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NOISE AS A BOOLEAN ALGEBRA OF σ-FIELDS

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A noise is a kind of homomorphism from a Boolean algebra of domains to the lattice of σ-fields. Leaving aside the homomorphism we examine its image, a Boolean algebra of σ-fields. The largest extension of such Boolean algebra of σ-fields, being well-defined always, is a complete Boolean algebra if and only if the noise is classical, which answers an old question of J. Feldman.

Introduction. The product of two measure spaces, widely known among mathematicians, leads to the tensor product of the corresponding Hilbert spaces L_2. The less widely known product of an infinite sequence of probability spaces leads to the so-called infinite tensor product space. A continuous product of probability spaces, used in the theory of noises, leads to a continuous tensor product of Hilbert spaces, used in noncommutative dynamics. Remarkable parallelism and fruitful interrelations between the two theories of continuous products, commutative (probability) and noncommutative (operator algebras) are noted [17, 19, 20].

The classical theory, developed in the 20th century, deals with independent increments (Lévy processes) in the commutative case, and quasi-free representations of canonical commutation relations (Fock spaces) in the noncommutative case. These classical continuous products are well understood, except for one condition of classicality, whose sufficiency was conjectured by H. Araki and E. J. Woods in 1966 ([1], page 210), in the noncommutative case (still open), and by J. Feldman in 1971 ([8], Problem 1.9), in the commutative case (now proved).

Araki and Woods note ([1], pages 161–162), that lattices of von Neumann algebras occur in quantum field theory and quantum statistical mechanics; these algebras correspond to domains in space–time or space; in most interesting cases they fail to be a Boolean algebra of type I factors. As a first step toward an understanding of such structures, Araki and Woods investigate "factorizations," complete Boolean algebras of type I factors, leaving aside their relation to the domains in space(–time), and conjecture that all such factorizations contain sufficiently many factorizable vectors.

Feldman defines "factored probability spaces" that are in fact complete Boolean algebras of sub-σ-fields (corresponding to Borel subsets of a parameter space,

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which does not really matter), investigates them assuming sufficiently many "decomposable processes" (basically the same as factorizable vectors) and asks whether this assumption holds always, or not.

In both cases the authors failed to prove that the completeness of the Boolean algebra implies classicality (via sufficiently many factorizable vectors).

In both cases the authors did not find any nonclassical factorizations, and did not formulate an appropriate framework for these. This challenge in the noncommutative case was met in 1987 by Powers [13] ("type III product system"), and in the commutative case in 1998 by Vershik and myself [20] ("black noise"). In both cases the framework was an incomplete Boolean algebra indexed by one-dimensional intervals and their finite unions. More interesting nonclassical noises were found soon (see the survey [19]), but the first highly important example is given recently by Schramm, Smirnov and Garban [14]—the noise of percolation, a conformally invariant black noise over the plane.

Being indexed by planar domains (whose needed regularity depends on some properties of the noise), such a noise exceeds the limits of the existing framework based on one-dimensional intervals. Abandoning the intervals, it is natural to return to the Boolean algebras, leaving aside (once again!) their relations to planar (or more general) domains; this time, however, the Boolean algebra is generally incomplete.

The present article provides a remake of the theory of noises, treated here as Boolean algebras of σ-fields. Completeness of the Boolean algebra implies classicality, which answers the question of Feldman.

The noncommutative case is still waiting for a similar treatment.

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1. Main results.

1.1. *Definitions.* Let $(\Omega, \mathcal{F}, P)$ be a probability space; that is, Ω is a set, $\mathcal{F}$ a σ-field (in other words, σ-algebra) of its subsets (throughout, every σ-field is assumed to contain all null sets), and P a probability measure on $(\Omega, \mathcal{F})$. We assume that $L_2(\Omega, \mathcal{F}, P)$ is separable. The set Λ of all sub-σ-fields of $\mathcal{F}$ is partially ordered (by inclusion: $x \le y$ means $x \subset y$ for $x, y \in \Lambda$), and is a lattice:

$$x \wedge y = x \cap y, \qquad x \vee y = \sigma(x, y) \qquad \text{for } x, y \in \Lambda;$$

here $\sigma(x, y)$ is the least σ-field containing both x and y. (See [4] for basics about lattices and Boolean algebras.) The greatest element 1_Λ of Λ is $\mathcal{F}$; the smallest element 0_Λ is the trivial σ-field (only null sets and their complements).

A subset $B \subset \Lambda$ is called a sublattice if $x \wedge y, x \vee y \in B$ for all $x, y \in B$. The sublattice is called distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in B$.

Let $B \subset \Lambda$ be a distributive sublattice, $0_\Lambda \in B$, $1_\Lambda \in B$. An element x of B is called complemented (in B), if $x \wedge y = 0_\Lambda$, $x \vee y = 1_\Lambda$ for some (necessarily unique) $y \in B$; in this case one says that y is the complement of x, and writes $y = x'$.

DEFINITION 1.1. A *noise-type Boolean algebra* is a distributive sublattice $B \subset \Lambda$ such that $0_\Lambda \in B$, $1_\Lambda \in B$, all elements of B are complemented (in B), and for every $x \in B$ the σ-fields x, x' are independent [i.e., $P(X \cap Y) = P(X)P(Y)$ for all $X \in x$, $Y \in y$].

From now on $B \subset \Lambda$ is a noise-type Boolean algebra.

DEFINITION 1.2. The *first chaos space* $H^{(1)}(B)$ is a (closed linear) subspace of the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ consisting of all $f \in H$ such that

$$f = \mathbb{E}(f|x) + \mathbb{E}(f|x') \qquad \text{for all } x \in B.$$

Here $\mathbb{E}(\cdot|x)$ is the conditional expectation, that is, the orthogonal projection onto the subspace H_x of all x-measurable elements of H.

DEFINITION 1.3. (a) B is called *classical* if the first chaos space generates the whole σ-field $\mathcal{F}$.

(b) B is called *black* if the first chaos space contains only 0 (but $0_\Lambda \neq 1_\Lambda$).

The lattice Λ is complete; that is, every subset $X \subset \Lambda$ has an infimum and a supremum,

$$\inf X = \bigcap_{x \in X} x, \qquad \sup X = \sigma\Bigl(\bigcup_{x \in X} x\Bigr).$$

A noise-type Boolean algebra B is called complete if

$$(\inf X) \in B \quad \text{and} \quad (\sup X) \in B \qquad \text{for every } X \subset B.$$

1.2. *The simplest nonclassical example.* Let $\Omega = \{-1, 1\}^\infty$ (all infinite sequences of ± 1) with the product measure μ^∞ where $\mu(\{-1\}) = \mu(\{1\}) = 1/2$. The coordinate projections $\xi_n : \Omega \to \{-1, 1\}$, $\xi_n(s_1, s_2, \dots) = s_n$, treated as random variables, are independent random signs. The products $\xi_1\xi_2, \xi_2\xi_3, \xi_3\xi_4, \dots$ are also independent random signs.

We introduce σ-fields

$$x_n = \sigma(\xi_n, \xi_{n+1}, \dots) \quad \text{and} \quad y_n = \sigma(\xi_n\xi_{n+1}) \qquad \text{for } n = 1, 2, \dots.$$

Then

$$1_\Lambda = x_1 \geq x_2 \geq \cdots;$$

$$y_n \leq x_n;$$

$$y_1, \dots, y_n, x_{n+1} \text{ are independent};$$

$$y_n \vee x_{n+1} = x_n.$$

The independent σ-fields $y_1, \dots, y_n, x_{n+1}$ are atoms of a finite noise-type Boolean algebra B_n (containing 2^{n+1} elements), and $B_n \subset B_{n+1}$. The union

$$B = B_1 \cup B_2 \cup \cdots$$

is an infinite noise-type Boolean algebra. As a Boolean algebra, B is isomorphic to the finite/cofinite Boolean algebra, that is, the algebra of all finite subsets of $\{1, 2, \dots\}$ and their complements; $x_n \in B$ corresponds to the cofinite set $\{n, n+1, \dots\}$, while $y_n \in B$ corresponds to the single-element set $\{n\}$. The first chaos space $H^{(1)}(B) = H^{(1)}(B_1) \cap H^{(1)}(B_2) \cap \cdots$ consists of linear combinations

$$c_1\xi_1\xi_2 + c_2\xi_2\xi_3 + c_3\xi_3\xi_4 + \cdots$$

for all $c_1, c_2, \dots \in \mathbb{R}$ such that $c_1^2 + c_2^2 + \cdots < \infty$. It is not $\{0\}$, which shows that B is not black. On the other hand, all elements of $H^{(1)}(B)$ are invariant under the measure preserving transformation $(s_1, s_2, \dots) \mapsto (-s_1, -s_2, \dots)$; therefore $\sigma(H^{(1)}(B))$ is not the whole 1_Λ, which shows that B is not classical.

The complement x_n' of x_n in B is $y_1 \vee \cdots \vee y_{n-1} = \sigma(\xi_1\xi_2, \xi_2\xi_3, \dots, \xi_{n-1}\xi_n)$. Clearly, $x_n \downarrow 0_\Lambda$ (i.e., $\inf_n x_n = 0_\Lambda$). Strangely, the relation $x_n' \uparrow 1_\Lambda$ fails; $x_n' \uparrow \sup_n y_n = \sigma(\xi_1\xi_2, \xi_2\xi_3, \dots) \neq 1_\Lambda$. "The phenomenon …tripped up even Kolmogorov and Wiener" [22], Section 4.12.

This example goes back to an unpublished dissertation of Vershik [21]. According to Emery and Schachermayer ([7], page 291), it is a paradigmatic example, well known in ergodic theory, independently discovered by several authors. See also [22], Section 4.12, [19], Section 1b.

1.3. *On Feldman's question.*

THEOREM 1.4. *If a noise-type Boolean algebra is complete, then it is classical.*

THEOREM 1.5. *The following conditions on a noise-type Boolean algebra B are equivalent*:

(a) *B is classical*;
(b) *there exists a complete noise-type Boolean algebra $\hat{B}$ such that $B \subset \hat{B}$*;
(c) $(\sup_n x_n) \vee (\inf_n x_n') = 1_\Lambda$ *for all $x_n \in B$ such that $x_1 \le x_2 \le \cdots$.*

See also Theorem 7.7 for another important condition of classicality.

1.4. *On completion.* Bad news: a noise-type Boolean algebra cannot be extended to a complete one unless it is classical. (See Theorem 1.5. True, every Boolean algebra admits a completion [9], Section 21, but not within Λ.)

Good news: an appropriate notion of completion exists and is described below (Definition 1.8).

The lower limit

$$\liminf_n x_n = \sup_n \inf_k x_{n+k}$$

is well defined for arbitrary $x_1, x_2, \ldots \in \Lambda$. (The upper limit is defined similarly.)

THEOREM 1.6. *Let B be a noise-type Boolean algebra and*

$$\mathrm{Cl}(B) = \Big\{\liminf_n x_n : x_1, x_2, \ldots \in B\Big\}$$

(*the set of lower limits of all sequences of elements of B*). *Then*:

(a) $(\inf_n x_n) \in \mathrm{Cl}(B)$ *whenever* $x_1, x_2, \ldots \in \mathrm{Cl}(B)$;
(b) $(\sup_n x_n) \in \mathrm{Cl}(B)$ *whenever* $x_1, x_2, \ldots \in \mathrm{Cl}(B)$, $x_1 \le x_2 \le \cdots$.

Thus, we add to B limits of all monotone sequences, iterate this operation until stabilization and get $\mathrm{Cl}(B)$, call it the closure of B. (It is not a noise-type Boolean algebra, unless B is classical.)

THEOREM 1.7. *Let B and* $\mathrm{Cl}(B)$ *be as in Theorem* 1.6, *and*

$$C = \{x \in \mathrm{Cl}(B) : \exists y \in \mathrm{Cl}(B)\; x \wedge y = 0_\Lambda, x \vee y = 1_\Lambda\}$$

[*the set of all complemented elements of* $\mathrm{Cl}(B)$]. *Then*

(a) *C is a noise-type Boolean algebra such that $B \subset C \subset \mathrm{Cl}(B)$*;
(b) *C contains every noise-type Boolean algebra C_1 satisfying $B \subset C_1 \subset \mathrm{Cl}(B)$*.

DEFINITION 1.8. The noise-type Boolean algebra C of Theorem 1.7 is called the *noise-type completion* of a noise-type Boolean algebra B.

EXAMPLE 1.9. Let B, y_n and ξ_n be as in Section 1.2. Then $\mathrm{Cl}(B) \setminus B$ consists of σ-fields of the form $\sup_{n\in I} y_n = \sigma(\{\xi_n \xi_{n+1} : n \in I\})$ where I runs over all infinite subsets of $\{1, 2, \ldots\}$. The noise-type completion of B is B itself.

If two noise-type Boolean algebras have the same closure, then clearly they have the same completion.

PROPOSITION 1.10. *If two noise-type Boolean algebras have the same closure*, *then they have the same first chaos space*.

Thus if $\mathrm{Cl}(B_1) = \mathrm{Cl}(B_2)$, then classicality of B_1 is equivalent to classicality of B_2, and blackness of B_1 is equivalent to blackness of B_2.

QUESTION 1.11. It follows from Theorem 1.6 that the following conditions are equivalent: $\mathrm{Cl}(B)$ is a lattice; $\mathrm{Cl}(B)$ is a complete lattice; $x \vee y \in \mathrm{Cl}(B)$ for all $x, y \in \mathrm{Cl}(B)$. These conditions are satisfied by every classical B. Are they satisfied by some nonclassical B? By all nonclassical B?

1.5. *On sufficient subalgebras.* Let B, B_0 be noise-type Boolean algebras such that $B_0 \subset B$. Clearly, $\mathrm{Cl}(B_0) \subset \mathrm{Cl}(B)$ and $H^{(1)}(B_0) \supset H^{(1)}(B)$. We say that:

- B_0 is dense in B if $\mathrm{Cl}(B_0) = \mathrm{Cl}(B)$;
- B_0 is sufficient in B if $H^{(1)}(B_0) = H^{(1)}(B)$.

If B_0 is sufficient in B, then clearly, classicality of B_0 is equivalent to classicality of B, and blackness of B_0 is equivalent to blackness of B.

A dense subalgebra is sufficient by Proposition 1.10. Surprisingly, a nondense subalgebra can be sufficient.

DEFINITION 1.12. A noise-type Boolean algebra B is *atomless* if

$$\inf_{x \in F} x = 0_\Lambda$$

for every ultrafilter $F \subset B$.

Recall that a set $F \subset B$ is called a filter if for all $x, y \in B$

$$x \in F, \qquad x \le y \quad \Longrightarrow \quad y \in F,$$

$$x, y \in F \quad \Longrightarrow \quad x \wedge y \in F,$$

$$0_\Lambda \notin F;$$

a filter F is called ultrafilter if it is a maximal filter; equivalently, if

$$\forall x \in B \qquad \bigl(x \notin F \Longrightarrow x' \in F\bigr).$$

THEOREM 1.13. *If a noise-type subalgebra is atomless, then it is sufficient.*

Some applications of this result are mentioned in the end of Section 1.6.

1.6. *On available examples and frameworks.* Several examples of nonclassical noise-type Boolean algebras are available in the literature but described in somewhat different frameworks.

According to Tsirelson and Vershik ([20], Definition 1.2), a *measure factorization* over a Boolean algebra $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \Lambda$ such that $\varphi(a_1 \wedge a_2) = \varphi(a_1) \wedge \varphi(a_2)$, $\varphi(a_1 \vee a_2) = \varphi(a_1) \vee \varphi(a_2)$, $\varphi(0_{\mathcal{A}}) = 0_\Lambda$, $\varphi(1_{\mathcal{A}}) = 1_\Lambda$, and two σ-fields $\varphi(a)$, $\varphi(a')$ are independent (for all $a, a_1, a_2 \in \mathcal{A}$). In this case the image $B = \varphi(\mathcal{A}) \subset \Lambda$ evidently is a noise-type Boolean algebra. A measure factorization over $\mathcal{A}$ may be defined equivalently as a homomorphism φ from $\mathcal{A}$ onto some noise-type Boolean algebra. Assuming that φ is an isomorphism (which usually holds) we may apply several notions introduced in [20] to noise-type Boolean algebras.

In particular, an element of the first chaos space $H^{(1)}(B)$ is the same as a square integrable real-valued *additive integral* [20], Definition 1.3 and Theorem 1.7.

Complex-valued *multiplicative integrals* are also examined in [20], Theorem 1.7; these generate a σ-field that contains the σ-field generated by $H^{(1)}(B)$. These two σ-fields differ in the "simplest nonclassical example" of Section 1.2. Namely, the latter σ-field consists of all measurable sets invariant under the sign change, while the former σ-field is the whole 1_Λ, since the coordinates $\xi_1, \xi_2, \dots$ are multiplicative integrals [indeed, $\xi_1 = (\xi_1\xi_2)(\xi_2\xi_3)\cdots(\xi_n\xi_{n+1})\xi_{n+1}$]. A sufficient condition for equality of the two σ-fields, given by [20], Theorem 1.7, is the *minimal up continuity condition* [20], Definition 1.6: $\sup_{x\in F} x' = 1_\Lambda$ for every ultrafilter $F \subset B$. This is stronger than the condition $\inf_{x\in F} x = 0_\Lambda$ called *minimal down continuity* in [20], Definition 1.6, and just *atomless* here (Definition 1.12). The "continuous example" in [19], Section 1b, is atomless but violates the minimal up continuity condition. The seemingly evident relation $\sup_{x\in F} x' = (\inf_{x\in F} x)'$ may fail (see Section 1.2), since sup and inf are taken in Λ rather than B; see also Remark 4.1.

