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RANDOM NORMAL MATRICES AND WARD IDENTITIES

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We consider the random normal matrix ensemble associated with a potential in the plane of sufficient growth near infinity. It is known that asymptotically as the order of the random matrix increases indefinitely, the eigenvalues approach a certain equilibrium density, given in terms of Frostman's solution to the minimum energy problem of weighted logarithmic potential theory. At a finer scale, we may consider fluctuations of eigenvalues about the equilibrium. In the present paper, we give the correction to the expectation of the fluctuations, and we show that the potential field of the corrected fluctuations converge on smooth test functions to a Gaussian free field with free boundary conditions on the droplet associated with the potential.

1. Summary. Given a suitable real-valued "weight function" Q in the plane, it is understood how to associate a corresponding (weighted) random normal matrix ensemble (in short: RNM-ensemble). Under reasonable conditions on Q, the eigenvalues of matrices picked randomly from the ensemble will condensate on a certain compact subset $S = S_Q$ of the complex plane, as the order of the matrices tends to infinity. The set S is called the *droplet* of the ensemble. It is well known that the droplet may be obtained in terms of weighted logarithmic potential theory and, that in its turn, the droplet determines the classical equilibrium distribution of the eigenvalues (Frostman's equilibrium measure).

In this paper, we obtain a central limit theorem for the fluctuations about the equilibrium distribution of linear statistics for the eigenvalues of random normal matrices. We also prove the convergence of the potential fields corresponding to corrected fluctuations to a Gaussian free field on S with free boundary conditions.

Our proof is based on the application of the Ward identities for the density field of the point-process of the eigenvalues. These identities are derived from the reparameterization invariance of the partition function and involve the joint intensities of the process; they are also known as the "loop equation." In their exact form, the Ward identities do not provide a closed system of equations, but combined

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with suitable a priori estimates of "error terms," we may derive the limit form of the equation, which in fact characterizes the entire fluctuation field as $n \to +\infty$, where n is the number of eigenvalues. This approach is certainly well known in physics literature, but its mathematical implementation—namely the justification of the a priori bounds—is a rather delicate matter. Our main source of inspiration is Johansson's paper [15], which treats the case of Hermitian matrices (and more general one-component plasma β-ensembles on the line).

While this paper follows the same general strategy as Johansson [15], we should emphasize that the nature of the estimates (and the resulting formulae) in the complex case is very different than the Hermitian case. On the one hand, Johansson's argument uses the Christoffel–Darboux formula for the reproducing kernel, which is a consequence of the three-term recursion formula for orthogonal polynomials on the real line (see, e.g., [20]) and has no analogue in the complex case; moreover, he assumes that the weight is polynomial. On the other hand, we have to deal with the fact that the droplet now has a nonempty interior in the complex plane. To this end, we may apply (as we did in [3]), the techniques of Bergman kernel asymptotics, but the control of the Bergman kernel is not so good near the boundary of the droplet, so the main focus of this paper will be on the study of the boundary terms, where we have to combine, in a nontrivial way, the global relations derived from the Ward identities with the local interior information from the Bergman kernel expansion theory. We remark that in [3], only the fluctuations in the interior of the droplet were studied.

2. Random normal matrix ensembles.

2.1. *Notational conventions.* By $\mathbb{D}(a, r)$ we mean the open Euclidean disk with center a and radius r. The special case $\mathbb{D}(0, 1)$ is simplified to $\mathbb{D}$. By $\mathrm{dist}_{\mathbb{C}}$, we mean the Euclidean distance in the complex plane $\mathbb{C}$. If A_n and B_n are expressions depending on a positive integer n, we write $A_n \lesssim B_n$ to indicate that $A_n \le C B_n$ for all n large enough where C is independent of n, and usually also independent of other relevant parameters. The notation $A_n \asymp B_n$ means that $A_n \lesssim B_n$ and $B_n \lesssim A_n$. If $z = x + \mathrm{i}y$ is the decomposition of a complex number into real and imaginary parts, we introduce the following notational conventions. We write $\partial = \partial_z := \frac{1}{2}(\partial/\partial x - \mathrm{i}\partial/\partial y)$ and $\bar\partial = \bar\partial_z := \frac{1}{2}(\partial/\partial x + \mathrm{i}\partial/\partial y)$ for the usual complex derivatives. We also write $\Delta := \Delta_z = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for the Laplacian, and introduce the notation $\mathit{\Delta} = \mathit{\Delta}_z := \frac{1}{4}\Delta_z$ for a quarter of the usual Laplacian, because it will appear many times naturally as a consequence of $\mathit{\Delta}_z = \partial_z\bar\partial_z$. The ∇ operator is defined by $\nabla f := (\partial f/\partial x, \partial f/\partial y)$, so that ∇f becomes $\mathbb{C}^2$-valued when f is $\mathbb{C}$-valued and differentiable. It is easy to check that

$$|\nabla f|^2 = 2\big(|\partial f|^2 + |\bar\partial f|^2\big).$$

We let $\mathrm{d}A(z) = \mathrm{d}^2 z = \mathrm{d}x\,\mathrm{d}y$ denote the area measure in the complex plane $\mathbb{C}$. Given suitable functions f and g, such that $fg \in L^1(\mathbb{C})$, we write

$$\langle f, g\rangle_{\mathbb{C}} := \int_{\mathbb{C}} fg\,\mathrm{d}A.$$

We will at times understand this bilinear form $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ more liberally, and think of f as a test function and g as a distribution. In particular, when f is continuous and μ is a Borel measure with $f \in L^1(\mathbb{C}, |\mu|)$, we write

$$\langle f, \mu\rangle_{\mathbb{C}} := \int_{\mathbb{C}} f\,\mathrm{d}\mu.$$

If Γ is a rectifiable curve in $\mathbb{C}$, we let $\mathrm{d}s = \mathrm{d}s_\Gamma$ denote arc length measure along Γ, and for suitable functions f, g we write

$$\langle f, g\rangle_\Gamma := \int_\Gamma fg\,\mathrm{d}s.$$

2.2. *The distribution of eigenvalues.* Let $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ be a suitable lower semi-continuous function subject to the growth condition

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 1. \tag{2.1}$$

We refer to Q as the *weight function* or the *potential*.

Let $\mathrm{NM}[n]$ be the set of all $n \times n$ normal matrices M, that is, matrices with $MM^* = M^*M$. The *partition function* on $\mathrm{NM}[n]$ associated with Q is the function

$$\mathcal{Z}_n = \int_{\mathrm{NM}[n]} \mathrm{e}^{-2n\,\mathrm{trace}[Q(M)]}\,\mathrm{d}M_n,$$

where $\mathrm{d}M_n$ is the Riemannian volume form on $\mathrm{NM}[n]$ inherited from the space $\mathbb{C}^{n^2}$ of all $n \times n$ matrices, and where $\mathrm{trace}[Q] : \mathrm{NM}[n] \to \mathbb{R} \cup \{+\infty\}$ is the random variable

$$\mathrm{trace}[Q](M) := \sum_{\lambda_j \in \mathrm{spec}(M)} Q(\lambda_j),$$

that is, it is the usual trace of the matrix $Q(M)$. We equip $\mathrm{NM}[n]$ with the probability measure

$$\mathrm{d}\,\mathrm{Prob}_{\mathrm{NM}[n]} := \frac{1}{\mathcal{Z}_n}\mathrm{e}^{-2n\,\mathrm{trace}[Q](M)}\,\mathrm{d}M_n,$$

and speak of the *random normal matrix ensemble* or "RNM-ensemble" associated with Q. The measure $\mathrm{Prob}_{\mathrm{NM}[n]}$ induces a probability measure Prob_n on the space $\mathbb{C}^n$ of eigenvalues, which is known as the *density of states* in external field Q; it is given by

$$\mathrm{d}\,\mathrm{Prob}_n(\lambda) := \frac{1}{Z_n}\mathrm{e}^{-H_n(\lambda)}\,\mathrm{d}A^{\otimes n}(\lambda), \qquad \lambda = (\lambda_j)_1^n \in \mathbb{C}^n. \tag{2.2}$$

Here, we have put

$$H_n(\lambda) := \sum_{j,k:j\neq k} \log \frac{1}{|\lambda_j - \lambda_k|} + 2n \sum_{j=1}^{n} Q(\lambda_j), \tag{2.3}$$

while $\mathrm{d}A^{\otimes n}(\lambda) = \mathrm{d}A(\lambda_1) \cdots \mathrm{d}A(\lambda_n)$ denotes Lebesgue measure in $\mathbb{C}^n$ and Z_n is the normalization constant giving Prob_n unit mass. By a slight abuse of language, we will refer to Z_n as the partition function of the ensemble.

We notice that H_n is the energy (Hamiltonian) of a system of n identical point charges in the plane located at the points λ_j, under the influence of the external field $2nQ$. In this interpretation, Prob_n is the law of the Coulomb gas in the external magnetic field $2nQ$ (at inverse temperature $\beta = 2$). In particular, this explains the repelling nature of the eigenvalues of random normal matrices; they tend to be very spread out in the vicinity of the droplet, just like point charges would.

Let us consider the n-point configuration ("set" with possible repeated elements) $\{\lambda_j\}_1^n$ of eigenvalues of a normal matrix picked randomly with respect to $\mathrm{Prob}_{\mathrm{NM}[n]}$. In an obvious manner, the measure Prob_n induces a probability law on the n-point configuration space; this is the law of the *n-point process* $\Lambda_n = \{\lambda_j\}_1^n$ associated to Q.

It is well known that the process Λ_n is *determinantal*. This means that there exists an Hermitian function K_n, called the *correlation kernel of the process* such that the density of states can be represented in the form

$$\mathrm{d}\,\mathrm{Prob}_n(\lambda) = \frac{1}{n!} \det\big[\mathrm{K}_n(\lambda_j, \lambda_k)\big]_{j,k=1}^n \,\mathrm{d}A^{\otimes n}(\lambda), \qquad \lambda \in \mathbb{C}^n.$$

Here, we have

$$\mathrm{K}_n(z, w) = \mathrm{k}_n(z, w)\mathrm{e}^{-n(Q(z)+Q(w))},$$

where k_n is the reproducing kernel of the space $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$, that is, the space of all analytic polynomials of degree at most $n-1$ with norm induced from the usual L^2-space on $\mathbb{C}$ associated with the weight function e^{-2nQ}. Alternatively, we can regard K_n as the reproducing kernel for the subspace

$$L^2_{n,Q}(\mathbb{C}) := \{p\mathrm{e}^{-nQ} : p \text{ is an analytic polynomial of degree less than } n\} \subset L^2(\mathbb{C});$$

in particular, we have the frequently useful identities

$$f(z) = \int_{\mathbb{C}} f(w)\overline{\mathrm{K}_n(w, z)}\,\mathrm{d}A(w), \qquad f \in L^2_{n,Q}(\mathbb{C}),$$

and

$$\int_{\mathbb{C}} \mathrm{K}_n(z, z)\,\mathrm{d}A(z) = n.$$

We refer to [8, 10, 16, 21] for more details on point-processes and random matrices.

2.3. *The equilibrium measure and the droplet.* We are interested in the asymptotic distribution of eigenvalues as n, the order of the random matrix, increases indefinitely. Let u_n denote the one-point function of Prob_n:

$$u_n(\lambda) := \frac{1}{n}\mathrm{K}_n(\lambda,\lambda), \qquad \lambda \in \mathbb{C}.$$

Given a suitable function f on $\mathbb{C}$, we associate the random variable $\mathrm{Tr}_n[f]$ on the probability space $(\mathbb{C}^n, \mathrm{Prob}_n)$ via

$$\mathrm{Tr}_n[f](\lambda) := \sum_{i=1}^{n} f(\lambda_i).$$

We reserve the notation $\mathbb{E}_n$ for the expectation with respect to Prob_n; then, for example,

$$\mathbb{E}_n\big(\mathrm{Tr}_n[f]\big) = n\int_{\mathbb{C}} f u_n \,\mathrm{d}A.$$

According to Johansson (see [11, 15]), we have the weak-star convergence of the measures

$$\mathrm{d}\sigma_n(z) := u_n(z)\,\mathrm{d}A(z)$$

to some compactly supported probability measure $\sigma = \sigma_Q$ on $\mathbb{C}$. This probability measure σ is the Frostman equilibrium measure of the logarithmic potential theory with external field Q. We briefly recall the definition and some basic properties of this probability measure; cf. [18] and [11] for a more detailed exposition.

We write $S = S_Q := \operatorname{supp}\sigma_Q$ and assume that Q is $\mathcal{C}^2$-smooth in some neighborhood of S. Then S is compact and $\Delta Q \geq 0$ holds on S; moreover, $\sigma = \sigma_Q$ is absolutely continuous with density (we recall that $\varDelta = \frac{1}{4}\Delta$)

$$u := \frac{1}{2\pi}1_S \Delta Q = \frac{2}{\pi}1_S \varDelta Q. \tag{2.4}$$

We refer to the compact set S_Q as the *droplet* corresponding to the external field Q. We will write $\check{Q}$ for the maximal subharmonic function $\leq Q$ which grows like $\log|z| + \mathrm{O}(1)$ when $|z| \to +\infty$. The *predroplet* is the super-coincidence set $S^* = S^*_Q$ given by

$$S^* := \big\{z \in \mathbb{C} : \check{Q}(z) \geq Q(z)\big\}.$$

Then S^* is compact with $S \subset S^*$, and if Q is C^2-smooth on $S^* \setminus S$, we have $\Delta Q = 0$ area-a.e. on $S^* \setminus S$ (cf. [11]). If we let δ_w stand for the unit point mass at a point $w \in \mathbb{C}$, we may form the empirical measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$. Here, as before, the λ_j are the eigenvalues of a random normal matrix, so the empirical measure is a stochastic probability measure. As $n \to +\infty$, we have almost surely that the empirical measure converges to the Frostman equilibrium measure σ.

Our present goal is to describe the fluctuations of the density field $\mu_n = \sum_{j=1}^{n} \delta_{\lambda_j}$ around the equilibrium. More precisely, we will study the distribution (linear statistic)

$$f \mapsto \langle f, \mu_n \rangle_{\mathbb{C}} - n\langle f, \sigma \rangle_{\mathbb{C}} = \mathrm{Tr}_n[f] - n\langle f, \sigma \rangle_{\mathbb{C}}, \qquad f \in \mathcal{C}_0^{\infty}(\mathbb{C}).$$

We will denote by ν_n the measure with density $n(u_n - u)$, that is,

$$\langle f, \nu_n \rangle_{\mathbb{C}} := \mathbb{E}_n\big(\mathrm{Tr}_n[f]\big) - n\langle f, \sigma \rangle_{\mathbb{C}} = n\langle f, \sigma_n - \sigma \rangle_{\mathbb{C}}, \qquad f \in \mathcal{C}_0^{\infty}(\mathbb{C}).$$

2.4. *Assumptions on the potential.* To state the main results of the paper, we make the following four assumptions:

(A1) (smoothness) Q is real-analytic (written $Q \in C^{\omega}$) in some neighborhood of the droplet $S = S_Q$;
(A2) (regularity) $\Delta Q \neq 0$ in S;
(A3) (topology) ∂S is a C^{ω}-smooth Jordan curve;
(A4) (potential theory) $S^* = S$ (the droplet equals the predroplet).

We will comment on the nature and consequences of these assumptions later. Let us agree to write

$$L = L_Q := \log \Delta Q.$$

This function is well defined and C^{ω}-smooth in a neighborhood of the droplet S.

2.5. *The Neumann jump operator.* We will use the following general system of notation. If g is a continuous function defined in a neighborhood of S, then we write g^S for the continuous and bounded function in $\mathbb{C}$ with the following properties: in S, g^S equals g, while in the complement $\mathbb{C} \setminus S$, g^S is harmonic. It is clear that this determines g^S uniquely.

If g is smooth on S, then

$$\mathcal{N}_{\Omega} g := -\frac{\partial g|_S}{\partial \mathrm{n}}, \qquad \Omega := \mathrm{int}(S),$$

where n is the (exterior) unit normal of Ω. We define the normal derivative $\mathcal{N}_{\Omega^{\circledast}} g$ for the complementary domain $\Omega^{\circledast} := \mathbb{C} \setminus S$ analogously. If both normal derivatives exist, then we define the *Neumann jump*:

$$\mathcal{N} g \equiv \mathcal{N}_{\partial S} g := \mathcal{N}_{\Omega} g + \mathcal{N}_{\Omega^{\circledast}} g.$$

By Green's formula, we have the identity (of distributions)

$$\Delta g^S = 1_{\Omega} \Delta g \, \mathrm{d}A + \mathcal{N}\big[g^S\big] \, \mathrm{d}s, \tag{2.5}$$

where ds is the arc-length measure on ∂S. Here, Δg^S is understood in the sense of distribution theory, and the measure on the right-hand side is understood as a distribution as well.

We now verify (2.5). Let ϕ be a test function. The left-hand side in (2.5) applied to ϕ is

$$\langle \phi, \Delta g^S\rangle_{\mathbb{C}} = \langle \Delta\phi, g^S\rangle_{\mathbb{C}} = \int_S g\Delta\phi\,\mathrm{d}A + \int_{\mathbb{C}\setminus S} g^S\Delta\phi\,\mathrm{d}A,$$

while the right-hand side applied to ϕ equals

$$\langle \phi, 1_\Omega \Delta g\rangle_{\mathbb{C}} + \langle \phi, \mathcal{N}[g^S]\rangle_{\partial S} = \int_S \phi\Delta g\,\mathrm{d}A + \int_{\partial S}\phi\mathcal{N}[g^S]\,\mathrm{d}s.$$

So, we need to check that

$$\int_S (g\Delta\phi - \phi\Delta g)\,\mathrm{d}A + \int_{\mathbb{C}\setminus S}(g^S\Delta\phi - \phi\Delta g^S)\,\mathrm{d}A = \int_{\partial S}\phi\mathcal{N}(g^S)\,\mathrm{d}s.$$

But this is an immediate consequence of Green's formula applied to the regions Ω and $\Omega^\circledast$ separately, and (2.5) follows.

2.6. *Main results*. We shall prove the following results, which were announced in [3].

THEOREM 2.1. *For all test functions $f \in C_0^\infty(\mathbb{C})$, the limit*

$$\langle f, \nu\rangle_{\mathbb{C}} := \lim_{n\to+\infty}\langle f, \nu_n\rangle_{\mathbb{C}}$$

exists, *and*

$$\langle f, \nu\rangle_{\mathbb{C}} = \frac{1}{8\pi}\left\{\int_S (\Delta f + f\Delta L)\,\mathrm{d}A + \int_{\partial S} f\mathcal{N}(L^S)\,\mathrm{d}s\right\}.$$

Equivalently, we have the convergence as $n \to +\infty$

$$\mathrm{d}\nu_n \to \mathrm{d}\nu = \frac{1}{8\pi}\Delta(1_S + L^S),$$

in the sense of distribution theory.

