0/(2\alpha_0+1)}(\log n)^{\kappa_2}$ for constants $\kappa_1, \kappa_2 > 0$.*

3.2. *Adaptive dimension reduction.* We next consider the smoothness class in (ii), namely $C^{\alpha}[0, 1]^I$ for $I \subset \{1, \ldots, d\}$ and $\alpha > 0$. If the true function has isotropic smoothness on the dimensions it depends on, it is intuitively clear that one does not need a separate scaling for each of the dimensions. Indeed, had we known the true coordinates $I \subset \{1, \ldots, d\}$, we could have only scaled the dimensions in I by a positive random variable A, and a slight modification of the results in van der Vaart and van Zanten (2009) would imply that a gamma prior on $A^{|I|}$ would lead to adaptation.

With that motivation, consider a joint prior $\pi_{\mathbf{A}}$ on $\mathbf{A}$ induced through the following hierarchical scheme:

(PD1) draw $\tilde{d}$ uniformly on $\{0, \ldots, d\}$,
(PD2) given $\tilde{d}$, draw a subset $S \subset \{1, \ldots, d\}$ with $|S| = \tilde{d}$ uniformly from all subsets of size $\tilde{d}$,
(PD3) generate a pair of random variables (A, B) with $A^{\tilde{d}} \sim$ gamma(b_1, b_2) and B drawn from a fixed compactly supported distribution,
(PD4) set $A_j = A$ for $j \in S$ and $A_j = B$ for $j \notin S$.

In particular, one can fix B to be any constant $c > 0$ in (PD3), which corresponds to the $\delta_c(\cdot)$ distribution. We next state our main result on adaptive dimension reduction. The proof of the following Theorem 3.2 has elements in common with the proof of the Theorem 3.1, and hence only a sketch of the proof is provided in Section 5.

THEOREM 3.2. *Let W be a centered homogeneous Gaussian random field on $\mathbb{R}^d$ with spectral measure ν that satisfies* (3.1) *and let $W^{\mathbf{A}}$ denote the multibandwidth process with $\mathbf{A} \sim \pi_{\mathbf{A}}$ as in* (PD1)–(PD4). *Suppose w_0 belongs to the Hölder space $C^\alpha[0,1]^I$ for some subset I of $\{1,\ldots,d\}$ and $\alpha > 0$. Then for every constant $C > 1$, there exist Borel measurable subsets B_n of $C[0,1]^d$ and a constant $D > 0$ such that, for every sufficiently large n, the conditions* (3.3)–(3.5) *are satisfied by $W^{\mathbf{A}}$ with $\varepsilon_n = n^{-\alpha/(2\alpha+d_0)}(\log n)^{\kappa_1}, \bar{\varepsilon}_n = n^{-\alpha/(2\alpha+d_0)}(\log n)^{\kappa_2}$ for constants $\kappa_1, \kappa_2 > 0$ and $d_0 = |I|$.*

3.3. *Connections between cases* (i) *and* (ii). The joint distributions on $\mathbf{A}$ specified in (PA1)–(PA2) and (PD1)–(PD4) are closely connected. To begin with, note that if we set $A_j = A, \theta_j = 1/d, j = 1, \ldots, d$ in (PA1)–(PA2), one reduces to a gamma prior on A^d; the optimal prior choice in the isotropic case [van der Vaart and van Zanten (2009)]. In the anisotropic case, our proposed prior can be motivated as follows. Recall that the purpose of rescaling is to traverse the sample paths of an infinite smooth stochastic process on a larger domain to make it more suitable for less smooth functions. If the true function has anisotropic smoothness, then we would like to stretch those directions more where the function is less smooth. For smaller θ_j's, the marginal distribution of a_j has lighter tails compared to larger values of θ_j. We would thus like θ_j to assume smaller values for the directions j where the function is more smooth and larger values corresponding to the less smooth directions. Without further constraints on θ, it is not possible to separate the scale of $\mathbf{A}$ from θ. This motivates us to constrain θ to the simplex which serves as a weak identifiability condition.

In the limit as $\theta_j \to 0$, the distribution of a_j converges to a point mass at zero. Accordingly, if the true function does not depend on a set of $(d - d^*)$ dimensions, we would set $\theta_j = 0$ for those dimensions and choose the remaining θ_j's from a $d^* - 1$-dimensional simplex. In particular, if the function has isotropic smoothness in the remaining d^* coordinates, one can simply choose $\theta_j = 1/d^*$ for those dimensions. This reduces to our prior choice in (PD1)–(PD4).

The dimensionality reduction in Section 3.2 deals with finitely many models and can be alternatively studied as a model selection problem [Ghosal, Lember and van der Vaart (2008)]. However, the connection established above allows us to treat anisotropy and dimension reduction under a single framework with the dimension reduction paradigm recognized as a limiting case of the anisotropic framework. We exploit this connection to propose a unified framework for adaptively estimating functions which possibly depend on fewer coordinates and have anisotropic

smoothness in the remaining ones, that is, functions in $C^{\boldsymbol{\alpha}}[0, 1]^I$ for $\boldsymbol{\alpha} \in \mathbb{R}_*^d$ and $I \subset \{1, \ldots, d\}$. In particular, we continue to consider rescaled Gaussian processes $W^{\mathbf{A}}$ with the following prior on $\mathbf{A}$:

(P1) draw $\tilde{d}$ uniformly on $\{0, \ldots, d\}$,

(P2) given $\tilde{d}$, draw a subset $S \subset \{1, \ldots, d\}$ with $|S| = \tilde{d}$ uniformly from all subsets of size $\tilde{d}$,

(P3) draw $\Theta = (\theta_1, \ldots, \theta_{\tilde{d}}) \in \mathcal{S}_{\tilde{d}-1}$ from a $\mathrm{Dir}(\beta_1, \ldots, \beta_{\tilde{d}})$ distribution,

(P4) given S and $\Theta = \theta$, draw $A_j^{1/\theta_j} \sim \mathrm{gamma}(b_1, b_2)$ independently for $j \in S$, and fix $A_j = c$ for some $c > 0$ and $j \notin S$.

A unification of Theorems 3.1 and 3.2 is provided in the following Theorem 3.3, with the proof omitted as it is similar to the previous theorems.

THEOREM 3.3. *Consider $W^{\mathbf{A}}$ with $\mathbf{A} \sim \pi_{\mathbf{A}}$ as in* (P1)–(P4). *Suppose w_0 belongs to $C^{\boldsymbol{\alpha}}[0, 1]^I$ for some subset I of $\{1, \ldots, d\}$ and $\boldsymbol{\alpha} \in \mathbb{R}_*^d$. Let $\alpha_{0I}^{-1} = \sum_{j \in I} \alpha_j^{-1}$. Then for every constant $C > 1$, the conclusions of Theorem 3.1 are satisfied by $W^{\mathbf{A}}$ with $\varepsilon_n = n^{-\alpha_{0I}/(2\alpha_{0I}+1)} (\log n)^{\kappa_1}, \bar{\varepsilon}_n = n^{-\alpha_{0I}/(2\alpha_{0I}+1)} (\log n)^{\kappa_2}$ for constants $\kappa_1, \kappa_2 > 0$.*

In Theorem 3.3, the exponents of the logarithmic terms increase linearly with the dimension and decrease with α_{0I}. In particular, κ_1 and κ_2 can be estimated by $\frac{d+1}{\alpha_{0I}+2}$ and $\frac{(\alpha_{0I}+3)(d+1)}{2(\alpha_{0I}+2)}$, respectively.

3.4. *Rates of convergence in specific settings.* Theorem 3.3 is in the same spirit as Theorem 3.1 of van der Vaart and van Zanten (2009) [see also Theorem 2.2 of de Jonge and van Zanten (2010)] and can be used to derive rates of posterior contraction in a variety of statistical problems involving Gaussian random fields. We shall consider a couple of specific problems with the message that similar results can be obtained for a large class of problems.

We first consider a regression problem where given independent response variable y_i and covariates $x_i \in [0, 1]^d$, the response is modeled as random perturbations around a smooth regression surface, that is,

$$(3.7) \qquad y_i = \mu(x_i) + \varepsilon_i, \qquad \varepsilon_i \sim \mathrm{N}(0, \sigma^2).$$

As motivated before, the true regression surface μ_0 might depend only on a subset of variables I and have anisotropic smoothness in the remaining variables. Accordingly, we assume $\mu_0 \in C^{\boldsymbol{\alpha}}[0, 1]^I$ for some I and α. Also, assume the true value σ_0 of σ lies in an interval $[a, b] \subset [0, \infty)$.

We use the law of $W^{\mathbf{A}}$ as a prior on μ, with the prior on $\mathbf{A}$ as in (P1)–(P4). We also assume a prior on σ supported on $[a, b]$. Denote the posterior distribution by $\Pi(\cdot \mid y_1, \ldots, y_n)$. Let $\|\mu\|_n^2 = n^{-1} \sum_{i=1}^n \mu^2(x_i)$ denote the L_2 norm corresponding

to the empirical distribution of the design points. The posterior is said to contract at a rate ε_n, if for every sufficiently large M,

$$(3.8) \quad E_{\mu_0,\sigma_0} \Pi\big[(\mu, \sigma): \|\mu - \mu_0\|_n + |\sigma - \sigma_0| > M\varepsilon_n \mid y_1, \ldots, y_n\big] \to 0.$$

THEOREM 3.4. *Consider the nonparametric regression model* (3.7) *with* $W^{\mathbf{A}}$ *as a prior on* μ, *with the prior on* $\mathbf{A}$ *as in* (P1)–(P4). *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ *be a vector of positive numbers and* I *be a subset of* $\{1, \ldots, d\}$. *If* $\mu_0 \in C^{\boldsymbol{\alpha}}[0, 1]^I$, *then* (3.8) *holds with* $\varepsilon_n = n^{-\alpha_{0I}/(2\alpha_{0I}+1)} \log^\kappa n$, *where* $\alpha_{0I}^{-1} = \sum_{j \in I} \alpha_j^{-1}$ *and* $\kappa > 0$ *is a positive constant.*

Thus, one obtains the minimax optimal rate up to a log factor adapting to the unknown dimensionality and anisotropic smoothness.

REMARK 3.5. Theorem 3.4 guarantees adaptive estimation of the regression surface. It is often of interest additionally to select the important variables affecting the response y. In the context of variable selection in a normal linear model, Barbieri and Berger (2004) advocated using the median probability model consisting of the variables with posterior marginal inclusion probability greater than or equal to half. They also proved the predictive optimality of such models. The same approach could be followed here for variable selection; however, the issue of optimality needs to be studied.