A wide class of countable atomless black noise-type Boolean algebras is obtained in [20], Section 4a, via combinatorial models on trees.

According to [19], Definition 3c1, a *continuous product of probability spaces* (over $\mathbb{R}$) is a family $(x_{s,t})_{s<t}$ of σ-fields $x_{s,t} \in \Lambda$ given for all $s, t \in \mathbb{R}$, $s < t$, such that $\sup_{s,t} x_{s,t} = 1_\Lambda$ and

$$x_{r,s} \otimes x_{s,t} = x_{r,t} \qquad \text{whenever } r < s < t$$

in the sense that $x_{r,s}$ and $x_{s,t}$ are independent and generate $x_{r,t}$. This is basically the same as a measure factorization over the Boolean algebra $\mathcal{A}$ of all finite unions of intervals (s,t) treated modulo finite sets (see [19], Section 11a, for details).

According to Tsirelson [19], Definition 3d1, a *noise* (over $\mathbb{R}$) is a *homogeneous* continuous product of probability spaces; "homogeneous" means existence of a measurable action $(T_h)_{h\in\mathbb{R}}$ of $\mathbb{R}$ on Ω such that

$$T_h \text{ sends } x_{s,t} \text{ to } x_{s+h,t+h} \qquad \text{whenever } s < t \text{ and } h \in \mathbb{R}$$

(see [19], Section 3d for details). It follows from homogeneity (and separability of H) that [19], Proposition 3d3 and Corollary 3d5

$$\inf_{\varepsilon>0} x_{s-\varepsilon,t+\varepsilon} = x_{s,t} = \sup_{\varepsilon>0} x_{s+\varepsilon,t-\varepsilon}, \tag{1.1}$$

which implies the minimal up continuity condition (since an ultrafilter must contain all neighborhoods of some point from $[-\infty, +\infty]$). Thus, additive and multiplicative integrals generate the same sub-σ-field, called the *stable σ-field* in [19], Section 4c, where it is defined in a completely different but equivalent way. Note also that every noise leads to an *atomless* noise-type Boolean algebra.

Two examples of a nonclassical, but not black, noise were published in 1999 and 2002 by J. Warren (see [19], Sections 2c, 2d).

Existence of a black noise was proved first in 1998 ([20], Section 5), via projective limit; see also [18], Section 8.2. However, this was not quite a *construction* of a specific noise; existence of a subsequence limit was proved, uniqueness was not.

All other black noise examples available for now use random configurations over $\mathbb{R}^{1+d}$ for some $d \ge 1$ (in most cases $d = 1$); the σ-field $x_{s,t}$ consists of all events "observable" within the domain $(s,t) \times \mathbb{R}^d \subset \mathbb{R}^{1+d}$.

Examples based on stochastic flows were published in 2001 by Watanabe and in 2004 by the author Le Jan, O. Raimond and S. Lemaire. In these examples the first coordinate of $\mathbb{R}^{1+d}$ is interpreted as time, the other d coordinates as space. Blackness is deduced from the relation $\|\mathbb{E}(f|x_{t,t+\varepsilon})\|^2 = o(\varepsilon)$ as $\varepsilon \to 0+$ for all $f \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbb{E} f = 0$. For details and references see [19], Section 7.

The first highly important example is the *black noise of percolation.* The corresponding random configuration over $\mathbb{R}^2$ is the full scaling limit of critical site percolation on the triangular lattice. This example was conjectured in 2004 ([18], Question 8.1 and Remark 8.2, [19], Question 11b1). It was rather clear that the noise of percolation must be black; it was less clear how to define its probability space and σ-fields $x_{s,t}$, and it was utterly unclear whether $x_{r,s}$ and $x_{s,t}$ generate $x_{r,t}$, or not. (It is not sufficient to know that $x_{r,s+\varepsilon}$ and $x_{s,t}$ generate $x_{r,t}$.) The affirmative answer was published in 2011 [14].

In order to say that the noise of percolation is a conformally invariant black noise over $\mathbb{R}^2$ we must first define a noise over $\mathbb{R}^2$. Recall that a noise over $\mathbb{R}$ is related to the Boolean algebra of all finite unions of intervals modulo finite sets. Its two-dimensional counterpart, according to Schramm and Smirnov [14], Corollary 1.20, is "an appropriate algebra of piecewise-smooth planar domains (e.g., generated by rectangles)." However, the algebra generated by rectangles hides the conformal invariance of this noise. The class of all piecewise-smooth domains is conformally invariant, however, two C^k-smooth curves may have a nondiscrete intersection. Piecewise analytic boundaries could be appropriate for this noise.

Stochastic flows on $\mathbb{R}^{1+d}$, mentioned above, lead to noises over $\mathbb{R}$, generally not $\mathbb{R}^{1+d}$ since, being uncorrelated in time, they may be correlated in space. However, two of them are also uncorrelated in (one-dimensional) space: Arratia's coalescing flow, or the Brownian web (see [19], Section 7f), and its sticky counterpart (see [19], Section 7j). For such flow it is natural to conjecture that a σ-field $y_{a,b}$ consisting of all events "observable" within the domain $\mathbb{R} \times (a,b) \subset \mathbb{R}^2$ is well defined whenever $a < b$, and $y_{a,b} \otimes y_{b,c} = y_{a,c}$. Then $(y_{a,b})_{a<b}$ is the second noise (over $\mathbb{R}$) obtained from this flow. Moreover, the σ-fields $x_{s,t} \wedge y_{a,b}$ indexed by rectangles $(s,t) \times (a,b)$ should form a noise over $\mathbb{R}^2$. For Arratia's flow this conjecture was proved in 2011 [6]. It appears that the relation $y_{a,b} \otimes y_{b,c} = y_{a,c}$ is harder to prove than the relation $x_{r,s} \otimes x_{s,t} = x_{r,t}$. Unlike percolation, Arratia's flow, being translation-invariant (in time and space), is not rotation-invariant, and the two noises $(x_{s,t})_{s<t}$, $(y_{a,b})_{a<b}$ are probably nonisomorphic.

Still, the notion of a noise over $\mathbb{R}^2$ is obscure because of nonuniqueness of an appropriate Boolean algebra of planar domains. Surely, a single "noise of percolation" is more satisfactory than "the noise of percolation on rectangles" different from "the noise of percolation on piecewise analytic domains" etc. These should be treated as different generators of the same object. On the level of noise-type

Boolean algebras the problem is solved by the noise-type completion (Section 1.4). However, it remains unclear how to relate the σ-fields belonging to the completion to something like planar domains.

Any reasonable definition of a noise over $\mathbb{R}^2$ leads to a noise-type Boolean algebra B, two noises $(x_{s,t})_{s<t}$, $(y_{a,b})_{a<b}$ over $\mathbb{R}$, their noise-type Boolean algebras $B_1 \subset B$, $B_2 \subset B$, and the corresponding first chaos spaces $H^{(1)}(B)$, $H^{(1)}(B_1)$, $H^{(1)}(B_2)$. As was noted after (1.1), B_1 and B_2 are atomless. By Theorem 1.13 they are sufficient, that is,

$$H^{(1)}(B_1) = H^{(1)}(B) = H^{(1)}(B_2).$$

Thus, if one of these three noises (one over $\mathbb{R}^2$ and two over $\mathbb{R}$) is classical, then the other two are classical; if one is black, then the other two are black.

For the noise of percolation we know that the noise over $\mathbb{R}^2$ is black and conclude that the corresponding two (evidently isomorphic) noises over $\mathbb{R}$ are black.

For the Arratia's flow we know that the first noise over $\mathbb{R}$ is black and conclude that the second noise over $\mathbb{R}$ is also black.

2. Preliminaries.

This section is a collection of useful facts (mostly folk-lore, I guess), more general than noise-type Boolean algebras.

Throughout, the probability space $(\Omega, \mathcal{F}, P)$, the complete lattice Λ of sub-σ-fields and the separable Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ are as in Section 1.1. Complex numbers are not used; H is a Hilbert space over $\mathbb{R}$. A "subspace" of H always means a closed linear subset. Recall also $0_\Lambda, 1_\Lambda, x \wedge y, x \vee y$ for $x, y \in \Lambda$, the notion of independent σ-fields, operators $\mathbb{E}(\cdot|x)$ of conditional expectation, and $\inf X, \sup X \in \Lambda$ for $X \subset \Lambda$ (Section 1.1).

2.1. *Type L_2 subspaces.*

FACT 2.1 ([15], Theorem 3). *The following two conditions on a subspace H_1 of H are equivalent*:

(a) *there exists a sub-σ-field $x \in \Lambda$ such that $H_1 = L_2(x)$, the space of all x-measurable functions of H*;

(b) *H_1 is a sublattice of H, containing constants. That is, H_1 contains $f \vee g$ and $f \wedge g$ for all $f, g \in H_1$, where $(f \vee g)(\omega) = \max(f(\omega), g(\omega))$ and $(f \wedge g)(\omega) = \min(f(\omega), g(\omega))$, and H_1 contains the one-dimensional space of constant functions.*

HINT TO THE PROOF THAT (B) $\Longrightarrow$ (A). $\mathbb{1}_{(0,\infty)}(f) = \lim_n((0 \vee nf) \wedge 1) \in H_1$ for $f \in H_1$. $\square$

Such subspaces H_1 will be called type L_2 (sub)spaces. (In [15] they are called measurable, which can be confusing.)

Due to linearity of H_1 the condition $f \vee g, f \wedge g \in H_1$ boils down to $|f| \in H_1$ for all $f \in H_1$. [Hint: $f \vee g = f + (0 \vee (g - f))$ and $0 \vee f = 0.5(f + |f|)$.]

FACT 2.2. *If $A \subset L_\infty(\Omega, \mathcal{F}, P)$ is a subalgebra containing constants, then the closure of A in H is a type L_2 space.*

("Subalgebra" means $fg \in A$ for all $f, g \in A$, in addition to linearity.)

HINT. Approximating the absolute value by polynomials we get $|f| \in H_1$ (the closure of A) for $f \in A$, and by continuity, for $f \in H_1$. $\square$

NOTATION 2.3. We denote the type L_2 space $L_2(x)$ corresponding to $x \in \Lambda$ by H_x, and the orthogonal projection $\mathbb{E}(\cdot|x)$ by Q_x. In particular, $H_0 = \{c\mathbb{1} : c \in \mathbb{R}\}$ is the one-dimensional subspace of constant functions on Ω, and $Q_0 f = (\mathbb{E} f)\mathbb{1} = \langle f, \mathbb{1}\rangle\mathbb{1}$. Also, $H_1 = H$, and $Q_1 = I$ is the identity operator.

Thus:

$$H_x \subset H; \qquad Q_x : H \to H; \qquad Q_x H = H_x \qquad \text{for } x \in \Lambda; \tag{2.1}$$

$$H_x \subset H_y \iff Q_x \le Q_y \iff x \le y; \tag{2.2}$$

$$Q_x Q_y = Q_x = Q_y Q_x \qquad \text{whenever } x \le y; \tag{2.3}$$

$$H_x = H_y \iff Q_x = Q_y \iff x = y; \tag{2.4}$$

$$H_{x \wedge y} = H_x \cap H_y; \tag{2.5}$$

(2.3) and (2.4) follow from (2.2); (2.5) is a special case of Fact 2.4.

FACT 2.4. $H_{\inf X} = \bigcap_{x \in X} H_x$ *for* $X \subset \Lambda$.

HINT. Measurability w.r.t. the intersection of σ-fields is equivalent to measurability w.r.t. each one of these σ-fields. $\square$

However, $H_{x \vee y}$ is generally much larger than the closure of $H_x + H_y$.

FACT 2.5 ([11], Theorem 3.5.1). *$H_{x \vee y}$ is the subspace spanned by pointwise products fg for $f \in H_x \cap L_\infty(\Omega, \mathcal{F}, P)$ and $g \in H_y \cap L_\infty(\Omega, \mathcal{F}, P)$.*

HINT. Linear combinations of these products are an algebra; by Fact 2.2 its closure is H_z for some $z \in \Lambda$; note that $z \ge x$, $z \ge y$, but also $z \le x \vee y$. $\square$

FACT 2.6. *Let $x, x_1, x_2, \ldots \in \Lambda$, $x_1 \le x_2 \le \cdots$ and $x = \sup_n x_n$. Then H_x is the closure of $H_{x_1} \cup H_{x_2} \cup \cdots$.*

HINT. By Fact 2.1, the closure of $\bigcup_n H_{x_n}$ is H_z for some $z \in \Lambda$; note that $z \ge x_n$ for all n, but also $z \le x$. $\square$

That is,

$$x_n \uparrow x \quad \Longrightarrow \quad H_{x_n} \uparrow H_x; \tag{2.6}$$

$$x_n \downarrow x \quad \Longrightarrow \quad H_{x_n} \downarrow H_x \tag{2.7}$$

(the latter holds by Fact 2.4).

2.2. *Strong operator convergence.* Let H be a Hilbert space and $A, A_1, A_2, \dots : H \to H$ operators (linear, bounded). Strong operator convergence of A_n to A is defined by

$$(A_n \to A) \quad \Longleftrightarrow \quad \big(\forall \psi \in H \ \|A_n\psi - A\psi\| \xrightarrow{n\to\infty} 0\big).$$

We write just $A_n \to A$, since we do not need other types of convergence for operators.

FACT 2.7 ([12], Remark 2.2.11). *$A_n \to A$ if and only if $\|A_n\psi - A\psi\| \to 0$ for a dense set of vectors ψ and $\sup_n \|A_n\| < \infty$.*

FACT 2.8 ([10], Problem 93; [12], Section 4.6.1). *If $A_n \to A$ and $B_n \to B$, then $A_n B_n \to AB$.*

FACT 2.9. *If $A_n \to A$, $B_n \to B$ and $A_n B_n = B_n A_n$ for all n, then $AB = BA$.*

HINT. Use Fact 2.8. $\square$

The following fact allows us to write $A_n \uparrow A$ (or $A_n \downarrow A$) unambiguously. We need it only for commuting orthogonal projections.

FACT 2.10 ([3], Proposition 43.1). *Let $A, A_1, A_2, \dots : H \to H$ be Hermitian operators, $A_1 \le A_2 \le \cdots$, then*

$$A = \sup_n A_n \quad \Longleftrightarrow \quad A_n \to A.$$

The natural bijective correspondence between subspaces of H and orthogonal projections $H \to H$ is order preserving, therefore

$$H_n \downarrow H_\infty \quad \Longleftrightarrow \quad Q_n \downarrow Q_\infty \quad \text{also} \quad H_n \uparrow H_\infty \quad \Longleftrightarrow \quad Q_n \uparrow Q_\infty \tag{2.8}$$

whenever $H_1, H_2, \dots, H_\infty \subset H$ are subspaces and $Q_1, Q_2, \dots, Q_\infty : H \to H$ the corresponding orthogonal projections.

In combination with (2.6), (2.7) it gives

$$x_n \downarrow x \text{ implies } Q_{x_n} \downarrow Q_x; \text{ also, } x_n \uparrow x \text{ implies } Q_{x_n} \uparrow Q_x. \tag{2.9}$$

Let H_1, H_2 be Hilbert spaces, and $H = H_1 \otimes H_2$ their tensor product.

FACT 2.11. *Let* $A, A_1, A_2, \dots : H_1 \to H_1$, $B, B_1, B_2, \dots : H_2 \to H_2$. *If* $A_n \to A$ *and* $B_n \to B$, *then* $A_n \otimes B_n \to A \otimes B$.

HINT. The operators are uniformly bounded, and converge on a dense set; use Fact 2.7. □

2.3. *Independence and tensor products.*

FACT 2.12. *If* $x, y \in \Lambda$ *are independent, then* $H_{x\vee y} = H_x \otimes H_y$ *up to the natural unitary equivalence*:

$$H_x \otimes H_y \ni f \otimes g \quad \longleftrightarrow \quad fg \in H_{x\vee y}.$$

HINT. By the independence, $\langle f_1 g_1, f_2 g_2 \rangle = \mathbb{E}(f_1 g_1 f_2 g_2) = \mathbb{E}(f_1 f_2) \times \mathbb{E}(g_1 g_2) = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle = \langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle$, thus, $H_x \otimes H_y$ is isometrically embedded into $H_{x\vee y}$; by Fact 2.5 the embedding is "onto." □

It may be puzzling that H_x is both a subspace of H and a tensor factor of H (which never happens in the general theory of Hilbert spaces). Here is an explanation. All spaces H_x contain the one-dimensional space H_0 of constant functions (on Ω). Multiplying an x-measurable function $f \in H_x$ by the constant function $g \in H_{x'}$, $g(\cdot) = 1$, we get the (puzzling) equality $f \otimes g = f$.

NOTATION 2.13. For $u, x \in \Lambda$ such that $u \le x$ we denote by $Q_u^{(x)}$ the restriction of Q_u to H_x.

Thus

$$Q_u^{(x)} : H_x \to H_x, \qquad Q_u^{(x)} H_x = H_u \qquad \text{for } u \le x.$$

FACT 2.14. *If* $x, y \in \Lambda$ *are independent*, $u \le x$, $v \le y$, *then treating* $H_{x\vee y}$ *as* $H_x \otimes H_y$, *we have*

$$Q_{u\vee v} = Q_u^{(x)} \otimes Q_v^{(y)}.$$

HINT. By Fact 2.12, $H_{u\vee v} = H_u \otimes H_v$, and this factorization may be treated as embedded into the factorization $H_{x\vee y} = H_x \otimes H_y$; the projection onto $H_u \otimes H_v \subset H_x \otimes H_y$ factorizes. □

In a more probabilistic language,

$$\mathbb{E}(fg | u \vee v) = \mathbb{E}(f|u)\mathbb{E}(g|v) \qquad \text{for } f \in L_2(x), g \in L_2(y).$$

Here is a very general fact (no σ-fields, no tensor products, just Hilbert spaces).