THEOREM 2.2. *Let $h \in C_0^\infty(\mathbb{C})$ be a real-valued test function. Then, as $n \to +\infty$, we have the convergence in distribution*

$$\mathrm{Tr}_n\, h - \mathbb{E}_n \,\mathrm{Tr}_n\, h \to N\left(0, \frac{1}{4\pi}\int_{\mathbb{C}} |\nabla h^S|^2\,\mathrm{d}A\right).$$

The last formula is to be understood in the sense of convergence of the random variables to a normal law in distribution. As noted in [3], the result may be restated in terms of convergence of random fields to a Gaussian field with free boundary conditions.

2.7. *Derivation of Theorem* 2.2. By appealing to the variational approach employed by Johansson in [15], we now show that the Gaussian convergence in Theorem 2.2 follows from a generalized version of Theorem 2.1, which we now state.

We fix a real-valued test function $h \in \mathcal{C}_0^\infty(\mathbb{C})$ and consider the perturbed potential

$$Q_n^h := Q - \frac{1}{n}h.$$

We denote by Prob_n^h the density of states associated with the perturbed potential Q_n^h [cf. (2.2)] given by

$$\text{(2.6)} \qquad \mathrm{d}\,\mathrm{Prob}_n^h(\lambda) := \frac{1}{Z_n^h}\mathrm{e}^{-H_n^h(\lambda)}\,\mathrm{d}A^{\otimes n}(\lambda), \qquad \lambda = (\lambda_j)_1^n \in \mathbb{C}^n,$$

where Z_n^h is the appropriate normalization constant ("partition function") and

$$H_n^h(\lambda) = \sum_{j,k:j\neq k} \log\frac{1}{|\lambda_j - \lambda_k|} + 2n\sum_{j=1}^n Q(\lambda_j) - 2\sum_{j=1}^n h(\lambda_j).$$

We let $\mathbb{E}_n^h$ denote expectation with respect to the perturbed law Prob_n^h. We also write u_n^h for the one-point function associated with the density of states Prob_n^h, and σ_n^h for the probability measure with density u_n^h (i.e., $\mathrm{d}\sigma_n^h = u_n^h\,\mathrm{d}A$). We let ν_n^h denote the measure $n(\sigma_n^h - \sigma)$, that is,

$$\text{(2.7)} \qquad \langle f, \nu_n^h\rangle_{\mathbb{C}} := n\langle f, \sigma_n^h - \sigma\rangle_{\mathbb{C}} = \mathbb{E}_n^h \mathrm{Tr}_n[f] - n\langle f, \sigma\rangle_{\mathbb{C}}.$$

THEOREM 2.3. *For all $f \in C_0^\infty(\mathbb{C})$, we have the convergence as $n \to +\infty$*

$$\langle f, \nu_n^h - \nu_n\rangle_{\mathbb{C}} \to \frac{1}{2\pi}\int_{\mathbb{C}} \nabla f^S \cdot \nabla h^S\,\mathrm{d}A.$$

Here, the dot stands for the inner product of vectors. We supply a proof of Theorem 2.3 in Section 5.

LEMMA 2.4. *Theorem 2.2 is a consequence of Theorem 2.3.*

PROOF. We follow the argument of Johansson [14, 15].

We write $X_n := \mathrm{Tr}_n[h] - \mathbb{E}_n\,\mathrm{Tr}_n[h]$ and let $a_n^h(\tau) := \mathbb{E}_n^{\tau h} X_n$. Here, $\tau \geq 0$ is a parameter, and $\mathbb{E}_n^{\tau h}$ denotes expectation corresponding to the potential $Q - \frac{1}{n}\tau h$. We note that $a_n^h(0) = 0$ because $\mathbb{E}_n X_n = 0$. In view of Theorem 2.3, we have that as $n \to +\infty$,

$$\text{(2.8)} \qquad a_n^h(\tau) \to \tau a, \qquad \text{where } a = \frac{1}{2\pi}\int_{\mathbb{C}} |\nabla h^S|^2\,\mathrm{d}A.$$

The convergence is for fixed τ. Next, we put

$$F_n(\tau) := \log \mathbb{E}_n\big[\mathrm{e}^{2\tau X_n}\big],$$

and observe that $F_n(0) = 0$ and that F_n is convex. Then Johansson's calculation (see [15]; the argument is reproduced in [3], pages 66–67) shows that we have

$$F_n'(\tau) = 2\mathbb{E}_n^{\tau h} X_n = 2a_n^h(\tau), \tag{2.9}$$

so the convexity of F_n means that $\tau \mapsto a_n^h(\tau)$ is increasing. In particular,

$$0 = 2a_n^h(0) = F_n'(0) \le F_n'(t) \le F_n'(\tau) = 2a_n^h(\tau), \qquad 0 \le t \le \tau,$$

so that by dominated convergence, integration of (2.9) gives that

$$\begin{aligned} \log \mathbb{E}_n\big[\mathrm{e}^{2\tau X_n}\big] &= F_n(\tau) = F_n(\tau) - F_n(0) \\ &= \int_0^\tau F_n'(t)\,\mathrm{d}t \to \int_0^\tau 2ta\,\mathrm{d}t = a\tau^2 \qquad \text{as } n \to +\infty. \end{aligned} \tag{2.10}$$

This calculates the limit of the moment generating function, and we see from (2.10) that all the moments of X_n converge to the moments of the normal $N(0, \frac{1}{2}a)$ distribution. It is well known that this implies convergence in distribution, namely Theorem 2.2 follows. $\square$

2.8. *Comments.*

2.8.1. *Related work.* The one-dimensional analogue of the weighted RNM theory is the random Hermitian matrix theory. As we mentioned, the convergence of the fluctuations to a Gaussian field was studied by Johansson in the important paper [15]. In the case of normal matrices, the convergence in Theorems 2.1 and 2.2 for test functions supported in the interior of the droplet was obtained in [3]; see also [5]. Also, in [3], we announced Theorems 2.1 and 2.2 and obtained several consequences of them, for example, the convergence of the Berezin measures, rooted at a point in the exterior to S, to harmonic measure. Earlier, Rider and Virág [17] proved Theorems 2.1 and 2.2 in the special case $Q(z) = \frac{1}{2}|z|^2$ (the *Ginibre ensemble*). In [9] and [4], the fluctuations of eigenvalues near the boundary are studied from a different perspective. Finally, we should mention the work of Wiegmann et al. [22–25], who pioneered—on the physical level—the application of the method of Ward identities to various aspects or RNM theory.

2.8.2. *Assumptions on the potential.* Here, we comment on the assumptions (A1)–(A4) which are made on the potential Q.

The C^ω-smoothness assumption (A1) is natural for the study of fluctuation properties near the boundary of the droplet (For test functions supported in the interior, one can do with less regularity).

By extending Sakai's theory [19] to general real-analytic weights, it was shown in [12] that the conditions (A1) and (A2) imply that ∂S is a union of finitely many C^ω-smooth curves with a finite number of singularities of known types. We rule out the singularities by the smoothness assumption (A3). What happens in the presence of singularities is probably an interesting topic, which we have not approached. Without singularities, the boundary of the droplet is a union of finitely many C^ω-smooth Jordan curves. The topological ingredient in assumption (A3) means that we only consider the case of a single boundary component. Our methods extend without difficulty to the case of a multiply connected droplet. The disconnected case requires further analysis, and is not considered in this paper.

The potential theoretic assumption (A4) is needed to ensure exponentially rapid decay of the (perturbed) one-point function u_n^h off the droplet S. We remark that the assumptions (A1)–(A3) entail that the set $S^* \setminus S$ has positive distance to S, but without (A4), $S^* \setminus S$ could nevertheless be rather big.

2.8.3. *Droplets and potential theory.* Here, we state the properties of the droplet that will be needed for our analysis. Proofs for these properties can be found in [11, 18].

We recall that $\check{Q}$ is the maximal subharmonic function $\le Q$ which grows like $\log|z| + \mathrm{O}(1)$ when $|z| \to +\infty$. We have that $\check{Q} = Q$ on S while $\check{Q}$ is $C^{1,1}$-smooth in $\mathbb{C}$ and

$$\check{Q}(z) = Q^S(z) + G(z, \infty), \qquad z \in \mathbb{C} \setminus S,$$

where G is the classical Green's function of $\mathbb{C} \setminus S$. In particular, if

$$U^\sigma(z) = \int_{\mathbb{C}} \log \frac{1}{|z - \zeta|} \, \mathrm{d}\sigma(\zeta)$$

denotes the logarithmic potential of the equilibrium measure, then

$$\check{Q} + U^\sigma \equiv c_Q, \tag{2.11}$$

where c_Q is a Robin-type constant.

The following proposition sums up the basic properties of the droplet and the function $\check{Q}$ (compare with [11]). We write $W^{2,\infty}$ for the usual (local) Sobolev space of functions with locally bounded second-order partial derivatives.

PROPOSITION 2.5. *Suppose Q satisfies* (A1)–(A3). *Then ∂S is a C^ω-smooth Jordan curve, and $\check{Q} \in W^{2,\infty}(\mathbb{C})$. Moreover, we have that*

$$\partial \check{Q} = [\partial Q]^S,$$

and

$$Q(z) - \check{Q}(z) \asymp \delta_{\partial S}(z)^2, \qquad z \in \mathbb{C} \setminus S, \ \text{as } \delta_{\partial S}(z) \to 0, \tag{2.12}$$

where $\delta_{\partial S}(z)$ denotes the distance from z to the droplet.

2.8.4. *Joint intensities*. We will occasionally use the intensity k-point function of the process Λ_n. This is the function defined by

$$R_n^{(k)}(z_1,\dots,z_k) := \lim_{\varepsilon\to 0}\frac{\mathrm{Prob}_n(\bigcap_{j=1}^k\{\Lambda_n\cap\mathbb{D}(z_j,\varepsilon)\neq\varnothing\})}{(\pi\varepsilon^2)^k} = \det\bigl[\mathrm{K}_n(z_i,z_j)\bigr]_{i,j=1}^k.$$

In particular, $R_n^{(1)} = nu_n$.

2.8.5. *Organization of the paper.* We will derive the following statement which combines Theorems 2.1 and 2.3.

MASTER FORMULA. Let ν_n^h be the measure defined in (2.7). Then

$$\lim_{n\to+\infty}\langle f,\nu_n^h\rangle_{\mathbb{C}} = \frac{1}{8\pi}\biggl\{\int_S(\Delta f + f\Delta L)\,\mathrm{d}A + \int_{\partial S} f\mathcal{N}\bigl(L^S\bigr)\,\mathrm{d}s\biggr\} + \frac{1}{2\pi}\int_{\mathbb{C}}\nabla f^S\cdot\nabla h^S\,\mathrm{d}A. \tag{2.13}$$

Our proof of this formula is based on the limit form of the Ward identities which we discuss in the next section. To justify this limit form, we need to estimate certain error terms; this is done in Section 4. In the proof, we refer to some basic estimates of polynomial Bergman kernels, which we collect in the Appendix. The proof of the master formula is completed in Section 5.

3. Ward identities.

3.1. *Exact identities*. For an appropriate function v on $\mathbb{C}$, we define a random variable $W_n^+[v]$ on the probability space $(\mathbb{C}^n,\mathrm{Prob}_n)$ by

$$W_n^+[v] := \frac{1}{2}\sum_{j,k:j\neq k}\frac{v(\lambda_j)-v(\lambda_k)}{\lambda_j-\lambda_k} - 2n\,\mathrm{Tr}_n[v\partial Q] + \mathrm{Tr}_n[\partial v].$$

The minimal requirement on v is that the above expression should be well defined.

PROPOSITION 3.1. *Let $v:\mathbb{C}\to\mathbb{C}$ be Lipschitz-continuous with compact support. Then*

$$\mathbb{E}_n W_n^+[v] = 0.$$

PROOF. The proof is based on the observation that the value of the partition function

$$Z_n := \int_{\mathbb{C}^n}\mathrm{e}^{-H_n(z)}\,\mathrm{d}A^{\otimes n}(z)$$

is unchanged under a change of variables in the integral. Here, H_n is the Hamiltonian given by (2.3). We will need to analyze the change of the volume element as well as the change of the Hamiltonian under the change of variables. To simplify the notation, we write

$$W_n^+[v] = \mathrm{I}_n[v] - \mathrm{II}_n[v] + \mathrm{III}_n[v],$$

where (a.e.)

$$\mathrm{I}_n[v](z) := \frac{1}{2}\sum_{j,k:j\neq k}\frac{v(z_j)-v(z_k)}{z_j - z_k}, \qquad \mathrm{II}_n[v](z) = 2n\sum_{j=1}^n \partial Q(z_j)v(z_j),$$

$$\mathrm{III}_n[v](z) \,=\, \sum_{j=1}^n \partial v(z_j).$$

We consider the change of variables $z_j = \phi(\zeta_j) := \zeta_j + \xi v(\zeta_j)$, for $1 \le j \le n$. Here, $\xi \in \mathbb{C}$ is assumed to be close to 0. The corresponding area element is

$$\begin{aligned}\mathrm{d}A(z_j) &= \big(|\partial\phi(\zeta_j)|^2 - |\bar\partial\phi(\zeta_j)|^2\big)\,\mathrm{d}A(\zeta_j)\\ &= \big\{1 + 2\,\mathrm{Re}\big[\xi\,\partial v(\zeta_j)\big] + \mathrm{O}\big(|\xi|^2\big)\big\}\,\mathrm{d}A(\zeta_j),\end{aligned}$$

so that the corresponding volume element becomes

$$\mathrm{d}A^{\otimes n}(z) = \big\{1 + 2\,\mathrm{Re}\big(\xi\,\mathrm{III}_n[v](\zeta)\big) + \mathrm{O}\big(|\xi|^2\big)\big\}\,\mathrm{d}A^{\otimes n}(\zeta).$$

We turn to the Hamiltonian after the change of variables. We note that

$$\begin{aligned}\log|z_i - z_j|^2 &= \log|\zeta_i - \zeta_j|^2 + \log\left|1 + \xi\,\frac{v(\zeta_i)-v(\zeta_j)}{\zeta_i-\zeta_j}\right|^2\\ &= \log|\zeta_i - \zeta_j|^2 + 2\,\mathrm{Re}\left(\xi\,\frac{v(\zeta_i)-v(\zeta_j)}{\zeta_i-\zeta_j}\right) + \mathrm{O}\big(|\xi|^2\big),\end{aligned}$$

so that

$$\sum_{j,k:j\neq k}\log\frac{1}{|z_j - z_k|} = \sum_{j,k:j\neq k}\log\frac{1}{|\zeta_j-\zeta_k|} - 2\,\mathrm{Re}\big[\xi\,\mathrm{I}_n(\zeta)\big] + \mathrm{O}\big(|\xi|^2\big) \tag{3.1}$$

as $|\xi| \to 0$. The external potential Q changes according to

$$Q(z_j) = Q\big(\zeta_j + \xi v(\zeta_j)\big) = Q(\zeta_j) + 2\,\mathrm{Re}\big(\xi\,\partial Q(\zeta_j)v(\zeta_j)\big) + \mathrm{O}\big(|\xi|^2\big),$$

so that

$$2n\sum_{j=1}^n Q(z_j) = 2n\sum_{j=1}^n Q(\zeta_j) + 2\,\mathrm{Re}\big[\xi\,\mathrm{II}_n(\zeta)\big] + \mathrm{O}\big(|\xi|^2\big). \tag{3.2}$$

Putting things together, we see that (3.1) and (3.2) imply that the Hamiltonian H_n given by (2.3) changes according to

$$H_n(z) = H_n(\zeta) + 2\,\mathrm{Re}\big(-\xi\,\mathrm{I}_n(\zeta) + \xi\,\mathrm{II}_n(\zeta)\big) + \mathrm{O}\big(|\xi|^2\big). \tag{3.3}$$

We find that after the change of variables, the partition function equals

$$\begin{aligned}Z_n &= \int_{\mathbb{C}^n} \mathrm{e}^{-H_n(z)}\,\mathrm{d}A^{\otimes n}(z)\\ &= \int_{\mathbb{C}^n} \mathrm{e}^{-H_n(\zeta)-2\,\mathrm{Re}[-\xi \mathrm{I}_n(\zeta)+\xi \mathrm{II}_n(\zeta)]+\mathrm{O}(|\xi|^2)}\\ &\qquad\times\big(1+2\,\mathrm{Re}\big[\xi \mathrm{III}_n(\zeta)\big]+\mathrm{O}\big(|\xi|^2\big)\big)\,\mathrm{d}A^{\otimes n}(\zeta).\end{aligned}$$

As the value of Z_n does not depend on the value of the small complex parameter ξ, a simple argument based on Taylor's formula gives that

$$\mathrm{Re}\bigg\{\xi \int_{\mathbb{C}^n} \big(\mathrm{III}_n(\zeta)+\mathrm{I}_n(\zeta)-\mathrm{II}_n(\zeta)\big)\mathrm{e}^{-H_n(\zeta)}\,\mathrm{d}A^{\otimes n}(\zeta)\bigg\}=0, \tag{3.4}$$

that is, $\mathrm{Re}(\xi \mathbb{E}_n W_n^+[v])=0$. Considering that ξ is an arbitrary complex number which is close enough to 0, the claimed assertion $\mathbb{E}_n(W_n^+[v])=0$ is immediate. □

By applying Proposition 3.1 to the perturbed potential $Q_n^h = Q - \frac{1}{n}h$, we obtain the identity

$$\mathbb{E}_n^h W_{n,h}^+[v]=0, \tag{3.5}$$

where $\mathbb{E}_n^h$ is the expectation operation with respect to the weight Q_n^h, and

$$W_{n,h}^+[v] := W_n^+[v]+2\,\mathrm{Tr}_n[v\partial h]. \tag{3.6}$$

If we write

$$V_n[v]=\frac{1}{2n}\sum_{i,j:i\neq j}\frac{v(\lambda_i)-v(\lambda_j)}{\lambda_i-\lambda_j},$$

we may reformulate (3.5) and (3.6) in the following fashion:

$$\begin{aligned}\mathbb{E}_n^h V_n[v] &= 2\mathbb{E}_n^h\,\mathrm{Tr}_n[v\partial Q]-\langle\partial v+2v\partial h,\sigma_n^h\rangle_{\mathbb{C}}\\ &= \langle 2nv\partial Q-2v\partial h-\partial v,\sigma_n^h\rangle_{\mathbb{C}}.\end{aligned} \tag{3.7}$$

Here, we recall that σ_n^h is the measure with density u_n^h.