A similar result on adaptation holds for density estimation using the logistic Gaussian process, where an unknown density g on the hypercube $[0, 1]^d$ is modeled as

$$(3.9) \qquad g(t) = \frac{e^{\mu(t)}}{\int_{[0,1]^d} e^{\mu(s)} \, ds}$$

for a function $\mu: \mathbb{R}^d \to \mathbb{R}$. Suppose $X_1, \ldots, X_n$ are drawn i.i.d. from a continuous, everywhere positive density $g_0 = \log \mu_0$ on $[0, 1]^d$.

THEOREM 3.6. *Suppose one uses the law of* $W^{\mathbf{A}}$ *as a prior on* μ *in* (3.9), *with the prior on* $\mathbf{A}$ *as in* (P1)–(P4). *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ *be a vector of positive numbers and* I *be a subset of* $\{1, \ldots, d\}$. *If* $\mu_0 = \log g_0 \in C^{\boldsymbol{\alpha}}[0, 1]^I$, *then the posterior contracts at the rate* $\varepsilon_n = n^{-\alpha_{0I}/(2\alpha_{0I}+1)} \log^\kappa n$ *with respect to the Hellinger distance, where* $\alpha_{0I}^{-1} = \sum_{j \in I} \alpha_j^{-1}$.

The proofs of Theorems 3.4 and 3.6 follow in a straightforward manner from our main results in Theorems 3.1 and 3.2. We do not provide a proof here since the steps are very similar to those in Section 3 of van der Vaart and van Zanten (2008a).

3.5. *Lower bounds on posterior contraction rates.* In this section, we derive a lower bound to the posterior convergence rate when the true function has anisotropic smoothness and a single bandwidth GP is used, exhibiting the necessity of the multi-bandwidth process. Consider once again the regression function estimation setting (3.7); we assume σ to be fixed and known here. For technical simplicity, we will formulate our lower bounds results in a slightly different setting; we assume a random design with $x_i \sim Q$, where Q is a distribution on $[0, 1]^d$ admitting a density ω with respect to the Lebesgue measure on $[0, 1]^d$. Let $\|\mu\|_{2,\omega}^2 = \int_{[0,1]^d} \mu^2(t)\omega(t)\,dt$. Recall that ξ_n is a lower bound [Castillo (2008)] to the posterior convergence rate around μ_0 in $\|\cdot\|_{2,\omega}$ if

$$(3.10) \qquad P\big(\|\mu - \mu_0\|_{2,\omega} \le \xi_n \mid Y_1, \ldots, Y_n, x_1, \ldots, x_n\big) \to 0 \qquad \text{a.s.}$$

as $n \to \infty$.

In the following, for a positive random variable A stochastically independent of W, we shall consider a rescaled Gaussian process W^A as the prior on the regression function μ, with A assigned the following prior (LBP):

(LBP) Consider $A^d \sim h$, where h is a density on the positive real line satisfying for any $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small,

$$(3.11) \qquad\qquad\qquad h(x) \le B_0 \exp(-B_1/x), \qquad x < \varepsilon_1,$$

$$
(3.12) \qquad C_1 x^p \exp\big(-D_1 x \log^q x\big) \le h(x) \le C_2 x^p \exp\big(-D_2 x \log^q x\big),
$$
$$x > 1/\varepsilon_2$$

for positive constants $B_0, B_1, C_1, C_2, D_1, D_2$ and constants $p, q \ge 0$.

The tail behavior of A implied by (3.12) is exactly the same as equation (3.4) in van der Vaart and van Zanten (2009). To establish the lower bound result, we further need to control the behavior of h near zero, which is specified in (3.11). In particular, letting h to be a gamma density truncated to $[c, \infty)$ for any constant $c > 0$ would satisfy (LBP). Examples of h supported on $(0, \infty)$ satisfying (LBP) include the three-parameter generalized inverse Gaussian (gIG) family with probability density function

$$h(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-(ax+b/x)/2}, \qquad x > 0,$$

where K_p is a modified Bessel function of the second kind, $a > 0, b > 0$ and p is a real parameter. Thus, being a subclass of prior distributions on A considered by van der Vaart and van Zanten (2009), (LBP) results in the minimax rate of posterior contraction upto a logarithmic term adaptively over isotropic α-Hölder functions of d-variables for any $\alpha > 0$ with respect to the empirical L_2 norm.

Next, consider a class of functions (LBT) for the true regression function μ_0:

(LBT) For integers $d \geq 3$, $1 \leq d_1 < d$ and $d_2 = d - d_1$, define $I_1 = \{1, \ldots, d_1\}$ and $I_2 = \{d_1 + 1, \ldots, d\}$. Assume $\mu_0(t) = \zeta(t_{I_1})\eta(t_{I_2})$, where $\zeta \in C^\alpha[0, 1]^{d_1}$, has support $[v, w]^{d_1}$ for $0 < v < w < 1$ and $\eta : [0, 1]^{d_2} \to \mathbb{R}$ is infinitely differentiable with support $[v, w]^{d_2}$. Assume further that there exists $0 < u < 1$, $\beta \geq \alpha$, such that for any constants K_1 and K_2 sufficiently large, the average (in L_2 sense) tail of the Fourier transform $\hat{\mu}_0$ of μ_0 satisfies

$$(3.13) \qquad \int_{\|\lambda_{I_1}\| > K_1, \|\lambda_{I_2}\| \in [K_2, 2K_2]} |\hat{\mu}_0(\lambda)|^2 \, d\lambda \geq K_1^{-2\beta} \exp\{-2K_2^u\}.$$

Since $\hat{\mu}_0(\lambda) = \hat{\zeta}(\lambda_{I_1})\hat{\eta}(\lambda_{I_2})$, it suffices to have $\zeta \in C^\alpha[0, 1]^{d_1}$ and η infinitely smooth such that $\int_{\|\lambda_1\| > K_1} |\hat{\zeta}(\lambda_1)|^2 \, d\lambda_1 \geq K_1^{-2\beta}$ and $\int_{\|\lambda_2\| \in [K_2, 2K_2]} |\hat{\eta}(\lambda_2)|^2 \, d\lambda_2 \geq \exp\{-2K_2^u\}$. We provide examples of functions ζ and η satisfying (LBT) in Appendix A for appropriate α, β and u; refer also to the discussion in the last paragraph of this section.

In the following Theorem 3.7, we demonstrate that (LBP) can lead to a slower (in an exponent of n) rate of convergence compared to the multi-bandwidth case with respect to the $\|\cdot\|_{2,\omega}$ norm when the true regression function μ_0 belongs to the anisotropic class (LBT).

THEOREM 3.7. *Consider the setting* (3.7) *with* (LBP) *as the prior and assume the true regression function μ_0 satisfies* (LBT). *Let $\gamma_1 = \alpha/(2\alpha + d)$, $\gamma_2 = \alpha/(2\alpha + d_1)$. Assume that the exponent β appearing in* (3.13) *satisfies $\beta < \alpha\gamma_2/\gamma_1$. Then* (3.10) *holds with $\xi_n = n^{-\gamma}$ for any $\gamma \in (\gamma_1, \gamma_2)$ satisfying $\beta < \alpha\gamma/\gamma_1$.*

Theorem 3.1 implies that the posterior rate of convergence for estimating μ_0 satisfying (LBT) under the proposed multi-bandwidth GP is faster than $n^{-\psi/(2\psi+1)}$ with $1/\psi = d_1/\alpha + d_2/m$ for any large integer m. Hence, this rate can be made arbitrarily close to $n^{-\gamma_2}$ (in an exponent of n). On the other hand, since μ_0 satisfying (LBT) belongs to the isotropic Hölder class $C^\alpha[0, 1]^d$, the posterior convergence rate under (LBP) is bounded above by $n^{-\gamma_1}$ up to a logarithmic term [van der Vaart and van Zanten (2009)].[2] Theorem 3.7 shows that under suitable conditions, one can find $\gamma \in (\gamma_1, \gamma_2)$ such that $n^{-\gamma}$ is a lower bound (3.10) to the posterior convergence rate under (LBP) in the $\|\cdot\|_{2,\omega}$ norm. In other words, the posterior is concentrated in the annulus with outer radius $n^{-\gamma_1}$ and inner radius $n^{-\gamma}$. This exhibits the lack of efficiency incurred by using a single bandwidth,[3] since $n^{-\gamma}$ is slower than $n^{-\gamma_2}$ by a genuine power of n.

The exponent β in the L_2 averaged Fourier tail of ζ dictates the lower bound ξ_n in Theorem 3.7, and hence the assumption $\beta < \alpha\gamma_2/\gamma_1$ merits a discussion. Fix α, d_1. Since $\gamma_2/\gamma_1 = 1 + d_2/(2\alpha + d_1)$, one can allow $\beta = K\alpha$ for any positive

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[2]We note once again that the upper bound results are for the empirical L_2 norm.

[3]Albeit in a slightly different norm.

constant $K > 1$ by choosing d_2 large enough in Theorem 3.7. Intuitively, more the number of dimensions that contribute to the anisotropy, less stringent becomes the requirement on ζ. In Appendix A, we present a specific example of ζ in the case $d_1 = 1$ with $\alpha = 1$ and $\beta = 3/2$. In this case, $\beta < \alpha\gamma_2/\gamma_1$ is equivalent to $3/2 < 1 + d_2/3$, which is satisfied for any $d_2 \geq 2$.

4. Auxiliary results. In this section, we present a number of auxiliary results that are crucially used to prove the main results.

4.1. *Properties of the multi-bandwidth Gaussian process.* We first summarize some properties of the RKHS of the scaled process $W^{\mathbf{a}}$ for a fixed vector of scales $\mathbf{a}$. Lemmas 4.1–4.4 generalize the results in Section 4 of van der Vaart and van Zanten (2009) from a single scaling to a vector of scales; we briefly sketch the proofs emphasizing the places we differ substantially from van der Vaart and van Zanten (2009).

Assume that the spectral measure ν of W has a spectral density f. For $\mathbf{a} \in \mathbb{R}_*^d$, the rescaled process $W^{\mathbf{a}}$ has a spectral measure $\nu_{\mathbf{a}}$ given by $\nu_{\mathbf{a}}(B) = \nu(B./\mathbf{a})$. Further, $\nu_{\mathbf{a}}$ admits a spectral density $f_{\mathbf{a}}$, with $f_{\mathbf{a}}(\lambda) = \frac{1}{\mathbf{a}^*}f(\lambda./\mathbf{a})$. For $w_0 \in C[0, 1]^d$, define $\phi_{w_0}^{\mathbf{a}}(\varepsilon)$ to be the concentration function of the rescaled Gaussian process $W^{\mathbf{a}}$.

As a straightforward extension of Lemmas 4.1 and 4.2 in van der Vaart and van Zanten (2009), it turns out that the RKHS of the process $W^{\mathbf{a}}$ can be characterized as below.