FACT 2.15 ([10], Problem 96, [3], Exercise 45.4). *Let Q_1, Q_2 be orthogonal projections in a Hilbert space H. Then $(Q_1 Q_2)^n$ converges strongly (as $n \to \infty$) to the orthogonal projection onto $(Q_1 H) \cap (Q_2 H)$.*

FACT 2.16. *$(Q_x Q_y)^n \to Q_{x \wedge y}$ strongly (as $n \to \infty$) whenever $x, y \in \Lambda$.*

HINT. $(Q_x H) \cap (Q_y H) = Q_{x \wedge y} H$ by (2.5); use Fact 2.15. □

FACT 2.17. *$(Q^{(x)}_{u_1} Q^{(x)}_{u_2})^n \to Q^{(x)}_{u_1 \wedge u_2}$ strongly (as $n \to \infty$) whenever $u_1, u_2 \le x$.*

HINT. Similar to Fact 2.16. □

FACT 2.18. *If $x, y \in \Lambda$ are independent, $u_1, u_2 \le x$ and $v_1, v_2 \le y$, then*

$$(u_1 \vee v_1) \wedge (u_2 \vee v_2) = (u_1 \wedge u_2) \vee (v_1 \wedge v_2).$$

HINT. By Fact 2.16, $(Q_{u_1 \vee v_1} Q_{u_2 \vee v_2})^n \to Q_{(u_1 \vee v_1) \wedge (u_2 \vee v_2)}$. By Fact 2.17, $(Q^{(x)}_{u_1} Q^{(x)}_{u_2})^n \to Q^{(x)}_{u_1 \wedge u_2}$ and $(Q^{(y)}_{v_1} Q^{(y)}_{v_2})^n \to Q^{(y)}_{v_1 \wedge v_2}$. By Fact 2.14, $Q_{u_1 \vee v_1} \times Q_{u_2 \vee v_2} = (Q^{(x)}_{u_1} \otimes Q^{(y)}_{v_1})(Q^{(x)}_{u_2} \otimes Q^{(y)}_{v_2}) = (Q^{(x)}_{u_1} Q^{(x)}_{u_2}) \otimes (Q^{(y)}_{v_1} Q^{(y)}_{v_2})$ and $Q_{(u_1 \wedge u_2) \vee (v_1 \wedge v_2)} = Q^{(x)}_{u_1 \wedge u_2} \otimes Q^{(y)}_{v_1 \wedge v_2}$; use (2.4). □

REMARK. In a distributive lattice the equality stated by Fact 2.18 is easy to check (assuming $x \wedge y = 0$ instead of independence). However, the lattice Λ is not distributive.

Useful special cases of Fact 2.18 (assuming that x, y are independent, $u \le x$ and $v \le y$):

$$(u \vee v) \wedge x = u, \qquad (u \vee v) \wedge y = v; \tag{2.10}$$

$$(u \vee y) \wedge (x \vee v) = u \vee v. \tag{2.11}$$

Here is another very general fact (no σ-fields, no tensor products, just random variables).

FACT 2.19. *Assume that $X, X_1, X_2, \dots$ and $Y, Y_1, Y_2, \dots$ are random variables (on a given probability space), and for every n the two random variables X_n, Y_n are independent; if $X_n \to X$, $Y_n \to Y$ in probability, then X, Y are independent.*

HINT. If $f, g : \mathbb{R} \to \mathbb{R}$ are bounded continuous functions, then $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$, $\mathbb{E}(g(Y_n)) \to \mathbb{E}(g(Y))$, $\mathbb{E}(f(X_n))\mathbb{E}(g(Y_n)) = \mathbb{E}(f(X_n)g(Y_n)) \to \mathbb{E}(f(X)g(Y))$, thus, $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$. □

The same holds for vector-valued random variables.

2.4. *Measure class spaces and commutative von Neumann algebras*. See [5, 16] or [3] for basics about von Neumann algebras; we need only the commutative case.

FACT 2.20 ([5], Section I.7.3, [16], Theorem III.1.22, [12], E4.7.2). *Every commutative von Neumann algebra $\mathcal{A}$ of operators on a separable Hilbert space H is isomorphic to the algebra $L_\infty(S, \Sigma, \mu)$ on some measure space (S, Σ, μ).*

Here and henceforth all measures are positive, finite and such that the corresponding L_2 spaces are separable. The isomorphism $\alpha : \mathcal{A} \to L_\infty(S, \Sigma, \mu)$ preserves linear operations, multiplication and norm. Hermitian operators of $\mathcal{A}$ correspond to real-valued functions of L_∞; we restrict ourselves to these and observe an order isomorphism,

$$A \le B \iff \alpha(A) \le \alpha(B); \qquad A = \sup_n A_n \iff \alpha(A) = \sup_n \alpha(A_n). \tag{2.12}$$

FACT 2.21 ([5], Section I.4.3, Corollary 1, [3], Section 46, Proposition 46.6 and Exercise 1). *Every isomorphism of von Neumann algebras preserves the strong operator convergence* (*of sequences*, *not nets*).

The measure μ may be replaced with any equivalent (i.e., mutually absolutely continuous) measure μ_1. Thus we may turn to a measure class space (see [2], Section 14.4) $(S, \Sigma, \mathcal{M})$ where $\mathcal{M}$ is an equivalence class of measures, and write $L_\infty(S, \Sigma, \mathcal{M})$; we have an isomorphism

$$\alpha : \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M}) \tag{2.13}$$

of von Neumann algebras. (See [2], Section 14.4, for the Hilbert space $L_2(S, \Sigma, \mathcal{M})$ on which $L_\infty(S, \Sigma, \mathcal{M})$ acts by multiplication.)

FACT 2.22. *Let $\mathcal{A}$ and α be as in* (2.13), $A, A_1, A_2, \ldots \in \mathcal{A}$, $\sup_n \|A_n\| < \infty$. *Then the following two conditions are equivalent*:

(a) $A_n \to A$ *in the strong operator topology*;
(b) $\alpha(A_n) \to \alpha(A)$ *in measure.*

HINT. ($A_n \to A$ strongly) $\iff$ ($\alpha(A_n) \to \alpha(A)$ strongly) $\iff$ ($\|\alpha(A_n) f - \alpha(A) f\|_2 \to 0$ for every bounded f) $\iff$ ($\alpha(A_n) \to \alpha(A)$ in measure). □

Let $\Sigma_1 \subset \Sigma$ be a sub-σ-field. Restrictions $\mu|_{\Sigma_1}$ of measures $\mu \in \mathcal{M}$ are mutually equivalent; denoting their equivalence class by $\mathcal{M}|_{\Sigma_1}$ we get a measure class space $(S, \Sigma_1, \mathcal{M}|_{\Sigma_1})$. Clearly, $L_\infty(S, \Sigma_1, \mathcal{M}|_{\Sigma_1}) \subset L_\infty(S, \Sigma, \mathcal{M})$ or, in shorter notation, $L_\infty(\Sigma_1) \subset L_\infty(\Sigma)$; this is also a von Neumann algebra.

FACT 2.23. *Every von Neumann subalgebra of $L_\infty(\Sigma)$ is $L_\infty(\Sigma_1)$ for some sub-σ-field $\Sigma_1 \subset \Sigma$.*

HINT. Similar to Fact 2.2. $\square$

We have $L_\infty(\Sigma_1) = \alpha(\mathcal{A}_1)$ where $\mathcal{A}_1 = \alpha^{-1}(L_\infty(\Sigma_1)) \subset \mathcal{A}$ is a von Neumann algebra. And conversely, if $\mathcal{A}_1 \subset \mathcal{A}$ is a von Neumann algebra, then $\alpha(\mathcal{A}_1) = L_\infty(\Sigma_1)$ for some sub-σ-field $\Sigma_1 \subset \Sigma$.

Given two von Neumann algebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, we denote by $\mathcal{A}_1 \vee \mathcal{A}_2$ the von Neumann algebra generated by $\mathcal{A}_1, \mathcal{A}_2$. Similarly, for two σ-fields $\Sigma_1, \Sigma_2 \subset \Sigma$ we denote by $\Sigma_1 \vee \Sigma_2$ the σ-field generated by Σ_1, Σ_2.

FACT 2.24. $L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) = L_\infty(\Sigma_1 \vee \Sigma_2)$.

HINT. By Fact 2.23, $L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) = L_\infty(\Sigma_3)$ for some Σ_3; note that $\Sigma_3 \supset \Sigma_1$, $\Sigma_3 \supset \Sigma_2$, but also $\Sigma_3 \subset \Sigma_1 \vee \Sigma_2$. $\square$

FACT 2.25. *If $\alpha(\mathcal{A}_1) = L_\infty(\Sigma_1)$ and $\alpha(\mathcal{A}_2) = L_\infty(\Sigma_2)$, then $\alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = L_\infty(\Sigma_1 \vee \Sigma_2)$.*

HINT. $\alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = \alpha(\mathcal{A}_1) \vee \alpha(\mathcal{A}_2)$, since α is an isomorphism; use Fact 2.24. $\square$

The product $(S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2)$ of two measure class spaces is a measure class space [2], 14.4; namely, $(S, \Sigma) = (S_1, \Sigma_1) \times (S_2, \Sigma_2)$, and $\mathcal{M}$ is the equivalence class containing $\mu_1 \times \mu_2$ for some (therefore all) $\mu_1 \in \mathcal{M}_1$, $\mu_2 \in \mathcal{M}_2$. In this case $L_\infty(S, \Sigma, \mathcal{M}) = L_\infty(S_1, \Sigma_1, \mathcal{M}_1) \otimes L_\infty(S_2, \Sigma_2, \mathcal{M}_2)$.

Given two commutative von Neumann algebras $\mathcal{A}_1$ on H_1 and $\mathcal{A}_2$ on H_2, their tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ is a von Neumann algebra on $H = H_1 \otimes H_2$. Given isomorphisms $\alpha_1 : \mathcal{A}_1 \to L_\infty(S_1, \Sigma_1, \mathcal{M}_1)$ and $\alpha_2 : \mathcal{A}_2 \to L_\infty(S_2, \Sigma_2, \mathcal{M}_2)$, we get an isomorphism $\alpha = \alpha_1 \otimes \alpha_2 : \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M})$, where $(S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2)$; namely, $\alpha(A_1 \otimes A_2) = \alpha_1(A_1) \otimes \alpha_2(A_2)$ for $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$. Note that $\alpha(\mathcal{A}_1 \otimes I) = L_\infty(\tilde\Sigma_1)$ and $\alpha(I \otimes \mathcal{A}_2) = L_\infty(\tilde\Sigma_2)$, where $\tilde\Sigma_1 = \{A_1 \times S_2 : A_1 \in \Sigma_1\}$ and $\tilde\Sigma_2 = \{S_1 \times A_2 : A_2 \in \Sigma_2\}$ are $\mathcal{M}$-independent sub-σ-fields of Σ, and $\tilde\Sigma_1 \vee \tilde\Sigma_2 = \Sigma$.

DEFINITION 2.26. Let $(S, \Sigma, \mathcal{M})$ be a measure class space. Two sub-σ-fields $\Sigma_1, \Sigma_2 \subset \Sigma$ are *$\mathcal{M}$-independent*, if they are μ-independent for some $\mu \in \mathcal{M}$, that is, $\mu(X \cap Y)\mu(S) = \mu(X)\mu(Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$.

FACT 2.27. *If σ-fields $\Sigma_1, \Sigma_2 \subset \Sigma$ are independent, then $L_\infty(\Sigma_1 \vee \Sigma_2) = L_\infty(\Sigma_1) \otimes L_\infty(\Sigma_2)$ up to the natural isomorphism*

$$L_\infty(\Sigma_1) \otimes L_\infty(\Sigma_2) \ni f \otimes g \quad \longleftrightarrow \quad fg \in L_\infty(\Sigma_1 \vee \Sigma_2).$$

HINT. Recall Fact 2.12. $\square$

FACT 2.28. *For every isomorphism* $\alpha : \mathcal{A}_1 \otimes \mathcal{A}_2 \to L_2(S, \Sigma, \mathcal{M})$, *there exist* $\mathcal{M}$*-independent* $\Sigma_1, \Sigma_2 \subset \Sigma$ *such that* $\alpha(\mathcal{A}_1 \otimes I) = L_\infty(\Sigma_1)$, $\alpha(I \otimes \mathcal{A}_2) = L_\infty(\Sigma_2)$, *and* $\Sigma_1 \vee \Sigma_2 = \Sigma$.

HINT. We get Σ_1, Σ_2 from Fact 2.23; $\Sigma_1 \vee \Sigma_2 = \Sigma$ by Fact 2.24; for proving independence we choose $\mu_1 \in \mathcal{M}_1$, $\mu_2 \in \mathcal{M}_2$, take isomorphisms $\alpha_1 : \mathcal{A}_1 \to L_\infty(S_1, \Sigma_1', \mathcal{M}_1)$, $\alpha_2 : \mathcal{A}_2 \to L_\infty(S_2, \Sigma_2', \mathcal{M}_2)$ and use the isomorphism $\beta = (\alpha_1 \otimes \alpha_2)\alpha^{-1} : L_\infty(S, \Sigma, \mathcal{M}) \to L_\infty((S_1, \Sigma_1', \mathcal{M}_1) \times (S_2, \Sigma_2', \mathcal{M}_2))$ for defining $\mu \in \mathcal{M}$ by $\int f \, \mathrm{d}\mu = \int (\beta f) \, \mathrm{d}(\mu_1 \times \mu_2)$; then Σ_1, Σ_2 are μ-independent. $\square$

Given an isomorphism $\alpha : \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M})$ of von Neumann algebras, we have subspaces $H(E)$, for $E \in \Sigma$, of the space H on which acts $\mathcal{A}$:

$$
\begin{aligned}
&H(E) = \alpha^{-1}(\mathbb{1}_E)H \subset H;\\
&H(E_1 \cap E_2) = H(E_1) \cap H(E_2);\\
&H(E_1 \uplus E_2) = H(E_1) \oplus H(E_2);\\
&H(E_1 \cup E_2) = H(E_1) + H(E_2)
\end{aligned}
\tag{2.14}
$$

[the third line differs from the fourth line by assuming that E_1, E_2 are disjoint and concluding that $H(E_1), H(E_2)$ are orthogonal]. By (2.12), (2.8)

$$
\begin{aligned}
&E_n \uparrow E \text{ implies } H(E_n) \uparrow H(E),\\
&E_n \downarrow E \text{ implies } H(E_n) \downarrow H(E).
\end{aligned}
\tag{2.15}
$$

2.5. *Boolean algebras.* Every finite Boolean algebra b has 2^n elements, where n is the number of the atoms $a_1, \dots, a_n$ of b; these atoms satisfy $a_k \wedge a_l = 0_b$ for $k \ne l$, and $a_1 \vee \dots \vee a_n = 1_b$. All elements of b are of the form

$$
a_{i_1} \vee \dots \vee a_{i_k}, \qquad 1 \le i_1 < \dots < i_k \le n.
\tag{2.16}
$$

We denote by Atoms(b) the set of all atoms of b and rewrite (2.16) as

$$
\forall x \in b \qquad x = \bigvee_{a \in \mathrm{Atoms}(b), a \le x} a.
\tag{2.17}
$$

FACT 2.29. *Let B be a Boolean algebra, $b_1, b_2 \subset B$ two finite Boolean subalgebras and $b \subset B$ the Boolean subalgebra generated by b_1, b_2. Then b is finite. If $a_1 \in$ Atoms(b_1), $a_2 \in$ Atoms(b_2) and $a_1 \wedge a_2 \ne 0_B$, then $a_1 \wedge a_2 \in$ Atoms(b), and all atoms of b are of this form.*

HINT. These $a_1 \wedge a_2$ are the atoms of some finite Boolean subalgebra b_3; note that $b_1 \subset b_3$ and $b_2 \subset b_3$, but also $b_3 \subset b$. $\square$

FACT 2.30. *The following four conditions on a Boolean algebra B are equivalent*:

$$\sup_n x_n \text{ exists for all } x_1, x_2, \ldots \in B;$$

$$\inf_n x_n \text{ exists for all } x_1, x_2, \ldots \in B;$$

$$\sup_n x_n \text{ exists for all } x_1, x_2, \ldots \in B \text{ satisfying } x_1 \le x_2 \le \cdots;$$

$$\inf_n x_n \text{ exists for all } x_1, x_2, \ldots \in B \text{ satisfying } x_1 \ge x_2 \ge \cdots.$$

HINT. First, $\inf_n x_n = (\sup_n x_n')'$; second, $\sup_n x_n = \sup_n (x_1 \vee \cdots \vee x_n)$. □

A Boolean algebra B satisfying these equivalent conditions is called *σ-complete* (in other words, a Boolean σ-algebra).

FACT 2.31 ([9], Section 14, Lemma 1). *The following two conditions on a Boolean algebra B are equivalent*:

(a) *no uncountable subset $X \subset B$ satisfies $x \wedge y = 0_B$ for all $x, y \in B$ ("the* countable chain condition*")*;

(b) *every subset X of B has a countable subset Y such that X and Y have the same set of upper bounds*.

FACT 2.32 ([9], Section 14, Corollary). *If a σ-complete Boolean algebra satisfies the countable chain condition*, *then it is complete*.

HINT. Use Fact 2.31(b). □

2.6. *Measurable functions and equivalence classes*. Let (S, Σ, μ) be a measure space, $\mu(S) < \infty$. As usual, we often treat equivalence classes of measurable functions on S as just measurable functions, which is harmless as long as only countably many equivalence classes are considered simultaneously. Otherwise, dealing with uncountable sets of equivalence classes, we must be cautious.

