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THE OUTLIERS OF A DEFORMED WIGNER MATRIX

BY ANTTI KNOWLES[1] AND JUN YIN[2]

New York University and University of Wisconsin

We derive the joint asymptotic distribution of the outlier eigenvalues of an additively deformed Wigner matrix H. Our only assumptions on the deformation are that its rank be fixed and its norm bounded. Our results extend those of [The isotropic semicircle law and deformation of Wigner matrices. Preprint] by admitting overlapping outliers and by computing the joint distribution of all outliers. In particular, we give a complete description of the failure of universality first observed in [*Ann. Probab.* **37** (2009) 1–47; *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (1013) 107–133; Free convolution with a semi-circular distribution and eigenvalues of spiked deformations of Wigner matrices. Preprint]. We also show that, under suitable conditions, outliers may be strongly correlated even if they are far from each other. Our proof relies on the isotropic local semicircle law established in [The isotropic semicircle law and deformation of Wigner matrices. Preprint]. The main technical achievement of the current paper is the joint asymptotics of an arbitrary finite family of random variables of the form $\langle \mathbf{v}, (H-z)^{-1}\mathbf{w}\rangle$.

1. Introduction. In this paper, we study a Wigner matrix H—a random $N \times N$ matrix whose entries are independent up to symmetry constraints—that has been deformed by the addition of a finite-rank matrix A belonging to the same symmetry class as H. By Weyl's eigenvalue interlacing inequalities, such a deformation does not influence the global statistics of the eigenvalues as $N \to \infty$. Thus, the empirical eigenvalue densities of the deformed matrix $H + A$ and the undeformed matrix H have the same large-scale asymptotics, and are governed by Wigner's famous semicircle law. However, the behavior of individual eigenvalues may change dramatically under such a deformation. In particular, deformed Wigner matrices may exhibit *outliers*—eigenvalues detached from the bulk spectrum. They were first investigated in [20] for a particular rank-one deformation. Subsequently, much progress [2–4, 8–10, 19, 21, 24, 25] has been made in the understanding of the outliers of deformed Wigner matrices. We refer to [21, 24, 25] for a more detailed review of recent developments.

We normalize H so that its spectrum is asymptotically given by the interval $[-2, 2]$. The creation of an outlier is associated with a sharp transition, where the

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magnitude of an eigenvalue d_i of A exceeds the threshold 1. As d_i (resp., $-d_i$) becomes larger than 1, the largest (resp., smallest) nonoutlier eigenvalue of $H + A$ detaches itself from the bulk spectrum and becomes an outlier. This transition is conjectured to take place on the scale $|d_i| - 1 \sim N^{-1/3}$. In fact, this scale was established in [1, 6, 7, 23] for the special cases where H is Gaussian—the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Unitary Ensemble (GUE). We sketch the results of [1, 6, 7, 23] in the case of additive deformations of GOE/GUE. For simplicity, we consider rank-one deformations, although the results of [1, 6, 7, 23] cover arbitrary finite-rank deformations. Let the eigenvalue d of A be of the form $d = 1 + wN^{-1/3}$ for some fixed $w \in \mathbb{R}$. In [1, 6, 7, 23], the authors proved for any fixed w the weak convergence

$$N^{2/3}(\lambda_N(H + A) - 2) \quad \Longrightarrow \quad \Lambda_w,$$

where $\lambda_N(H + A)$ denotes the largest eigenvalue of $H + A$. In particular, the largest eigenvalue of $H + A$ fluctuates on the scale $N^{-2/3}$. Moreover, the asymptotics in w of the law Λ_w was analysed in [1, 5–7, 23]: as $w \to +\infty$ (and after an appropriate affine scaling), the law Λ_w converges to a Gaussian; as $w \to -\infty$, the law Λ_w converges to the Tracy–Widom-β distribution (where $\beta = 1$ for GOE and $\beta = 2$ for GUE), which famously governs the distribution of the largest eigenvalue of the underformed matrix H [28, 29].

The proofs of [1, 23] use an asymptotic analysis of Fredholm determinants, while those of [5–7] use an explicit tridiagonal representation of H; both of these approaches rely heavily on the Gaussian nature of H. In order to study the phase transition for non-Gaussian matrix ensembles, and in particular address the question of spectral universality, a different approach is needed. Interestingly, it was observed in [8–10] that the distribution of the outliers is not universal, and may depend on the law of H as well as the geometry of the eigenvectors of A. The nonuniversality of the outliers was further investigated in [21, 24, 25].

In a recent paper [21], we considered finite-rank deformations of a Wigner matrix whose entries have subexponential decay. The two main results of [21] may be informally summarized as follows.

(a) We proved that the nonoutliers of $H + A$ *stick* to the extremal eigenvalues of the original Wigner matrix H with high precision, provided that each eigenvalue d_i of A satisfies $||d_i| - 1| \geq (\log N)^{C \log\log N} N^{-1/3}$.

(b) We identified the asymptotic distribution of a single outlier, provided that (i) it is separated from the asymptotic bulk spectrum $[-2, 2]$ by at least $(\log N)^{C \log\log N} N^{-2/3}$ and (ii) it does not overlap with any other outlier of $H + A$. Here, two outliers are said to *overlap* if their separation is comparable to the scale on which they fluctuate; see Section 2.2 below for a precise definition.

Note that the assumption (i) of (b) is optimal, up to the logarithmic factor $(\log N)^{C \log\log N}$. Indeed, the extremal bulk eigenvalues of $H + A$ are known [21], Theorem 2.7, to fluctuate on the scale $N^{-2/3}$; for an eigenvalue of $H + A$ to be an outlier, therefore, we require that its distance from the asymptotic bulk spectrum $[-2, 2]$ be much greater than $N^{-2/3}$. See Section 2.2 below for more details.

The goal of this paper is to extend the result (b) by obtaining a complete description of the asymptotic distribution of the outliers. Our only assumptions on the deformation $A \equiv A_N$ are that its rank be fixed and its norm bounded. (In particular, the eigenvalues of A may depend on N in an arbitrary fashion, provided they remain bounded, and its eigenvectors may be an arbitrary orthonormal family.) Our main result gives the asymptotic joint distribution of all outliers. Here, an outlier is by definition an eigenvalue of $H + A$ whose classical location [see (2.5) below] is separated from the asymptotic bulk spectrum $[-2, 2]$ by at least $(\log N)^{C \log \log N} N^{-2/3}$ for some (large) constant C. Our main result is given in Theorem 2.11 below.

Thus, in this paper we extend the result (b) in two directions: we allow overlapping outliers, and we derive the joint asymptotic distribution of all outliers. The distribution of overlapping outliers is more complicated than that of nonoverlapping outliers, as overlapping outliers exhibit a level repulsion similar to that among the bulk eigenvalues of Wigner matrices. This repulsion manifests itself by the joint distribution of a group of overlapping outliers being given by the distribution of eigenvalues of a small (explicit) random matrix [see (2.15) below]. The mechanism underlying the repulsion among outliers is therefore the same as that for the eigenvalues of GUE: the Jacobian relating the eigenvalue–eigenvector entries to the matrix entries has a Vandermonde determinant structure, and vanishes if two eigenvalues coincide. Several special cases of overlapping outliers have already been studied in the works [8–10, 24, 25], which in particular exhibited the level repulsion mechanism described above.

Due to this level repulsion, overlapping outliers are obviously not asymptotically independent. A novel observation, which follows from our main result, is that in general nonoverlapping outliers are not asymptotically independent either; in this case the lack of independence does not arise from level repulsion, but from a more subtle interplay between the distribution of H and the geometry of the eigenvectors of A. In some special cases, such as GOE/GUE, nonoverlapping outliers are, however, asymptotically independent. More precisely, our main result (Theorem 2.11 below) shows that two outliers may, under suitable conditions on H and A, be strongly correlated in the limit $N \to \infty$, even if they are far from each other (e.g., on opposite sides of the bulk spectrum).

Finally, we note that throughout this paper we assume that the entries of H have subexponential decay. We need this assumption because our proof relies heavily on the local semicircle law and eigenvalue rigidity estimates for H, proved in [18] under the assumption of subexponential decay. However, this assumption is not fundamental to our approach, which may be combined with the recent methods for dealing with heavy-tailed Wigner matrices developed in [11, 12, 22]. Moreover, the assumption that the norm of A be bounded may be easily removed; in fact, large eigenvalues of A are easier to treat than small ones.

We remark that recently Pizzo, Renfrew and Soshnikov [24, 25] took a different approach, and derived the asymptotic distribution of a single group of overlapping outliers under optimal tail assumptions on H. On the other hand, in [24, 25] it is assumed that the eigenvalues of A are independent of N and that its eigenvectors satisfy a condition which roughly constrains them to be either strongly localized or delocalized.

1.1. *Outline of the proof.* As in [21], our proof relies on the *isotropic local semicircle law*, proved in [21], Theorems 2.2 and 2.3. The isotropic local semicircle law is an extension of the *local semicircle law*, whose study was initiated in [14, 15]. The local semicircle law has since become a cornerstone of random matrix theory, in particular in establishing the universality of Wigner matrices [13, 16–18, 26, 27]. The strongest versions of the local semicircle law, proved in [12, 18], give precise estimates on the local eigenvalue density, down to scales containing N^{ε} eigenvalues. In fact, as formulated in [18], the local semicircle law gives optimal high-probability estimates on the quantity

$$G_{ij}(z) - \delta_{ij} m(z), \tag{1.1}$$

where $m(z)$ denotes the Stieltjes transform of Wigner's semicircle law and $G(z) := (H - z)^{-1}$ is the resolvent of H.

The isotropic local semicircle law is a generalization of the local semicircle law, in that it gives optimal high-probability estimates on the quantity

$$\langle \mathbf{v}, (G(z) - m(z)\mathbb{1})\mathbf{w}\rangle, \tag{1.2}$$

where $\mathbf{v}$ and $\mathbf{w}$ are arbitrary deterministic vectors. Clearly, (1.1) is a special case obtained from (1.2) by setting $\mathbf{v} = \mathbf{e}_i$ and $\mathbf{w} = \mathbf{e}_j$, where $\mathbf{e}_i$ denotes ith standard basis vector of $\mathbb{C}^N$.

As in the works [21, 24, 25], a major part of our proof consists in deriving the asymptotic distribution of the entries of $G(z)$. The main technical achievement of this paper is to obtain the *joint* asymptotics of an arbitrary finite family of variables of the form $\langle \mathbf{v}, G(z)\mathbf{w}\rangle$, whereby the spectral parameters z of different entries may differ, and are assumed to satisfy $2 + (\log N)^{C \log\log N} N^{-2/3} \leq |\operatorname{Re} z| \leq C$ for some positive constant C. The question of the joint asymptotics of the resolvent entries occurs more generally in several problems on deformed random matrix models, and we therefore believe that the techniques of this paper are also of interest for other problems on deformed matrix ensembles.

An important ingredient in our proof is the four-step strategy introduced in [21]. It may be summarized as follows: (i) reduction to the distribution of the resolvent G, (ii) the case of Gaussian H, (iii) the case of almost Gaussian H, (iv) the case of general H. Steps (i)–(iii) in the current paper are substantially different from their counterparts in [21]; this results from treating an entire overlapping group of outliers simultaneously, as well as from the need to develop an argument that admits an analysis of the joint law of different groups. In fact, for pedagogical reasons, first—in Sections 4–7—we give the proof for the case of a single group of overlapping outliers,[3] and then—in Section 9.1—extend it to yield the full joint distribution. In contrast to the steps (i)–(iii), step (iv) survives almost unchanged from [21], and in Section 7 we give an explanation of the required modifications.

Another ingredient of our proof is a two-level partitioning of the outliers combined with near-degenerate perturbation theory for eigenvalues. Roughly, outliers are partitioned into blocks depending on whether they overlap. In the finer partition, denoted by Π below (see Definition 2.10), we regroup two outliers into the same block if their mean separation is bounded by some large constant (denoted by s below) times the magnitude of their fluctuations. Due to logarithmic error factors of the form $(\log N)^{C \log\log N}$ that appear naturally in high-probability estimates pervading our proof, we shall require a second, coarser, partition, denoted by Γ below (see Definition 9.1). In Γ, we regroup two outliers into the same block if their mean separation is bounded by $(\log N)^{C \log\log N}$ times the magnitude of their fluctuations. The link between Γ and Π is provided by perturbation theory, and is performed in Sections 8 (for a single group) and 9 (for the full joint distribution).

2. Formulation of results.

2.1. *The setup.* Let $H = (h_{ij})_{i,j=1}^N$ be an $N \times N$ random matrix. We assume that the upper-triangular entries $(h_{ij} : i \le j)$ are independent complex-valued random variables. The remaining entries of H are given by imposing $H = H^*$. Here H^* denotes the Hermitian conjugate of H. We assume that all entries are centred, $\mathbb{E}h_{ij} = 0$. In addition, we assume that one of the two following conditions holds.

(i) *Real symmetric Wigner matrix*: $h_{ij} \in \mathbb{R}$ for all i, j and

$$\mathbb{E}h_{ii}^2 = \frac{2}{N}, \qquad \mathbb{E}h_{ij}^2 = \frac{1}{N} \qquad (i \ne j).$$

(ii) *Complex Hermitian Wigner matrix*:

$$\mathbb{E}h_{ii}^2 = \frac{1}{N}, \qquad \mathbb{E}|h_{ij}|^2 = \frac{1}{N}, \qquad \mathbb{E}h_{ij}^2 = 0 \qquad (i \ne j).$$

We introduce the usual index β of random matrix theory, defined to be 1 in the real symmetric case and 2 in the complex Hermitian case. We use the abbreviation GOE/GUE to mean GOE if H is a real symmetric Wigner matrix with Gaussian entries and GUE if H is a complex Hermitian Wigner matrix with Gaussian entries. We assume that the entries of H have uniformly subexponential decay, that is, that there exists a constant $\vartheta > 0$ such that

$$\mathbb{P}\big(\sqrt{N}|h_{ij}| \ge x\big) \le \vartheta^{-1} \exp\big(-x^{\vartheta}\big) \tag{2.1}$$

[3]In the resolvent language, this means that the spectral parameters z of all the resolvent entries coincide.

for all i, j and N. Note that we do not assume the entries of H to be identically distributed, and we do not require any smoothness in the distribution of the entries of H.

We consider a deformation of fixed, finite rank $r \in \mathbb{N}$. Let $V \equiv V_N$ be a deterministic $N \times r$ matrix satisfying $V^* V = \mathbb{1}_r$, and $D \equiv D_N$ be a deterministic $r \times r$ diagonal matrix whose eigenvalues are nonzero. Both V and D depend on N. We sometimes also use the notation $V = [\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(r)}]$, where $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(r)} \in \mathbb{C}^N$ are orthonormal, as well as $D = \operatorname{diag}(d_1, \dots, d_r)$. We always assume that the eigenvalues of D satisfy

$$-\Sigma + 1 \le d_1 \le d_2 \le \cdots \le d_r \le \Sigma - 1, \tag{2.2}$$

where Σ is some fixed positive constant. We are interested in the spectrum of the deformed matrix

$$\widetilde{H} := H + VDV^* = H + \sum_{i=1}^{r} d_i \mathbf{v}^{(i)} (\mathbf{v}^{(i)})^*.$$

The following definition summarizes our conventions for the spectrum of a matrix. For our purposes, it is important to allow the matrix entries and its eigenvalues to be indexed by an arbitrary subset of positive integers.

DEFINITION 2.1. Let π be a finite set of positive integers, and let $A = (A_{ij})_{i,j \in \pi}$ be a $|\pi| \times |\pi|$ Hermitian matrix whose entries are indexed by elements of π. We denote by

$$\sigma(A) := (\lambda_i(A))_{i \in \pi} \in \mathbb{R}^{\pi}$$

the family of eigenvalues of A. We always order the eigenvalues so that $\lambda_i(A) \le \lambda_j(A)$ if $i \le j$.

By a slight abuse of notation, we sometimes identify $\sigma(A)$ with the set $\{\lambda_i(A)\}_{i \in \pi} \subset \mathbb{R}$. Thus, for instance, $\operatorname{dist}(\sigma(A), \sigma(B)) := \min_{i,j} |\lambda_i(A) - \lambda_j(B)|$ denotes the distance between $\sigma(A)$ and $\sigma(B)$ viewed as subsets of $\mathbb{R}$.

We abbreviate the (random) eigenvalues of H and $\widetilde{H}$ by

$$\lambda_\alpha := \lambda_\alpha(H), \qquad \mu_\alpha := \lambda_\alpha(\widetilde{H}).$$

The following definition introduces a convenient notation for minors of matrices.

DEFINITION 2.2 (Minors). For an $r \times r$ matrix $A = (A_{ij})_{i,j=1}^{r}$ and a subset $\pi \subset \{1, \dots, r\}$ of integers, we define the $|\pi| \times |\pi|$ matrix

$$A_{[\pi]} = (A_{ij})_{i,j \in \pi}.$$

We shall frequently make use of the logarithmic control parameter

$$\varphi \equiv \varphi_N := (\log N)^{\log \log N}. \tag{2.3}$$

The interpretation of φ is that of a slowly growing parameter [note that $\varphi \le N^{\varepsilon}$ for any $\varepsilon > 0$ and large enough $N \ge N_0(\varepsilon)$]. Throughout this paper, every quantity that is not explicitly a constant may depend on N, with the sole exception of the rank r of the deformation, which is required to be fixed. Unless needed, we consistently drop the argument N from such quantities.

We denote by C a generic positive large constant, whose value may change from one expression to the next. For two positive quantities A_N and B_N, we use the notation $A_N \asymp B_N$ to mean $C^{-1}A_N \le B_N \le CA_N$ for some positive constant C. Moreover, we write $A_N \ll B_N$ if $A_N/B_N \to 0$ and $A_N \gg B_N$ if $B_N \ll A_N$. Finally, for $a < b$ we set $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.

2.2. *Heuristics of outliers*. Before stating our results, we give a heuristic description of the behavior of the outliers. An eigenvalue d_i of D satisfying

$$|d_i| - 1 \gg N^{-1/3} \tag{2.4}$$

gives rise to an outlier $\mu_{\alpha(i)}$ located around its classical location $\theta(d_i)$, where we defined, for $d \in \mathbb{R} \setminus (-1, 1)$,

$$\theta(d) := d + \frac{1}{d} \tag{2.5}$$

and

$$\alpha(i) := \begin{cases} i, & \text{if } d_i < 0, \\ N - r + i, & \text{if } d_i > 0. \end{cases} \tag{2.6}$$

Condition (2.4) may be heuristically understood as follows; for simplicity set $r = 1$ and $D = d > 1$. The extremal eigenvalues of $\widetilde{H}$ that are not outliers fluctuate on the scale $N^{-2/3}$ (see [21], Theorem 2.7), the same scale as the extremal eigenvalues of the undeformed matrix H. For the largest eigenvalue μ_N of $\widetilde{H}$ to be an outlier, we require that its separation from the asymptotic bulk spectrum $[-2, 2]$, which is of the order $\theta(d) - 2$, be much greater than $N^{-2/3}$. This leads to condition (2.4) by a simple expansion of θ around 1.

The outlier $\mu_{\alpha(i)}$ associated with d_i fluctuates on the scale $N^{-1/2}(|d_i| - 1)^{1/2}$. Thus, $\mu_{\alpha(i)}$ fluctuates on the scale $N^{-1/2}$ if d_i is well-separated from the critical point 1, and on the scale $N^{-2/3}$ if d_i is critical, that is, $d_i = 1 + aN^{-1/3}$ for some fixed $a > 0$. The outliers associated with d_i and d_j *overlap* if their separation is comparable to or less than the scale on which they fluctuate. The overlapping condition thus reads

$$\left|\theta(d_i) - \theta(d_j)\right| \le CN^{-1/2}\left(|d_i| - 1\right)^{1/2} \tag{2.7}$$

for some (typically large) constant $C > 0$. Note that the factor $|d_i| - 1$ on the right-hand side could be replaced with $|d_j| - 1$. Indeed, recalling (2.4), it is not hard to

FIG. 1. *A general outlier configuration. We draw the outlier* $\mu_{\alpha(i)}$ *associated with* d_i *using a black line marking its mean location* $\theta(d_i)$ *and a grey curve indicating its probability density. The breadth of the curve associated with* d_i *is of the order* $N^{-1/2}(|d_i|-1)^{1/2}$. *Outliers whose probability densities overlap satisfy* (2.7) [*or, equivalently*, (2.8)]. *We do not draw the bulk eigenvalues, which are contained in the grey bar.*

check that (2.7) for some $C > 0$ is equivalent to (2.7) with d_i on the right-hand side replaced with d_j and the constant C replaced with a constant $C' \asymp C$. Using (2.5) and recalling (2.4), we may rewrite the overlapping condition (2.7) as

$$N^{1/2}\big(|d_i|-1\big)^{1/2}|d_i - d_j| \le C \tag{2.8}$$

for some $C > 0$. As in (2.7), $|d_i| - 1$ may be replaced with $|d_j| - 1$. Figure 1 summarizes the general picture of outliers.