LEMMA 4.1. *The RKHS $\mathbb{H}^{\mathbf{a}}$ of the process $\{W_t^{\mathbf{a}} : t \in [0, 1]^d\}$ consists of real parts of the functions*

$$t \mapsto \int e^{i(\lambda, t)} g(\lambda)\nu_{\mathbf{a}}(d\lambda),$$

where g runs over the complex Hilbert space $L_2(\nu_{\mathbf{a}})$. Further, the RKHS norm of the element in the above display is given by $\|g\|_{L_2(\nu_{\mathbf{a}})}$.

Lemma 4.3 of van der Vaart and van Zanten (2009) shows that for any isotropic Hölder smooth function w, convolutions with an appropriately chosen class of higher order kernels indexed by the scaling parameter a belong to the RKHS. This suggests that driving the bandwidth $1/a$ to zero, one can obtain improved approximations to any Hölder smooth function. The following Lemma 4.2 illustrates the usefulness of using separate bandwidths for each dimension for approximating anisotropic Hölder functions from the RKHS.

LEMMA 4.2. *Assume ν has a density with respect to the Lebesgue measure which is bounded away from zero on a neighborhood of the origin. Let $\boldsymbol{\alpha} \in \mathbb{R}_*^d$ be*

given. Then, for any subset I of $\{1, \ldots, d\}$ and $w \in C^{\boldsymbol{\alpha}}[0, 1]^I$, there exists constants C and D depending only on ν and w such that, for a_i's large enough,

$$\inf\left\{\|h\|_{\mathbb{H}^{\mathbf{a}}}^2 : \|h - w\|_\infty \le C \sum_{i \in I} a_i^{-\alpha_i}\right\} \le D\mathbf{a}^*.$$

PROOF. We shall prove the result for $w \in C^{\boldsymbol{\alpha}}[0, 1]^d$ and sketch an argument for extending the proof to any $w \in C^{\boldsymbol{\alpha}}[0, 1]^I$.

Let $\psi_j, j = 1, \ldots, d$, be a set of higher order kernels which satisfy $\int \psi_j(t_j)\,dt_j = 1$, $\int t_j^k \psi_j(t_j)\,dt_j = 0$ for any positive integer k and $\int |t_j|^{\alpha_j} |\psi_j(t_j)|\,dt_j \le 1$. Define $\psi : \mathbb{R}^d \to \mathbb{C}$ by $\psi(t) = \psi_1(t_1) \cdots \psi_d(t_d)$ so that one has $\int_{\mathbb{R}^d} \psi(t)\,dt = 1$, $\int_{\mathbb{R}^d} t^k \psi(t)\,dt = 0$ for any nonzero multiindex $k = (k_1, \ldots, k_d)$, and the functions $|\hat{\psi}|/f$ and $|\hat{\psi}|^2/f$ are uniformly bounded, where $\hat{\psi}$ denotes the Fourier transform of ψ.

For a vector of positive numbers $\mathbf{a} = (a_1, \ldots, a_d)$, let $\psi_{\mathbf{a}}(t) = \mathbf{a}^* \psi(\mathbf{a} \cdot t)$, where $\mathbf{a}^* = \prod_{j=1}^d a_j$. Proceeding as in Lemma 4.3 of van der Vaart and van Zanten (2009), one can show that the convolution $\psi_{\mathbf{a}} * w$ is contained in the RKHS $\mathbb{H}^{\mathbf{a}}$ and the squared RKHS norm of $\psi_{\mathbf{a}} * w$ is bounded by $D\mathbf{a}^*$, with D depending only on ν and w. Thus, the proof of Lemma 4.2 would be completed if we can show that $\|\psi_{\mathbf{a}} * w - w\|_\infty \le C \sum_{j=1}^d a_j^{-\alpha_j}$.

We have, for any $t \in \mathbb{R}^d$,

$$\psi_{\mathbf{a}} * w(t) - w(t) = \int \psi(s)\{w(t - s./\mathbf{a}) - w(t)\}\,ds.$$

For $1 \le j \le d - 1$, let $u^{(j)}$ denote the vector in $\mathbb{R}^d$ with $u_i^{(j)} = 0$ for $i = 1, \ldots, j$ and $u_i^{(j)} = 1$ for $i = j + 1, \ldots, d$. For any two vectors $x, y \in \mathbb{R}^d$, we can navigate from x to y in a piecewise linear fashion traveling parallel to one of the coordinate axes at a time. The vertices of the path will be given by $x^{(0)} = x$, $x^{(j)} = u^{(j)} \cdot x + (1 - u^{(j)}) \cdot y$ for $j = 1, \ldots, d - 1$ and $x^{(d)} = y$.

A multivariate Taylor expansion of $w(t - s./a)$ around $w(t)$ cannot take advantage of the anisotropic smoothness of w across different coordinate axes. Letting $x = t$, $y = t - s./a$ and $x^{(j)}, j = 0, 1, \ldots, d$ as above, let us write $w(y) - w(x)$ in the following telescoping form:

$$w(y) - w(x) = \sum_{j=1}^d w\big(x^{(j)}\big) - w\big(x^{(j-1)}\big)$$

$$= \sum_{j=1}^d w_j\big(t_j - s_j/a_j \mid x^{(j)}\big) - w_j\big(t_j \mid x^{(j)}\big),$$

where the functions w_j are as defined in Section 2, with $w_j(t \mid x) = w(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d)$ for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Thus,

$$w(t - s./a) - w(t) = \sum_{j=1}^{d} \left[\sum_{i=1}^{\lfloor \alpha_j \rfloor} D^i w_j(t_j \mid x^{(j)}) \frac{(-s_j/a_j)^i}{i!} + S_j(t_j, -s_j/a_j) \right],$$

where $|S_j(t_j, -s_j/a_j)| \leq K s_j^{\alpha_j} a_j^{-\alpha_j}$ by (2.2), for a constant K depending on ν and w but not on t and s. Combining the above, we have

$$\left| \int \psi(s)\{w(t - s./\mathbf{a}) - w(t)\} \, ds \right| = \left| \sum_{j=1}^{d} \int \psi(s_j) S_j(t_j, -s_j/a_j) \, ds_j \right|$$

$$\leq C \sum_{j=1}^{d} a_j^{-\alpha_j}.$$

If, $w \in C^{\boldsymbol{\alpha}}[0, 1]^I$ for some subset I of $\{1, \ldots, d\}$ with $|I| = \tilde{d}$, so that $w(t) = w_0(t_I)$ for some $w_0 \in C^{\boldsymbol{\alpha}_I}[0, 1]^{\tilde{d}}$, then the conclusion follows trivially follows from the observation $\psi_{\mathbf{a}} * w = \psi_{\mathbf{a}_I} * w_0$. $\quad \square$

We next study the centered small ball probability of the rescaled process and the metric entropy of the unit RKHS ball.

LEMMA 4.3.   *For any a_0 positive, there exists constants C and $\varepsilon_0 > 0$ such that for $\mathbf{a} \geq a_0$ and $\varepsilon < \varepsilon_0$,*

$$-\log \mathrm{P}(\|W^{\mathbf{a}}\|_{\infty} \leq \varepsilon) \leq C\mathbf{a}^* \left(\log \frac{\bar{\mathbf{a}}}{\varepsilon} \right)^{d+1}.$$

PROOF.   This follows from Theorem 2 in Kuelbs and Li (1993) and Lemma 4.6 in van der Vaart and van Zanten (2009). Proceeding as in Lemma 4.6 in van der Vaart and van Zanten (2009) and Lemma 4.4, we obtain

$$(4.1) \qquad\qquad \phi^{\mathbf{a}}(\varepsilon) + \log 0.5 \leq K_1 \mathbf{a}^* \left(\log \frac{\phi_0^{\mathbf{a}}(\varepsilon)}{\varepsilon} \right)^{1+d}$$

for some constant $K_1 > 0$. Note that with $L = [0, a_1] \times \cdots \times [0, a_d]$,

$$\phi_0^{\mathbf{a}}(\varepsilon) = -\log P(\|W^{\mathbf{a}}\|_{\infty} \leq \varepsilon) = -\log P\left(\sup_{t \in L} |W_t| \leq \varepsilon \right)$$

$$\leq -\log P\left(\sup_{t \in [0, \bar{\mathbf{a}}]^d} |W_t| \leq \varepsilon \right) \leq K_2 \left(\frac{\bar{\mathbf{a}}}{\varepsilon} \right)^{\tau}$$

for some constant K_2 and $\tau > 0$, where the last inequality follows from the proof of Lemma 4.6 in van der Vaart and van Zanten (2009). Inserting this bound in (4.1), we obtain the desired result. $\quad \square$

Let $\mathbb{H}_1^{\mathbf{a}}$ denote the unit ball in the RKHS of $W^{\mathbf{a}}$.

LEMMA 4.4. *There exists a constant K, depending only on v and d, such that, for $\varepsilon < 1/2$,*

$$\log N\big(\varepsilon, \mathbb{H}_1^{\mathbf{a}}, \|\cdot\|_\infty\big) \le K\mathbf{a}^*\left(\log \frac{1}{\varepsilon}\right)^{d+1}.$$

PROOF. By Lemma 4.1, an element of $\mathbb{H}_1^{\mathbf{a}}$ can be written as the real part of the function $h:[0,1]^d \to \mathbb{C}$ given by

$$(4.2) \qquad h(t) = \int e^{i(\lambda,t)} \psi(\lambda) v_{\mathbf{a}}(d\lambda)$$

for $\psi : \mathbb{R}^d \to \mathbb{C}$ a function with $\int |\psi(\lambda)|^2 v_{\mathbf{a}}(d\lambda) \le 1$.

For $z \in \mathbb{C}^d$, continue to denote the function $z \mapsto \int e^{(\lambda,z)} \psi(\lambda) v_a(d\lambda)$ by h. Using the Cauchy–Schwarz inequality and the change of variable theorem,

$$(4.3) \qquad |h(z)|^2 \le \int e^{(\lambda, 2\mathbf{a}\cdot\mathrm{Re}(z))} v(d\lambda),$$

where $\mathrm{Re}(z)$ denotes the vector whose jth element is the real part of z_j for $j = 1, \ldots, d$, and $\mathbf{a} \cdot \mathrm{Re}(z) = (a_1 \mathrm{Re}(z_1), \ldots, a_d \mathrm{Re}(z_d))^{\mathrm{T}}$. From (4.3) and the dominated convergence theorem, any $h \in \mathbb{H}_1^{\mathbf{a}}$ can be analytically extended to $\Gamma = \{z \in \mathbb{C}^d : \|2\mathbf{a}\cdot\mathrm{Re}(z)\|_2 < \delta\}$. Clearly, Γ contains a strip Ω in $\mathbb{C}^d$ given by $\Omega = \{z \in \mathbb{C}^d : |\mathrm{Re}(z_j)| \le R_j, j = 1, \ldots, d\}$ with $R_j = \delta/(6a_j\sqrt{d})$. Also, for every $z \in \Omega$, h satisfies the uniform bound $|h(z)|^2 \le \int e^{\delta\|\lambda\|} v(d\lambda) = C^2$.