All equivalence classes of measurable functions $S \to [0, 1]$ are a complete lattice. Let Z be some set of such classes. If Z is countable, then its supremum, $\sup Z$, may be treated naively (as the pointwise supremum of functions). For an uncountable Z we have $\sup Z = \sup Z_0$ for some countable $Z_0 \subset Z$. In particular, the equality holds whenever Z_0 is dense in Z according to the L_1 metric.

The same holds for functions $S \to \{0, 1\}$ or, equivalently, measurable sets. Functions $S \to [0, \infty]$ are also a complete lattice, since $[0, \infty]$ can be transformed into $[0, 1]$ by an increasing bijection.

In the context of (2.14), (2.15) we have

$$H\Big(\inf_{i \in I} E_i\Big) = \bigcap_{i \in I} H(E_i) \tag{2.18}$$

for an arbitrary (not just countable) family of equivalence classes E_i of measurable sets. Similarly,

$$H\Big(\sup_{i\in I} E_i\Big) = \sup_{i\in I} H(E_i), \tag{2.19}$$

the closure of the sum of all $H(E_i)$.

FACT 2.33. *For every increasing sequence of measurable functions $f_n : S \to [0,\infty)$ there exist $n_1 < n_2 < \cdots$ such that almost every $s \in S$ satisfies one of two incompatible conditions*:

$$\textit{either } \lim_n f_n(s) < \infty \textit{ or } f_{n_k}(s) \ge k \textit{ for all } k \textit{ large enough}$$

[*here "k large enough" means $k \ge k_0(s)$*].

HINT. Take n_k such that

$$\sum_k \mu\Big(\{s : f_{n_k} < k\} \cap \Big\{s : \lim_n f_n(s) = \infty\Big\}\Big) < \infty. \qquad \square$$

All said above holds also for a measure class space $(S, \Sigma, \mathcal{M})$ (see Section 2.4) in place of the measure space (S, Σ, μ).

3. Convergence of σ-fields and independence. Throughout this section $(\Omega, \mathcal{F}, P)$, Λ, H and Q_x are as in Section 2.

3.1. *Definition of the convergence*. The strong operator topology on the projection operators Q_x induces a topology on Λ; we call it the strong operator topology on Λ. It is metrizable (since the strong operator topology is metrizable on operators of norm ≤ 1; see [3], Section 8, Exercise 1). Thus, for $x, x_1, x_2, \ldots \in \Lambda$,

$$x_n \to x \text{ means } \forall f \in H \, \|Q_{x_n} f - Q_x f\| \xrightarrow{n\to\infty} 0.$$

On the other hand we have the monotone convergence derived from the partial order on Λ,

$$x_n \downarrow x \text{ means } x_1 \ge x_2 \ge \cdots \text{ and } \inf_n x_n = x,$$
$$x_n \uparrow x \text{ means } x_1 \le x_2 \le \cdots \text{ and } \sup_n x_n = x.$$

By Fact 2.10,

$$x_n \downarrow x \text{ implies } x_n \to x; \text{ also, } x_n \uparrow x \text{ implies } x_n \to x. \tag{3.1}$$

3.2. *Commuting σ-fields.*

DEFINITION 3.1. Elements $x, y \in \Lambda$ are *commuting*, if $Q_x Q_y = Q_y Q_x$. A subset of Λ is *commutative*, if its elements are pairwise commuting.

By (2.3),

(3.2) every linearly ordered subset of Λ is commutative.

By Fact 2.9,

(3.3) if $x_n \to x$, $y_n \to y$, and for every n the two elements x_n, y_n are commuting, then x, y are commuting.

In particular,

(3.4) the closure of a commutative set is commutative.

It follows from Fact 2.16, or just (2.5), that

(3.5) if $x, y \in \Lambda$ are commuting then $Q_x Q_y = Q_{x \wedge y}$.

Recall $\liminf_n x_n$ for $x_n \in \Lambda$ defined in Section 1.3.

LEMMA 3.2. *If $x_n \in \Lambda$ are pairwise commuting and $x_n \to x$, then $\liminf_k x_{n_k} = x$ for some $n_1 < n_2 < \cdots$.*

PROOF. The commuting projection operators Q_{x_n} generate a commutative von Neumann algebra; by Fact 2.20 this algebra is isomorphic to the algebra L_∞ on some measure space (of finite measure). Denoting the isomorphism by α we have $\alpha(Q_{x_n}) = \mathbb{1}_{E_n}$, $\alpha(Q_x) = \mathbb{1}_E$ (indicators of some measurable sets E_n, E). Using (3.5) we get

$$\alpha(Q_{x_m \wedge x_n}) = \mathbb{1}_{E_m \cap E_n}$$

for all m, n; the same holds for more than two indices.

The strong convergence of operators $Q_{x_n} \to Q_x$ implies by Fact 2.22 convergence in measure of indicators, $\mathbb{1}_{E_n} \to \mathbb{1}_E$. We choose a subsequence convergent almost everywhere, $\mathbb{1}_{E_{n_k}} \to \mathbb{1}_E$, then $\liminf_k \mathbb{1}_{E_{n_k}} = \mathbb{1}_E$, that is,

$$\sup_k \inf_i \mathbb{1}_{E_{n_{k+i}}} = \mathbb{1}_E.$$

We have $\alpha(Q_{x_{n_k} \wedge x_{n_{k+1}} \wedge \cdots \wedge x_{n_{k+i}}}) = \mathbb{1}_{E_{n_k} \cap E_{n_{k+1}} \cap \cdots \cap E_{n_{k+i}}}$, therefore (for $i \to \infty$), $\alpha(Q_{\inf_i x_{n_{k+i}}}) = \inf_i \mathbb{1}_{E_{n_{k+i}}}$, and further (for $k \to \infty$), $\alpha(Q_{\sup_k \inf_i x_{n_{k+i}}}) = \sup_k \inf_i \mathbb{1}_{E_{n_{k+i}}}$. We get $\alpha(Q_{\liminf_k x_{n_k}}) = \liminf_k \mathbb{1}_{E_{n_k}} = \mathbb{1}_E = \alpha(Q_x)$, therefore $\liminf_k x_{n_k} = x$. $\square$

PROPOSITION 3.3. *Assume that a set $B \subset \Lambda$ is commutative, and $x \wedge y \in B$ for all $x, y \in B$. Then the set*

$$\mathrm{Cl}(B) = \Big\{\liminf_n x_n : x_1, x_2, \ldots \in B\Big\}$$

(*lower limits of all sequences of elements of B*) *is equal to the topological closure of B.*

PROOF. On one hand, if $x_n \to x$, then $x \in \mathrm{Cl}(B)$ by Lemma 3.2. On the other hand, $\liminf x_n = \sup_n \inf_k x_{n+k}$ belongs to the topological closure by (3.1). □

PROPOSITION 3.4. *Let $x_n, y_n, x, y \in \Lambda$, $x_n \to x$, $y_n \to y$, and for each n* (*separately*), *x_n, y_n commute. Then $x_n \wedge y_n \to x \wedge y$.*

PROOF. By (3.3), $Q_x Q_y = Q_y Q_x$. By (3.5), $Q_{x\wedge y} = Q_x Q_y$. Similarly, $Q_{x_n \wedge y_n} = Q_{x_n} Q_{y_n}$. Using Fact 2.8 we get $Q_{x_n\wedge y_n} \to Q_{x\wedge y}$, that is, $x_n \wedge y_n \to x \wedge y$. □

3.3. *Independent σ-fields.*

PROPOSITION 3.5. *The following two conditions on $x, y \in \Lambda$ are equivalent*:

(a) *x, y are independent*;
(b) *x, y are commuting, and $x \wedge y = 0_\Lambda$.*

PROOF. (a) $\Longrightarrow$ (b): independence of x, y implies $\mathbb{E}(f|y) = \mathbb{E}f$ for all $f \in L_2(x)$, that is, $Q_y f = \langle f, \mathbb{1}\rangle\mathbb{1}$ for $f \in H_x$, and therefore $Q_y Q_x = Q_0 = Q_x Q_y$; use (3.5).

(b) $\Longrightarrow$ (a): by (3.5), $Q_y Q_x = Q_0 = Q_x Q_y$; thus $Q_y f = \langle f, \mathbb{1}\rangle\mathbb{1}$ for $f \in H_x$, and therefore $P(A \cap B) = \langle \mathbb{1}_A, \mathbb{1}_B\rangle = \langle \mathbb{1}_A, Q_y \mathbb{1}_B\rangle = \langle Q_y \mathbb{1}_A, \mathbb{1}_B\rangle = \langle \mathbb{1}_A, \mathbb{1}\rangle \times \langle \mathbb{1}, \mathbb{1}_B\rangle = P(A)P(B)$ for all $A \in x$, $B \in y$. □

It may happen that $x \wedge y = 0$ but x, y are not commuting. (In particular, it may happen that x, y are independent w.r.t. some measure equivalent to P, but not w.r.t. P.)

COROLLARY 3.6. *If $x_n \to x$, $y_n \to y$, and x_n, y_n are independent for each n* (*separately*), *then x, y are independent.*

PROOF. By Proposition 3.5, x_n, y_n are commuting, and $x_n \wedge y_n = 0_\Lambda$. By (3.3), x, y are commuting. By Proposition 3.4, $x \wedge y = 0_\Lambda$. By Proposition 3.5 (again), x, y are independent. □

3.4. *Product σ-fields.* For every $x \in \Lambda$ the triple $(\Omega, x, P|_x)$ is also a probability space, and it may be used similarly to $(\Omega, \mathcal{F}, P)$, giving the complete lattice $\Lambda(\Omega, x, P|_x)$, endowed with the topology, etc. This lattice is naturally embedded into Λ,

$$\Lambda(\Omega, x, P|_x) = \{y \in \Lambda : y \le x\}.$$

The lattice operations $(\wedge, \vee)$, defined on $\Lambda(\Omega, x, P|_x)$, do not differ from these induced from Λ (which is evident); also the topology, defined on $\Lambda(\Omega, x, P|_x)$, does not differ from the topology induced from Λ (which follows easily from the equality $Q_y = Q_y^{(x)} Q_x$ for $y \le x$; see Notation 2.13 for $Q_y^{(x)}$). Thus it is correct to define Λ_x, as a lattice and topological space,[1] by

$$\Lambda(\Omega, x, P|_x) = \Lambda_x = \{y \in \Lambda : y \le x\} \subset \Lambda.$$

Given $x, y \in \Lambda$, the product set $\Lambda_x \times \Lambda_y$ carries the product topology and the product partial order, and is again a lattice (see [4], Section 2.15, for the product of two lattices), moreover, a complete lattice (see [4], Exercise 2.26(ii)).

On the other hand, for independent $x, y \in \Lambda$ we introduce

$$\Lambda_{x,y} = \{u \vee v : u \le x, v \le y\} \subset \Lambda_{x \vee y}.$$

Generally, $\Lambda_{x,y}$ is only a small part of $\Lambda_{x \vee y}$; indeed, a sub-σ-field on the product of two probability spaces is generally not a product of two sub-σ-fields. This fact is a manifestation of nondistributivity of the lattice Λ; the equality

$$(x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge z$$

fails whenever $z \in \Lambda_{x \vee y} \setminus \Lambda_{x,y}$.

LEMMA 3.7. *Every element of $\Lambda_{x,y}$ is commuting with x (and y).*

PROOF. By Fact 2.14, treating $H_{x \vee y}$ as $H_x \otimes H_y$ we have $Q_{u \vee v} = Q_u^{(x)} \otimes Q_v^{(y)}$ whenever $u \le x$, $v \le y$. Also, $Q_x = Q_x^{(x)} \otimes Q_0^{(y)}$. By (3.2), $Q_u^{(x)}$ and $Q_x^{(x)}$ are commuting; the same holds for $Q_v^{(y)}$ and $Q_0^{(y)}$. Therefore $Q_{u \vee v}$ and Q_x are commuting. □

THEOREM 3.8. *If $x, y \in \Lambda$ are independent, then $\Lambda_{x,y}$ is a closed subset of Λ, the maps*

$$\Lambda_x \times \Lambda_y \ni (u, v) \mapsto u \vee v \in \Lambda_{x,y},$$

$$\Lambda_{x,y} \ni z \mapsto (x \wedge z, y \wedge z) \in \Lambda_x \times \Lambda_y$$

are mutually inverse bijections, and each of them is both an isomorphism of lattices and a homeomorphism of topological spaces.

[1] Not "topological lattice" since the lattice operations are generally not continuous.

PROOF. The composition map $\Lambda_x \times \Lambda_y \to \Lambda_{x,y} \to \Lambda_x \times \Lambda_y$ is the identity by (2.10). Taking into account that the map $\Lambda_x \times \Lambda_y \to \Lambda_{x,y}$ is surjective we get mutually inverse bijections.

The map $\Lambda_x \times \Lambda_y \to \Lambda_{x,y}$ preserves lattice operations: "$\wedge$" by Fact 2.18, and "$\vee$" trivially. It is a bijective homomorphism, therefore, isomorphism of lattices.

Let $u, u_1, u_2, \ldots \in \Lambda_x$, $u_n \to u$, and $v, v_1, v_2, \ldots \in \Lambda_y$, $v_n \to v$. Then $Q^{(x)}_{u_n} \to Q^{(x)}_u$ and $Q^{(y)}_{v_n} \to Q^{(y)}_v$. By Fact 2.11, $Q^{(x)}_{u_n} \otimes Q^{(y)}_{v_n} \to Q^{(x)}_u \otimes Q^{(y)}_v$. By Fact 2.14, $Q_{u_n \vee v_n} \to Q_{u \vee v}$, that is, $u_n \vee v_n \to u \vee v$. The map $\Lambda_x \times \Lambda_y \to \Lambda_{x,y}$ is thus continuous.

Let $z_1, z_2, \ldots \in \Lambda_{x,y}$, $z_n \to z \in \Lambda$. By Lemma 3.7 and Proposition 3.4, $x \wedge z_n \to x \wedge z$. Similarly, $y \wedge z_n \to y \wedge z$. In particular, taking $z \in \Lambda_{x,y}$ we see that the map $\Lambda_{x,y} \to \Lambda_x \times \Lambda_y$ is continuous. In general (for $z \in \Lambda$) we get $z_n = (x \wedge z_n) \vee (y \wedge z_n) \to (x \wedge z) \vee (y \wedge z)$, therefore $z = (x \wedge z) \vee (y \wedge z) \in \Lambda_{x,y}$; we see that $\Lambda_{x,y}$ is closed. $\square$

It follows that

$$\Lambda_{x,y} = \{z \in \Lambda : z = (x \wedge z) \vee (y \wedge z)\}. \tag{3.6}$$

REMARK 3.9. By Theorem 3.8, any relation between elements of $\Lambda_{x,y}$ expressed in terms of lattice operations (and limits) is equivalent to the conjunction of two similar relations "restricted" to x and y. For example, the relation

$$(z_1 \vee z_2) \wedge z_3 = z_4 \vee z_5$$

between $z_1, z_2, z_3, z_4, z_5 \in \Lambda_{x,y}$ splits in two; first,

$$\big((x \wedge z_1) \vee (x \wedge z_2)\big) \wedge (x \wedge z_3) = (x \wedge z_4) \vee (x \wedge z_5),$$

and second, a similar relation with y in place of x.

4. Noise-type completion. Throughout Sections 4–7, $B \subset \Lambda$ is a noise-type Boolean algebra (as defined by Definition 1.1); Λ, H and Q_x are as in Section 2.

4.1. *The closure; proving Theorem* 1.6. By separability of H,

$$B \text{ satisfies the countable chain condition,} \tag{4.1}$$

since otherwise there exists an uncountable set of pairwise orthogonal nontrivial subspaces of H. By Fact 2.32,

$$B \text{ is complete if and only if it is } \sigma\text{-complete.} \tag{4.2}$$

Recall that every $x \in B$ has its complement $x' \in B$,

$$x \wedge x' = 0_\Lambda, \qquad x \vee x' = 1_\Lambda; \qquad x, x' \text{ are independent.}$$

(The complement in B is unique, however, many other independent complements may exist in Λ.)

By distributivity of B, $y = (x \wedge y) \vee (x' \wedge y)$ for all $x, y \in B$; by (3.6),

$$B \subset \Lambda_{x,x'} \qquad \text{for every } x \in B. \tag{4.3}$$

By Lemma 3.7,

$$B \text{ is a commutative subset of } \Lambda. \tag{4.4}$$

Recall $\mathrm{Cl}(B)$ introduced in Theorem 1.6; by Proposition 3.3,

$$\text{the topological closure of } B \text{ is } \mathrm{Cl}(B) = \Big\{ \liminf_n x_n : x_1, x_2, \ldots \in B \Big\}. \tag{4.5}$$

Taking into account that $\Lambda_{x,x'}$ is closed by Theorem 3.8, we get from (4.3)

$$\mathrm{Cl}(B) \subset \Lambda_{x,x'} \qquad \text{for every } x \in B. \tag{4.6}$$

By (4.4) and (3.4),

$$\mathrm{Cl}(B) \text{ is a commutative subset of } \Lambda. \tag{4.7}$$

By Proposition 3.4,

$$x \wedge y \in \mathrm{Cl}(B) \qquad \text{for all } x, y \in \mathrm{Cl}(B). \tag{4.8}$$

By (3.5),

$$Q_x Q_y = Q_{x \wedge y} \qquad \text{for all } x, y \in \mathrm{Cl}(B). \tag{4.9}$$

PROOF OF THEOREM 1.6. If $x_n \in \mathrm{Cl}(B)$ and $x_n \uparrow x$, then $x_n \to x$ by (3.1), therefore $x \in \mathrm{Cl}(B)$, which proves item (b) of the theorem.

