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# NO ZERO-CROSSINGS FOR RANDOM POLYNOMIALS AND THE HEAT EQUATION

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Consider random polynomial $\sum_{i=0}^{n} a_i x^i$ of independent mean-zero normal coefficients $a_i$, whose variance is a regularly varying function (in $i$) of order $\alpha$. We derive general criteria for continuity of persistence exponents for centered Gaussian processes, and use these to show that such polynomial has no roots in $[0, 1]$ with probability $n^{-b_\alpha+o(1)}$, and no roots in $(1, \infty)$ with probability $n^{-b_0+o(1)}$, hence for $n$ even, it has no real roots with probability $n^{-2b_\alpha-2b_0+o(1)}$. Here, $b_\alpha = 0$ when $\alpha \le -1$ and otherwise $b_\alpha \in (0, \infty)$ is independent of the detailed regularly varying variance function and corresponds to persistence probabilities for an explicit stationary Gaussian process of smooth sample path. Further, making precise the solution $\phi_d(\mathbf{x}, t)$ to the $d$-dimensional heat equation initiated by a Gaussian white noise $\phi_d(\mathbf{x}, 0)$, we confirm that the probability of $\phi_d(\mathbf{x}, t) \neq 0$ for all $t \in [1, T]$, is $T^{-b_\alpha+o(1)}$, for $\alpha = d/2 - 1$.

## 1. Introduction.

Algebraic polynomials of the form

$$Q_n(x) = \sum_{i=0}^{n} a_i x^i \tag{1.1}$$

with $x \in \mathbb{R}$ and independent, zero-mean random coefficients $a_i$ are objects of much interest in probability theory. In particular, for i.i.d. normal $\{a_i\}$, the number $N_n$ of real roots has been studied in some detail, starting with Littlewood and Offord work [13–15] that provides upper and lower bounds on $E_n = \mathbb{E}[N_n]$ as well as on both tails of the law of $N_n$. Among its consequences is the upper bound $\mathbb{P}(N_n = 0) = O(\frac{1}{\log n})$, much refined in [5], which proved that for $n$ even $\mathbb{P}(N_n = 0) = n^{-4b_0+o(1)}$ decays polynomially and that the same positive, finite, power exponent $b_0$ applies for any i.i.d. $\{a_i\}$ of finite moments of all orders.

In another direction, Kac [10] provides an explicit formula for $E_n$ in case of i.i.d. normal $\{a_i\}$, yielding also the sharp asymptotics $E_n \sim \frac{2}{\pi} \log n$, whereas [17] shows that $N_n$ is asymptotically normal of mean $E_n$ and $\mathrm{Var}(N_n) \sim \frac{4}{\pi}(1 - \frac{2}{\pi}) \log n$. Most of these results extend to other distributions of the i.i.d. $\{a_i\}$ (see the historical

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account in [5], Section 2). We also note in passing the rich asymptotic theory for location of *complex* zeros of $z \mapsto Q_n(z)$ and related random analytic functions (cf. [8, 9] and the references therein).

Our focus here is on persistence probabilities

$$p_J(n) = \mathbb{P}\big(Q_n(x) < 0,\ \forall x \in J\big). \tag{1.2}$$

Such probabilities have been extensively studied, for other stochastic processes, also in reliability theory and in the physics literature, cf. the surveys [2, 16] and references therein. Specifically, we study the asymptotics of $p_J(n)$ for $J = [0, 1]$, $J = (1, \infty)$, $J = [0, \infty)$ and $J = \mathbb{R}$, where $\{a_i\}$ are independent, centered normal with $\mathbb{E}(a_0^2) = 1$ and $i \mapsto \mathbb{E}(a_i^2) = i^\alpha L(i)$ forms a regularly varying sequence of order $\alpha$, at $i \to \infty$. Equivalently, we consider any $i \mapsto L(i)$ slowly varying at infinity (namely, such that $L([\mu i])/L(i) \to 1$ when $i \to \infty$, for any fixed $\mu > 0$, cf. [3]). To this end, deriving in Theorem 1.6 a new, general flexible criteria for continuity of persistence probability tail exponential rates, we show in Theorem 1.3 that for *any* slowly varying $L(\cdot)$,

$$p_{[0,1]}(n) = n^{-b_\alpha + o(1)}, \qquad p_{(1,\infty)}(n) = n^{-b_0 + o(1)},$$
$$p_{[0,\infty)}(n) = n^{-b_\alpha - b_0 + o(1)}.$$

Subject to a mild regularity condition on $L(2k)/L(2k + 1)$, we further deduce that $p_{\mathbb{R}}(2n) = n^{-2b_\alpha - 2b_0 + o(1)}$ [clearly, $p_{\mathbb{R}}(2n + 1) = 0$ and we note in passing that $\mathbb{P}(N_n = 0) = 2p_{\mathbb{R}}(n)$].

The power exponent $b_\alpha$ is thus universal, that is, independent of the specific slowly varying function $L(\cdot)$, and the asymptotics of $p_{(1,\infty)}(n)$ is further independent of the order $\alpha$ of the regularly varying variance of $a_i$ (as already noted in [20] for the case of $L(\cdot) \equiv 1$).

1.1. *Nonzero crossings for random polynomials.* Hereafter, let $F(s, t) := \operatorname{sech}((t - s)/2)$, $\{\widehat{Z}_t, t \geq 0\}$ denote the centered stationary Gaussian process of covariance function $\exp\{-(t - s)^2/8)\}$ and for each $\alpha > -1$, consider the centered Gaussian process

$$Y_t^{(\alpha)} = \frac{\int_0^\infty g_t(r)\, dW_r}{(\int_0^\infty g_t(r)^2\, dr)^{1/2}}, \tag{1.3}$$

where $g_t(r) := r^{\alpha/2} \exp(-e^{-t} r)$ (see [5], (1.4), for $\alpha = 0$). We start with some preliminary facts about these processes and their persistence exponents.

LEMMA 1.1. *For any $\alpha > -1$, the $C^\infty(\mathbb{R})$-valued stochastic process $t \mapsto Y_t^{(\alpha)}$ of* (1.3) *has covariance function $F(s, t)^{\alpha+1}$. Further, its persistence exponent*

$$b_\alpha := -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Y_t^{(\alpha)} \leq \delta_T\Big), \tag{1.4}$$

*exists and is independent of the precise choice of* $\delta_T \to 0$. *These persistence exponents are such that the nonincreasing* $(\alpha+1)^{-1} b_\alpha \uparrow 1/2$ *when* $\alpha \downarrow -1$ *and the nondecreasing* $(\alpha+1)^{-1/2} b_\alpha \uparrow \hat{b}_\infty$ *when* $\alpha \uparrow \infty$, *where* $\hat{b}_\infty$ *denotes the finite persistence exponent of* $\{\widehat{Z}_t\}$.

REMARK 1.2. Accurate numerical values are known for some values of $b_\alpha$ (see [20] and references therein), but no analytic prediction for it has ever been given. The best rigorously proved lower and upper bounds at $\alpha = 0$ are $b_0 \in (1/(4\sqrt{3}), 1/4]$, derived in [18], Proposition 2 and [11], Theorem 3.2, respectively. From Lemma 1.1, we have that $b_\alpha$ is between $\sqrt{\alpha+1}\, b_0$ and $(\alpha+1) b_0$. Hence, $b_\alpha \in (0, \infty)$ admits the corresponding lower and upper bounds. It further has linear asymptotics at $\alpha \downarrow -1$ and square-root growth for $\alpha \to \infty$, thereby confirming the predictions of [20].

Here is our first main result.

THEOREM 1.3. *Consider random algebraic polynomials* $Q_n(\cdot)$ *of independent, centered normal coefficients* $\{a_i\}$ *such that* $\mathbb{E}[a_0^2] = 1$ *and let* $L(i) := i^{-\alpha} \mathbb{E}[a_i^2]$, $i \geq 1$, *for some* $\alpha \in \mathbb{R}$.

(a) *Setting hereafter* $b_\alpha \equiv 0$ *when* $\alpha \leq -1$ *and* $T_n := \log n$, *we have that for any slowly varying sequence* $L(\cdot)$,

$$\lim_{n\to\infty} \frac{1}{T_n} \log p_{[0,1]}(n) = -b_\alpha, \tag{1.5}$$

$$\lim_{n\to\infty} \frac{1}{T_n} \log p_{(1,\infty)}(n) = -b_0, \tag{1.6}$$

$$\lim_{n\to\infty} \frac{1}{T_n} \log p_{[0,\infty)}(n) = -b_\alpha - b_0. \tag{1.7}$$

(b) *If in addition*

$$\lim_{n\to\infty} n \left| \frac{L(n+1)}{L(n)} - 1 \right| = 0, \tag{1.8}$$

*then further*,

$$\lim_{n\to\infty} \frac{1}{T_n} \log p_{\mathbb{R}}(2n) = -2b_\alpha - 2b_0. \tag{1.9}$$

REMARK 1.4. The rate condition (1.8) is the discrete version of the condition $x \frac{d}{dx}(\log L(x)) \to 0$ as $x \to \infty$. For example, (1.8) holds when $L(x) = (\log x)^\gamma$, for any $\gamma \in \mathbb{R}$, or when $L(x) = \exp\{(\log x)^\lambda\}$ for any $|\lambda| < 1$, but fails in case of the slowly varying $L(n) = 1 + n^{-1}(1 + (-1)^n)$.

1.2. *Heat equation initiated by white noise.* Setting $K_t(\mathbf{x}) := (4\pi t)^{-d/2} \times \exp\{-\frac{\|\mathbf{x}\|_2^2}{4t}\}$, recall that for any *smooth enough* $\psi(\cdot)$, the function

$$\phi_d(\mathbf{x}, t) = \int_{\mathbb{R}^d} K_t(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\, d\mathbf{y} \tag{1.10}$$

is a classical solution of the $d$-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x}, t)}{\partial t} = \Delta \phi_d(\mathbf{x}, t) \tag{1.11}$$

on $\mathbb{D}_0 = \mathbb{R}^d \times (0, \infty)$ with initial condition $\phi_d(\cdot, 0) = \psi(\cdot)$. It is *formally* argued in [20] that taking for $\psi(\cdot)$ a centered Gaussian field of covariance $\delta_d(\mathbf{x} - \mathbf{y})$, should yield by (1.10) a centered Gaussian field $\phi_d(\mathbf{x}, t)$ with covariance $\mathbb{E}[\phi_d(\mathbf{x}_1, t)\phi_d(\mathbf{x}_2, s)] = K_{t+s}(\mathbf{x}_1 - \mathbf{x}_2)$. Assuming the existence of such a process, it would have for each fixed $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x} \in \mathbb{R}^d$, the time covariance $K_{t+s}(\mathbf{0})$. Thus, taking $\alpha = d/2 - 1$, it follows that

$$\phi_d(\mathbf{x}, e^t) \stackrel{\mathcal{L}}{=} \sqrt{K_{2e^t}(\mathbf{0})}\, Y_t^{(\alpha)}$$

for $\{Y_t^{(\alpha)}\}$ of Lemma 1.1. Consequently,

$$\lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}(\phi_d(\mathbf{x}, t) \neq 0,\ \forall t \in [1, T]) = -b_\alpha, \tag{1.12}$$

$$\lim_{R\to\infty} \frac{1}{R} \log \mathbb{P}(\phi_1(x, 1) \neq 0,\ \forall |x| \le R/2) = -\hat{b}_\infty \tag{1.13}$$

for $b_\alpha$ of (1.4) and $\hat{b}_\infty$ of Lemma 1.1. That is, the seemingly unrelated random polynomials $\{Q_n(x)_{x\in[0,1]}\}$ have the same persistence power exponent $b_\alpha$ as these solutions $\{\phi_{2(\alpha+1)}(\mathbf{x}, t)_{t\in[1,T]}\}$ of the heat equation.

While on a set of full measure the random function $\mathbf{x} \mapsto \psi(\mathbf{x})$ is not Lebesgue measurable [hence the integral (1.10) ill-defined], we make precise the notion of solution $\phi_d(\mathbf{x}, t) \in \mathcal{C}^\infty(\mathbb{D}_0)$ of (1.11) such that $\phi_d(\mathbf{x}, t)$ is a centered Gaussian field of covariance $K_{t+s}(\mathbf{x}_1 - \mathbf{x}_2)$. (Added in galleys: after our article was accepted for publication we realized that this is already done in Section 8 of [4].) Of course, upon rigorously constructing such a field we immediately get the confirmation of both (1.12) and (1.13).

THEOREM 1.5. *Equip $\mathcal{C}_0 = \mathcal{C}^{2,1}(\mathbb{D}_0)$ with the topology of uniform convergence on compacts of function and its relevant partial derivatives of first and second order. There exists a $(\mathcal{C}_0, \mathcal{B}_{\mathcal{C}_0})$-valued, centered Gaussian field $\phi_d(\mathbf{x}, t)$ of covariance function $C((\mathbf{x}_1, t), (\mathbf{x}_2, s)) = K_{s+t}(\mathbf{x}_1 - \mathbf{x}_2)$, which satisfies (1.11) on $\mathbb{D}_0$. Further, $\phi_d \in \mathcal{C}^\infty(\mathbb{D}_0)$ and for any $0 < t_1 < t_2$,*

$$\phi_d(\mathbf{x}, t_2) = \int_{\mathbb{R}^d} K_{t_2 - t_1}(\mathbf{x} - \mathbf{y})\phi_d(\mathbf{y}, t_1)\, d\mathbf{y}. \tag{1.14}$$

1.3. *Continuity of persistence exponents for Gaussian processes.* The motivation for this work lies in the prediction of [19, 20] for much of our results, but the persistence asymptotics of Theorem 1.3 has been rigorously derived before only for i.i.d. $\{a_i\}$ [namely, $\alpha = 0$ and $L(\cdot) \equiv 1$], where [5] relies on an explicitly simple closed form of $\mathrm{Cov}(Q_n(x), Q_n(y))$ for handling this case. In contrast, no such closed form expression exist for $\alpha \neq 0$ and especially for $L(\cdot) \not\equiv 1$, henceforth requiring a more delicate treatment of the covariance in various domains of $x$, $y$, to which much of our effort is devoted.

Indeed, beware that the convergence of covariance functions for smooth centered Gaussian processes [such as $Q_n(\cdot)$], while implying weak convergence of the corresponding laws, falls short of relating their large deviations (and in particular the relevant persistence power exponents). For example, with $Z$ standard normal independent of $\{Y_\cdot^{(\alpha)}\}$, the positive autocorrelation of the smooth, stationary, centered Gaussian process $\sqrt{1-\epsilon_n}Y_\cdot^{(\alpha)} + \sqrt{\epsilon_n}Z$ is within $\epsilon_n \to 0$ of the autocorrelation of $\{Y_\cdot^{(\alpha)}\}$ but for $\epsilon_n \log n \to \infty$, the corresponding persistence exponent is easily shown to be $0 \neq b_\alpha$. Our second main result shows that in contrast, persistence power exponent is continuous for any collection of centered Gaussian processes whose maxima over compact intervals converge *pointwise*, *arbitrarily slowly*, to those of the limit process [see (1.17) below], provided their nonnegative auto-correlations satisfy a mild uniform integrability condition [see (1.15)], and the persistence exponent of the limiting process is somewhat stable [see (1.16)].