2.3. *The distribution of a single group.* After these preparations, we state our results. We begin by defining a reference matrix which will describe the distribution of a group of overlapping outliers. Define the moment matrices $\mu^{(3)} = (\mu_{ij}^{(3)})$ and $\mu^{(4)} = (\mu_{ij}^{(4)})$ of H through

$$\mu_{ij}^{(3)} := N^{3/2}\mathbb{E}\big(|h_{ij}|^2 h_{ij}\big), \qquad \mu_{ij}^{(4)} := N^2\mathbb{E}|h_{ij}|^4.$$

Using the matrices $\mu^{(3)}$ and $\mu^{(4)}$, we define the deterministic functions

$$\mathcal{P}_{ij,kl}(R) := R_{il}R_{kj} + \mathbf{1}(\beta = 1)R_{ik}R_{jl},$$

$$\mathcal{Q}_{ij,kl}(V) := \frac{1}{\sqrt{N}}\sum_{a,b}\big(\overline{V}_{ai}\overline{V}_{ak}V_{al}\mu_{ab}^{(3)}V_{bj} + \overline{V}_{ia}\mu_{ab}^{(3)}V_{bj}\overline{V}_{bk}V_{bl}$$

$$+ \overline{V}_{ak}\overline{V}_{ai}V_{aj}\mu_{ab}^{(3)}V_{bl} + \overline{V}_{ka}\mu_{ab}^{(3)}V_{bl}\overline{V}_{bi}V_{bj}\big),$$

$$\mathcal{R}_{ij,kl}(V) := \frac{1}{N}\sum_{a,b}\big(\mu_{ab}^{(4)} - 4 + \beta\big)\overline{V}_{bi}V_{bj}\overline{V}_{bk}V_{bl},$$

where $i, j, k, l \in [\![1, r]\!]$, R is an $r \times r$ matrix, and V an $N \times r$ matrix. Moreover, we define the deterministic $r \times r$ matrix

$$\mathcal{S}(V) := \frac{1}{N}V^*\mu^{(3)}V.$$

REMARK 2.3. Using Cauchy–Schwarz and assumption (2.1), it is easy to check that $\mathcal{P}(V^*V)$, $\mathcal{Q}(V)$, $\mathcal{R}(V)$ and $\mathcal{S}(V)$ are uniformly bounded for V satisfying $0 \le V^*V \le \mathbb{1}$ (in the sense of quadratic forms).

Next, let $\delta \equiv \delta_N$ be a positive sequence satisfying $\varphi^{-1} \leq \delta \ll 1$. (Our result will be independent of δ provided it satisfies this condition; see Remark 2.4 below.) The sequence δ will serve as a cutoff in the size of the entries of V when computing the law of $V^* H V$: entries of V smaller than δ give rise to an asymptotically Gaussian random variable by the central limit theorem; the remaining entries are treated separately, and the associated random variable is in general not Gaussian. Thus, we define the matrix $V_\delta = (V_{ij}^\delta)$ through

$$V_{ij}^\delta := V_{ij} \mathbf{1}(|V_{ij}| > \delta).$$

For $\ell \in [\![1, r]\!]$ satisfying $|d_\ell| > 1$ we define the $r \times r$ matrix

$$\Upsilon^\ell := (|d_\ell| + 1)(|d_\ell| - 1)^{1/2} \biggl(\frac{N^{1/2} V_\delta^* H V_\delta}{d_\ell^2} + \frac{\mathcal{S}(V)}{d_\ell^4} \biggr). \tag{2.9}$$

Abbreviate

$$\Delta_{ij,kl} := \mathcal{P}_{ij,kl}(\mathbb{1}) = \delta_{il}\delta_{kj} + \mathbf{1}(\beta = 1)\delta_{ik}\delta_{jl}. \tag{2.10}$$

Note that Δ is nothing but the covariance matrix of a GOE/GUE matrix: if $r^{-1/2}\Phi$ is an $r \times r$ GOE/GUE matrix then $\mathbb{E}\Phi_{ij}\Phi_{kl} = \Delta_{ij,kl}$. We introduce an $r \times r$ Gaussian matrix Ψ^ℓ, independent of H, which is complex Hermitian for $\beta = 2$ and real symmetric for $\beta = 1$. The entries of Ψ^ℓ are centred, and their law is determined by the covariance

$$\begin{aligned} \mathbb{E}\Psi_{ij}^\ell \Psi_{kl}^\ell = {} & \frac{|d_\ell| + 1}{d_\ell^2} \Delta_{ij,kl} + (|d_\ell| + 1)^2 (|d_\ell| - 1) \\ & \times \biggl(-\frac{\mathcal{P}_{ij,kl}(V_\delta^* V_\delta)}{d_\ell^4} + \frac{\mathcal{Q}_{ij,kl}(V)}{d_\ell^5} + \frac{\mathcal{R}_{ij,kl}(V)}{d_\ell^6} \biggr) + E_{ij,kl}. \end{aligned} \tag{2.11}$$

Here $E_{ij,kl} := \varphi^{-1}\Delta_{ij,kl}$ is a term, that is, needed to ensure that the right-hand side of (2.11) is a nonnegative $r^2 \times r^2$ matrix. This nonnegativity follows as a by-product of our proof, in which the right-hand side of (2.11) is obtained from the covariance of an explicit random matrix; see Proposition 6.1 below for more details. Note that the term $E_{ij,kl}$ does not influence the asymptotic distribution of Ψ^ℓ.

REMARK 2.4. A different choice of δ, subject to $\varphi^{-1} \leq \delta \ll 1$, leads to the same asymptotic distribution for $\Upsilon^\ell + \Psi^\ell$. This is an easy consequence of the central limit theorem and the observation that the matrix entries

$$\biggl((|d_\ell| + 1)(|d_\ell| - 1)^{1/2} \frac{N^{1/2} V_\delta^* H V_\delta}{d_\ell^2} \biggr)_{ij}$$

have covariance matrix $(|d_\ell| + 1)^2 (|d_\ell| - 1) d_\ell^{-4} \mathcal{P}_{ij,kl}(V_\delta^* V_\delta)$.

Before stating our result in full generality, we give a special case which captures its essence and whose statement is somewhat simpler.

THEOREM 2.5. *For large enough K the following holds. Let $\pi \subset [\![1, r]\!]$ be a subset of consecutive integers, and fix $\ell \in \pi$. Suppose that $|d_\ell| \geq 1 + \varphi^K N^{-1/3}$. Suppose moreover that there is a constant C such that*

$$N^{1/2}\big(|d_\ell| - 1\big)^{1/2}|d_i - d_\ell| \leq C \tag{2.12}$$

for all $i \in \pi$ and, as $N \to \infty$,

$$N^{1/2}\big(|d_\ell| - 1\big)^{1/2}|d_i - d_\ell| \to \infty \tag{2.13}$$

for all $i \in [\![1, r]\!] \setminus \pi$.

Define the rescaled eigenvalues $\boldsymbol{\zeta} = (\zeta_i)_{i \in \pi}$ through

$$\zeta_i := N^{1/2}\big(|d_\ell| - 1\big)^{-1/2}\big(\mu_{\alpha(i)} - \theta(d_\ell)\big), \tag{2.14}$$

where we recall the definition (2.6) of $\alpha(i)$. Let $\boldsymbol{\xi} = (\xi_i)_{i \in \pi}$ denote the eigenvalues of the random $|\pi| \times |\pi|$ matrix

$$\Upsilon^\ell_{[\pi]} + \Psi^\ell_{[\pi]} + N^{1/2}\big(|d_\ell| - 1\big)^{1/2}\big(|d_\ell| + 1\big)\big(d_\ell^{-1} - D_{[\pi]}^{-1}\big). \tag{2.15}$$

Then for any bounded and continuous function f we have

$$\lim_N\big(\mathbb{E} f(\boldsymbol{\zeta}) - \mathbb{E} f(\boldsymbol{\xi})\big) = 0.$$

The subset π indexes outliers that belong to the same group of overlapping outliers, as required by (2.12) [see also (2.8) in the preceding discussion]. As required by (2.13), the remaining outliers do not overlap with the outliers indexed by π.

REMARK 2.6. The reference point ℓ for the block π is arbitrary and unimportant. See Lemma 4.6 below and the comment preceding it for a more detailed discussion.

REMARK 2.7. For the special case $\pi = \{\ell\}$, Theorem 2.5 essentially[4] reduces to Theorem 2.14 of [21]. In addition, Theorem 2.5 corrects a minor issue in the statement of Theorem 2.14 of [21], where the variance of Υ was not necessarily positive. Indeed, in the language of the current paper, in [21] the term $V_\delta^* H V_\delta$ in (2.9) was of the form $V^* H V$, which amounted to transferring a large Gaussian component from Ψ to Υ. This transfer was ill-advised as it sometimes resulted in a negative variance for Ψ (which would however be compensated in the sum $\Upsilon + \Psi$ by a large asymptotically Gaussian component in Υ).

[4] In fact, condition of [21] analogous to (2.13), equation (2.24) in [21], is slightly stronger than (2.13).

The functions $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ in (2.9) and (2.11) are in general nonzero in the limit $N \to \infty$. They encode the *nonuniversality* of the distribution of the outliers. Thus, the distribution of the outliers may depend on the law of the entries of H as well as on the geometry of the eigenvectors V.

In the GOE/GUE case, it is easy to check that $\Upsilon^\ell + \Psi^\ell$ is asymptotically Gaussian with covariance matrix

$$\frac{|d_\ell| + 1}{d_\ell^2} \Delta_{ij,kl}. \tag{2.16}$$

Moreover, if $\lim_N |d_\ell| = 1$ then the matrix $\Upsilon^\ell + \Psi^\ell$ converges weakly to a Gaussian matrix with covariance given by (2.16). In this case, therefore, the nonuniversality is washed out. Thus, only outliers separated from the bulk spectrum $[-2, 2]$ by a distance of order one may exhibit nonuniversality.

If $\lim_N \max_{i,j} |V_{ij}| = 0$, then an appropriate choice of δ yields $\Upsilon^\ell = (|d_\ell| + 1)(|d_\ell| - 1)^{1/2} d_\ell^{-4} \mathcal{S}(V)$ as well as a matrix Ψ^ℓ whose covariance is asymptotically that of the GOE/GUE case, that is, (2.16). Hence, in this case the only manifestation of nonuniversality is the deterministic shift given by Υ^ℓ.

It is possible to find scenarios in which each term of (2.9) and (2.11) [apart from the trivial error term E in (2.11)] contributes in the limit $N \to \infty$. This is, for instance, the case if $\mu_{ij}^{(3)}$ and $\mu_{ij}^{(4)}$ do not depend on i and j, $\mu_{ij}^{(4)}$ is not asymptotically $4 - \beta$, and an eigenvector $\mathbf{v}^{(i)}$ satisfies $\|\mathbf{v}^{(i)}\|_\infty \geq c$ as well as $\|\mathbf{v}^{(i)}\|_1 \geq cN^{1/2}$ for some constant $c > 0$. We refer to [21], Remarks 2.17–2.21, for analogous remarks, where more details are given for the case $\pi = \{\ell\}$.

Next, we give the asymptotic distribution of a group of overlapping outliers in full generality. Thus, Theorem 2.9 below holds for arbitrary sequences $V \equiv V_N$ and $D \equiv D_N$ satisfying $V^* V = \mathbb{1}$ and (2.2).

DEFINITION 2.8. Let N and D be given. For $s > 0$ and $\ell \in [\![1, r]\!]$ satisfying $|d_\ell| > 1$, define $\pi(\ell, s) \equiv \pi_{N,D}(\ell, s)$ as the smallest subset of $[\![1, r]\!]$ with the two following properties.

(i) $\ell \in \pi(\ell, s)$.
(ii) If for $i, j \in [\![1, r]\!]$ we have $|d_i| > 1$ and

$$N^{1/2} \big(|d_i| - 1\big)^{1/2} |d_i - d_j| \leq s, \tag{2.17}$$

then either $i, j \in \pi(\ell, s)$ or $i, j \in [\![1, r]\!] \setminus \pi(\ell, s)$.

The subset $\pi(\ell, s)$ indexes those outliers that belong to the same group of overlapping outliers as ℓ, where s is a cutoff distance used to determine whether two outliers are considered overlapping. Note that $\pi(\ell, s)$ is a set of consecutive integers.

THEOREM 2.9. *For large enough K the following holds. Let $\varepsilon > 0$ be arbitrary, and let $f_1, \dots, f_r$ be bounded continuous functions, where f_k is a function on $\mathbb{R}^k$. Then there exist $N_0 \in \mathbb{N}$ and $s_0 > 0$ such that for all $N \geq N_0$ and $s \geq s_0$ the following holds.*

Suppose that $\ell \in [\![1, r]\!]$ satisfies

$$|d_\ell| \geq 1 + \varphi^K N^{-1/3} \tag{2.18}$$

and set $\pi := \pi(\ell, s)$. Then

$$\left|\mathbb{E} f_{|\pi|}(\boldsymbol{\zeta}) - \mathbb{E} f_{|\pi|}(\boldsymbol{\xi})\right| \leq \varepsilon, \tag{2.19}$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ were defined Theorem 2.5.

2.4. *The joint distribution.* In order to describe the joint distribution of all outliers, we organize them into groups of overlapping outliers, using a partition Π whose blocks π are defined using the subsets $\pi(\ell, s)$ from Definition 2.8.

DEFINITION 2.10. Let N and D be given, and fix $K > 0$ and $s > 0$. We introduce a partition[5] $\Pi \equiv \Pi(N, K, s, D)$ on a subset of $[\![1, r]\!]$, defined as

$$\Pi := \left\{\pi(\ell, s) : \ell \in [\![1, r]\!], |d_\ell| \geq 1 + \varphi^K N^{-1/3}\right\}.$$

We also use the notation $\Pi = \{\pi\}_{\pi \in \Pi}$ and $[\Pi] := \bigcup_{\pi \in \Pi} \pi$.

The indices in $[\Pi]$ give rise to outliers, which are grouped into the blocks of Π. Indices in $[\![1, r]\!] \setminus [\Pi]$ do not give rise to outliers.

For $\pi \in \Pi$, we define

$$d_\pi := \min\{d_i : i \in \pi\}. \tag{2.20}$$

We chose this value for definiteness, although any other choice of d_i with $i \in \pi$ would do equally well.

Next, in analogy to (2.15), we define a $|[\Pi]| \times |[\Pi]|$ reference matrix whose eigenvalues will have the same asymptotic distribution as the appropriately rescaled outliers $(\mu_{\alpha(i)})_{i \in [\Pi]}$. Define the block diagonal $|[\Pi]| \times |[\Pi]|$ matrix $\Upsilon = \bigoplus_{\pi \in \Pi} \Upsilon^\pi$, where

$$\Upsilon^\pi := \left(|d_\pi| + 1\right)\left(|d_\pi| - 1\right)^{1/2}\left(\frac{N^{1/2} V_\delta^* H V_\delta}{d_\pi^2} + \frac{\mathcal{S}(V)}{d_\pi^4}\right)_{[\pi]}.$$

In addition, we introduce a Hermitian, Gaussian $|[\Pi]| \times |[\Pi]|$ matrix Ψ, that is, independent of H and whose entries have mean zero. It is block diagonal,

[5]That Π is a partition follows from the observation that $\ell' \in \pi(\ell, s)$ if and only if $\ell \in \pi(\ell', s)$. Therefore if ℓ and ℓ' satisfy $|d_\ell| \geq 1 + \varphi^K N^{-2/3}$ and $|d_{\ell'}| \geq 1 + \varphi^K N^{-2/3}$ then either $\pi(\ell, s) = \pi(\ell', s)$ or $\pi(\ell, s) \cap \pi(\ell', s) = \varnothing$.

$\Psi = \bigoplus_{\pi\in\Pi} \Psi^\pi$, where the block $\Psi^\pi = (\Psi^\pi_{ij})_{i,j\in\pi}$ is a $|\pi| \times |\pi|$ matrix. The law of Ψ is determined by the covariance

$$
\begin{aligned}
\mathbb{E}\Psi^\pi_{ij}\Psi^{\pi'}_{kl} &= \frac{|d_\pi|+1}{d_\pi^2}\delta_{\pi\pi'}\Delta_{ij,kl} + \delta_{\pi\pi'}E_{ij,kl} \\
&\quad + \biggl(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^2}\biggr) \\
&\quad \times \biggl(-\mathcal{P}_{ij,kl}\bigl(V_\delta^* V_\delta\bigr) + \frac{1}{d_\pi d_{\pi'}}\mathcal{R}_{ij,kl}(V) \\
&\qquad + \frac{\mathcal{W}_{ij,kl}(V)}{d_{\pi'}} + \frac{\mathcal{W}_{kl,ij}(V)}{d_\pi}\biggr),
\end{aligned}
\tag{2.21}
$$

where we defined

$$
\mathcal{W}_{ij,kl}(V) := \frac{1}{\sqrt{N}}\sum_{a,b}\bigl(\overline{V}_{ai}\overline{V}_{ak}V_{al}\mu^{(3)}_{ab}V_{bj} + \overline{V}_{ia}\mu^{(3)}_{ab}V_{bj}\overline{V}_{bk}V_{bl}\bigr).
$$

(Note that $\mathcal{Q}_{ij,kl} = \mathcal{W}_{ij,kl} + \mathcal{W}_{kl,ij}$.) As in (2.11), the factor $E_{ij,kl} = \varphi^{-1}\Delta_{ij,kl}$, whose contribution vanishes in the limit $N \to \infty$, simply ensures that the right-hand side of (2.21) defines a nonnegative matrix; this nonnegativity is an immediate corollary of our proof in Section 9.1.

Next, in analogy to (2.14), we introduce the rescaled family of outliers $\boldsymbol{\zeta} = (\zeta_i^\pi : \pi \in \Pi, i \in \pi) \in \mathbb{R}^{[\Pi]}$ whose entries are defined by

$$
\zeta_i^\pi := N^{1/2}\bigl(|d_\pi|-1\bigr)^{-1/2}\bigl(\mu_{\alpha(i)} - \theta(d_\pi)\bigr),
\tag{2.22}
$$

where we recall the definition (2.6) of $\alpha(i)$. Moreover, for $\pi \in \Pi$ let $\boldsymbol{\xi}^\pi = (\xi_i^\pi : i \in \pi)$ denote the eigenvalues of the random $|\pi| \times |\pi|$ matrix

$$
\Upsilon^\pi + \Psi^\pi + N^{1/2}\bigl(|d_\pi|-1\bigr)^{1/2}(|d_\pi|+1)\bigl(d_\pi^{-1} - D_{[\pi]}^{-1}\bigr)
$$

and write $\boldsymbol{\xi} = (\boldsymbol{\xi}^\pi : \pi \in \Pi) = (\xi_i^\pi : \pi \in \Pi, i \in \pi) \in \mathbb{R}^{[\Pi]}$. We may now state our main result in its greatest generality.

THEOREM 2.11. *For large enough K the following holds. Let $\varepsilon > 0$ be arbitrary, and let $f_1, \ldots, f_r$ be bounded continuous functions, where f_k is a function on $\mathbb{R}^k$. Then there exist $N_0 \in \mathbb{N}$ and $s_0 > 0$ such that for all $N \geq N_0$ and $s \geq s_0$ we have*

$$
\bigl|\mathbb{E}f_{|[\Pi]|}(\boldsymbol{\zeta}) - \mathbb{E}f_{|[\Pi]|}(\boldsymbol{\xi})\bigr| \leq \varepsilon.
$$

We conclude this section by drawing some consequences from Theorem 2.11. In the GOE/GUE case, it is easy to see that the law of the block matrix $\Upsilon + \Psi$ is asymptotically Gaussian with covariance

$$
\frac{|d_\pi|+1}{d_\pi^2}\delta_{\pi\pi'}\Delta_{ij,kl}.
$$

In particular, we find that overlapping outliers repel each other according to the usual random matrix level repulsion, while nonoverlapping outliers are asymptotically independent.