If $x_n \in \mathrm{Cl}(B)$ and $x = \inf_n x_n$, then $x_1 \wedge \cdots \wedge x_n = y_n \in \mathrm{Cl}(B)$ by (4.8) and $y_n \downarrow x$, thus $y_n \to x$ by (3.1) (again) and $x \in \mathrm{Cl}(B)$, which proves item (a) of the theorem. $\square$

By Proposition 3.5 and (4.7), for $x, y \in \mathrm{Cl}(B)$,

$$x \wedge y = 0_\Lambda \text{ if and only if } x, y \text{ are independent.} \tag{4.10}$$

By Proposition 3.4 and (4.7), for $x, x_n, y, y_n \in \mathrm{Cl}(B)$,

$$\text{if } x_n \to x,\ y_n \to y \text{ then } x_n \wedge y_n \to x \wedge y. \tag{4.11}$$

REMARK 4.1. In contrast, $x_n \vee y_n$ need not converge to $x \vee y$, even if $x_n \in B$, $x_n \downarrow 0_\Lambda$, $y_n = x_n'$; it may happen that $y_n \uparrow y$, $y \neq 1_\Lambda$. This situation appears already in the (simplest nonclassical) example given in Section 1.2.

On the other hand, if $x_n \in B$, $x_n \to 1_\Lambda$, then necessarily $x_n' \to 0_\Lambda$ (but we do not need this fact).

By Theorem 3.8, for every $z \in B$ the map $x \mapsto x \wedge z$ is a lattice homomorphism $\Lambda_{z,z'} \to \Lambda_z$, thus, $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y \in \Lambda_{z,z'}$; in particular, it holds for all $x, y \in \mathrm{Cl}(B)$ by (4.6). If $x \vee y = 1_\Lambda$, then $z = (x \wedge z) \vee (y \wedge z)$. If in addition $x \wedge y = 0_\Lambda$, then x, y are independent by (4.10), and $z \in \Lambda_{x,y}$ by (3.6). Thus $B \subset \Lambda_{x,y}$. By Theorem 3.8 $\Lambda_{x,y}$ is closed, and we conclude.

PROPOSITION 4.2. *If* $x, y \in \mathrm{Cl}(B)$, $x \wedge y = 0_\Lambda$, $x \vee y = 1_\Lambda$, *then* $\mathrm{Cl}(B) \subset \Lambda_{x,y}$.

COROLLARY 4.3. *For every* $x \in \mathrm{Cl}(B)$ *there exists at most one* $y \in \mathrm{Cl}(B)$ *such that* $x \wedge y = 0_\Lambda$ *and* $x \vee y = 1_\Lambda$.

PROOF. Assume that $y_1, y_2 \in \mathrm{Cl}(B)$, $x \wedge y_k = 0_\Lambda$ and $x \vee y_k = 1_\Lambda$ for $k = 1, 2$. By Proposition 4.2, $y_2 \in \Lambda_{x,y_1}$, that is, $y_2 = (x \wedge y_2) \vee (y_1 \wedge y_2) = y_1 \wedge y_2$. Similarly, $y_1 = y_2 \wedge y_1$. □

4.2. *The completion*; *proving Theorem* 1.7. Let B and $\mathrm{Cl}(B)$ be as in Section 4.1, and

$$C = \{x \in \mathrm{Cl}(B) : \exists y \in \mathrm{Cl}(B)\ x \wedge y = 0_\Lambda, x \vee y = 1_\Lambda\}$$

as in Theorem 1.7; clearly,

$$B \subset C \subset \mathrm{Cl}(B). \tag{4.12}$$

Taking Corollary 4.3 into account, we extend the complement operation, $x \mapsto x'$, from B to C:

$$x' \in C \text{ for } x \in C; \qquad (x')' = x;$$
$$x \wedge x' = 0_\Lambda; \qquad x \vee x' = 1_\Lambda.$$

By (4.10), x, x' are independent; and by Proposition 4.2,

$$\forall x \in C \qquad \mathrm{Cl}(B) \subset \Lambda_{x,x'}. \tag{4.13}$$

LEMMA 4.4. *For every* $x \in C$ *the map*

$$\mathrm{Cl}(B) \ni y \mapsto x \vee y \in \Lambda$$

is continuous.

PROOF. Let $y_n, y \in \mathrm{Cl}(B)$, $y_n \to y$; we have to prove that $x \vee y_n \to x \vee y$. By (4.11), $x' \wedge y_n \to x' \wedge y$. Applying Theorem 3.8 to $(x, x' \wedge y_n) \in \Lambda_x \times \Lambda_{x'}$ we get $x \vee (x' \wedge y_n) \to x \vee (x' \wedge y)$. It remains to prove that $x \vee (x' \wedge y_n) = x \vee y_n$ and $x \vee (x' \wedge y) = x \vee y$. We prove the latter; the former is similar. Note that $y \in \mathrm{Cl}(B) \subset \Lambda_{x,x'}$ by (4.13). The lattice isomorphism $\Lambda_{x,x'} \to \Lambda_x \times \Lambda_{x'}$ of Theorem 3.8 maps x into $(x, 0)$ and y into $(x \wedge y, x' \wedge y)$; therefore it maps $x \vee y$ into $(x \vee (x \wedge y), 0 \vee (x' \wedge y)) = (x, x' \wedge y)$, which implies $x \vee (x' \wedge y) = x \vee y$. □

LEMMA 4.5.

$$\forall x \in C \ \forall y \in \mathrm{Cl}(B) \qquad x \vee y \in \mathrm{Cl}(B).$$

PROOF. By Lemma 4.4 it is sufficient to consider $y \in B$. Applying Lemma 4.4 (again) to $y \in B \subset C$ we see that the map $\mathrm{Cl}(B) \ni z \mapsto y \vee z \in \Lambda$ is continuous. This map sends B into B, and therefore it sends $x \in C \subset \mathrm{Cl}(B)$ into $\mathrm{Cl}(B)$. □

LEMMA 4.6. *For all* $x, y \in C$,

$$x \vee y \in C \quad \textit{and} \quad (x \vee y)' = x' \wedge y'.$$

PROOF. By Lemma 4.5, $x \vee y \in \mathrm{Cl}(B)$. By (4.8), $x' \wedge y' \in \mathrm{Cl}(B)$. We have to prove that $(x \vee y) \wedge (x' \wedge y') = 0_\Lambda$ and $(x \vee y) \vee (x' \wedge y') = 1_\Lambda$. We do it using Remark 3.9.

First, $x, y, x', y' \in C \subset \mathrm{Cl}(B) \subset \Lambda_{x,x'}$.

Second, we consider $z = (x \vee y) \wedge (x' \wedge y')$ and "restrict" it first to x: $x \wedge z = (x \vee (x \wedge y)) \wedge (0_\Lambda \wedge (x \wedge y')) = 0_\Lambda$, and second, to x': $x' \wedge z = (0_\Lambda \vee (x' \wedge y)) \wedge x' \wedge (x' \wedge y') \le y \wedge y' = 0_\Lambda$. We get $z = 0_\Lambda$, that is, $(x \vee y) \wedge (x' \wedge y') = 0_\Lambda$.

Third, we consider $z = (x \vee y) \vee (x' \wedge y')$ and get $x \wedge z = x \vee (x \wedge y) \vee (x \wedge x' \wedge y') = x$ and $x' \wedge z = (x' \wedge x) \vee (x' \wedge y) \vee (x' \wedge x' \wedge y') = (x' \wedge y) \vee (x' \wedge y') = x' \wedge (y \vee y') = x'$. Therefore $z = x \vee x' = 1_\Lambda$, that is, $(x \vee y) \vee (x' \wedge y') = 1_\Lambda$. □

In addition, $x \wedge y = (x' \vee y')' \in C$ for all $x, y \in C$; thus C is a sublattice of Λ. The lattice C is distributive, that is, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in C$, since $C \subset \Lambda_{x,x'}$ by (4.12), (4.13), and the map $\Lambda_{x,x'} \ni y \mapsto x \wedge y \in \Lambda_x$ is a lattice homomorphism by Theorem 3.8. Also, $0_\Lambda \in C$, $1_\Lambda \in C$, and each $x \in C$ has a complement x' in C. By (4.12) and (4.10), x, x' are independent for every $x \in C$. Thus C is a noise-type Boolean algebra satisfying (4.12), which proves item (a) of Theorem 1.7.

If C_1 is also a noise-type Boolean algebra satisfying $B \subset C_1 \subset \mathrm{Cl}(B)$, then every element of C_1 belongs to C, since its complement in C_1 is also its complement in $\mathrm{Cl}(B)$. Thus $C_1 \subset C$, which proves item (b) of Theorem 1.7.

COROLLARY 4.7. *The following two conditions on a noise-type Boolean algebra B are equivalent*:

(a) $C = \mathrm{Cl}(B)$ (*where C is the completion of B*);
(b) *there exists a complete noise-type Boolean algebra $\hat{B}$ such that $B \subset \hat{B}$.*

PROOF. (a) $\Longrightarrow$ (b): the noise-type Boolean algebra $C = \mathrm{Cl}(B)$ is closed; by (3.1) it is σ-complete (recall Section 2.5); by (4.2) it is complete.

(b) $\Longrightarrow$ (a): Given $x \in \mathrm{Cl}(B)$, we take $x_n \in B$ such that $x = \liminf_n x_n$ [recall (4.5)]; $x \in \hat{B}$. The complement x' of x in $\hat{B}$ belongs to $\mathrm{Cl}(B)$, since $(\liminf_n x_n)' = \limsup_n x'_n$ in $\hat{B}$. Thus, x is complemented in $\mathrm{Cl}(B)$, that is, $x \in C$. □

5. Classicality and blackness.

5.1. *Atomless algebras*. Recall Section 1.5.

PROPOSITION 5.1. *If B is atomless, then for every $f \in H$ satisfying $Q_0 f = 0$ and $\varepsilon > 0$ there exist n and $x_1, \dots, x_n \in B$ such that*

$$x_1 \vee \cdots \vee x_n = 1_\Lambda \quad \textit{and} \quad \|Q_{x_1} f\| \le \varepsilon, \dots, \|Q_{x_n} f\| \le \varepsilon.$$

The proof is given after three lemmas.

LEMMA 5.2. *Let $F \subset B$ be a filter such that $\inf_{x\in F} x = 0_\Lambda$. Then $\inf_{x\in F} \|Q_x f\| = 0$ for all $f \in H$ satisfying $Q_0 f = 0$.*

PROOF. Given such f, we denote $c = \inf_{x\in F} \|Q_x f\|$, assume that $c > 0$ and seek a contradiction.

We choose $x_n \in F$ such that $x_1 \ge x_2 \ge \cdots$ and $\|Q_{x_n} f\| \downarrow c$. Necessarily, $x_n \downarrow x$ for some $x \in \Lambda$; by (2.9), $Q_{x_n} \to Q_x$, thus $\|Q_x f\| = c$.

For arbitrary $y \in F$ we have $\|Q_y Q_{x_n} f\| \ge c$ [since $Q_y Q_{x_n} = Q_{y\wedge x_n}$ by (4.9), and $y \wedge x_n \in F$], therefore $\|Q_y Q_x f\| \ge c = \|Q_x f\|$, which implies $Q_y Q_x f = Q_x f$, that is, $Q_x f \in H_y$ for all $y \in F$. By Fact 2.4, $\bigcap_{y\in F} H_y = H_0$. We get $Q_x f \in H_0$, $Q_0 Q_x f = 0$ and $\|Q_x f\| \ne 0$; a contradiction. □

LEMMA 5.3. *Let a function $m : B \to [0,\infty)$ satisfy $m(x \vee y) + m(x \wedge y) \ge m(x) + m(y)$ for all $x, y \in B$, and $m(0_\Lambda) = 0$. Then the following two conditions on m are equivalent*:

(a) *for every $\varepsilon > 0$ there exist n and $x_1, \dots, x_n \in B$ such that $x_1 \vee \cdots \vee x_n = 1_\Lambda$ and $m(x_1) \le \varepsilon, \dots, m(x_n) \le \varepsilon$;*

(b) $\inf_{x\in F} m(x) = 0$ *for every ultrafilter $F \subset B$.*

PROOF. (a) $\Longrightarrow$ (b): the ultrafilter must contain at least one x_k, thus $\inf_{x\in F} m(x) \le \varepsilon$ for every ε.

(b) $\Longrightarrow$ (a): we assume that (a) is violated and prove that (b) is violated.

Note that $m(x \vee y) \ge m(x) + m(y) \ge m(x)$ whenever $x \wedge y = 0_\Lambda$, and therefore $m(x) \ge m(y)$ whenever $x \ge y$.

We define $\gamma : B \to [0,\infty)$ by

$$\gamma(x) = \inf_{x_1\vee\cdots\vee x_n = x} \max\bigl(m(x_1), \dots, m(x_n)\bigr),$$

the infimum being taken over all n and all $x_1, \dots, x_n \in B$ such that $x_1 \vee \cdots \vee x_n = x$. We denote $c = \gamma(1_\Lambda)$ and note that $c > 0$ [since (a) is violated]. Clearly, $\gamma(x) \le m(x)$, and $\gamma(x \vee y) = \max(\gamma(x), \gamma(y))$ for all $x, y \in B$.

CLAIM. *For every $x \in B$ and $\varepsilon > 0$ there exists $y \in B$ such that $y \le x$ and $\gamma(x) = \gamma(y) \le m(y) \le \gamma(x) + \varepsilon$.*

PROOF. Take $x_1, \dots, x_n$ such that $x_1 \vee \dots \vee x_n = x$ and $\max_k m(x_k) \le \gamma(x) + \varepsilon$; note that $\gamma(x) = \max_k \gamma(x_k)$, choose k such that $\gamma(x) = \gamma(x_k)$, and then $y = x_k$ fits. □

Iterating the transition from x to y we construct $x_0, x_1, x_2, \dots \in B$ such that $1_\Lambda = x_0 \ge x_1 \ge x_2 \ge \cdots$, $\gamma(x_n) = c$ for all n, and $m(x_n) \downarrow c$ as $n \to \infty$.

We introduce

$$F = \Bigl\{ y \in B : \lim_n m(x_n \wedge y) \ge c \Bigr\} = \{ y \in B : m(x_n \wedge y) \downarrow c \}$$

and note that $\inf_{y \in F} m(y) \ge c > 0$ [just because $m(y) \ge m(x_n \wedge y)$]. It is sufficient to prove that F is an ultrafilter.

If $y \in F$ and $y \le z$, then $z \in F$ [just because $m(x_n \wedge y) \le m(x_n \wedge z)$].

If $y, z \in F$, then $m(x_n) \ge m((x_n \wedge y) \vee (x_n \wedge z)) \ge m(x_n \wedge y) + m(x_n \wedge z) - m((x_n \wedge y) \wedge (x_n \wedge z))$, therefore $\lim_n m(x_n \wedge y \wedge z) \ge \lim_n m(x_n \wedge y) + \lim_n m(x_n \wedge z) - \lim_n m(x_n) = c$, thus $y \wedge z \in F$. We conclude that F is a filter.

For arbitrary $y \in B$ we have $c = \gamma(x_n) \le \max(m(x_n \wedge y), m(x_n \wedge y'))$ for all n; thus $c \le \lim_n \max(m(x_n \wedge y), m(x_n \wedge y')) = \max(\lim_n m(x_n \wedge y), \lim_n m(x_n \wedge y'))$, which shows that $y \notin F \Longrightarrow y' \in F$. We conclude that F is an ultrafilter, which completes the proof. □

LEMMA 5.4. *$Q_x + Q_y \le Q_{x \vee y} + Q_{x \wedge y}$ for all $x, y \in B$.*

PROOF. By (4.4), Q_x and Q_y are commuting projections, which implies $Q_x + Q_y = Q_x \vee Q_y + Q_x \wedge Q_y$, where $Q_x \vee Q_y$ and $Q_x \wedge Q_y$ are projections onto $Q_x H + Q_y H$ and $Q_x H \cap Q_y H$, respectively. Using (4.9), $Q_x \wedge Q_y = Q_x Q_y = Q_{x \wedge y}$. It remains to note that $Q_x \vee Q_y \le Q_{x \vee y}$ just because $Q_x \le Q_{x \vee y}$ and $Q_y \le Q_{x \vee y}$. □

Taking into account that $\| Q_x \psi \|^2 = \langle Q_x \psi, \psi \rangle$ we get

$$\| Q_x f \|^2 + \| Q_y f \|^2 \le \| Q_{x \vee y} f \|^2 + \| Q_{x \wedge y} f \|^2 \tag{5.1}$$

for all $x, y \in B$ and $f \in H$. Thus, the function $m : x \mapsto \| Q_x f \|^2$ satisfies the condition $m(x \vee y) + m(x \wedge y) \ge m(x) + m(y)$ of Lemma 5.3; the other condition, $m(0_\Lambda) = 0$, is also satisfied if $Q_0 f = 0$.

PROOF OF PROPOSITION 5.1. Let $f \in H$, $Q_0 f = 0$. By Lemma 5.2, $\inf_{x \in F} \| Q_x f \| = 0$ for every ultrafilter $F \subset B$. It remains to apply Lemma 5.3 to $m : x \mapsto \| Q_x f \|^2$. □

5.2. *The first chaos*; *proving Proposition* 1.10. Let C be the completion of B; see Definition 1.8. Recall the first chaos space $H^{(1)}(B) \subset H$ (Definition 1.2).