THEOREM 1.6. *Let $\mathcal{S}$ denote the class of all stationary, autocorrelation functions $A : [0,\infty) \mapsto [-1,1]$ with $\mathcal{S}_+$ denoting the subset of nonnegative $A \in \mathcal{S}$. For centered stationary Gaussian process $\{Z_t\}_{t\geq 0}$ of autocorrelation $A(s,t) = A(0,t-s) \in \mathcal{S}_+$, the nonnegative, possibly infinite, limit*

$$b(A) := -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Z_t < 0\Big),$$

*exists. Consider centered Gaussian processes $\{Z_t^{(k)}\}_{t\geq 0}$, $1 \leq k \leq \infty$ (normalized to have $\mathbb{E}[(Z_t^{(k)})^2] = 1$), of nonnegative autocorrelations $A_k(s,t)$, such that $A_\infty(s,t) \in \mathcal{S}_+$. Suppose that the following three conditions hold*:

$$\limsup_{k,\tau\to\infty} \sup_{s\geq 0} \left\{ \frac{\log A_k(s,s+\tau)}{\log \tau} \right\} < -1. \tag{1.15}$$

$$\limsup_{M\to\infty} \frac{1}{M} \log \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < M^{-\eta}\Big) = -b(A_\infty) \qquad \forall \eta > 0 \tag{1.16}$$

*and there exist $\zeta > 0$ and $M_1 < \infty$ such that for any $z \in [0,\zeta]$ and $M \geq M_1$*,

$$\begin{aligned} \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < z\Big) &\leq \liminf_{k\to\infty} \inf_{s\geq 0} \mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big) \\ &\leq \limsup_{k\to\infty} \sup_{s\geq 0} \mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big) \\ &\leq \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} \leq z\Big). \end{aligned} \tag{1.17}$$

*Then*

$$\lim_{k,T\to\infty}\frac{1}{T}\log\mathbb{P}\Big(\sup_{t\in[0,T]} Z_t^{(k)}<0\Big)=-b(A_\infty). \tag{1.18}$$

REMARK 1.7. Theorem 1.6 only requires that (1.17) holds for $z=0$ and $z_M=CM^{-\eta}\downarrow 0$. Further, its proof applies even when $A_k(\cdot,\cdot)$ and $Z_t^{(k)}$ are defined only on $[0,T_k^\star]$, for some given $T_k^\star\to\infty$, with the conclusion (1.18) valid then for any unbounded $T_k\le T_k^\star$. We also note in passing that when dealing with *stationary* $A_k\in\mathcal{S}_+$ for all $k$ large enough, it suffices to consider only $s=0$ in (1.15) and (1.17), with (1.18) implying in particular that, in such setting,

$$\lim_{k\to\infty} b(A_k)=b(A_\infty). \tag{1.19}$$

The first of the three conditions of Theorem 1.6, namely (1.15), is usually easy to check. Its second condition, (1.16), is relatively mild, and in particular applies whenever $Z_t^{(\infty)}$ of continuous sample path has decreasing autocorrelation $A_\infty(0,t)$ such that

$$a_{h,\theta}^2:=\inf_{0<t\le h}\left\{\frac{A_\infty(0,\theta t)-A_\infty(0,t)}{1-A_\infty(0,t)}\right\}>0 \tag{1.20}$$

for any finite $h>0$ and $\theta\in(0,1)$ (see [12], Theorem 3.1(iii), and its proof).

Our next lemma provides explicit sufficient conditions that yield the last condition, (1.17), of Theorem 1.6 [and which we utilize when proving Lemma 1.1 and part (a) of Theorem 1.3].

LEMMA 1.8. *Condition* (1.17) *holds if to $D\in\mathcal{S}$ corresponds a Gaussian process of continuous sample paths and for any finite $M$ there exist positive $\epsilon_k\to 0$ such that whenever $\tau\in[0,M]$ (and $s\in[0,T_k^\star]$),*

$$\begin{aligned}(1-\epsilon_k)A_\infty(0,\tau)+\epsilon_k D(0,\tau)&\le A_k(s,s+\tau)\\ &\le(1-\epsilon_k)A_\infty(0,\tau)+\epsilon_k.\end{aligned} \tag{1.21}$$

*Alternatively, setting $p_k^2(u):=2-2\inf_{s\ge 0,\tau\in[0,u]}A_k(s,s+\tau)$, if $A_k(s,s+\tau)\to A_\infty(0,\tau)$ pointwise and*

$$\lim_{\delta\downarrow 0}\sup_{1\le k\le\infty}\int_0^\infty[p_k(e^{-v^2})\wedge\delta]\,dv=0, \tag{1.22}$$

*then the corresponding laws of $\{Z_{s+\cdot}^{(k)}:s\ge 0,1\le k\le\infty\}$ are uniformly tight with respect to supremum norm on $\mathcal{C}[0,M]$, which for $A_k\in\mathcal{S}$ implies that* (1.17) *holds for any $z\in\mathbb{R}$.*

For example, by dominated convergence, (1.22) holds whenever for some $\eta>1$,

$$\limsup_{u\downarrow 0}|\log u|^\eta\sup_{1\le k\le\infty}\{p_k^2(u)\}<\infty. \tag{1.23}$$

REMARK 1.9. To demonstrate the flexibility of our approach, we utilize Remark 1.7 to confirm the persistence exponent values predicted by [20] for the so called Binomial random polynomials. That is, with $\hat{b}_\infty$ as in Lemma 1.1, if $\mathbb{E}[a_i^2] = \frac{n}{i}$ for $i = 0, \dots, n$, then

$$\lim_{n\to\infty} n^{-1/2} \log p_{[0,\infty)}(n) = -\pi \hat{b}_\infty, \tag{1.24}$$

$$\lim_{n\to\infty} (2n)^{-1/2} \log p_{\mathbb{R}}(2n) = -2\pi \hat{b}_\infty. \tag{1.25}$$

Indeed, the parameterization $x := \tan(s/(2\sqrt{n}))$, with $s \in [0, \pi\sqrt{n})$ for $x \in \mathbb{R}_+$ and $s \in (-\pi\sqrt{n}, \pi\sqrt{n})$ in case $x \in \mathbb{R}$, translates the Binomial random polynomials, into stationary, centered Gaussian processes whose autocorrelations

$$A_n(s,t) := \left[\cos\left(\frac{t-s}{2\sqrt{n}}\right)\right]^n$$

are nonnegative when either $s, t \in [0, \pi\sqrt{n})$ or $n$ is even. Recall that the continuous, symmetric function $f(u) := u^2/2 + \log\cos(u)$ on $|u| \le \pi/2$, decreases in $u \ge 0$; hence $A_n(0,\tau) \uparrow e^{-\tau^2/8} := A_\infty(0,\tau)$ as $n \to \infty$, per fixed $\tau \in \mathbb{R}$ [out of which uniform super-exponential decay in $\tau$, hence condition (1.15) follows]. With $A_\infty(0,\tau) \in \mathcal{S}_+$ both (1.24) and (1.25) are specializations to this context of conclusion (1.19) of Theorem 1.6, so it remains only to verify that (1.20) and (1.23) hold here. Now, condition (1.20) holds, for example, by [12], Remark 3.1, whereas (1.23) holds since $p_n^2(u) \le p_2^2(u) \le u^2/4$ for all $n \ge 2$ and $u$.

1.4. *Theorem 1.3: Proof outline and extensions.* We proceed to outline the intuition, following [5] and [20], which governs our proof of Theorem 1.3. First, since $x \mapsto Q_n(x)$ is continuous, for $x \in [0,1]$ not too close to 1, the sign of $Q_n(x)$ can be controlled by the value of $Q_n(0)$; hence, the asymptotics of $p_{[0,1]}(n)$ is dominated by the behavior of $Q_n(x)$ for $x \approx 1$. To handle the latter, setting $x = e^{-u}$ allows for approximating

$$\operatorname{Cov}\big(Q_n(e^{-u}), Q_n(e^{-v})\big) = 1 + \sum_{i=1}^{n} L(i) i^\alpha e^{-i(u+v)} := h_{\alpha,n}(u+v) \tag{1.26}$$

for $\alpha > -1$ and small, but not too small values of $u, v$ [namely, in range of $(w_\ell, w_h)$, for $nw_\ell \to \infty$ and $w_h \to 0$], by

$$\int_0^\infty L(r) r^\alpha e^{-r(u+v)}\,dr \sim \Gamma(\alpha+1)(u+v)^{-(\alpha+1)} L\left(\frac{1}{u+v}\right).$$

The correlation between $Q_n(e^{-u})$ and $Q_n(e^{-v})$ is then approximately $S(u,v) \times R(u,v)^{\alpha+1}$ where

$$R(u,v) := \frac{2\sqrt{uv}}{u+v}, \qquad S(u,v) := \frac{L(1/(u+v))}{\sqrt{L(1/(2u))L(1/(2v))}} \tag{1.27}$$

and for small $u, v$ the slowly varying nature of $L(\cdot)$ at infinity implies that $S(u, v)$ is nearly one. Consequently, replacing $S(u, v)$ by 1, upon setting $s := -\log u$ and $t := -\log v$ we arrive at the correlation between $Y_t^{(\alpha)}$ and $Y_s^{(\alpha)}$ with relevant range $t, s \in [\delta T_n, (1-\delta)T_n]$ (for $w_\ell = n^{-(1-\delta)}$ and $w_h = n^{-\delta}$), yielding the persistence power exponent $b_\alpha$ of (1.4). On a more technical note, as long as the ratio $u/v$ is bounded, we have indeed that $S(u, v) \approx 1$ for any slowly varying $L(\cdot)$, but the supremum of $u/v$ over the domain of $(u, v)$ relevant to the asymptotics of $p_{[0,1]}(n)$ is $O(n)$, requiring us to rely on Theorem 1.6.

Similarly, the main contribution to $p_{(1,\infty)}(n)$ comes from $x \approx 1$. However, setting $x = e^u$, even at the relevant range of small $u, v \in (n^{-(1-\delta)}, n^{-\delta})$, here the large values of $i$ dominate the covariance function of $Q_n(e^u)$ resulting, for any $\alpha \in \mathbb{R}$, with

$$\operatorname{Cov}\bigl(Q_n(e^u), Q_n(e^v)\bigr) = 1 + \sum_{i=1}^{n} L(i) i^\alpha e^{i(u+v)} \sim (u+v)^{-1} L(n) n^\alpha e^{n(u+v)}.$$

The limiting correlation is now approximately independent of $\alpha$ and $L(\cdot)$, given for $s = -\log u$ and $t = -\log v$ by $R(u, v) = F(s, t)$ [we note in passing that for $\alpha < -1$ this approximation breaks down at $C(\alpha)\log n/n$, a threshold which $w_\ell$ must thus exceed, causing further technical challenge, as seen in proof of Lemma 3.1].

Finally, part (b) of Theorem 1.3 then follows upon showing that, for even values of $n$, the events of having $Q_n(x)$ negative throughout each of the four intervals $\pm[0, 1]$ and $\pm(1, \infty)$, are approximately independent of each other [with (1.8) utilized for controlling the dependence between $Q_n(x)$ and $Q_n(-x)$].

REMARK 1.10. We show, in part (b) of Lemma 3.1, that the sequence $n \mapsto p_{[0,1]}(n)$ is bounded away from zero whenever $\sum_i L(i) i^\alpha$ converges (in particular, for any $\alpha < -1$). Things are more involved when $\alpha = -1$, as it is easy to check that for $L(x) = (\log x)^\gamma$, $\gamma \geq 0$ and $n$ large $h_{-1,n}(e^{-t} + e^{-s}) = (\gamma+1)^{-1}[\min(t, s)]^{\gamma+1}[1 + O(1/\min(t, s))]$ when $t, s \in [1, \log n]$. Hence, for the relevant (large) values of $t$, the asymptotic autocorrelation of $Q_n(e^{-e^{-t}})$ is that of Brownian motion, raised to power $\gamma + 1$, suggesting that in this case $p_{[0,1]}(n) = (\log n)^{-(\gamma+1)/2+o(1)}$ is sensitive to the choice of $L(\cdot)$. The lower bound of (4.16) may be improved to $(|\log v|/|\log u|)^r$, yielding the persistence lower bound $(\log n)^{-(\gamma+1)+o(1)}$ [by the same reasoning as in proof of (4.18)].

REMARK 1.11. As we briefly outline next, Theorem 1.6 can also deal with the main contribution to persistence probabilities for Weyl random polynomials. Namely, the case of $\mathbb{E}[a_i^2] = 1/i!$, $i \geq 0$ and intervals $\overline{J} = [0, \sqrt{n} - \Gamma_n]$ with $\Gamma_n \to \infty$. In this setting, we have that

$$h_n(st) := \operatorname{Cov}\bigl(Q_n(s), Q_n(t)\bigr) = \sum_{i=0}^{n} \frac{(st)^i}{i!} \sim e^{st}$$

for $s,t \in \overline{J}$, with uniform relative error $\eta_n := 1 - e^{-z}h_n(z) = \mathbb{P}(N_z > n)$, where $N_z$ denotes a Poisson random variable of parameter $z = n - \sqrt{n}\Gamma_n$. Considering $A_n(s,t) := \operatorname{corr}(Q_n(s), Q_n(t))$ and $A_\infty(s,t) = e^{-(t-s)^2/2}$, this yields the bound (1.21) for $D(s,t) = A_\infty(s,t)^2$, some $\epsilon_n \to 0$ and all $s,t \in \overline{J}$, so from Lemma 1.8 we have that (1.17) holds when $s \in \overline{J}$. The covariance estimate further implies that $A_n(s,t) \le 4A_\infty(s,t)$ for all $s,t \in \overline{J}$ and $n$ large enough, from which (1.15) follows. We have seen already that (1.16) holds for $\widehat{Z}_{2t}$ (see Remark 1.9), so taking $n^{-1/2}\Gamma_n \to 0$ we deduce from Theorem 1.6 that

$$\lim_{n\to\infty} n^{-1/2} \log p_{\overline{J}}(n) = -2\hat{b}_\infty$$

as predicted in [20]. The upper bound $p_{\mathbb{R}_+}(n) \le \exp(-2\hat{b}_\infty n^{1/2}(1+o(1)))$ follows and to confirm, as predicted there, that it is sharp, one needs only to show that $n^{-1/2}\log p_{[\sqrt{n}-\Gamma_n,\infty)}(n) \to 0$.

REMARK 1.12. While we do not pursue this here, by a strong approximation argument like the one done in [5], the conclusions of Theorem 1.3 should extend to nonnormal $\{a_i\}$ with all moments finite.

REMARK 1.13. Changing from mean-zero coefficients to regularly varying negative mean of order $\alpha_\star$ can alter persistence power exponents associated with $Q_n(\cdot)$, depending on the relation between $\alpha$ and $\alpha_\star$. Indeed, setting $\mathbb{E}[a_i] = -i^{\alpha_\star}L_\star(i)$ for some $\alpha_\star \in \mathbb{R}$, some slowly varying $L_\star(\cdot)$ and all $i \ge 1$, results with $\mathbb{E}[Q_n(e^{-u})]$ having the same form as $-h_{\alpha_\star,n}(u)$ in the regime of small, but not too small values of $u$ of relevance here. The relevant persistence power exponent is thus reduced, or eliminated all together, when $h_{\alpha_\star,n}(u) \gg \sqrt{h_{\alpha,n}(2u)}$ and expected to remain intact when $h_{\alpha_\star,n}(u) \ll \sqrt{h_{\alpha,n}(2u)}$. The same applies for the persistence power exponents associated with the neighborhood of $-1$, except for $\mathbb{E}[Q_n(-e^{-u})]$ having the form of $h_{\alpha_\star-1,n}(u)$, due to cancellations between mean values for even coefficients and those for odd coefficients. For example, $p_{[0,1]}(n) = n^{-o(1)}$ even for $\alpha > -1$ as soon as $(\alpha_\star + 1) > (\alpha+1)/2$, whereas for $p_{[-1,0]}(n)$ this requires $\alpha_\star > (\alpha+1)/2$. Similarly, we get the prediction $p_{(1,\infty)}(n) = n^{-\lambda b_0}$ when $\alpha_\star = (\alpha - \lambda)/2$ for $\lambda \in [0,1]$ [and upon reducing $\alpha_\star$ by one, same applies for $p_{(-\infty,-1)}(n)$]. We prove none of these predictions, but note in passing their agreement in case $\alpha_\star = \alpha = 0$ with the rigorous analysis of [5].

