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BIASED RANDOM WALK IN POSITIVE RANDOM CONDUCTANCES ON $\mathbb{Z}^d$

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We study the biased random walk in positive random conductances on $\mathbb{Z}^d$. This walk is transient in the direction of the bias. Our main result is that the random walk is ballistic if, and only if, the conductances have finite mean. Moreover, in the sub-ballistic regime we find the polynomial order of the distance moved by the particle. This extends results obtained by Shen [*Ann. Appl. Probab.* **12** (2002) 477–510], who proved positivity of the speed in the uniformly elliptic setting.

1. Introduction. One of the most fundamental questions in random walks in random media is understanding the long-term behavior of the random walk. This topic has been intensively studied, and we refer the reader to [29] for a general survey of the field. An interesting feature of random walks in random environments (RWRE) is that several models exhibit anomalous behaviors. One of the main reasons for such behaviors is trapping, a phenomenon observed by physicists long ago [19] and which is a central topic in RWRE. The importance of trapping in several physical models (including RWRE) motivated the introduction of the Bouchaud trap model (BTM). This is an idealized model that received a lot of mathematical attention. A review of the main results can be found in [5], a survey which conjectures that the type of results obtained in the BTM should extend to a wide variety of models, including RWRE.

One very characteristic behavior associated to trapping is the existence of a zero asymptotic speed for RWRE with directional transience. In the last few years, several articles have analyzed such models from a trapping perspective, such as [13] and [14] on $\mathbb{Z}$ and [2, 3] and [17] on trees. The results on the d-dimensional lattice (with $d \geq 2$) are much more rare, since RWRE on $\mathbb{Z}^d$ are harder to analyze. Among the most natural examples of directionally transient RWRE in $\mathbb{Z}^d$ are biased random walks in random conductances. So far, mathematically, only two models of biased random walks in $\mathbb{Z}^d$ have been studied from a trapping perspective: one is on a supercritical percolation cluster and the other is in environments assumed to be uniformly elliptic. Before further discussing trapping issues, we wish to mention that biased random walks in random environment also raise many other interesting questions, such as the Einstein relation which has led to many new developments, see [4, 16] and [21].

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In the case of biased random walks on a percolation cluster in $\mathbb{Z}^d$, it was shown in [6] (for $d = 2$) and in [26] that the walk is directionally transient and, more interestingly, there exists a zero-speed regime. More recently in [15] a characterization of the zero-speed regime has been achieved. Those results confirmed the predictions of the physicists that trapping occurs in the model; see [8] and [9].

In the case of a biased random walk in random conductances which are uniformly elliptic, it has been shown in [25] that the walk is directionally transient and has always positive speed and verifies an annealed central limit theorem. These results are coherent with the conjecture that, a directionally transient random walk in random environment which is uniformly elliptic, should have positive speed; see [27]. Hence, trapping does not seem to appear under uniform ellipticity conditions.

The results on those two models do not bring any understanding of the behavior of the random walk in positive conductances that might be arbitrarily close to zero. In such a model, we truly lose the uniform elliptic assumption, as opposed to the biased random walk on the percolation cluster, where the walk is still uniformly elliptic on the graph where the walk is restricted.

Our purpose in this paper is to understand the ballistic-regime of a biased random walk in positive i.i.d. conductances and how trapping arises in such a model.

2. Model. We introduce $\mathbf{P}[\cdot] = P_*^{\otimes E(\mathbb{Z}^d)}$, where P_* is the law of a positive random variable $c_* \in (0, \infty)$. This measure gives a random environment usually denoted ω.