LEMMA 5.5. *The following three conditions on* $f \in H$ *are equivalent*:

(a) $f \in H^{(1)}(B)$, *that is*, $f = Q_x f + Q_{x'} f$ *for all* $x \in B$;
(b) $Q_{x\vee y} f = Q_x f + Q_y f$ *for all* $x, y \in B$ *satisfying* $x \wedge y = 0_\Lambda$;
(c) $Q_{x\vee y} f + Q_{x\wedge y} f = Q_x f + Q_y f$ *for all* $x, y \in B$, *and* $Q_0 f = 0$.

PROOF. Condition (a) for $x = 0_\Lambda$ gives $f = Q_0 f + f$, that is, $Q_0 f = 0$. Condition (b) for $x = y = 0_\Lambda$ gives $Q_0 f = Q_0 f + Q_0 f$, that is, $Q_0 f = 0$ (again). Condition (c) requires $Q_0 f = 0$ explicitly. Thus, we restrict ourselves to f satisfying $Q_0 f = 0$.

Clearly, (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a); we'll prove that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Recall (4.9): $Q_x Q_y = Q_{x\wedge y}$.

(a) $\Longrightarrow$ (b): If $x \wedge y = 0_\Lambda$, then $Q_{x\vee y} f = Q_{x\vee y}(Q_x f + Q_{x'} f) = Q_{x\vee y} Q_x f + Q_{x\vee y} Q_{x'} f = Q_{(x\vee y)\wedge x} f + Q_{(x\vee y)\wedge x'} f = Q_x f + Q_y f$.

(b) $\Longrightarrow$ (c): we apply (b) twice; first, to x and $x' \wedge y$, getting $Q_{x\vee y} f = Q_x f + Q_{x'\wedge y} f$, and second, to $x \wedge y$ and $x' \wedge y$, getting $Q_y f = Q_{x\wedge y} f + Q_{x'\wedge y} f$. It remains to eliminate $Q_{x'\wedge y} f$. □

PROOF OF PROPOSITION 1.10. It is sufficient to prove that $H^{(1)}(B) = H^{(1)}(C)$. The inclusion $H^{(1)}(B) \supset H^{(1)}(C)$ follows readily from the inclusion $B \subset C$. We have to prove that $H^{(1)}(B) \subset H^{(1)}(C)$. Let $f \in H^{(1)}(B)$. By Lemma 5.5, $Q_0 f = 0$ and $Q_{x\vee y} f + Q_{x\wedge y} f = Q_x f + Q_y f$ for all $x, y \in B$; it is sufficient to extend this equality to all $x, y \in C$. We do it in two steps: first, we extend it to $x \in B$, $y \in C$ by separate continuity in y for fixed x; and second, we extend it to $x, y \in C$ by separate continuity in x for fixed y. The separate continuity of $x \vee y$ is ensured by Lemma 4.4. Continuity of $x \wedge y$ is ensured by (4.11). □

From now on we often abbreviate $H^{(1)}(B)$ to $H^{(1)}$.

CLAIM. *The space* $H^{(1)}$ *is invariant under projections* Q_x *for* $x \in B$ *and moreover, for* $x \in \mathrm{Cl}(B)$.

PROOF. For $f \in H^{(1)}$, $x \in \mathrm{Cl}(B)$ and $g = Q_x f$ we have, using (4.7), $Q_y g + Q_{y'} g = (Q_y + Q_{y'}) Q_x f = Q_x (Q_y + Q_{y'}) f = Q_x f = g$ for all $y \in B$, which means $g \in H^{(1)}$. □

We denote the restriction of Q_x to $H^{(1)}$ by $Q_x^{(1)}$; using (4.9) and Lemma 5.5 we have for all $x, y \in B$,

$$Q_x^{(1)} : H^{(1)} \to H^{(1)}; \qquad Q_x^{(1)} f = Q_x f; \qquad Q_0^{(1)} = 0, \qquad Q_1^{(1)} = I; \tag{5.2}$$

$$Q_{x\wedge y}^{(1)} = Q_x^{(1)} Q_y^{(1)}; \tag{5.3}$$

(5.4) $$Q^{(1)}_{x\vee y} + Q^{(1)}_{x\wedge y} = Q^{(1)}_x + Q^{(1)}_y;$$

(5.5) $$Q^{(1)}_{x\vee y} = Q^{(1)}_x + Q^{(1)}_y \qquad \text{whenever } x\wedge y = 0_\Lambda;$$

(5.6) $$Q^{(1)}_x + Q^{(1)}_{x'} = I;$$

here I is the identity operator on $H^{(1)}$.

5.3. *Sufficient subalgebras; proving Theorem 1.13.*

LEMMA 5.6. *The following two conditions on $x \in B$ and $f \in H$ are equivalent*:

(a) $f = Q_x f + Q_{x'} f$;
(b) $\mathbb{E} f = 0$, *and* $\mathbb{E}(fgh) = 0$ *for all* $g \in H_x$, $h \in H_{x'}$ *satisfying* $\mathbb{E}g = 0$, $\mathbb{E}h = 0$.

PROOF. Treating H as $H_x \otimes H_{x'}$ according to Fact 2.12 we have

$$\begin{aligned} H &= \big((H_x \ominus H_0) \oplus H_0\big) \otimes \big((H_{x'} \ominus H_0) \oplus H_0\big) \\ &= (H_x \ominus H_0) \otimes (H_{x'} \ominus H_0) \oplus (H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0) \\ &\quad \oplus H_0 \otimes H_0; \end{aligned}$$

here $H_x \ominus H_0$ is the orthogonal complement of H_0 in H_x (it consists of all zero-mean functions of H_x). In this notation $Q_x + Q_{x'}$ becomes

$$\begin{aligned} &I \otimes Q_0^{(x')} + Q_0^{(x)} \otimes I \\ &\qquad = \big((I - Q_0^{(x)}) + Q_0^{(x)}\big) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes \big((I - Q_0^{(x')}) + Q_0^{(x')}\big) \\ &\qquad = (I - Q_0^{(x)}) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes (I - Q_0^{(x')}) + 2Q_0^{(x)} \otimes Q_0^{(x')}, \end{aligned}$$

the projection onto $(H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0)$ plus twice the projection onto $H_0 \otimes H_0$ $(= H_0)$. Thus, the equality $f = (Q_x + Q_{x'})f$ [item (a)] becomes $f \in (H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0)$, or equivalently, orthogonality of f to H_0 and $(H_x \ominus H_0) \otimes (H_{x'} \ominus H_0)$, which is item (b). □

REMARK 5.7. The proof given above shows also that

$$\{f \in H : f = Q_x f + Q_{x'} f\} = (H_x \ominus H_0) \oplus (H_{x'} \ominus H_0)$$

for all $x \in B$.

Let $B_0 \subset B$ be a noise-type subalgebra, and $f \in H^{(1)}(B_0)$. We say that f is B_0-atomless, if for every $\varepsilon > 0$ there exist n and $x_1, \dots, x_n \in B_0$ such that $x_1 \vee \dots \vee x_n = 1_\Lambda$ and $\|Q_{x_1} f\| \le \varepsilon, \dots, \|Q_{x_n} f\| \le \varepsilon$.

PROPOSITION 5.8. *If $f \in H^{(1)}(B_0)$ is B_0-atomless, then $f \in H^{(1)}(B)$.*

PROOF. Given $x \in B$, we have to prove that $f = Q_x f + Q_{x'} f$. Let $g \in H_x \ominus H_0$, $h \in H_{x'} \ominus H_0$; by Lemma 5.6 it is sufficient to prove that $\mathbb{E}(fgh) = 0$.

Given $\varepsilon > 0$, we take $y_1, \dots, y_n$ in B_0 such that $y_1 \vee \dots \vee y_n = 1_\Lambda$, $\|Q_{y_i} f\| \le \varepsilon$ for all i, and in addition, $y_i \wedge y_j = 0_\Lambda$ whenever $i \ne j$. We have $f = \sum_i Q_{y_i} f$ by Lemma 5.5, thus, $\mathbb{E}(fgh) = \sum_i \mathbb{E}((Q_{y_i} f)gh)$. Further, $\mathbb{E}((Q_{y_i} f)gh) = \langle Q_{y_i} f, g \otimes h\rangle = \langle Q_{y_i} f, Q_{y_i}(g \otimes h)\rangle = \langle Q_{y_i} f, (Q_{u_i}^{(x)} \otimes Q_{v_i}^{(x')})(g \otimes h)\rangle = \langle Q_{y_i} f, (Q_{u_i}^{(x)} g) \otimes (Q_{v_i}^{(x')} h)\rangle$, where $u_i = y_i \wedge x$ and $v_i = y_i \wedge x'$; it follows that $|\mathbb{E}(fgh)| \le \sum_i \|Q_{y_i} f\| \cdot \|Q_{u_i}^{(x)} g\| \cdot \|Q_{v_i}^{(x')} h\|$. By (5.1), $\sum_i \|Q_{u_i}^{(x)} g\|^2 \le \|g\|^2$ and $\sum_i \|Q_{v_i}^{(x')} h\|^2 \le \|h\|^2$. We get $|\mathbb{E}(fgh)| \le (\max_i \|Q_{y_i} f\|)(\sum_i \|Q_{u_i}^{(x)} g\| \cdot \|Q_{v_i}^{(x')} h\|) \le \varepsilon \|g\| \|h\|$ for all ε. $\square$

PROOF OF THEOREM 1.13. Given an atomless noise-type subalgebra $B_0 \subset B$, we have to prove that $H^{(1)}(B_0) \subset H^{(1)}(B)$. Applying Proposition 5.1 to B_0 we see that every $f \in H^{(1)}(B_0)$ is B_0-atomless. By Proposition 5.8, $f \in H^{(1)}(B)$. $\square$

6. The easy part of Theorem 1.5.

6.1. *From* (a) *to* (b). In this subsection we assume that B is a *classical* noise-type Boolean algebra and prove that its completion, C, is equal to its closure, $\mathrm{Cl}(B)$; in combination with Corollary 4.7 it gives the implication (a) $\Longrightarrow$ (b) of Theorem 1.5.

The first chaos space $H^{(1)}$ is invariant under Q_x for $x \in B$ and moreover, for $x \in \mathrm{Cl}(B)$, as noted in Section 5.2. We denote by $\mathrm{Down}(x)$, for $x \in \mathrm{Cl}(B)$, the restriction of Q_x to $H^{(1)}$ (treated as an operator $H^{(1)} \to H^{(1)}$), recall Section 5.2 and note that

$$\mathrm{Down}(x) : H^{(1)} \to H^{(1)}, \qquad \mathrm{Down}(x) f = Q_x f \qquad \text{for } x \in \mathrm{Cl}(B); \tag{6.1}$$

$$\mathrm{Down}(x) = Q_x^{(1)} \qquad \text{for } x \in B; \tag{6.2}$$

$$\mathrm{Down}(x) + \mathrm{Down}(x') = I \qquad \text{for } x \in B; \tag{6.3}$$

$$x \le y \text{ implies } \mathrm{Down}(x) \le \mathrm{Down}(y) \qquad \text{for } x, y \in \mathrm{Cl}(B). \tag{6.4}$$

We denote by $\mathbf{Q}$ the closure of $\{\mathrm{Down}(x) : x \in B\}$ in the strong operator topology; $\mathbf{Q}$ is a closed set of commuting projections on $H^{(1)}$; we have $\mathrm{Down}(x) \in \mathbf{Q}$ for $x \in B$, and by continuity for $x \in \mathrm{Cl}(B)$ as well.

Note that $q \in \mathbf{Q}$ implies $I - q \in \mathbf{Q}$ [since $\mathrm{Down}(x_n) \to q$ implies $\mathrm{Down}(x_n') \to I - q$ by (6.3)].

For $q \in \mathbf{Q}$ we define $\mathrm{Up}(q) = \sigma(qH^{(1)}) \in \Lambda$ (the σ-field generated by qf for all $f \in H^{(1)}$) and note that

$$q_1 \le q_2 \qquad \text{implies } \mathrm{Up}(q_1) \le \mathrm{Up}(q_2); \tag{6.5}$$

$$\mathrm{Up}(q) \vee \mathrm{Up}(I - q) = 1_\Lambda \qquad \text{for } q \in \mathbf{Q}; \tag{6.6}$$

in general, $\mathrm{Up}(q) \vee \mathrm{Up}(I-q) = \sigma(H^{(1)})$, since $qH^{(1)} + (I-q)H^{(1)} = H^{(1)}$; and the equality $\sigma(H^{(1)}) = 1_\Lambda$ is the classicality (Definition 1.3).

LEMMA 6.1. *$\mathrm{Up}(q)$ and $\mathrm{Up}(I-q)$ are independent (for each $q \in \mathbf{Q}$).*

PROOF. We take $x_n \in B$ such that $\mathrm{Down}(x_n) \to q$, then $\mathrm{Down}(x_n') \to I - q$. We have to prove that $\sigma(qH^{(1)})$ and $\sigma((I-q)H^{(1)})$ are independent, that is, two random vectors $(qf_1, \dots, qf_k)$ and $((I-q)g_1, \dots, (I-q)g_l)$ are independent for all k, l and all $f_1, \dots, f_k, g_1, \dots, g_l \in H^{(1)}$. It follows by Fact 2.19 from the similar claim for $\mathrm{Down}(x_n)$ in place of q. $\square$

LEMMA 6.2. *$\mathrm{Up}(\mathrm{Down}(x)) = x$ for every $x \in B$.*

PROOF. Denote $q = \mathrm{Down}(x)$, then $\mathrm{Down}(x') = I - q$ by (6.3). We have $\mathrm{Up}(q) \le x$ (since $qf = Q_x f$ is x-measurable for $f \in H^{(1)}$); similarly, $\mathrm{Up}(I - q) \le x'$. By (6.6) and (2.10), $\mathrm{Up}(q) = (\mathrm{Up}(q) \vee \mathrm{Up}(I-q)) \wedge x = x$. $\square$

LEMMA 6.3. *If $q, q_1, q_2, \dots \in \mathbf{Q}$ satisfy $q_n \uparrow q$, then $\mathrm{Up}(q_n) \uparrow \mathrm{Up}(q)$.*

PROOF. $q_n H^{(1)} \uparrow qH^{(1)}$ implies $\sigma(q_n H^{(1)}) \uparrow \sigma(qH^{(1)})$. $\square$

LEMMA 6.4. *If $q, q_1, q_2, \dots \in \mathbf{Q}$ satisfy $q_n \downarrow q$, then $\mathrm{Up}(q_n) \downarrow \mathrm{Up}(q)$.*

PROOF. We have $\mathrm{Up}(q_n) \downarrow x$ for some $x \in \Lambda$, $x \ge \mathrm{Up}(q)$. By Lemma 6.1, $\mathrm{Up}(q_n)$ and $\mathrm{Up}(I - q_n)$ are independent; thus, x and $\mathrm{Up}(I - q_n)$ are independent for all n. By Lemma 6.3, $\mathrm{Up}(I - q_n) \uparrow \mathrm{Up}(I - q)$. Therefore x and $\mathrm{Up}(I - q)$ are independent. By (6.6) and (2.10), $\mathrm{Up}(q) = (\mathrm{Up}(q) \vee \mathrm{Up}(I-q)) \wedge x = x$. $\square$

Now we prove that $C = \mathrm{Cl}(B)$. By (4.5), every $x \in \mathrm{Cl}(B)$ is of the form

$$x = \liminf_n x_n = \sup_n \inf_k x_{n+k}$$

for some $x_n \in B$. It follows that $\mathrm{Down}(x) = \liminf_n \mathrm{Down}(x_n)$; by Lemmas 6.3, 6.4, $\mathrm{Up}(\mathrm{Down}(x)) = \liminf_n \mathrm{Up}(\mathrm{Down}(x_n))$; using Lemma 6.2 we get $\mathrm{Up}(\mathrm{Down}(x)) = \liminf_n x_n = x$.

On the other hand, $I - \mathrm{Down}(x) = \limsup_n (I - \mathrm{Down}(x_n)) = \limsup_n \mathrm{Down}(x_n')$ by (6.3), thus the element $y = \mathrm{Up}(I - \mathrm{Down}(x))$ satisfies (by Lemmas 6.3, 6.4 and 6.2 again) $y = \limsup_n \mathrm{Up}(\mathrm{Down}(x_n')) = \limsup_n x_n' \in \mathrm{Cl}(B)$.

By Lemma 6.1, x and y are independent. By (6.6), $x \vee y = 1_\Lambda$. Therefore, y is the complement of x in $\mathrm{Cl}(B)$, and we conclude that $x \in C$. Thus, $C = \mathrm{Cl}(B)$.

6.2. *From* (b) *to* (c). As before, C stands for the completion of B. Let $x \in \mathrm{Cl}(B)$ be such that $x_n \uparrow x$ for some $x_n \in B$.

PROPOSITION 6.5. *The following five conditions on x are equivalent*:

(a) $x \in C$;
(b) $x \vee \lim_n x'_n = 1_\Lambda$ *for some* $x_n \in B$ *satisfying* $x_n \uparrow x$;
(c) $x \vee \lim_n x'_n = 1_\Lambda$ *for all* $x_n \in B$ *satisfying* $x_n \uparrow x$;
(d) $\lim_m \lim_n (x_m \vee x'_n) = 1_\Lambda$ *for some* $x_n \in B$ *satisfying* $x_n \uparrow x$;
(e) $\lim_m \lim_n (x_m \vee x'_n) = 1_\Lambda$ *for all* $x_n \in B$ *satisfying* $x_n \uparrow x$.

LEMMA 6.6. $(\sup_n x_n) \wedge (\inf_n x'_n) = 0_\Lambda$ *for every increasing sequence* $(x_n)_n$ *of elements of* B.