We prove Theorem 1.6, Lemmas 1.1 and 1.8 in Section 2, Theorem 1.3 in Section 3 and Theorem 1.5 in Section 5, devoting Section 4 to proofs of the auxiliary lemmas we use for proving Theorem 1.3.

## 2. Proofs of Lemma 1.1, Theorem 1.6 and Lemma 1.8.

2.1. *Proof of Theorem* 1.6. By subadditivity lemma, the existence of the limit $b(A)$ follows from Slepian's inequality (see [1], Theorem 2.2.1), and nonnegativity of the autocorrelation $A \in \mathcal{S}_+$.

Considering (1.17) for $z = 0$ and fixed $M$ large enough, there exist $\xi_k \downarrow 0$ such that for all $k$,

$$\inf_{s\geq 0} \mathbb{P}\Big(\sup_{t\in[0,M]} Z^{(k)}_{s+t} < 0\Big) \geq \mathbb{P}\Big(\sup_{t\in[0,M]} Z^{(\infty)}_t < 0\Big) - \xi_k.$$

Thus, by Slepian's inequality and the nonnegativity of $A_k(\cdot,\cdot)$, we conclude that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Z^{(k)}_t < 0\Big) \geq \Big[\mathbb{P}\Big(\sup_{t\in[0,M]} Z^{(\infty)}_t < 0\Big) - \xi_k\Big]^{\lceil T/M\rceil},$$

which upon taking log, dividing by $T$ and letting $k, T \to \infty$ gives

$$\liminf_{k,T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Z^{(k)}_t < 0\Big) \geq \frac{1}{M} \log \mathbb{P}\Big(\sup_{t\in[0,M]} Z^{(\infty)}_t < 0\Big).$$

So, considering $M \to \infty$ completes the proof of the lower bound in (1.18).

To get the matching upper bound, note that by (1.15), there exist $\eta > 1$ and $M_0$ finite, such that for all large $k$ and any $s, t$,

$$A_k(s,t) \leq M_0^{\eta} |t-s|^{-\eta}. \tag{2.1}$$

For such $\eta$ and $M_0$, set $0 < \delta < (1-\eta^{-1})/2$ small enough for

$$4(M_0\delta)^{\eta} \sum_{i=1}^{\infty} i^{-\eta} < 1. \tag{2.2}$$

Next, fixing finite $M$ large enough for $\gamma := (M\delta^2)^{-\eta} \leq 3/4$, let $s_i = (1+\delta)Mi$, $i \geq 1$, and consider the $\delta M$-separated intervals $I_i := [s_i - M, s_i]$. Since $|s-t| \geq \delta M|i-j|$ whenever $s \in I_i$, $t \in I_j$, it follows from (2.1) that then $A_k(s,t) \leq \gamma (M_0\delta)^{\eta} |i-j|^{-\eta}$. Thus, setting $I(t) := i$ for $t \in I_i$ we have that for any $s, t \in \bigcup_i I_i$,

$$A_k(s,t) \leq (1-\gamma) A_k(s,t) 1_{\{I(s)=I(t)\}} + \gamma B\big(I(s), I(t)\big), \tag{2.3}$$

where $B(i,i) = 1$ and $B(i,j) := (M_0\delta)^{\eta}|i-j|^{-\eta}$ for $i \neq j$. Setting $N := \lfloor T/(M(1+\delta))\rfloor$ and

$$\mathcal{J}_T := \bigcup_{i=1}^{N} I_i \subset [0,T],$$

it follows from (2.2) and the Gershgorin circle theorem, that all the eigenvalues of the symmetric $N$-dimensional matrix $\mathbf{B} = \{B(i,j)\}_{i,j=1}^{N}$ lie within $[1/2, 3/2]$.

In particular, $\mathbf{B}$ is positive definite and the RHS of (2.3) is the autocorrelation of the centered Gaussian process $\sqrt{1-\gamma}\overline{Z}_t^{(k)} + \sqrt{\gamma}X_{I(t)}$ on $\mathcal{J}_T$, where the centered, stationary, Gaussian sequence $\{X_i\}_{i=1}^{\infty}$ of autocorrelation $B(i,j)$, is independent of the mutually independent restrictions of $\overline{Z}_t^{(k)}$ to intervals $I_i$, having the same law as $Z_t^{(k)}$ within each $I_i$. Thus, by Slepian's inequality for some $\xi_k \downarrow 0$, any $k$ large enough and all $T$,

$$
\begin{aligned}
\mathbb{P}\Bigl(\sup_{t\in[0,T]} Z_t^{(k)} < 0\Bigr) &\le \mathbb{P}\Bigl(\sup_{t\in\mathcal{J}_T} Z_t^{(k)} < 0\Bigr) \\
&\le \mathbb{P}\Bigl(\sup_{t\in[0,T]}\{\sqrt{1-\gamma}\overline{Z}_t^{(k)} + \sqrt{\gamma}X_{I(t)}\} < 0\Bigr) \\
&= \mathbb{E}\Biggl[\prod_{i=1}^{N}\mathbb{P}\Bigl(\sup_{t\in I_i} Z_t^{(k)} \le -\frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}X_i \Big| \mathbf{X}\Bigr)\Biggr] \\
&\le \mathbb{E}\prod_{i=1}^{N}\Bigl[\mathbb{P}\Bigl(\sup_{t\in I_i} Z_t^{(k)} < 2\gamma^{\delta}\Bigr) + 1_{\{X_i\le -\gamma^{\delta-1/2}\}}\Bigr] \\
&\le \mathbb{E}\prod_{i=1}^{N}\Bigl[\mathbb{P}\Bigl(\sup_{t\in[0,M]} Z_t^{(\infty)} \le 2\gamma^{\delta}\Bigr) + \xi_k + 1_{\{X_i\le -\gamma^{\delta-1/2}\}}\Bigr], \qquad (2.4)
\end{aligned}
$$

where in the last inequality we use (1.17) for $z = 2\gamma^{\delta} \le \zeta$ (provided $M$ is large enough). Since $B(i,j)$ is nonincreasing in $|i-j|$, by Slepian's inequality the last term is in turn further bounded above by

$$
\sum_{j=0}^{N}\binom{N}{j}\Bigl(\mathbb{P}\Bigl(\sup_{t\in[0,M]} Z_t^{(\infty)} < 3\gamma^{\delta}\Bigr) + \xi_k\Bigr)^{N-j}\mathbb{P}(X_i \ge \gamma^{\delta-1/2}, 1\le i\le j). \qquad (2.5)
$$

Proceeding to bound $\mathbb{P}(X_i \ge \gamma^{\delta-1/2}, 1\le i\le j)$, recall that all eigenvalues of $\mathbf{B}$ lie within $[1/2, 3/2]$, and so the quadratic form $\mathbf{x}'\mathbf{B}^{-1}\mathbf{x}$ is bounded bellow by $\frac{2}{3}\|\mathbf{x}\|_2^2$, yielding the bound

$$
\begin{aligned}
\mathbb{P}\bigl(X_i \ge \gamma^{\delta-1/2}, 1\le i\le j\bigr) &= \det(\mathbf{B})^{-1/2}(2\pi)^{-j/2}\int_{[\gamma^{\delta-1/2},\infty)^j} e^{-(1/2)\mathbf{x}'\mathbf{B}^{-1}\mathbf{x}}\,d\mathbf{x} \\
&\le \frac{2^{j/2}}{(2\pi)^{j/2}}\int_{[\gamma^{\delta-1/2},\infty)^j} e^{-1/3\|\mathbf{x}\|_2^2}\,d\mathbf{x} \\
&= 3^{j/2}\mathbb{P}\bigl(X_1 \ge \sqrt{2/3}\gamma^{\delta-1/2}\bigr)^j.
\end{aligned}
$$

Combining this with (2.4) and (2.5), we deduce that

$$
\mathbb{P}\Bigl(\sup_{t\in[0,T]} Z_t^{(k)} < 0\Bigr) \le \Bigl[\mathbb{P}\Bigl(\sup_{t\in[0,M]} Z_t^{(\infty)} < 3\gamma^{\delta}\Bigr) + \xi_k + \sqrt{3}\mathbb{P}\bigl(X_1 \ge \sqrt{2/3}\gamma^{\delta-1/2}\bigr)\Bigr]^N.
$$

Considering $T^{-1}$ log of this inequality in the limit $T, k \to \infty$ results with

$$\limsup_{k,T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{t\in[0,T]} Z_t^{(k)} < 0\Big)$$
$$\le \frac{1}{M(1+\delta)} \log\Big[\mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < 3\gamma^{\delta}\Big) + \sqrt{3}\mathbb{P}\big(X_1 \ge \sqrt{2/3}\gamma^{\delta-1/2}\big)\Big]. \tag{2.6}$$

Next, note that with $X_1$ a standard normal variable and $\eta(1-2\delta) > 1$,

$$\limsup_{M\to\infty} \frac{1}{M} \log \mathbb{P}\big(X_1 \ge \sqrt{2/3}\gamma^{\delta-1/2}\big) \le -\liminf_{M\to\infty} \big(3M\gamma^{1-2\delta}\big)^{-1} = -\infty,$$

whereas by (1.16) we have

$$\limsup_{M\to\infty} \frac{1}{M} \log \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < 3\gamma^{\delta}\Big) = -b(A_\infty).$$

Thus, considering the RHS of (2.6) as $M \to \infty$, then $\delta \downarrow 0$, yields the upper bound in (1.18).

2.2. *Proof of Lemma* 1.8. Let $V_t$ denote the stationary, centered Gaussian process of auto-correlation $D(\cdot,\cdot) \in \mathcal{S}$. Assuming without loss of generality that $\epsilon_k \in [0, 3/4]$ (so $1 - \sqrt{1-\epsilon_k} \le \sqrt{\epsilon_k} \wedge 1/2$), per fixed $M$ and $z$, by Slepian's inequality and the LHS of (1.21), for any $s \ge 0$ and $k$,

$$\mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big)$$
$$\ge \mathbb{P}\Big(\sup_{t\in[0,M]} \{\sqrt{1-\epsilon_k} Z_t^{(\infty)} + \sqrt{\epsilon_k} V_t\} < z\Big)$$
$$\ge \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < z - 2\epsilon_k^{1/4}\Big) - \mathbb{P}\Big(\sup_{t\in[0,M]} V_t \ge \epsilon_k^{-1/4} - |z|\Big).$$

By sample path continuity, $\sup_{t\in[0,M]} V_t$ is finite almost surely, so with $\epsilon_k \to 0$ it follows from the preceding that for any $z$ and $M$ finite,

$$\liminf_{k\to\infty} \inf_{s\ge 0} \mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big) \ge \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < z\Big).$$

Similarly, from the RHS of (1.21) we have that for any $s \ge 0$ and $k$,

$$\mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big)$$
$$\le \mathbb{P}\Big(\sup_{t\in[0,M]} \{\sqrt{1-\epsilon_k} Z_t^{(\infty)} + \sqrt{\epsilon_k} X_1\} < z\Big)$$
$$\le \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} < z + 2\epsilon_k^{1/4}\Big) + \mathbb{P}\big(X_1 \le -\epsilon_k^{-1/4} + |z|\big),$$

hence for any $z$ and $M$ finite,

$$\limsup_{k\to\infty}\sup_{s\geq 0}\mathbb{P}\Big(\sup_{t\in[0,M]} Z_{s+t}^{(k)} < z\Big) \leq \mathbb{P}\Big(\sup_{t\in[0,M]} Z_t^{(\infty)} \leq z\Big).$$

Turning to the second part of the lemma, recall [1], Theorem 1.4.1, that for some universal constant $C$ and all $s$, $M$, $k$ and $\delta > 0$,

$$\mathbb{E}\Big[\sup_{|t-t'|\leq\delta, t,t'\leq M} \big|Z_{s+t}^{(k)} - Z_{s+t'}^{(k)}\big|\Big] \leq C\int_0^\infty \big[p_k(e^{-v^2}) \wedge \delta\big]\,dv$$

(using integration by parts, one easily confirms that the preceeding is equivalent to [1], (1.4.5)). Thus, as $Z_s^{(k)}$ has a standard normal law, for any $k$, the condition (1.22) guarantees (by an application of Arzela–Ascoli theorem), the stated uniform tightness of the laws of $Z_{s+\cdot}^{(k)}$ on $\mathcal{C}[0, M]$. As such, by Prohorov's theorem it is a precompact collection of laws (with respect to weak convergence on $\mathcal{C}[0, M]$). Clearly, pointwise convergence of $A_k(s, s+\tau)$ to $A_\infty(0, \tau)$ implies, per fixed $s$ and finite $M$, convergence as $k \to \infty$ of the f.d.d. of $Z_{s+\cdot}^{(k)}$ on $[0, M]$ to those of $Z_\cdot^{(\infty)}$. In combination with the preceding precompactness, this verifies the convergence of $Z_{s+\cdot}^{(k)}$ to $Z_\cdot^{(\infty)}$ in distribution on $\mathcal{C}[0, M]$ (per $s$ and $M$). The convergence in law of $\sup_{t\in[0,M]} Z_{s+t}^{(k)}$ to $\sup_{t\in[0,M]} Z_t^{(\infty)}$ which follows (by continuity of $z_\cdot \mapsto \sup_{t\in[0,M]} z_t$ on $\mathcal{C}[0, M]$), implies, by definition, the validity of (1.17) in case $A_k \in \mathcal{S}$ (where such convergence is by default uniform in $s$).

2.3. *Proof of Lemma* 1.1. The centered Gaussian process $Y_t^{(\alpha)}$ of (1.3) is well defined [since the nonrandom, nonzero $g_t \in L_2(\mathbb{R}_+)$ for all $t \in \mathbb{R}$ and $\alpha > -1$]. Further, since $\|g_t\|_2 = e^{t(\alpha+1)/2}\|g_0\|_2$ and

$$(g_t, g_s) := \int_0^\infty g_t(r)g_s(r)\,dr = \left(\frac{e^{-t}+e^{-s}}{2}\right)^{-(\alpha+1)}\|g_0\|_2^2,$$

it follows that

$$\operatorname{Cov}\big(Y_t^{(\alpha)}, Y_s^{(\alpha)}\big) = \frac{(g_t, g_s)}{\|g_t\|_2\|g_s\|_2} = \left[\operatorname{sech}\left(\frac{t-s}{2}\right)\right]^{\alpha+1},$$

so $\{Y_t^{(\alpha)}, t \in \mathbb{R}\}$ is stationary and of the specified nonnegative covariance function. Next, since

$$\hat{g}_t(r) := \frac{g_t(r)}{\|g_t\|_2} = \frac{r^{\alpha/2}}{\|g_0\|_2}\exp\big(-t(\alpha+1)/2 - e^{-t}r\big),$$

is infinitely differentiable in $t$ with $\|\frac{d^k\hat{g}_t}{dt^k}\|_2$ finite for all $k \in \mathbb{N}$, the sample functions $t \mapsto Y_t^{(\alpha)} = \int_0^\infty \hat{g}_t(r)\,dW_r$ of (1.3) are $\mathcal{C}^\infty(\mathbb{R})$-valued.

The limit (1.4) for $\delta_T \equiv 0$ is merely $b(F^{\alpha+1})$ for covariance function $F^{\alpha+1} \in \mathcal{S}_+$. Further, with $\tau \mapsto \rho_\alpha(\tau) := [\operatorname{sech}(\tau/2)]^{\alpha+1}$ decreasing and satisfying the condition of [12], Remark 3.1, it follows from [12], Theorem 3.1(iii), that (1.4) extends to any $\delta_T \to 0$.