Let $R = (R_1, \ldots, R_d)^{\mathrm{T}}$. Partition $T = [0,1]^d$ into rectangles $\Gamma_1, \ldots, \Gamma_m$ with centers $\{t_1, t_2, \ldots, t_m\}$ such that given any $z \in T$, there exists Γ_j with center $t_j = (t_{j1}, \ldots, t_{jd})^{\mathrm{T}}$ with $|z_i - t_{ji}| \le R_i/4, i = 1, \ldots, d$. Consider the piecewise polynomials $P = \sum_{j=1}^m P_{j,\gamma_j} 1_{\Gamma_j}$ with

$$P_{j,\gamma_j}(t) = \sum_{n.\le k} \gamma_{j,n}(t - t_j)^n.$$

A finite set of functions $\mathcal{P}_{\mathbf{a}}$ is obtained by discretizing the coefficients $\gamma_{j,n}$ for each j and n over a grid of mesh width ε/R^n in the interval $[-C/R^n, C/R^n]$, with $R^n = R_1^{n_1} \cdots R_d^{n_d}$ and C defined as above. Choosing $m \precsim 1/R^*$ and k such that $k \precsim \log(1/\varepsilon)$ and $(2/3)^k \le K\varepsilon$ for some constant K, the collection $\mathcal{P}_{\mathbf{a}}$ can be shown to form a $K\varepsilon$-net to $\mathbb{H}_1^{\mathbf{a}}$ using (4.4) and (4.5):

$$(4.4) \qquad \left|\sum_{n.>k} \frac{D^n h_\psi(t_i)}{n!}(z - t_i)^n\right| \le \sum_{n.>k} \frac{C}{R^n}(R/2)^n \le KC\left(\frac{2}{3}\right)^k,$$

$$(4.5) \qquad \left|\sum_{n.\le k} \frac{D^n h_\psi(t_i)}{n!}(z - t_i)^n - P_{i,\gamma_i}(z)\right| \le K\varepsilon.$$

The details are similar to Lemma 4.5 in van der Vaart and van Zanten (2009), and hence omitted. □

Lemma 4.7 of van der Vaart and van Zanten (2009) exploited a containment relation among unit RKHS balls with different scalings to construct the sieves B_n. Such a result sufficed in the single bandwidth case exploiting the ordering of $\mathbb{R}_+$. However, the result can only be generalized with respect to the partial order on $\mathbb{R}_+^d$ and one does not obtain a straightforward generalization of their sieve construction in the multi-bandwidth case since the entropy of their sieve blows up in trying to control the joint probability of the rescaling vector $\mathbf{a}$ outside a hyperrectangle in $\mathbb{R}_+^d$.

The problem mentioned above is fundamentally due to the curse of dimensionality and one needs a more careful construction of the sieve to avoid this problem. In the proof of Lemma 4.4, a collection of piece-wise polynomials is used to cover the unit RKHS ball $\mathbb{H}_1^{\mathbf{a}}$. The main idea in the next Lemma 4.5 is to exploit the fact that the same set of piecewise polynomials can also be used to cover $\mathbb{H}_1^{\mathbf{b}}$ for $\mathbf{b}$ sufficiently close to $\mathbf{a}$. We then come up with a careful choice of a compact subset $\mathcal{Q}$ of $\mathbb{R}_+^d$ that balances the metric entropy of the collection of unit RKHS balls $\mathbb{H}_1^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{Q}$ and the complement probability of $\mathcal{Q}$ under the joint prior on $\mathbf{a}$.

For $\mathbf{u} \in \mathbb{R}_+^d$, let $\mathcal{C}_{\mathbf{u}}$ denote the rectangle in the positive quadrant given by $\mathbf{a} \le \mathbf{u}$, that is, $0 \le a_j \le u_j$ for all $j = 1, \ldots, d$. For a fixed $r > 1$, let $\mathcal{Q}^{(r)}$ consist of vectors $\mathbf{a}$ with $a_j \le r^{\theta_j}$ for some $\theta \in \mathcal{S}_{d-1}$. It is easy to see that $\mathcal{Q}^{(r)}$ is a union of rectangles $\mathcal{C}_{r^\theta}$ with θ varying over $\mathcal{S}_{d-1}$,

$$\mathcal{Q}^{(r)} = \bigcup_{\theta \in \mathcal{S}_{d-1}} \mathcal{C}_{r^\theta}.$$

The outer boundary of $\mathcal{Q}^{(r)}$ consists of points $\mathbf{a}$ with $0 < a_j \le r$ for all $j = 1, \ldots, d$ and $\mathbf{a}^* = r$ (see Figure 1). By Lemma 4.4, for any such $\mathbf{a}$ in the outer boundary of $\mathcal{Q}^{(r)}$, the metric entropy of $\mathbb{H}_1^{\mathbf{a}}$ is bounded by a constant multiple of $r \log^{d+1}(1/\varepsilon)$. Our techniques were motivated by our observation that the entropy
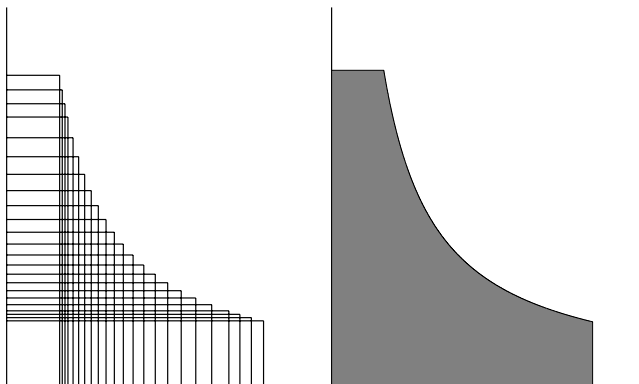

FIG. 1. *Left panel: for $d = 2$ and fixed $r > 1$, rectangles $\mathcal{C}_{r^\theta} = \{0 \le \mathbf{a} \le r^\theta\}$ for different values of $\theta \in \mathcal{S}_{d-1}$. Right panel: the region $\mathcal{Q}$ (shaded) resulting from the union of all such rectangles.*

remains of the same order even if one considers a union over the outer boundary of $\mathcal{Q}^{(r)}$. We present a stronger result in Lemma 4.5 which further states that the entropy remains of the same order even if the union is considered over all of $\mathcal{Q}^{(r)}$.

LEMMA 4.5.   *Let v satisfy* (3.1). *Fix $r \geq 1$. Then there exists a constant K depending on v and d only, so that, for $\varepsilon < 1/2$,*

$$\log N\left(\varepsilon, \bigcup_{\mathbf{a} \in \mathcal{Q}^{(r)}} \mathbb{H}_1^{\mathbf{a}}, \|\cdot\|_\infty\right) \leq Kr\left(\log\frac{1}{\varepsilon}\right)^{d+1}.$$

PROOF.   Let $\mathcal{Q} = \mathcal{Q}^{(r)}$. Fix $\mathbf{a} \in \mathcal{Q}$. Let I denote the subset of $\{1, \ldots, d\}$ such that $a_j \leq 1$ for all $j \in I$ and $a_j > 1$ for all $j \notin I$. Let $\Omega^{\mathbf{a}} = \{z \in \mathbb{C}^d : |\operatorname{Re}(z_j)| \leq R_j, j = 1, \ldots, d\}$ with $R_j = \delta/(6a_j\sqrt{d})$ if $j \notin I$ and $R_j = \delta/(6\sqrt{d})$ if $j \in I$. Note that for $z \in \Omega^{\mathbf{a}}$,

$$\|2\mathbf{a} \cdot \operatorname{Re}(z)\|^2 = \sum_{j=1}^d 4a_j^2|\operatorname{Re}(z_j)|^2$$

(4.6)

$$\leq \sum_{j \in I} 4R_j^2 + \sum_{j \notin I} 4a_j^2 R_j^2 \leq \frac{\delta^2}{9}.$$

Hence, $\|2\mathbf{a} \cdot \operatorname{Re}(z)\| \leq \delta/3$ for $z \in \Omega^{\mathbf{a}}$. Following the argument after the display in (4.3), it thus follows that any function $h \in \mathbb{H}_1^{\mathbf{a}}$ has an analytic extension to $\Omega^{\mathbf{a}}$. Let $\mathbf{b} \in \mathcal{Q}$ satisfy $\max_j |a_j - b_j| \leq 1$. We shall exhibit that any $h \in \mathbb{H}_1^{\mathbf{b}}$ can also be extended analytically to the same strip $\Omega^{\mathbf{a}}$ by showing that $\|2\mathbf{b} \cdot \operatorname{Re}(z)\| < \delta$ on $\Omega^{\mathbf{a}}$. To that end, for $z \in \Omega^{\mathbf{a}}$, first observe that

(4.7) $$\|2\mathbf{b} \cdot \operatorname{Re}(z)\| \leq \|2\mathbf{a} \cdot \operatorname{Re}(z)\| + \|2(\mathbf{b} - \mathbf{a}) \cdot \operatorname{Re}(z)\|.$$

The first term in the right-hand side above is bounded by $\delta/3$ following (4.6). To tackle the second term, we use $|b_j - a_j| \leq 1 \leq a_j$ for $j \notin I$ to conclude

$$\|2(\mathbf{b} - \mathbf{a}) \cdot \operatorname{Re}(z)\|^2 = \sum_{j=1}^d 4(b_j - a_j)^2|\operatorname{Re}(z_j)|^2$$

(4.8)

$$\leq \sum_{j \in I} 4R_j^2 + \sum_{j \notin I} 4a_j^2 R_j^2 \leq \frac{\delta^2}{9}.$$

Combining (4.7) and (4.8), $\|2\mathbf{b} \cdot \operatorname{Re}(z)\| \leq 2\delta/3$ on $\Omega^{\mathbf{a}}$, proving our claim. Since the same tail estimate as in (4.4) works for any $h \in \mathbb{H}_1^{\mathbf{b}}$, it follows from (4.5) that the set of functions $\mathcal{P}_{\mathbf{a}}$ form a $K\varepsilon$-net for $\mathbb{H}_1^{\mathbf{b}}$.