In order to define the random walk, we introduce a bias $\ell = \lambda\vec{\ell}$ of strength $\lambda > 0$ and a direction $\vec{\ell}$ which is in the unit sphere with respect to the Euclidian metric of $\mathbb{R}^d$. In an environment ω, we consider the Markov chain of law P_x^ω on $\mathbb{Z}^d$ with transition probabilities $p^\omega(x, y)$ for $x, y \in \mathbb{Z}^d$ defined by:

(1) $X_0 = x$, P_x^ω-a.s.,
(2) $p^\omega(x, y) = \frac{c^\omega(x,y)}{\sum_{z\sim x} c^\omega(x,z)}$,

where $x \sim y$ means that x and y are adjacent in $\mathbb{Z}^d$, and also we set

$$\text{for all } x \sim y \in \mathbb{Z}^d \qquad c^\omega(x, y) = c_*^\omega([x, y])e^{(y+x)\cdot\ell}. \tag{2.1}$$

This Markov chain is reversible with invariant measure given by

$$\pi^\omega(x) = \sum_{y\sim x} c^\omega(x, y).$$

The random variable $c^\omega(x, y)$ is called the conductance between x and y in the configuration ω. This comes from the links existing between reversible Markov chains and electrical networks. We refer the reader to [10] and [23] for a further background on this relation, which we will use extensively. Moreover for an edge $e = [x, y] \in E(\mathbb{Z}^d)$, we denote $c^\omega(e) = c^\omega(x, y)$.

Finally the annealed law of the biased random walk will be the semi-direct product $\mathbb{P} = \mathbf{P}[\cdot] \times P_0^\omega[\cdot]$.

In the case where $c_* \in (1/K, K)$ for some $K < \infty$, the walk is uniformly elliptic, and this model is the one previously studied in [25].

3. Results. First, we prove that the walk is directionally transient.

PROPOSITION 3.1. *We have*

$$\lim X_n \cdot \vec{\ell} = \infty, \qquad \mathbb{P}\text{-}a.s.$$

This proposition is a consequence of Proposition 7.1.

Our main result is:

THEOREM 3.1. *For $d \geq 2$, we have*

$$\lim \frac{X_n}{n} = v, \qquad \mathbb{P}\text{-}a.s.,$$

where:

(1) *if $E_*[c_*] < \infty$, then $v \cdot \vec{\ell} > 0$;*
(2) *if $E_*[c_*] = \infty$, then $v = \vec{0}$.*

Moreover, if $\lim \frac{\ln P_*[c_* > n]}{\ln n} = -\gamma$ with $\gamma < 1$, then

$$\lim \frac{\ln X_n \cdot \vec{\ell}}{\ln n} = \gamma, \qquad \mathbb{P}\text{-}a.s.$$

This theorem follows from Propositions 8.1, 9.1, Lemma 9.2 and Proposition 9.2.

This result proves that trapping phenomena may occur in an elliptic regime, that is, when all transition probabilities are positive.

Let us rapidly discuss the different main ways the walk may be trapped (see Figure 1):

(1) an edge with high conductance surrounded by normal conductances;
(2) a normal edge surrounded by very small conductances.

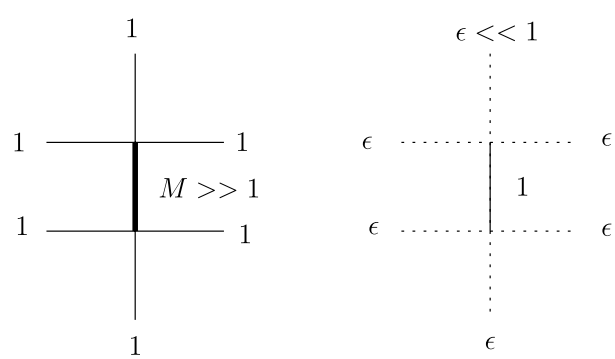

FIG. 1. *The two main types of traps.*

Let us discuss how the first type of traps function. Assume that we have an edge e of conductance $c_*(e)$ surrounded by edges of fixed conductances, say 1. A simple computation shows that the walk will need a time of the order of $c_*(e)$ to leave the endpoints of e. Hence, if the expectation of c_* is infinite, then the annealed exit time of the e is infinite. Heuristically, one edge is enough to trap the walk strongly. This phenomenon is enough to explain the zero-speed regime.