By yet another application of Slepian's inequality, the stated monotonicity properties of $\alpha \mapsto b_\alpha$ are immediate consequence of the monotonicity of $\alpha \mapsto \rho_\alpha(\tau/(\alpha+1))$ and $\alpha \mapsto \rho_\alpha(\tau/\sqrt{\alpha+1})$, per fixed $\tau$. Applying the monotone transformation $-\log(\cdot)$ to these two functions of $\alpha+1$ and setting $f(u) := \log\cosh(u)$, the preceding is in turn equivalent to $u \mapsto u^{-1}f(u)$ nondecreasing and $u \mapsto u^{-2}f(u)$ nonincreasing on $(0,\infty)$. The former holds since

$$\psi_1(u) := u^2\big(u^{-1}f(u)\big)' = uf'(u) - f(u)$$

is such that $\psi_1'(u) = uf''(u) = u\,\mathrm{sech}^2(u) \geq 0$, hence $u \mapsto \psi_1(u)$ is nondecreasing, starting at $\psi_1(0) = -f(0) = 0$. So, necessarily both $\psi_1(u)$ and $u^{-2}\psi_1(u) = (u^{-1}f(u))'$ are nonnegative for $u > 0$, from which it follows that $u^{-1}f(u)$ is nondecreasing. Similarly, setting

$$\psi_2(u) := u^3\big(u^{-2}f(u)\big)' = uf'(u) - 2f(u)$$

and noting that $f'(0) = \tanh(0) = 0$, results with

$$\psi_2'(u) = uf''(u) - f'(u) = \int_0^u \big(f''(u) - f''(r)\big)\,dr \leq 0,$$

due to the monotonicity of $f''(u) = \mathrm{sech}^2(u)$. So, with $u \mapsto \psi_2(u)$ nonincreasing on $(0,\infty)$ and starting at $\psi_2(0) = -2f(0) = 0$, we deduce that $\psi_2(u) \leq 0$, and hence also $u^{-3}\psi_2(u) = (u^{-2}f(u))' \leq 0$, as claimed.

With $u^{-1}f(u) \uparrow 1$ as $u \uparrow \infty$, when $\alpha \downarrow -1$ the autocorrelation $\widetilde{A}_\alpha(0,\tau) := \rho_\alpha(|\tau|/(\alpha+1))$ of $Y^{(\alpha)}_{t/(\alpha+1)}$ converges downward to the autocorrelation function $\widetilde{A}_{-1}(0,\tau) := \exp(-|\tau|/2)$ of the standard, stationary Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$, whose persistence exponent is $1/2$ (cf. [5], Lemma 2.5). In view of (1.4) and Slepian's inequality, this results with

$$(\alpha+1)^{-1}b_\alpha = b(\widetilde{A}_\alpha) \leq b(\widetilde{A}_{-1}) = 1/2,$$

whereas the convergence of $b(\widetilde{A}_\alpha)$ to $b(\widetilde{A}_{-1})$ is established by applying Theorem 1.6, as in (1.19). Indeed, condition (1.15) of the theorem holds since $\widetilde{A}_\alpha(0,\tau) \leq \widetilde{A}_0(0,\tau) = \rho_0(\tau)$ decays exponentially in $\tau$, uniformly in $\alpha \leq 0$, while by Lemma 1.8, condition (1.17) holds for all $z \in \mathbb{R}$ since in this setting $p_\alpha^2(u) = 2(1 - \widetilde{A}_\alpha(0,u)) \leq 2(1 - e^{-u/2}) \leq u$ satisfies (1.23), and the limiting Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$ of continuous sample path satisfies condition (1.16) since, for example, it satisfies (1.20) by [12], Remark 3.1.

Similarly, since $u^{-2}f(u) \uparrow 1/2$ for $u \downarrow 0$, the correlation functions $\widehat{A}_\alpha(0,\tau) := \rho_\alpha(|\tau|/\sqrt{\alpha+1})$ of $Y^{(\alpha)}_{t/\sqrt{\alpha+1}}$, $\alpha > -1$, converge downward to $\widehat{A}_\infty(0,\tau) := \exp(-\tau^2/8)$ when $\alpha \uparrow \infty$. Consequently, $\widehat{A}_\infty \in \mathcal{S}_+$ is the covariance function of some centered, stationary Gaussian process $\{\widehat{Z}_t, t \geq 0\}$, having nonnegative persistence exponent $\hat{b}_\infty := b(\widehat{A}_\infty)$. By Slepian's inequality and (1.4),

$$(\alpha+1)^{-1/2} b_\alpha = b(\widehat{A}_\alpha) \leq b(\widehat{A}_\infty) = \hat{b}_\infty$$

and $b(\widehat{A}_\alpha) \to b(\widehat{A}_\infty)$ as a consequence of applying Theorem 1.6 for $\widehat{A}_\alpha \in \mathcal{S}_+$. Indeed, in this setting we have the uniform (over $\alpha \geq 0$), exponential decay of $\widehat{A}_\alpha(0,\tau) \leq \rho_0(\tau)$, condition (1.23) of Lemma 1.8 holds as $p_\alpha^2(u) = 2(1 - \widehat{A}_\alpha(0,u)) \leq 2(1-e^{-u^2/8}) \leq u^2/4$ and we dealt already in Remark 1.9 with condition (1.20), and thereby (1.16). Finally, noting that $\exp(-|\tau|/8) \leq \exp(-\tau^2/8)$ for $|\tau| \leq 1$ and applying Slepian's inequality twice, we find that for all $T$,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \widehat{Z}_t \leq 0\Big) \geq \mathbb{P}\Big(\sup_{t\in[0,1]} \widehat{Z}_t \leq 0\Big)^{\lceil T\rceil} \geq \mathbb{P}\Big(\sup_{t\in[0,1]} X_{t/4} \leq 0\Big)^{\lceil T\rceil}.$$

Clearly, $\mathbb{P}(\sup_{t\in[0,1/4]} X_t \leq 0) > 0$, hence $\hat{b}_\infty$ is finite.

## 3. Proof of Theorem 1.3.

3.1. *Asymptotics for $p_{[0,1]}(n)$ and $p_{(1,\infty)}(n)$.* We start by stating the three lemmas used in proving part (a) of Theorem 1.3 (deferring their proofs to Section 4). First, due to smoothness of $Q_n(\cdot)$, for $\delta > 0$ small, $\operatorname{sgn}\{Q_n(e^{-u})\}$ is controlled by the value of $Q_n(1)$ when $|u| \leq n^{-(1-\delta)}$ and by the values of $a_0$ or $a_n$ when $|u| \geq n^{-\delta}$. Hence, as our next lemma states, the contribution of this range of arguments to persistence exponents is negligible.

LEMMA 3.1. *In the setting of Theorem 1.3*:

(a) *For any $\alpha \in \mathbb{R}$ and slowly varying $L(\cdot)$*,

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Big(\sup_{|u|\leq n^{-(1-\delta)}} \{Q_n(e^{-u})\} < 0\Big) = 0, \tag{3.1}$$

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\big(Q_n(e^{-u}) < 0,\ \forall |u| \geq n^{-\delta}\big) = 0. \tag{3.2}$$

(b) *If $\sum_i L(i) i^\alpha$ converges then $n \mapsto p_{[0,1]}(n)$ is bounded away from zero. More generally, if $\alpha \leq -1$ then*

$$\lim_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Big(\sup_{u\geq 0}\{Q_n(e^{-u})\} < 0\Big) = 0. \tag{3.3}$$

Hereafter, for positive functions $f, g$ of common domain, $f(x) \lesssim g(x)$ stands for existence of finite uniform bound $\sup_x f(x)/g(x) \leq C(\alpha, L(\cdot))$.

From (3.3), we have that $p_{[0,1]}(n) = n^{-o(1)}$ when $\alpha \leq -1$, and our next lemma is key to finding the contribution of $u \in (n^{-(1-\delta)}, n^{-\delta})$ to the asymptotics of $p_{[0,1]}(n)$, in case $\alpha > -1$.

LEMMA 3.2. *For any $\alpha > -1$, $\delta > 0$, slowly varying $L(\cdot)$ and $h_{\alpha,n}(\cdot)$ as in* (1.26),

$$\lim_{n\to\infty} \sup_{w\in(2n^{-(1-\delta)},2n^{-\delta})} \left| \frac{w^{\alpha+1} h_{\alpha,n}(w)}{L(1/w)} - \Gamma(\alpha+1) \right| = 0. \tag{3.4}$$

*Consequently, in the setting of Theorem* 1.3, *for $u, v \in (n^{-(1-\delta)}, n^{-\delta})$,*

$$\bar{c}_n(u,v) := \operatorname{corr}[Q_n(e^{-u}), Q_n(e^{-v})] \lesssim e^{-(\alpha+1)/4|\log v - \log u|} \tag{3.5}$$

*and for any $M$ finite there exist $\epsilon_n = \epsilon_n(M) \downarrow 0$ such that if in addition $u/v \in [1/M, M]$, then*

$$\begin{aligned} (1-\epsilon_n)R(u,v)^{\alpha+1} + \epsilon_n R(u,v)^{\alpha+2} &\le \bar{c}_n(u,v) \\ &\le (1-\epsilon_n)R(u,v)^{\alpha+1} + \epsilon_n \end{aligned} \tag{3.6}$$

[*for $R(\cdot,\cdot)$ of* (1.27)].

Similarly, the following lemma controls the contribution of $x \in (e^{n^{-(1-\delta)}}, e^{n^{-\delta}})$ to $p_{(1,\infty)}(n)$.

LEMMA 3.3. *For $h_{\alpha,n}(\cdot)$ of* (1.26), *any $\alpha \in \mathbb{R}$, $\delta > 0$ and slowly varying $L(\cdot)$, as $n \to \infty$,*

$$\sup_{w\in(2n^{-(1-\delta)},2n^{-\delta})} \left| \frac{we^{-nw} h_{\alpha,n}(-w)}{L(n)n^{\alpha}} - 1 \right| \to 0. \tag{3.7}$$

*Consequently, for all $u, v \in (n^{-(1-\delta)}, n^{-\delta})$,*

$$\tilde{c}_n(u,v) := \operatorname{corr}[Q_n(e^{u}), Q_n(e^{v})] \lesssim e^{-1/2|\log v - \log u|} \tag{3.8}$$

*and for any $M$ finite there exist $\epsilon_n = \epsilon_n(M) \downarrow 0$ such that if in addition $u/v \in [1/M, M]$, then*

$$(1-\epsilon_n)R(u,v) + \epsilon_n R(u,v)^2 \le \tilde{c}_n(u,v) \le (1-\epsilon_n)R(u,v) + \epsilon_n. \tag{3.9}$$

PROOF OF PART (a) OF THEOREM 1.3. Starting with the proof of (1.5), we fix $\delta > 0$ and partition $\mathbb{R}_+$ into three disjoint intervals $\overline{J}_H = [n^{-\delta}, \infty)$, $\overline{J} = (n^{-(1-\delta)}, n^{-\delta})$ and $\overline{J}_L = [0, n^{-(1-\delta)}]$. Then, with $\overline{Q}_n(u) := Q_n(e^{-u})/\sqrt{h_{\alpha,n}(2u)}$, by Slepian's inequality and the nonnegativity of the covariance of $Q_n(\cdot)$, we have that

$$\begin{aligned} &\mathbb{P}\Big(\sup_{u\in\overline{J}}\{\overline{Q}_n(u)\} < 0\Big) \\ &\quad \ge \mathbb{P}\Big(\sup_{x\in[0,1]}\{Q_n(x)\} < 0\Big) \\ &\quad \ge \mathbb{P}\Big(\sup_{u\in\overline{J}}\{\overline{Q}_n(u)\} < 0\Big)\mathbb{P}\Big(\sup_{u\in\overline{J}_L}\{\overline{Q}_n(u)\} < 0\Big)\mathbb{P}\Big(\sup_{u\in\overline{J}_H}\{\overline{Q}_n(u)\} < 0\Big). \end{aligned}$$

Considering the limit of $\frac{1}{T_n}\log(\cdot)$ of these probabilities as $n \to \infty$ followed by $\delta \downarrow 0$, we have by Lemma 3.1 that suffices to consider $\alpha > -1$, and only the term involving $u \in \overline{J}$ is relevant for the asymptotics of $p_{[0,1]}(n)$. To deal with the latter term, let

$$A_n(s,t) := \bar{c}_n\bigl(\exp\{-e^{-s}/n^{\delta}\}, \exp\{-e^{-t}/n^{\delta}\}\bigr)$$

so that $u, v \in \overline{J}$ correspond to $s := -\log u - \delta T_n$ and $t := -\log v - \delta T_n$, in $[0, (1-2\delta)T_n]$. Upon this change of variables, the inequalities (3.6) of Lemma 3.2 translates into (1.21) holding for $A_\infty(s,t) := F(s,t)^{\alpha+1}$ and $D(s,t) := F(s,t)^{\alpha+2}$ in $\mathcal{S}_+$, the covariance functions of processes $Y_t^{(\alpha)}$ and $Y_t^{(\alpha+1)}$ of continuous sample path. Hence, by Lemma 1.8 condition (1.17) of Theorem 1.6 holds, whereas by (1.4) of Lemma 1.1 so does condition (1.16), and from (3.5) we have that $A_n(s,t) \le C\exp(-\frac{\alpha+1}{4}|t-s|)$ for some $C$ finite, any $n$ and all $s, t \in [0, (1-2\delta)T_n]$, which is much stronger than condition (1.15). We thus conclude from Theorem 1.6 (for $T = T_n \to \infty$, as in Remark 1.7), that

$$\lim_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Bigl(\sup_{u\in\overline{J}}\{\overline{Q}_n(u)\} < 0\Bigr) = -(1-2\delta)b_\alpha \tag{3.10}$$

from which (1.5) follows upon taking $\delta \downarrow 0$.

Similarly, for proving (1.6) we fix $\delta > 0$ and considering $\widehat{Q}_n(w) := Q_n(e^w)/\sqrt{h_{\alpha,n}(-2w)}$, split the supremum over $w \in \mathbb{R}_+$ into the disjoint $\overline{J}_L$, $\overline{J}$ and $\overline{J}_H$, of which by Lemma 3.1 only the supremum over $w \in \overline{J}$ matters. Same change of variable yields covariance functions $A_n(s,t) := \tilde{c}_n(\exp\{-e^{-s}/n^{\delta}\}, \exp\{-e^{-t}/n^{\delta}\})$ for $s, t \in [0, (1-2\delta)T_n]$, which in view of (3.9) of Lemma 3.3 satisfy (1.21) for $A_\infty(s,t) = F(s,t)$ and $D(s,t) = F(s,t)^2$, whereas the bound (3.8) of that lemma provides uniform exponential decay $A_n(s,t) \le C\exp(-|t-s|/2)$. Put together, by yet another application of Lemmas 1.8 and 1.1, and Theorem 1.6, we conclude that

$$\lim_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Bigl(\sup_{u\in\overline{J}}\{\widehat{Q}_n(u)\} < 0\Bigr) = -(1-2\delta)b_0, \tag{3.11}$$

so letting $\delta \downarrow 0$ we arrive at (1.6).

Turning to prove (1.7), since $Q_n(x)$ has nonnegative correlation on $[0,\infty)$, by Slepian's inequality, for any slowly varying $L(\cdot)$ and all $n$, the lower bound

$$p_{[0,\infty)}(n) \ge n^{-b_\alpha - b_0 - o(1)} \tag{3.12}$$

as in (1.7), is a direct consequence of the corresponding lower bounds of (1.6) and (1.5), and the matching upper bound for (1.7) is derived in the sequel [while upper bounding $p_{\mathbb{R}}(n)$]. $\square$

3.2. *Lower bound on* $p_{\mathbb{R}}(n)$. Having centered Gaussian coefficients, the joint law of $\{Q_n(x) : x \in \mathbb{R}\}$ is invariant under $x \mapsto -x$, hence same lower bound applies for $p_{(-\infty,0]}(n)$. Consequently, for the stated lower bound on $p_{\mathbb{R}}(2n)$, it suffices to establish strong control on $\mathrm{corr}[Q_n(x), Q_n(-y)]$ for $x, y > 0$.

Unfortunately, in case $x = y \in (0, 1)$ fixed, these correlations *do not* decay with $n$. However, the nonnegligible correlation comes from lower order coefficients of $Q_n(\cdot)$, so our first order of business is to show that suffices to consider only the higher order part of $Q_n(\cdot)$.