Let $\mathcal{A}$ be a set of points in $\mathcal{Q}$ such that for any $\mathbf{b} \in \mathcal{Q}$, there exists $\mathbf{a} \in \mathcal{A}$ such that $\max_j |a_j - b_j| \leq 1$. One can clearly find an $\mathcal{A}$ with $|\mathcal{A}| \leq r^d$. The proof is completed by observing that $\bigcup_{\mathbf{a} \in \mathcal{A}} \mathcal{P}_{\mathbf{a}}$ form a $K\varepsilon$ net for $\bigcup_{\mathbf{a} \in \mathcal{Q}} \mathbb{H}_1^{\mathbf{a}}$. $\square$

4.2. *Results for lower bound.*   We now state and prove Lemma 4.6 which enables us to derive a lower bound to the concentration function $\phi^a(\varepsilon)$ for a fixed $a > 0$. This lower bound coupled with the model identifiability of (3.7) results in a lower bound to the posterior concentration rate.

Denote by $\mathbb{H}^a$ the reproducing kernel Hilbert space of the Gaussian process W^a. The key to obtaining a lower bound to the concentration function $\phi^a_{\mu_0}(\varepsilon)$ when μ_0 has anisotropic smoothness (LBT) is to find a lower bound to $\inf_{h \in \mathbb{H}^a : \|h - \mu_0\|_2 \leq \varepsilon} \|h\|^2_{\mathbb{H}^a}/2$.

Let h^a_ψ be the real part of the function $\int e^{i(\lambda, t)} \psi(\lambda) f_a(\lambda) \, d\lambda$, where $f_a(\lambda)$ denotes the spectral density corresponding to the spectral measure ν_a of W^a, so that $f_a(\lambda) = a^{-d} f(\lambda/a)$ with $f(\lambda) = e^{-\|\lambda\|^2/4}/(2^d \pi^{d/2})$. From van der Vaart and van Zanten (2009), the RKHS of $\mathbb{H}^a$ consists of functions h^a_ψ for $\psi \in L_2(\nu_a)$.

LEMMA 4.6.   *If μ_0 satisfies (LBT) for some $\beta > 0$ and $0 < u < 1$, then for some constant $K > 0$,*

$$
\inf_{h \in \mathbb{H}^a : \|h - \mu_0\|_2 \leq \varepsilon} \frac{\|h\|^2_{\mathbb{H}^a}}{2}
$$
$$
\geq \begin{cases} C\varepsilon^2 a^d \exp\{K\varepsilon^{-2/\beta}/(4a^2)\}, & \text{if } a > \varepsilon^{-(2-u)/(2\beta)}, \\ C\varepsilon^2 a^d \exp\{K\varepsilon^{-u/\beta}\}, & \text{if } a \leq \varepsilon^{-(2-u)/(2\beta)}. \end{cases}
$$

PROOF.   Let r be a function such that r is equal to 1 on the support of μ_0, has itself support inside $[0, 1]^d$, $|r(\lambda)| \leq 1$ and the Fourier transform $|\hat{r}(\lambda)| \leq e^{-K\|\lambda\|^u}$ for large $\|\lambda\|$. For any $0 < u < 1$, such an r exists; we provide a construction for $u = 1/2$ in Proposition B.1 in Appendix B. Let $\psi \in L_2(\nu_a)$ be such that $\|h^a_\psi - \mu_0\|_2 < \varepsilon$. By construction, $h^a_\psi r$ has support inside $[0, 1]^d$ and $\mu_0 r = \mu_0$, so that $\|h^a_\psi r - \mu_0\|_{2,\mathbb{R}} \leq \|h^a_\psi - \mu_0\|_2$, where $\|\cdot\|_{2,\mathbb{R}}$ is the norm of $L_2(\mathbb{R}^d)$ and $\|\cdot\|_2$ the norm of $L_2[0, 1]^d$. The function $h^a_\psi r$ has Fourier transform[4] $(\psi_1 f_a) * \hat{r}$, where $\psi_1(\lambda) = \psi(-\lambda)$. Hence, by Parseval's identity $\|(\psi_1 f_a) * \hat{r} - \hat{\mu}_0\|_{2,\mathbb{R}} < \varepsilon$. Defining $\chi_{K_1, K_2} = I_{\{\lambda \in \mathbb{R}^d : \|\lambda_{I_1}\| > K_1, \|\lambda_{I_2}\| \in [K_2, 2K_2]\}}$, we have

$$
\|\psi_1 f_a * \hat{r} \chi_{2K_1, K_2}\|_{2,\mathbb{R}} \geq \|\hat{\mu}_0 \chi_{2K_1, K_2}\|_{2,\mathbb{R}} - \varepsilon.
$$

We also have from (LBT),

$$
\|\hat{\mu}_0 \chi_{2K_1, K_2}\|^2_{2,\mathbb{R}} \geq C\left(\frac{1}{K_1}\right)^{2\beta} \exp\{-2K_2^u\}.
$$

---

[4]According to our convention, $\widehat{f * g} = (2\pi)^d \hat{f} \hat{g}$. We drop the constant $(2\pi)^d$ for notational simplicity.

Using the inequalities in the previous two displays and Proposition C.1 in Appendix C with $2K_1$ instead of K_1, $A_1 = K_1$, $f = \psi_1 f_a$ and $g = \hat{r}$,

$$(4.9) \quad \|\psi_1 f_a \chi_{K_1,\cdot}\|_{2,\mathbb{R}} \int_{\|t_{I_1}\| \le K_1} |\hat{r}(t)| \, dt$$

$$\ge C\left(\frac{1}{K_1}\right)^\beta e^{-K_2^u} - \varepsilon - \|\psi_1 f_a\|_{2,\mathbb{R}} \int_{\|t_{I_1}\| > K_1} |\hat{r}(t)| \, dt.$$

Since $\|h_\psi^a\|_{\mathbb{H}^a} = \|\psi \sqrt{f_a}\|_{2,\mathbb{R}}$ and f_a is symmetric about zero,

$$(4.10) \qquad \|\psi_1 f_a\|_{2,\mathbb{R}} \le C a^{-d/2} \|h_\psi^a\|_{\mathbb{H}^a}.$$

Also,

$$(4.11) \quad \|\psi_1 f_a \chi_{K_1,\cdot}\|_{2,\mathbb{R}}^2 = \int_{\|\lambda_{I_1}\| > K_1} (\psi_1^2 f_a) f_a \le \left(\sup_{\|\lambda_{I_1}\| > K_1} f_a\right) \|h_\psi^a\|_{\mathbb{H}^a}^2$$

$$\le C a^{-d} e^{-K_1^2/(4a^2)} \|h_\psi^a\|_{\mathbb{H}^a}^2.$$

Equations (4.10) and (4.11) imply

$$(4.12) \qquad \|h_\psi^a\|_{\mathbb{H}^a} \ge \frac{a^{d/2}(C(1/K_1)^\beta e^{-K_2^u} - \varepsilon)}{e^{-K_1^2/(8a^2)} \int_{\|t_{I_1}\| \le K_1} |\hat{r}(t)| \, dt + \int_{\|t_{I_1}\| > K_1} |\hat{r}(t)| \, dt}.$$

For large K_1, $\int_{\|t_{I_1}\| > K_1} |\hat{r}(t)| \, dt \le \int_{\|t\| > K_1} |\hat{r}(t)| \, dt \le \int_{\|t\| > K_1} e^{-K\|t\|^u} \, dt = C \int_{K_1}^\infty r^{d-1} e^{-Kr^u} \, dr \le e^{-KK_1^u}$ and $\int_{\|t_{I_1}\| \le K_1} |\hat{r}(t)| \, dt \le \int |\hat{r}(t)| \, dt \le C$. We can thus bound the denominator in (4.12) from above by $Ce^{-K_1^2/(8a^2)} + e^{-KK_1^u}$. For fixed K_1, $Ce^{-K_1^2/(8a^2)} < e^{-KK_1^u} \Leftrightarrow a \le C'K_1^{(2-u)/2}$. Hence, the denominator can be bounded above by $2Ce^{-K_1^2/(8a^2)}$ or $2e^{-KK_1^u}$ depending on whether a is, respectively, larger or smaller than $C'K_1^{(2-u)/2}$.

Fix K_2 to be a large number. Choose K_1 large enough depending on ε such that $C(1/K_1)^\beta e^{-cK_2^u} = 2\varepsilon$; this implies $K_1 = (C/\varepsilon)^{1/\beta}$ for some $C > 0$. With this choice, $K_1^{(2-u)/2} = C(1/\varepsilon)^{(2-u)/(2\beta)}$, $e^{-KK_1^u} = \exp\{-K(1/\varepsilon)^{u/\beta}\}$ and $e^{-K_1^2/(8a^2)} = \exp\{-K(1/\varepsilon)^{2/\beta}/(8a^2)\}$. Substituting the sequence of bounds in (4.12), we have the desired result. $\quad\square$

5. Proof of main results. We shall only provide a detailed proof of Theorem 3.1 and sketch the main steps in the proof of Theorem 3.2.

5.1. *Proof of Theorem* 3.1. Let us begin by observing that

$$P(\|W^{\mathbf{A}} - w_0\|_\infty \le 2\varepsilon) = \int P(\|W^{\mathbf{a}} - w_0\|_\infty \le 2\varepsilon) \pi_{\mathbf{A}}(d\mathbf{a})$$

$$= \int \left\{ \int P(\|W^{\mathbf{a}} - w_0\|_\infty \le 2\varepsilon) \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} \right\} \pi(\theta) \, d\theta.$$

As in van der Vaart and van Zanten (2009), we first derive bounds on the non-centered small ball probability for a fixed rescaling $\mathbf{a}$, and then integrate over the distribution of $\mathbf{a}$ to derive the same for $W^{\mathbf{A}}$.