3.2. *Some logarithmic potentials.* We recall from (2.11) that $\check{Q}$ may be written as

$$\check{Q}=c_Q-U^\sigma,$$

where c_Q is a Robin-type constant and U^σ is the usual logarithmic potential associated with σ. More generally, if μ is a finite Borel measure in $\mathbb{C}$, with finite moment

$$\int_{\mathbb{C}}|\zeta|\,\mathrm{d}|\mu|(\zeta)<+\infty,$$

its logarithmic potential U^μ is the function

$$U^\mu(z) = \int_{\mathbb{C}} \log \frac{1}{|z-\zeta|} \, \mathrm{d}\mu(\zeta), \qquad z \in \mathbb{C}.$$

We introduce the following function, associated with the measure σ_n^h:

$$Q^{\otimes}_{n,h} := c_Q - U^{\sigma_n^h}. \tag{3.8}$$

We write $Q^{\otimes}_n := Q^{\otimes}_{n,0}$ in case $h = 0$. Then, as $n \to +\infty$, $Q^{\otimes}_{n,h} \to \check{Q}$, uniformly in $\mathbb{C}$. Moreover, the way things are set up, we have

$$\Delta Q^{\otimes}_{n,h} = 2\pi u_n^h.$$

Using the estimates of the one-point function u_n^h developed in Lemma 4.1 and Theorem 4.2, it is not difficult to show that $\nabla Q^{\otimes}_{n,h} \to \nabla \check{Q}$, uniformly in $\mathbb{C}$.

3.3. *Cauchy kernels*. For each $z \in \mathbb{C}$, let κ_z denote the function

$$\kappa_z(\lambda) = \frac{1}{z-\lambda},$$

so that $z \mapsto \langle \kappa_z, \sigma \rangle_{\mathbb{C}}$ is the Cauchy transform of the measure σ. By Proposition 2.5, we have

$$\langle \kappa_z, \sigma \rangle_{\mathbb{C}} = 2\partial \check{Q}(z).$$

We will also need the Cauchy integral $\langle \kappa_z, \sigma_n^h \rangle_{\mathbb{C}}$. We observe that

$$\langle \kappa_z, \sigma_n^h \rangle_{\mathbb{C}} = 2\partial Q^{\otimes}_{n,h}(z), \qquad z \in \mathbb{C}. \tag{3.9}$$

We now introduce the function

$$D_n^h(z) := \langle \kappa_z, \nu_n^h \rangle_{\mathbb{C}},$$

and write $D_n := D_n^0$ in case $h = 0$. In terms of the function $Q^{\otimes}_{n,h}$, we have

$$D_n^h = 2n\partial\big[Q^{\otimes}_{n,h} - \check{Q}\big] \quad \text{and} \quad \bar\partial D_n^h = n\pi\big(u_n^h - u\big), \tag{3.10}$$

and if f is a test function, then

$$\langle f, \nu_n^h \rangle_{\mathbb{C}} = \frac{1}{\pi} \int_{\mathbb{C}} f \,\bar\partial D_n^h \, \mathrm{d}A = -\frac{1}{\pi} \int_{\mathbb{C}} D_n^h \bar\partial f \, \mathrm{d}A. \tag{3.11}$$

Let K_n^h denote the correlation kernel with respect to the weight $Q_n^h = Q - \frac{1}{n}h$. In terms of D_n^h, we may rewrite the $V_n[v]$ term which appears in the Ward identity as follows.

LEMMA 3.2. *We have that*

$$\mathbb{E}_n^h V_n[v] = 2n \int_{\mathbb{C}} v u_n^h \partial \check{Q} \, \mathrm{d}A + \int_{\mathbb{C}} v u_n^h D_n^h \, \mathrm{d}A - \frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} \left|\mathrm{K}_n^h(z, w)\right|^2 \mathrm{d}A^{\otimes 2}(z, w).$$

PROOF. After all, we have

$$\mathbb{E}_n^h V_n[v] = \frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} R_{n,h}^{(2)}(z, w) \, \mathrm{d}A^{\otimes 2}(z, w),$$

where

$$R_{n,h}^{(2)}(z, w) := \mathrm{K}_n^h(z, z)\mathrm{K}_n^h(w, w) - \left|\mathrm{K}_n^h(z, w)\right|^2.$$

Next, we see that

$$\begin{aligned} &\frac{1}{2n} \int_{\mathbb{C}^2} \frac{v(z) - v(w)}{z - w} \mathrm{K}_n^h(z, z)\mathrm{K}_n^h(w, w) \, \mathrm{d}A^{\otimes 2}(z, w) \\ &\quad = \frac{1}{n} \int_{\mathbb{C}^2} \frac{v(z)}{z - w} \mathrm{K}_n^h(z, z)\mathrm{K}_n^h(w, w) \, \mathrm{d}A^{\otimes 2}(z, w) \\ &\quad = n \int_{\mathbb{C}^2} \frac{v(z)}{z - w} u_n^h(z) u_n^h(w) \, \mathrm{d}A^{\otimes 2}(z, w) = n \int_{\mathbb{C}} v(z) u_n^h(z) \langle \kappa_z, \sigma_n^h \rangle_{\mathbb{C}} \, \mathrm{d}A(z) \\ &\quad = 2n \int_{\mathbb{C}} v u_n^h \partial Q_{n,h}^{\otimes} \, \mathrm{d}A = 2n \int_{\mathbb{C}} v u_n^h \partial \check{Q} \, \mathrm{d}A + \int_{\mathbb{C}} v u_n^h D_n^h \, \mathrm{d}A, \end{aligned}$$

where we first used symmetry, second the identity $\mathrm{K}_n^h(z, z) \equiv n u_n^h(z)$, third, the equality (3.9) and fourth, the relation (3.10). The proof is complete. □

3.4. *Limit form of the Ward identity.* The main formula (2.13) will be derived from Theorem 3.3 below. In this theorem, we make the following assumptions on the vector field v:

(3.4-i) v is bounded in $\mathbb{C}$;
(3.4-ii) v is Lipschitz-continuous in $\mathbb{C}$;
(3.4-iii) v is uniformly C^2-smooth in $\mathbb{C} \setminus \partial S$.

[The last condition means that the restriction of v to S and the restriction to $(\mathbb{C} \setminus S) \cup \partial S$ are both C^2-smooth.]

THEOREM 3.3. *If v satisfies* (3.4-i)–(3.4-iii), *then as $n \to +\infty$,*

$$\frac{2}{\pi} \int_S v D_n^h \Delta Q \, \mathrm{d}A + \frac{2}{\pi} \int_{\mathbb{C} \setminus S} v \partial(\check{Q} - Q) \bar{\partial} D_n^h \, \mathrm{d}A \to -\frac{1}{2} \langle \partial v, \sigma \rangle_{\mathbb{C}} - 2 \langle v \partial h, \sigma \rangle_{\mathbb{C}}.$$

We postpone the proof to remark that it will be convenient to integrate by parts in the second integral in Theorem 3.3. To control the boundary term, we can use the next lemma.

LEMMA 3.4. *For big n, we have the estimate*

$$|D_n^h(z)| = \mathrm{O}\left(\frac{n}{|z|^2}\right) \qquad \text{as } |z| \to +\infty,$$

where the implied constant is independent of n.

PROOF. As u_n^h and u are both probability densities, we see that

$$\frac{D_n^h(z)}{n} = \int_{\mathbb{C}} \frac{u_n^h(\lambda) - u(\lambda)}{z - \lambda}\, \mathrm{d}A(\lambda) = \int \left(\frac{1}{z-\lambda} - \frac{1}{z}\right)\bigl(u_n^h(\lambda) - u(\lambda)\bigr)\, \mathrm{d}A(\lambda).$$

Next, since

$$\frac{1}{z-\lambda} - \frac{1}{z} = \frac{1}{z^2}\frac{\lambda}{1-\lambda/z},$$

we need to show that the integrals

$$\int_{\mathbb{C}} \frac{|u_n^h(\lambda) - u(\lambda)|}{|1-\lambda/z|}|\lambda|\, \mathrm{d}A(\lambda)$$

are uniformly bounded. We use that for some positive constant C_1,

$$|u_n^h(\lambda) - u(\lambda)| \le u(\lambda) + u_n^h(\lambda) \le \frac{C_1}{1+|\lambda|^4}, \qquad \lambda \in \mathbb{C}, \tag{3.12}$$

which may be justified by appealing to the basic estimate (cf. Lemma 4.1 below)

$$u_n^h(\lambda) \le C_2 \mathrm{e}^{-2n(Q(\lambda) - \check{Q}(\lambda))}, \qquad \lambda \in \mathbb{C},$$

together with the growth assumption (2.1). It is here that we need n to be big enough. Next, in view of (3.12),

$$\int_{\mathbb{C}} \frac{|u_n^h(\lambda) - u(\lambda)|}{|1-\lambda/z|}|\lambda|\, \mathrm{d}A(\lambda) \le C_1 \int_{\mathbb{C}} \frac{(1+|\lambda|)^{-4}}{|1-\lambda/z|}|\lambda|\, \mathrm{d}A(\lambda) \le 100 C_1,$$

where the estimate of the integral can be achieved by splitting the plane into the disk $\mathbb{D}(z, \frac{1}{2}|z|)$ and its complement, and by suitably estimating the integrand in each region. □

Since $\partial Q = \partial \check{Q}$ on S, an integration by parts argument leads to the following reformulation of Theorem 3.3.

COROLLARY 3.5 ("Limit Ward identity"). *Suppose that v meets the conditions (3.4-i)–(3.4-iii). Then as $n \to +\infty$, we have the convergence*

$$\frac{2}{\pi}\int_{\mathbb{C}} \bigl(v\Delta Q + \bar{\partial} v \partial (Q - \check{Q})\bigr) D_n^h\, \mathrm{d}A \to -\frac{1}{2}\langle \partial v, \sigma\rangle_{\mathbb{C}} - 2\langle v \partial h, \sigma\rangle_{\mathbb{C}}.$$

3.5. *Error terms and the proof of Theorem* 3.3. We recall that the Ward identity (3.7) states that

$$\mathbb{E}_n^h V_n[v] = \langle 2nv\partial Q - 2v\partial h - \partial v, \sigma_n^h \rangle_{\mathbb{C}},$$

while Lemma 3.2 supplies the formula

$$\mathbb{E}_n^h V_n[v] = \langle 2nv\partial \check{Q} + vD_n^h, \sigma_n^h \rangle_{\mathbb{C}} - \frac{1}{2n}\int_{\mathbb{C}^2} \frac{v(z)-v(w)}{z-w} \big|\mathrm{K}_n^h(z,w)\big|^2 \,\mathrm{d}A^{\otimes 2}(z,w).$$

As we equate the two, and perform some rearrangement, we arrive at

$$\begin{aligned}(3.13)\qquad &\langle 2nv\partial(\check{Q}-Q), \sigma_n^h - \sigma\rangle_{\mathbb{C}} + \langle vD_n^h, \sigma\rangle_{\mathbb{C}} \\ &\quad = -\Big\langle 2v\partial h + \frac{1}{2}\partial v, \sigma_n^h \Big\rangle_{\mathbb{C}} - \langle vD_n^h, \sigma_n^h - \sigma\rangle_{\mathbb{C}} \\ &\qquad + \frac{1}{2n}\int_{\mathbb{C}^2} \frac{v(z)-v(w)}{z-w}\big|\mathrm{K}_n^h(z,w)\big|^2 \,\mathrm{d}A^{\otimes 2}(z,w) - \frac{1}{2}\langle \partial v, \sigma_n^h\rangle_{\mathbb{C}}.\end{aligned}$$

In the rearrangement, we used the facts that σ is supported on S and that $\partial(\check{Q}-Q)=0$ on S. Let us introduce the *first error term* by

$$(3.14)\quad \epsilon_{n,h}^1[v] := \frac{1}{n}\int_{\mathbb{C}^2} \frac{v(z)-v(w)}{z-w}\big|\mathrm{K}_n^h(z,w)\big|^2 \,\mathrm{d}A^{\otimes 2}(z,w) - \langle \partial v, \sigma_n^h\rangle_{\mathbb{C}},$$

and the *second error term* by

$$\begin{aligned}(3.15)\qquad \epsilon_{n,h}^2[v] &:= \langle vD_n^h, \sigma_n^h - \sigma\rangle_{\mathbb{C}} = \int_{\mathbb{C}} vD_n^h(u_n^h - u)\,\mathrm{d}A \\ &= -\frac{1}{2\pi n}\int_{\mathbb{C}} [D_n^h]^2 \bar\partial v \,\mathrm{d}A.\end{aligned}$$

We insert these error terms into (3.13), to obtain

$$\begin{aligned}&\langle 2nv\partial(\check{Q}-Q), \sigma_n^h - \sigma\rangle_{\mathbb{C}} + \langle vD_n^h, \sigma\rangle_{\mathbb{C}} \\ &\quad = -\langle 2v\partial h + \tfrac{1}{2}\partial v, \sigma_n^h\rangle_{\mathbb{C}} - \epsilon_{n,h}^2[v] + \tfrac{1}{2}\epsilon_{n,h}^1[v].\end{aligned}$$

We next rewrite this relation in integral form using (3.10) and (2.4):

$$\begin{aligned}(3.16)\qquad &\frac{2}{\pi}\int_{\mathbb{C}\setminus S} v\partial(\check{Q}-Q)\bar\partial D_n^h \,\mathrm{d}A + \frac{2}{\pi}\int_S vD_n^h \Delta Q\,\mathrm{d}A \\ &\quad = -\Big\langle 2v\partial h + \frac{1}{2}\partial v, \sigma_n^h\Big\rangle_{\mathbb{C}} + \frac{1}{2}\epsilon_{n,h}^1[v] - \epsilon_{n,h}^2[v].\end{aligned}$$

As $n \to +\infty$, we have the convergence $\sigma_n^h \to \sigma$ in the weak-star sense of measures. For $h=0$, this is Johansson's theorem (see [11, 15]), while in this more general setting, it follows from the one-point function estimates in Lemma 4.1 and Theorem 4.2 below. We see from (3.16) that once we have established that $\epsilon_{n,h}^j[v] \to 0$ as $n \to +\infty$ for $j=1,2$, the assertion of Theorem 3.3 is immediate.

In the next section, we will show that for each v satisfying conditions (3.4-i)–(3.4-iii), the error terms $\epsilon^j_{n,h}[v]$ tend to zero as $n \to +\infty$, for $j = 1, 2$, which completes the proof of Theorem 3.3.

4. Estimates of the error terms.

4.1. *Estimates of the kernel* K_n^h. We will use two different estimates, one which gives control in the interior $\mathrm{int}(S)$ of the droplet S, and another which gives control in the exterior domain $\mathbb{C} \setminus S$.

4.1.1. *The exterior estimate.* We recall that $\mathrm{K}_n^h(z, w)$ is the correlation kernel of the n-point process associated with potential $Q_n^h = Q - \frac{1}{n}h$. We have the following global estimate, which is particularly useful in the exterior of the droplet.

LEMMA 4.1. *There exists a positive constant C which only depends on Q, h such that*

$$\mathrm{K}_n^h(z, z) \le Cn\mathrm{e}^{-2n(Q-\check{Q})(z)}, \qquad z \in \mathbb{C}.$$

This estimate has been recorded (see, e.g., [2], Section 3) for the kernels K_n, that is, in the case $h = 0$. Since obviously

$$\int_{\mathbb{C}} |p|^2 \mathrm{e}^{-2nQ_n^h}\,\mathrm{d}A \asymp \int_{\mathbb{C}} |p|^2 \mathrm{e}^{-2nQ}\,\mathrm{d}A,$$

the norms of the point evaluation functionals are equivalent in the spaces $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$ and $\mathrm{Pol}_n(\mathrm{e}^{-2nQ_n^h})$. In terms of reproducing kernels, this means that $\mathrm{k}_n(z, z) \asymp \mathrm{k}_n^h(z, z)$, and so $\mathrm{K}_n(z, z) \asymp \mathrm{K}_n^h(z, z)$ as well. It follows that the case $h \ne 0$ does not require any separate treatment.

In the following, we shall use the notation

$$\delta_{\partial S}(z) := \mathrm{dist}_{\mathbb{C}}(z, \partial S)$$

and

$$\delta_n := n^{-1/2}[\log n]^2.$$

In view of our assumptions on the droplet (see Proposition 2.5), and the growth control (2.1) together with our assumption (A4), we have, for some small but positive real parameter ε,

$$Q(z) - \check{Q}(z) \ge \varepsilon \min\{\log(2 + |z|), \delta_{\partial S}(z)^2\}, \qquad z \in \mathbb{C} \setminus S. \tag{4.1}$$

For big n, it follows that for any $N > 0$ there exists a constant C_N such that

$$\mathrm{K}_n^h(z, z) \le C_N n^{-N}(1 + |z|)^{-3} \qquad \text{for } z \in \mathbb{C} \setminus S \text{ with } \delta_{\partial S}(z) \ge \delta_n. \tag{4.2}$$

4.1.2. *The interior estimate.* Let us recall that we assume that Q is real-analytic in some neighborhood of S. This means that we can lift Q to a complex analytic function of two variables in some neighborhood in $\mathbb{C}^2$ of the conjugate-diagonal

$$\{(z,\bar{z}): z \in S\} \subset \mathbb{C}^2.$$

We will use the same letter Q for this extension, so that, for example,

$$Q(z) = Q(z,\bar{z}).$$

We have

$$Q(z,w) = \overline{Q(\bar{w},\bar{z})}$$

and

$$\partial_1 Q(z,\bar{z}) = \partial Q(z), \qquad \partial_1\partial_2 Q(z,\bar{z}) = \partial\bar{\partial} Q(z) = \Delta Q(z),$$
$$\partial_1^2 Q(z,\bar{z}) = \partial^2 Q(z), \qquad \text{etc.}$$

Using this extension and some technical mathematical machinery, one can show that for z, w confined to the interior of the droplet S, the leading contribution to the perturbed correlation kernel K_n^h is of the form

$$\mathrm{K}_{n,h}^{\sharp}(z,w) = \frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\mathrm{e}^{n[2Q(z,\bar{w})-Q(z)-Q(w)]}\mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}[(z-w)\partial h(w)]}. \tag{4.3}$$

The diagonal restriction of this approximate correlation kernel is

$$\mathrm{K}_{n,h}^{\sharp}(w,w) = \frac{2n}{\pi}\Delta Q(w) = \frac{n}{2\pi}\Delta Q(w).$$

THEOREM 4.2. *Suppose that $z, w \in S$, with $\delta_{\partial S}(z) > 2\delta_n$ and $|z-w| < \delta_n$. Then*

$$\left|\mathrm{K}_n^h(z,w) - \mathrm{K}_{n,h}^{\sharp}(z,w)\right| = \mathrm{O}(1)$$

as $n \to +\infty$, where the implied constant in $\mathrm{O}(1)$ *depends on Q, h, but not on n.*

Similar types of expansions are discussed, for example, in [1, 2, 7]. As there is no convenient reference for this particular result, and to make the paper self-contained, we include a proof in the Appendix.