At first glance it is surprising that, in Theorem 3.1, there is only a condition on the tail of c_* at infinity. Indeed, if the tail of c_* at 0 is sufficiently big, more precisely such that $E[\min_{i=1,\dots,4d-2} 1/c_*^{(i)}] = \infty$ for $c_*^{(i)}$ i.i.d. chosen under P_*, then the second type of traps are such that the annealed exit time of the central edge is infinite. This condition does not appear in Theorem 3.1 because the walk is unlikely to reach such an edge. Indeed, it needs to cross an edge with extremely low conductance to enter the trap. This type of trapping is barely strong enough to create a zero-speed regime (see Remark 9.1), nevertheless it forces us to be very careful in our analysis of the model.

One may try to create traps similar to those encountered in the biased random walk on the percolation cluster. In this model, if the bias is high enough, a long dead-end in the direction of the drift can trap the walk strongly enough to force zero-speed. In our context, we are not allowed to use zero conductances, but we may use extremely low conductances, forcing the walk to exit the dead-end at the same place it entered. Nevertheless, this type of trap is very inefficient. Indeed, most edges forming a dead-end have to verify $c_*(e) < \varepsilon$ to be able to contain the walk for a long period, and this for any fixed $\varepsilon > 0$. The probabilistic cost of creating such a trap is way too high.

Hence, small conductances cannot create zero-speed, but high conductances can. To conclude, we give an idealized version of the two most important types of traps in this model,

$$X_1 = \mathrm{Geom}\big((1/c_*) \wedge 1\big)$$

or

$$X_2 = \begin{cases} \mathrm{Geom}(c'_* \wedge 1), & \text{with probability } c'_* \wedge 1, \\ 0, & \text{else,} \end{cases}$$

where c_* is chosen according to the law P_*, and c'_* has the law of

$$\max_{i=1,\dots,4d-2} c_*^{(i)}$$

where $c_*^{(i)}$ are i.i.d. chosen under the law P_* and independent of the geometric random variables. Intuitively, one should be able to understand anything related to trapping with biased random walks in an elliptic setting using those idealized traps.

Before moving on, let us say a word about the central limit theorem, which is known to hold in the uniformly elliptic case (see [25]), at least in the annealed

setting. Using the idealized model we just described above, we are led to believe that if the two following conditions hold:

(1) $P_*[1/c_* \geq x] \leq x^{-1/(4d-2)-\varepsilon}$ for some $\varepsilon > 0$,
(2) $P_*[c_* \geq x] \leq x^{-2-\varepsilon}$ for some $\varepsilon > 0$,

then an annealed central theorem should hold. These conditions should in some weak sense be necessary as indicated in Remark 9.2. It is interesting to note that even though the tail of $1/c_*$ at 0 does not affect the law of large numbers, it is, however, important for the central limit theorem. Although strongly related to the estimates made in this article, the central limit theorem is not a directed consequence of them, and due to the length of this paper, we choose not to pursue this issue further.

Let us explain the organization of the paper. We begin by studying exit probabilities of large boxes; the main point of Section 5 is to prove Theorem 5.1 which is a property similar to Sznitman's conditions (T) and $(T)_\gamma$; see [27]. This property is one of the key estimates for studying directionally transient RWRE. It allows us to define regeneration times, similar to the ones introduced in [28], and study them; this is done in Section 7. The construction of regeneration times in this model is complicated by the fact that we lack any type of uniform ellipticity. This issue is explained in more details and dealt with in Section 6. The law of large numbers in the positive speed regime is obtained in Section 8. The zero-speed regime is studied in Section 9. The next section is devoted to notations which will be used throughout this paper.

4. Notations. Let us denote by $(e_i)_{i=1,\dots,d}$ an orthonormal basis of $\mathbb{Z}^d$ such that $e_1 \cdot \vec{\ell} \geq e_2 \cdot \vec{\ell} \geq \cdots \geq e_d \cdot \vec{\ell} \geq 0$; in particular we have $e_1 \cdot \vec{\ell} \geq 1/\sqrt{d}$. The set $\{\pm e_1, \dots, \pm e_d\}$ will be denoted by ν. Moreover, we complete $f_1 := \vec{\ell}$ into an orthonormal basis $(f_i)_{1 \leq i \leq d}$ of $\mathbb{R}^d$.