In general outliers are not asymptotically independent, even if they do not overlap. Such correlations arise from correlations between different blocks of $\Upsilon + \Psi$. There are two possible sources for these correlations: the term $V_\delta^* H V_\delta$ in the definition of Υ, and the terms $\mathcal{R}$ and $\mathcal{W}$ in the covariance (2.21) of the Gaussian matrix Ψ. Thus, two outliers may be strongly correlated even if they are located on opposite sides of the bulk spectrum.

3. Tools. The rest of this paper is devoted to the proofs of Theorems 2.5, 2.9 and 2.11. Sections 3–8 are devoted to the proof of Theorem 2.9; Theorem 2.5 is an easy corollary of Theorem 2.9. Finally, Theorem 2.11 is proved in Section 9 by an extension of the arguments of Sections 3–8.

We begin with a preliminary section that collects tools we shall use in the proof. We introduce the spectral parameter

$$z = E + \mathrm{i}\eta,$$

which will be used as the argument of Stieltjes transforms and resolvents. In the following, we often use the notation $E = \operatorname{Re} z$ and $\eta = \operatorname{Im} z$ without further comment. Let

$$\varrho(x) := \frac{1}{2\pi}\sqrt{[4 - x^2]_+} \qquad (x \in \mathbb{R})$$

denote the density of the local semicircle law, and

$$m(z) := \int \frac{\varrho(x)}{x - z}\,\mathrm{d}x \qquad \bigl(z \notin [-2, 2]\bigr) \tag{3.1}$$

its Stieltjes transform. It is well known that the Stieltjes transform m satisfies the identity

$$m(z) + \frac{1}{m(z)} + z = 0. \tag{3.2}$$

It is easy to see that (3.2) and the definition (2.5) imply

$$m\bigl(\theta(d)\bigr) = -\frac{1}{d}. \tag{3.3}$$

For $E \in \mathbb{R}$, define

$$\kappa_E := \bigl||E| - 2\bigr|, \tag{3.4}$$

the distance from E to the spectral edges ± 2. We have the simple estimate

$$\kappa_{\theta(d)} \asymp \bigl(|d| - 1\bigr)^2 \tag{3.5}$$

for $|d| > 1$. The following lemma collects some useful properties of m.

LEMMA 3.1. *For* $|z| \leq 2\Sigma$, *we have*

$$|m(z)| \asymp 1, \qquad |1 - m(z)^2| \asymp \sqrt{\kappa + \eta}. \tag{3.6}$$

Moreover,

$$\operatorname{Im} m(z) \asymp \begin{cases} \sqrt{\kappa + \eta}, & \text{if } |E| \leq 2, \\ \dfrac{\eta}{\sqrt{\kappa + \eta}}, & \text{if } |E| \geq 2. \end{cases}$$

(*Here the implicit constants depend on* Σ.)

PROOF. The proof is an elementary calculation; see Lemma 4.2 in [17]. □

The following definition introduces a notion of high probability that is suitable for our needs.

DEFINITION 3.2 (High probability events). We say that an N-dependent event Ξ holds with *high probability* if there is some constant C such that

$$\mathbb{P}(\Xi^c) \leq N^C \exp(-\varphi) \tag{3.7}$$

for large enough N.

Next, we give the key tool behind the proof of Theorem 2.9: the *Isotropic local semicircle law*. We use the notation $\mathbf{v} = (v_i)_{i=1}^N \in \mathbb{C}^N$ for the components of a vector. We introduce the standard scalar product $\langle \mathbf{v}, \mathbf{w} \rangle := \sum_i \bar{v}_i w_i$. For $\eta > 0$, we define the resolvent of H through

$$G(z) := (H - z)^{-1}.$$

The following result was proved in [21], Theorem 2.3.

THEOREM 3.3 (Isotropic local semicircle law outside of the spectrum). *Fix* $\Sigma \geq 3$. *There exists a constant* C *such that for large enough* K *and any deterministic* $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$ *we have with high probability*

$$|\langle \mathbf{v}, G(z)\mathbf{w} \rangle - m(z)\langle \mathbf{v}, \mathbf{w} \rangle| \leq \varphi^C \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \|\mathbf{v}\| \|\mathbf{w}\| \tag{3.8}$$

for all

$$E \in [-\Sigma, -2 - \varphi^K N^{-2/3}] \cup [2 + \varphi^K N^{-2/3}, \Sigma] \quad \text{and} \quad \eta \in (0, \Sigma].$$

Using (3.5) and Lemma 3.1, we find that the control parameter in (3.8) may be written as

$$\sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \asymp N^{-1/2}(\kappa_E + \eta)^{-1/4} \leq N^{-1/2}\kappa_E^{-1/4}. \tag{3.9}$$

The following result provides sharp (up to logarithmic factors) large deviations bounds on the locations of the outliers.

THEOREM 3.4 (Locations of the deformed eigenvalues). *There exists a constant C such that, for large enough K and under condition* (2.2), *we have*

$$\big|\mu_{\alpha(i)} - \theta(d_i)\big| \le \varphi^C N^{-1/2}\big(|d_i| - 1\big)^{1/2} \tag{3.10}$$

with high probability provided that $|d_i| \ge 1 + \varphi^K N^{-1/3}$.

PROOF. This was essentially proved in [21], Theorem 2.7, by setting $\psi = 1$ there; see equation (2.20) of [21]. Note that Theorem 2.7 of [21] has slightly stronger assumptions than Theorem 3.4, requiring in addition that there be no eigenvalues d_j of D satisfying $||d_j| - 1| < \varphi^K N^{-1/3}$. However, this assumption was only needed for equation (2.21) of [21], and the proof from Section 6 of [21] may be applied verbatim to (3.10) under the assumptions of Theorem 3.4. □

We shall often need to consider minors of H, which are the content of the following definition. It is a convenient extension of Definition 2.2.

DEFINITION 3.5 (Minors and partial expectation). (i) For $U \subset [\![1, N]\!]$, we define

$$H^{(U)} := H_{[U^c]} = (h_{ij})_{i,j \in U^c},$$

where $U^c := [\![1, N]\!] \setminus U$. Moreover, we define the resolvent of $H^{(U)}$ through

$$G^{(U)}(z) := \big(H^{(U)} - z\big)^{-1}.$$

(ii) Set

$$\sum_i^{(U)} := \sum_{i : i \notin U}.$$

When $U = \{a\}$, we abbreviate $(\{a\})$ by (a) in the above definitions; similarly, we write (ab) instead of $(\{a, b\})$.

(iii) For $U \subset [\![1, N]\!]$ define the partial expectation $\mathbb{E}_U(X) := \mathbb{E}(X | H^{(U)})$.

Next, we record some basic large deviations estimates from [21], Lemma 3.5.

LEMMA 3.6 (Large deviations estimates). *Let $a_1, \dots, a_N, b_1, \dots, b_M$ be independent random variables with zero mean and unit variance. Assume that there is a constant $\vartheta > 0$ such that*

$$\begin{aligned} &\mathbb{P}\big(|a_i| \ge x\big) \le \vartheta^{-1} \exp\big(-x^\vartheta\big) \qquad (i = 1, \dots, N), \\ &\mathbb{P}\big(|b_i| \ge x\big) \le \vartheta^{-1} \exp\big(-x^\vartheta\big) \qquad (i = 1, \dots, M). \end{aligned} \tag{3.11}$$

Then there exists a constant $\rho \equiv \rho(\vartheta) > 1$ *such that, for any* $\xi > 0$ *and any deterministic complex numbers* A_i *and* B_{ij}, *we have with high probability*

$$\left|\sum_i A_i |a_i|^2 - \sum_i A_i\right| \le \varphi^{\rho\xi}\left(\sum_i |A_i|^2\right)^{1/2}, \tag{3.12}$$

$$\left|\sum_{i \neq j} \bar{a}_i B_{ij} a_j\right| \le \varphi^{\rho\xi}\left(\sum_{i \neq j} |B_{ij}|^2\right)^{1/2}, \tag{3.13}$$

$$\left|\sum_{i,j} a_i B_{ij} b_j\right| \le \varphi^{\rho\xi}\left(\sum_{i,j} |B_{ij}|^2\right)^{1/2}. \tag{3.14}$$

We conclude this preliminary section by quoting a result on the eigenvalue rigidity of H. Denote by $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_N$ the classical locations of the eigenvalues of H, defined through

$$N \int_{-\infty}^{\gamma_\alpha} \varrho(x)\,\mathrm{d}x = \alpha \qquad (1 \le \alpha \le N). \tag{3.15}$$

The following result was proved in [18], Theorem 2.2.

THEOREM 3.7 (Rigidity of eigenvalues). *There exists a constant* C *such that we have with high probability*

$$|\lambda_\alpha - \gamma_\alpha| \le \varphi^C \bigl(\min\{\alpha, N+1-\alpha\}\bigr)^{-1/3} N^{-2/3}$$

for all $\alpha \in [\![1, N]\!]$.

4. Coarser grouping of outliers and reduction to the law of G. For the following, we fix the sequences $(V_N)_N$ and $(D_N)_N$. It will sometimes be convenient to assume that

$$\lim_N d_i^{(N)} \qquad \text{exists for all } i \in [\![1, r]\!]. \tag{4.1}$$

To that end, we invoke the following elementary result.

LEMMA 4.1. *Let* $(a_N)_N$ *be a sequence of nonnegative numbers and* $\varepsilon > 0$. *The following statements are equivalent.*

(i) $a_N \le \varepsilon$ *for large enough* N.
(ii) *Each subsequence has a further subsequence along which* $a_N \le \varepsilon$.

We use Lemma 4.1 by setting a_N to be the left-hand side of (2.19). Using Lemma 4.1, we therefore find that Theorem 2.9 holds for arbitrary D if it holds for D satisfying (4.1). From now on, we therefore assume without loss of generality that (4.1) holds.

For the proof of Theorem 2.9, we need a new subset of $[\![1, r]\!]$, denoted by $\gamma(\ell)$, which is larger than or equal to the subset $\pi(\ell, s)$ from Definition 2.8.

DEFINITION 4.2. For $\ell \in [\![1, r]\!]$ satisfying (2.18), define $\gamma(\ell) \equiv \gamma_{N,D,K}(\ell)$ as the smallest subset of $[\![1, r]\!]$ with the two following properties.

(i) $\ell \in \gamma(\ell)$.
(ii) If for $i, j \in [\![1, r]\!]$ we have $|d_i| > 1$ and

$$N^{1/2}\big(|d_i| - 1\big)^{1/2}|d_i - d_j| \le \varphi^{K/2}, \tag{4.2}$$

then either $i, j \in \gamma(\ell)$ or $i, j \in \bar{\gamma}(\ell)$.

Here we use the notation $\bar{\gamma}(\ell) := [\![1, r]\!] \setminus \gamma(\ell)$.

Note that $\gamma(\ell)$ is a set of consecutive integers. Similar to $\pi(\ell, s)$, the set $\gamma(\ell)$ indexes outliers that are close to that indexed by ℓ, except that now the threshold used to determine whether two outliers overlap is larger ($\varphi^{K/2}$ instead of the N-independent s). This need to regroup outliers into larger subsets arises from the perturbation theory argument in Proposition 4.5 below. At the end of the proof, in Section 8, we shall use perturbation theory a second time to obtain a statement involving outliers in $\pi(\ell, s)$ instead of $\gamma(\ell)$.

For the following, we introduce the abbreviation

$$\delta_\rho(d) := \varphi^\rho N^{-1/2}\big(|d| - 1\big)^{-1/2},$$

so that (4.2) reads $|d_i - d_j| \le \delta_{K/2}(d_i)$. We have the following elementary result.

LEMMA 4.3. *Let $\rho > 0$. If $|d| \ge 1 + \varphi^\rho N^{-1/3}$ and $|d - d'| \le \delta_\rho(d)$, then*

$$\big|d'\big| - 1 = \big(|d| - 1\big)\big(1 + O\big(\varphi^{-\rho/2}\big)\big).$$

For brevity, we fix ℓ satisfying (2.18), and abbreviate $\gamma \equiv \gamma(\ell)$ and $\bar{\gamma} \equiv \bar{\gamma}(\ell)$ when there is no risk of confusion. The indices of γ and $\bar{\gamma}$ are separated in the following sense.

LEMMA 4.4. *If $i \in \gamma$ and $j \in \bar{\gamma}$, then*

$$|d_i - d_j| > \delta_{K/2}(d_i). \tag{4.3}$$

If $i, j \in \gamma$, then

$$|d_i - d_j| \le 2r\delta_{K/2}(d_i). \tag{4.4}$$

PROOF. The bound (4.3) follows immediately from the definition of γ. The bound (4.4) follows immediately from Lemma 4.3 and the fact that γ is a set of at most r consecutive integers. □

Since D is diagonal, we may write

$$D = D_{[\gamma]} \oplus D_{[\bar{\gamma}]}.$$

The matrix $D_{[\gamma]}$ has dimensions $|\gamma| \times |\gamma|$ and eigenvalues $(d_i)_{i\in\gamma}$. Define the region

$$\mathcal{B} := \Big[\min_{i\in\gamma}\big(d_i - \delta_{K/4}(d_i)\big), \max_{i\in\gamma}\big(d_i + \delta_{K/4}(d_i)\big)\Big]. \tag{4.5}$$

From (2.18), (4.4) and Lemma 4.3 we get, for any $i \in \gamma$, that

$$\begin{aligned}|d_i| - \delta_{K/4}(d_i) &\geq |d_\ell| - |d_i - d_\ell| - 2\varphi^{K/4} N^{-1/2}\big(|d_\ell| - 1\big)^{-1/2} \\ &\geq 1 + \varphi^K N^{-1/3} - (2r+2)\varphi^{K/2} N^{-1/2}\big(|d_\ell| - 1\big)^{-1/2} \\ &\geq 1 + \varphi^K N^{-1/3} - (2r+2) N^{-1/3} \\ &> 1.\end{aligned}$$

We therefore conclude that $\mathcal{B} \subset \mathbb{R} \setminus [-1, 1]$. For large enough K a simple estimate using the definition of θ and the bound (3.10) yields for all $i \in \gamma$

$$\sigma(\widetilde{H}) \cap \theta(\mathcal{B}) = \{\mu_{\alpha(i)}\}_{i\in\gamma} \tag{4.6}$$

with high probability. In other words, $\theta(\mathcal{B})$ houses with high probability all of the outliers indexed by γ, and no other eigenvalues of $\widetilde{H}$. Moreover, from Theorem 3.7 we find that for large enough K the region $\theta(\mathcal{B})$ contains with high probability no eigenvalues of H.

We may now state the main result of this section. Introduce the $r \times r$ matrix

$$M(z) := V^* G(z) V.$$

To shorten notation, for i satisfying $|d_i| > 1$ we often abbreviate

$$\theta_i := \theta(d_i).$$

PROPOSITION 4.5. *The following holds for large enough K. Let $\ell \in [\![1, r]\!]$ satisfy* (2.18), *and write $\gamma \equiv \gamma(\ell)$. Then for all $i \in \gamma$ we have*

$$\left|\mu_{\alpha(i)} - \lambda_i\bigg(\theta_\ell - \frac{1}{m'(\theta_\ell)}\big(M(\theta_\ell) + D^{-1}\big)_{[\gamma]}\bigg)\right| \leq \varphi^{-1} N^{-1/2}\big(|d_\ell| - 1\big)^{1/2} \tag{4.7}$$

with high probability. [*Recall Definitions* 2.1 *and* 2.2 *for the meaning of $\lambda_i(\cdot)$ on the left-hand side*.]

PROOF. Our strategy for locating the outliers is based on the well-known fact that $x \notin \sigma(H)$ is an eigenvalue of $\widetilde{H}$ if and only if $M(x) + D^{-1}$ has a zero eigenvalue (see, e.g., Lemma 6.1 of [21]). Below, we develop a counting argument that finds the eigenvalues of $\widetilde{H}$ by analysing the behavior of each eigenvalue of $M(x) + D^{-1}$ as x varies. For our argument to work, it is important that no two eigenvalues of $M(x) + D^{-1}$ simultaneously cross the origin. [This condition is made precise in the claim (∗) below.] In order to rule out such coincidences, we introduce additional randomness, by adding a small perturbation $\varepsilon\Delta$, where Δ has an absolutely continuous law. The sole purpose of this perturbation is to exclude these coincidences almost surely in the randomness of Δ. This perturbation is purely *qualitative* in the sense that $\varepsilon > 0$ may be arbitrarily small; once the counting argument is concluded, we may easily take $\varepsilon \to 0$ and recover the claim for $\varepsilon = 0$ by a trivial continuity argument.

Thus, let Δ be an $r \times r$ Hermitian random matrix whose upper-triangular entries are independent and have an absolutely continuous law supported in the unit disc. Moreover, let Δ be independent of H. Let $\varepsilon > 0$. We shall prove the claim of Proposition 4.5 for the matrix $\widetilde{H}^\varepsilon := H + V(D^{-1} + \varepsilon\Delta)^{-1}V^*$ for small enough ε (depending on N), instead of $\widetilde{H} = H + VDV^*$.

Define the $r \times r$ matrix

$$A^\varepsilon(x) := M(x) - m(x) + D^{-1} + \varepsilon\Delta. \tag{4.8}$$

From [21], Lemma 6.1, we get that $x \notin \sigma(H)$ is an eigenvalue of $\widetilde{H}^\varepsilon$ if and only if $A^\varepsilon(x) + m(x)$ has a zero eigenvalue. Similar to Proposition 7.1 in [21], we use perturbation theory to compare the eigenvalues of $A^\varepsilon(x)$ with those of the block matrix

$$\tilde{A}^\varepsilon(x) := A^\varepsilon_{[\gamma]}(x) \oplus A^\varepsilon_{[\bar{\gamma}]}(x).$$

In order to apply perturbation theory, we must establish a lower bound on the spectral gap

$$\operatorname{dist}\big(\sigma\big(A^\varepsilon_{[\gamma]}(\theta_\ell)\big), \sigma\big(A^\varepsilon_{[\bar{\gamma}]}(\theta_\ell)\big)\big).$$

We find, for large enough K and small enough ε (depending on N), that with high probability

$$\begin{aligned} &\operatorname{dist}\big(\sigma\big(A^\varepsilon_{[\gamma]}(\theta_\ell)\big), \sigma\big(A^\varepsilon_{[\bar{\gamma}]}(\theta_\ell)\big)\big) \\ &\quad \geq \operatorname{dist}\big(\sigma\big(D^{-1}_{[\gamma]}\big), \sigma\big(D^{-1}_{[\bar{\gamma}]}\big)\big) - \delta_C(d_\ell) - r\varepsilon \\ &\quad \geq c\delta_{K/2}(d_\ell) - \delta_C(d_\ell) \geq \delta_{K/2-1}(d_\ell); \end{aligned} \tag{4.9}$$

in the first step we used Lemma A.2, $\|\varepsilon\Delta\| \leq r\varepsilon$ and

$$\big\|M(\theta_\ell) - m(\theta_\ell)\big\| \leq \delta_C(d_\ell) \tag{4.10}$$

by Theorem 3.3, (3.5), (3.9) and (2.18); in the second step we used (4.3) and chose ε to be small enough (depending on N); in the last step we chose K to be large enough (depending on C).

Next, Theorem 3.3, (3.5) and (3.9) yield, with high probability,

$$\big\|A^\varepsilon(\theta_\ell) - \tilde{A}^\varepsilon(\theta_\ell)\big\| \leq \delta_{K/4-2}(d_\ell) \tag{4.11}$$

for large enough K and small enough ε (depending on N).

Define the regions

$$\mathcal{D} := \bigcup_{i\in\gamma}[d_i^{-1} - \delta_{K/4}(d_\ell), d_i^{-1} + \delta_{K/4}(d_\ell)],$$

$$\overline{\mathcal{D}} := \bigcup_{i\in\bar{\gamma}}[d_i^{-1} - \delta_{K/4}(d_\ell), d_i^{-1} + \delta_{K/4}(d_\ell)],$$

which are disjoint by (4.3). Using (4.10), we find that for large enough K and small enough ε (depending on N) we have, with high probability,

$$\sigma\big(A^\varepsilon_{[\gamma]}(\theta_\ell)\big) \subset \mathcal{D}, \qquad \sigma\big(A^\varepsilon(\theta_\ell)\big) \subset \mathcal{D} \cup \overline{\mathcal{D}}.$$

Moreover, both $A^\varepsilon(\theta_\ell)$ and $A^\varepsilon_{[\gamma]}(\theta_\ell)$ have exactly $|\gamma|$ eigenvalues in $\mathcal{D}$; we denote these eigenvalues by $(a_i^\varepsilon)_{i\in\gamma}$ and $(\tilde{a}_i^\varepsilon)_{i\in\gamma}$, respectively.