We now turn to the proof that the error terms $\epsilon_{n,h}^1[v]$ and $\epsilon_{n,h}^2[v]$ [cf. (3.14)–(3.15)] are negligible. Our proof is based on the above mentioned estimates of the correlation kernels K_n^h.

4.2. *The first error term.* We start with the observation that if $w \in S$ and $\delta_{\partial S}(w) > 2\delta_n$ then at short distances the so-called Berezin kernel rooted at w

$$Б_{n,h}^{\langle w\rangle}(z) = \frac{|\mathrm{K}_n^h(z,w)|^2}{\mathrm{K}_n^h(w,w)}$$

is close to the heat kernel

$$H_n^{\langle w\rangle}(z) = \frac{1}{\pi} an \mathrm{e}^{-an|z-w|^2}, \qquad a := 2\Delta Q(w) > 0.$$

Both kernels determine probability measures indexed by w. Most of the heat kernel measure is concentrated in the disc $\mathbb{D}(w,\delta_n)$,

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} H_n^{\langle w\rangle}(z)\,\mathrm{d}A(z) = \mathrm{O}\big(n^{-N}\big) \qquad \text{as } n \to +\infty, \tag{4.4}$$

where N denotes an arbitrary (large) positive number.

LEMMA 4.3. *Suppose that $z, w \in S$, with $\delta_{\partial S}(w) > 2\delta_n$ and $|z-w| < \delta_n$. Then*

$$\big|Б_{n,h}^{\langle w\rangle}(z) - H_n^{\langle w\rangle}(z)\big| = \mathrm{O}\big(n^{1/2}\big) \qquad \textit{as } n \to +\infty,$$

where the implied constant only depends on Q, h.

PROOF. In view of Theorem 4.2, we have

$$Б_n^{\langle w\rangle}(z) = \frac{|\mathrm{K}_{n,h}^\sharp(z,w)|^2}{\mathrm{K}_{n,h}^\sharp(w,w)} + \mathrm{O}(1),$$

where $\mathrm{K}_{n,h}^\sharp$ is as in (4.3). Next, we fix w and apply Taylor's formula to get that

$$\begin{aligned}
&\mathrm{Re}\big\{2Q(z,\bar{w}) - Q(z) - Q(w)\big\}\\
&\quad = 2\,\mathrm{Re}\,Q(z,\bar{w}) - Q(z) - Q(w)\\
&\quad = 2\,\mathrm{Re}\big\{Q(w) + (z-w)\partial Q(w) + \tfrac{1}{2}(z-w)^2\partial^2 Q(w)\big\}\\
&\qquad - \big\{Q(w) + 2\,\mathrm{Re}\big[(z-w)\partial Q(w)\big]\\
&\qquad\qquad + \mathrm{Re}\big[(z-w)^2\partial^2 Q(w)\big] + |z-w|^2\Delta Q(w)\big\}\\
&\qquad - Q(w) + \mathrm{O}\big(|z-w|^3\big)\\
&\quad = -|z-w|^2\Delta Q(w) + \mathrm{O}\big(|z-w|^3\big).
\end{aligned}$$

Note that for $0 \le t < +\infty$,

$$t^{1/2}\mathrm{e}^{-nt} \le n^{-1/2}, \qquad t^{3/2}\mathrm{e}^{-nt} \le n^{-3/2}. \tag{4.5}$$

Using the explicit formula (4.3), we find that with $a = 2\Delta Q(w)$,

$$\frac{|\mathrm{K}^{\sharp}_{n,h}(z,w)|^2}{\mathrm{K}^{\sharp}_{n,h}(w,w)} = \frac{n}{\pi}\big[a + \mathrm{O}(|z-w|)\big]\mathrm{e}^{-\alpha n|z-w|^2+\mathrm{O}(n|z-w|^3)} = H_n^{\langle w\rangle}(z) + \mathrm{O}(n^{1/2}),$$

where in the last step we rely on (4.5). This does it. □

COROLLARY 4.4. *If $w \in S$ and $\delta_{\partial S}(w) > 2\delta_n$, then*

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} \text{Б}^{\langle w\rangle}_{n,h}(z)\,\mathrm{d}A(z) = \mathrm{O}(n^{1/2}\delta_n^2) = \mathrm{O}(n^{-1/2}[\log n]^4).$$

PROOF. We notice that

$$\begin{aligned}\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} \text{Б}^{\langle w\rangle}_{n,h}\,\mathrm{d}A &= 1 - \int_{\mathbb{D}(w,\delta_n)} \text{Б}^{\langle w\rangle}_{n,h}\,\mathrm{d}A \\ &= 1 - \int_{\mathbb{D}(w,\delta_n)} H_n^{\langle w\rangle}\,\mathrm{d}A + \int_{\mathbb{D}(w,\delta_n)} \big(H_n^{\langle w\rangle} - \text{Б}^{\langle w\rangle}_{n,h}\big)\,\mathrm{d}A \\ &= \int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)} H_n^{\langle w\rangle}\,\mathrm{d}A + \int_{\mathbb{D}(w,\delta_n)} \big(H_n^{\langle w\rangle} - \text{Б}^{\langle w\rangle}_{n,h}\big)\,\mathrm{d}A.\end{aligned}$$

The assertion now follows from Lemma 4.3 and the decay (4.4) of the heat kernel. □

PROPOSITION 4.5. *Suppose that v meets the conditions* (3.4-i)–(3.4-iii). *Then* $\epsilon^1_{n,h}[v] = \mathrm{O}(n^{1/2}\delta_n^2) = \mathrm{O}(n^{-1/2}[\log n]^4)$ *as $n \to +\infty$.*

PROOF. We consider the auxiliary function

$$F_n^h[v](w) := \int_{\mathbb{C}} \left\{\frac{v(z)-v(w)}{z-w} - \partial v(w)\right\} \text{Б}^{\langle w\rangle}_{n,h}(z)\,\mathrm{d}A(z), \qquad w \in \mathbb{C}.$$

Then the error term defined by (3.14) may be expressed in the form

$$\epsilon^1_{n,h}[v] = \int_{\mathbb{C}} u_n^h(w) F_n^h[v](w)\,\mathrm{d}A(w). \tag{4.6}$$

As v is globally Lipschitz-continuous, $F_n^h[v]$ is uniformly bounded; indeed, we have the estimate

$$\|F_n^h[v]\|_{L^\infty(\mathbb{C})} \le 2\|\nabla v\|_{L^\infty(\mathbb{C})}. \tag{4.7}$$

Let $\mathcal{B}_n$ be the thin "tube" or "belt" around ∂S given by

$$\mathcal{B}_n := \{z \in \mathbb{C} : \delta_{\partial S}(z) < 2\delta_n\}. \tag{4.8}$$

Since the area of $\mathcal{B}_n$ is $\asymp \delta_n$, it follows that

$$\int_{\mathcal{B}_n} u_n^h(w)\big|F_n^h(w)\big|\,\mathrm{d}A \le 2\|\nabla v\|_{L^\infty(\mathbb{C})}\int_{\mathcal{B}_n} u_n^h(w)\,\mathrm{d}A \\ \le 2C\|\nabla v\|_{L^\infty(\mathbb{C})}\int_{\mathcal{B}_n}\mathrm{d}A = \mathrm{O}(\delta_n)$$

as $n \to +\infty$, where C is the constant of Lemma 4.1, which only depends on Q, h. We turn to the estimation of the same integrand over the complementary set $\mathbb{C} \setminus \mathcal{B}_n$. For $w \in \mathbb{C} \setminus \mathcal{B}_n$ and $z \in \mathbb{C}$ with $|z - w| < \delta_n$, then both z, w lie in the same component of $\mathbb{C} \setminus \partial S$ where the assumptions on v tell us that v is uniformly C^2-smooth. By Taylor's formula, then we have

$$v(z) = v(w) + (z - w)\partial v(w) + (\bar{z} - \bar{w})\bar{\partial} v(w) + \mathrm{O}\big(|z - w|^2\big),$$

where the implied constant is uniform. As a consequence, we get that

$$\frac{v(z) - v(w)}{z - w} - \partial v(w) = \bar{\partial} v(w)\frac{\bar{z} - \bar{w}}{z - w} + \mathrm{O}\big(|z - w|\big),$$

with a uniform implied constant. This leads to

$$\int_{\mathbb{D}(w,\delta_n)}\left\{\frac{v(z) - v(w)}{z - w} - \partial v(w)\right\} H_n^{\langle w\rangle}(z)\,\mathrm{d}A(z) \\ = \bar{\partial} v(w)\int_{\mathbb{D}(w,\delta_n)}\frac{\bar{z} - \bar{w}}{z - w}H_n^{\langle w\rangle}(z)\,\mathrm{d}A(z) + \mathrm{O}\big(n^{-1/2}\big) = \mathrm{O}\big(n^{-1/2}\big),$$

by the radial symmetry of the heat kernel; the error term may be obtained by integration against the heat kernel. Next, we use Lemma 4.3 and the global Lipschitz-continuity of v to see that

$$\left|\int_{\mathbb{D}(w,\delta_n)}\left\{\frac{v(z) - v(w)}{z - w} - \partial v(w)\right\}\big[\mathrm{B}_{n,h}^{\langle w\rangle}(z) - H_n^{\langle w\rangle}(z)\big]\,\mathrm{d}A(z)\right| \\ = \mathrm{O}\big(n^{1/2}\delta_n^2\big) = \mathrm{O}\big(n^{-1/2}[\log n]^4\big),$$

uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. Finally, we use the global Lipschitz-continuity of v together with Corollary 4.4 to see that

$$\int_{\mathbb{C}\setminus\mathbb{D}(w,\delta_n)}\left|\frac{v(z) - v(w)}{z - w} - \partial v(w)\right|\mathrm{B}_{n,h}^{\langle w\rangle}(z)\,\mathrm{d}A(z) \\ = \mathrm{O}\big(n^{-1/2}[\log n]^4\big) \qquad \text{as } n \to +\infty,$$

uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. Putting the above ingredients together, we realize that we have obtained that

$$\big|F_n^h(w)\big| = \left|\int_{\mathbb{C}}\left\{\frac{v(z) - v(w)}{z - w} - \partial v(w)\right\}\mathrm{B}_{n,h}^{\langle w\rangle}(z)\,\mathrm{d}A(z)\right| = \mathrm{O}\big(n^{-1/2}[\log n]^4\big)$$

as $n \to +\infty$, uniformly in $w \in \mathbb{C} \setminus \mathcal{B}_n$. This entails that

$$\int_{\mathbb{C}\setminus\mathcal{B}_n} |F_n^h(w)| u_n^h(w)\,\mathrm{d}A(w) = \mathrm{O}\big(n^{-1/2}[\log n]^4\big),$$

which combined with the previous estimate of the integral over $\mathcal{B}_n$ gives the assertion of the proposition. $\square$

4.3. *The second error term.* We shall prove the following proposition.

PROPOSITION 4.6. *We have that for some small* $\beta > 0$,

$$\begin{aligned}\epsilon_{n,h}^2[v] &= -\frac{1}{2\pi n}\int_{\mathbb{C}} [D_n^h]^2 \bar\partial v\,\mathrm{d}A \\ &= \mathrm{O}\big(n^{-\beta/2}\big(\|v\|_{L^\infty(\mathbb{C})} + \|\nabla v\|_{L^\infty(\mathbb{C})}\big)\big) = \mathrm{o}(1) \qquad \textit{as } n \to +\infty.\end{aligned}$$

The proof will involve certain estimates of the function

$$D_n^h(z) = \langle \kappa_z, \nu_n^h\rangle_{\mathbb{C}} = n\langle \kappa_z, \sigma_n^h - \sigma\rangle_{\mathbb{C}} = n\int_{\mathbb{C}} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\,\mathrm{d}A(\zeta).$$

It is convenient to split the integral into two parts:

$$D_n^h(z) = D_{n,\mathrm{I}}^h(z) + D_{n,\mathrm{II}}^h(z),$$

where

$$D_{n,\mathrm{I}}^h(z) := n\int_{\mathcal{B}_n} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\,\mathrm{d}A(\zeta),$$

$$D_{n,\mathrm{II}}^h(z) := n\int_{\mathbb{C}\setminus\mathcal{B}_n} \frac{u_n^h(\zeta) - u(\zeta)}{z-\zeta}\,\mathrm{d}A(\zeta);$$

here, $\mathcal{B}_n$ is the thin "belt" around ∂S given by (4.8). Since

$$n\big[u_n^h(\zeta) - u(\zeta)\big] = K_n^h(\zeta,\zeta) - \frac{2n}{\pi} 1_S(\zeta)\Delta Q(\zeta),$$

we get from Theorem 4.2 that

$$n\big|u_n^h(\zeta) - u(\zeta)\big| = \mathrm{O}(1), \qquad \zeta \in S \setminus \mathcal{B}_n, \tag{4.9}$$

uniformly in ζ, as $n \to +\infty$. The estimate (4.2) supplies fast decay of $n|u_n^h - u|$ in $\mathbb{C} \setminus (S \cup \mathcal{B}_n)$, and together with the above estimate (4.9), this leads to the conclusion that

$$\begin{aligned}\|D_{n,\mathrm{II}}^h\|_{L^\infty(\mathbb{C})} &\le n \sup_{z\in\mathbb{C}} \int_{\mathbb{C}\setminus\mathcal{B}_n} \frac{|u_n^h(\zeta) - u(\zeta)|}{|z-\zeta|}\,\mathrm{d}A(\zeta) \\ &= \mathrm{O}(1) \qquad \text{as } n \to +\infty.\end{aligned} \tag{4.10}$$

We turn to the estimation of $D_{n,\mathrm{I}}^h$.

LEMMA 4.7. *We have*

$$\|D^h_{n,\mathrm{I}}\|_{L^\infty(\mathbb{C})} = \mathrm{O}\big(n^{1/2}[\log n]^3\big) \qquad \text{as } n \to +\infty.$$

PROOF. As we shall see, this follows from the trivial bound $\|u^h_n - u\|_{L^\infty(\mathbb{C})} = \mathrm{O}(1)$ as $n \to +\infty$, which is a consequence of Lemma 4.1. We just need to estimate the integral

$$\int_{\mathcal{B}_n} \frac{\mathrm{d}A(\zeta)}{|z-\zeta|}.$$

Without loss of generality, we can take $z = 0$ and replace $\mathcal{B}_n$ by the rectangle $|x| < 1$, $|y| < \delta_n$ (with $z = x + \mathrm{i}y$). We have

$$\int_{\mathcal{B}_n} \frac{\mathrm{d}A(\zeta)}{|\zeta|} = \int_{-1}^{1} \mathrm{d}x \int_{-\delta_n}^{\delta_n} \frac{\mathrm{d}y}{\sqrt{x^2+y^2}} = \mathrm{I} + \mathrm{II},$$

where I is the integral where x is confined to the interval $-\delta_n < x < \delta_n$, and II is the remaining term. By passing to polar coordinates, we see that

$$\mathrm{I} \asymp \int_0^{\delta_n} \frac{r\,\mathrm{d}r}{r} = \delta_n = n^{-1/2}[\log n]^2$$

and

$$\mathrm{II} \asymp \delta_n \int_{\delta_n}^{1} \frac{\mathrm{d}x}{x} \asymp \delta_n \log\frac{1}{\delta_n} = \mathrm{O}\big(n^{-1/2}[\log n]^3\big) \qquad \text{as } n \to +\infty.$$

The assertion of the lemma is immediate. $\square$

Lemma 4.7 and (4.10) together give us the following estimate of the second error term:

$$\epsilon^2_{n,h}[v] \le C_3[\log n]^6 \|\nabla v\|_{L^1(\mathbb{C})} \tag{4.11}$$

for some positive constant C_3. This comes rather close but is still weaker than what we want. Our strategy will be to use (4.11) and iterate the argument with the Ward identity. This will supply a better estimate in the *interior* of the droplet.