Indeed, by definition, for any slowly varying $L(\cdot)$ there exists $r \in \mathbb{N}$ such that $L(i) > 0$ for all $i \geq 2r$. Further, as $\rho \downarrow 0$, uniformly in $|x| \leq 1$

$$f_\rho(x) := 1 + x^{2r} - \rho \sum_{i=1}^{r} |x|^{2i-1} \to f_0(x) \geq 1$$

and $f_\rho(x)$ is nondecreasing in $|x| \geq 1$ for all $\rho$ small enough, hence $\inf_x f_{\rho_0}(x) > 0$ for some $\rho_0 > 0$. Fixing $\delta > 0$, set $m = m_n := \lceil \delta T_n \rceil$ and with $\hat{a}_i$ denoting independent centered Gaussian variables of variances $(3/4)\mathbb{E}[a_i^2]$, independent of the sequence $\{a_i\}$, note that $Q_n(\cdot) = Q_n^L(\cdot) + Q_n^M(\cdot) + Q_n^H(\cdot)$, for the independent algebraic polynomials,

$$Q_n^L(x) := \hat{a}_0 + \sum_{i=1}^{2r-1} a_i x^i + \hat{a}_{2r} x^{2r},$$

$$Q_n^M(x) := 0.5 \sum_{i=r}^{m-1} x^{2i} \big[a_{2i} + 2a_{2i+1}x + a_{2i+2}x^2\big],$$

$$Q_n^H(x) := 0.5 a_0 + \hat{a}_{2m} x^{2m} + \sum_{i=2m+1}^{n} a_i x^i.$$

For any $\rho > 0$, the event

$$\Gamma_\rho := \Big\{\hat{a}_0 \leq -1, \sup_{i=1}^{r-1}\{a_{2i}\} \leq 0, \sup_{i=1}^{r}\big\{|a_{2i-1}|\big\} \leq \rho, \hat{a}_{2r} \leq -1\Big\},$$

of positive probability [as $\mathbb{E}[a_0^2]L(2r) > 0$], results with $Q_n^L(\cdot) \leq -f_\rho(\cdot)$. Hence,

$$\mathbb{P}\Big(\sup_{x \in \mathbb{R}}\big\{Q_n^L(x)\big\} < 0\Big) \geq \mathbb{P}(\Gamma_{\rho_0}) > 0.$$

Next, if $a_{2i} \leq 0$ and $a_{2i}a_{2i+2} \geq a_{2i+1}^2$ for all $r \leq i \leq m-1$, then necessarily $Q_n^M(x) \leq 0$ for all $x \in \mathbb{R}$. Due to strict positivity of the slowly varying $L(2i)$ for $i \geq r$,

$$c_{2i} := \frac{L(2i+1)}{\sqrt{L(2i)L(2i+2)}} \left(\frac{(2i+1)^2}{(2i)(2i+2)}\right)^{\alpha/2}$$

is uniformly bounded for $i \ge r$, for example, $C := \sup_{i \ge r}\{c_{2i}\}$ is finite and with $a_i = \sqrt{i^\alpha L(i)} Z_i$ for standard i.i.d. Gaussian $\{Z_i\}$, the preceding event occurs whenever $Z_{2i} \le -\sqrt{C}$ and $|Z_{2i+1}| \le 1$ for all $r \le i \le m$. That is, for some positive $\lambda = \lambda(C) < \mathbb{P}(\Gamma_{\rho_0})$ and all $n$ large

$$\mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{Q_n^M(x)\} \le 0\Big) \ge \lambda^m.$$

By the preceding and independence of these three polynomials,

$$\begin{aligned} p_{\mathbb{R}}(n) &\ge \mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{Q_n^L(x)\} < 0, \sup_{x\in\mathbb{R}}\{Q_n^M(x)\} \le 0, \sup_{x\in\mathbb{R}}\{Q_n^H(x)\} \le 0\Big) \\ &\ge \lambda^{m+1}\mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{\widetilde{Q}_n(x)\} \le 0\Big), \end{aligned} \tag{3.13}$$

where $\widetilde{Q}_n(x) := \frac{Q_n^H(x)}{\sqrt{\operatorname{var}(Q_n^H(x))}}$ and $d_n(x, y) := \operatorname{corr}[Q_n^H(x), Q_n^H(y)]$. Note that the covariance of $Q_n^H(e^{-\cdot})$ is $0.25 + h_{\alpha,n}(\cdot) - h_{\alpha,2m-1}(\cdot)$ and $m = m_n = O(\log n)$ is small enough that both (3.6) and (3.9) apply for $d_n(e^{-u}, e^{-v})$. It is further not hard to check that Lemma 3.1 holds for $Q_n^H(\cdot)$. Thus, by a rerun of the proof of part (a) of Theorem 1.3 we arrive at the analog of (3.12) for $Q_n^H(\cdot)$. Namely, that if $\xi_n \to 0$ as $n \to \infty$, then

$$\mathbb{P}\Big(\sup_{x\ge 0}\{\widetilde{Q}_n(x)\} \le \xi_n\Big) \ge n^{-b_\alpha - b_0 - o(1)}. \tag{3.14}$$

We show in the sequel that subject to condition (1.8) on $L(\cdot)$, for even values of $n \to \infty$,

$$\gamma_n := -m_n \inf_{xy>0}\{d_n(x, -y) \wedge 0\} \to 0. \tag{3.15}$$

This implies that for $\epsilon_n = 2\gamma_n/m_n$,

$$(1 - \epsilon_n)\, d_n(x, y) + \epsilon_n \ge d_n(x, y) 1_{\{xy\ge 0\}},$$

hence with $\xi_n := -\gamma_n^{1/4}$ [so $\xi_n^2/\epsilon_n = m_n/(2\sqrt{\gamma_n})$], and $Z$ a standard Gaussian independent of $\widetilde{Q}_n(\cdot)$, it follows from Slepian's inequality and the union bound that

$$\begin{aligned} \mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{\widetilde{Q}_n(x)\} \le 0\Big) &\ge \mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{\sqrt{1-\epsilon_n}\,\widetilde{Q}_n(x) + \sqrt{\epsilon_n} Z\} \le \xi_n\Big) - \mathbb{P}(\sqrt{\epsilon_n} Z \le \xi_n) \\ &\ge \Big[\mathbb{P}\Big(\sup_{x\ge 0}\{\widetilde{Q}_n(x)\} \le \xi_n\Big)\Big]^2 - e^{-m_n/(4\sqrt{\gamma_n})}. \end{aligned}$$

Considering $T_n^{-1}\log(\cdot)$ of both sides and taking $n \to \infty$ followed by $\delta \downarrow 0$, we conclude in view of (3.13), (3.14) and our choice of $m = m_n = \lceil \delta T_n \rceil$, that

$$\liminf_{n\to\infty} \frac{1}{T_n} \log p_{\mathbb{R}}(n) \ge 2 \lim_{\delta\downarrow 0} \liminf_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Big(\sup_{x\ge 0}\{\widetilde{Q}_n(x)\} \le \xi_n\Big) \ge -2(b_\alpha + b_0).$$

Proceeding to prove (3.15), note that for $x, y \geq 0$,

$$d_n(x, -y) = d_n(x, y)\left[\frac{0.25 + h_e^\delta(xy) - h_o^\delta(xy)}{0.25 + h_e^\delta(xy) + h_o^\delta(xy)}\right],$$

where, assuming hereafter that $n$ is an *even* integer,

$$h_e^\delta(z) := \sum_{i=m+1}^{n/2} L(2i)(2i)^\alpha z^{2i} + \frac{3}{4}L(2m)(2m)^\alpha z^{2m},$$

$$h_o^\delta(z) := \sum_{i=m+1}^{n/2} L(2i-1)(2i-1)^\alpha z^{2i-1}.$$

With $d_n(x, y) \in [0, 1]$, we thus get (3.15) by showing that for some $\gamma_n \to 0$,

$$h_e^\delta(z) \geq \left(1 - \gamma_n m_n^{-1}\right) h_o^\delta(z) \qquad \forall z \geq 0. \tag{3.16}$$

To this end, setting $C_{2i-1} := \sqrt{L(2i)L(2i-2)(2i)^\alpha(2i-2)^\alpha}$, observe that with $n$ even [and $L(\cdot)$ nonnegative], by discriminant calculations similar to those we used for bounding $Q_n^M(\cdot)$,

$$h_e^\delta(z) \geq \sum_{i=m+1}^{n/2} C_{2i-1} z^{2i-1} \qquad \forall z \in \mathbb{R}.$$

Hence, (3.16) follows from

$$\limsup_{i\to\infty}(2i-1)\left|\frac{C_{2i-1}}{L(2i-1)(2i-1)^\alpha} - 1\right| = 0,$$

which for $\alpha$ finite is a direct consequence of our assumption (1.8).

3.3. *Upper bound on $p_{\mathbb{R}}(n)$.* Considering first the case of $\alpha > -1$, we fix $\delta > 0$ and have that

$$p_{\mathbb{R}}(n) \leq \mathbb{P}\Big(\sup_{x \in I_n(\delta)} \{Q_n(x)\} < 0\Big),$$

where

$$I_n(\delta) := \pm\{(e^{-n^{-(1-\delta)}}, e^{-n^{-\delta}}) \cup (e^{n^{-(1-\delta)}}, e^{n^{-\delta}})\} =: \bigcup_{i=1}^{4} J_i(\delta).$$

The asymptotic of $p_{J_3(\delta)}(n)$ and $p_{J_4(\delta)}(n)$, provided in (3.10), and (3.11), respectively, extend to any crossing levels $\xi_n \to 0$. In view of these and the invariance of law of $Q_n(\cdot)$ to change of sign, by the usual argument based on Slepian's inequality, it remains only to show that the autocorrelation $c_n(x, y) := \mathrm{corr}[Q_n(x), Q_n(y)]$ satisfies

$$c_n(x, y) \leq \epsilon_n + (1 - \epsilon_n) c_n(x, y) 1_{\{(x,y) \in J_i(\delta), 1 \leq i \leq 4\}} \tag{3.17}$$

for some $\epsilon_n T_n \to 0$. This amounts to confirming that

$$T_n c_n(x, -y) \lesssim o(1) \qquad \forall x, y \in (e^{-n^{-\delta}}, e^{n^{-\delta}}), \tag{3.18}$$

$$T_n c_n(x, y^{-1}) \lesssim o(1) \qquad \forall x, y \in (e^{-n^{-\delta}}, e^{-n^{-(1-\delta)}}). \tag{3.19}$$

Turning to prove (3.18), note that

$$\mathrm{Cov}(Q_n(x), Q_n(y)) = h_e(xy) + h_o(xy)$$

for

$$h_e(z) := 1 + \sum_{i=1}^{n/2} L(2i)(2i)^\alpha z^{2i}, \qquad h_o(z) := \sum_{i=1}^{n/2} L(2i-1)(2i-1)^\alpha z^{2i-1}.$$

Thus,

$$|c_n(x, -y)| = c_n(x, y)\frac{|h_e(xy) - h_o(xy)|}{h_e(xy) + h_o(xy)} \le \frac{|h_e(xy) - h_o(xy)|}{h_e(xy) + h_o(xy)}$$

and it suffices to show that as $n \to \infty$,

$$T_n \sup_{|\log z| \le 2n^{-\delta}} \frac{|h_e(z) - h_o(z)|}{h_e(z) + h_o(z)} \to 0. \tag{3.20}$$

To this end, setting $m = m_n := \lfloor T_n^2 \rfloor$ we have by (1.8) that

$$\begin{aligned}
&|h_e(z) - h_o(z)| \\
&\quad \le 1 + \sum_{i=1}^{2m} L(i) i^\alpha z^i + \sum_{i=m+1}^{n/2} L(2i)(2i)^\alpha z^{2i} \left| \frac{L(2i-1)(2i-1)^\alpha}{L(2i)(2i)^\alpha} z^{-1} - 1 \right| \\
&\quad \lesssim \sum_{i=1}^{2m} i^{\alpha+\delta} + \sum_{i=m+1}^{n/2} \left[ \left| 1 - \frac{1}{z} \right| + \sup_{i \ge m} \left| \frac{L(2i-1)(2i-1)^\alpha}{L(2i)(2i)^\alpha} - 1 \right| \right] L(2i)(2i)^\alpha z^{2i} \\
&\quad \lesssim T_n^{2(\alpha+2)_+} + [n^{-\delta} + m_n^{-1}] h_e(z).
\end{aligned}$$

Noting that $z \mapsto [h_e(z) + h_o(z)]$ is nondecreasing on $\mathbb{R}_+$, we get from (3.4) that

$$\inf_{|\log z| \le 2n^{-\delta}} [h_e(z) + h_o(z)] \gtrsim L(n^\delta) n^{\delta(\alpha+1)} \gtrsim n^{\delta(\alpha+1)/2}$$

and (3.20) follows. Proceeding to prove (3.19), note that $\max(x, y)^n \le e^{-n^\delta}$ for $x, y \in J_3(\delta)$, hence

$$\begin{aligned}
c_n(x, y^{-1}) &= \frac{y^n + \sum_{i=1}^n L(i) i^\alpha x^i y^{n-i}}{[(1 + \sum_{i=1}^n L(i) i^\alpha x^{2i})(y^{2n} + \sum_{i=1}^n L(i) i^\alpha y^{2(n-i)})]^{1/2}} \\
&\lesssim \frac{n^{\alpha+2} \max(x, y)^n}{\sqrt{L(n) n^\alpha}} \lesssim e^{-n^{\delta/2}}.
\end{aligned}$$

Finally, in case $\alpha \le -1$ it suffices to consider the event of no-crossing in intervals $J_1(\delta) \cup J_4(\delta)$ outside $[-1, 1]$. Consequently, suffices to confirm only (3.18), the first of our two claims, and only for $x, y \in J_4(\delta) := (e^{n^{-(1-\delta)}}, e^{n^{-\delta}})$. We proceed as before via (3.20), now needing it only for $\sqrt{z} \in J_4(\delta)$, so at end of its proof we rely here on the bound (3.7) at $w = 2n^{-(1-\delta)}$ (which hold for all $\alpha \in \mathbb{R}$), to get that uniformly in $\sqrt{z} \in J_4(\delta)$,

$$h_e(z) + h_o(z) \gtrsim n^{1-\delta} L(n) n^{\alpha} e^{2n^{\delta}} \gtrsim e^{n^{\delta}}.$$

**4. Proofs of Lemmas 3.1–3.3.** We begin by proving Lemmas 3.2 and 3.3 regarding asymptotic covariances in intervals which dominate the persistence probabilities of Theorem 1.3.

PROOF OF LEMMA 3.2. We set $\overline{J} := (n^{-(1-\delta)}, n^{-\delta})$ and make frequent use of the following obvious estimates, valid for all $l > -1$ and $y > 1 > w > 0$:

$$w^{l+1} \sum_{i \geq y/w} i^l e^{-iw} \lesssim e^{-y/2}, \qquad w^{l+1} \int_{x \geq y/w} x^l e^{-xw}\, dx \lesssim e^{-y/2},$$

$$w^{l+1} \sum_{i=1}^{1/w} i^l \lesssim 1.$$

Here, the constants implied by $\lesssim$ are allowed to depend on $l$ (in any case we use these bounds only for $l = \alpha$, $l = \alpha + 1$ and $l = \alpha + 2$).