We may now apply perturbation theory. Invoking Proposition A.1 using (4.9) and (4.11) yields with high probability

$$a_i^\varepsilon = \tilde{a}_i^\varepsilon + O\left(\frac{\delta_{K/4-2}(d_\ell)^2}{\delta_{K/2-1}(d_\ell)}\right) = \tilde{a}_i^\varepsilon + O\big(\delta_{-3}(d_\ell)\big) \tag{4.12}$$

for $i \in \gamma$.

Next, we allow the argument x of $A^\varepsilon(x)$ to vary in order to locate the eigenvalues of $\widetilde{H}^\varepsilon$. We recall the following derivative bound from [21], Lemma 7.2: there is a constant C such that for large enough K we have for all ℓ^2-normalised $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$, with high probability,

$$\begin{aligned} &\big|\partial_x G_{\mathbf{vw}}(x) - \partial_x m(x)\langle \mathbf{v}, \mathbf{w}\rangle\big| \le \varphi^C N^{-1/3}\kappa_x^{-1} \\ &\qquad \text{for } x \in \big[-\Sigma, -2 - \varphi^{K/2}N^{-2/3}\big] \cup \big[2 + \varphi^{K/2}N^{-1/3}, \Sigma\big]. \end{aligned} \tag{4.13}$$

By the definition (4.5) of $\mathcal{B}$, we find from Lemma 4.3, (2.18) and (4.4) that

$$x \in \theta(\mathcal{B}) \quad \Longrightarrow \quad \theta\big(d_\ell - 3r\delta_{K/2}(d_\ell)\big) \le x \le \theta\big(d_\ell + 3r\delta_{K/2}(d_\ell)\big). \tag{4.14}$$

We deduce using Lemma 4.3, (2.18) and (3.5) that

$$\kappa_x \asymp \big(|d_\ell| - 1\big)^2 \qquad \text{for } x \in \theta(\mathcal{B}). \tag{4.15}$$

Therefore from Theorem 3.3, we conclude with high probability

$$M(x) = m(x) + O\big(\delta_C(d_\ell)\big) \qquad \text{for } x \in \theta(\mathcal{B}). \tag{4.16}$$

Similarly, from (4.13) we get with high probability

$$M'(x) = m'(x) + O\big(\varphi^C N^{-1/3}\big(|d_\ell| - 1\big)^{-2}\big) \qquad \text{for } x \in \theta(\mathcal{B}). \tag{4.17}$$

With these preliminary bounds, we may vary $x \in \theta(\mathcal{B})$. Let $(a_i(x))_{i\in\gamma}$ denote the continuous family of eigenvalues of $A^\varepsilon(x)$ satisfying $a_i^\varepsilon(\theta_\ell) = a_i^\varepsilon$ for $i \in \gamma$.

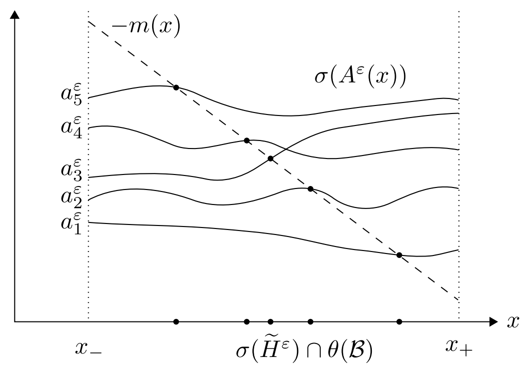

FIG. 2. *The spectrum of $A^\varepsilon(x)$ for $x \in \theta(\mathcal{B})$. For definiteness, we chose $\gamma = [\![1, 5]\!]$. The region $x \in \theta(\mathcal{B})$ is delimited by dotted lines. The eigenvalues of $\widetilde{H}^\varepsilon$ are labelled by black dots on the x-axis.*

For the following argument, it is helpful to keep Figure 2 in mind. We make the following claim:

$$(*) \qquad \begin{gathered} \text{Almost surely, for all } x \in \theta(\mathcal{B}) \text{ we have that} \\ a_i^\varepsilon(x) = -m(x) \text{ for at most one } i \in \gamma. \end{gathered}$$

We omit the details of the proof[6] of $(*)$. Note that the necessity for $(*)$ to hold is the only reason we had to introduce the additional randomness Δ into $\widetilde{H}^\varepsilon$.

For definiteness, suppose for the following that $d_\ell > 1$. We claim that for all $i \in \gamma$ we have with high probability

$$a_i^\varepsilon(x_-) \le -m(x_-), \qquad -m(x_+) \le a_i^\varepsilon(x_+), \tag{4.18}$$

where $x_\pm$ denote the endpoints of the interval $\theta(\mathcal{B})$. Let us focus on the first estimate; the second one is proved similarly. Let $i := \min \gamma$. Since $d \mapsto d - \delta_{K/4}(d)$ is increasing, we find that the left endpoint of $\mathcal{B}$ is $d_i - \delta_{K/4}(d_i)$. From (4.16) and Lemma A.2, we find with high probability

$$\begin{aligned} \max_{x \in \theta(\mathcal{B})} \max_{j \in \gamma} a_j^\varepsilon(x) &\le \frac{1}{d_i} + \delta_C(d_\ell) + r\varepsilon \\ &\le \frac{1}{d_i - \delta_{K/4}(d_i)} - c\delta_{K/4}(d_i) + \delta_C(d_\ell) + r\varepsilon \\ &\le \frac{1}{d_i - \delta_{K/4}(d_i)} = -m(x_-); \end{aligned}$$

[6]The claim $(*)$ reduces to the following statement. Let $B(x)$ with $x \in I$ and Δ be Hermitian matrices such that $B(x)$ is deterministic and depends smoothly on x, and Δ has an absolutely continuous law; then, almost surely in Δ, for all $x \in I$ the matrix $B(x) + \Delta$ has at most one zero eigenvalue. Let S denote the subset of matrices with multiple eigenvalues at zero, so that S is an algebraic variety of codimension two. The claim therefore reduces to the statement that the path $\{B(x)\}_{x \in I} + \Delta$ almost surely does not intersect S, which is standard.

in the second step we used $1 \le d_i \le \Sigma - 1$; the third step holds for large enough K and small enough ε (depending on N), by Lemma 4.3; the last step follows from (3.3). This concludes the proof of (4.18).

Recall that $\widetilde{H}^\varepsilon$ has with high probability exactly $|\gamma|$ eigenvalues in $\theta(\mathcal{B})$. By continuity of $a_i^\varepsilon(x)$ the property $(*)$ and (4.18), we therefore get that the function $-m(x)$ intersects each function $a_i^\varepsilon(x)$, $i \in \gamma$, exactly once in $\theta(\mathcal{B})$. Let $i \in \gamma$ and denote by x_i^ε the unique point (with high probability) in $\theta(\mathcal{B})$ at which $a_i^\varepsilon(x_i^\varepsilon) = -m(x_i^\varepsilon)$.

From the definition of A^ε and (4.17) we get, with high probability,

$$\begin{aligned} -m(x_i^\varepsilon) &= a_i^\varepsilon(\theta_\ell) + O\bigl(\varphi^C N^{-1/3}(|d_\ell| - 1)^{-2}|x_i^\varepsilon - \theta_\ell|\bigr) \\ &= a_i^\varepsilon + O\bigl(\varphi^{K/2+C} N^{-5/6}(|d_\ell| - 1)^{-3/2}\bigr), \end{aligned} \tag{4.19}$$

where in the second step we used (4.14), the fact that $x_i^\varepsilon \in \theta(\mathcal{B})$, and the elementary bound $|\theta'(d)| \asymp |d| - 1$. [Recall that by definition $a_i^\varepsilon(\theta_\ell) = a_i^\varepsilon$.] Now we may use (4.12) and (4.19) to get

$$-m(x_i^\varepsilon) = \tilde{a}_i^\varepsilon + O\bigl(\delta_{-3}(d_\ell) + \varphi^{K/2+C} N^{-5/6}(|d_\ell| - 1)^{-3/2}\bigr) \tag{4.20}$$

with high probability. Now we expand the left-hand side using the identity

$$m' = \frac{m^2}{1 - m^2} \asymp \kappa_x^{-1/2}, \tag{4.21}$$

which follows easily from (3.2); in the second step we used Lemma 3.1. Differentiating again, we get $m''(x) \asymp \kappa_x^{-3/2}$. From (4.15), we therefore get

$$\begin{aligned} m(x_i^\varepsilon) &= m(\theta_\ell) + m'(\theta_\ell)(x_i^\varepsilon - \theta_\ell) \\ &\quad + O\bigl((|d_\ell| - 1)^{-3}\bigl((|d_\ell| - 1)\delta_{K/2}(d_\ell)\bigr)^2\bigr) \\ &= m(\theta_\ell) + m'(\theta_\ell)(x_i^\varepsilon - \theta_\ell) + O\bigl(\varphi^K (|d_\ell| - 1)^{-2} N^{-1}\bigr) \end{aligned} \tag{4.22}$$

with high probability. Solving x_i^ε from (4.22) and $-m(x_i^\varepsilon)$ from (4.20), we find for large enough K with high probability

$$\begin{aligned} x_i^\varepsilon &= \theta_\ell - \frac{1}{m'(\theta_\ell)}\bigl(\tilde{a}_i^\varepsilon + m(\theta_\ell)\bigr) \\ &\quad + O\bigl(\varphi^{-3} N^{-1/2}(|d_\ell| - 1)^{1/2} \\ &\qquad + \varphi^{K/2+C} N^{-5/6}(|d_\ell| - 1)^{-1/2} + \varphi^K N^{-1}(|d_\ell| - 1)^{-1}\bigr) \\ &= \theta_\ell - \frac{1}{m'(\theta_\ell)}\bigl(\tilde{a}_i^\varepsilon + m(\theta_\ell)\bigr) \\ &\quad + O\bigl(\varphi^{-3} N^{-1/2}(|d_\ell| - 1)^{1/2} \\ &\qquad + \varphi^{-K/2+C} N^{-1/2}(|d_\ell| - 1)^{1/2} + \varphi^{-K/2} N^{-1/2}(|d_\ell| - 1)^{1/2}\bigr) \\ &= \theta_\ell - \frac{1}{m'(\theta_\ell)}\bigl(\tilde{a}_i^\varepsilon + m(\theta_\ell)\bigr) + O\bigl(\varphi^{-2} N^{-1/2}(|d_\ell| - 1)^{1/2}\bigr); \end{aligned}$$

in the first step we estimated the error terms using $m'(\theta_\ell) \asymp (|d_\ell| - 1)^{-1}$ by (4.21) and (4.15); in the second step we used (2.18); the last step follows by choosing K large enough. Thus, we conclude that

$$x_i^\varepsilon = \lambda_i\biggl(\theta_\ell - \frac{1}{m'(\theta_\ell)}\bigl(M_{[\gamma]}(\theta_\ell) + D_{[\gamma]}^{-1} + \varepsilon\Delta_{[\gamma]}\bigr)\biggr) + O\bigl(\varphi^{-2}N^{-1/2}\bigl(|d_\ell| - 1\bigr)^{1/2}\bigr)$$

with high probability for small enough ε (depending on N). Taking $\varepsilon \to 0$ completes the proof. $\square$

We conclude this section with a remark on the choice of the reference point θ_ℓ in Proposition 4.5. By definition of γ, if $i \in \gamma(\ell)$ then $\gamma(i) = \gamma(\ell)$. Obviously, the distribution of the overlapping group of outliers $(\mu_{\alpha(i)})_{i\in\gamma}$ cannot depend on the particular choice of $\ell \in \gamma$. Nevertheless, the reference matrix $\theta_\ell - \frac{1}{m'(\theta_\ell)}(M_{[\gamma]}(\theta_\ell) + D_{[\gamma]}^{-1})$ in (4.7) depends explicitly on $\ell \in \gamma$ via θ_ℓ. This is not a contradiction, however, since a different choice of ℓ leads to a reference matrix which only differs from the original one by an error term of order $O(\varphi^{-1}N^{-1/2}(|d_\ell| - 1)^{1/2})$; this difference may be absorbed into the error term on the right-hand side of (4.7). We shall need this fact in Section 9. The precise statement is as follows. (To simplify notation, we state it without loss of generality for the case $\gamma = [\![1, r]\!]$.)

LEMMA 4.6. *Suppose that* $\gamma(1) = [\![1, r]\!]$ *and that* $|d_1| \ge 1 + \varphi^K N^{-1/3}$. *Let*

$$d, \tilde{d} \in \bigl[d_1 - \delta_{K/2+1}(d_1), d_1 + \delta_{K/2+1}(d_1)\bigr].$$

Then for large enough K *we have*

$$\biggl\|\biggl(\theta - \frac{1}{m'(\theta)}\bigl(M(\theta) + D^{-1}\bigr)\biggr) - \biggl(\tilde{\theta} - \frac{1}{m'(\tilde{\theta})}\bigl(M(\tilde{\theta}) + D^{-1}\bigr)\biggr)\biggr\| \\ \le \varphi^{-1}N^{-1/2}\bigl(|d_1| - 1\bigr)^{1/2},$$

where we abbreviated $\theta \equiv \theta(d)$ *and* $\tilde{\theta} \equiv \theta(\tilde{d})$.

PROOF. We write

$$\begin{aligned}
&\biggl(\theta_i - \frac{1}{m'(\theta)}\bigl(M(\theta) + D^{-1}\bigr)\biggr) - \biggl(\tilde{\theta} - \frac{1}{m'(\tilde{\theta})}\bigl(M(\tilde{\theta}) + D^{-1}\bigr)\biggr) \\
&\quad = \theta - \tilde{\theta} + \frac{1}{m'(\theta)}\bigl(M(\tilde{\theta}) - M(\theta)\bigr) + \biggl(\frac{1}{m'(\tilde{\theta})} - \frac{1}{m'(\theta)}\biggr)\bigl(M(\tilde{\theta}) + D^{-1}\bigr) \\
&\quad = \theta - \tilde{\theta} + \frac{1}{m'(\theta)}\bigl(m(\tilde{\theta}) - m(\theta)\bigr) + \biggl(\frac{1}{m'(\tilde{\theta})} - \frac{1}{m'(\theta)}\biggr)\bigl(m(\tilde{\theta}) + \tilde{d}^{-1}\bigr) \\
&\qquad + O\bigl(\varphi^{K/2+C}N^{-5/6}\bigl(|d_1| - 1\bigr)^{-1/2} + \varphi^{K/2+C}N^{-1}\bigl(|d_1| - 1\bigr)^{-1}\bigr)
\end{aligned}$$

$$= d + \frac{1}{d} - \tilde{d} - \frac{1}{\tilde{d}} + (d^2 - 1)\left(\frac{1}{d} - \frac{1}{\tilde{d}}\right) + O\big(\varphi^{-2}N^{-1/2}(|d_1| - 1)^{1/2}\big)$$
$$= O\big(\varphi^{-2}N^{-1/2}(|d_1| - 1)^{1/2}\big)$$

with high probability; in the second step we wrote $M(\tilde{\theta}) - M(\theta) = \int_\theta^{\tilde{\theta}} M'(\xi)\,\mathrm{d}\xi$ and used (4.17) and Lemma 4.3, as well as Theorem 3.3, (3.9), (3.5), (4.21), and the fact that $m''(x) \asymp \kappa_x^{-3/2}$; in the third step we used (2.5), (3.3), and the assumption that K is large enough; in the last step we used that $(d - \tilde{d})^2 \le 4\varphi^{K+1}N^{-1}(|d_1| - 1)^{-1}$. □

5. The Gaussian case. Suppose that ℓ satisfies (2.18). By Proposition 4.5, in order to analyse the joint distribution of the outliers $(\mu_{\alpha(i)})_{i\in\gamma}$ with $\gamma \equiv \gamma(\ell)$, it suffices to analyse the distribution of the eigenvalues of the $|\gamma| \times |\gamma|$ matrix $M_{[\gamma]}(\theta_\ell)$. In this section, we do this under the assumption that the entries of H are Gaussian, that is, that H is a GOE/GUE matrix.

Recall that γ may depend on N. To simplify notation, in Sections 5–7 we take $\gamma = [\![1, r]\!]$, which allows us to drop subscripts $[\gamma]$ and avoid minor nuisances arising from the fact that γ may depend on N. In fact, this special case will easily imply the case of general γ; see Section 8.

The following definition is a convenient shorthand for the equivalence relation defined by two random matrices of fixed size having the same asymptotic distribution.

DEFINITION 5.1. For two sequences X_N and Y_N of random $k \times k$ matrices, where $k \in \mathbb{N}$ is fixed, we write $X \overset{d}{\sim} Y$ if

$$\lim_N \big(\mathbb{E} f(X_N) - \mathbb{E} f(Y_N)\big) = 0$$

for all continuous and bounded f.

Let $\Phi = (\Phi_{ij})_{i,j=1}^r$ be an $r \times r$ GOE/GUE matrix multiplied by $\sqrt{r}$. In other words, the covariances of Φ are given by

$$\mathbb{E}\Phi_{ij}\Phi_{kl} = \Delta_{ij,kl}, \tag{5.1}$$

where $\Delta_{ij,kl}$ was defined in (2.10). The following proposition is the main result of this section. It provides the joint distribution of the eigenvalues of $M(\theta)$, which, by Proposition 4.5, immediately yields the distribution of the γ-group of outliers under the assumption that H is a GOE/GUE matrix. However, since we are ultimately interested in non-Gaussian H, we shall not combine it Proposition 4.5 directly, but instead use it as an input for the more general case covered in Section 6.

PROPOSITION 5.2. *The following holds for large enough K. Let $\theta \equiv \theta(d)$ for some d satisfying $|d| \geq 1 + \varphi^K N^{-1/3}$. Suppose moreover that H is a GOE/GUE matrix. Then*

$$N^{1/2}\big(|d|-1\big)^{1/2}\big(M(\theta) - m(\theta)\big) \overset{d}{\sim} \frac{1}{|d|\sqrt{|d|+1}}\Phi.$$

PROOF. Throughout the proof, we drop the spectral parameter $z = \theta$ from quantities such as $M(\theta)$. By unitary invariance of H, we may assume that $V_{ij} = \delta_{ij}$, that is, $\mathbf{v}^{(i)}$ is the ith standard basis vector of $\mathbb{C}^N$. By Schur's complement formula, we therefore get $M = B^{-1}$ where $B = (B_{ij})_{i,j=1}^r$ is the Hermitian $r \times r$ matrix defined by

$$B_{ij} := h_{ij} - \theta - \sum_{a,b}^{(1\cdots r)} h_{ia} G_{ab}^{(1\cdots r)} h_{bj}.$$

We now claim that

$$\left| \frac{1}{N} \sum_{a}^{(1\cdots r)} G_{aa}^{(1\cdots r)} - m \right| \leq \varphi^C N^{-1} \kappa_\theta^{-1}. \tag{5.2}$$

Bearing later applications in mind, we in fact prove, for any $\ell \in \mathbb{N}$, that

$$\left| \operatorname{Tr} G^\ell - N \int \frac{\varrho(x)}{(x-\theta)^\ell} \, \mathrm{d}x \right| \leq \varphi^C \kappa_\theta^{-\ell} \tag{5.3}$$

with high probability. Applying (5.3) with $\ell = 1$ to the minor $H^{(1\cdots r)}$ immediately yields (5.2). In order to prove (5.3), we use Theorem 3.7 to get with high probability

$$\begin{aligned}
&\left| \sum_\alpha \frac{1}{(\lambda_\alpha - \theta)^\ell} - \sum_\alpha \frac{1}{(\gamma_\alpha - \theta)^\ell} \right| \\
&\quad \leq \varphi^C \sum_{\alpha=1}^{N/2} \frac{\alpha^{-1/3} N^{-2/3}}{(|\theta| - |\gamma_\alpha|)^{\ell+1}} \leq \frac{\varphi^C}{N} \sum_{\alpha=1}^{N/2} \frac{(\alpha/N)^{-1/3}}{((\alpha/N)^{2/3} + \kappa_\theta)^{\ell+1}} \\
&\quad \leq \varphi^C \int_0^\infty \frac{x^{-1/3}}{(x^{2/3} + \kappa_\theta)^{\ell+1}} \, \mathrm{d}x \leq \frac{\varphi^C}{\kappa_\theta^\ell};
\end{aligned} \tag{5.4}$$

in the first step we estimated the contribution of $\alpha > N/2$ by the contribution of $N + 1 - \alpha$, and used that $|\lambda_\alpha - \gamma_\alpha| \ll |\theta| - |\gamma_\alpha|$ with high probability by Theorem 3.7 and the assumption on θ (for large enough K); in the second step we used the estimate

$$2 - |\gamma_\alpha| \asymp \alpha^{2/3} N^{-2/3} \tag{5.5}$$

for $\alpha \le N/2$, as follows from the definition of γ_α. Similarly, setting $\gamma_0 := -2$, we find

$$N \int \frac{\varrho(x)}{(x-\theta)^\ell} \,\mathrm{d}x = N \sum_{\alpha=1}^{N} \int_{\gamma_{\alpha-1}}^{\gamma_\alpha} \frac{\varrho(x)}{(x-\theta)^\ell} \,\mathrm{d}x$$
$$= \sum_{\alpha=1}^{N} \frac{1}{(\gamma_\alpha - \theta)^\ell} + O\biggl(\sum_{\alpha=1}^{N/2} \frac{\alpha^{-1/3} N^{-2/3}}{(|\theta| - |\gamma_\alpha|)^{\ell+1}}\biggr) \tag{5.6}$$
$$= \sum_{\alpha=1}^{N} \frac{1}{(\gamma_\alpha - \theta)^\ell} + O\biggl(\frac{1}{\kappa_\theta^\ell}\biggr).$$

Now (5.3) follows from (5.4) and (5.6).