LEMMA 4.8. *For big n, we have that for some positive constant C_4,*

$$\big|D^h_{n,\mathrm{I}}(z)\big| \le C_4 \frac{[\log n]^6}{\delta_{\partial S}(z)^3}, \qquad z \in S.$$

PROOF. Let ψ be a function of Lipschitz norm ≤ 1 supported inside the droplet S, that is, $\|\nabla\psi\|_{L^\infty(\mathbb{C})} \le 1$. Then we have

$$\big|\epsilon^1_{n,h}[\psi]\big| \le 2, \qquad \big|\epsilon^2_{n,h}[\psi]\big| \le C_3[\log n]^6,$$

where the constants do not depend on ψ; the first estimate follows from a combination of (4.6) and (4.7), and the second one is just (4.11). By (3.16) applied to the function ψ, we have

$$\frac{2}{\pi}\int_S \psi D_n^h \Delta Q \,\mathrm{d}A = -\left\langle 2\psi\partial h + \frac{1}{2}\partial\psi, \sigma_n^h\right\rangle_{\mathbb{C}} + \frac{1}{2}\epsilon_{n,h}^1[\psi] - \epsilon_{n,h}^2[\psi],$$

and, therefore,

$$\begin{aligned}\left|\frac{2}{\pi}\int_S \psi D_n^h \Delta Q \,\mathrm{d}A\right| &\le 2\|\psi\|_{L^\infty(\mathbb{C})}\|\nabla h\|_{L^\infty(\mathbb{C})} + \frac{1}{2}\|\nabla\psi\|_{L^\infty(\mathbb{C})} \\ &\quad + \frac{1}{2}\left|\epsilon_{n,h}^1[\psi]\right| + \left|\epsilon_{n,h}^2[\psi]\right| \\ &\le C_5[\log n]^6\end{aligned} \tag{4.12}$$

for a suitable positive constant C_5. The claimed estimate is trivial for $z \in S$ with $\delta_{\partial S}(z) \le n^{-1/3}$, as it is a consequence of the global estimate of Lemma 4.7. In the remaining case when $z \in S$ has $\delta_{\partial S}(z) \ge n^{-1/3}$, we consider the function

$$\psi(\zeta) = \max\left\{\frac{(1/2)\delta_{\partial S}(z) - |\zeta - z|}{\Delta Q(\zeta)}, 0\right\}.$$

Then ψ has Lipschitz norm $\asymp 1$, and by the analyticity of $D_{n,\mathrm{I}}^h$ in $S \setminus \mathcal{B}_n$, we get the mean value identity

$$\begin{aligned}\int_S \psi(\zeta) D_{n,\mathrm{I}}^h(\zeta)\Delta Q(\zeta)\,\mathrm{d}A(\zeta) &= 2\pi D_{n,\mathrm{I}}^h(z)\int_0^{(1/2)\delta_{\partial S}(z)}\left(\frac{1}{2}\delta_{\partial S}(z) - r\right) r\,\mathrm{d}r \\ &= \frac{\pi}{24}\left[\delta_{\partial S}(z)\right]^3 D_{n,\mathrm{I}}^h(z).\end{aligned}$$

Combined with (4.12), this gives the claimed estimate. □

4.3.1. *A flow of curves*. We need to introduce a family of curves $\Gamma[\varepsilon]$, where $\Gamma[0] = \Gamma = \partial S$, for $0 \le \varepsilon \ll 1$. Let $\mathrm{n}_\Gamma(\zeta)$ denote the *exterior* unit normal vector to Γ at the point $\zeta \in \Gamma$. The curve $\Gamma[\varepsilon]$ consists of the points

$$\zeta[\varepsilon] = \zeta[\varepsilon, \Gamma] = \zeta + \varepsilon \mathrm{n}_\Gamma(\zeta), \qquad \zeta \in \Gamma.$$

As Γ is a real-analytically smooth Jordan curve, so is $\Gamma[\varepsilon]$ for small ε. We now check that the normal vector is preserved under the flow of curves $\Gamma[\varepsilon]$:

$$\mathrm{n}_{\Gamma[\varepsilon]}\big(\zeta[\varepsilon]\big) = \mathrm{n}_\Gamma(\zeta), \qquad \zeta \in \Gamma. \tag{4.13}$$

To this end, let Γ be parametrized with positive orientation by $\zeta(t)$, $0 \le t \le 1$, where $\zeta'(t) \ne 0$ and $\zeta(0) = \zeta(1)$. Then $\mathrm{n}_\Gamma(\zeta(t)) = -\mathrm{i}\zeta'(t)/|\zeta'(t)|$, so that

$$\zeta[\varepsilon](t) = \zeta(t) + \varepsilon \mathrm{n}_\Gamma\big(\zeta(t)\big) = \zeta(t) - \mathrm{i}\varepsilon\frac{\zeta'(t)}{|\zeta'(t)|}$$

parameterizes $\Gamma[\varepsilon]$. A calculation gives that

$$\big(\zeta[\varepsilon]\big)'(t)=\zeta'(t)\bigg(1+\varepsilon\frac{\operatorname{Im}[\bar{\zeta}'(t)\zeta''(t)]}{|\zeta'(t)|^3}\bigg),$$

which means that the two tangents point in the same direction. As a consequence, the two normals point in the same direction as well.

Finally, we need an estimate of $D^h_{n,\mathrm{I}}$ in the *exterior* of the droplet S. This will be done in the next subsection by reflecting the previous interior estimate in the curve $\Gamma:=\partial S$. We then use the following lemma. Let us fix some sufficiently small positive number, for example, $\beta=\frac{1}{10}$ will do, and define $\Gamma_n:=\Gamma[n^{-\beta}]$, in the above notation. For big n, Γ_n is then a C^ω-smooth curve in $\mathbb{C}\setminus S$ which is very close to $\Gamma=\partial S$. The complement $\mathbb{C}\setminus\Gamma_n$ has two connectivity components; let Ω_n be component which is bounded, and $\Omega_n^\circledast$ the remaining component, which is unbounded.

Let $L^2(\Gamma_n)$ denote the usual L^2 space of functions on Γ_n with respect to arc-length measure.

LEMMA 4.9. *We have that*

$$\|D^h_{n,\mathrm{I}}\|^2_{L^2(\Gamma_n)}=\mathrm{O}\big(n^{1-(1/2)\beta}\big)\qquad \textit{as } n\to+\infty.$$

Given this estimate, we can complete the proof of Proposition 4.6 as follows.

PROOF OF PROPOSITION 4.6. By the correlation kernel decay in (4.2) and the uniform estimate of D^h_n supplied by (4.10) and Lemma 4.7, we have that

$$\begin{aligned}\epsilon^2_{n,h}[v]&=\int_{\mathbb{C}}vD^h_n(u^h_n-u)\,\mathrm{d}A=\int_{\Omega_n}vD^h_n(u^h_n-u)\,\mathrm{d}A+\mathrm{O}\big(n^{-100}\|v\|_{L^\infty(\mathbb{C})}\big)\\&=\frac{1}{2\pi n}\int_{\Omega_n}v\bar\partial\big([D^h_n]^2\big)\,\mathrm{d}A+\mathrm{O}\big(n^{-100}\|v\|_{L^\infty(\mathbb{C})}\big)\\&=-\frac{1}{2\pi n}\int_{\Omega_n}[D^h_n]^2\bar\partial v\,\mathrm{d}A+\frac{1}{4\pi n}\int_{\Gamma_n}[D^h_n(z)]^2v(z)\,\mathrm{d}z\\&\quad+\mathrm{O}\big(n^{-100}\|v\|_{L^\infty(\mathbb{C})}\big),\end{aligned}$$

if we use the Cauchy–Green formula. As a consequence, we find that

$$\begin{aligned}\big|\epsilon^2_{n,h}[v]\big|\le{}&\frac{1}{2\pi n}\|D^h_n\|^2_{L^2(\Omega_n)}\|\nabla v\|_{L^\infty(\mathbb{C})}+\frac{1}{4\pi n}\|D^h_n\|^2_{L^2(\Gamma_n)}\|v\|_{L^\infty(\mathbb{C})}\\&+\mathrm{O}\big(n^{-100}\|v\|_{L^\infty(\mathbb{C})}\big).\end{aligned}$$

The second term is taken care of by Lemma 4.9. To estimate the first term, we consider the set

$$\mathcal{A}_n=\{z\in\mathbb{C}:\delta_{\partial S}(z)<n^{-\beta}\}.$$

The area of $\mathcal{A}_n$ is $\asymp n^{-\beta}$, and in $S \setminus \mathcal{A}_n$ we have $|D^h_{n,\mathrm{I}}| = \mathrm{O}(n^{3\beta}[\log n]^6)$ (cf. Lemma 4.8). Inside $\mathcal{A}_n$, we apply the uniform bound of Lemma 4.7. Since we have that $\Omega_n \subset S \cup \mathcal{A}_n$, we find that

$$\begin{aligned}\|D^h_n\|^2_{L^2(\Omega_n)} &= \int_{\Omega_n} |D^h_n|^2 \,\mathrm{d}A \le \int_{S\cup\mathcal{A}_n} |D^h_n|^2 \mathrm{d}A \\ &= \int_{\mathcal{A}_n} |D^h_n|^2 \,\mathrm{d}A + \int_{S\setminus A_n} |D^h_n|^2 \,\mathrm{d}A = \mathrm{O}\big(n^{1-\beta}[\log n]^6 + n^{6\beta}[\log n]^{12}\big),\end{aligned}$$

so that

$$\|D^h_n\|^2_{L^2(\Omega_n)} = \mathrm{O}\big(n^{1-(1/2)\beta}\big).$$

This finishes the proof of the proposition. $\square$

4.4. *The proof of Lemma* 4.9. We first establish the following fact, uniformly as $n \to +\infty$:

$$\big|\mathrm{Im}\{\mathrm{n}_\Gamma(\zeta) D^h_{n,\mathrm{I}}(\zeta[n^{-\beta}])\}\big| = \mathrm{O}\big(n^{1/2-(1/4)\beta}\big), \qquad \zeta \in \Gamma. \tag{4.14}$$

PROOF. Without loss of generality, we may take $\zeta = 0$ and $\mathrm{n}_\Gamma(\zeta) = \mathrm{i}$. Then the tangent to Γ at 0 is horizontal, so Γ is the graph of a function $y = y(x)$ where $y(x) = \mathrm{O}(x^2)$ as $x \to 0$. We will show that

$$\big|\mathrm{Re}\{D^h_{n,\mathrm{I}}(\mathrm{i}n^{-\beta}) - D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})\}\big| = \mathrm{O}\big(n^{1/2-(1/4)\beta}\big). \tag{4.15}$$

This implies the desired estimate (4.14), because by Lemma 4.8, there exists a positive constant C_6 such that

$$\big|D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})\big| \le C_6 n^{3\beta}[\log n]^6 \le n^{1/2-(1/3)\beta},$$

where the right-hand side estimate is valid for big n, provided $\beta < \frac{3}{20}$. To obtain (4.15), we notice that

$$\begin{aligned} I &:= \mathrm{Re}\{D^h_{n,\mathrm{I}}(\mathrm{i}n^{-\beta}) - D^h_{n,\mathrm{I}}(-\mathrm{i}n^{-\beta})\} \\ &= n\int_{\mathcal{B}_n} \mathrm{Re}\left\{\frac{1}{z+\mathrm{i}n^{-\beta}} - \frac{1}{z-\mathrm{i}n^{-\beta}}\right\}\big(u^h_n(z) - u(z)\big)\,\mathrm{d}A(z).\end{aligned}$$

We next subdivide the thin belt $\mathcal{B}_n$ into two parts:

$$\mathcal{B}^1_n := \mathcal{B}_n \cap \{x + \mathrm{i}y : \max\{|x|, |y|\} \le n^{-\gamma}\}, \qquad \mathcal{B}^2_n := \mathcal{B}_n \setminus \mathcal{B}^1_n,$$

where γ is a parameter with $0 < \gamma < \beta < \frac{1}{2}$ (we have some freedom here). The part $\mathcal{B}^1_n$ is the local part, and $\mathcal{B}^2_n$ is the remainder. This allows us to split the integral I accordingly: $I = I^1 + I^2$, where

$$I^1 := n\int_{\mathcal{B}^1_n} \mathrm{Re}\left\{\frac{1}{\bar{z}-\mathrm{i}n^{-\beta}} - \frac{1}{z-\mathrm{i}n^{-\beta}}\right\}\big(u^h_n(z) - u(z)\big)\,\mathrm{d}A(z)$$

and

$$I^2 := n \int_{\mathcal{B}_n^2} \mathrm{Re}\left\{ \frac{1}{z + \mathrm{i}n^{-\beta}} - \frac{1}{z - \mathrm{i}n^{-\beta}} \right\} \big(u_n^h(z) - u(z)\big)\, \mathrm{d}A(z);$$

note that in the first formula, we use that for complex numbers ξ, we have $\mathrm{Re}\,\bar{\xi} = \mathrm{Re}\,\xi$. By Lemma 4.1, the function $u_n^h - u$ is uniformly bounded, say $|u_n^h - u| \le C_7$, so that

$$|I^1| \le C_7 n \int_{\mathcal{B}_n^1} \frac{2|\operatorname{Im} z|}{|(z - \mathrm{i}n^{-\beta})(z + \mathrm{i}n^{-\beta})|}\, \mathrm{d}A(z). \tag{4.16}$$

The analogous estimate involving I^2 reads

$$|I^2| \le C_7 n \int_{\mathcal{B}_n^2} \frac{2n^{-\beta}}{|(z - \mathrm{i}n^{-\beta})(z + \mathrm{i}n^{-\beta})|}\, \mathrm{d}A(z). \tag{4.17}$$

Since curve Γ is parameterized by $y = y(x)$ with $y(x) = \mathrm{O}(x^2)$ as $x \to 0$, we see that $|\operatorname{Im} z| = |y| = \mathrm{O}(n^{-2\gamma})$ on $\mathcal{B}_n^1$. Moreover, geometric considerations lead to

$$\big|(z - \mathrm{i}n^{-\beta})(z + \mathrm{i}n^{-\beta})\big| \gtrsim |\operatorname{Re} z|^2 + n^{-2\beta} = x^2 + n^{-2\beta}, \qquad z = x + \mathrm{i}y \in \mathcal{B}_n^1,$$

and

$$\big|(z - \mathrm{i}n^{-\beta})(z + \mathrm{i}n^{-\beta})\big| \asymp |z|^2, \qquad z \in \mathcal{B}_n^2.$$

As we combine the above estimates with (4.16) and (4.17), and recall that $\mathcal{B}_n$ is a thin belt of width $\asymp \delta_n = n^{-1/2}[\log n]^2$ around $\Gamma = \partial S$, we realize that

$$\begin{aligned} |I^1| &\lesssim n^{1-2\gamma}\delta_n \int_{-n^{-\gamma}}^{n^{\gamma}} \frac{\mathrm{d}t}{t^2 + n^{-2\beta}} = n^{1+\beta-2\gamma}\delta_n \int_{-n^{\beta-\gamma}}^{n^{\beta-\gamma}} \frac{\mathrm{d}\tau}{1+\tau^2} \\ &\le \pi n^{1/2+\beta-2\gamma}[\log n]^2 \end{aligned} \tag{4.18}$$

and

$$|I^2| \lesssim n^{1-\beta}\delta_n \int_{n^{-\gamma}}^{1} \frac{\mathrm{d}t}{t^2} \asymp n^{1+\gamma-\beta}\delta_n = n^{1/2+\gamma-\beta}[\log n]^2. \tag{4.19}$$

If we now pick $\gamma := \frac{2}{3}\beta$, we obtain

$$I = I^1 + I^2 = \mathrm{O}\big(n^{1/2-(1/3)\beta}[\log n]^2\big) \qquad \text{as } n \to +\infty,$$

which is even better than claimed. As a consequence, (4.14) follows. $\square$

To complete the proof of Lemma 4.9, we let n_{Γ_n} be the exterior unit normal of Γ_n. By (4.13), n_{Γ_n} at the point $\zeta[n^{-\beta}] \in \Gamma_n$ for $\zeta \in \Gamma$ is the same as $\mathrm{n}_\Gamma(\zeta)$. So, from (4.14) we may derive that

$$\big|\operatorname{Im}[\mathrm{n}_{\Gamma_n} D_{n,\mathrm{I}}^h]\big| = \mathrm{O}\big(n^{1/2-(1/4)\beta}\big) \qquad \text{as } n \to +\infty, \tag{4.20}$$

uniformly on Γ_n. Next, let $\mathbb{D}^e := \{z : |z| > 1\}$ denote the exterior disk, and consider the conformal map

$$\phi_n : \Omega_n^{\circledast} \to \mathbb{D}^e,$$

which fixes the point at infinity ($\phi_n(\infty) = \infty$). We put

$$G_n^h := \frac{\phi_n}{\phi_n'} D_{n,\mathrm{I}}^h.$$

Being the Cauchy transform of a density supported in $\mathcal{B}_n$, the function $D_{n,\mathrm{I}}^h(z)$ is holomorphic in $\mathbb{C} \setminus \mathcal{B}_n$, with decay rate $\mathrm{O}(|z|^{-1})$ as $|z| \to +\infty$. More precisely,

$$D_{n,\mathrm{I}}^h(z) = \frac{n}{z} \int_{\mathcal{B}_n} (u_n^h - u)\, \mathrm{d}A + \mathrm{O}\big(n|z|^{-2}\big) \qquad \text{as } |z| \to +\infty. \tag{4.21}$$

The quotient $\phi_n(z)/\phi_n'(z)$ grows like $z + \mathrm{O}(1)$ as $|z| \to +\infty$, so the function G_n^h gets to be bounded near infinity. Since $\Omega_n^{\circledast} \subset \mathbb{C} \setminus \mathcal{B}_n$ for big n, G_n^h is holomorphic and bounded in $\Omega_n^{\circledast}$ for big n. The conformal mappings ϕ_n are uniformly smooth in $\Omega_n^{\circledast}$, as a consequence of the smoothness assumptions on Γ which lead to the corresponding uniform smoothness of Γ_n. In particular, we have $|\phi_n'| \asymp 1$ in $\Omega_n^{\circledast}$ uniformly in n for big n. The Green function for the point at infinity in $\Omega_n^{\circledast}$ may be expressed as $\log|\phi_n|$, and at the boundary Γ_n its gradient points in the outward normal direction. This means that the outward unit normal n_{Γ_n} is

$$\mathrm{n}_{\Gamma_n}(z) = \frac{\bar\partial \log|\phi_n(z)|}{|(\bar\partial \log|\phi_n(z)|)|} = \frac{\bar\phi_n'(z)|\phi_n(z)|}{\bar\phi_n(z)|\phi_n'(z)|} = \frac{\phi_n(z)|\phi_n'(z)|}{\phi_n'(z)|\phi_n(z)|}, \qquad z \in \Gamma_n.$$

It follows that

$$\mathrm{Im}\big[\mathrm{n}_{\Gamma_n} D_{n,\mathrm{I}}^h\big] = \frac{|\phi_n'|}{|\phi_n|} \mathrm{Im}\big[G_n^h\big] \qquad \text{on } \Gamma_n,$$

so we see that (4.20) asserts that

$$\|\mathrm{Im}[G_n^h]\|_{L^\infty(\Gamma_n)} = \mathrm{O}\big(n^{1/2-(1/4)\beta}\big) \qquad \text{as } n \to +\infty. \tag{4.22}$$

Moreover,

$$G_n^h(\infty) = \mathrm{O}(1) \qquad \text{as } n \to +\infty, \tag{4.23}$$

as we easily see from the decay information (4.21) and from the identity

$$n \int_{\mathcal{B}_n} (u_n^h - u)\, \mathrm{d}A = \int_{S \setminus \mathcal{B}_n} (\mathrm{K}_{n,h}^\sharp - \mathrm{K}_n^h)\, \mathrm{d}A - \int_{\mathbb{C}\setminus(S \cup \mathcal{B}_n)} \mathrm{K}_n^h\, \mathrm{d}A,$$

if we recall the estimate (4.2) and Theorem 4.2. The harmonic conjugation operator is bounded in the setting of H^p spaces on the unit disk (or on the exterior disk if we like), and as we transfer this result to the context of $\Omega_n^{\circledast}$, we get from (4.22) that

$$\|G_n^h\|_{L^p(\Gamma_n)} \le C(p) \|\mathrm{Im}\, G_n\|_{L^p(\Gamma_n)} + \mathrm{O}(1) = \mathrm{O}_p\big(n^{1/2-(1/4)\beta}\big) \qquad \text{as } n \to +\infty$$

for any fixed p, $1 < p < +\infty$, where the positive constant $C(p)$ depends on p. The special case $p = 2$ gives us the assertion of Lemma 4.9. $\square$

5. Proof of the main formula. In this section, we will use the limit form of the Ward identity (Corollary 3.5) to derive our main formula (2.13): for every test function f the limit

$$\langle f, \nu^h\rangle_{\mathbb{C}} := \lim_{n\to+\infty}\langle f, \nu_n^h\rangle_{\mathbb{C}}$$

exists and equals

$$\langle f, \nu^h\rangle_{\mathbb{C}} = \frac{1}{8\pi}\left\{\int_S (\Delta f + f\,\Delta L)\,\mathrm{d}A + \int_{\partial S} f\mathcal{N}(L^S)\,\mathrm{d}s\right\} + \frac{1}{2\pi}\int_{\mathbb{C}} \nabla f^S\cdot\nabla h^S\,\mathrm{d}A. \tag{5.1}$$

5.1. *Decomposition of the test function.* The following statement uses our assumption that ∂S is a C^ω-smooth Jordan curve.