PROOF. Note that $x_m \wedge (\inf_n x'_n) \le x_m \wedge x'_m = 0_\Lambda$, and use (4.11). □

PROOF OF PROPOSITION 6.5. (c) $\Longrightarrow$ (b): trivial.
(b) $\Longrightarrow$ (a): by Lemma 6.6, $x \wedge \lim_n x'_n = 0_\Lambda$, thus, x has the complement $\lim_n x'_n$ and therefore belongs to C.
(a) $\Longrightarrow$ (c): if $x_n \uparrow x$, then (taking complements in the Boolean algebra C) $x'_n \ge x'$, therefore $\lim_n x'_n \ge x'$ and $x \vee \lim_n x'_n \ge x \vee x' = 1_\Lambda$.
We see that (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c); Lemma 6.7 below gives (b) $\Longleftrightarrow$ (d) and (c) $\Longleftrightarrow$ (e). □

LEMMA 6.7. *For every increasing sequence $(x_n)_n$ of elements of B*,

$$\Bigl(\lim_n x_n\Bigr) \vee \Bigl(\lim_n x'_n\Bigr) = \lim_m \lim_n (x_m \vee x'_n).$$

PROOF. Denote for convenience $y = \lim_n x_n$ and $z = \lim_n x'_n$. We have $x'_n \le x'_m$ for $n \ge m$. Applying Theorem 3.8 to the pairs $(x_m, x'_n) \in \Lambda_{x_m} \times \Lambda_{x'_m}$ for a fixed m and all $n \ge m$ we get $x_m \vee x'_n \to x_m \vee z$ as $n \to \infty$. Further, $x_m \wedge z \le x_m \wedge x'_m = 0$ for all m; by (4.11), $y \wedge z = 0$, and by (4.10), y and z are independent. Applying Theorem 3.8 (again) to $(x_m, z) \in \Lambda_y \times \Lambda_z$ we get $x_m \vee z \to y \vee z$ as $m \to \infty$. Finally, $\lim_m \lim_n (x_m \vee x'_n) = \lim_m (x_m \vee z) = y \vee z = (\lim_n x_n) \vee (\lim_n x'_n)$. □

By Corollary 4.7, condition (b) of Theorem 1.5 is equivalent to $C = \mathrm{Cl}(B)$. If it is satisfied, then Proposition 6.5 gives $(\sup_n x_n) \vee (\inf_n x'_n) = 1_\Lambda$ for all $x_n \in B$ such that $x_1 \le x_2 \le \cdots$, which is condition (c) of Theorem 1.5.

7. The difficult part of Theorem 1.5. The proof of the implication (c) $\Longrightarrow$ (a) of Theorem 1.5, given in this section, is a remake of [18], Sections 6c/6.3. In both cases spectrum is crucial. The one-dimensional framework used in [18] leads to "spectral sets"—random compact subsets of the parameter space $\mathbb{R}$. The

Boolean framework used here, being free of any parameter space, leads to a more abstract "spectral space"; see Section 7.2. The number of points in a spectral set, used in [18], becomes here a special function (denoted by K in Section 7.4) on the spectral space.

7.1. *A random supremum.* By Proposition 6.5, condition (c) of Theorem 1.5 may be reformulated as follows:

$$\sup_n x_n \in C \qquad \text{for all } x_n \in B \text{ such that } x_1 \le x_2 \le \cdots \tag{7.1}$$

or equivalently,

$$\lim_m \lim_n \big(x_1 \vee \cdots \vee x_m \vee (x_1 \vee \cdots \vee x_n)'\big) = 1 \qquad \text{for all } x_n \in B. \tag{7.2}$$

In order to effectively use this condition we choose a sequence $(x_n)_n$, $x_n \in B$, whose supremum is unlikely to belong to C. Ultimately it will be proved that $\sup_n x_n \in C$ only if B is classical.

However, we do not construct $(x_n)_n$ explicitly. Instead we use probabilistic method: construct a random sequence that has the needed property with a nonzero probability.

Our noise-type Boolean algebra B consists of sub-σ-fields on a probability space $(\Omega, \mathcal{F}, P)$. However, randomness of x_n does not mean that x_n is a function on Ω. Another probability space, unrelated to $(\Omega, \mathcal{F}, P)$, is involved. It may be thought of as the space of sequences $(x_n)_n$ endowed with a probability measure described below.

A measure on a Boolean algebra b is defined as a countably additive function $b \to [0, \infty)$ ([9], Section 15). However, the distribution of a random element of b (assuming that b is finite) is rather a probability measure ν on the set of all elements of b, that is, a countably additive function $\nu : 2^b \to [0, \infty)$, $\nu(b) = 1$. It boils down to a function $b \to [0, \infty)$, $x \mapsto \nu(\{x\})$, such that $\sum_{x \in b} \nu(\{x\}) = 1$.

Given a finite Boolean algebra b and a number $p \in (0, 1)$, we introduce a probability measure $\nu_{b,p}$ on the set of elements of b by

$$\nu_{b,p}\big(\{a_{i_1} \vee \cdots \vee a_{i_k}\}\big) = p^k (1-p)^{n-k} \qquad \text{for } 1 \le i_1 < \cdots < i_k \le n \tag{7.3}$$

[using the notation of (2.16)]. That is, each atom is included with probability p, independently of others.

Given finite Boolean subalgebras $b_1 \subset b_2 \subset \cdots \subset B$ and numbers $p_1, p_2, \ldots \in (0, 1)$, we consider probability measures $\nu_n = \nu_{b_n, p_n}$ and their product, the probability measure $\nu = \nu_1 \times \nu_2 \times \cdots$ on the set $b_1 \times b_2 \times \cdots$ of sequences $(x_n)_n$, $x_n \in b_n$. We note that $\sup_n x_n \in \mathrm{Cl}(B)$ for all such sequences and ask, whether or not

$$\sup_n x_n \in C \qquad \text{for } \nu\text{-almost all sequences } (x_n)_n, \tag{7.4}$$

or equivalently,

$$\lim_m \lim_n \bigl(x_1 \vee \dots \vee x_m \vee (x_1 \vee \dots \vee x_n)'\bigr) = 1_\Lambda \qquad \text{for } \nu\text{-almost all sequences } (x_n)_n. \tag{7.5}$$

PROPOSITION 7.1. *If (7.5) holds for all such $b_1, b_2, \dots$ and $p_1, p_2, \dots$, then B is classical.*

In order to prove the implication (c) $\Longrightarrow$ (a) of Theorem 1.5 it is sufficient to prove Proposition 7.1. To this end we need spectral theory.

7.2. *Spectrum as a measure class factorization.* The projections Q_x for $x \in \mathrm{Cl}(B)$ commute by (4.7), and generate a commutative von Neumann algebra $\mathcal{A}$. Section 2.4 gives us a measure class space $(S, \Sigma, \mathcal{M})$ and an isomorphism

$$\alpha : \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M}). \tag{7.6}$$

We call $(S, \Sigma, \mathcal{M})$ (endowed with α) the *spectral space* of B. Projections Q_x turn into indicators

$$\alpha(Q_x) = \mathbb{1}_{S_x}, \qquad S_x \in \Sigma \qquad \text{for } x \in \mathrm{Cl}(B) \tag{7.7}$$

(of course, S_x is an equivalence class rather than a set); (4.9) gives

$$S_x \cap S_y = S_{x \wedge y} \qquad \text{for } x, y \in \mathrm{Cl}(B). \tag{7.8}$$

(In contrast, the evident inclusion $S_x \cup S_y \subset S_{x \vee y}$ is generally strict.)

CLAIM.

$$x_n \downarrow x \textit{ implies } S_{x_n} \downarrow S_x; \qquad \textit{also } x_n \uparrow x \textit{ implies } S_{x_n} \uparrow S_x; \tag{7.9}$$

here $x, x_1, x_2, \dots \in \mathrm{Cl}(B)$.

PROOF. let $x_n \uparrow x$, then $Q_{x_n} \uparrow Q_x$, thus $\alpha(Q_{x_n}) \uparrow \alpha(Q_x)$ by (2.12), which means $S_{x_n} \uparrow S_x$; the case $x_n \downarrow x$ is similar. $\square$

The subspaces $H_x = Q_x H \subset H$ for $x \in \mathrm{Cl}(B)$ are a special case of the subspaces $H(E) = \alpha^{-1}(\mathbb{1}_E)H \subset H$ for $E \in \Sigma$ [recall (2.14)]; by (7.7),

$$H(S_x) = H_x \qquad \text{for } x \in \mathrm{Cl}(B). \tag{7.10}$$

Every subset of B leads to a subalgebra of $\mathcal{A}$. In particular, for every $x \in B$ we introduce the von Neumann algebra

$$\mathcal{A}_x \subset \mathcal{A} \quad \text{generated by } \{Q_y : y \in B, x \vee y = 1_\Lambda\} = \{Q_{u \vee x'} : u \in B, u \le x\} \tag{7.11}$$

and the σ-field $\Sigma_x \subset \Sigma$ such that

$$\alpha(\mathcal{A}_x) = L_\infty(\Sigma_x) \qquad \text{for } x \in B \tag{7.12}$$

(see Fact 2.23). Note that

$$x \le y \qquad \text{implies } \mathcal{A}_x \subset \mathcal{A}_y \text{ and } \Sigma_x \subset \Sigma_y \qquad \text{for } x, y \in B. \tag{7.13}$$

Recall Notation 2.13: $Q_u^{(x)} : H_x \to H_x$ for $u \le x$, and Fact 2.14: given independent x, y, treating $H_{x\vee y}$ as $H_x \otimes H_y$ we have $Q_{u\vee v} = Q_u^{(x)} \otimes Q_v^{(y)}$ for all $u \le x$, $v \le y$. Introducing von Neumann algebras $\mathcal{A}^{(x)}$ of operators on H_x,

$$\mathcal{A}^{(x)} \text{ generated by } \{Q_u^{(x)} : u \in B, u \le x\}, \tag{7.14}$$

we get

$$\mathcal{A}^{(x\vee y)} = \mathcal{A}^{(x)} \otimes \mathcal{A}^{(y)} \qquad \text{whenever } x \wedge y = 0, x, y \in B. \tag{7.15}$$

In the case $y = x'$, treating H as $H_x \otimes H_{x'}$ we have

$$\mathcal{A} = \mathcal{A}^{(x)} \otimes \mathcal{A}^{(x')} \quad \text{and} \quad \mathcal{A}_x = \mathcal{A}^{(x)} \otimes I \qquad \text{for } x \in B \tag{7.16}$$

(for the latter, fix $v = x'$),—a natural isomorphism between $\mathcal{A}_x$ and $\mathcal{A}^{(x)}$. Thus, $\alpha(\mathcal{A}^{(x)} \otimes I) = L_\infty(\Sigma_x)$, $\alpha(I \otimes \mathcal{A}^{(x')}) = L_\infty(\Sigma_{x'})$ and $\alpha(\mathcal{A}^{(x)} \otimes \mathcal{A}^{(x')}) = L_\infty(\Sigma)$. By Fact 2.28, for all $x \in B$,

$$\Sigma_x \text{ and } \Sigma_{x'} \text{ are } \mathcal{M}\text{-independent,} \tag{7.17}$$

$$\Sigma_x \vee \Sigma_{x'} = \Sigma. \tag{7.18}$$

(Thus, the spectral space is a measure class (or "type") factorization as defined in [20], Section 1c and discussed in [2], Section 14.4, [19], Section 10.)

REMARK 7.2. The closure of B determines uniquely the algebra $\mathcal{A}$ and therefore also the spectral space.

EXAMPLE 7.3. Let a noise-type Boolean algebra B be finite, with n atoms. Then $\mathcal{A}$ is of dimension 2^n; $(S, \Sigma, \mathcal{M})$ is the discrete space with 2^n points. Up to isomorphism we may treat both B and S as consisting of all subsets of Atoms(B), and then S_x consists of all subsets of x.

EXAMPLE 7.4. Let B and y_n be as in Section 1.2 and Example 1.9. The sign change transformation $\Omega \to \Omega$ decomposes the Hilbert space: $H = H_{\text{even}} \oplus H_{\text{odd}}$. Introducing $y = \sup_n y_n \in \text{Cl}(B) \setminus B$ we have $H_y = H_{\text{even}}$; the projection Q_y onto H_{even} corresponds to the indicator of S_y. Up to isomorphism we may treat S_y as consisting of all finite subsets of $\{1, 2, \dots\}$, and $S \setminus S_y$ as consisting of their complements, the cofinite subsets of $\{1, 2, \dots\}$. Both B and S become the same countable set, and S_x consists of all finite/cofinite subsets of x (i.e., finite subsets of a finite x, but finite/cofinite subsets of a cofinite x). See also [19], Section 9a (for $m = 2$).

EXAMPLE 7.5. Let B correspond to a noise over $\mathbb{R}$ (see Section 1.6), and assume that the noise is classical, which is equivalent to classicality of B (as defined by Definition 1.3); it is also equivalent to existence of Lévy processes whose increments generate the noise. Assume that the noise is not trivial, that is, $1_B \ne 0_B$. Then B as a Boolean algebra is isomorphic to the Boolean algebra of all finite unions of intervals (on $\mathbb{R}$) modulo finite sets. Up to isomorphism we may treat $(S, \Sigma, \mathcal{M})$ as the space of all finite subsets of $\mathbb{R}$; a measure μ on S belongs to $\mathcal{M}$ if and only if μ is equivalent (i.e., mutually absolutely continuous) to the (symmetrized) n-dimensional Lebesgue measure on the subset $S_n \subset S$ of all n-point sets, for every $n = 0, 1, 2, \dots$; for $n = 0$ it means an atom: $\mu(\{\emptyset\}) > 0$. As before, S_x consists of all $s \in S$ such that $s \subset x$; but now S and B are quite different collections of sets. See also [19], Example 9b9.

In contrast, for a black noise the elements of S may be thought of as some perfect compact subsets of $\mathbb{R}$ (including the empty set), of Lebesgue measure zero. And if a noise is neither classical nor black, then all finite sets belong to S, but also some infinite compact sets of Lebesgue measure zero belong to S. These may be countable or not, depending on the noise. See also [19], Sections 9b, 9c.

7.3. *Restriction to a sub-σ-field.* As was noted in Section 3.4, for an arbitrary $x \in \Lambda$ the triple $(\Omega, x, P|_x)$ is also a probability space, and its lattice of σ-fields is naturally embedded into Λ,

$$\Lambda(\Omega, x, P|_x) = \Lambda_x = \{y \in \Lambda : y \le x\} \subset \Lambda.$$

Dealing with a noise-type Boolean algebra $B \subset \Lambda$ over $(\Omega, \mathcal{F}, P)$, we introduce

$$B_x = B \cap \Lambda_x = \{u \in B : u \le x\} \subset B \qquad \text{for } x \in B$$

and note that

$$B_x \subset \Lambda_x \text{ is a noise-type Boolean algebra over } (\Omega, x, P|_x);$$

thus, notions introduced for B have their counterparts for B_x. We mark them by the *left* index x. Some of these counterparts were used in previous (sub)sections. For $x \in B$:

$$_xH = H_x; \qquad \text{see Section 2.3,}$$

$$_xQ_u = Q_u^{(x)} \qquad \text{for } u \in B_x; \qquad \text{see Notation 2.13,}$$

$$_x\mathcal{A} = \mathcal{A}^{(x)}; \qquad \text{see (7.14),}$$

$$_xS = S;\ _x\Sigma = \Sigma_x; \qquad \text{see (7.12),}$$

$$_x\alpha : \mathcal{A}^{(x)} \to L_\infty(\Sigma_x), \qquad _x\alpha(A) = \alpha(A \otimes I); \qquad \text{see (7.16),}$$

$$_xS_u = S_{u \vee x'} \qquad \text{for } u \in B_x; \qquad \text{see (7.7),}$$

$$_xH^{(1)} = H^{(1)} \cap H_x;$$

the last line follows easily from Lemma 5.5; the next to the last line holds, since ${}_x\alpha({}_xQ_u) = \alpha(Q_u^{(x)} \otimes I) = \alpha(Q_u^{(x)} \otimes Q_{x'}^{(x')}) = \alpha(Q_{u \vee x'})$. The counterpart of $H(E) = \alpha^{-1}(\mathbb{1}_E)H$ for $E \in \Sigma$ is ${}_xH(E) = {}_x\alpha^{-1}(\mathbb{1}_E)H_x$ for $E \in \Sigma_x$.

LEMMA 7.6. *For every $x \in B$, treating H as $H_x \otimes H_{x'}$ we have $H(E \cap F) = ({}_xH(E)) \otimes ({}_{x'}H(F))$ for all $E \in \Sigma_x$, $F \in \Sigma_{x'}$.*

PROOF. We take $A \in \mathcal{A}^{(x)}$, $B \in \mathcal{A}^{(x')}$ such that $\alpha(A \otimes I) = \mathbb{1}_E$, $\alpha(I \otimes B) = \mathbb{1}_F$, then $\alpha(A \otimes B) = \mathbb{1}_E \mathbb{1}_F = \mathbb{1}_{E \cap F}$ and $H(E \cap F) = (A \otimes B)(H_x \otimes H_{x'}) = (AH_x) \otimes (BH_{x'}) = ({}_xH(E)) \otimes ({}_{x'}H(F))$. $\square$

7.4. *Classicality via spectrum.* Let $b \subset B$ be a finite Boolean subalgebra. For almost every $s \in S$ the set $\{x \in b : s \in S_x\}$ is a filter on b due to (7.8); like every filter on a finite Boolean algebra, it is generated by some $x_b(s) \in b$,

$$\forall x \in b \qquad \big(s \in S_x \Longleftrightarrow x \ge x_b(s)\big). \tag{7.19}$$

Like every element of b, $x_b(s)$ is the union of some of the atoms of b [recall (2.17)]; the number of these atoms will be denoted by $K_b(s)$,

$$K_b(s) = \big|\{a \in \mathrm{Atoms}(b) : a \le x_b(s)\}\big|.$$

For two finite Boolean subalgebras,

$$\text{if } b_1 \subset b_2 \text{ then } K_{b_1}(\cdot) \le K_{b_2}(\cdot) \text{ and } x_{b_1}(s) \ge x_{b_2}(s). \tag{7.20}$$

Each K_b is an equivalence class (rather than a function), and the set of all b need not be countable. We take supremum in the complete lattice of all equivalence classes of measurable functions $S \to [0, +\infty]$ (recall Section 2.6):

$$K = \sup_b K_b, \qquad K : S \to [0, +\infty], \tag{7.21}$$

where b runs over all finite Boolean subalgebras $b \subset B$.