Using $\mathbb{E} h_{ia} h_{bj} = \delta_{ij}\delta_{ab} N^{-1}$ and (3.5), we therefore get from (5.2)

$$\sum_{a,b}^{(1\cdots r)} h_{ia} G_{ab}^{(1\cdots r)} h_{bj} - \delta_{ij} m$$
$$= (\mathbb{1} - \mathbb{E}_{1\cdots r}) \sum_{a,b}^{(1\cdots r)} h_{ia} G_{ab}^{(1\cdots r)} h_{bj} + O\bigl(\varphi^C N^{-1} (d-1)^{-2}\bigr)$$

with high probability. We may therefore write

$$B_{ij} = -\theta - m - (-h_{ij} + W_{ij} + R_{ij}),$$

where

$$W_{ij} := (\mathbb{1} - \mathbb{E}_{1\cdots r}) \sum_{a,b}^{(1\cdots r)} h_{ia} G_{ab}^{(1\cdots r)} h_{bj} \quad \text{and} \quad R_{ij} = O\bigl(\varphi^C N^{-1} \bigl(|d|-1\bigr)^{-2}\bigr)$$

with high probability.

Next, we claim that

$$W_{ij} = O\bigl(\varphi^C N^{-1/2} \bigl(|d|-1\bigr)^{-1/2}\bigr) \tag{5.7}$$

with high probability. Indeed, using Lemma 3.6 we get

$$|W_{ij}| \le \varphi^C \biggl(\frac{1}{N^2} \sum_{a,b}^{(1\cdots r)} \bigl|G_{ab}^{(1\cdots r)}\bigr|^2\biggr)^{1/2}$$
$$= \varphi^C \biggl(\frac{1}{N^2} \operatorname{Tr}\bigl(G^{(1\cdots r)*} G^{(1\cdots r)}\bigr)\biggr)^{1/2} \le \varphi^C N^{-1/2} \bigl(|d|-1\bigr)^{-1/2}$$

with high probability. In the last step we used (5.3), (4.13), and $G = G^*$ to get (dropping the upper indices to simplify notation)

$$\frac{1}{N^2} \operatorname{Tr}\bigl(G^* G\bigr) = N^{-1} m' + O\bigl(\varphi^C N^{-2} \kappa_\theta^{-2}\bigr)$$
$$= O\bigl(N^{-1} \kappa_\theta^{-1/2} + \varphi^C N^{-2} \kappa_\theta^{-2}\bigr) = O\bigl(N^{-1} \bigl(|d|-1\bigr)^{-1}\bigr)$$

with high probability.

Using the bounds (5.7) and $|h_{ij}| \le \varphi^C N^{-1/2}$ with high probability [as follows from (2.1)], we may expand with (3.2) to get

$$M_{ij} = m\delta_{ij} + m^2(-h_{ij} + W_{ij}) + O(\varphi^C N^{-1}(|d|-1)^{-2})$$

with high probability. Let $H_{[1\cdots r]} = H^{(r+1\cdots N)}$ denote the upper $r \times r$ block of H. Thus we get

$$\begin{aligned} &N^{1/2}(|d|-1)^{1/2}(M-m) \\ &\quad = m^2 N^{1/2}(|d|-1)^{1/2}(-H_{[1\cdots r]} + W) + O(\varphi^C N^{-1/2}(|d|-1)^{-3/2}) \end{aligned} \tag{5.8}$$

with high probability. In particular, for large enough K we get

$$N^{1/2}(|d|-1)^{1/2}(M-m) \overset{d}{\sim} m^2 N^{1/2}(|d|-1)^{1/2}(-H_{[1\cdots r]} + W). \tag{5.9}$$

By definition, $H_{[1\cdots r]}$ and W are independent. What therefore remains is to compute the asymptotic distribution of W. We claim that W converges in law to an $r \times r$ Gaussian matrix:

$$N^{1/2}(|d|-1)^{1/2} W \overset{d}{\sim} \frac{1}{\sqrt{|d|+1}} \Phi. \tag{5.10}$$

By the Cramér–Wold device, it suffices to show that

$$N^{1/2}(|d|-1)^{1/2} \sum_{i,j} Q_{ij} W_{ij} \overset{d}{\sim} \frac{1}{\sqrt{|d|+1}} \sum_{i,j} Q_{ij} \Phi_{ij}$$

for any deterministic matrix $Q = (Q_{ij})$ satisfying $Q = Q^*$ and $Q_{ij} \in \mathbb{R}$ if $\beta = 1$. To that end, we diagonalize $G^{(1\cdots r)}$ by writing

$$N^{-1/2}(|d|-1)^{1/2} G^{(1\cdots r)} = U^* \Lambda U,$$

where U is a unitary $(N-r) \times (N-r)$ matrix and $\Lambda = \operatorname{diag}(\Lambda_{r+1}, \ldots, \Lambda_N)$. Moreover, we introduce the $r \times (N-r)$ matrix $h := (h_{ia} : i \le r, a \ge r+1)$. Since the entries of h are i.i.d. Gaussians, U is orthogonal/unitary, and H is independent of (Λ, U), we find that $(\Lambda, Uh) \overset{d}{=} (\Lambda, h)$. We conclude that

$$\begin{aligned} N^{1/2}(|d|-1)^{1/2} \sum_{i,j=1}^r Q_{ij} W_{ij} &= N(1 - \mathbb{E}_{1\cdots r}) \operatorname{Tr}(Q h^* U^* \Lambda U h) \\ &\overset{d}{=} N(1 - \mathbb{E}_{1\cdots r}) \operatorname{Tr}(Q h^* \Lambda h) \\ &= \sum_a^{(1\cdots r)} \Lambda_a \sum_{i,j=1}^r Q_{ij} N(h_{ia} h_{aj} - \mathbb{E} h_{ia} h_{aj}) \\ &=: X. \end{aligned}$$

Note that $(\sum_{i,j} Q_{ij} N(h_{ia}h_{aj} - \mathbb{E}h_{ia}h_{aj}))_{a=r+1}^{N}$ is a family of i.i.d. random variables, independent of Λ, with variance $2\beta^{-1}\operatorname{Tr} Q^2$. Therefore,

$$\begin{aligned}
\mathbb{E}X^2 &= \frac{2}{\beta}\operatorname{Tr} Q^2 \sum_{a}^{(1\cdots r)} \Lambda_a^2 \\
&= \frac{2}{\beta}\operatorname{Tr} Q^2 N^{-1}(|d|-1)\operatorname{Tr}(G^{(1\cdots r)})^2 \\
&= \frac{2}{\beta}\operatorname{Tr} Q^2\bigl((|d|-1)m' + O(\varphi^C N^{-1}(|d|-1)^{-3})\bigr) \\
&= \frac{2}{\beta}\operatorname{Tr} Q^2\bigl((|d|-1)m' + O(\varphi^{-1})\bigr)
\end{aligned}$$

with high probability for large enough K, where we used (5.3). Moreover, we have

$$\begin{aligned}
\sum_{a}^{(1\cdots r)} \Lambda_a^4 &= N^{-2}(|d|-1)^2\operatorname{Tr}(G^{(1\cdots r)})^4 \\
&= N^{-2}(|d|-1)^2\bigl(Nm'''/6 + O(\varphi^C(|d|-1)^{-8})\bigr) \\
&= O\bigl(N^{-1}(|d|-1)^{-3} + N^{-2}(|d|-1)^{-6}\bigr) = O(\varphi^{-1})
\end{aligned}$$

with high probability for large enough K, where in the second step we used (5.3) and in the third step the estimate $m''' \asymp \kappa_\theta^{-5/2}$ as follows by differentiating (4.21) twice and from Lemma 3.1.

We conclude from the central limit theorem that

$$X \overset{d}{\sim} \mathcal{N}\biggl(0, \frac{2}{\beta(|d|+1)}\operatorname{Tr} Q^2\biggr),$$

where we used the identity

$$(|d|-1)m' = \frac{1}{|d|+1}$$

as follows from (4.21) and (3.3). Thus, (5.10) follows the identity

$$\frac{1}{\sqrt{|d|+1}}\sum_{i,j} Q_{ij}\Phi_{ij} \overset{d}{=} \mathcal{N}\biggl(0, \frac{2}{\beta(|d|+1)}\operatorname{Tr} Q^2\biggr)$$

as follows from a from a simple variance calculation.

Next, by definition of $H_{[1\cdots r]}$ we have

$$-N^{1/2}(|d|-1)^{1/2} H_{[1\cdots r]} \overset{d}{=} (|d|-1)^{1/2}\Phi.$$

Thus, we find

$$N^{1/2}(|d|-1)^{1/2}(-H_{[1\cdots r]} + W) \overset{d}{\sim} \frac{|d|}{\sqrt{|d|+1}}\Phi.$$

The claim now follows from (5.9) and (3.3). □

6. The almost Gaussian case. The next step of the proof is to consider the case where most entries of H are Gaussian. The exponent $\rho \geq 2$ is used to define a cutoff scale in the entries of V, below which the corresponding entries of H are assumed to be Gaussian. Proposition 6.1 will ultimately be fed into Lemma 7.1 below, at which time we shall choose ρ to be large enough.

PROPOSITION 6.1. *The following holds for large enough K. Let $\theta \equiv \theta(d)$ for some d satisfying $|d| \geq 1 + \varphi^K N^{-1/3}$. Let $\rho \geq 2$. Suppose that the Wigner matrix H satisfies*

$$\max_{1 \leq l \leq r} \max\{|V_{il}|, |V_{jl}|\} \leq \varphi^{-\rho} \quad \Longrightarrow \quad h_{ij} \text{ is Gaussian.} \tag{6.1}$$

Then

$$N^{1/2}(|d|-1)^{1/2}(M(\theta) - m(\theta)) \overset{d}{\sim} -N^{1/2}(|d|-1)^{1/2} d^{-2} V_\delta^* H V_\delta + \Psi_0,$$

where $\Psi_0 = \Psi_0^*$ is a Gaussian matrix, independent of H, with centred entries and covariance

$$\begin{aligned} \mathbb{E}(\Psi_0)_{ij}(\Psi_0)_{kl} = {} & \frac{|d|-1}{d^4}\big(\Delta_{ij,kl} - \mathcal{P}_{ij,kl}(V_\delta^* V_\delta)\big) + \frac{1}{d^4(|d|+1)} \Delta_{ij,kl} \\ & + \frac{|d|-1}{d^5} \mathcal{Q}_{ij,kl}(V) + \frac{|d|-1}{d^6} \mathcal{R}_{ij,kl}(V). \end{aligned}$$

PROOF. Throughout the proof, we drop the spectral parameter $z = \theta$ from our notation.

Step 1. We start with some linear algebra in order to write the matrix M in a form amenable to analysis. Since $\|\mathbf{v}^{(l)}\| = 1$ for all l we find that

$$\big|\{i : |V_{il}| > \varphi^{-\rho}\}\big| \leq \varphi^{2\rho}.$$

We shall permute the rows of V by using an $N \times N$ permutation matrix O according to $M = V^* G V = (OV)^* O G O^* O V$. It is easy to see that we may permute the rows of V by setting $V \mapsto OV$ so that after the permutation we have

$$V = \begin{pmatrix} U \\ W \end{pmatrix},$$

where:

(i) U is a $\mu \times r$ matrix and W an $(N - \mu) \times r$ matrix,
(ii) $|W_{il}| \leq \varphi^{-\rho}$ for all i and l,
(iii) $\mu \leq r\varphi^{2\rho}$.

After the permutation $H \mapsto OHO^*$, we may write H as

$$H = \begin{pmatrix} A & B^* \\ B & H_0 \end{pmatrix},$$

where A is a $\mu \times \mu$ matrix, B an $(N-\mu) \times \mu$ matrix, and H_0 an $(N-\mu) \times (N-\mu)$ matrix with Gaussian entries [as follows from (6.1)].

Next, we rotate the rows of W by choosing a unitary $(N-\mu) \times (N-\mu)$ matrix $\widetilde{S}$ such that

$$\widetilde{S}W = \begin{pmatrix} \widetilde{W} \\ 0 \end{pmatrix},$$

where $\widetilde{W}$ is an $r \times r$ matrix that satisfies

$$U^*U + W^*W = U^*U + \widetilde{W}^*\widetilde{W} = \mathbb{1}_r. \tag{6.2}$$

Thus, we get

$$\begin{aligned} M &= V^* \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{S}^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{S} \end{pmatrix} \begin{pmatrix} A-\theta & B^* \\ B & H_0-\theta \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{S}^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{S} \end{pmatrix} V \\ &\overset{d}{=} \begin{pmatrix} U \\ \widetilde{W} \\ 0 \end{pmatrix}^* \begin{pmatrix} A-\theta & B^*\widetilde{S}^* \\ \widetilde{S}B & H_0-\theta \end{pmatrix}^{-1} \begin{pmatrix} U \\ \widetilde{W} \\ 0 \end{pmatrix}, \end{aligned}$$

where $\overset{d}{=}$ denotes equality in distribution. Here we used the unitary invariance of the Gaussian matrix H_0.

Next, we decompose

$$H_0 = \begin{pmatrix} H_1 & Z^* \\ Z & H_2 \end{pmatrix}, \qquad \widetilde{S} = \begin{pmatrix} R \\ S \end{pmatrix},$$

where H_1 is an $r \times r$ Gaussian matrix, Z an $(N-\mu-r) \times r$ Gaussian matrix, and H_2 an $(N-\mu-r) \times (N-\mu-r)$ Gaussian matrix. Moreover, R is an $r \times (N-\mu)$ matrix and we have

$$RR^* = \mathbb{1}_r, \qquad SS^* = \mathbb{1}_{N-\mu-r}, \qquad RS^* = 0, \qquad R^*R + S^*S = \mathbb{1}_{N-\mu}.$$

Thus, we find

$$\begin{aligned} M &\overset{d}{=} \begin{pmatrix} U \\ \widetilde{W} \\ 0 \end{pmatrix}^* \begin{pmatrix} A-\theta & B^*R^* & B^*S^* \\ RB & H_1-\theta & Z^* \\ SB & Z & H_2-\theta \end{pmatrix}^{-1} \begin{pmatrix} U \\ \widetilde{W} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} Y \\ 0 \end{pmatrix}^* \begin{pmatrix} \tilde{A}-\theta & F^* \\ F & H_2-\theta \end{pmatrix}^{-1} \begin{pmatrix} Y \\ 0 \end{pmatrix} =: \Theta, \end{aligned}$$

where

$$Y := \begin{pmatrix} U \\ \widetilde{W} \end{pmatrix}, \qquad F := (SB, Z), \qquad \tilde{A} := \begin{pmatrix} A & B^*R^* \\ RB & H_1 \end{pmatrix}.$$

Here Y is a $(\mu + r) \times r$ matrix satisfying $Y^*Y = \mathbb{1}_r$, and F is an $(N-\mu-r) \times (\mu + r)$ matrix.

Step 2. We claim that

$$F^*F = \mathbb{1}_{\mu+r} + O(\varphi^C N^{-1/2}) \tag{6.3}$$

with high probability (in the sense of matrix entries). In order to prove (6.3), we write

$$F^*F = \begin{pmatrix} B^*S^*SB & B^*S^*Z \\ Z^*SB & Z^*Z \end{pmatrix}$$

and consider each block separately. For $i \neq j$, we get using (3.14)

$$\begin{aligned} |(B^*S^*SB)_{ij}| &= \left|\sum_{k,l} \overline{B}_{ki}(S^*S)_{kl}B_{lj}\right| \\ &\leq \frac{\varphi^C}{N}\left(\sum_{k,l}|(S^*S)_{kl}|^2\right)^{1/2} = \frac{\varphi^C}{N}(\operatorname{Tr}(S^*S)^2)^{1/2} \leq \varphi^C N^{-1/2} \end{aligned}$$

with high probability. Similarly, (3.12) and (3.13) yield

$$(B^*S^*SB)_{ii} = \sum_k (S^*S)_{kk}|B_{ki}|^2 + \sum_{k\neq l} \overline{B}_{ki}(S^*S)_{kl}B_{li} = 1 + O(\varphi^C N^{-1/2})$$

with high probability, where we used that $N^{-1}\operatorname{Tr} S^*S = 1 - (\mu + r)N^{-1}$. Next, from (3.12), (3.13) and (3.14) we easily get

$$Z^*Z = \mathbb{1}_r + O(\varphi^C N^{-1/2}) \tag{6.4}$$

with high probability. Finally, (3.14) yields

$$|(B^*S^*Z)_{ij}| = \left|\sum_{k,l} \overline{B}_{ki}S^*_{kl}Z_{lj}\right| \leq \frac{\varphi^C}{N}\left(\sum_{k,l}|S^*_{kl}|\right)^{1/2} = \frac{\varphi^C}{N}(\operatorname{Tr} S^*S)^{1/2} \leq \varphi^C N^{-1/2}$$

with high probability. This concludes the proof of (6.3).

Next, we define

$$G_2 := (H_2 - \theta)^{-1}$$

and claim that

$$F^*G_2F = m + O(\varphi^C N^{-1/2}(|d| - 1)^{-1/2}) \tag{6.5}$$

with high probability (in the sense of matrix entries). Since $N^{1/2}(N - \mu - r)^{-1/2}H_2$ is an $(N - \mu - r) \times (N - \mu - r)$ GOE/GUE matrix that is independent of F, (6.5) follows from Theorem 3.3, (3.5), (3.9) and (6.3).