LEMMA 5.1. *Let $f\in C^\infty(\mathbb{C})$ be bounded. Then f has the following representation*:

$$f = f_+ + f_- + f_0,$$

where:

(i) *all three functions are C^∞-smooth and bounded in $\mathbb{C}$*,
(ii) $\bar\partial f_+ = 0$ *and* $\partial f_- = 0$ *in* $\mathbb{C}\setminus S$,
(iii) $f_0 = 0$ *on* $\Gamma = \partial S$.

PROOF. Let us consider a conformal map

$$\phi:\mathbb{D}^e\to\mathbb{C}\setminus S,$$

which preserves the point at infinity; here, $\mathbb{D}^e = \{z\in\mathbb{C}: |z|>1\}$ is the exterior disk. The smoothness assumptions on Γ imply that ϕ is extremely smooth (e.g., real-analytic on $\mathbb{T}$). The restriction of the function $F := f\circ\phi$ to $\mathbb{T}$ is in $C^\infty(\mathbb{T})$, and so it has a Fourier series representation

$$F(\zeta) = \sum_{j=-\infty}^{+\infty} a_j\zeta^j,\qquad \zeta\in\mathbb{T}.$$

The functions

$$F_+(z) = \sum_{j=0}^{+\infty} a_{-j}z^{-j},\qquad F_-(z) = \sum_{j=1}^{+\infty}\frac{a_j}{\bar z^j},$$

are then well defined in the closed exterior disk $\bar{\mathbb{D}}^e$, and C^∞-smooth up to the boundary. It is easy to extend F_+, F_- to C^∞-smooth functions on all of $\mathbb{C}$. Likewise, we may extend ϕ to C^∞-smooth diffeomorphism $\phi:\mathbb{C}\to\mathbb{C}$. We put

$$f_+ := F_+\circ\phi^{-1},\qquad f_- := F_-\circ\phi^{-1},$$

and realize that f_+, f_- are both C^∞-smooth and bounded, and that (ii) holds. Finally, we put

$$f_0 := f - f_+ - f_-.$$

It is automatic that f_0 is C^∞-smooth and bounded, and that f_0 vanishes on $\Gamma = \partial S$. $\square$

CONCLUSION. It is enough to prove the main formula (5.1) only for functions of the form $f = f_+ + f_- + f_0$ as in the last lemma with an *additional* assumption that f_0 is supported inside any given neighborhood of the droplet S. Indeed, either side of the formula (5.1) will not change if we "kill" f_0 outside the neighborhood. The justification is immediate by Lemma 4.1 (exterior decay).

In what follows, we will choose a neighborhood O of S such that the potential Q is real-analytic, strictly subharmonic in O, and

$$\partial Q \neq \partial \check{Q} \qquad \text{in } O \setminus S,$$

and will assume $\operatorname{supp}(f_0) \subset O$.

5.2. *The choice of the vector field in the Ward identity.* We will now compute the limit

$$\langle f, \nu^h \rangle_{\mathbb{C}} := \lim_{n\to+\infty} \langle f, \nu_n^h \rangle_{\mathbb{C}}$$

(and prove its existence) in the case where

$$f = f_+ + f_0.$$

To apply the limit Ward identity (see Corollary 3.5),

$$\begin{aligned} &\frac{2}{\pi}\int_{\mathbb{C}} \{v\Delta Q + \bar\partial v \partial(Q - \check{Q})\} D_n^h \, dA \\ &\quad \to -\left\langle \frac{1}{2}\partial v + 2v\partial h, \sigma \right\rangle_{\mathbb{C}} \qquad \text{as } n \to +\infty, \end{aligned} \tag{5.2}$$

we set

$$v = v_+ + v_0,$$

where

$$v_0 = \frac{\bar\partial f_0}{\Delta Q} 1_S + \frac{f_0}{\partial(Q - \check{Q})} 1_{\mathbb{C}\setminus S} \tag{5.3}$$

and

$$v_+ = \frac{\bar\partial f_+}{\Delta Q} 1_S. \tag{5.4}$$

Here, we need the additional assumption made on the support of f_0. We may combine the above to

$$v = \frac{\bar\partial f}{\Delta Q} 1_S + \frac{f_0}{\partial(Q - \check{Q})} 1_{\mathbb{C}\setminus S}.$$

We calculate that

$$v \Delta Q + \bar\partial v \partial(Q - \check{Q}) = \bar\partial f \qquad \text{on } \mathbb{C} \setminus \partial S,$$

but to plug this information into (5.2), we need to that it is an identity in the sense of distribution theory on all of $\mathbb{C}$. This will be all right if, for example, v is Lipschitz-continuous near ∂S. We would then also need to know that v satisfies the conditions (3.4-i)–(3.4-iii).

LEMMA 5.2. *The vector field v defined above is bounded and globally Lipschitz-continuous. Moreover, the restrictions of v to S and to $S^{\circledast} := (\mathbb{C} \setminus S) \cup \partial S$ are both C^∞-smooth.*

PROOF. The vector field v_+ is C^∞-smooth and supported on S, as $\bar\partial f_+$ is C^∞-smooth and supported on S, and $\Delta Q \neq 0$ on S. It remains to handle the vector field v_0. We need to check the following items:

(i) $v_0|_S$ and $v_0|_{S^\circledast}$ are both C^∞-smooth, and
(ii) v_0 is continuous across ∂S.

PROOF OF (i). It is clear from the defining formula that $v_0|_S$ is C^∞-smooth. As for $v_0|_{S^\circledast}$, we have $v_0 = f_0/g$ in $\mathbb{C} \setminus S$ where $g = \partial(Q - \check{Q})$. Since the statement is local, we consider a conformal map ψ that takes a neighbourhood of a boundary point in ∂S onto a neighbourhood of a point in $\mathbb{R}$ and takes (parts of) ∂S to $\mathbb{R}$. We fix the map so that (locally) $S^\circledast$ is mapped into the upper half plane $y \geq 0$. If we denote $F = f_0 \circ \psi$ and $G = g \circ \psi$, then $F = 0$ and $G = 0$ on $\mathbb{R}$. Moreover, locally, G is the restriction to $y \geq 0$ of a real-analytic function, with non-vanishing partial derivative $\partial_y G$ along the real line. Thus it is enough to check that

$$\frac{F(x + \mathrm{i}y)}{y} = \int_0^1 \frac{\partial F}{\partial y}(x + \mathrm{i}y\tau)\, \mathrm{d}\tau$$

has bounded derivatives of all orders. But this is pretty obvious, since we may differentiate under the integral sign.

PROOF OF (ii). Let $\mathrm{n} = \mathrm{n}_\Gamma(\zeta)$ be the exterior unit normal with respect to S. By Taylor's formula, we have

$$f_0(\zeta + \delta \mathrm{n}) = \delta \frac{\partial f_0}{\partial \mathrm{n}}(\zeta) + \mathrm{O}(\delta^2) = 2\delta \bar\partial f_0(\zeta)\overline{\mathrm{n}(\zeta)} + \mathrm{O}(\delta^2).$$

Similarly, if $g := \partial(Q - \check{Q})$ on $\mathbb{C} \setminus S$, then g extends to a real-analytically smooth function on $S^{\circledast}$. This real-analytic function has a unique extension to a neighborhood of $S^{\circledast}$, which we also denote by g. Since $g = 0$ on ∂S and $\bar\partial g = \Delta Q$ in $\mathbb{C} \setminus S$, Taylor's formula gives that

$$g(\zeta + \delta \mathrm{n}) = \delta \frac{\partial g}{\partial \mathrm{n}}(\zeta) + \mathrm{O}(\delta^2) = 2\delta \bar\partial g(\zeta)\overline{n(\zeta)} + \mathrm{O}(\delta^2).$$

It follows that

$$\frac{f_0(\zeta + \delta \mathrm{n})}{g(\zeta + \delta \mathrm{n})} = \frac{\bar\partial f_0(\zeta)}{\Delta Q(\zeta)} + \mathrm{O}(\delta),$$

and an inspection shows that the implied constant is locally uniform in $\zeta \in \Gamma = \partial S$. This shows that v_0 is continuous across Γ. $\square$

We have now established that the vector field $v = v_0 + v_+$ meets the conditions (3.4-i)–(3.4-iii). This gives us the following result.

COROLLARY 5.3. *If $f = f_0 + f_+$, then*

$$\langle f, \nu^h \rangle_{\mathbb{C}} = \tfrac{1}{4}\langle \partial v, \sigma \rangle_{\mathbb{C}} + \langle v \partial h, \sigma \rangle_{\mathbb{C}}.$$

PROOF. The conclusion is immediate from (5.2) and (3.11). $\square$

5.3. *The conclusion of the proof.*

5.3.1. *General test functions.* We now turn to the general case

$$f = f_+ + f_0 + f_-.$$

In view of Corollary 5.3, we have

$$\langle f_+, \nu^h \rangle_{\mathbb{C}} = \tfrac{1}{4}\langle \partial v_+, \sigma \rangle_{\mathbb{C}} + \langle v_+ \partial h, \sigma \rangle_{\mathbb{C}},$$

where v_+ is given by (5.4). By applying complex conjugation to this relation, while using the fact that the perturbation h is real-valued and the measures ν_n^h are all real-valued (and so the limit ν^h is real-valued, too), we get a similar expression for f_-:

$$\langle f_-, \nu^h \rangle_{\mathbb{C}} = \tfrac{1}{4}\langle \bar\partial v_-, \sigma \rangle_{\mathbb{C}} + \langle v_- \bar\partial h, \sigma \rangle_{\mathbb{C}},$$

where

$$v_- := \frac{\partial f_-}{\partial \bar\partial Q} \cdot 1_S. \tag{5.5}$$

Adding up the three contributions from f_+, f_-, f_0, we find that

$$\begin{aligned}\langle f, \nu^h \rangle_{\mathbb{C}} &= \tfrac{1}{4}\langle \partial v_0, \sigma \rangle_{\mathbb{C}} + \langle v_0 \partial h, \sigma \rangle_{\mathbb{C}} + \tfrac{1}{4}\langle \partial v_+, \sigma \rangle_{\mathbb{C}} + \langle v_+ \partial h, \sigma \rangle_{\mathbb{C}} \\ &\quad + \tfrac{1}{4}\langle \bar\partial v_-, \sigma \rangle_{\mathbb{C}} + \langle v_- \bar\partial h, \sigma \rangle_{\mathbb{C}}.\end{aligned}$$

The expression we get when we put $h=0$ is $\nu=\nu^0$, so that

$$\langle f,\nu\rangle_{\mathbb{C}}=\tfrac{1}{4}\langle\partial v_0+\partial v_+ +\bar\partial v_-,\sigma\rangle_{\mathbb{C}} \tag{5.6}$$

and

$$\langle f,\nu^h-\nu\rangle_{\mathbb{C}}=\langle v_0\partial h+v_+\partial h+v_-\bar\partial h,\sigma\rangle_{\mathbb{C}}. \tag{5.7}$$

5.3.2. *The computation of* ν. We recall that

$$\mathrm{d}\sigma=\frac{1}{2\pi}1_S\Delta Q\,\mathrm{d}A=\frac{2}{\pi}1_S\varDelta Q\,\mathrm{d}A\quad\text{and}\quad L=\log\Delta Q.$$

Using (5.6), we compute

$$\begin{aligned}\langle f,\nu\rangle_{\mathbb{C}}&=\frac{1}{2\pi}\int_S\left\{\partial\left(\frac{\bar\partial f_0+\bar\partial f_+}{\partial\bar\partial Q}\right)+\bar\partial\left(\frac{\partial f_-}{\partial\bar\partial Q}\right)\right\}\varDelta Q\,\mathrm{d}A\\&=\frac{1}{2\pi}\int_S\{\varDelta(f_0+f_++f_-)-\bar\partial f_0\partial\log\varDelta Q\\&\qquad-\bar\partial f_+\partial\log\varDelta Q-\partial f_-\bar\partial\log\varDelta Q\}\,\mathrm{d}A\\&=\frac{1}{2\pi}\int_S\{\varDelta f-\bar\partial f_0\partial L-\bar\partial f_+\partial L-\partial f_-\bar\partial L\}\,\mathrm{d}A.\end{aligned}$$

At this point, we modify L outside some neighborhood of S to get a smooth function with compact support. We will still use the notation L for the modified function. The last expression clearly does not change as a result of this modification. We can now transform the part of the integral which involves L as follows using the Cauchy–Green formula:

$$\begin{aligned}-\int_S\{\bar\partial f_0\partial L+\bar\partial f_+\partial L+\partial f_-\bar\partial L\}\,\mathrm{d}A&=\int_S f_0\varDelta L\,\mathrm{d}A-\int_{\mathbb{C}}\{\bar\partial f_+\partial L+\partial f_-\bar\partial L\}\,\mathrm{d}A\\&=\int_S f_0\varDelta L\,\mathrm{d}A+\int_{\mathbb{C}}(f_++f_-)\varDelta L\,\mathrm{d}A\\&=\int_S f\varDelta L\,\mathrm{d}A+\int_{\mathbb{C}\setminus S}f^S\varDelta L\,\mathrm{d}A.\end{aligned}$$

In other words, we have that

$$\langle f,\nu\rangle_{\mathbb{C}}=\frac{1}{8\pi}\left\{\int_S(\Delta f+f\Delta L)\,\mathrm{d}A+\int_{\mathbb{C}\setminus S}f^S\Delta L\,\mathrm{d}A\right\}.$$

We remark that the formula for $\langle f,\nu\rangle_{\mathbb{C}}$ was stated in this form in [3].

Finally, we express the last integral in terms of the Neumann jump. By Green's formula, we have

$$\begin{aligned}\int_{\mathbb{C}\setminus S}f^S\Delta L\,\mathrm{d}A&=\int_{\mathbb{C}\setminus S}(f^S\Delta L-L\Delta f^S)\,\mathrm{d}A=\int_{\partial S}\left(f^S\frac{\partial L}{\partial n^{\circledast}}-L^S\frac{\partial f^S}{\partial n^{\circledast}}\right)\mathrm{d}s\\&=\int_{\partial S}\left(f^S\frac{\partial L}{\partial n^{\circledast}}-f^S\frac{\partial L^S}{\partial n^{\circledast}}\right)\mathrm{d}s=\int_{\partial S}f\mathcal{N}(L^S)\,\mathrm{d}s,\end{aligned}$$

where the last step just involved the definition of the Neumann jump. In an intermediate step, we used that

$$\int_{\partial S}\left(f^S\frac{\partial L^S}{\partial \mathrm{n}^{\circledast}}-L^S\frac{\partial f^S}{\partial \mathrm{n}^{\circledast}}\right)\mathrm{d}s=\int_{\mathbb{C}\setminus S}(f^S\Delta L^S-L^S\Delta f^S)\,\mathrm{d}A=0,$$

which comes from Green's formula. Here, $\mathrm{n}^{\circledast}$ is the unit normal vector which point into S. In conclusion, we arrive at

$$\langle f,\nu\rangle_{\mathbb{C}}=\frac{1}{8\pi}\left\{\int_S(\Delta f+f\Delta L)\,\mathrm{d}A+\int_{\partial S}f\mathcal{N}(L^S)\,\mathrm{d}s\right\}. \tag{5.8}$$

5.3.3. *The computation of* $\nu^h-\nu$. Using the identity (5.7), we can deduce that

$$\begin{aligned}\langle f,\nu^h-\nu\rangle_{\mathbb{C}}&=\frac{2}{\pi}\int_S\{\bar\partial f_+\partial h+\partial f_-\bar\partial h+\bar\partial f_0\partial h\}\,\mathrm{d}A\\&=\frac{1}{2\pi}\int\nabla f^S\cdot\nabla h^S\,\mathrm{d}A.\end{aligned} \tag{5.9}$$

This is because of the following calculations:

$$\int_S\bar\partial f_+\partial h\,\mathrm{d}A=\int_{\mathbb{C}}\bar\partial f_+\partial h\,\mathrm{d}A=-\frac14\int_{\mathbb{C}}f_+\Delta h\,\mathrm{d}A=\frac14\int_{\mathbb{C}}\nabla f_+\cdot\nabla h\,\mathrm{d}A,$$

and analogously

$$\int_S\partial f_-\bar\partial h\,\mathrm{d}A=\frac14\int_{\mathbb{C}}\nabla f_-\cdot\nabla h\,\mathrm{d}A;$$

moreover, on the other hand, we have

$$\int_S\bar\partial f_0\partial h\,\mathrm{d}A=-\frac14\int_S f_0\Delta h\,\mathrm{d}A=\frac14\int_S\nabla f_0\cdot\nabla h\,\mathrm{d}A.$$

The above three identities lead to

$$\langle f,\nu^h-\nu\rangle_{\mathbb{C}}=\frac{1}{2\pi}\left\{\int_S\nabla f\cdot\nabla h\,\mathrm{d}A+\int_{\mathbb{C}\setminus S}\nabla f^S\cdot\nabla h\,\mathrm{d}A\right\},$$

and if we use that

$$\int_{\mathbb{C}\setminus S}\nabla f^S\cdot\nabla h\,\mathrm{d}A=\int_{\mathbb{C}\setminus S}\nabla f^S\cdot\nabla h^S\,\mathrm{d}A,$$

which is a consequence of the fact that harmonic functions minimize the Dirichlet norm, we arrive at (5.9) right away. By combining (5.8) and (5.9), we see that

$$\langle f,\nu^h\rangle_{\mathbb{C}}=\frac{1}{8\pi}\left\{\int_S(\Delta f+f\Delta L)\,\mathrm{d}A+\int_{\partial S}f\mathcal{N}(L^S)\,\mathrm{d}s\right\}+\frac{1}{2\pi}\int_{\mathbb{C}}\nabla f^S\cdot\nabla h^S\,\mathrm{d}A,$$

and the main formula (5.1) has been completely established. □

APPENDIX: THE PROOF OF THEOREM 4.2

A.1. Polynomial Bergman spaces. For a suitable (extended) real-valued function ϕ, we denote by $L^2(\mathrm{e}^{-2\phi})$ the space normed by

$$\|f\|^2_{\mathrm{e}^{-2\phi}} := \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-2\phi}\, \mathrm{d}A.$$

We denote by $A^2(\mathrm{e}^{-2\phi})$ the subspace of L^2_ϕ consisting of a.e. entire functions; $\mathrm{Pol}_n(\mathrm{e}^{-2\phi})$ denotes the subspace consisting of analytic polynomials of degree at most $n-1$.