We set

$$\mathcal{H}^+(k) = \{x \in \mathbb{Z}^d; x \cdot \vec{\ell} > k\} \quad \text{and} \quad \mathcal{H}^-(k) = \{x \in \mathbb{Z}^d; x \cdot \vec{\ell} \leq k\}$$

and

$$\mathcal{H}_x^+ = \mathcal{H}^+(x \cdot \vec{\ell}) \quad \text{and} \quad \mathcal{H}_x^- = \mathcal{H}^-(x \cdot \vec{\ell}).$$

For any graph G, let us introduce $d_G(x, y)$ the graph distance in G between x and y. Define for $x \in G$ and $r > 0$

$$B_G(x, r) = \{y \in G; d_G(x, y) \leq r\}.$$

Given a set V of vertices of $\mathbb{Z}^d$, we denote by $|V|$ its cardinality, by $E(V) = \{[x, y] \in E(\mathbb{Z}^d); x, y \in V\}$ its edges and

$$\partial V = \{x \notin V; y \in V \text{ and } x \sim y\}$$

as well as

$$\partial_E V = \{[x, y] \in E(\mathbb{Z}^d); x \in V \text{ and } y \notin V\},$$

its borders.

Given a set E of edges of $\mathbb{Z}^d$, we denote $V(E) = \{x \in \mathbb{Z}^d; x$ is an endpoint of $e \in E\}$ its vertices.

Denote for any $L, L' \geq 1$

$$B(L, L') = \{z \in \mathbb{Z}^d; |z \cdot \vec{\ell}| \leq L \text{ and } |z \cdot f_i| \leq L' \text{ for } i \in [2, d]\}$$

and

$$\partial^+ B(L, L') = \{z \in \partial B(L, L'); z \cdot \vec{\ell} > L\}.$$

We introduce the following notation. For any set of vertices A of a certain graph on which a random walk X_n is defined, we denote

$$T_A = \inf\{n \geq 0; X_n \in A\}, \qquad T_A^+ = \inf\{n \geq 1; X_n \in A\}$$

and

$$T_A^{\text{ex}} = \inf\{n \geq 0; X_n \notin A\}.$$

We will use a slight abuse of notation and write x instead of $\{x\}$ when the set is a point x.

This allows us to define the hitting time of "level" n by

$$\Delta_n = T_{\mathcal{H}^+(n)}.$$

Also, θ_n will denote the time shift by n units of time.

Finally, we will use the notation $P[A, B]$ to designated $P[A \cap B]$, when we are given a probability measure P and two sets A and B. Similarly given a random variable X and an event A, we may use the notation $E[X, A]$ to designate $E[X\mathbf{1}\{A\}]$ when there is no confusion possible.

In this paper constants are denoted by $c \in (0, \infty)$ or $C \in (0, \infty)$ without emphasizing their dependence on d and the law P_*. Moreover the value of those constants may change from line to line.

5. Exit probability of large boxes. Our first goal is to obtain estimates on the exit probabilities of large boxes, which will allow us to prove directional transience and is key to analyzing this model. In particular, we aim at showing:

THEOREM 5.1. *For $\alpha > d + 3$*

$$\mathbb{P}[T_{\partial B(L, L^\alpha)} \neq T_{\partial^+ B(L, L^\alpha)}] \leq Ce^{-cL}.$$

After this section α will be fixed, greater than $d+3$.

We will adapt a strategy of proof used in [15]. For the most part, the technical details and notations are simpler in our context. We will go over the parts of the proof which can be simplified, but we will eventually refer the reader to [15] for the conclusion of the proof which is exactly similar in both cases. The notations have been chosen so that the reader can follow the needed proofs in [15] to the word.

First, let us describe the strategy we will follow.

The fundamental idea is to partition the space into a good part where the walk is well-behaved and a bad part consisting of small connected components where we have very little control over the random walk.