Given $\mathbf{a} \in \mathbb{R}^d_*$, recall the definition of the centered and noncentered concentration functions of the process $W^{\mathbf{a}}$,

$$
(5.1) \qquad
\begin{aligned}
\phi_0^{\mathbf{a}}(\varepsilon) &= -\log P(\|W^{\mathbf{a}}\|_\infty \le \varepsilon), \\
\phi_{w_0}^{\mathbf{a}}(\varepsilon) &= \inf_{h \in \mathbb{H}^{\mathbf{a}} : \|h - w_0\|_\infty \le \varepsilon} \|h\|_{\mathbb{H}^{\mathbf{a}}}^2 - \log P(\|W^{\mathbf{a}}\|_\infty \le \varepsilon).
\end{aligned}
$$

For a fixed $\mathbf{a}$, the noncentered small ball probability of $W^{\mathbf{a}}$ can be bound in terms of the concentration function as follows [van der Vaart and van Zanten (2008b)]:

$$
P(\|W^{\mathbf{a}} - w_0\|_\infty \le 2\varepsilon) \ge e^{-\phi_{w_0}^{\mathbf{a}}(\varepsilon)}.
$$

Now, suppose that $w_0 \in C^{\boldsymbol{\alpha}}[0,1]^d$ for some $\boldsymbol{\alpha} \in \mathbb{R}^d_*$. From Lemmas 4.2 and 4.3, it follows that for every $a_0 > 0$, there exist positive constants $\varepsilon_0 < 1/2$, C, D and E that depend only on w_0 and ν such that, for $\mathbf{a} > a_0$, $\varepsilon < \varepsilon_0$ and $C \sum_{i=1}^d a_i^{-\alpha_i} < \varepsilon$,

$$
\phi_{w_0}^{\mathbf{a}}(\varepsilon) \le D\mathbf{a}^* + E\mathbf{a}^* \left(\log \frac{\bar{\mathbf{a}}}{\varepsilon} \right)^{1+d} \le K_1 \mathbf{a}^* \left(\log \frac{\bar{\mathbf{a}}}{\varepsilon} \right)^{1+d}
$$

with K_1 depending only on a_0, ν and d. Thus, for $\varepsilon < \min\{\varepsilon_0, C_1 a_0^{-\bar{\alpha}}\}$, by (5.1), for constants $K_2, \ldots, K_6 > 0$ and $C_2, \ldots, C_6 > 0$,

$$
\begin{aligned}
P(\|W^{\mathbf{A}} &- w_0\|_\infty \le 2\varepsilon) \\
&\ge \int_\theta \left\{ \int e^{-\phi_{w_0}^{\mathbf{a}}(\varepsilon)} \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} \right\} \pi(\theta) \, d\theta \\
&\ge \int_\theta \left\{ \int_{a_1 = (C_1/\varepsilon)^{1/\alpha_1}}^{2(C_1/\varepsilon)^{1/\alpha_1}} \cdots \int_{a_d = (C_1/\varepsilon)^{1/\alpha_d}}^{2(C_1/\varepsilon)^{1/\alpha_d}} e^{-K_1 \mathbf{a}^* \log^{1+d}(\bar{\mathbf{a}}/\varepsilon)} \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} \right\} \pi(\theta) \, d\theta \\
&\ge C_2 e^{-K_2 (1/\varepsilon)^{1/\alpha_0} \log^{1+d}(1/\varepsilon)} \\
&\quad \times \int_\theta \left\{ \int_{a_1 = (C_1/\varepsilon)^{1/\alpha_1}}^{2(C_1/\varepsilon)^{1/\alpha_1}} \cdots \int_{a_d = (C_1/\varepsilon)^{1/\alpha_d}}^{2(C_1/\varepsilon)^{1/\alpha_d}} \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} \right\} \pi(\theta) \, d\theta.
\end{aligned}
$$

Let Γ denote the region in the simplex $\mathcal{S}_{d-1}$ given by $\Gamma = \{\theta \in \mathcal{S}_{d-1} : \tau < \theta_j - \frac{\alpha_0}{\alpha_j} < 2\tau, j = 1, \ldots, d-1\}$. Since $\sum_{j=1}^d \alpha_0/\alpha_j = 1$, we can choose $\tau > 0$ small enough to guarantee that any θ satisfying the set of inequalities lies inside the simplex. Moreover, with $\theta_d = 1 - \sum_{j=1}^{d-1} \theta_j$, one has $(d-1)\tau < \theta_d < 2(d-1)\tau$. Choosing $\tau = C_3/\log(1/\varepsilon)$, one can show that $\sum_{j=1}^d (1/\varepsilon)^{1/(\alpha_j \theta_j)} \le C_4 (1/\varepsilon)^{1/\alpha_0}$

for any $\theta \in \Gamma$. Now,

$$\int \left\{ \int_{a_1=(C_1/\varepsilon)^{1/\alpha_1}}^{2(C_1/\varepsilon)^{1/\alpha_1}} \cdots \int_{a_d=(C_1/\varepsilon)^{1/\alpha_d}}^{2(C_1/\varepsilon)^{1/\alpha_d}} \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} \right\} \pi(\theta) \, d\theta$$

$$\geq \int \left\{ \int_{a_1=(C_1/\varepsilon)^{1/\alpha_1}}^{2(C_1/\varepsilon)^{1/\alpha_1}} \cdots \int_{a_d=(C_1/\varepsilon)^{1/\alpha_d}}^{2(C_1/\varepsilon)^{1/\alpha_d}} e^{-\sum_{j=1}^{d} a_j^{1/\theta_j}} \, d\mathbf{a} \right\} \pi(\theta) \, d\theta$$

$$\geq \int e^{-K_3 \sum_{j=1}^{d}(1/\varepsilon)^{1/\alpha_j \theta_j}} \pi(\theta) \, d\theta$$

$$\geq \int_{\theta \in \Gamma} e^{-K_4(1/\varepsilon)^{1/\alpha_0}} \pi(\theta) \, d\theta \geq C_5 e^{-K_5(1/\varepsilon)^{1/\alpha_0}}.$$

The last inequality in the above display uses that Γ contains a hyper-cube of width τ so that its π-mass is at least polynomial in τ. Hence,

$$(5.2) \qquad \mathrm{P}\big(\|W^{\mathbf{A}} - w_0\|_\infty \leq 2\varepsilon\big) \geq C_6 e^{-K_6(1/\varepsilon)^{1/\alpha_0} \log^{1+d}(1/\varepsilon)}.$$

Let $\mathbb{B}_1$ denote the unit sup-norm ball of $C[0,1]^d$. For a vector $\theta \in \mathcal{S}_{d-1}$ and positive constants $M, r > 1, \varepsilon$, let $B^\theta = B^\theta(M, r, \varepsilon)$ denote the set,

$$B^\theta = \bigcup_{\mathbf{a} \leq r^\theta} (M\mathbb{H}_1^{\mathbf{a}}) + \varepsilon\mathbb{B}_1,$$

where r^θ denotes the vector whose jth element is r^{θ_j}. We further let

$$B = \bigcup_{\mathbf{a} \in \mathcal{Q}^{(r)}} (M\mathbb{H}_1^{\mathbf{a}}) + \varepsilon\mathbb{B}_1 = \bigcup_{\theta \in \mathcal{S}_{d-1}} \bigcup_{\mathbf{a} \leq r^\theta} (M\mathbb{H}_1^{\mathbf{a}}) + \varepsilon\mathbb{B}_1.$$

Let us first calculate the probability $\mathrm{P}(W^{\mathbf{A}} \notin B^\theta \mid \theta)$. Note that

$$\mathrm{P}\big(W^{\mathbf{A}} \notin B^\theta \mid \theta\big) = \int \mathrm{P}\big(W^\theta \notin B^\theta\big) \pi(\mathbf{a} \mid \theta) \, d\mathbf{a}$$

$$\leq \int_{\mathbf{a} \leq r^\theta} \mathrm{P}\big(W^{\mathbf{a}} \notin B^\theta\big) \pi(\mathbf{a} \mid \theta) \, d\mathbf{a} + \mathrm{P}\big(\mathbf{A} \not\leq r^\theta \mid \theta\big),$$

where $\mathrm{P}(W^{\mathbf{a}} \not\leq r \mid \theta)$ is a shorthand notation for $\mathrm{P}(\text{at least one } A_j > r^{\theta_j} \mid \theta)$.

To tackle the first term in the last display, note that B^θ contains the set $M\mathbb{H}_1^{\mathbf{a}} + \varepsilon\mathbb{B}_1$ for any $\mathbf{a} \leq r^\theta$. Hence, for any $\mathbf{a} \leq r^\theta$, by Borell's inequality,

$$\mathrm{P}\big(W^{\mathbf{a}} \notin B^\theta\big) \leq \mathrm{P}\big(W^{\mathbf{a}} \notin M\mathbb{H}_1^{\mathbf{a}} + \varepsilon\mathbb{B}_1\big)$$

$$\leq 1 - \Phi\big\{M + \Phi^{-1}\big(e^{-\phi_0^{\mathbf{a}}(\varepsilon)}\big)\big\}$$

$$\leq 1 - \Phi\big\{M + \Phi^{-1}\big(e^{-\phi_0^{r^\theta}(\varepsilon)}\big)\big\} \leq e^{-\phi_0^{r^\theta}(\varepsilon)},$$

if $M \geq -2\Phi^{-1}(e^{-\phi_0^{r^\theta}(\varepsilon)})$. The penultimate inequality in the above display follows from the fact that, with $T = [0,1]^d$,

$$e^{-\phi_0^{\mathbf{a}}(\varepsilon)} = \mathrm{P}\Big(\sup_{t \in \mathbf{a} \cdot T} |W_t| \leq \varepsilon\Big) \geq \mathrm{P}\Big(\sup_{t \in r^\theta \cdot T} |W_t| \leq \varepsilon\Big) = e^{-\phi_0^{r^\theta}(\varepsilon)}.$$

By Lemma 4.10 of van der Vaart and van Zanten (2009), $\Phi^{-1}(u) \geq -\{2\log(1/u)\}^{1/2}$ for $u \in (0,1)$. Hence, the last inequality in the above display remains valid if we choose

$$M \geq 4\sqrt{\phi_0^{r^\theta}(\varepsilon)}.$$

Since A_j^{1/θ_j} follows a gamma distribution given θ_j, in view of Lemma 4.9 of van der Vaart and van Zanten (2009), for r larger than a positive constant depending only on the parameters of the gamma distribution,

$$P(A_j > r^{\theta_j} \mid \theta) \leq C_1 r^{D_1} e^{-D_2 r}.$$

Combining the above, since B contains B^θ for every $\theta \in \mathcal{S}_{d-1}$,

$$P(W^{\mathbf{A}} \notin B) = \int_\theta \left\{ \int P(W^{\mathbf{a}} \notin B \mid \theta) g(\mathbf{a} \mid \theta) \right\}$$

(5.3)
$$\leq \int_\theta \left\{ \int P(W^{\mathbf{a}} \notin B^\theta \mid \theta) g(\mathbf{a} \mid \theta) \right\}$$

$$\leq C_2 r^{D_1} e^{-D_2 r} + e^{-D_3 r \log(r/\varepsilon)^{d+1}}.$$

From Lemma 4.5, the entropy of B can be estimated as

$$\log N(2\varepsilon, B, \|\cdot\|_\infty) \leq \log N\left(\varepsilon, \bigcup_{\theta \in \mathcal{S}_{d-1}} \bigcup_{\mathbf{a} \leq r^\theta} (M\mathbb{H}_1^{\mathbf{a}}), \|\cdot\|_\infty\right)$$

(5.4)
$$\leq Kr \log\left(\frac{M}{\varepsilon}\right)^{d+1}.$$

Thus, (5.2), (5.3) and (5.4) can be simultaneously satisfied if we choose, for constants $\kappa, \kappa_1, \kappa_2 > 0$,

$$\varepsilon_n = n^{-\alpha_0/(2\alpha_0+1)} \log^\kappa(n),$$

$$r_n = n^{1/(2\alpha_0+1)} \log^{\kappa_1}(n),$$

$$M_n = r_n \log^{\kappa_2}(n).$$

5.2. *Proof of Theorem* 3.2. For ease of notation, we shall make the simplifying assumption that the random variable B is degenerate at 1. For $a > 0$ and $S \subset \{1, \ldots, d\}$, let $\mathbb{H}^{a,S}$ denote the RKHS of $W^{\mathbf{a}}$, where $a_j = a$ for $j \in S$ and $a_j = 1$ for $j \notin S$.