Starting with the proof of (3.4), from the representation theorem [3], Theorem 1.3.1, it follows that $L(x) \sim \tilde{L}(x)$ and $x^{\eta}\tilde{L}(x)$ is eventually increasing (decreasing), if $\eta > 0$ (or $\eta < 0$, resp.). Hence, to simplify the presentation we can assume hereafter that $x^{\eta} L(x)$ is eventually increasing (decreasing) if $\eta > 0$ (or $\eta < 0$, resp.). Thus, for $\eta := (l+1)/2 > 0$ there exists $x_1 < \infty$ such that $L(i) \leq L(1/w)/(wi)^{\eta}$ for all $x_1 \leq i \leq 1/w$. Consequently, for all $a \geq wx_1$,

$$\frac{w^{l+1}}{L(1/w)} \sum_{i=x_1}^{a/w} L(i) i^l e^{-iw} \leq w^{l+1-\eta} \sum_{i=x_1}^{a/w} i^{l-\eta} e^{-iw} \lesssim a^{(l+1)/2}. \tag{4.1}$$

Likewise, there exists $x_2 < \infty$ such that $L(i) \leq iwL(1/w)$ for $x_2 \leq 1/w \leq i$; hence, for $b \geq wx_2$,

$$\frac{w^{l+1}}{L(1/w)} \sum_{i \geq b/w} L(i) i^l e^{-iw} \leq w^{l+2} \sum_{i \geq b/w} i^{l+1} e^{-iw} \lesssim e^{-b/2}. \tag{4.2}$$

Combining the bounds (4.1) and (4.2) with those corresponding to $L(\cdot) \equiv 1$, results with

$$\begin{aligned} &\frac{w^{l+1}}{L(1/w)} \left| \sum_{i=x_1}^{\infty} \left[ L(i) - L\left(\frac{1}{w}\right)\right] i^l e^{-iw} \right| \\ &\qquad \lesssim a^{(l+1)/2} + e^{-b/2} + \left\{ \sup_{\lambda \in [a,b]} \left| \frac{L(\lambda/w)}{L(1/w)} - 1 \right| \right\} w^{l+1} \sum_{i=x_1}^{\infty} i^l e^{-iw}. \end{aligned}$$

Since for $l+1>0$ and $w>0$,

$$\left|w^{l+1}\sum_{i=x_1}^{\infty} i^l e^{-iw} - \Gamma(l+1)\right| \lesssim w^{\min(l+1,1)},$$

it follows that for any $n \geq b/w$,

$$\left|\frac{w^{l+1}h_{l,n}(w)}{L(1/w)} - \Gamma(l+1)\right| \tag{4.3}$$
$$\lesssim a^{(l+1)/2} + e^{-b/2} + \sup_{\lambda\in[a,b]}\left|\frac{L(\lambda/w)}{L(1/w)} - 1\right| + w^{\min(l+1,1)/2}.$$

To deduce (3.4), consider $l=\alpha>-1$ and fixing $\epsilon>0$, choose $a=a(\epsilon)$ small and $b=b(\epsilon)$ large such that for all $w\in 2\overline{J}$ the first two terms on the right-hand side are bounded by $\epsilon$. Then recall that for $w\downarrow 0$, the convergence $|L(\lambda/w)/L(1/w)-1|\to 0$ is uniform over $\lambda$ in compacts (cf. [3], Theorem 1.2.1).

Turning to prove (3.5), we have by (3.4) that for $u,v\in\overline{J}$,

$$\bar{c}_n(u,v) = \frac{h_{\alpha,n}(u+v)}{\sqrt{h_{\alpha,n}(2u)h_{\alpha,n}(2v)}} \lesssim S(u,v)R(u,v)^{\alpha+1}$$

with $S(\cdot,\cdot)$ and $R(\cdot,\cdot)$ of (1.27). By the eventual monotonicity of $x\mapsto x^{\pm 2\eta}L(x)$, we further have for $n^{-\delta}\geq v\geq u>0$ and all large $n$,

$$\sqrt{\frac{L(1/(u+v))}{L(1/(2u))}} \leq \left(\frac{u+v}{2u}\right)^{\eta}, \qquad \sqrt{\frac{L(1/(u+v))}{L(1/(2v))}} \leq \left(\frac{2v}{u+v}\right)^{\eta},$$

resulting with $S(u,v)\leq (v/u)^{\eta}$. Clearly, $R(u,v)\leq 2(v/u)^{-1/2}$, so taking $\eta=(\alpha+1)/4$ we arrive at (3.5). Next, fixing $M>1$ and setting $\bar{g}_{\alpha,n}(w):=w^{\alpha+1}h_{\alpha,n}(w)$,

$$\overline{G}_{\alpha,n}(u,v) := \frac{\bar{c}_n(u,v)}{R(u,v)^{\alpha+1}} = \frac{\bar{g}_{\alpha,n}(u+v)}{\sqrt{\bar{g}_{\alpha,n}(2u)\bar{g}_{\alpha,n}(2v)}}$$

[by (1.27) and the preceding expression for $\bar{c}_n(u,v)$], our claim (3.6) amounts to

$$-\epsilon_n\bigl(1-R(u,v)\bigr) \leq \overline{G}_{\alpha,n}(u,v) - 1 \leq \epsilon_n\bigl(R(u,v)^{-(\alpha+1)} - 1\bigr) \tag{4.4}$$

for some $\epsilon_n\to 0$, any $v\in[u,Mu]$ and all $u\in\overline{J}$. Since $z-1-\log z\geq 0$ on $\mathbb{R}_+$ and $\epsilon p(1-r)\leq \log(1+\epsilon(r^{-p}-1))$ whenever $p\geq 0$ and $r,\epsilon\in[0,1]$, the inequality (4.4) follows in turn from

$$-\epsilon_n\bigl(1-R(u,v)\bigr) \leq G_{\alpha,n}(u,v) := \log\overline{G}_{\alpha,n}(u,v) \leq \epsilon_n(\alpha+1)\bigl(1-R(u,v)\bigr).$$

To this end, setting $\epsilon_n := (1+\alpha\wedge 0)^{-1}(1+M)^2\tilde{\epsilon}_n$ and noting that

$$1-R(u,v) = \frac{(\sqrt{v}-\sqrt{u})^2}{v+u} \geq \frac{(v-u)^2}{2(v+u)^2} \geq \frac{(v-u)^2}{2(1+M)^2u^2},$$

it suffices to show that for some $\tilde{\epsilon}_n \to 0$,

$$\big|G_{\alpha,n}(u,v)\big| \le \tilde{\epsilon}_n \frac{(v-u)^2}{2u^2}. \tag{4.5}$$

Now, fixing $u$, we expand the function $v \mapsto G_{\alpha,n}(u,v)$ in Taylor's series about $v = u$, to get

$$G_{\alpha,n}(u,v) = G_{\alpha,n}(u,u) + (v-u)G'_{\alpha,n}(u,u) + \frac{(v-u)^2}{2} G''_{\alpha,n}(u,\xi) \tag{4.6}$$

for some $\xi = \xi_n(u,v) \in [u,v]$. With

$$G_{\alpha,n}(u,v) = g_{\alpha,n}(u+v) - \tfrac{1}{2} g_{\alpha,n}(2u) - \tfrac{1}{2} g_{\alpha,n}(2v), \qquad g_{\alpha,n}(w) := \log \bar{g}_{\alpha,n}(w),$$

clearly $G_{\alpha,n}(u,u) = G'_{\alpha,n}(u,u) = 0$ and

$$\begin{aligned} u^2\big|G''_{\alpha,n}(u,\xi)\big| &= u^2\big|g''_{\alpha,n}(u+\xi) - 2g''_{\alpha,n}(2\xi)\big| \\ &\le 3 \sup_{w \in 2\overline{J}} \{w^2 |g''_{\alpha,n}(w)|\} := \tilde{\epsilon}_n. \end{aligned} \tag{4.7}$$

Thus, to complete the proof of (4.5), and thereby that of (3.6), it suffices to show that $w^2|g''_{\alpha,n}(w)| \to 0$ uniformly in $w \in 2\overline{J}$. For this task, setting $h^0_{l,n}(w) := h_{l,n}(w) - 1$, we have that $h'_{l,n}(w) = -h^0_{l+1,n}(w)$ and consequently,

$$w^2 g''_{\alpha,n}(w) = -(\alpha+1) + \frac{w^2 h^0_{\alpha+2,n}(w)}{h_{\alpha,n}(w)} - \left(\frac{w h^0_{\alpha+1,n}(w)}{h_{\alpha,n}(w)}\right)^2. \tag{4.8}$$

From (3.4), we know that for $l = 1, 2$, uniformly in $w \in 2\overline{J}$, as $n \to \infty$,

$$\frac{w^l h^0_{\alpha+l,n}(w)}{h_{\alpha,n}(w)} \to \frac{\Gamma(\alpha+l+1)}{\Gamma(\alpha+1)}$$

and we are done since

$$-(\alpha+1) + \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)} - \left(\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\right)^2 = 0. \tag{4.9}$$

□

PROOF OF LEMMA 3.3. To prove (3.7), fix $\delta \in (0,1)$ and setting $\kappa_n := n - n^{1-\delta/2}$, note that for $w \in 2\overline{J}$

$$\begin{aligned} &(1-e^{-w})e^{-nw} \sum_{i=\kappa_n+1}^{n} \left| \frac{L(i) i^\alpha}{L(n) n^\alpha} - 1 \right| e^{iw} \\ &\quad \lesssim n^{-\delta/2} + \sup_{\mu \in [1-n^{-\delta/2},1]} \left| \frac{L(\mu n)}{L(n)} - 1 \right| =: \gamma_n, \end{aligned}$$

$e^{-nw}h_{\alpha,\kappa_n}(-w) \lesssim e^{-n^{\delta/3}}$ and

$$\left|(1-e^{-w})e^{-nw}\sum_{i=\kappa_n+1}^{n} e^{iw} - 1\right| \lesssim e^{-n^{\delta/2}}.$$

Combining these bounds, we find that for any $\alpha \in \mathbb{R}$ and $w \in 2\overline{J}$,

$$\left|\frac{we^{-nw}h_{\alpha,n}(-w)}{L(n)n^{\alpha}} - 1\right| \lesssim \gamma_n \tag{4.10}$$

from which (3.7) follows, since $\gamma_n \to 0$ for any fixed slowly varying $L(\cdot)$ and $\delta > 0$.

We now confirm (3.8) by noting that

$$\tilde{c}_n(u,v) = h_{\alpha,n}(-u-v)/\sqrt{h_{\alpha,n}(-2u)h_{\alpha,n}(-2v)},$$

which by (3.7) converges as $n \to \infty$, uniformly in $u, v \in \overline{J}$, to $R(u,v) \le 2(v \vee u/v \wedge u)^{-1/2}$.

Next, proceeding along the same lines as the proof of (3.6), now with $\overline{G}_{\alpha,n}(u,v) := \tilde{c}_n(u,v)/R(u,v)$ and $g_{\alpha,n}(w) := \log[wh_{\alpha,n}(-w)]$, reduces the proof of (3.9) to $w^2|g''_{\alpha,n}(w)| \to 0$, uniformly in $w \in 2\overline{J}$. To this end, it is not hard to check that (4.8) is replaced here by

$$w^2 g''_{\alpha,n}(w) = -1 + \frac{w^2 h^0_{\alpha+2,n}(-w)}{h_{\alpha,n}(-w)} - \left(\frac{wh^0_{\alpha+1,n}(-w)}{h_{\alpha,n}(-w)}\right)^2 = -1 + \operatorname{Var}(wH_{n,w}),$$

where [adopting the convention $L(0)0^{\alpha} = 1$], for $j = 0, 1, \dots, n$,

$$\mathbb{P}(H_{n,w} = j) = \frac{L(n-j)(n-j)^{\alpha}e^{-jw}}{\sum_{k=0}^{n} L(n-k)(n-k)^{\alpha}e^{-kw}}.$$

The variance of the Geometric$(e^{-w})$ random variable $H_{\infty,w}$ is $\frac{1}{4}[\sinh(w/2)]^{-2}$, hence $\operatorname{Var}(wH_{\infty,w}) \to 1$ when $w \downarrow 0$. Further, as we have already seen, truncating $wH_{\infty,w}$ and $wH_{n,w}$ at $wn^{1-\delta/2}$ changes the corresponding variances by at most $e^{-n^{\delta/3}}$, uniformly over $w \in 2\overline{J}$ and from the estimates leading to (4.10), we easily deduce that

$$\sup_{w\in 2\overline{J}, j \le n^{1-\delta/2}} \left|\frac{\mathbb{P}(H_{n,w} = j)}{\mathbb{P}(H_{\infty,w} = j)} - 1\right| \lesssim \gamma_n.$$

Combining these facts, we conclude that

$$\sup_{w\in 2\overline{J}} \left|\frac{\operatorname{Var}(wH_{n,w})}{\operatorname{Var}(wH_{\infty,w})} - 1\right| \lesssim \gamma_n,$$

thereby completing the proof of (3.9). $\square$

We proceed with a regularity lemma that is used in the sequel for proving Lemma 3.1 (and Lemma 5.1).

LEMMA 4.1. *There exist finite universal constants $K_d$, such that if centered Gaussian process $\{Z_t, t \in T\}$, indexed on $T = [a, b]^d \subset \mathbb{R}^d$, satisfies*

$$D(s,t)^2 := \mathbb{E}\big[(Z_t - Z_s)^2\big] \le M^2 \|t - s\|_2^2 \qquad \forall s, t \in T \tag{4.11}$$

*for some $M < \infty$, then*

$$\mathbb{E}\Big[\sup_{t \in T} Z_t\Big] \le K_d M |b - a|. \tag{4.12}$$

*Further, if for $d = 1$ we have that $t \mapsto Z_t \in \mathcal{C}^1$ and*

$$2(b-a)^2 \sup_{t \in T} \mathbb{E}\big[Z_t'^2\big] \le \sup_{t \in T} \mathbb{E}\big[Z_t^2\big], \tag{4.13}$$

*then for some universal constant $\mu > 0$,*

$$\mathbb{P}\Big(\sup_{t \in T}\{Z_t\} < 0\Big) \ge \mu. \tag{4.14}$$

PROOF. For proving (4.12) note that there exist $C_d < \infty$ such that $T$ is covered by at most $N(\epsilon) = \min\{1, \epsilon^{-d}(C_d M|b-a|)^d\}$ Euclidean balls of radius $\epsilon/M$. With $B_D(s,r) = \{t \in T : D(s,t) \le \epsilon\}$ denoting the ball in pseudo-metric $D(\cdot,\cdot)$ of radius $\epsilon \ge 0$ and center $s \in T$ and $B(s,\epsilon)$ the Euclidean ball of same radius and center, our assumption (4.11) implies that $B(s, \epsilon/M) \subseteq B_D(s,\epsilon)$ for any $s \in T$, thereby inducing a cover of $T$ by at most $N(\epsilon)$ balls of radius $\epsilon$ in pseudo-metric $D(\cdot,\cdot)$. Recall [1], Theorem 1.3.3, that there exist universal finite $K_0$ such that

$$\mathbb{E}\Big[\sup_{t \in T} Z_t\Big] \le K_0 \int_0^{C_d M|b-a|} \sqrt{\log N(\epsilon)}\, d\epsilon.$$

Our thesis follows upon change of variable $y = \sqrt{d^{-1} \log N(\epsilon)}$, with $K_d := 2\sqrt{d} K_0 C_d \int_0^\infty y^2 e^{-y^2}\, dy$.