The strategy is two-fold:

(1) We may study the behavior of the random walk at times where it is in the good part of the space, in which it can easily be controlled. We will refer to this object as the modified walk. We show that the modified walk behaves nicely, that is, verifies Theorem 5.1. This is essentially achieved using a combination of spectral gap estimates and the Carne–Varopoulos formula [7].

(2) We need to show that information on the exit probabilities for this modified random walk allows us to derive interesting statements on the actual random walk. This is a natural thing to expect, since the bad parts of the environment are small.

A more detailed discussion of the strategy of proof can be found at the beginning of Section 7 in [15].

5.1. *Bad areas*. We say that an edge e is K-normal if $c_*(e) \in [1/K, K]$, where K will be taken to be very large in the sequel. If an edge is not K-normal, we will say it is K-abnormal which occurs with arbitrarily small probability $\varepsilon(K) := P_*[c_* \notin [1/K, K]]$, since $c_* \in (0, \infty)$.

In relation to this, we will say that a vertex x is K-open if for all $y \sim x$ the edge $[x, y]$ is K-normal. If a vertex is not K-open, we will say it is K-closed. By taking K large enough, the probability that a vertex is K-open goes to 1. Finally a vertex $x \in \mathbb{Z}^d$ is K-good if there exists an infinite directed K-open path starting at x; that is, we have $\{x = x_0, x_1, x_2, x_3, \dots\}$ with $x_0 = x$ such that for all $i \geq 0$:

(1) we have $x_{2i+1} - x_{2i} = e_1$ and $x_{2i+2} - x_{2i+1} \in \{e_1, \dots, e_d\}$;
(2) x_i is K-open.

If a vertex is not K-good, it is said to be K-bad. The key property of a good point will be that there exists a open path $(x_i)_{i \geq 0}$ such that $x_0 \cdot \vec{\ell} < x_1 \cdot \vec{\ell} \leq x_2 \cdot \vec{\ell} < x_3 \cdot \vec{\ell} \leq \cdots$, and also this path verifies $(x_i - x_0) \cdot \vec{\ell} \geq c(d)i$.

To ease the notation, which will be used repeatedly throughout the article, we will not always mention the K-dependences.

The first of these results is stated in terms of the width of a subset $A \subseteq \mathbb{Z}^d$, which we define to be

$$W(A) = \max_{1 \le i \le d} \Big(\max_{y \in A} y \cdot e_i - \min_{y \in A} y \cdot e_i \Big).$$

Let us denote $\mathrm{BAD}_K(x)$ the connected component of K-bad vertices containing x, in case x is good then $\mathrm{BAD}_K(x) = \varnothing$.

LEMMA 5.1. *There exists K_0 such that, for any $K \ge K_0$ and for any $x \in \mathbb{Z}^d$, we have that the cluster $\mathrm{BAD}_K(x)$ is finite $\mathbf{P}_p[\cdot]$-a.s. and*

$$\mathbf{P}_p\big[W\big(\mathrm{BAD}_K(x)\big) \ge n\big] \le C \exp\big(-\xi_1(K) n\big),$$

where $\xi_1(K) \to \infty$ as K tends to infinity.

PROOF. We call two vertices 2-connected if $\|u - v\|_1 = 2$, so that we may define $\mathrm{BAD}_K^e(x)$ as the 2-connected component of bad vertices containing x. Any element of $\mathrm{BAD}(x)$ is a neighbor of $\mathrm{BAD}_K^e(x)$ so that $W(\mathrm{BAD}_K(x)) \le W(\mathrm{BAD}_K^e(x)) + 2$.

Consider now the site percolation model on the even lattice $\mathbb{Z}^d_{\mathrm{even}} = \{v \in \mathbb{Z}^d, \|v\|_{t_1} \text{ is even}\}$ where y is even-open if and only if, in the original model, the vertices y, $y + e_1$ and $y + e_1 + e_i$ $(i \le d)$ are open. An edge $[y, z]$ is even-open if, and only if, y and z are even-open. This last model is a 4-dependent oriented bond percolation model, which has a measure that we denote by $P_{p,\mathrm{orient}}$.