THEOREM 7.7. *B is classical if and only if $K(\cdot) < \infty$ almost everywhere.*

We split this theorem in two propositions as follows. Recall that classicality is defined by Definition 1.3 as the equality $\sigma(H^{(1)}) = 1_\Lambda$. Introducing

$$E_k = \{s \in S : K(s) = k\} \quad \text{and} \quad H^{(k)} = H(E_k) \qquad \text{for } k = 0, 1, 2, \ldots$$

[recall (2.14)] we reformulate the condition $K(\cdot) < \infty$ as $S = \biguplus_k E_k$ and further, by (2.15), as $H = \bigoplus_k H^{(k)}$. For $k = 1$ the new notation conforms to the old one in the following sense.

PROPOSITION 7.8. *$H(E_1)$ is equal to the first chaos space $H^{(1)}$ (defined by Definition 1.2).*

PROPOSITION 7.9. *$\sigma(H^{(k)}) \subset \sigma(H^{(1)})$ for all $k = 2, 3, \dots$.*

Thus, $\bigoplus_k H^{(k)} = H \iff \sigma(H^{(1)}) = \sigma(H) \iff \sigma(H^{(1)}) = 1_\Lambda$. We see that Theorem 7.7 follows from Propositions 7.8, 7.9.

The proof of Proposition 7.8 is given after three lemmas.

We introduce minimal nontrivial finite Boolean subalgebras $b_x = \{0, x, x', 1\}$ for $x \in B$.

LEMMA 7.10. *For every $x \in B$,*

$$\{f \in H : f = Q_x f + Q_{x'} f\} = H\bigl(\{s : K_{b_x}(s) = 1\}\bigr).$$

PROOF. $\{s : K_{b_x}(s) = 1\} = \{s : K_{b_x}(s) \le 1\} \setminus \{s : K_{b_x}(s) = 0\} = (S_x \cup S_{x'}) \setminus S_0 = (S_x \setminus S_0) \uplus (S_{x'} \setminus S_0)$ (since $S_x \cap S_{x'} = S_0$), thus $H(\{s : K_{b_x}(s) = 1\}) = H(S_x \setminus S_0) \oplus H(S_{x'} \setminus S_0) = (H_x \ominus H_0) \oplus (H_{x'} \ominus H_0)$; use Remark 5.7. $\square$

LEMMA 7.11. *Assume that $b_1, b_2 \subset B$ are finite Boolean subalgebras, and $b \subset B$ is the (finite by Fact 2.29) Boolean subalgebra generated by b_1, b_2. Then*

$$\bigl\{s : K_{b_1}(s) \le 1\bigr\} \cap \bigl\{s : K_{b_2}(s) \le 1\bigr\} \subset \bigl\{s : K_b(s) \le 1\bigr\}.$$

PROOF. If $K_{b_1}(s) \le 1$, $K_{b_2}(s) \le 1$ and $s \notin S_0$, then $x_{b_1}(s) \in \mathrm{Atoms}(b_1)$, $x_{b_2}(s) \in \mathrm{Atoms}(b_2)$, thus $x_b(s) \le x_{b_1}(s) \wedge x_{b_2}(s) \in \mathrm{Atoms}(b)$ by Fact 2.29, therefore $K_b(s) \le 1$. $\square$

LEMMA 7.12. *$\{s : K(s) \le 1\} = \inf_{x \in B}\{s : K_{b_x}(s) \le 1\}$, and $\{s : K(s) = 1\} = \inf_{x \in B}\{s : K_{b_x}(s) = 1\}$ (the infimum of equivalence classes).*

PROOF. Every finite Boolean subalgebra b is generated by the Boolean subalgebras b_x for $x \in b$; by Lemma 7.11, $\{s : K_b(s) \le 1\} \supset \bigcap_{x \in b}\{s : K_{b_x}(s) \le 1\}$; the infimum over all b gives $\{s : K(s) \le 1\} \supset \inf_{x \in B}\{s : K_{b_x}(s) \le 1\}$. The converse inclusion being trivial, we get the first equality. The second equality follows, since the set $\{s : K_b(s) = 0\}$ is equal to S_0, irrespective of b. $\square$

PROOF OF PROPOSITION 7.8. It follows from the second equality of Lemma 7.12, using (2.18), that $H(E_1) = \bigcap_{x \in B} H(\{s : K_{b_x}(s) = 1\})$. Using Lemma 7.10 we get $H(E_1) = \bigcap_{x \in B}\{f \in H : f = Q_x f + Q_{x'} f\} = H^{(1)}$. $\square$

In order to prove Theorem 7.7 it remains to prove Proposition 7.9.

We have K introduced for B by (7.21), but also for B_x we have ${}_xK$, the counterpart of K in the sense of Section 7.3;

$${}_xK = \sup_b {}_xK_b, \qquad {}_xK : S \to [0, \infty] \qquad \text{for } x \in B,$$

where b runs over all finite Boolean subalgebras $b \subset B_x$; ${}_xK$ is an equivalence class of Σ_x-measurable functions $S \to [0, \infty]$.

LEMMA 7.13. *${}_{x\vee y}K = {}_xK + {}_yK$ for all $x, y \in B$ such that $x \wedge y = 0_\Lambda$.*

PROOF. When calculating ${}_{x\vee y}K$ we may restrict ourselves to finite subalgebras $b \subset B_{x\vee y}$ that contain x and y; recall (7.20). Each such b may be thought of as a pair of $b_1 \subset B_x$ and $b_2 \subset B_y$. We have $\mathrm{Atoms}(b) = \mathrm{Atoms}(b_1) \uplus \mathrm{Atoms}(b_2)$, $x_b(s) = x_{b_1}(s) \vee x_{b_2}(s)$ (recall that ${}_xS_u = S_{u\vee x'}$ for $u \le x$), thus ${}_{x\vee y}K_b = {}_xK_{b_1} + {}_yK_{b_2}$; take the supremum in b_1, b_2. $\square$

LEMMA 7.14. $\{s \in S : K(s) = 2\} = \sup_{x\in B}\{s \in S : {}_xK(s) = {}_{x'}K(s) = 1\}$ (*the supremum of equivalence classes*).

PROOF. The "$\supset$" inclusion follows from Lemma 7.13; it is sufficient to prove that $\{s \in S : K(s) = 2\} \subset \bigcup_{x\in b_1\cup b_2\cup\cdots}\{s \in S : {}_xK(s) = {}_{x'}K(s) = 1\}$ if $b_1 \subset b_2 \subset \cdots$ satisfy $K_{b_n} \uparrow K$.

Given s such that $K(s) = 2$, we take n such that $K_{b_n}(s) = 2$, that is, $x_{b_n}(s)$ contains exactly two atoms of b_n. We choose $x \in b_n$ that contains exactly one of these two atoms; then ${}_xK_{b_n}(s) = {}_{x'}K_{b_n}(s) = 1$, therefore ${}_xK(s) = {}_{x'}K(s) = 1$, since $1 = {}_xK_{b_n}(s) \le {}_xK(s) = K(s) - {}_{x'}K(s) \le 2 - {}_{x'}K_{b_n}(s) = 1$. $\square$

We use the counterpart (in the sense of Section 7.3) of Proposition 7.8: ${}_xH({}_xE_1) = {}_xH^{(1)}$, that is, for every $x \in B$,

$$ {}_xH\big(\{s \in S : {}_xK(s) = 1\}\big) = H^{(1)} \cap H_x. \tag{7.22} $$

PROOF OF PROPOSITION 7.9 FOR $k = 2$. It follows from Lemma 7.14 and (2.19) that $H^{(2)}$ is generated (as a closed linear subspace of H) by the union, over all $x \in B$, of the subspaces $H(\{s \in S : {}_xK(s) = {}_{x'}K(s) = 1\})$. In order to get $\sigma(H^{(2)}) \subset \sigma(H^{(1)})$ it is sufficient to prove that

$$ \sigma\big(H\big(\{s \in S : {}_xK(s) = {}_{x'}K(s) = 1\}\big)\big) \subset \sigma\big(H^{(1)}\big) \qquad \text{for all } x \in B. \tag{7.23} $$

By Lemma 7.6 and (7.22), $H(\{s \in S : {}_xK(s) = {}_{x'}K(s) = 1\}) = {}_xH(\{s \in S : {}_xK(s) = 1\}) \otimes {}_{x'}H(\{s \in S : {}_{x'}K(s) = 1\}) = (H_x \cap H^{(1)}) \otimes (H_{x'} \cap H^{(1)})$, which implies (7.23). $\square$

The proof of Proposition 7.9 for higher k is similar. Lemma 7.14 is generalized to

$$ \{s \in S : K(s) = k\} = \sup_{x\in B}\{s \in S : {}_xK(s) = k-1, {}_{x'}K(s) = 1\}, $$

and (7.23) to

$$ \sigma\big(H\big(\{s \in S : {}_xK(s) = k-1, {}_{x'}K(s) = 1\}\big)\big) \subset \sigma\big(H^{(k-1)} \cup H^{(1)}\big). $$

Thus, $\sigma(H^{(k)}) \subset \sigma(H^{(k-1)} \cup H^{(1)})$. By induction in k, $\sigma(H^{(k)}) \subset \sigma(H^{(1)})$, which completes the proof of Proposition 7.9 and Theorem 7.7.

7.5. *Finishing the proof.*

PROPOSITION 7.15. *If* (7.5) *holds for all* $b_1 \subset b_2 \subset \cdots$ *and* $p_1, p_2, \ldots \in (0, 1)$, *then* $K(\cdot) < \infty$ *almost everywhere.*

By Theorem 7.7, in order to prove Proposition 7.1 it is sufficient to prove Proposition 7.15.

The relation $\lim_m \lim_n (y_m \vee y_n') = 1_\Lambda$ for $y_1 \le y_2 \le \cdots$ [appearing in (7.5) with $y_n = x_1 \vee \cdots \vee x_n$] may be reformulated in spectral terms using (7.9); it turns into $\bigcup_m \bigcap_n S_{y_m \vee y_n'} = S$, in other words, almost every $s \in S$ satisfies $\exists m \forall n \; s \in S_{y_m \vee y_n'}$. Accordingly, (7.5) may be rewritten as follows:

$$\text{for } \nu\text{-almost all sequences } (x_n)_n, \text{ for almost all } s \in S, \exists m \forall n \quad s \in S_{x_1 \vee \cdots \vee x_m \vee (x_1 \vee \cdots \vee x_n)'}. \tag{7.24}$$

We choose $p_1, p_2, \ldots \in (0, 1)$ and $c_1, c_2, \ldots \in \{1, 2, 3, \ldots\}$ such that

$$\sum_n p_n < 1, \tag{7.25}$$

$$(1 - p_n)^{c_n} \to 0 \qquad \text{as } n \to \infty. \tag{7.26}$$

We also choose finite Boolean subalgebras $b_1 \subset b_2 \subset \cdots \subset B$ such that $K_{b_n} \uparrow K$ and introduce $b = b_1 \cup b_2 \cup \cdots \subset B$ (a countable Boolean subalgebra).

CLAIM.

$${}_x K_{b_n} \uparrow {}_x K \qquad \textit{for every } x \in b. \tag{7.27}$$

PROOF. ${}_x K \ge \lim_n {}_x K_{b_n} = \lim_n (K_{b_n} - {}_{x'} K_{b_n}) \ge K - {}_{x'} K = {}_x K$. $\square$

REMARK. For $x \in b_n$, by ${}_x K_{b_n}$ we mean ${}_x K_{b_n \cap B_x}$. Thus ${}_x K_{b_n}$ is well defined for all n large enough, provided that $x \in b$.

Using Fact 2.33 we take $n_1 < n_2 < \cdots$ such that for almost every $s \in S$

$$\begin{gathered} \text{either } {}_x K(s) < \infty, \\ \text{or } {}_x K_{b_{n_k}}(s) \ge c_k \text{ for all } k \text{ large enough.} \end{gathered} \tag{7.28}$$

These n_k depend on $x \in b$. However, countably many x can be served by a single sequence $(n_k)_k$ using the well-known diagonal argument. This way we ensure (7.28) with a single $(n_k)_k$ for all $x \in b$. Now we rename b_{n_k} into b_k, discard a null set of bad points $s \in S$ and get

$$\begin{gathered} \text{either } {}_x K(s) < \infty, \\ \text{or } {}_x K_{b_n}(s) \ge c_n \text{ for all } n \text{ large enough} \end{gathered} \tag{7.29}$$

for all $x \in b$ and $s \in S$; here "n large enough" means $n \ge n_0(x, s)$.

We recall the product measure $\nu = \nu_1 \times \nu_2 \times \cdots$ introduced in Section 7.1 on the product set $b_1 \times b_2 \times \cdots$; as before, $\nu_n = \nu_{b_n, p_n}$. For notational convenience we treat the coordinate maps $X_n : (b_1 \times b_2 \times \cdots, \nu) \to b_n$, $X_n(x_1, x_2, \ldots) = x_n$, as independent b_n-valued random variables; X_n is distributed ν_n, that is, $\mathbb{P}(X_n = x) = \nu_n(\{x\})$ for $x \in b_n$. We introduce b_n-valued random variables

$$Y_n = X_1 \vee \cdots \vee X_n.$$

LEMMA 7.16. *$\mathbb{P}({}_{Y_n'}K(s) < \infty) \le p_1 + \cdots + p_n$ for all $s \in S$ such that $K(s) = \infty$ and all n.*

PROOF. There exists $a \in \mathrm{Atoms}(b_n)$ such that ${}_aK(s) = \infty$ [since $\sum_a {}_aK(s) = K(s) = \infty$]. We have ${}_{Y_n'}K(s) < \infty \Longrightarrow a \le Y_n \Longrightarrow \exists k \in \{1, \ldots, n\}\ a \le X_k$, therefore $\mathbb{P}({}_{Y_n'}K(s) < \infty) \le \sum_{k=1}^n \mathbb{P}(a \le X_k) = \sum_{k=1}^n p_k$. $\square$

LEMMA 7.17. *If $x \in b_m$ and $s \in S$ satisfy ${}_xK(s) = \infty$, then*

$$\mathbb{P}\big(\forall n > m\ X_n \wedge x \wedge x_{b_n}(s) = 0_\Lambda\big) = 0.$$

PROOF. For $n > m$,

$$\mathbb{P}\big(X_n \wedge x \wedge x_{b_n}(s) = 0_\Lambda\big) = (1 - p_n)^{{}_xK_{b_n}(s)},$$

since $x \wedge x_{b_n}(s)$ contains ${}_xK_{b_n}(s)$ atoms of b_n. By (7.29), ${}_xK_{b_n}(s) \ge c_n$ for all n large enough. Thus, $\mathbb{P}(X_n \wedge x \wedge x_{b_n}(s) = 0_\Lambda) \le (1 - p_n)^{c_n} \to 0$ as $n \to \infty$ by (7.26). $\square$

LEMMA 7.18.

$$\mathbb{P}\big({}_{Y_m'}K(s) = \infty \text{ and } \forall n > m\ s \in S_{Y_m \vee Y_n'}\big) = 0$$

for all $s \in S$ and m.

PROOF. By (7.19), $s \in S_{Y_m \vee Y_n'} \Longleftrightarrow Y_m \vee Y_n' \ge x_{b_n}(s)$ for $n > m$. We have to prove that

$$\mathbb{P}\big(Y_m' = y \text{ and } \forall n > m\ y' \vee Y_n' \ge x_{b_n}(s)\big) = 0$$

for every $y \in b_m$ satisfying ${}_yK(s) = \infty$. By Lemma 7.17,

$$\mathbb{P}\big(\forall n > m\ X_n \wedge y \wedge x_{b_n}(s) = 0_\Lambda\big) = 0.$$

It remains to note that $y' \vee Y_n' \ge x_{b_n}(s) \Longleftrightarrow (y \wedge Y_n)' \ge x_{b_n}(s) \Longleftrightarrow (y \wedge Y_n) \wedge x_{b_n}(s) = 0_\Lambda \Longrightarrow y \wedge X_n \wedge x_{b_n}(s) = 0_\Lambda$. $\square$

Now we prove Proposition 7.15. We use (7.24) for $b_1, b_2, \ldots$ and $p_1, p_2, \ldots$ satisfying (7.25), (7.26),

$$\exists m\ \forall n \qquad s \in S_{Y_m \vee Y_n'}$$

almost surely, for almost all $s \in S$. In combination with Lemma 7.18 it gives

$$\mathbb{P}\big(\exists m\ {}_{Y'_m}K(s) < \infty\big) = 1$$

for almost all $s \in S$. On the other hand, by Lemma 7.16 and (7.25),

$$\mathbb{P}\big(\exists m\ {}_{Y'_m}K(s) < \infty\big) = \lim_m \mathbb{P}\big({}_{Y'_m}K(s) < \infty\big) \le p_1 + p_2 + \cdots < 1$$

for all $s \in S$ such that $K(s) = \infty$. Therefore $K(s) < \infty$ for almost all s, which completes the proof of Propositions 7.15, 7.1 and finally, Theorem 1.5. Theorem 1.4 follows immediately.

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