For a subset $S \subset \{1, \ldots, d\}$ with $|S| = \tilde{d}$, and given positive constants M, r, ξ, ε, let

$$B_S = B_S(M, r, \xi, \varepsilon)$$

$$= \left[M\left(\frac{r}{\xi}\right)^{\tilde{d}} \mathbb{H}_1^{r,S} + \varepsilon\mathbb{B}_1 \right] \cup \left[\bigcup_{a < \xi} (M\mathbb{H}_1^{a,S}) + \varepsilon\mathbb{B}_1 \right].$$

Since, given S, $A^{\tilde{d}} \sim \text{gamma}(b_1, b_2)$, it can be shown that, for some constant $C_1 > 0$,

$$P(W^{\mathbf{A}} \notin B_S \mid S) \precsim e^{-C_1 r^{\tilde{d}*}}.$$

The dominating term in the ε entropy of B_S is bounded by

$$C_2 r^{d*} \log^{1+\tilde{d}}\left(\frac{C_3 M}{\varepsilon}\right).$$

While calculating the concentration probability around $w_0 \in C^{\boldsymbol{\alpha}}[0, 1]^I$, simply use the fact that $\text{pr}(S = I) > 0$.

Combining the above, the sieves B_n are constructed as

$$B_n = \bigcup_{\tilde{d}=1}^{d} \bigcup_{S\,:\,|S|=\tilde{d}} B_S(M_n^S, r_n^S, \xi_n, \varepsilon_n),$$

where, for constants $\kappa, \kappa_1 > 0$, $\varepsilon_n = n^{-\alpha/(2\alpha+d_0)} \log^\kappa n$, $r_n^S = (n^{d_0/(2\alpha+d_0)})^{1/|S|} \times \log^{\kappa_1}(n)$ and $(M_n^S)^2 = (r_n^S)^{\tilde{d}} \log(r_n^S/\varepsilon_n)$.

5.3. *Proof of Theorem* 3.7. Let $\gamma \in (\gamma_1, \gamma_2)$ satisfy $\beta < \alpha\gamma/\gamma_1$. We show that

$$(5.5) \qquad \frac{P(\|W^A - \mu_0\|_2 < \xi_n)}{P(\|W^A - \mu_0\|_\infty < \varepsilon_n)} \precsim \exp\{-2n\varepsilon_n^2\}$$

for $\xi_n = n^{-\gamma}$ and $\varepsilon_n = n^{-\alpha/(2\alpha+d)} \log^{t_1} n$ for some appropriate constant $t_1 > 0$. It then follows from the proof of Theorem 8 in van der Vaart and van Zanten (2011) that ξ_n is a lower bound to the rate of posterior contraction around μ_0.

We first derive an upper bound to $P(\|W^A - \mu_0\|_2 < \xi)$ for ξ small. Let g denote the density of A induced from (LBP). Clearly, $P(\|W^A - \mu_0\|_2 < \xi) = \int_{a=0}^{\infty} P(\|W^a - \mu_0\|_2 < \xi) g(a)\,da$. First, find $u(\xi)$ sufficiently small and $v(\xi)$ sufficiently large such that

$$(5.6) \qquad \begin{aligned} P(A < u(\xi)) &\le \exp(-C\xi^{-d/\beta}), \\ P(A > v(\xi)) &\le \exp\{-C\xi^{-d/\beta} \log^s(1/\xi)\} \end{aligned}$$

for some $s \in \mathbb{R}$. From (LBP), a simple calculation yields

$$(5.7) \qquad \begin{aligned} P(A < u(\xi)) &\le \exp(-Cu(\xi)^{-d}), \\ P(A > v(\xi)) &\le \exp(-Cv(\xi)^d \log^q v(\xi)). \end{aligned}$$

Hence, if we choose $u(\xi) = \xi^{1/\beta}$, and $v(\xi) = \frac{\xi^{-1/\beta}}{2\sqrt{2}\log^{1/2}(1/\xi)}$, then (5.6) is satisfied with $s = q - d/2$.

For $a \in (u(\xi), v(\xi))$, we bound the noncentered small ball probability $P(\|W^a - \mu_0\|_2 < \xi)$ above by $\exp\{-\phi_{\mu_0}^a(\xi)\}$ [Lemma 5.3 of van der Vaart and van Zanten (2008b)] and further invoke the lower bound to the concentration function $\phi_{\mu_0}^a(\xi)$ developed in Lemma 4.6. Specifically, we subdivide $(u(\xi), v(\xi))$ into two disjoint regions based on the conclusion of Lemma 4.6 with $u = 1/2$. If

$a \in (u(\xi), \xi^{-3/(4\beta)})$, Lemma 4.6 implies

$$(5.8) \qquad \phi_{\mu_0}^a(\xi) \geq \xi^{2+d/\beta} \exp\{K\xi^{-1/(2\beta)}\} \geq \xi^{-d/\beta}.$$

If $a \in (\xi^{-3/(4\beta)}, v(\xi))$, then again from Lemma 4.6,

$$(5.9) \qquad \phi_{\mu_0}^a(\xi) \geq C\xi^2 a^d \exp\{\xi^{-2/\beta}/(4a^2)\}.$$

Putting together all the bounds, and noting that $s \geq -d/2$,

$$\int_{a=0}^{\infty} P(\|W^a - \mu_0\|_2 < \xi)g(a)\,da$$

$$(5.10) \qquad \leq 2\exp(-C\xi^{-d/\beta}) + \exp\{-C\xi^{-d/\beta}\log^{-d/2}(1/\xi)\}$$

$$+ \int_{a=\xi^{-3/(4\beta)}}^{v(\xi)} \exp[-C\xi^2 a^d \exp\{\xi^{-2/\beta}/(4a^2)\}]g(a)\,da.$$

Observe that $\psi(x) = x^d \exp\{\xi^{-2/\beta}/(4x^2)\}$ is decreasing if $x \in (0, \xi^{-1/\beta}/\sqrt{2d}]$ and increasing if $x > \xi^{-1/\beta}/\sqrt{2d}$. Since $v(\xi) \leq \xi^{-1/\beta}/\sqrt{2d}$, the third term in the r.h.s. of (5.10) is bounded above by $\exp[-C\xi^2\psi\{v(\xi)\}]$. Since $\psi\{v(\xi)\} = \xi^{-2}v(\xi)^d$, we get

$$\exp[-C\xi^2\psi\{v(\xi)\}] = \exp\{-Cv(\xi)^d\} = \exp\{-C\xi^{-d/\beta}\log^{-d/2}(1/\xi)\}.$$

Substituting this bound in (5.10), we finally obtain

$$(5.11) \qquad P(\|W^A - \mu_0\|_2 \leq \xi) \leq \exp\{-C\xi^{-d/\beta}\log^{-d/2}(1/\xi)\}.$$

From van der Vaart and van Zanten (2009), it also follows that

$$(5.12) \qquad P(\|W^A - \mu_0\|_\infty < \varepsilon_n) \geq \exp\{-n\varepsilon_n^2\}$$

for $\varepsilon_n = n^{-\alpha/(2\alpha+d)}\log^{t_1} n$. Recall $\gamma_1 = \alpha/(2\alpha + d)$. Using (5.11) and (5.12), we obtain with ξ replaced by $\xi_n = n^{-\gamma}$,

$$\frac{P(\|W^A - \mu_0\|_2 < \xi_n)}{P(\|W^A - \mu_0\|_\infty < \varepsilon_n)} \leq \exp\{-(n^{\gamma d/\beta}\log^{-d/2} n - n^{\gamma_1 d/\alpha}\log^{2t_1} n)\}.$$

Since $\gamma/\beta > \gamma_1/\alpha$ by assumption, $n^{\gamma d/\beta} > 3n^{\gamma_1 d/\alpha}\log^{2t_1+d/2} n$ for large enough n. Hence, for large enough n,

$$\frac{P(\|W^A - \mu_0\|_2 < \xi_n)}{P(\|W^A - \mu_0\|_\infty < \varepsilon_n)} \leq \exp\{-2n^{\gamma_1 d/\alpha}\log^{2t_1} n\} = \exp\{-2n\varepsilon_n^2\},$$

proving (5.5).

6. Discussion. We showed that a Gaussian process model with dimensional specific scalings equipped with an appropriately chosen joint prior on the scales can adapt to the true dimensionality or different smoothness levels along different

coordinates of the true function. In some situations, it might be more reasonable to assume the true function to be supported on a smaller dimensional linear subspace. In such cases, a minor modification of our approach can achieve dimension adaptability by incorporating an orthogonal projection of the covariate space as $W^{\mathbf{a},Q}(t) = W(\mathbf{a} \cdot Qt)$ for a $d \times d$ orthogonal matrix Q. Such an approach is recently pursued by Tokdar (2011) which assumes isotropy in the ambient dimensions and uses the same prior as in Section 3.2. One could easily allow anisotropy in the rotated coordinate system using the unified prior in Section 3.3. As a topic for future research, we would like to explore consistent estimation of the dimension of the true subspace and the subspace itself.

A salient feature of our prior (PD1)–(PD4) compared to Zou et al. (2010), Savitsky, Vannucci and Sha (2011) is that the tail heaviness of A is related to the subset size of S. For larger subsets, the tails of A get lighter, resulting in downweighted scalings for larger subsets compared to smaller ones. It would be interesting to explore the implied difference in practical performance from Zou et al. (2010).

APPENDIX A

In this appendix, we provide examples of functions η and ζ which satisfy condition (LBT) in Section 3.5.