Fix p' close to 1. For K large enough, the probability that a vertex is K-open can be made arbitrarily close to 1. This means that we can make the probability of an edge being even-open arbitrarily close to 1, so, by Theorem 0.0 in [22], the law $P_{p,\mathrm{orient}}$ dominates an i.i.d. bond percolation with parameter p'.

Let us introduce the outer edge-boundary $\partial_E \mathrm{BAD}_K^e(x)$ of $\mathrm{BAD}_K^e(x)$ in the graph $\mathbb{Z}^d_{\mathrm{even}}$ with the following notion of adjacency: x and y are adjacent if $x - y \in \{\pm(e_i \pm e_j) \text{ with } i \ne j \text{ and } i, j \le d\}$.

We describe how to do the proof for $d = 2$. We will assign an arrow to any edge $[y, z] \in \partial_E \mathrm{BAD}_K^e(x)$, and assuming $y \in \mathrm{BAD}_K^e(x)$ and $z \notin \mathrm{BAD}_K^e(x)$, we set:

(1) $\swarrow$, if $y - x = e_1 - e_2$;
(2) $\nwarrow$, if $y - x = e_1 + e_2$;
(3) $\nearrow$, if $y - x = -e_1 + e_2$;
(4) $\searrow$, if $y - x = -e_1 - e_2$.

This boundary is represented dually in Figure 2. By an argument similar to that of Durrett [11], page 1026, we see that $n_{\nearrow} + n_{\searrow} = n_{\swarrow} + n_{\nwarrow}$, where $n_{\nearrow}$, for example, is the number of edges labeled $\nearrow$ in $\partial_E \mathrm{BAD}_K^e(x)$.

(1) Any $\nwarrow$ edge of $\partial_E \mathrm{BAD}_K^e(x)$ has one endpoint, say y, which is bad, and one, $y + e_1 + e_2$, which is good. This implies that y is even-closed. Now, we will

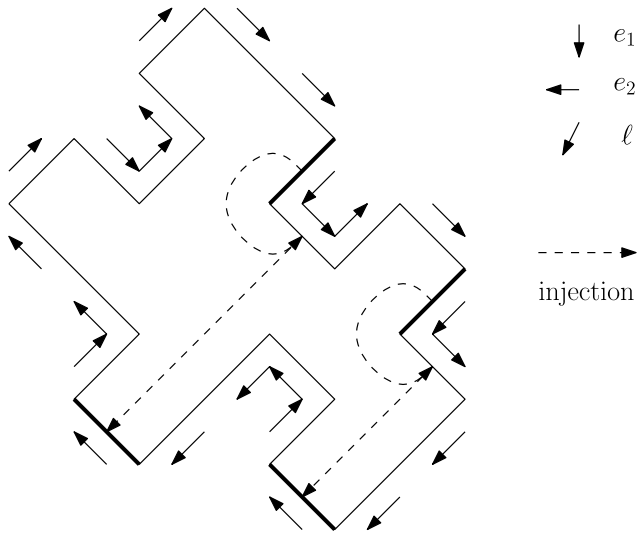

FIG. 2. *The outer edge-boundary* $\partial_E \mathrm{BAD}_K^e(x)$ *of* $\mathrm{BAD}_K^e(x)$ *represented dually on the even lattice when* $d = 2$.

argue that if $n_{\nwarrow} \geq n_{\swarrow}/2$, then at least one sixth of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are $\nwarrow$ edges, and hence even-closed. Indeed, since $n_{\nearrow} + n_{\searrow} = n_{\swarrow} + n_{\nwarrow}$, we know that $n_{\swarrow} + n_{\nwarrow} = |\partial_E \mathrm{BAD}_K^e(x)|/2$ and also, using our hypothesis, $n_{\swarrow} + n_{\nwarrow} \leq 3n_{\nwarrow}$. Those two inequalities imply that $n_{\nwarrow} \geq |\partial_E \mathrm{BAD}_K^e(x)|/6$. Hence, at least one sixth of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are even-closed.