Step 3. For the following, we use the letter $\mathcal{E}$ to denote any (random) error term satisfying $|\mathcal{E}| \leq \varphi^C N^{-1}(|d| - 1)^{-1}$ with high probability for some constant C. We apply Schur's complement formula to get

$$\begin{aligned} \Theta &= Y^*(-\theta - m - (-\tilde{A} + F^*G_2F - m))^{-1}Y \\ &= mY^*Y - m^2Y^*\tilde{A}Y + m^2(Y^*F^*G_2FY - mY^*Y) + \mathcal{E} \\ &= m - m^2Y^*\tilde{A}Y + m^2(Y^*F^*G_2FY - m) + \mathcal{E}, \end{aligned}$$

where in the second step we expanded using (3.2) and estimated the error term using (6.5), $\mu \le \varphi^C$, and $\|\tilde{A}\| \le \varphi^C N^{-1/2}$ with high probability. Using $R^* \widetilde{W} = W$, we get

$$\Theta = m - m^2\bigl(U^*AU + U^*B^*W + W^*BU + \widetilde{W}^*H_1\widetilde{W}\bigr) + m^2 Y^*F^*(G_2 - m)FY + m^3\bigl(Y^*F^*FY - \mathbb{1}\bigr) + \mathcal{E}.$$

Next, we rewrite the term $Y^*F^*(G_2 - m)FY$ so as to decouple the randomness of H_2 from that of F. From (6.3), we find

$$Y^*F^*FY = \mathbb{1}_r + O\bigl(\varphi^C N^{-1/2}\bigr) \tag{6.6}$$

with high probability. Define the deterministic $(N - \mu - r) \times r$ matrix

$$E_1 := \begin{pmatrix} \mathbb{1}_r \\ 0_{(N-\mu-2r)\times r} \end{pmatrix}.$$

Next, we claim that there is a unitary $(N - \mu - r) \times (N - \mu - r)$ matrix O_1, which is F-measurable, such that

$$\|O_1FY - E_1\| \le \varphi^C N^{-1/2} \tag{6.7}$$

with high probability. In order to prove (6.7), write $(\mathbf{x}_1, \dots, \mathbf{x}_r) := FY$. Then (6.6) simply states that the vectors $\mathbf{x}_1, \dots, \mathbf{x}_r$ form a basis of an r-dimensional subspace, which is orthonormal up to errors of order $\varphi^C N^{-1/2}$ with high probability. More precisely, we choose a unitary matrix U_1 such that $U_1\mathbf{x}_1$ lies in the direction of $\mathbf{e}_1$. Hence, by (6.6), we have $U_1FY = (\mathbf{e}_1, U_1\mathbf{x}_2, \dots, U_1\mathbf{x}_r) + O(\varphi^C N^{-1/2})$ with high probability. Note moreover that by (6.6) we have $\langle \mathbf{e}_1, U_1\mathbf{x}_i\rangle = O(\varphi^C N^{-1/2})$ with high probability for $i \ge 2$. Next, we choose a unitary matrix U_2 that leaves $\mathbf{e}_1$ invariant and maximizes $\langle \mathbf{e}_2, U_2U_1\mathbf{x}_2\rangle$. Hence, again by (6.6), we have $U_2U_1FY = (\mathbf{e}_1, \mathbf{e}_2, U_2U_1\mathbf{e}_3, \dots, U_2U_1\mathbf{e}_r) + O(\varphi^C N^{-1/2})$ with high probability. We continue in this manner, at the kth step choosing a unitary matrix U_k that leaves $\mathbf{e}_1, \dots, \mathbf{e}_{k-1}$ invariant and maximizes $\langle \mathbf{e}_k, U_k \cdots U_1\mathbf{x}_k\rangle$. Finally, we define $O_1 := U_r \cdots U_1$. By construction, the estimate in (6.7) holds. Moreover, since Y is deterministic, O_1 is clearly F-measurable. This concludes the proof of (6.7).

Using Theorem 3.3 and the fact that F and H_2 are independent, we therefore get from (6.7)

$$(O_1FY)^*(G_2 - m)O_1FY = E_1^*(G_2 - m)E_1 + \mathcal{E}.$$

We conclude that

$$M \overset{d}{=} m - m^2\bigl(U^*AU + U^*B^*W + W^*BU + \widetilde{W}^*H_1\widetilde{W}\bigr) + m^2E_1^*(G_2 - m)E_1 + m^3\bigl(Y^*F^*FY - \mathbb{1}\bigr) + \mathcal{E},$$

where we used that $O_1G_2O_1^* \overset{d}{=} G_2$ and that all terms apart from $m^2E_1^*(G_2 - m)E_1$ are independent of H_2.

Next, we compute

$$\begin{aligned}
Y^*F^*FY \\
&= U^*B^*S^*SBU + U^*B^*S^*Z\widetilde{W} + \widetilde{W}^*Z^*SBU + \widetilde{W}^*Z^*Z\widetilde{W} \\
&= U^*B^*BU - U^*B^*R^*RBU + U^*B^*S^*Z\widetilde{W} + \widetilde{W}^*Z^*SBU + \widetilde{W}^*Z^*Z\widetilde{W} \\
&= U^*B^*BU + U^*B^*S^*Z\widetilde{W} + \widetilde{W}^*Z^*SBU + \widetilde{W}^*Z^*Z\widetilde{W} + O(\varphi^C N^{-1})
\end{aligned}$$

with high probability, where in the last step we used Lemma 3.6 and $\operatorname{Tr}(R^*R)^2 = r$. Using (6.2), we rewrite

$$\begin{aligned}
U^*B^*BU + \widetilde{W}^*Z^*Z\widetilde{W} - \mathbb{1} \\
&= \mathbb{IE}(U^*B^*BU + \widetilde{W}^*Z^*Z\widetilde{W}) - \frac{\mu}{N}U^*U - \frac{\mu + r}{N}\widetilde{W}^*\widetilde{W},
\end{aligned}$$

where we introduced the notation $\mathbb{IE}X := X - \mathbb{E}X$.

Thus, we conclude that

$$M - m \overset{d}{=} \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \mathcal{E}, \tag{6.8}$$

where

$$\begin{aligned}
\Theta_1 &:= m^2 E_1^*(G_2 - m)E_1, \\
\Theta_2 &:= -m^2 U^*AU, \\
\Theta_3 &:= -m^2 \widetilde{W}^*H_1\widetilde{W}, \\
\Theta_4 &:= -m^2(U^*B^*W + W^*BU) \\
&\quad + m^3\mathbb{IE}(U^*B^*BU + U^*B^*S^*Z\widetilde{W} + \widetilde{W}^*Z^*SBU + \widetilde{W}^*Z^*Z\widetilde{W}).
\end{aligned}$$

By definition, the random variables Θ_1, Θ_2, Θ_3 and Θ_4 are independent.

Step 4. We compute the asymptotics of Θ_1, Θ_2, and Θ_3. We begin with Θ_1. We shall apply Proposition 5.2 to the $(N-\mu-r)\times(N-\mu-r)$ Gaussian matrix H_2. Thus, in Proposition 5.2 we replace N with $N-\mu-r$, H with H_2, and $M(\theta) = V^*(H-\theta)^{-1}V$ by $V^*(H_2-\theta)^{-1}V$ with $V := E_1$. Since $\mu + r \le \varphi^C$ we find that $N - \mu - r \asymp N$. We therefore conclude from Proposition 5.2 that

$$N^{1/2}(|d|-1)^{1/2}\Theta_1 \overset{d}{\sim} \frac{1}{|d|^3\sqrt{|d|+1}}\Phi.$$

Here we used (3.3). Recall that Φ is the rescaled GOE/GUE matrix satisfying (5.1).

In order to deal with Θ_2, we introduce, in analogy to V_δ, the matrix $U_\delta = (U_{il}^\delta)$ whose entries are defined by $U_{il}^\delta := U_{il}\mathbf{1}(|U_{il}| > \delta)$. In particular, since $\delta \ge \varphi^{-1} \ge \varphi^{-\rho}$, we have $V_\delta = \binom{U_\delta}{0}$. Writing $\widehat{U}_\delta = (\widehat{U}_{il}^\delta) := U - U_\delta$, we get

$$U^*AU = U_\delta^*AU_\delta + \widehat{U}_\delta^*AU_\delta + U_\delta^*A\widehat{U}_\delta + \widehat{U}_\delta^*A\widehat{U}_\delta.$$

Next, we define the matrices

$$\Psi_1 := \widehat{U}_\delta^* A U_\delta + U_\delta^* A \widehat{U}_\delta, \qquad \Psi_2 := \widehat{U}_\delta^* A \widehat{U}_\delta.$$

Note that, by definition, Ψ_1, Ψ_2 and $U_\delta^* A U_\delta$ are independent. We now compute the covariances of the matrices Ψ_1 and Ψ_2. A simple calculation yields

$$\begin{aligned} N\mathbb{E}(\Psi_1)_{ij}(\Psi_1)_{kl} &= 2\mathcal{T}_{ij,kl}\big(U_\delta^* U_\delta, \widehat{U}_\delta^* \widehat{U}_\delta\big), \\ N\mathbb{E}(\Psi_2)_{ij}(\Psi_2)_{kl} &= \mathcal{T}_{ij,kl}\big(\widehat{U}_\delta^* \widehat{U}_\delta, \widehat{U}_\delta^* \widehat{U}_\delta\big), \end{aligned} \tag{6.9}$$

where we defined

$$\mathcal{T}_{ij,kl}(R,T) := \tfrac{1}{2}\big(R_{il}T_{kj} + R_{kj}T_{il} + \mathbf{1}(\beta = 1)(R_{ik}T_{jl} + R_{jl}T_{ik})\big).$$

For example, let us prove the second identity for the case $\beta = 2$. Using $N\mathbb{E}h_{ab}h_{cd} = \delta_{ad}\delta_{bc}$ we find

$$\begin{aligned} N\mathbb{E}(\Psi_2)_{ij}(\Psi_2)_{kl} &= N\mathbb{E}\sum_{a,b,c,d=1}^{\mu} \widehat{U}_{ia}^{\delta *} h_{ab} \widehat{U}_{bj}^{\delta} \widehat{U}_{kc}^{\delta *} h_{cd} \widehat{U}_{dl}^{\delta} \\ &= \sum_{a,b=1}^{\mu} \widehat{U}_{ia}^{\delta *} \widehat{U}_{bj}^{\delta} \widehat{U}_{kb}^{\delta *} \widehat{U}_{al}^{\delta} = \big(\widehat{U}_\delta^* \widehat{U}_\delta\big)_{il} \big(\widehat{U}_\delta^* \widehat{U}_\delta\big)_{kj}. \end{aligned}$$

The other cases are handled similarly. Moreover, since by definition we have $|\widehat{U}_{il}^{\delta}| \leq \delta \ll 1$, the central limit theorem implies that $N^{1/2}\Psi_1$ and $N^{1/2}\Psi_2$ converge to a Gaussian random matrix. Hence, the asymptotics of Ψ_1 and Ψ_2 are governed entirely by their covariances (6.9).

Similarly, Θ_3 is Gaussian with covariance

$$N\mathbb{E}(\Theta_3)_{ij}(\Theta_3)_{kl} = d^{-4}\mathcal{T}_{ij,kl}\big(W^*W, W^*W\big),$$

where we used (3.3). Using $U_\delta^* A U_\delta = V_\delta^* H V_\delta$, we therefore conclude that

$$\begin{aligned} &N^{1/2}\big(|d|-1\big)^{1/2}(\Theta_1 + \Theta_2 + \Theta_3) \\ &\quad\overset{d}{\sim} \frac{1}{|d|^3\sqrt{|d|+1}}\Phi - N^{1/2}\frac{(|d|-1)^{1/2}}{d^2} V_\delta^* H V_\delta + \Psi_3, \end{aligned} \tag{6.10}$$

where Ψ_3 is Gaussian with covariance

$$\begin{aligned} &\mathbb{E}(\Psi_3)_{ij}(\Psi_3)_{kl} \\ &\quad= \frac{|d|-1}{d^4}\big(2\mathcal{T}_{ij,kl}\big(U_\delta^* U_\delta, \widehat{U}_\delta^* \widehat{U}_\delta\big) + \mathcal{T}_{ij,kl}\big(\widehat{U}_\delta^* \widehat{U}_\delta, \widehat{U}_\delta^* \widehat{U}_\delta\big) \\ &\qquad + \mathcal{T}_{ij,kl}\big(W^*W, W^*W\big)\big). \end{aligned} \tag{6.11}$$

Step 5. Next, we compute the asymptotics of Θ_4. We shall prove that $N^{1/2}(|d|-1)^{1/2}\Theta_4$ is asymptotically Gaussian, and compute its covariance matrix.

Using Lemma 3.6, we find

$$
\begin{aligned}
(B^*S^*SB)_{ij} &= \sum_{k\neq l}(S^*S)_{kl}\overline{B}_{ki}B_{lj} + \sum_k (S^*S)_{kk}\overline{B}_{ki}B_{kj} \\
&= \sum_{k\neq l}(S^*S)_{kl}\overline{B}_{ki}B_{lj} + \sum_k (S^*S)_{kk}\left(\overline{B}_{ki}B_{kj} - \frac{\delta_{ij}}{N}\right) \\
&\quad + \frac{N-\mu-r}{N}\delta_{ij} \\
&= \delta_{ij} + O(\varphi^C N^{-1/2})
\end{aligned}
\tag{6.12}
$$

with high probability. Define the deterministic $(N-\mu-r)\times\mu$ matrix

$$
E_2 := \begin{pmatrix} \mathbb{1}_\mu \\ 0_{(N-2\mu-r)\times\mu} \end{pmatrix}.
$$

Exactly as after (6.12) we find that (6.12) and Gaussian elimination imply that there is a unitary $(N-\mu-r)\times(N-\mu-r)$ matrix O_2, which is B-measurable, such that

$$
\|O_2SB - E_2\| \le \varphi^C N^{-1/2}
$$

with high probability. Thus, we get

$$
\begin{aligned}
\bigl|(\widetilde{W}^*Z^*(O_2SB-E_2)U)_{ij}\bigr| &= \left|\sum_k \widetilde{W}^*_{ik}\sum_l ((O_2SB-E_2)U)_{lj}\overline{Z}_{lk}\right| \\
&\le \varphi^C N^{-1/2}\bigl(U^*(O_2SB-E_2)^*(O_2SB-E_2)U\bigr)_{ii}^{1/2} \\
&\le \varphi^C N^{-1}
\end{aligned}
$$

with high probability. Using that Z is independent of B and O_2, we therefore find

$$
\begin{aligned}
\Theta_4 \overset{d}{=} &-m^2(U^*B^*W + W^*BU) \\
&+ m^3 \mathbb{IE}\bigl(U^*B^*BU + U^*E_2^*Z\widetilde{W} + \widetilde{W}^*Z^*E_2U + \widetilde{W}^*Z^*Z\widetilde{W}\bigr) + \mathcal{E}.
\end{aligned}
$$

Defining the $(N-\mu-r)\times r$ matrix

$$
\widetilde{U} := E_2U = \begin{pmatrix} U \\ 0_{(N-\mu-2r)\times r} \end{pmatrix},
$$

we therefore have

$$
\Theta_4 \overset{d}{=} \Theta_4' + \Theta_4'' + \mathcal{E},
$$

where

$$\Theta_4' := -m^2(U^*B^*W + W^*BU) + m^3\mathbb{IE}(U^*B^*BU),$$
$$\Theta_4'' := m^3(\widetilde{U}^*Z\widetilde{W} + \widetilde{W}^*Z^*\widetilde{U} + \mathbb{IE}(\widetilde{W}Z^*Z\widetilde{W})).$$

By definition, Θ_4' and Θ_4'' are independent. Recalling that $|W_{il}| \le \varphi^{-\rho}$, we find from the central limit theorem that $N^{1/2}\Theta_4'$ and $N^{1/2}\Theta_4''$ are each asymptotically Gaussian. Hence, it suffices to compute their covariances. A straightforward computation yields

$$N\mathbb{E}(\Theta_4')_{ij}(\Theta_4')_{kl} = 2m^4\mathcal{T}_{ij,kl}(U^*U, W^*W) - m^5\mathcal{Q}_{ij,kl}(U, W) + m^6(\mathcal{T}_{ij,kl}(U^*U, U^*U) + \mathcal{R}_{ij,kl}(U)),$$

where we defined

$$\mathcal{Q}_{ij,kl}(U, W) := N^{-1/2}\sum_{a,b}(\overline{U}_{ai}\overline{U}_{ak}U_{al}\mu_{ab}^{(3)}W_{bj} + \overline{W}_{ia}\mu_{ab}^{(3)}U_{bj}\overline{U}_{bk}U_{bl} + \overline{U}_{ak}\overline{U}_{ai}U_{aj}\mu_{ab}^{(3)}W_{bl} + \overline{W}_{ka}\mu_{ab}^{(3)}U_{bl}\overline{U}_{bi}U_{bj}).$$

[By a slight abuse of notation, we write $\mathcal{R}_{ij,kl}(U)$ by identifying U with the $N \times r$ vector $\binom{U}{0}$.]

We may similarly deal with Θ_4''. Using $\widetilde{U}^*\widetilde{U} = U^*U$ and $\widetilde{W}^*\widetilde{W} = W^*W$ we find

$$N\mathbb{E}(\Theta_4'')_{ij}(\Theta_4'')_{kl} = 2m^6\mathcal{T}_{ij,kl}(U^*U, W^*W) + m^6\mathcal{T}_{ij,kl}(W^*W, W^*W).$$

Combining Θ_4' and Θ_4'', and recalling (3.3), we find

$$N\mathbb{E}(\Theta_4)_{ij}(\Theta_4)_{kl} = 2d^{-4}\mathcal{T}_{ij,kl}(U^*U, W^*W) + d^{-5}\mathcal{Q}_{ij,kl}(U, W) + d^{-6}(\Delta_{ij,kl} + \mathcal{R}_{ij,kl}(U)), \tag{6.13}$$

where we used that

$$\mathcal{T}_{ij,kl}(U^*U, U^*U) + \mathcal{T}_{ij,kl}(W^*W, W^*W) + 2\mathcal{T}_{ij,kl}(U^*U, W^*W) = \mathcal{T}_{ij,kl}(\mathbb{1}, \mathbb{1}) = \Delta_{ij,kl}$$

as follows from $W^*W + U^*U = \mathbb{1}$.

Step 6. We may now consider the sum $\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4$. From (6.8), (6.10), (6.11), (6.13), and the definition of $\mathcal{E}$, we get

$$N^{1/2}(|d|-1)^{1/2}(M-m) \overset{d}{\sim} -N^{1/2}(|d|-1)^{1/2}d^{-2}V_\delta^*HV_\delta + \Psi_4,$$

where $\Psi_4 = \Psi_4^*$ is a Gaussian matrix, independent of H, with covariance

$$\mathbb{E}(\Psi_4)_{ij}(\Psi_4)_{kl} = \frac{|d|-1}{d^4}(\Delta_{ij,kl} - \mathcal{P}_{ij,kl}(V_\delta^*V_\delta)) + \frac{|d|-1}{d^5}\mathcal{Q}_{ij,kl}(U, W) + \frac{|d|-1}{d^6}(\Delta_{ij,kl} + \mathcal{R}_{ij,kl}(U)) + \frac{1}{d^6(|d|+1)}\Delta_{ij,kl}.$$

Here we used that

$$
\begin{aligned}
&2\mathcal{T}_{ij,kl}(U_\delta^* U_\delta, \widehat{U}_\delta^* \widehat{U}_\delta) + \mathcal{T}_{ij,kl}(\widehat{U}_\delta^* \widehat{U}_\delta, \widehat{U}_\delta^* \widehat{U}_\delta) \\
&\quad + \mathcal{T}_{ij,kl}(W^* W, W^* W) + 2\mathcal{T}_{ij,kl}(U^* U, W^* W) \\
&\quad = \Delta_{ij,kl} - \mathcal{T}_{ij,kl}(U_\delta^* U_\delta, U_\delta^* U_\delta) = \Delta_{ij,kl} - \mathcal{P}_{ij,kl}(V_\delta^* V_\delta)
\end{aligned}
$$

as follows from the bilinearity of $\mathcal{T}_{ij,kl}(\cdot,\cdot)$ as well as the identities $\mathcal{T}_{ij,kl}(\mathbb{1},\mathbb{1}) = \Delta_{ij,kl}$, $\mathbb{1} = U_\delta^* U_\delta + \widehat{U}_\delta^* \widehat{U}_\delta + W^* W$ and $U_\delta^* U_\delta = V_\delta^* V_\delta$.

Using that U is a $\mu \times r$ matrix with $\mu \le r\varphi^{2\rho}$ and $|W_{il}| \le \varphi^{-\rho}$, we easily find that

$$
\begin{aligned}
\mathcal{Q}_{ij,kl}(U, W) &= \mathcal{Q}_{ij,kl}(V) + O(\varphi^{-\rho}), \\
\mathcal{R}_{ij,kl}(U) &= \mathcal{R}_{ij,kl}(V) + O(\varphi^{-2\rho}).
\end{aligned} \tag{6.14}
$$

Since $\rho \ge 2$, it is not hard to see that the errors on the right-hand side of (6.14) are bounded from above (in the sense of matrices) by the matrix $E_{ij,kl} = \varphi^{-1}\Delta_{ij,kl}$. In particular, from (6.13) we get that the matrix

$$
2d^{-4}\mathcal{T}_{ij,kl}(U^* U, W^* W) + d^{-5}\mathcal{Q}_{ij,kl}(V) + d^{-6}(\Delta_{ij,kl} + \mathcal{R}_{ij,kl}(V)) + E_{ij,kl}
$$

is nonnegative, from which we conclude that the right-hand side of (2.11) is nonnegative. This completes the proof. □

7. The general case. The general case follows from Proposition 6.1 and Green function comparison. The argument is almost identical to that of Section 7.4 in [21], and we only sketch the differences.