EXAMPLE OF η.   We first provide an example of η with $u = 1/2$. Existence of *bump functions* or infinitely smooth compactly supported functions are well known; for example, $\Psi(t) = \exp\{-1/(1 - t^2)\}\chi_{|t|<1}$ is an example of a C^∞ function with support $[-1, 1]$ [Section 13 of Tu (2011)]. From Johnson (2007), $|\hat{\Psi}(\lambda)| \asymp \exp(-C\sqrt{|\lambda|})$ for large $|\lambda|$. Define $\eta : [0, 1] \to \mathbb{R}$ by $\eta(x) = \Psi\{2(x - v)/(w - v) - 1\}$ if $x \in (v, w)$ and zero otherwise, that is, shift and scale Ψ to have support on $[v, w]$. A simple calculation yields $\hat{\eta}(\lambda) = (2\pi a)^{-1}e^{-i\lambda/a}\hat{\Psi}(\lambda/a)$, where $a = 2/(w - v) - 1$ and $b = -2v/(w - v)$. Hence, one also has $|\hat{\eta}(\lambda)| \asymp \exp(-C\sqrt{|\lambda|})$ for large $|\lambda|$. Continue to denote by η the function on $[0, 1]^{d_2}$ given by $\eta(x) = \prod_{j=1}^{d_2} \eta(x_j)$. Clearly, η has support $[v, w]^{d_2}$ and $|\hat{\eta}(\lambda)| \asymp \exp(-C\sqrt{\|\lambda\|})$, since[5] $d_2^{3/4}\|\lambda\|^{1/2} \ge \sum_{j=1}^{d_2} \sqrt{|\lambda_j|} \ge \|\lambda\|^{1/2}$. Transforming to polar coordinates,

$$\int_{\|\lambda\|\in[K_2,2K_2]} |\hat{\eta}(\lambda)|^2 \, d\lambda = C' \int_{K_2}^{2K_2} r^{d_2-1}e^{-2C\sqrt{r}} \, dr$$

$$= C' \int_{\sqrt{K_2}}^{\sqrt{2K_2}} z^{2d_2-1}e^{-2Cz} \, dz \ge e^{-2C\sqrt{K_2}}$$

for large K_2.

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[5]With $a_j = |\lambda_j|^{1/2}$, $(\sum a_j)^4 \ge \sum a_j^4$ and using Cauchy–Schwarz inequality twice, $(\sum a_j)^4 \le d_2^3 \sum a_j^4$.

EXAMPLE OF ζ.   As mentioned in the final paragraph of Section 3.5, we present a concrete example of ζ in the case $d_1 = 1$ with $\alpha = 1$ and $\beta = 3/2$. For ease of notation, we present the example on $[-2, 2]$ with support $[-1, 1]$; linearly transforming to any compact interval which is a subset of $[0, 1]$ does not affect the tail behavior of the Fourier transform.

Define $\zeta : [-2, 2] \to \mathbb{R}$ as $\zeta(x) = (1 - |x|^2)\chi_{|x| \leq 1}$. Observe that $\zeta \in C^1[-2, 2]$, that is, ζ is Lipschitz continuous, since ζ is absolutely continuous with an a.e. bounded derivative. However, $\zeta \notin C^{1+s}[-2, 2]$ for any $s > 0$ as ζ is not differentiable at ± 1. By equation 9.1.20 in Abramowitz and Stegun (1992), $\hat{\zeta}(\lambda) = \sqrt{2/\pi}|\lambda|^{-3/2}J_{3/2}(|\lambda|)$, where $J_v(x)$ is the Bessel function of the first kind of order v. Further, combining equations 10.1.1 and 10.1.11 in Abramowitz and Stegun (1992), $J_{3/2}(x) = \sqrt{2/(\pi x)}(\sin x/x - \cos x)$ for $x > 0$, so that

$$\hat{\zeta}(\lambda) = \left(\frac{2}{\pi}\right)|\lambda|^{-3}\left(\sin|\lambda| - |\lambda|\cos|\lambda|\right).$$

Using $\int_M^\infty x^{-l}\cos^2(x)\,dx$, $\int_M^\infty x^{-l}\sin^2(x)\,dx \asymp M^{-l+1}$ for any $l > 2$ and $M > 0$ large,[6] we have for sufficiently large $K_1 > 0$,

$$\int_{|\lambda| \geq K_1} |\hat{\zeta}(\lambda)|^2\,d\lambda$$
$$= C_1\left\{\int_{K_1}^\infty \lambda^{-6}\sin^2(\lambda)\,d\lambda + \int_{K_1}^\infty \lambda^{-4}\cos^2(\lambda)\,d\lambda - \int_{K_1}^\infty \lambda^{-5}\sin(2\lambda)\,d\lambda\right\}$$
$$\geq C_2 K_1^{-5} + C_3 K_1^{-3} - C_4 K_1^{-4}$$
$$\geq C_5 K_1^{-3}$$

for constants $C_i > 0$, $i = 1, \ldots, 5$. Thus, (LBT) is satisfied with $\beta = 3/2$.

APPENDIX B

PROPOSITION B.1.   *Let $0 < v < w < 1$. There exists a function $r : [0, 1]^d \to \mathbb{R}$ such that $r(t) = 1$ for all $t \in [v, w]^d$, r has support within $[0, 1]^d$ and $|\hat{r}(\lambda)| \leq e^{-K\sqrt{\|\lambda\|}}$ for large $\|\lambda\|$.*

PROOF.   We first construct a C^∞ function $\rho : \mathbb{R} \to [0, 1]$ which is identically 1 on $[-a, a]$ and has support in $[-b, b]$ for some $0 < a < b < 1$. Recall the C^∞ bump function Ψ from Appendix A and set $g(t) = (1/c)\Psi(t/c)$ with $c = (b - a)/2$. Clearly, g is an infinitely smooth function with support $[-c, c]$. Let $\xi : \mathbb{R} \to [0, 1]$ be the indicator function of the interval $[-(b + a)/2, (b + a)/2]$. Define $\rho = g * \xi$. We claim that ρ is a smooth function identically 1 on $[-a, a]$ and has

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[6]See Appendix D.

support in $[-b, b]$. Observe that if $|x| > b$, $\rho(y) = \int_{-\infty}^{\infty} g(y)\xi(x - y)\,dy = 0$ as $|x - y| > (b + a)/2$ if $|y| < c = (b - a)/2$. Also, if $|x| < a$, $|y| \le c$, $|x - y| \le a + c = (a + b)/2$. Hence, $\rho(x) = 1$ if $|x| < a$. It is also easy to show that $\rho : \mathbb{R} \to [0, 1]$ is a C^{∞} function.

Now, map the interval $[-a, a]$ linearly to $[v, w]$ through $x \mapsto px + q$, with $p = (w - v)/(2a)$ and $q = (w + v)/2$. Also, let $w_1 = pb + q$ and $v_1 = -pb + q$. By suitable choice of a and b, we can ensure that $0 < v_1 < w_1 < 1$. Then the function $\tilde{\rho}(t) = \rho\{(t - q)/p\}$ is infinitely smooth, equals 1 on $[v, w]$ and has support in $[v_1, w_1]$. Defining $r(t) = \tilde{\rho}(t_1) \cdots \tilde{\rho}(t_d)$, all assertions of Proposition B.1 are satisfied barring the tail behavior of the Fourier transform which we prove below. Proceeding as in Appendix A, one has

$$|\hat{r}(\lambda)| = \left| \prod_{j=1}^{d} \hat{\tilde{\rho}}(\lambda_j) \right| = C \prod_{j=1}^{d} |\hat{g}(p\lambda_j)| |\hat{\xi}(p\lambda_j)|.$$

Using $|\hat{\Psi}(\lambda)| \asymp \exp(-C\sqrt{|\lambda|})$ and $|\hat{\xi}(\lambda)| \le 1$ for large $|\lambda_j|$, one has $|\hat{r}(\lambda)| \le C \prod_{j=1}^{d} \exp(-K_1\sqrt{|\lambda_j|})$ for constants C, $K_1 > 0$ and for large $|\lambda_j|$. The proof is completed by observing $\sum_{j=1}^{d} \sqrt{|\lambda_j|} \ge \|\lambda\|^{1/2}$. $\quad\square$

APPENDIX C

This is a modified version of Lemma 16 in van der Vaart and van Zanten (2011); the proof is a simple extension, and hence omitted.

PROPOSITION C.1. *For arbitrary functions* $f, g : \mathbb{R}^d \to \mathbb{R}$, $I \subset \{1, \ldots, d\}$, $J = \{1, \ldots, d\} \setminus I$; χ_{K_1, K_2} *and* $\chi_{K_1, \cdot}$ *the indicator functions of* $\{\lambda \in \mathbb{R}^d : \|\lambda_I\| > K_1, \|\lambda_J\| \in [K_2, 2K_2]\}$ *and* $\{\lambda \in \mathbb{R}^d : \|\lambda_I\| > K_1\}$, *respectively, and* $0 < A_1 < K_1$,

(C.1)
$$\|f \chi_{K_1 - A_1, \cdot}\|_{2,\mathbb{R}} \int_{\|t_I\| \le A_1} |g(t)|\,dt$$
$$\ge \|(f * g)\chi_{K_1, K_2}\|_{2,\mathbb{R}} - \|f\|_{2,\mathbb{R}} \int_{\|t_I\| > A_1} |g(t)|\,dt.$$

APPENDIX D

We show that for any $l > 2$ and $M > 0$ large, $\int_M^{\infty} x^{-l} \sin^2(x)\,dx \asymp M^{-l+1}$. The upper bound is immediate and we focus on the lower bound. Without loss of generality, assume $M = \pi m$ for some positive integer m, so that it is enough to consider $\mathrm{I} = \int_m^{\infty} x^{-l} \sin^2(\pi x)\,dx$. Write $\mathrm{I} = \sum_{j=m}^{\infty} \int_j^{j+1} x^{-l} \sin^2(\pi x)\,dx$. Noting that $\sin^2(\pi x)$ can be bounded below by $1/2$ on $[j + 1/4, j + 3/4]$ for any $j \ge 1$, we have $\mathrm{I} \gtrsim \sum_{j=m}^{\infty} b_j$, where $b_j = (j + 1/4)^{-l+1} - (j + $

$3/4)^{-l+1}$. Noting that $(j + 3/4)^{l-1} - (j + 1/4)^{l-1} \geq (j + 3/4)^{l-2}/2$, we have $b_j \geq (j + 3/4)^{-l}/2 \geq (2j)^{-l}/2$. Hence, $\sum_{j=m}^{\infty} b_j \gtrsim \sum_{j=m}^{\infty} \int_j^{j+1} j^{-l} \, dx \geq \sum_{j=m}^{\infty} \int_j^{j+1} x^{-l} \, dx \geq \int_m^{\infty} x^{-l} \, dx = m^{-l+1}/(l - 1)$.

Along similar lines, we can show that $\int_M^{\infty} x^{-l} \cos^2(x) \, dx \asymp M^{-l+1}$.

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