(2) Otherwise, let us assume that $n_{\swarrow} \geq 2n_{\nwarrow}$. We may notice any $\swarrow$ edge followed (in the sense of the arrows) by an $\searrow$ edge can be mapped in an injective manner to an $\nwarrow$ edge. This injection is indicated in Figure 2 (by considering the bold edges). This injection, with $n_{\swarrow} \geq 2n_{\nwarrow}$, means that at least half of the $\swarrow$ edges are not followed by an $\searrow$ edge. So, using that $n_{\nearrow} + n_{\searrow} = n_{\swarrow} + n_{\nwarrow}$, we see that at least one sixth of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are $\swarrow$ edges that are not followed by an $\searrow$ edge. Consider such an $\swarrow$ edge, and we can see that the endpoint y of the $\swarrow$ edge which is inside $\mathrm{BAD}_K^e(x)$ verifies that $y + 2e_1$ is not in $\mathrm{BAD}_K^e(x)$. Hence for any such $\swarrow$, there is one endpoint y which is bad and such that $y + 2e_1$ is good, and hence y is even-closed. Once again at least one sixth of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are even-closed.

This means that at least one sixth of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are even-closed. The outer boundary is a minimal cutset, as described in [1]. The number of such boundaries of size n is bounded (by Corollary 9 in [1]) by $\exp(Cn)$. Hence, if p' is close enough to 1, a counting argument allows us to obtain the desired exponential tail for $W(\mathrm{BAD}_K^e(x))$ under $P_{p'}$, and hence under $P_{p,\mathrm{orient}}$ (since the latter is dominated by the former).

For general dimensions, we note that there exists $i_0 \in [2, d]$ such that a proportion at least $1/d$ of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ are edges of the form $[y, y \pm e_{i_0}]$ and

$[y, y \pm e_1]$ for some $y \in \mathbb{Z}^d$. We may then apply the previous reasoning in every plane $y + \mathbb{Z}e_1 + \mathbb{Z}e_{i_0}$ containing edges of $\partial_E \mathrm{BAD}_K^e(x)$ to show that at least a proportion $1/6$ of those edges are even-closed. Thus, at least a proportion $1/6d$ of the edges of $\partial_E \mathrm{BAD}_K^e(x)$ verify the same property. By repeating the same counting argument as in the previous paragraph we can infer the lemma. $\square$

Let us define $\mathrm{BAD}_K = \bigcup_{x\in\mathbb{Z}} \mathrm{BAD}_K(x)$ which is a union of finite sets. Also we set $\mathrm{GOOD}_K = \mathbb{Z}^d \setminus \mathrm{BAD}_K$. We may notice that

$$\text{for any } x \in \mathrm{BAD}_K \qquad \partial\mathrm{BAD}_K(x) \subset \mathrm{GOOD}_K, \tag{5.1}$$

since $\mathrm{BAD}_K(x)$ is a connected component of bad points.

In the sequel K will always be large enough so that $\mathrm{BAD}_K(x)$ are finite for any $x \in \mathbb{Z}^d$.

5.2. *A graph transformation to seal off big traps.* Given a certain configuration ω, we construct a graph ω_K (with conductances) such that the random walk induced by recording only the steps of the original random walk in ω outside of large traps has the same law as the random walk in ω_K.

We denote ω_K the graph obtained from ω by the following transformation. The vertices of ω_K are the vertices of GOOD_K, and the edges of ω_K are:

(1) $\{[x, y], x, y \in \mathrm{GOOD}_K$, with $x \notin \partial\mathrm{BAD}_K$ or $y \notin \partial\mathrm{BAD}_K\}$ and have conductance $c^{\omega_K}([x, y]) := c^{\omega}([x, y])$,

(2) $\{[x, y], x, y \in \partial\mathrm{BAD}_K\}$ (including loops) which have conductance

$$\begin{aligned} c^{\omega_K}\big([x, y]\big) &:= \pi^{\omega}(x) P_x^{\omega}\big[X_1 \in \mathrm{BAD}_K \cup \partial\mathrm{BAD}_K, T_y^+ = T^+_{\partial\mathrm{BAD}_K}\big] \\ &= \pi^{\omega}(y) P_y^{\omega}\big[X_1 \in \mathrm{BAD}_K \cup \partial\mathrm{BAD}_K, T_x^+ = T^+_{\partial\mathrm{BAD}_K}\big], \end{aligned}$$

the last equality being a consequence of reversibility and ensures symmetry for the conductances.