Let $H = (N^{-1/2} X_{ij})$ be an arbitrary real symmetric/complex Hermitian Wigner matrix and $(N^{-1/2} Y_{ij})$ a GOE/GUE matrix independent of H. For $\rho > 0$, define the subset

$$
I_\rho := \{ i \in [\![1, N]\!] : |V_{il}| \le \varphi^{-\rho} \text{ for all } l \in [\![1, r]\!] \}.
$$

Define a new Wigner matrix $\widehat{H} = (N^{-1/2} \widehat{X}_{ij})$ through

$$
\widehat{X}_{ij} := \begin{cases} Y_{ij}, & \text{if } i \in I_\rho \text{ and } j \in I_\rho, \\ X_{ij}, & \text{otherwise.} \end{cases}
$$

Thus, $\widehat{H}$ satisfies the assumptions of Proposition 6.1. Let

$$
J_\rho := \{ (i, j) : 1 \le i \le j \le N, i \in I_\rho \text{ and } j \in I_\rho \}.
$$

Choose a bijective map $\phi : J_\rho \to \{1, \dots, |J_\rho|\}$. For $1 \le \tau \le |J_\rho|$ denote by $H_\tau = (h_{ij}^\tau)$ the Hermitian matrix defined by

$$
h_{ij}^\tau := \begin{cases} N^{-1/2} X_{ij}, & \text{if } \phi(i, j) \le \tau \\ N^{-1/2} \widehat{X}_{ij}, & \text{otherwise} \end{cases} \qquad (i \le j).
$$

In particular, $H_0 = \widehat{H}$ and $H_{|J_\rho|} = H$. Let now $(a,b) \in J_\rho$ satisfy $\phi(a,b) = \tau$. We write

$$H_{\tau-1} = Q + N^{-1/2}\big(Y_{ab}E^{(ab)} + \mathbf{1}(a \neq b)Y_{ba}E^{(ba)}\big)$$

and

$$H_\tau = Q + N^{-1/2}\big(X_{ab}E^{(ab)} + \mathbf{1}(a \neq b)X_{ba}E^{(ba)}\big).$$

Here $E^{(ab)}$ denotes the matrix with entries $E^{(ab)}_{ij} := \delta_{ai}\delta_{bj}$. Hence we have $Q_{ab} = Q_{ba} = 0$, and the matrices $H_{\tau-1}$ and H_τ differ only in the entries (a,b) and (b,a).

Next, we introduce the resolvents

$$R(z) := \frac{1}{Q - z}, \qquad S(z) := \frac{1}{H_{\tau-1} - z}, \qquad T(z) := \frac{1}{H_\tau - z}.$$

Let $|d| \geq 1 + \varphi^K N^{-1/2}$. Set $z := \theta(d) + \mathrm{i}N^{-4}$ (as in [21], Section 7.4, we add a small imaginary part to z to ensure weak control on low-probability events) and define

$$x_R := N^{1/2}\big(|d| - 1\big)^{1/2}\big(V^* R(z) V - m(z)\big). \tag{7.1}$$

The quantities x_S and x_T are defined analogously with R replaced by S and T, respectively.

The following estimate is the main comparison estimate. It is very similar to Lemma 7.13 of [21].

LEMMA 7.1. *Provided ρ is a large enough constant, the following holds. Let $f \in C^3(\mathbb{C}^{r\times r})$ be bounded with bounded derivatives and $q \equiv q_N$ be an arbitrary deterministic sequence of $r \times r$ matrices. Then*

$$\begin{aligned}\mathbb{E}f(x_T + q) = \mathbb{E}f(x_R + q) + \sum_{i,j=1}^{r} Z^{(ab)}_{ij}\mathbb{E}\frac{\partial f}{\partial x_{ij}}(x_R + q)\\ + A_{ab} + O\big(\varphi^{-1}\mathcal{E}_{ab}\big),\end{aligned} \tag{7.2}$$

$$\mathbb{E}f(x_S + q) = \mathbb{E}f(x_R + q) + A_{ab} + O\big(\varphi^{-1}\mathcal{E}_{ab}\big), \tag{7.3}$$

where A_{ab} satisfies $|A_{ab}| \leq \varphi^{-1}$,

$$Z^{(ab)}_{ij} := -N^{-1}\big(|d| - 1\big)^{1/2}\big(m^4\mu^{(3)}_{ab}\overline{V}_{ai}V_{bj} + m^4\mu^{(3)}_{ba}\overline{V}_{bi}V_{aj}\big)$$

and

$$\mathcal{E}_{ab} := \sum_{i,j=1}^{r}\sum_{\sigma,\tau=0}^{2} N^{-2+\sigma/2+\tau/2}|V_{ai}|^\sigma|V_{bj}|^\tau + \delta_{ab}\sum_{i=1}^{r}\sum_{\sigma=0}^{2} N^{-1+\sigma/2}|V_{ai}|^\sigma. \tag{7.4}$$

PROOF. The proof follows the proof of Lemma 7.13 of [21] with cosmetic modifications whose details we omit. □

Using Lemma 7.1, we may now complete the proof in the general case. The following proposition is the main result of this section, and is the conclusion of the arguments from Sections 5–7.

PROPOSITION 7.2. *The following holds for large enough K. Let $\theta \equiv \theta(d)$ for some d satisfying $|d| \geq 1 + \varphi^K N^{-1/3}$. Then*

$$N^{1/2}(|d|-1)^{1/2}(M(\theta) - m(\theta)) \overset{d}{\sim} -\frac{N^{1/2}(|d|-1)^{1/2}}{d^2} V_\delta^* H V_\delta - \frac{(|d|-1)^{1/2}\mathcal{S}(V)}{d^4} + \Psi_0,$$

where Ψ_0 is the Gaussian matrix from Proposition 6.1.

PROOF. The proof follows the proof of Theorem 2.14 in Section 7.4 of [21] with cosmetic modifications whose details we omit. The main inputs are Proposition 6.1 and Lemma 7.1. The imaginary part of the spectral parameter $z = \theta(d) + \mathrm{i}N^{-4}$ is easily removed using the estimate $m(z) = -d + O(N^{-3})$. The condition $f \in C^3$ in Lemma 7.1 can be relaxed to $f \in C$ by standard properties of weak convergence of measures. □

8. Conclusion of the proof of Theorems 2.5 and 2.9. We may now conclude the proof of Theorems 2.5 and 2.9. First, we note that Theorem 2.5 is an easy corollary of Theorem 2.9. We focus therefore on the proof of Theorem 2.9.

Fix K to be the constant from Proposition 7.2. Fix $\ell \in [\![1, r]\!]$ and define the subset

$$\Lambda := \{N \in \mathbb{N} : |d_\ell^{(N)}| \geq 1 + \varphi^K N^{-1/3}\}.$$

We assume that Λ is a subsequence (i.e., infinite), for otherwise the claim of Theorem 2.9 is vacuous. For given $s > 0$, we introduce the partition

$$\Lambda = \bigcup_{\gamma,\pi} \Lambda_{\pi,\gamma}(s), \tag{8.1}$$

where the union ranges over subsets π, γ of $[\![1, r]\!]$ satisfying $\ell \in \pi \subset \gamma \subset [\![1, r]\!]$, and

$$\Lambda_{\pi,\gamma}(s) := \{N \in \Lambda : \gamma_N(\ell) = \gamma, \pi_N(\ell, s) = \pi\},$$

where $\pi_N(\ell, s) \equiv \pi(\ell, s)$ and $\gamma_N(\ell) \equiv \gamma(\ell)$ are the subsets from Definitions 2.8 and 4.2.

We shall prove the following result.

PROPOSITION 8.1. *Fix ℓ, π and γ satisfying $\ell \in \pi \subset \gamma \subset [\![1, r]\!]$. Let $\varepsilon > 0$ be given, and let $f_1, \ldots, f_r$ be bounded continuous functions, where f_k is a function* *on $\mathbb{R}^k$ satisfying $\|f_k\|_\infty \le 1$. Then there exist constants N_0 and s_0, both depending on ε and $f_1, \dots, f_r$, such that* (2.19) *holds for all $s \ge s_0$ and all $N \ge N_0$ satisfying $N \in \Lambda_{\pi,\gamma}(s)$.*

Before proving Proposition 8.1, we note that it immediately implies Theorem 2.9, since the partition (8.1) ranges over a finite family containing $O(1)$ elements.

PROOF OF PROPOSITION 8.1. From (4.6), we know that $\theta(\mathcal{B})$ contains with high probability precisely $|\gamma|$ outliers, namely $(\mu_{\alpha(i)})_{i\in\gamma}$. Following (2.14), for $i \in \gamma$ we introduce the rescaled eigenvalues

$$\zeta_i = N^{1/2}\big(|d_\ell| - 1\big)^{-1/2}(\mu_{\alpha(i)} - \theta_\ell).$$

In order to identify the asymptotics of ζ_i, we introduce the $|\gamma| \times |\gamma|$ matrices

$$X \equiv X_N := -N^{1/2}\big(|d_\ell| - 1\big)^{1/2}\big(|d_\ell| + 1\big)\big(M_{[\gamma]}(\theta_\ell) - m(\theta_\ell)\big),$$

$$Y \equiv Y_N := -N^{1/2}\big(|d_\ell| - 1\big)^{1/2}\big(|d_\ell| + 1\big)\big(D_{[\gamma]}^{-1} - d_\ell^{-1}\big).$$

Note that X is random and Y deterministic. From (4.7), (3.3) and (4.21), we get for all $i \in \gamma$ that

$$\big|\zeta_i - \lambda_i(X + Y)\big| \le \varphi^{-1} \tag{8.2}$$

with high probability. By Proposition 7.2 and Remark 2.3, the family $(X_N)_N$ is tight.

By definition of π and Lemma 4.3, if $i \in \pi$ and $j \in \gamma \setminus \pi$ then

$$|d_i - d_j| > sN^{-1/2}\big(|d_\ell| - 1\big)^{-1/2}/2. \tag{8.3}$$

We have the splitting

$$D_{[\gamma]} = D_{[\pi]} \oplus D_{[\gamma\setminus\pi]}.$$

We shall apply perturbation theory to the matrix $X + Y$. In order to do so, we truncate X by defining $X^t := X\mathbf{1}(\|X\| \le t)$ for $t > 0$. Then by tightness of X there exists a $t \equiv t(\varepsilon) > 0$ such that

$$\mathbb{P}\big(X_N \ne X_N^t\big) \le \frac{\varepsilon}{5} \tag{8.4}$$

for all N. For the truncated matrices, we find the spectral gap

$$\begin{aligned} &\operatorname{dist}\big(\sigma\big(X_{[\pi]}^t + Y_{[\pi]}\big), \sigma\big(X_{[\gamma\setminus\pi]}^t + Y_{[\gamma\setminus\pi]}\big)\big) \\ &\qquad \ge \operatorname{dist}\big(\sigma(Y_{[\pi]}), \sigma(Y_{[\gamma\setminus\pi]})\big) - 2t \ge cs - 2t, \end{aligned}$$

where the constant c only depends on Σ in (2.2); here in the last step we used (8.3). Proposition A.1 therefore yields

$$\big|\lambda_i(X^t + Y) - \lambda_i\big(X_{[\pi]}^t + Y_{[\pi]}\big)\big| \le \frac{t^2}{cs - 2t - 2t^2}. \tag{8.5}$$

We conclude that for there exists an s_0 and an N_0, both depending on ε and $f_{|\pi|}$, such that for $s \geq s_0$ and $N \geq N_0$ satisfying $N \in \Lambda_{\pi,\gamma}(s)$ we have

$$\begin{aligned}
&\big|\mathbb{E} f_{|\pi|}\big((\zeta_i)_{i\in\pi}\big) - \mathbb{E} f_{|\pi|}\big(\big(\lambda_i(X_{[\pi]} + Y_{[\pi]})\big)_{i\in\pi}\big)\big| \\
&\quad \leq \big|\mathbb{E} f_{|\pi|}\big((\zeta_i)_{i\in\pi}\big) - \mathbb{E} f_{|\pi|}\big(\big(\lambda_i(X^t_{[\pi]} + Y_{[\pi]})\big)_{i\in\pi}\big)\big| + \frac{\varepsilon}{5} \\
&\quad \leq \big|\mathbb{E} f_{|\pi|}\big((\zeta_i)_{i\in\pi}\big) - \mathbb{E} f_{|\pi|}\big(\big(\lambda_i(X^t + Y)\big)_{i\in\pi}\big)\big| + \frac{2\varepsilon}{5} \\
&\quad \leq \big|\mathbb{E} f_{|\pi|}\big((\zeta_i)_{i\in\pi}\big) - \mathbb{E} f_{|\pi|}\big(\big(\lambda_i(X + Y)\big)_{i\in\pi}\big)\big| + \frac{3\varepsilon}{5} \\
&\quad \leq \frac{4\varepsilon}{5},
\end{aligned}$$

where in the first step we used (8.4), in the second step (8.5) and dominated convergence, in the third step (8.4) again, and in the last step (8.2) and dominated convergence. Proposition 8.1 now follows from Proposition 7.2 applied to the $|\pi| \times |\pi|$ matrix

$$N^{1/2}\big(|d_\ell - 1|\big)^{1/2}\big(M(\theta_\ell) - m(\theta_\ell)\big)_{[\pi]} = -\big(|d_\ell| + 1\big)^{-1} X_{[\pi]}. \qquad \square$$

9. The joint distribution: Proof of Theorem 2.11. In this final section, we extend the arguments of Sections 4–8 to cover the joint distribution of all outliers, and hence prove Theorem 2.11.

We begin by introducing a coarser partition Γ, defined analogously to Π from Definition 2.10, except that $\pi(\ell, s)$ is replaced with $\gamma(\ell)$ from Definition 4.2.

DEFINITION 9.1. Let N and D be given, and fix $K > 0$. We introduce a partition[7] $\Gamma \equiv \Gamma(N, K, D)$ on a subset of $[\![1, r]\!]$, defined as

$$\Gamma := \big\{\gamma(\ell) : \ell \in [\![1, r]\!], |d_\ell| \geq 1 + \varphi^K N^{-1/3}\big\}.$$

We also use the notation $\Gamma = \{\gamma\}_{\gamma\in\Gamma}$.

It is immediate from Definitions 2.10 and 9.1 that $[\Pi] \subset \bigcup_{\gamma\in\Gamma} \gamma$ and that for each $\pi \in \Pi$ there is a (unique) $\gamma \in \Gamma$ such that $\pi \subset \gamma$. In analogy to (2.20), we set for definiteness

$$d_\gamma := \min\{d_i : i \in \gamma\}, \qquad \theta_\gamma := \theta(d_\gamma).$$

Note that for $\pi \in \gamma$ we have

$$\frac{d_\pi}{d_\gamma} = 1 + o(1), \qquad \frac{|d_\pi| - 1}{|d_\gamma| - 1} = 1 + o(1). \tag{9.1}$$

The following result follows from Proposition 4.5 and (4.21).

[7] As in the footnote to Definition 2.10, it is easy to see that Γ is a partition.

PROPOSITION 9.2. *The following holds for large enough K. For any $\gamma \in \Gamma$ and $i \in \gamma$ we have*

$$|\mu_{\alpha(i)} - \lambda_i(\theta_\gamma - (d_\gamma^2 - 1)(M(\theta_\gamma) + D^{-1})_{[\gamma]})| \le \varphi^{-1} N^{-1/2}(|d_\gamma| - 1)^{1/2} \tag{9.2}$$

with high probability.

As in Section 8, we may assume without loss of generality that the partitions Π and Γ are independent of N. [Otherwise partition

$$\mathbb{N} = \bigcup_{\Gamma,\Pi} \Lambda_{\Pi,\Gamma}(s),$$

$$\Lambda_{\Pi,\Gamma}(s) := \{N \in \mathbb{N} : \Gamma(N, K, D) = \Gamma, \Pi(N, K, s, D) = \Pi\}.$$

Since the union is over a finite family of $O(1)$ subsets of $\mathbb{N}$, we may first fix Γ and Π and then restrict ourselves to $N \in \Lambda_{\Gamma,\Pi}(s)$.] As in the proof of Proposition 8.1, we define for each $\pi \in \Gamma$ the $|\pi| \times |\pi|$ matrix

$$X^\pi := -N^{1/2}(|d_\pi| - 1)^{1/2}(|d_\pi| + 1)(M(\theta_\pi) - m(\theta_\pi))_{[\pi]}.$$

The joint distribution of $(X^\pi)_{\pi\in\Pi}$ is described by the following result, which is analogous to Proposition 7.2.

PROPOSITION 9.3. *For large enough K, we have*

$$\bigoplus_{\pi\in\Pi} X^\pi \overset{d}{\sim} \bigoplus_{\pi\in\Pi} (\Upsilon^\pi + \Psi^\pi), \tag{9.3}$$

where Υ^π and Ψ^π were defined in Section 2.4.

We postpone the proof of Proposition 9.3 to the next section, and finish the proof of Theorem 2.11 first. In order to identify the location of ζ_i^π, we invoke Proposition 9.2 and make use of the freedom provided by Lemma 4.6 in order to change the reference point θ_γ at will. Thus, Proposition 9.2 and Lemma 4.6 yield, for any $\pi \in \Pi$, $i \in \pi$, and $\gamma \in \Gamma$ containing π, that

$$\begin{aligned} \zeta_i^\pi &= N^{1/2}(|d_\pi| - 1)^{-1/2}(\mu_{\alpha(i)} - \theta_\pi) \\ &= -N^{1/2}(|d_\pi| - 1)^{1/2}(|d_\pi| + 1)\lambda_i((M(\theta_\pi) + D^{-1})_{[\gamma]}) + O(\varphi^{-1}) \end{aligned} \tag{9.4}$$

with high probability, where we used (4.21), (9.1) and Lemma A.2.

Next, for $\pi \in \Pi$ let $\gamma(\pi)$ denote the unique element of Γ that contains π. For each $\pi \in \Pi$, we introduce the $|\gamma(\pi)| \times |\gamma(\pi)|$ matrices

$$\widetilde{X}^\pi := -N^{1/2}(|d_\pi| - 1)^{1/2}(|d_\pi| + 1)(M(\theta_\pi) - m(\theta_\pi))_{[\gamma(\pi)]},$$

$$\widetilde{Y}^\pi := -N^{1/2}(|d_\pi| - 1)^{1/2}(|d_\pi| + 1)(D^{-1} - d_\pi^{-1})_{[\gamma(\pi)]}.$$

Thus, (9.4) reads

$$\zeta_i^\pi = \lambda_i(\widetilde{X}^\pi + \widetilde{Y}^\pi) + O(\varphi^{-1})$$

with high probability. By Proposition 7.2 and Remark 2.3, $\widetilde{X}^\pi$ is tight (in N). We may now repeat verbatim the truncation and perturbation theory argument from the proof of Proposition 8.1, following (8.3). The conclusion is that there exists an s_0 and an N_0, both depending on ε and $f_{|[\Pi]|}$, such that for $s \geq s_0$ and $N \geq N_0$ we have

$$\left|\mathbb{E} f_{|[\Pi]|}\big((\zeta_i^\pi)_{\pi\in\Pi, i\in\pi}\big) - \mathbb{E} f_{|[\Pi]|}\big((\lambda_i\big[(\widetilde{X}^\pi + \widetilde{Y}^\pi)_{[\pi]}\big])_{\pi\in\Pi, i\in\pi}\big)\right| \leq \frac{\varepsilon}{2}.$$

The claim now follows from Proposition 9.3 and the observation that $(\widetilde{X}^\pi)_{[\pi]} = X^\pi$. This concludes the proof of Theorem 2.11.

9.1. *Proof of Proposition* 9.3. What remains is to prove Proposition 9.3. Clearly, it is a generalization of Proposition 7.2. The proof of Proposition 9.3 relies on the same three-step strategy as that of Proposition 7.2: the Gaussian case, the almost Gaussian case and the general case.

We begin with the Gaussian case (generalization of Section 5).

PROPOSITION 9.4. *Suppose that H is a GOE/GUE matrix. Then for large enough K we have*

$$\bigoplus_{\pi\in\Pi} N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(M(\theta_\pi) - m(\theta_\pi)\big)_{[\pi]} \overset{d}{\sim} \bigoplus_{\pi\in\Pi} \frac{1}{|d_\pi|\sqrt{|d_\pi|+1}}\Phi_\pi;$$

here $(\Phi_\pi)_{\pi\in\Pi}$ is a family of independent Gaussian matrices, where each Φ_π is a $|\pi|\times|\pi|$ matrix whose covariance is given by (5.1).