Turning to prove (4.14), let ${\sigma_T}^2 := \sup_{t \in T} \mathbb{E}[Z_t^2]$ and $\overline{Z}_t := Z_t - Z_{t_0}$ for $t_0 \in T$ such that $\mathbb{E}[Z_{t_0}^2] = \sigma_T^2$. Then, by Cauchy–Schwarz we have that for any $s, t \in T$,

$$\mathbb{E}\big[(\overline{Z}_t - \overline{Z}_s)^2\big] = \mathbb{E}\big[(Z_t - Z_s)^2\big] \le (t-s)^2 \sup_{u \in [s,t]} \mathbb{E}\big[Z_u'^2\big].$$

Thus, (4.13) results with

$$\bar{\sigma}_T^2 := \sup_{t \in T} \mathbb{E}\big[\overline{Z}_t^2\big] \le \frac{1}{2} {\sigma_T}^2$$

and considering (4.12) for $\overline{Z}_t$, we further have that $\mathbb{E}[\sup_{t \in T} \overline{Z}_t] \le K_1 \sigma_T$. Clearly,

$$\sup_{t \in T} Z_t = Z_{t_0} + \sup_{t \in T} \overline{Z}_t,$$

so by a union bound we have for any $\lambda > 0$,

$$\mathbb{P}\Big(\sup_{t \in T}\{Z_t\} < 0\Big) \ge \mathbb{P}(Z_{t_0} < -\lambda \sigma_T) - \mathbb{P}\Big(\sup_{t \in T}\{\overline{Z}_t\} > \lambda \sigma_T\Big). \tag{4.15}$$

For $\lambda \geq K_1$, large enough the first term on the right-hand side is at least $0.5e^{-\lambda^2/2}$ and by Borell-TIS inequality, the second term is at most

$$2\exp\left\{-\frac{(\lambda-K_1)^2\sigma_T^2}{2\bar{\sigma}_T^2}\right\} \leq 2e^{-(\lambda-K_1)^2}.$$

This completes the proof, since $\mu := 0.5e^{-\lambda^2/2} - 2e^{-(\lambda-K_1)^2}$ is strictly positive for $\lambda$ large enough. $\square$

We establish part (a) of Lemma 3.1 by partitioning relevant domains of $Q_n(e^{-\cdot})$ to at most $\gamma(\delta)T_n$ subintervals, within each of which (4.13) holds [and where $\gamma(\delta) \to 0$], thereby combining Lemma 4.1 and Slepian's inequality. However, to provide the estimates of part (b) in *critical case* of $\alpha = -1$, we require the following comparison (after a change of argument), between $Q_n(e^{-\cdot})$ and the standard stationary Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$.

LEMMA 4.2. *For* $\alpha = -1$ *and any slowly varying* $L(\cdot)$, *there exist* $r(\gamma) \downarrow 0$ *when* $\gamma \downarrow 0$, *such that*

$$\bar{c}_n(u,v) \geq \left(\frac{u}{v}\right)^{r(\gamma)} \qquad \forall 0 < u \leq v \leq \gamma. \tag{4.16}$$

PROOF. First note that for $v \geq u \geq 0$, by the monotonicity of $u \mapsto h_{\alpha,n}(u)$,

$$\bar{c}_n(u,v) = \frac{h_{\alpha,n}(u+v)}{\sqrt{h_{\alpha,n}(2u)h_{\alpha,n}(2v)}} \geq \frac{h_{\alpha,n}(2v)}{h_{\alpha,n}(2u)} \geq \frac{h_{\alpha,\infty}(2v)}{h_{\alpha,\infty}(2u)},$$

where the second inequality follows by noting that $n \mapsto h_{\alpha,n}(2v)/h_{\alpha,n}(2u)$ is monotone decreasing [for $e^{-2(n+1)(v-u)} \leq h_{\alpha,n}(2v)/h_{\alpha,n}(2u)$ via term by term comparison]. We thus get (4.16) upon finding $r = r(\gamma) \downarrow 0$ for which $\xi_r(u) := u^r h_{-1,\infty}(u)$ is nondecreasing on $(0, 2\gamma]$. Since $\xi_r'(u) \geq 0$ if and only if

$$r \geq \zeta(u) := \frac{uh^0_{0,\infty}(u)}{h_{-1,\infty}(u)},$$

this amounts to showing that $\zeta(u) \downarrow 0$ for $u \downarrow 0$. To this end, recall (4.3) that $uh^0_{0,\infty}(u) \lesssim L(1/u)$ and moreover for any $\eta > 0$,

$$h_{-1,\infty}(u) \geq e^{-1} \sum_{i=\eta/u}^{1/u} L(i)i^{-1} \geq e^{-1}L(1/u)\bigl(1+o(1)\bigr)\log(1/\eta),$$

so considering $u \downarrow 0$ followed by $\eta \downarrow 0$ we conclude that also $\zeta(u) \to 0$ as $u \downarrow 0$. $\square$

PROOF OF LEMMA 3.1.

(a) We first consider $\alpha > -1$ and establish (3.1) by partitioning $[-n^{-(1-\delta)}, n^{-(1-\delta)}]$ to at most $\gamma(\delta)T_n$ intervals $\{I_k\}$, with $\gamma(\delta) \to 0$, such that $Z_u =$

$e^{n(u\wedge 0)}Q_n(e^{-u})$ satisfies (4.13) within each such subinterval $I_k$. Indeed, since $Q_n(e^{-u})$ has nonnegative autocorrelation, by Slepian's inequality and (4.14) we have then that

$$\mathbb{P}\Big(\sup_{|u|\le n^{-(1-\delta)}}\{Q_n(e^{-u})\}<0\Big)\ge\prod_k\mathbb{P}\Big(\sup_{u\in I_k}\{Z_u\}<0\Big)\ge\mu^{\gamma(\delta)T_n}$$

for some universal constant $\mu>0$, yielding (3.1) upon considering $T_n^{-1}\log(\cdot)$ of these probabilities in the limit $n\to\infty$ followed by $\delta\downarrow 0$.

To carry out this program, note first that both $\mathbb{E}[Q_n(e^{-u})^2]=h_{\alpha,n}(2u)$ and $\mathbb{E}[Q_n'(e^{-u})^2]=h_{\alpha+2,n}(2u)$ are monotone in $u\ge 0$, with (4.13) obviously satisfied within *any* subinterval of size $1/(2n)$.

Further, from (4.3) we have that for any $l>-1$ there exist finite $b=b_l$ and positive $w_l$, so that $u^{l+1}h_{l,n}(u)/L(1/u)$ is bounded (and bounded away from zero), uniformly in $u\in[0,w_l]$ and $n\ge b_l/u$. So, with $\alpha>-1$, the same applies for $u^2h^0_{\alpha+2,n}(2u)/h_{\alpha,n}(2u)$. This in turn implies that for some $\eta>0$, $u_\star>0$ and $b\ge 2$ finite [depending only on $\alpha$ and $L(\cdot)$], setting $u_{k,n}=k/(2n)$, $k=0,\dots,b$ and $u_{k+b,n}=be^{\eta k}/(2n)$, $k\ge 0$, the process $Z_u=Q_n(e^{-u})$ satisfies (4.13) in each interval $I_k=[u_{k-1,n},u_{k,n}]$, $k\ge 1$, provided $u_{k,n}\le u_\star$. Since $u_{k_\star+b,n}\ge n^{-(1-\delta)}$ for $k_\star:=(\delta/\eta)T_n$, this takes care of the part of $u\ge 0$ in (3.1). In case $u=-w<0$, we follow the same reasoning, just now applying Lemma 4.1 for the rescaled process $Z_w:=e^{-nw}Q_n(e^w)$, $w\ge 0$. Specifically, setting

$$\tilde h_{l,n}(w):=\sum_{j=0}^n L(n-j)(n-j)^\alpha j^l e^{-jw}$$

for $l=0,2$ [with $L(0)0^\alpha:=1$], it is easy to check that $\mathbb{E}[Z_w^2]=\tilde h_{0,n}(2w)$ and $\mathbb{E}[{Z_w'}^2]=\tilde h_{2,n}(2w)$. Thus, per $\alpha>-1$ and slowly varying $L(\cdot)$, the same partition takes care of $u<0$ in (3.1) provided $w^3\tilde h_{2,n}(w)/(L(n)n^\alpha)$ is bounded and $w\tilde h_{0,n}(w)/(L(n)n^\alpha)$ bounded away from zero, uniformly in $w\in[bn^{-1},w_\star]$, for some $b<\infty$ and $w_\star>0$. To this end, fixing $l\ge 0$ and $\epsilon\in(0,1)$, note that the ratio between $\sum_{j\le(1-\epsilon)n}L(n-j)(n-j)^\alpha j^l e^{-jw}$ and $L(n)n^\alpha\sum_{j\le(1-\epsilon)n}j^le^{-jw}$ is bounded and bounded away from zero, uniformly in $n$ and $w$ (for any $\alpha\in\mathbb{R}$), and the same applies for the ratio between the latter and $L(n)n^\alpha/w^{l+1}$, provided $(1-\epsilon)(nw)\ge b$ [as shown in the course of proving (3.4)]. Next, recall that $\sum_{i=0}^n L(i)i^\alpha\lesssim L(n)n^{\alpha+1}$ for $\alpha>-1$ and slowly varying $L(\cdot)$; hence, we are done, for

$$\begin{aligned}\sum_{j>(1-\epsilon)n}^n L(n-j)(n-j)^\alpha j^l e^{-jw}&\le e^{-(1-\epsilon)nw}n^l\sum_{i=0}^{\epsilon n}L(i)i^\alpha\\&\lesssim L(n)n^\alpha w^{-(l+1)}\xi_\epsilon(nw),\end{aligned}$$

where $\xi_\epsilon(b):=b^{l+1}e^{-(1-\epsilon)b}\to 0$ as $b\to\infty$.

Having dealt with (3.1) for $\alpha > -1$, we turn to $\alpha \leq -1$ and fixing $\gamma > 0$ set $b(\gamma) := \gamma - (\alpha + 1)$. Fixing $l \geq 0$, we claim that $w^{l+1}\tilde{h}_{l,n}(w)/(L(n)n^{\alpha})$ is bounded and bounded away from zero, uniformly in $w \in [b(\gamma)T_n n^{-1}, w_\star]$. Indeed, the only difference is that now $\sum_{i=0}^{n} L(i)i^{\alpha} \lesssim L(n)n^{\eta}$ for any fixed $\eta > 0$, so to neglect the contribution of $j > (1-\epsilon)n$ to $\tilde{h}_{l,n}(w)$ we need that

$$n^{\eta-(\alpha+1)}\xi_\epsilon(nw) \to 0,$$

which applies for any $nw \geq b(\gamma)T_n$ if $\epsilon > 0$ and $\eta > 0$ are small enough so that $\gamma(1-\epsilon) > 2\eta - \epsilon(\alpha+1)$. We further cover $[0, \gamma T_n/(2n)]$ and $[b(-\gamma)T_n/(2n), b(\gamma)T_n/(2n)]$ by at most $3\gamma T_n$ intervals of equal length $1/(2n)$, within each of which Lemma 4.1 applies for $Z_w = e^{-nw}Q_n(e^w)$. So, given that (3.3) handles the domain $u \geq 0$, by the same reasoning as before, we establish (3.1) by showing that for any fixed $\gamma > 0$, $\alpha < -1$ and $\eta > 0$ small enough, the process $w \mapsto Q_n(e^w)$ satisfies condition (4.13) within each subinterval of the partition of $[\gamma T_n/(2n), b(-\gamma)T_n/(2n)]$ given by $w_{k,n} = e^{\eta k}w_{0,n}$, $k \geq 1$, and $w_{0,n} = \gamma T_n/(2n)$. As $h_{\alpha,n}(-w) \geq 1$, this in turn amounts to proving that $w^2 h^0_{\alpha+2,n}(-w)$ is uniformly bounded on $(0, b(-\gamma)T_n/n]$. Indeed, adapting the calculation leading to (4.10), now for $\kappa_n = \epsilon n$ and with $L(i) \lesssim i^{\epsilon}$, we find that

$$h^0_{\alpha+2,n}(-w) \lesssim e^{nw}n^{\epsilon+\alpha+3} + e^{\epsilon nw}n^{\epsilon+(\alpha+3)_+},$$

which yields the stated uniform boundedness for $e^{nw} \leq n^{b(-\gamma)}$ upon choosing $\epsilon > 0$ small enough so that

$$b(-\gamma) + \epsilon + \alpha + 1 < 0, \qquad \epsilon b(-\gamma) + \epsilon + (\alpha+3)_+ - 2 < 0.$$

We proceed to confirm (3.2) where, by (3.3), if $\alpha \leq -1$ we only need to consider $u = -w \leq 0$. Setting $w_{k,n} := e^{\eta k}n^{-\delta}$, $k \geq 0$, recall that we have already seen that for any $\alpha \in \mathbb{R}$ and $\eta > 0$ small enough, the rescaled process $Z_w$ satisfies (4.14) within each subinterval $I_k := [w_{k-1,n}, w_{k,n}]$ [and when $\alpha > -1$ the same applies also for $Z_u = Q_n(e^{-u})$ with $u > 0$]. Hence, partitioning $\pm u \in [n^{-\delta}, u_\star]$ for fixed $u_\star \in (0,1]$ to at most $k_\star$ such subintervals, by the same reasoning we applied for (3.1) in case $\alpha > -1$, the proof of (3.2) reduces to showing that for all $\alpha \in \mathbb{R}$ and any fixed $u_\star > 0$,

$$\inf_n \mathbb{P}\big(Q_n\big(e^{-u}\big) < 0,\ \forall |u| \geq u_\star\big) > 0. \tag{4.17}$$

We deal with $u \leq -u_\star$ in (4.17) by equivalently, considering $\{R_n(x) := x^n Q_n(x^{-1}) < 0\}$ for $x \in (0, x_\star]$, with $x_\star := e^{-u_\star} < 1$. Specifically, note that for $x \in [0, x_\star]$,

$$\mathbb{E}\big[R_n'(x)^2\big] \lesssim \sum_{j=2}^{n} L(n-j)(n-j)^{\alpha} j^2 x_\star^{2j}$$

is bounded by $CL(n)n^{\alpha}$ for $C = C(\alpha, L(\cdot))$ finite and all $n$. Indeed, with $\sum_{j=0}^{\infty} j^2 x_\star^{2j}$ finite, such bound applies for the sum over $j \le (1-\epsilon)n$ on the right-hand side, whereas the remainder sum over $(1-\epsilon)n < j \le n$ contributes at most

$$n^2 x_\star^{2(1-\epsilon)n} \sum_{i=0}^{\epsilon n} L(i) i^{\alpha},$$

which is exponentially decaying in $n$, hence dominated by $L(n)n^{\alpha}$. Since $\mathbb{E}[R_n(x)^2] \ge L(n)n^{\alpha}$ for all $x > 0$ and $n$, the uniform partition of $[0, x_\star]$ to $r$ subintervals $\{I_k\}$ of length $x_\star/r$ each, results for $r$ large enough with $x \mapsto R_n(x)$ satisfying (4.13) within each subinterval $I_k$. Hence, by Slepian's inequality, we get that $\mathbb{P}(\sup_{x\in[0,x_\star]}\{R_n(x)\} < 0) \ge \mu^r$. The same argument applies for $u \ge u_\star$, since $\mathbb{E}[Q_n(x)^2] \ge 1$ for all $x \ge 0$ and

$$\mathbb{E}\big[Q_n'(x)^2\big] \le \sum_{i=1}^{\infty} L(i) i^{\alpha+2} x^{2(i-1)}$$

is uniformly bounded on $[0, x_\star]$ [for any fixed $\alpha \in \mathbb{R}$ and slowly varying $L(\cdot)$].

(b) Setting $v_n := \mathbb{E}[Q_n(1)^2] = 1 + \sum_{i=1}^{n} L(i) i^{\alpha}$ and $\overline{Q}_n(x) := Q_n(x) - Q_n(1)$, note that

$$\sup_{x\in[0,1]} \mathbb{E}\big[\overline{Q}_n(x)^2\big] = v_n - 1.$$

If the monotone limit $v_\infty$ of $v_n$ is finite, then $x \mapsto Q_\infty(x) = \sum_{i=0}^{\infty} a_i x^i$ is a well-defined centered Gaussian process on [0, 1] whose sample path are a.s. (uniformly) continuous; hence, $K_\infty := \mathbb{E}[\sup_{x\in[0,1]} Q_\infty(x)]$ is finite. Since $n \mapsto \mathbb{E}[(Q_n(x) - Q_n(y))^2]$ is nondecreasing, it follows from Sudakov–Fernique inequality that the (nondecreasing) sequence $K_n := \mathbb{E}[\sup_{x\in[0,1]} Q_n(x)]$ is bounded above by $K_\infty$. As argued around (4.15), by Borell-TIS inequality, for any $\lambda \ge K_\infty \ge \sup_n K_n$ large enough and all $n$,

$$\begin{aligned} p_{[0,1]}(n) &\ge \mathbb{P}\big(Q_n(1) < -\lambda\sqrt{v_n}\big) - \mathbb{P}\Big(\sup_{x\in[0,1]}\big\{\overline{Q}_n(x)\big\} > \lambda\sqrt{v_n}\Big) \\ &\ge 0.5 e^{-\lambda^2/2} - 2e^{-(\lambda-K_n)^2 v_n/(2(v_n-1))}, \end{aligned}$$

with $v_n \uparrow v_\infty \in [1, \infty)$, the right-hand side is bounded away from zero for some $\lambda$ and all $n$ large enough, and hence so is $n \mapsto p_{[0,1]}(n)$.