Next, we consider a potential Q, real-analytic with $\Delta Q > 0$ on the droplet S, which is subject to the usual growth condition. The smoothness assumption is excessive here, and one can do with less (e.g., C^∞-smoothness is enough, cf. [6]). We will discuss this issue below. We are interested in the perturbed weight

$$Q_n^h := Q - \frac{1}{n} h,$$

where h is a C^∞-smooth bounded real-valued function.

We denote by k_n the reproducing kernel for the space $\mathrm{Pol}_n(\mathrm{e}^{-2nQ})$. The corresponding orthogonal projection $\mathbf{P}_n : L^2(\mathrm{e}^{-2nQ}) \to \mathrm{Pol}_n(\mathrm{e}^{-2nQ})$ is then given by

$$\mathbf{P}_n[f](z) = \int_{\mathbb{C}} \mathrm{k}_n(z,w) f(w) \mathrm{e}^{-2nQ(w)}\, \mathrm{d}A(w), \qquad z \in \mathbb{C}.$$

Analogously, in the perturbed case, the orthogonal projection map $\mathbf{P}_n^h : L^2(\mathrm{e}^{-2nQ_n^h}) \to \mathrm{Pol}_n(\mathrm{e}^{-2nQ_n^h})$ is given in terms of the reproducing kernel k_n^h for the space $\mathrm{Pol}_n(\mathrm{e}^{-2nQ_n^h})$.

A.2. Polarization of smooth weights. The polarization of C^∞-smooth weights was outlined briefly in [6]. We will explain how this works in the simple case of one complex variable.

We begin with a C^∞-smooth function $F : \mathbb{R} \to \mathbb{C}$. To describe the natural extensions, we need the notation of *flat functions*. We say that a continuous function $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-*flat* if

$$|f(z)| \le C(N,R) |\operatorname{Im} z|^N, \qquad |z| \le R,$$

holds for all positive N, R, for some positive constant $C(N,R)$. It is well known that a C^∞-smooth function $F : \mathbb{R} \to \mathbb{C}$ has an extension $\tilde{F} : \mathbb{C} \to \mathbb{C}$ with $\bar\partial \tilde{F}$ which is $\mathbb{R}$-flat; we call such an extension *almost holomorphic*. Also, if $G : \mathbb{C} \to \mathbb{R}$ has the property that $\bar\partial G$ is $\mathbb{R}$-flat, then the restriction of G to $\mathbb{R}$ is C^∞-smooth. So the property of having an extension with $\mathbb{R}$-flat $\bar\partial$-derivative characterizes the C^∞-smooth functions. If the almost holomorphic extension $\tilde{F}$ has the property $\mathrm{conj}(\tilde{F}(z)) = \tilde{F}(z)$ ["conj" stands for complex conjugation], we say that $\tilde{F}$ is *symmetric*. By considering the function

$$\tfrac{1}{2}\big(\tilde{F}(z) + \overline{\tilde{F}(\bar z)}\big)$$

in place of $\tilde{F}$, we see that every C^∞-smooth real-valued function $F:\mathbb{R}\to\mathbb{R}$ has a symmetric almost holomorphic extension. The almost holomorphic extensions $\tilde{F}$ which are possible all differ by an $\mathbb{R}$-flat function. Next, if $F:\mathbb{R}^2\to\mathbb{C}$ is C^∞-smooth, it has an extension $\tilde{F}:\mathbb{C}^2\to\mathbb{C}$ such that $\bar{\partial}_1\tilde{F}$ and $\bar{\partial}_2\tilde{F}$ are both $\mathbb{R}^2$-flat in the natural sense; such an extension is said to be almost holomorphic in this setting as well. If F is real-valued, it has a symmetric almost holomorphic extension $\tilde{F}$; this means that $\operatorname{conj}(\tilde{F}(z_1,z_2))=\tilde{F}(\bar{z}_1,\bar{z}_2)$, as in the one-variable case. We apply this to the setting of a C^∞-smooth function $\Phi:\mathbb{C}\to\mathbb{R}$. By equating the complex plane with $\mathbb{R}^2$ in the standard fashion, we find a function $\Phi_0:\mathbb{R}^2\to\mathbb{R}$ with $\Phi_0(x_1,x_2)=\Phi(x_1+\mathrm{i}x_2)$. This function Φ_0 has a symmetric almost holomorphic extension $\tilde{\Phi}_0(z_1,z_2)$. Now, if we define $\tilde{\Phi}$ to be the function

$$\tilde{\Phi}(z,\bar{w}):=\tilde{\Phi}_0\left(\frac{z+\bar{w}}{2},\frac{z-\bar{w}}{2\mathrm{i}}\right),$$

we have a function with $\tilde{\Phi}(z,\bar{z})=\Phi(z)$ and the Hermitian property $\overline{\tilde{\Phi}(z,\bar{w})}=\tilde{\Phi}(w,\bar{z})$. Moreover, if Φ were real-analytic to start with, we would naturally choose $\tilde{\Phi}(z,\bar{w})$ to be holomorphic in $(z,\bar{w})$ near the diagonal $z=w$. We call $\tilde{\Phi}(z,\bar{w})$ the *polarization* of $\Phi(z)$. The theory of almost holomorphic functions has its roots in the independent work of Hörmander and of Dyn'kin.

A.3. Approximate Bergman kernels. We define approximate reproducing kernels and Bergman projection as follows. In the unperturbed case $h=0$, the well-known first-order approximation inside the droplet is given by the expression

$$\mathrm{k}^\sharp(z,w)=\frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\mathrm{e}^{2nQ(z,\bar{w})},$$

where $Q(z,\bar{w})$ is a polarization of Q (see Section A.2), which in our real-analytic situation means that near the diagonal $z=w$, $Q(z,\bar{w})$ is the unique holomorphic function in $(z,\bar{w})$ with

$$Q(w,\bar{w})=Q(w).$$

The polarization can be defined (up to flat functions) also for C^∞-smooth functions Q; see Section A.2 for details. In the perturbed case, we could do the same, and use a polarization $h(z,\bar{w})$ of h, which would give us the expression

$$\frac{2}{\pi}\{n(\partial_1\partial_2 Q)(z,\bar{w})-(\partial_1\partial_2 h)(z,\bar{w})\}\mathrm{e}^{2nQ(z,\bar{w})}\mathrm{e}^{-2h(z,\bar{w})}\tag{A.1}$$

for the approximate perturbed Bergman kernel. Now, we are going to throw away all terms in (A.1) that can go into the error term $\mathrm{O}(\mathrm{e}^{n[Q(z)+Q(w)]})$. This allows us to simplify the approximate perturbed Bergman kernel in (A.1) to

$$\frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\mathrm{e}^{2nQ(z,\bar{w})}\mathrm{e}^{-2h(z,\bar{w})}.\tag{A.2}$$

Moreover, Taylor's formula shows that the polarization of h looks like

$$h(z,\bar{w}) = h(w) + (z-w)\partial h(w) + \mathrm{O}\big(|z-w|^2\big),$$

and it is actually possible to throw the "O" term into the error term. This leaves us with the approximate perturbed Bergman kernel

$$\mathrm{k}^{\sharp}_{n,h}(z,w) := \frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\mathrm{e}^{2nQ(z,\bar{w})}\mathrm{e}^{-2h_w(z)}, \tag{A.3}$$

where $h_w(z) := h(w) + (z-w)\partial h(w)$ is the Taylor approximant of $h(z,w)$. This kernel is not Hermitian, which is the cost of replacing the polarization of h with its Taylor approximant. However, it is easy to show that it is almost Hermitian, in the sense that

$$\big|\mathrm{k}^{\sharp}_{n,h}(z,w) - \overline{\mathrm{k}^{\sharp}_{n,h}(w,z)}\big| = \mathrm{O}\big(\mathrm{e}^{n[Q(z)+Q(w)]}\big) \tag{A.4}$$

as $n \to +\infty$, where the implied constant only depends on Q, h, provided that z, w are confined to the droplet S and $|z-w| < \delta_n$. The corresponding approximate Bergman projection is

$$\mathbf{P}^{\sharp}_{n,h}[f](w) = \int_S \overline{\mathrm{k}^{\sharp}_{n,h}(\zeta,w)} f(\zeta)\mathrm{e}^{-2nQ^h_n(\zeta)}\,\mathrm{d}A(\zeta),$$

whenever the integral makes sense.

A.4. Local approximate Bergman projections. We have the following result, which says that the approximate perturbed Bergman kernel is indeed a good approximation.

LEMMA A.1. *If $z \in S$, $\delta_{\partial S}(z) > 2\delta_n$, and if $|z-w| < \delta_n$, then*

$$\big|\mathrm{k}^h_n(z,w) - \mathrm{k}^{\sharp}_{n,h}(z,w)\big| = \mathrm{O}\big(\mathrm{e}^{n[Q(z)+Q(w)]}\big)$$

as $n \to +\infty$, where the implied constant only depends on Q, h and not on n.

From Lemma A.1, Theorem 4.2 follows in a straightforward fashion.

PROOF OF THEOREM 4.2. We recall that the correlation kernel of the determinantal process associated with the perturbed potential Q^h_n is

$$\mathrm{K}^h_n(z,w) = \mathrm{k}^h_n(z,w)\mathrm{e}^{-n[Q^h_n(z)+Q^h_n(w)]} = \mathrm{k}^h_n(z)\mathrm{e}^{-n[Q(z)+Q(w)]}\mathrm{e}^{h(z)+h(w)}.$$

In view of Lemma A.1, we then have

$$\mathrm{K}^h_n(z,w) = \frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar{w})\mathrm{e}^{n[2Q(z,\bar{w})-Q(z)-Q(w)]}\mathrm{e}^{-2h_w(z)+h(z)+h(w)} + \mathrm{O}(1).$$

Moreover, since Taylor's formula gives

$$-2h_w(z) + h(z) + h(w) = -2\mathrm{i}\,\mathrm{Im}\big[(z-w)\partial h(w)\big] + \mathrm{O}\big(|z-w|^2\big),$$

we get that

$$\mathrm{K}_n^h(z,w) = \frac{2n}{\pi}\big\{(\partial_1\partial_2 Q)(z,\bar w) + \mathrm{O}\big(|z-w|^2\big)\big\}$$
$$\times\, \mathrm{e}^{n[2Q(z,\bar w)-Q(z)-Q(w)]}\mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}[(z-w)\partial h(w)]} + \mathrm{O}(1).$$

Next, an exercise involving Taylor's formula shows that for z, w close enough to one another,

$$\mathrm{Re}\big[2Q(z,\bar w) - Q(z) - Q(w)\big] \le -b|z-w|^2 \tag{A.5}$$

holds for some constant $b > 0$, as long as z, w are confined to S. This allows us to use the elementary estimate

$$t\mathrm{e}^{-nt} \le \frac{1}{n}, \qquad 0 \le t < +\infty,$$

to get rid of the $\mathrm{O}(|z-w|^2)$ term:

$$\mathrm{K}_n^h(z,w) = \frac{2n}{\pi}(\partial_1\partial_2 Q)(z,\bar w)\mathrm{e}^{n[2Q(z,\bar w)-Q(z)-Q(w)]}\mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}[(z-w)\partial h(w)]} + \mathrm{O}(1).$$

Theorem 4.2 is now immediate. $\square$

In the sequel, a lot of positive constants will appear, and we will denote them by C_j for positive integers j. We will usually not mention each appearance of such a constant.

We turn to the proof of Lemma A.1. The first ingredient is the following.

LEMMA A.2. *Let χ_z be a cut-off function with the following properties*: $0 \le \chi_z \le 1$ *on* $\mathbb{C}$, *while* $\chi_z = 1$ *on* $\mathbb{D}(z, \frac{3}{2}\delta_n)$ *and* $\chi_z = 0$ *off* $\mathbb{D}(z, 2\delta_n)$; *in addition*, *we require that* $\|\bar\partial\chi_z\|_{L^2(\mathbb{C})} \le C_1$, *for some positive absolute constant* C_1. *Then*, *if* f *is analytic and bounded in the disk* $\mathbb{D}(z, 2\delta_n)$, *we have*

$$\big|f(z) - \mathbf{P}^\sharp_{n,h}[\chi_z f](z)\big| \le C_2 n^{-1/2}\mathrm{e}^{nQ(z)}\bigg\{\int_{\mathbb{D}(z,2\delta_n)}|f|\mathrm{e}^{-2nQ}\,\mathrm{d}A\bigg\}^{1/2},$$

where C_2 *is a positive constant.*

PROOF. Without loss of generality, we may take $z = 0$; we then write $\chi_z = \chi_0$. We observe that

$$\mathbf{P}^\sharp_{n,h}[\chi_0 f](0)$$
$$= \frac{2n}{\pi}\int_S [\chi_0 f](\zeta)(\partial_1\partial_2 Q)(0,\bar\zeta)\mathrm{e}^{2[h(\zeta)-h(0)-\bar\zeta\bar\partial h(0)]}\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta)]}\,\mathrm{d}A(\zeta).$$

Here, we understand that $\chi_0 f$ is extended to vanish off $\mathbb{D}(z, 2\delta_n) = \mathbb{D}(0, 2\delta_n)$. Since

$$\bar\partial_\zeta\big\{\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta,\bar\zeta)]}\big\} = -2n\big\{(\partial_2 Q)(\zeta,\bar\zeta) - (\partial_2 Q)(0,\bar\zeta)\big\}\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta,\bar\zeta)]},$$

we may rewrite the expression as follows:

$$\begin{aligned}(\mathrm{A}.6)\qquad &\mathbf{P}^{\sharp}_{n,h}[\chi_0 f](0)\\ &\quad= -\frac{1}{\pi}\int_S \frac{1}{\zeta} f(\zeta)\chi_0(\zeta)A(\zeta)B(\zeta)\bar\partial\big\{\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta,\bar\zeta)]}\big\}\,\mathrm{d}A(\zeta),\end{aligned}$$

where

$$A(\zeta) = \frac{\zeta(\partial_1\partial_2 Q)(0,\bar\zeta)}{(\partial_2 Q)(\zeta,\bar\zeta)-(\partial_2 Q)(0,\bar\zeta)}, \qquad B(\zeta) = \mathrm{e}^{2[h(\zeta)-h(0)-\bar\zeta\bar\partial h(0)]}.$$

It is an elementary but important observation that

$$(\mathrm{A}.7)\qquad A(\zeta), B(\zeta) = \mathrm{O}(1), \qquad \bar\partial A(\zeta), \bar\partial B(\zeta) = \mathrm{O}(|\zeta|)$$

for $\zeta \in \bar{\mathbb{D}}(0, 2\delta_n)$, where the implicit constants are uniformly bounded throughout. Since $A(0) = B(0) = 1$, an integration by parts exercise based on (A.6) gives that

$$(\mathrm{A}.8)\qquad \mathbf{P}^{\sharp}_{n,h}[\chi_0 f](0) = f(0) + \beta_1 + \beta_2,$$

where

$$\beta_1 := \int_{\mathbb{C}} A(\zeta)B(\zeta)\bar\partial(f\chi_0)(\zeta)\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta)]}\frac{\mathrm{d}A(\zeta)}{\zeta}$$

and

$$\beta_2 := \int_{\mathbb{C}} (f\chi_0)(\zeta)\bar\partial\big(A(\zeta)B(\zeta)\big)\mathrm{e}^{2n[Q(0,\bar\zeta)-Q(\zeta)]}\frac{\mathrm{d}A(\zeta)}{\zeta}.$$

Since $\bar\partial(f\chi_0)$ is supported in the annulus $\bar{\mathbb{D}}(0, 2\delta_n) \setminus \mathbb{D}(0, \frac{3}{2}\delta_n)$, we may estimate β_1 by

$$\begin{aligned}(\mathrm{A}.9)\qquad |\beta_1| \le \frac{C_3}{\delta_n}\int_{\mathbb{D}(0,2\delta_n)\setminus\mathbb{D}(0,(3/2)\delta_n)} &\big|f(\zeta)\bar\partial\chi_0(\zeta)\big|\\ &\times \mathrm{e}^{2n[\operatorname{Re} Q(0,\bar\zeta)-Q(\zeta)]}\,\mathrm{d}A(\zeta),\end{aligned}$$

where C_3 is the product of the bounds of $|A(\zeta)|$ and $|B(\zeta)|$ in (A.7). Analogously, we see that since $f\chi_0$ is supported in the disk $\bar{\mathbb{D}}(0, 2\delta_n)$, and $\bar\partial(AB) = A\bar\partial B + B\bar\partial A$,

$$(\mathrm{A}.10)\qquad |\beta_2| \le C_4\int_{\mathbb{D}(0,2\delta_n)} \big|f(\zeta)\chi_0(\zeta)\big|\mathrm{e}^{2n[\operatorname{Re} Q(0,\bar\zeta)-Q(\zeta)]}\,\mathrm{d}A(\zeta),$$

where constant C_4 comes from (A.7). Next, we observe that by (A.5),

$$2n \operatorname{Re} Q(0, \bar\zeta) - nQ(\zeta) \le nQ(0) - nb|\zeta|^2,$$

so that the Cauchy–Schwarz inequality applied to (A.10) gives ($0 \le \chi_0 \le 1$)

$$
\begin{aligned}
|\beta_2|^2 &\le C_4^2 \mathrm{e}^{2nQ(0)} \int_{\mathbb{D}(0,2\delta_n)} |f|^2 \mathrm{e}^{-2nQ}\,\mathrm{d}A \int_{\mathbb{D}(0,2\delta_n)} \mathrm{e}^{-2nb|\zeta|^2}\,\mathrm{d}A(\zeta) \\
&\le C_4^2 \frac{\pi}{nb} \mathrm{e}^{2nQ(0)} \int_{\mathbb{D}(0,2\delta_n)} |f|^2 \mathrm{e}^{-2nQ}\,\mathrm{d}A.
\end{aligned}
\tag{A.11}
$$