PROOF. The proof is a straightforward extension of that of Proposition 5.2, and we only indicate the changes. For each argument θ_π, we use Schur's complement formula on the whole block $[\![1, r]\!]$. Thus, instead of (5.8), we get

$$\begin{aligned} &N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(M(\theta_\pi) - m(\theta_\pi)\big) \\ &\quad = d_\pi^{-2} N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(-H_{[1\cdots r]} + W(\theta_\pi)\big) + O\big(\varphi^C N^{-1/2}\big(|d_\pi|-1\big)^{-3/2}\big).\end{aligned}$$

This gives

$$\begin{aligned} &\bigoplus_{\pi\in\Pi} N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(M(\theta_\pi) - m(\theta_\pi)\big)_{[\pi]} \\ &\quad \overset{d}{\sim} \bigoplus_{\pi\in\Pi} d_\pi^{-2} N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(-H_{[1\cdots r]} + W(\theta_\pi)\big)_{[\pi]}, \end{aligned} \tag{9.5}$$

which is the appropriate generalization of (5.9). By definition, $H_{[1\cdots r]}$ is independent of the family of matrices $(W(\theta_\pi))_{\pi\in\Pi}$, and the submatrices $H_{[\pi]}$, $\pi \in \Pi$, are obviously independent. We may now repeat verbatim the proof of (5.10) to get

$$\bigoplus_{\pi\in\Pi} N^{1/2}(|d_\pi|-1)^{1/2} W_{[\pi]}(\theta_\pi) \overset{d}{\sim} \bigoplus_{\pi\in\Pi} \frac{1}{\sqrt{|d_\pi|+1}} \Phi_\pi. \tag{9.6}$$

The claim now follows from (9.5). □

Next, we consider the almost Gaussian case (generalization of Section 6).

PROPOSITION 9.5. *Let $\rho > 0$. Suppose that the Wigner matrix H satisfies*

$$\max_{1\le l\le r} \max\{|V_{il}|, |V_{jl}|\} \le \varphi^{-\rho} \quad \Longrightarrow \quad h_{ij} \text{ is Gaussian.} \tag{9.7}$$

Define $\widetilde{\Upsilon}$ to be the matrix Υ without the shift arising from $\mathcal{S}(V)$, that is, $\widetilde{\Upsilon} = \bigoplus_{\pi\in\Pi} \widetilde{\Upsilon}^\pi$ with

$$\widetilde{\Upsilon}^\pi := (|d_\pi|+1)(|d_\pi|-1)^{1/2}\left(\frac{N^{1/2} V_\delta^* H V_\delta}{d_\pi^2}\right)_{[\pi]}.$$

Then for large enough K we have

$$\bigoplus_{\pi\in\Pi} X^\pi \overset{d}{\sim} \bigoplus_{\pi\in\Pi} (\widetilde{\Upsilon}^\pi + \Psi^\pi). \tag{9.8}$$

PROOF. We start exactly as in the proof of Proposition 6.1. We repeat the steps up to (6.8) verbatim on the family of $r \times r$ matrices $(M(\theta_\pi) - m(\theta_\pi))_{\pi\in\Pi}$, whereby all of the reduction operations are performed simultaneously on each matrix $M(\theta_\pi) - m(\theta_\pi)$. Note that these matrices only differ in the argument θ_π; hence all steps of the reduction (and in particular the quantities O, O_1, U, W, $\widetilde{W}$, A, B, H_0, H_1, Z, etc.) are the same for all matrices $M(\theta_\pi) - m(\theta_\pi)$. We take over the notation from the proof of Proposition 6.1 without further comment. Thus, we are led to the following generalization of (6.8):

$$\bigoplus_{\pi\in\Pi} X^\pi \overset{d}{\sim} \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4' + \Theta_4'', \tag{9.9}$$

where

$$\Theta_1 := \bigoplus_{\pi\in\Pi} \left(-N^{1/2}(|d_\pi|-1)^{1/2}(|d_\pi|+1) d_\pi^{-2} [E_1^*(G_2(\theta_\pi) - m(\theta_\pi))E_1]_{[\pi]}\right),$$

$$\Theta_2 := \bigoplus_{\pi\in\Pi} \left(N^{1/2}(|d_\pi|-1)^{1/2}(|d_\pi|+1) d_\pi^{-2} [U^* A U]_{[\pi]}\right),$$

$$\Theta_3 := \bigoplus_{\pi\in\Pi} \left(N^{1/2}(|d_\pi|-1)^{1/2}(|d_\pi|+1) d_\pi^{-2} [\widetilde{W}^* H_1 \widetilde{W}]_{[\pi]}\right),$$

$$\Theta_4' := \bigoplus_{\pi\in\Pi} \big(N^{1/2}(|d_\pi|-1)^{1/2}(|d_\pi|+1) \times \big[d_\pi^{-2}(U^*B^*W + W^*BU) + d_\pi^{-3}\mathbb{IE}(U^*B^*BU)\big]_{[\pi]}\big),$$

$$\Theta_4'' := \bigoplus_{\pi\in\Pi} \big(N^{1/2}(|d_\pi|-1)^{1/2}(|d_\pi|+1)d_\pi^{-3} \times \big[\widetilde{U}^*Z\widetilde{W} + \widetilde{W}^*Z^*\widetilde{U} + \mathbb{IE}(\widetilde{W}Z^*Z\widetilde{W})\big]_{[\pi]}\big).$$

(We deviate somewhat from the convention of Section 6 in that, unlike there, we include the normalization factor, which depends on π, in the definition of the variables Θ.) By definition, the random matrices Θ_1, Θ_2, Θ_3, Θ_4' and Θ_4'' are independent. They are all block diagonal, and we sometimes use the notation $\Theta_1 = \bigoplus_{\pi\in\Pi}\Theta_1^\pi$, etc., for their blocks. What remains is to identify their individual asymptotic distributions.

The matrix is Θ_1 is easy: from Proposition 9.4 we immediately get

$$\Theta_1 \overset{d}{\sim} \bigoplus_{\pi\in\Pi} \frac{\sqrt{|d_\pi|+1}}{|d_\pi|^3}\Phi_\pi,$$

where $(\Phi_\pi)_{\pi\in\Pi}$ is defined as in Proposition 9.4. The matrix Θ_2 is dealt with in the same way as in the proof of Proposition 6.1; we omit the details. By definition, Θ_3 is Gaussian with mean zero. A short computation yields the covariance

$$\mathbb{E}(\Theta_3^\pi)_{ij}(\Theta_3^{\pi'})_{kl} = \bigg(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^2}\bigg)\mathcal{T}_{ij,kl}(W^*W, W^*W)$$

for $\pi,\pi'\in\Pi$, $i,j\in\pi$ and $k,l\in\pi'$. We may therefore conclude that, similar to (6.10) and (6.11), we have

$$(\Theta_1+\Theta_2+\Theta_3) \overset{d}{\sim} \bigoplus_{\pi\in\Pi}\frac{\sqrt{|d_\pi|+1}}{|d_\pi|^3}\Phi_\pi + \bigoplus_{\pi\in\Pi}\widetilde{\Upsilon}^\pi + \bigoplus_{\pi\in\Pi}\Psi_3^\pi, \tag{9.10}$$

where $\bigoplus_{\pi\in\Pi}\Psi_3^\pi$ is a block diagonal Gaussian matrix with mean zero and covariance

$$\mathbb{E}(\Psi_3^\pi)_{ij}(\Psi_3^{\pi'})_{kl} = \bigg(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^2}\bigg) \times \big(2\mathcal{T}_{ij,kl}(U_\delta^*U_\delta, \widehat{U}_\delta^*\widehat{U}_\delta) + \mathcal{T}_{ij,kl}(\widehat{U}_\delta^*\widehat{U}_\delta, \widehat{U}_\delta^*\widehat{U}_\delta) + \mathcal{T}_{ij,kl}(W^*W, W^*W)\big) \tag{9.11}$$

for $\pi,\pi'\in\Pi$, $i,j\in\pi$ and $k,l\in\pi'$.

Next, we deal with Θ_4' and Θ_4''. By the central limit theorem and the definition of W, as in the proof of Proposition 6.1, both of these matrices are asymptotically Gaussian (with mean zero). The variances may be computed along the same lines as in the proof of Proposition 6.1. The result is, for $\pi, \pi' \in \Pi$, $i, j \in \pi$ and $k, l \in \pi'$,

$$\begin{aligned}
&\mathbb{E}(\Theta_4')_{ij}(\Theta_4')_{kl} \\
&\quad = \biggl(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^2}\biggr) \\
&\qquad \times \biggl(2T_{ij,kl}(U^*U, W^*W) + \frac{1}{d_\pi d_{\pi'}}(\mathcal{T}_{ij,kl}(U^*U, U^*U) + \mathcal{R}_{ij,kl}(U)) \\
&\qquad\quad + \frac{N^{-1/2}}{d_{\pi'}} \sum_{a,b} (\overline{U}_{ai}\overline{U}_{ak}U_{al}\mu_{ab}^{(3)}W_{bj} + \overline{W}_{ia}\mu_{ab}^{(3)}U_{bj}\overline{U}_{bk}U_{bl}) \\
&\qquad\qquad + \frac{N^{-1/2}}{d_\pi} \sum_{a,b} (\overline{U}_{ak}\overline{U}_{ai}U_{aj}\mu_{ab}^{(3)}W_{bl} + \overline{W}_{ka}\mu_{ab}^{(3)}U_{bl}\overline{U}_{bi}U_{bj})\biggr)
\end{aligned}$$

as well as

$$\begin{aligned}
\mathbb{E}(\Theta_4'')_{ij}(\Theta_4'')_{kl} &= \biggl(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^3}\biggr) \\
&\qquad \times (2\mathcal{T}_{ij,kl}(U^*U, W^*W) + \mathcal{T}_{ij,kl}(W^*W, W^*W)).
\end{aligned}$$

Putting everything together, we get

$$\bigoplus_{\pi\in\Pi} X^\pi \overset{d}{\sim} \bigoplus_{\pi\in\Pi} \widetilde{\Upsilon}^\pi + \bigoplus_{\pi\in\Pi} \Psi_4^\pi, \tag{9.12}$$

where $\bigoplus_{\pi\in\Pi} \Psi_4^\pi$ is a Gaussian block diagonal matrix with mean zero that is independent of H, and whose covariance is given by

$$\begin{aligned}
&\mathbb{E}(\Psi_4^\pi)_{ij}(\Psi_4^{\pi'})_{kl} \\
&\quad = \frac{|d_\pi|+1}{d_\pi^2}\delta_{\pi\pi'}\Delta_{ij,kl} + \delta_{\pi\pi'}E_{ij,kl} \\
&\qquad + \biggl(\prod_{p=\pi,\pi'} \frac{(|d_p|-1)^{1/2}(|d_p|+1)}{d_p^2}\biggr) \\
&\qquad\quad \times \biggl(-\mathcal{P}_{ij,kl}(V_\delta^* V_\delta) + \frac{1}{d_\pi d_{\pi'}}\mathcal{R}_{ij,kl}(V) \\
&\qquad\qquad + \frac{N^{-1/2}}{d_{\pi'}} \sum_{a,b} (\overline{U}_{ai}\overline{U}_{ak}U_{al}\mu_{ab}^{(3)}W_{bj} + \overline{W}_{ia}\mu_{ab}^{(3)}U_{bj}\overline{U}_{bk}U_{bl}) \\
&\qquad\qquad + \frac{N^{-1/2}}{d_\pi} \sum_{a,b} (\overline{U}_{ak}\overline{U}_{ai}U_{aj}\mu_{ab}^{(3)}W_{bl} + \overline{W}_{ka}\mu_{ab}^{(3)}U_{bl}\overline{U}_{bi}U_{bj})\biggr).
\end{aligned}$$

Similar to (6.14), we find using the definition of U and W that the two last lines are asymptotic to $\frac{\mathcal{W}_{ij,kl}(V)}{d_{\pi'}} + \frac{\mathcal{W}_{kl,ij}(V)}{d_\pi}$. Thus, we get

$$\bigoplus_{\pi\in\Pi} \Psi_4^\pi \overset{d}{\sim} \bigoplus_{\pi\in\Pi} \Psi^\pi. \tag{9.13}$$

This concludes the proof. □

In order to conclude the proof of Proposition 9.3, we finally consider the general case (generalization of Section 7). As in Proposition 7.2, in the general case we get a deterministic shift $\bigoplus_{\pi\in\Pi} \mathcal{S}^\pi$, where

$$\mathcal{S}^\pi := \frac{(|d_\pi|+1)(|d_\pi|-1)^{1/2}}{d_\pi^4} \mathcal{S}_{[\pi]}(V). \tag{9.14}$$

The proof is similar to those of Lemma 7.1 and Proposition 7.2. We take over the setup and notation from Section 7 up to, but not including, (7.1). For each $\pi \in \Pi$, we define the spectral parameter $z_\pi := \theta_\pi + \mathrm{i}N^{-4}$ and the $|\pi| \times |\pi|$ matrix

$$x_R^\pi := N^{1/2}\big(|d_\pi|-1\big)^{1/2}\big(V^* R(z_\pi) V - m(z_\pi)\big)_{[\pi]}, \tag{9.15}$$

we well as the $|[\Pi]| \times |[\Pi]|$ block diagonal matrix $x_R := \bigoplus_{\pi\in\Pi} x_R^\pi$. The quantities x_S and x_T are defined analogously with R replaced by S and T, respectively. The following is the main comparison estimate, which generalizes Lemma 7.1.

LEMMA 9.6. *Provided ρ is a large enough constant, the following holds. Let $f \in C^3(\mathbb{C}^{|[\Pi]|\times|[\Pi]|})$ be bounded with bounded derivatives and $q \equiv q_N$ be an arbitrary deterministic sequence of $|[\Pi]| \times |[\Pi]|$ matrices. Then*

$$\begin{aligned}\mathbb{E} f(x_T+q) = \mathbb{E} f(x_R+q) &+ \sum_{i,j\in[\Pi]} Z_{ij}^{(ab)} \mathbb{E}\frac{\partial f}{\partial x_{ij}}(x_R+q)\\ &+ A_{ab} + O\big(\varphi^{-1}\mathcal{E}_{ab}\big),\end{aligned} \tag{9.16}$$

$$\mathbb{E} f(x_S+q) = \mathbb{E} f(x_R+q) + A_{ab} + O\big(\varphi^{-1}\mathcal{E}_{ab}\big), \tag{9.17}$$

where A_{ab} satisfies $|A_{ab}| \le \varphi^{-1}$, the error term $\mathcal{E}_{ab}$ is defined in (7.4), and $Z^{(ab)}$ is the $|[\Pi]| \times |[\Pi]|$ block diagonal matrix $Z^{(ab)} := \bigoplus_{\pi\in\Pi} Z^{(ab),\pi}$ with $|\pi| \times |\pi|$ blocks

$$Z_{ij}^{(ab),\pi} := -N^{-1}\big(|d_\pi|-1\big)^{1/2}\big(m(z_\pi)^4 \mu_{ab}^{(3)} \overline{V}_{ai} V_{bj} + m(z_\pi)^4 \mu_{ba}^{(3)} \overline{V}_{bi} V_{aj}\big) \qquad (i,j\in\pi).$$

PROOF. The proof of Lemma 7.1 may be taken over almost verbatim, following the proof of Lemma 7.13 of [21]. □

The comparison estimate from Lemma 9.6 yields the shift described by $\mathcal{S}$. The precise statement is given by the following proposition, which generalizes Proposition 7.2.

PROPOSITION 9.7. *For large enough K, we have*

$$\bigoplus_{\pi\in\Pi} X^\pi \overset{d}{\sim} \bigoplus_{\pi\in\Pi} (\widetilde{\Upsilon}^\pi + \Psi^\pi + \mathcal{S}^\pi),$$

where $\mathcal{S}^\pi$ was defined in (9.14).

PROOF. As in the proof of Proposition 7.2, we follow the proof of Theorem 2.14 in Section 7.4 of [21]. The inputs are Proposition 9.5 and Lemma 9.6. □

Now Proposition 9.3 follows immediately from Proposition 9.7 using $\Upsilon^\pi = \widetilde{\Upsilon}^\pi + \mathcal{S}^\pi$. This concludes the proof of Proposition 9.3.

APPENDIX: NEAR-DEGENERATE PERTURBATIONS

In this appendix, we record some basic results on the perturbation of near-degenerate spectra.

PROPOSITION A.1. *Let A and B be nonzero Hermitian matrices on $\mathbb{C}^N$. Let $n+m=N$, so that $\mathbb{C}^N = \mathbb{C}^n \oplus \mathbb{C}^m$, and assume that A and B are of the form*

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}$$

(*in self-explanatory notation*). *Define the spectral gap*

$$\Delta := \operatorname{dist}\bigl(\sigma(A_{11}), \sigma(A_{22})\bigr)$$

and assume that $\Delta \geq 3\|B\|$.

Define the domain

$$\mathcal{D} := \bigl\{\mu \in \mathbb{C} : \operatorname{dist}\bigl(\mu, \sigma(A_{11})\bigr) < 2\|B\|\bigr\}.$$

Then $A+B$ has exactly n eigenvalues $\mu_1 \leq \cdots \leq \mu_n$ in $\mathcal{D}$ (*counted with multiplicity*), *which satisfy*

$$\bigl|\mu_i - \lambda_i(A_{11})\bigr| \leq \frac{\|B\|^2}{\Delta - 2\|B\|} \qquad (i = 1, \dots, n).$$

PROOF. The eigenvalue–eigenvector equation reads $(A+B)\mathbf{x} = \mu\mathbf{x}$. Writing $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^n \oplus \mathbb{C}^m$ leads to the system

$$\begin{aligned} A_{11}\mathbf{x}_1 + B_{12}\mathbf{x}_2 &= \mu\mathbf{x}_1, \\ A_{22}\mathbf{x}_2 + B_{21}\mathbf{x}_1 &= \mu\mathbf{x}_2. \end{aligned} \tag{A.1}$$

By assumption, for $\mu \in \mathcal{D}$ we have

$$\operatorname{dist}\bigl(\mu, \sigma(A_{22})\bigr) \geq \Delta - 2\|B\|. \tag{A.2}$$

Since $\Delta - 2\|B\| \geq \|B\| > 0$, we find that (A.1) is equivalent to the system

$$\mathbf{x}_2 = -(A_{22} - \mu)^{-1} B_{21}\mathbf{x}_1, \qquad A_{11}\mathbf{x}_1 - \mu\mathbf{x}_1 - B_{12}(A_{22} - \mu)^{-1} B_{21}\mathbf{x}_1 = 0.$$

Replacing B with tB for $t \in [0, 1]$, we conclude that for $\mu \in \mathcal{D}$ we have the equivalence

$$\mu \in \sigma(A + tB) \quad \Longleftrightarrow \quad f_t(\mu) = 0,$$

where

$$f_t(\mu) := \det\bigl(A_{11} - \mu - t^2 B_{12}(A_{22} - \mu)^{-1} B_{21}\bigr).$$

Moreover, from Lemma A.2 below we find that $\mathcal{D}$ contains exactly n eigenvalues of $A + tB$, for all $t \in [0, 1]$. It is well known that the eigenvalues $\mu_i(t)$ of $A + tB$ are continuous in t. We now claim that each such continuous $\mu_i(t)$ is in fact Lipschitz continuous with Lipschitz constant

$$L := \frac{\|B\|^2}{\Delta - 2\|B\|}.$$

Assuming this is proved, the claim immediately follows from $|\mu_i - \lambda_i| = |\mu_i(1) - \mu_i(0)| \leq L$.

In order to prove the Lipschitz continuity of $\mu_i(t)$, note that $\mu_i(t)$ is an eigenvalue of the matrix

$$X_i(t) := A_{11} - t^2 B_{12}\bigl(A_{22} - \mu_i(t)\bigr)^{-1} B_{21}.$$

Then the Lipschitz continuity of $\mu_i(t)$ follows readily from Lemma A.2 below and the estimate

$$\bigl\|B_{12}\bigl(A_{22} - \mu_i(t)\bigr)^{-1} B_{21}\bigr\| \leq L$$

as follows from (A.2), the fact that $\mu_i(t) \in \mathcal{D}$ for all $t \in [0, 1]$, and the fact that A_{22} is Hermitian. □

LEMMA A.2. *Let A and B be square matrices, with A Hermitian. Then the spectrum of $A + B$ is contained in the closed $\|B\|$-neighborhood of the spectrum of A.*

PROOF. Using the identity $(A + B - z)^{-1} = (A - z)^{-1}(1 + B(A - z)^{-1})^{-1}$ we conclude that if $\operatorname{dist}(z, \sigma(A)) > \|B\|$ then $z \notin \sigma(A + B)$. □

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DEPARTMENT OF MATHEMATICS
ETH ZÜRICH
RÄMISTRASSE 101
8092 ZÜRICH
SWITZERLAND
E-MAIL: knowles@math.ethz.ch

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706
USA
E-MAIL: jyin@math.uwisc.edu