Assuming hereafter that $v_\infty = \infty$ and in particular that $\alpha = -1$, in view of Lemma 4.2, we get (3.3) once we show that

$$\liminf_{n\to\infty} \frac{1}{T_n} \log \mathbb{P}\Big(\sup_{u\in[\gamma n^{-1},\gamma]} \{Q_n(e^{-u})\} < 0\Big) \ge -r(\gamma) \tag{4.18}$$

(which per Lemma 4.2 converges to zero as $\gamma \downarrow 0$). This is done upon realizing that the auto-correlation function of $u \mapsto X_{-2r(\gamma)\log(u/\gamma)}$ matches the right-hand side of (4.16), hence by Slepian's inequality,

$$\mathbb{P}\Big(\sup_{u\in[\gamma n^{-1},\gamma]}\{Q_n(e^{-u})\}<0\Big)\geq \mathbb{P}\Big(\sup_{t\in[0,2r(\gamma)T_n]}\{X_t\}<0\Big)$$

and (4.18) follows, since $X_t$ has persistence exponent $1/2$. □

**5. Proof of Theorem 1.5.** We start with two lemmas, the first of which provides for each fixed positive time a smooth initial condition of the required law, while the second explicitly constructs a solution of the heat equation for such initial condition.

LEMMA 5.1. *Equip $\mathcal{A} = \mathcal{C}(\mathbb{R}^d)$ with the topology of uniform convergence on compact sets. For any $\varepsilon > 0$, there exists an $(\mathcal{A}, \mathcal{B}_{\mathcal{A}})$-valued centered Gaussian field $g_\varepsilon(\cdot)$ with covariance $C_\varepsilon(\mathbf{x}_1,\mathbf{x}_2) = K_{2\varepsilon}(\mathbf{x}_1-\mathbf{x}_2)$ such that $|g_\varepsilon(\mathbf{x})| \leq a\|\mathbf{x}\|+b$ for some $a, b$ (possibly random) and all $\mathbf{x}$.*

PROOF. Since $C_\varepsilon(\cdot,\cdot)$ is positive definite, there exists a centered Gaussian field $g_\varepsilon(\mathbf{x})$ indexed on $\mathbb{R}^d$ with covariance function $C_\varepsilon(\cdot,\cdot)$. Further, with $\delta = 2\varepsilon$ and utilizing the bound $1-e^{-r}\leq r$,

$$\mathbb{E}\big[\big(g_\varepsilon(\mathbf{x}_1)-g_\varepsilon(\mathbf{x}_2)\big)^2\big] = 2\big(K_\delta(\mathbf{0})-K_\delta(\mathbf{x}_1-\mathbf{x}_2)\big)\leq \frac{\|\mathbf{x}_1-\mathbf{x}_2\|^2}{(4\pi\delta)^{d/2}2\delta}. \tag{5.1}$$

Hence, using the induced bound on higher moments of $g_\varepsilon(\mathbf{x}_1) - g_\varepsilon(\mathbf{x}_2)$, by Kolmogorov–Centsov continuity theorem we can and shall consider hereafter the unique continuous modification of $g_\varepsilon(\cdot)$, which takes values in $\mathcal{A}$ and is measurable with respect to the corresponding Borel $\sigma$-algebra $\mathcal{B}_{\mathcal{A}}$.

Combining the bound (5.1) with Lemma 4.1, we have that $\mathbb{E}[\sup_{\|\mathbf{x}\|\leq n} g_\varepsilon(\mathbf{x})] \leq M'n$, for some finite $M' = M'(d,\eta)$ and all $n$. Further, with $\mathbb{E}[g_\varepsilon(\mathbf{x})^2] = K_{2\varepsilon}(\mathbf{0})$ uniformly bounded in $\mathbf{x}$, we have by Borell-TIS inequality and the symmetry of $g_\varepsilon(\cdot)$, that

$$\mathbb{P}\Big(\sup_{\|\mathbf{x}\|\leq n}|g_\varepsilon(\mathbf{x})|>2M'n\Big)\leq 2e^{-M'^2n^2/2K_{2\varepsilon}(\mathbf{0})}.$$

Hence, by the Borel–Cantelli lemma, almost surely $\sup_{\|\mathbf{x}\|\leq n}|g_\varepsilon(\mathbf{x})|\leq 2M'n$ for all $n\geq N(\omega)$ large enough, so $|g_\varepsilon(\mathbf{x})|\leq a\|\mathbf{x}\|+b$, for $a = 2M'$ and $b = b(\omega) = \sup_{\|\mathbf{x}\|\leq N(\omega)}|g_\varepsilon(\mathbf{x})|$ is a.s. finite [since $N(\omega)$ is a.s. finite and $g_\varepsilon \in \mathcal{A}$]. Finally, to have such growth condition hold for *all* $\omega$, let $g_\varepsilon(\cdot)\equiv 0$ on the null set where $N(\omega) = \infty$, which neither affects the law of $g_\varepsilon(\cdot)$ nor its sample path continuity. □

LEMMA 5.2. *Let $g \in \mathcal{A}$ satisfy $|g(\mathbf{x})| \le a\|\mathbf{x}\| + b$ for some $a, b$ finite. Then, for any $d = 1, \dots,$ and $\varepsilon > 0$, setting $\mathbb{D}_\varepsilon = \mathbb{R}^d \times (\eta, \infty)$, the function*

$$\phi(\mathbf{x}, t) = \int_{\mathbb{R}^d} K_{t-\varepsilon}(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^d} K_{t-\varepsilon}(\mathbf{y}) g(\mathbf{x} - \mathbf{y})\, d\mathbf{y} \tag{5.2}$$

*is a solution in $\mathcal{C}_\varepsilon := \mathcal{C}^{2,1}(\mathbb{D}_\varepsilon)$ of the heat equation (1.11), and the unique such solution which converges to $g(\mathbf{x})$ for $t \downarrow \eta$ and satisfies the growth condition $|\phi(\mathbf{x}, t)| \le p\|\mathbf{x}\| + q\sqrt{t} + r$ for some finite constants $p, q, r$.*

PROOF. Since $K_s(\cdot)$ is a probability density on $\mathbb{R}^d$ such that $\int \|\mathbf{u}\|^2 K_s(\mathbf{u})\, d\mathbf{u} = 2ds$, from the given growth condition of $g(\cdot)$ it follows that for any $t > \eta$,

$$\left|\phi(\mathbf{x}, t)\right| \le b + a\|\mathbf{x}\| + a \int_{\mathbb{R}^d} \|\mathbf{y}\| K_{t-\varepsilon}(\mathbf{y})\, d\mathbf{y} \le b + a\|\mathbf{x}\| + a\sqrt{2d(t - \varepsilon)}.$$

Thus, $\phi(\cdot, \cdot)$ of (5.2) is well defined and satisfies the growth condition (with $p = a$, $q = a\sqrt{2d}$ and $r = b$). With $\phi(\mathbf{x}, \varepsilon + s)$ alternatively being the expected value of $g(\mathbf{x} - \sqrt{s}\mathbf{U})$ for a standard multivariate normal $\mathbf{U}$, dominated convergence provides its convergence to $g(\mathbf{x})$ (uniformly on compacts), as $s \downarrow 0$.

To confirm that $\phi \in \mathcal{C}_\varepsilon$ satisfies the heat equation (1.11) on $\mathbb{D}_\varepsilon$, note that

$$\begin{aligned}
\phi(\mathbf{x}, t) &= K_{t-\varepsilon}(\mathbf{x}) F\left(\frac{\mathbf{x}}{2(t - \varepsilon)}, \frac{1}{4(t - \varepsilon)}\right), \\
F(\boldsymbol{\theta}_1, \theta_2) &:= \int_{\mathbb{R}^d} e^{\boldsymbol{\theta}_1' \mathbf{y} - \theta_2 \mathbf{y}'\mathbf{y}} g(\mathbf{y})\, d\mathbf{y}.
\end{aligned}$$

Clearly, $K_t(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{D}_0)$ and combining the assumed linear growth of $g(\cdot)$ with dominated convergence, we have that also $F \in \mathcal{C}^\infty(\mathbb{D}_0)$. Hence, $\phi \in \mathcal{C}_\varepsilon$ and by the same reasoning, each partial derivative of $\phi(\cdot, \cdot)$ can be taken within the integral (5.2) over $\mathbf{y}$. As $K_t(\mathbf{x})$ satisfies (1.11) on $\mathbb{D}_0$, it thus follows that $\phi(\cdot)$ satisfies this PDE on $\mathbb{D}_\varepsilon$. Finally, the uniqueness of solution of (1.11) in $\mathcal{C}_\varepsilon$ subject to the assumed linear growth condition and the given initial condition $g \in \mathcal{A}$ at $t = \varepsilon$, is well known (e.g., see [7], Theorem 2.3.7, for uniqueness on $[\varepsilon, T]$, any $T > 0$). $\square$

We now complete the proof of Theorem 1.5 by combining the preceding lemmas with Kolmogorov's extension theorem (to construct one measurable solution over all of $\mathbb{D}_0$).

PROOF OF THEOREM 1.5. Fixing $\delta = 2\varepsilon > 0$, by Lemma 5.1 there exists centered $(\mathcal{A}, \mathcal{B}_\mathcal{A})$-valued Gaussian field $g_\varepsilon(\cdot)$ of law $\mathbb{P}_\varepsilon$ corresponding to covariance function $K_\delta(\mathbf{x}_1 - \mathbf{x}_2)$. We claim that $\phi|_\varepsilon = \mathbb{T}_\varepsilon(g_\varepsilon)$ given by (5.2) for $t \ge \delta$, is

$(\mathcal{C}_\delta, \mathcal{B}_{\mathcal{C}_\delta})$-measurable. Indeed, consider smooth $\psi : \mathbb{R} \mapsto [0,1]$ supported on $\mathbb{R}_+$ such that $\psi(r) = 1$ for $r \geq 1$ and let $\hat{\phi}_n = \mathbb{T}_{\varepsilon,n}(g_\varepsilon)$, given by

$$\hat{\phi}_n(\mathbf{x}, t) = \int_{\mathbb{R}^d} \psi\big(n - \|\mathbf{x} - \mathbf{y}\|^2\big) K_{t-\varepsilon}(\mathbf{x} - \mathbf{y}) g_\varepsilon(\mathbf{y})\, d\mathbf{y}.$$

Since these integrals are over bounded domains of $\mathbf{y}$ values and $(\mathbf{x}, t) \mapsto K_{t-\varepsilon}(\mathbf{x})\psi(n - \|\mathbf{x}\|^2)$ is smooth for $t \geq \delta > \varepsilon$, each mapping $\mathbb{T}_{\varepsilon,n} : (\mathcal{A}, \mathcal{B}_{\mathcal{A}}) \mapsto (\mathcal{C}_\delta, \mathcal{B}_{\mathcal{C}_\delta})$ is continuous (with respect to the relevant uniform convergence on compacts). Further, by the growth condition of Lemma 5.1 on $g_\varepsilon$, for any $M < \infty$ and multi-index $(\mathbf{r}, \ell)$,

$$\sup_{\|\mathbf{x}\| \leq M, s \in [0,M]} \left| \frac{\partial}{\partial x_{r_1} \cdots \partial x_{r_k}\, \partial s^\ell} \int_{\mathbb{R}^d} K_{s+\varepsilon}(\mathbf{x} - \mathbf{y})\big(1 - \psi\big(n - \|\mathbf{x} - \mathbf{y}\|^2\big)\big) g_\varepsilon(\mathbf{y})\, d\mathbf{y} \right|$$
$$\stackrel{n\to\infty}{\longrightarrow} 0.$$

Consequently, we have that $\mathbb{T}_{\varepsilon,n}(g_\varepsilon) \to \phi|_\varepsilon$ in $\mathcal{C}_\delta$ as $n \to \infty$, yielding the Borel measurability of $\phi|_\varepsilon$.

Let $\mathbb{Q}_\delta = \mathbb{P}_\varepsilon \circ \mathbb{T}_\varepsilon^{-1}$ denote the centered Gaussian law of $\phi|_\varepsilon$ thus induced on $(\mathcal{C}_\delta, \mathcal{B}_{\mathcal{C}_\delta})$ by (5.2). For any $\delta' > \delta \geq 0$, clearly $\mathbb{D}_\delta \subset \mathbb{D}_{\delta'}$ making the identity map a projection $\pi_{\delta,\delta'} : \mathcal{C}_\delta \mapsto \mathcal{C}_{\delta'}$, with the complete, separable, metrizable space $\mathcal{C}_0$ being homeomorphic to the projective limit of $\{\mathcal{C}_\delta, \delta > 0\}$ (with respect to these projections). It is easy to check that for all $t, s \geq \delta$,

$$\begin{aligned}
&\mathbb{E}\big[\phi\big|_\varepsilon(\mathbf{x}_1, t)\phi\big|_\varepsilon(\mathbf{x}_2, s)\big] \\
&\quad = \iint K_{t-\varepsilon}(\mathbf{x}_1 - \mathbf{y}_1) K_{s-\varepsilon}(\mathbf{x}_2 - \mathbf{y}_2) C_\varepsilon(\mathbf{y}_1, \mathbf{y}_2)\, d\mathbf{y}_1\, d\mathbf{y}_2 \\
&\quad = K_{t+s}(\mathbf{x}_1 - \mathbf{x}_2),
\end{aligned}$$

is independent of $\varepsilon > 0$. In particular, for any $\delta' > \delta > 0$ the Borel probability measure $\mathbb{Q}_{\delta'}$ on $\mathcal{C}_{\delta'}$ is just the push-forward of $\mathbb{Q}_\delta$ via the projection $\pi_{\delta,\delta'}$. Consequently, setting the f.d.d. of $\{\phi|_{\varepsilon'}(\cdot) : \varepsilon' \geq \varepsilon\}$ on $(0, \infty)$ to match those of $\{\pi_{2\varepsilon,2\varepsilon'}(\phi|_\varepsilon) : \varepsilon' \geq \varepsilon\}$ yields a consistent collection, so Kolmogorov's extension theorem provides existence of Borel probability measure $\mathbb{Q}_0$ on $\mathcal{C}_0$ such that each $\mathbb{Q}_\delta$ is the push-forward of $\mathbb{Q}_0$ by $\pi_{0,\delta}$ (see, e.g., [6], Theorems 12.1.2 and 13.1.1). In particular, $\mathbb{Q}_0$ corresponds to a centered Gaussian field $\phi_d \in \mathcal{C}_0$ having the same covariance as its restrictions $\phi|_\varepsilon$ to subdomains $\mathbb{D}_{2\varepsilon}$. As each $\phi|_\varepsilon$ satisfies (1.11) on $\mathbb{D}_\varepsilon$, clearly $\phi_d$ satisfies it throughout $\mathbb{D}_0$ and the identity (1.14) further follows from our explicit construction via (5.2) of the restriction of $\phi_d$ to $\mathbb{D}_{t_1}$ [by utilizing Fubini's theorem, the growth condition of Lemma 5.1 and convolution properties of the Brownian semigroup $t \mapsto K_t(\cdot)$]. Finally, $\phi_d \in \mathcal{C}^\infty(\mathbb{D}_0)$ by the integral representation (1.14) and smoothness of $(\mathbf{x}, t) \mapsto K_t(\mathbf{x})$. $\square$

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