Similarly, we obtain from (A.9) that

$$
\begin{aligned}
|\beta_1|^2 \le \frac{C_3^2}{\delta_n^2} \mathrm{e}^{2nQ(0)} & \int_{\mathbb{D}(0,2\delta_n)\setminus\mathbb{D}(0,(3/2)\delta_n)} |f|^2 \mathrm{e}^{-2nQ}\,\mathrm{d}A \\
&\times \int_{\mathbb{D}(0,2\delta_n)\setminus\mathbb{D}(0,(3/2)\delta_n)} \left|\bar\partial\chi_0(\zeta)\right|^2 \mathrm{e}^{-2nb|\zeta|^2}\,\mathrm{d}A(\zeta) \\
&\le C_1^2 C_3^2 \mathrm{e}^{2nQ(0)} \frac{\mathrm{e}^{-2nb\delta_n^2}}{\delta_n^2} \int_{\mathbb{D}(0,2\delta_n)} |f|^2 \mathrm{e}^{-2nQ}\,\mathrm{d}A,
\end{aligned}
\tag{A.12}
$$

where we used the assumed bound on the L^2-norm of $\bar\partial\chi_0$. Finally, the claimed bound now follows from (A.8) combined with the estimates (A.11) and (A.12). □

PROOF OF LEMMA A.1. We are in the setting that $\mathrm{dist}_{\mathbb{C}}(z, \mathbb{C}\setminus S) > 2\delta_n$ and $|w-z| < \delta_n$. By applying Lemma A.2 to the function $f(\zeta) = \mathrm{k}_n^h(\zeta, w)$, we obtain that

$$
\begin{aligned}
&\left|\mathrm{k}_n^h(z,w) - \mathbf{P}_{n,h}^\sharp[\chi_z \mathrm{k}_n^h(\cdot,w)](z)\right| \\
&\qquad \le C_2 n^{-1/2} \mathrm{e}^{nQ(z)} \left\{\int_{\mathbb{C}} \left|\mathrm{k}_n^h(\zeta,w)\right|^2 \mathrm{e}^{-nQ(\zeta)}\,\mathrm{d}A(\zeta)\right\}^{1/2}.
\end{aligned}
\tag{A.13}
$$

Since h is bounded, the reproducing property of the kernel k_n^h gives that

$$
\begin{aligned}
\int_{\mathbb{C}} \left|\mathrm{k}_n^h(\zeta,w)\right|^2 \mathrm{e}^{-nQ(\zeta)}\,\mathrm{d}A(\zeta) &= \int_{\mathbb{C}} \left|\mathrm{k}_n^h(\zeta,w)\right|^2 \mathrm{e}^{-nQ_n^h(\zeta)} \mathrm{e}^{-h(\zeta)}\,\mathrm{d}A(\zeta) \\
&\le \mathrm{e}^{\|h\|_{L^\infty(\mathbb{C})}} \int_{\mathbb{C}} \left|\mathrm{k}_n^h(\zeta,w)\right|^2 \mathrm{e}^{-nQ_n^h(\zeta)}\,\mathrm{d}A(\zeta) \\
&= \mathrm{e}^{\|h\|_{L^\infty(\mathbb{C})}} \mathrm{k}_n^h(w,w).
\end{aligned}
\tag{A.14}
$$

In view of the global estimate (cf. [2], Section 3)

$$
\mathrm{k}_n^h(w,w) \le C_5 n \mathrm{e}^{n\check{Q}_n^h(w)} \le C_5 n \mathrm{e}^{nQ_n^h(w)} \le C_5 n \mathrm{e}^{\|h\|_{L^\infty(\mathbb{C})}} \mathrm{e}^{nQ(w)}
$$

for a suitable positive constant C_5 that only depends on Q, h, (A.13) and (A.14) combine to show that

$$
\left|\mathrm{k}_n^h(z,w) - \mathbf{P}_{n,h}^\sharp[\chi_z \mathrm{k}_n^h(\cdot,w)](z)\right| \le C_6 \mathrm{e}^{n[Q(z)+Q(w)]}.
\tag{A.15}
$$

Next, we observe that (k_n^h is an Hermitian kernel)

$$\text{(A.16)}\qquad \begin{aligned}\overline{\mathbf{P}_{n,h}^\sharp[\chi_z \mathrm{k}_n^h(\cdot,w)](z)} &= \int_{\mathbb{C}} \mathrm{k}_{n,h}^\sharp(\zeta,z)\chi_z(\zeta)\mathrm{k}_n^h(w,\zeta)\mathrm{e}^{-nQ_n^h(\zeta)}\,\mathrm{d}A(\zeta)\\ &= \mathbf{P}_n^h[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,z)](w).\end{aligned}$$

We shall obtain the estimate

$$\text{(A.17)}\qquad \left|\mathrm{k}_{n,h}^\sharp(w,z) - \mathbf{P}_n^h[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,z)](w)\right| \le C_7 \mathrm{e}^{n[Q(z)+Q(w)]}.$$

When we combine (A.15) with (A.17), and take into account the almost Hermitian property (A.4), the assertion of Lemma A.1 is immediate. The verification of (A.17) is the same as in [7] or [1], and depends on the observation that $L^2(\mathrm{e}^{-2nQ}) = L^2(\mathrm{e}^{-2nQ_n^h})$ as spaces, with equivalence of norms. To make the presentation as complete as possible, we supply a detailed argument.

For a given smooth function f, we consider U_0, the norm minimal solution in $L^2(\mathrm{e}^{-2nQ})$ to the problem

$$\text{(A.18)}\qquad \bar\partial U = \bar\partial f \quad \text{and} \quad u - f \in \mathrm{Pol}_n,$$

where Pol_n stands for the n-dimensional space of all polynomials of degree $\le n-1$. Let U_1 be the corresponding norm minimal solution in $L^2(\mathrm{e}^{-2nQ_n^h})$. We quickly argue that

$$\text{(A.19)}\qquad \mathrm{e}^{-\|h\|_{L^\infty(\mathbb{C})}}\|U_0\|_{\mathrm{e}^{-2nQ}} \le \|U_1\|_{\mathrm{e}^{-2nQ_n^h}} \le \mathrm{e}^{\|h\|_{L^\infty(\mathbb{C})}}\|U_0\|_{\mathrm{e}^{-2nQ}}.$$

Next, we note that

$$U_0 = f - \mathbf{P}_n[f], \qquad U_1 = f - \mathbf{P}_n^h[f].$$

We put $f := \chi_z \mathrm{k}_{n,h}^\sharp(\cdot,\xi)$, where $\xi \in \mathbb{C}$ will be determined later; then

$$U_0(\zeta) = \chi_z(\zeta)\mathrm{k}_{n,h}^\sharp(\zeta,\xi) - \mathbf{P}_n[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,\xi)](\zeta).$$

We shall obtain the estimate

$$\text{(A.20)}\qquad \|U_0\|_{\mathrm{e}^{-2nQ}} \le C_8 \left\|\bar\partial[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,\xi)]\right\|_{\mathrm{e}^{-2nQ}}.$$

To this end, we put

$$2\phi(\zeta) := 2n\check{Q}(\zeta) + \log(1+|\zeta|^2),$$

and consider the function V_0, the minimal norm solution in $L^2(\mathrm{e}^{-2\phi})$ to the problem

$$\bar\partial V = \bar\partial[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,\xi)].$$

As defined, the function ϕ is strictly subharmonic, and in fact, $\Delta\phi(\zeta) \ge 2(1+|\zeta|^2)^{-2}$. Hörmander's standard minimal norm estimate for the $\bar\partial$-equation gives (see, e.g., [13], page 250)

$$\|V_0\|_{\mathrm{e}^{-2\phi}}^2 \le 2\int_{\mathbb{C}} \left|\bar\partial[\chi_z \mathrm{k}_{n,h}^\sharp(\cdot,\xi)]\right|^2 \frac{\mathrm{e}^{-2\phi}}{\Delta\phi}\,\mathrm{d}A.$$

Since χ_z is supported inside the droplet S, where $\check{Q} = Q$, and $\Delta\phi \geq n\Delta Q \geq n\varepsilon$ holds for some constant $\varepsilon > 0$ in the interior of S, we find that

$$\|V_0\|_{\mathrm{e}^{-2\phi}} \leq C_9 n^{-1/2} \|\bar{\partial}[\chi_z \mathrm{k}^{\sharp}_{n,h}(\cdot, \xi)]\|_{\mathrm{e}^{-2nQ}}.$$

From the growth assumption (2.1) on Q near infinity, the inequality $2\phi \leq 2nQ + \mathrm{O}(1)$ holds in the whole complex plane, and hence

$$\|V_0\|_{\mathrm{e}^{-2nQ}} \leq C_{10} \|V_0\|_{\mathrm{e}^{-2\phi}}.$$

In view of the above two displayed equations, we have

$$\|V_0\|_{\mathrm{e}^{-2nQ}} \leq C_{11} n^{-1/2} \|\bar{\partial}[\chi_z \mathrm{k}^{\sharp}_{n,h}(\cdot, \xi)]\|_{\mathrm{e}^{-2nQ}}, \tag{A.21}$$

with $C_{11} := C_9 C_{10}$. The difference, the function $V_0 - \chi_z \mathrm{k}_{n,h}\sharp(\cdot, \xi)$, belongs to the weighted Bergman space $A^2(\mathrm{e}^{-2\phi})$ of all entire functions in $L^2(\mathrm{e}^{-2\phi})$. For fixed (big) n, we have that

$$2\phi(\zeta) = 2(n+1)\log|\zeta| + \mathrm{O}(1) \qquad \text{as } |\zeta| \to +\infty,$$

which leads to the conclusion that the Bergman space $A^2(\mathrm{e}^{-2\phi})$ coincides with the polynomial space Pol_n as a linear space. We now see that function V_0 is a solution to the problem (A.18). As U_0 is the norm minimal solution, (A.20) is a consequence of (A.21).

We see from the norm equivalence (A.19) that the estimate (A.20) implies that

$$\|U_1\|_{\mathrm{e}^{-2nQ^h_n}} \leq C_{12} n^{-1/2} \|\bar{\partial}[\chi_z \mathrm{k}^{\sharp}_{n,h}(\cdot, \xi)]\|_{\mathrm{e}^{-2nQ^h_n}}, \tag{A.22}$$

where

$$U_1(\zeta) = \chi_z(\zeta)\mathrm{k}^{\sharp}_{n,h}(\zeta, \xi) - \mathbf{P}^h_n[\chi_z \mathrm{k}^{\sharp}_{n,h}(\cdot, \xi)](\zeta)$$

is the norm minimal solution in $L^2(\mathrm{e}^{-2nQ^h_n})$ to the equation (A.18) with $f(\zeta) = \chi_z(\zeta)\mathrm{k}^{\sharp}_{n.h}(\zeta, \xi)$. We need to turn the norm estimate (A.22) into a pointwise estimate. Since

$$\bar{\partial} U_1(\zeta) = \bar{\partial}_\zeta[\chi_z(\zeta)\mathrm{k}^{\sharp}_{n,h}(\zeta, \xi)](\zeta) = \mathrm{k}^{\sharp}_{n,h}(\zeta, \xi)\bar{\partial}\chi_z(\zeta),$$

we obtain from (A.5) that

$$\begin{aligned} |\bar{\partial} U_1(\zeta)|^2 \mathrm{e}^{-2nQ(\zeta)} &= |\bar{\partial}\chi_z(\zeta)|^2 |\mathrm{k}^{\sharp}_{n,h}(\zeta, \xi)|^2 \mathrm{e}^{-2nQ(\zeta)} \\ &\leq C_{13} n^2 |\bar{\partial}\chi_z(\zeta)|^2 \mathrm{e}^{2n(Q(\xi) - b|\zeta - \xi|^2)} \end{aligned}$$

holds provided that ζ, ξ are sufficiently close to one another, and confined to a fixed compact set. We finally fix ξ; we put $\xi := z$. With this choice, $\delta_n \leq |\zeta - z| \leq 2\delta_n$ for ζ with $\bar{\partial}\chi_z(\zeta) \neq 0$, so the points are close enough, and confined to the droplet S, and from the above estimate, we get

$$|\bar{\partial} U_1(\zeta)|^2 \mathrm{e}^{-2nQ(\zeta)} \leq C_{13} n^2 \mathrm{e}^{-2nb\delta_n^2} |\bar{\partial}\chi_z(\zeta)|^2 \mathrm{e}^{2nQ(z)}.$$

We plug this into (A.22), and arrive at

$$\|U_1\|_{\mathrm{e}^{-2nQ_n^h}} \le C_{14} n^{1/2} \mathrm{e}^{-nb\delta_n^2} \mathrm{e}^{nQ(z)}, \tag{A.23}$$

if we use the assumed L^2-control on $\bar\partial\chi_z$ and the equivalence of the norms associated with Q and Q_n^h. Another appeal to norm equivalence gives us

$$\|U_1\|_{\mathrm{e}^{-2nQ}} \le C_{15} n^{1/2} \mathrm{e}^{-nb\delta_n^2} \mathrm{e}^{nQ(z)}. \tag{A.24}$$

The function $\chi_z(\zeta)\mathrm{k}^\sharp(\zeta, z)$ is holomorphic as a function of ζ in the disk $\mathbb{D}(z, \frac{3}{2}\delta_n)$, and subtraction of a polynomial does not change that. This permits us to invoke Lemma 3.2 of [2], which says that

$$|U_1(w)|^2 \mathrm{e}^{-2nQ(w)} \le C_{16} n \|U_1\|^2_{\mathrm{e}^{-2nQ}}, \qquad w \in \mathbb{D}(z, \delta_n).$$

In combination with (A.24), this gives (since $\chi_z(w) = 1$)

$$|\mathrm{k}^\sharp_{n,h}(w, z) - \mathbf{P}_n^h[\chi_z \mathrm{k}^\sharp_{n,h}(\cdot, z)](w)| = |U_1(w)| \le C_{17} n \mathrm{e}^{-nb\delta_n^2} \mathrm{e}^{n[Q(z)+Q(w)]}.$$

Given the strong exponential decay, this estimate is actually much stronger than required to obtain (A.17). This concludes the proof of Lemma A.1. □

REFERENCES

[1] AMEUR, Y. (2013). Near-boundary asymptotics for correlation kernels. *J. Geom. Anal.* **23** 73–95. MR3010273

[2] AMEUR, Y., HEDENMALM, H. and MAKAROV, N. (2010). Berezin transform in polynomial Bergman spaces. *Comm. Pure Appl. Math.* **63** 1533–1584. MR2742007

[3] AMEUR, Y., HEDENMALM, H. and MAKAROV, N. (2011). Fluctuations of eigenvalues of random normal matrices. *Duke Math. J.* **159** 31–81. MR2817648

[4] AMEUR, Y., KANG, N.-G. and MAKAROV, N. (2014). Rescaling Ward identities in the random normal matrix model. Available at arXiv:1410.4132.

[5] BERMAN, R. (2008). Determinantal point processes and fermions on complex manifolds: Bulk universality. Preprint. Available at arXiv:0811.3341.

[6] BERMAN, R., BERNDTSSON, B. and SJÖSTRAND, J. (2008). A direct approach to Bergman kernel asymptotics for positive line bundles. *Ark. Mat.* **46** 197–217. MR2430724

[7] BERMAN, R. J. (2009). Bergman kernels for weighted polynomials and weighted equilibrium measures of $\mathbb{C}^n$. *Indiana Univ. Math. J.* **58** 1921–1946. MR2542983

[8] BORODIN, A. (2011). Determinantal point processes. In *The Oxford Handbook of Random Matrix Theory* 231–249. Oxford Univ. Press, Oxford. MR2932631

[9] BORODIN, A. and SINCLAIR, C. D. (2009). The Ginibre ensemble of real random matrices and its scaling limits. *Comm. Math. Phys.* **291** 177–224. MR2530159

[10] ELBAU, P. and FELDER, G. (2005). Density of eigenvalues of random normal matrices. *Comm. Math. Phys.* **259** 433–450. MR2172690

[11] HEDENMALM, H. and MAKAROV, N. (2013). Coulomb gas ensembles and Laplacian growth. *Proc. Lond. Math. Soc.* (3) **106** 859–907. MR3056295

[12] HEDENMALM, H. and SHIMORIN, S. (2002). Hele-Shaw flow on hyperbolic surfaces. *J. Math. Pures Appl.* (9) **81** 187–222. MR1894061

[13] HÖRMANDER, L. (1994). *Notions of Convexity*. *Progress in Mathematics* **127**. Birkhäuser, Boston, MA. MR1301332

[14] JOHANSSON, K. (1988). On Szegő's asymptotic formula for Toeplitz determinants and generalizations. *Bull. Sci. Math.* (2) **112** 257–304. MR0975365

[15] JOHANSSON, K. (1998). On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** 151–204. MR1487983

[16] MEHTA, M. L. (2004). *Random Matrices*, 3rd ed. *Pure and Applied Mathematics* (*Amsterdam*) **142**. Elsevier, Amsterdam. MR2129906

[17] RIDER, B. and VIRÁG, B. (2007). The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN* **2007** Art. ID rnm006. MR2361453

[18] SAFF, E. B. and TOTIK, V. (1997). *Logarithmic Potentials with External Fields*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **316**. Springer, Berlin. MR1485778

[19] SAKAI, M. (1991). Regularity of a boundary having a Schwarz function. *Acta Math.* **166** 263–297. MR1097025

[20] SIMON, B. (2008). The Christoffel–Darboux kernel. In *Perspectives in Partial Differential Equations*, *Harmonic Analysis and Applications*. *Proc. Sympos. Pure Math.* **79** 295–335. Amer. Math. Soc., Providence, RI. MR2500498

[21] SOSHNIKOV, A. (2000). Determinantal random point fields. *Uspekhi Mat. Nauk* **55** 107–160. MR1799012

[22] WIEGMANN, P. and ZABRODIN, A. (2003). Large scale correlations in normal non-Hermitian matrix ensembles. *J. Phys. A* **36** 3411–3424. MR1986427

[23] WIEGMANN, P. and ZABRODIN, A. (2006). Large *N* expansion for normal and complex matrix ensembles. In *Frontiers in Number Theory*, *Physics*, *and Geometry I* 213–229. Springer, Berlin. MR2261096

[24] ZABRODIN, A. (2006). Matrix models and growth processes: From viscous flows to the quantum Hall effect. In *Applications of Random Matrices in Physics*. *NATO Sci. Ser. II Math. Phys. Chem.* **221** 261–318. Springer, Dordrecht. MR2232116

[25] ZABRODIN, A. and WIEGMANN, P. (2006). Large-*N* expansion for the 2D Dyson gas. *J. Phys. A* **39** 8933–8963. MR2240466

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