We call the walk induced by X_n on GOOD_K, the walk Y_n defined to be $Y_n = X_{\rho_n}$ where

$$\rho_0 = T_{\mathrm{GOOD}_K} \quad \text{and} \quad \rho_{i+1} = T^+_{\mathrm{GOOD}_K} \circ \theta_{\rho_i},$$

where we recall that $\circ\theta_i$ stands for the time shift by i units of time. This means that ρ_{i+1} is the first time (strictly) after ρ_i when the walk is in GOOD_K. This means that ρ_i are the successive times when X_n is in GOOD_K.

From [15], Proposition 7.2, we have the two following properties.

PROPOSITION 5.1. *The reversible walk defined by the conductances ω_K verifies the two following properties*:

(1) *It is reversible with respect to $\pi^{\omega}(\cdot)$.*

(2) *If started at $x \in \omega_K$, it has the same law as the walk induced by X_n on* GOOD_K *started at x.*

Furthermore, we have:

LEMMA 5.2. *For $x, y \in \mathrm{GOOD}_K$ which are nearest neighbors in $\mathbb{Z}^d$, we have $c^{\omega_K}([x, y]) \geq c^{\omega}([x, y])$.*

Hence, we may notice the following:

REMARK 5.1. We may notice that for any $x \in \mathrm{GOOD}_K$, we have

$$\frac{c}{K} e^{-2\lambda x \cdot \vec{\ell}} \leq \pi^{\omega_K}(x) \leq C K e^{-2\lambda x \cdot \vec{\ell}}$$

and for any $y \in \mathrm{GOOD}_K$ adjacent, in $\mathbb{Z}^d$, to x

$$\frac{c}{K} e^{-2\lambda x \cdot \vec{\ell}} \leq c^{\omega_K}\big([x, y]\big) \leq C K e^{-2\lambda x \cdot \vec{\ell}}.$$

5.3. *Spectral gap estimate in ω_K.* The following arguments are heavily inspired from [26] and use spectral gap estimates. After showing that the spectral gap in ω_K, we can deduce that the walk is likely to exit the box quickly in ω_K. Finally, we need to argue that exiting the box quickly, we should exit it in the direction of the drift. This allows us to obtain Theorem 5.1, once we have argued that the exit probabilities in ω and ω_K are strongly related. This paper only contains the first step of this reasoning, the following ones being treated in [15].

For technical reasons, we introduce the notation

$$\tilde{B}\big(L, L^{\alpha}\big) = \big\{x \in \mathbb{Z}^d, -L \leq x \cdot \vec{\ell} \leq 2L \text{ and } |x \cdot f_i| \leq L^{\alpha} \text{ for } i \geq 2\big\}.$$

Let us introduce the principal Dirichlet eigenvalue of $I - P^{\omega_K}$, in $\tilde{B}(L, L^{\alpha}) \cap \omega_K$

$$\Lambda_{\omega_K}\big(\tilde{B}\big(L, L^{\alpha}\big)\big) = \begin{cases} \inf\big\{\mathcal{E}_{\mathrm{GOOD}_K}(f, f),\ f_{|(\tilde{B}(L, L^{\alpha}) \cap \omega_K)^c} = 0, \\ \qquad\qquad \|f\|_{L^2(\pi(\omega_K))} = 1\big\}, \\ \qquad\qquad \text{when } \tilde{B}\big(L, L^{\alpha}\big) \cap \omega_K \neq \varnothing, \\ \infty, \qquad \text{by convention when } \tilde{B}\big(L, L^{\alpha}\big) \cap \omega_K = \varnothing, \end{cases} \tag{5.2}$$

where the Dirichlet form is defined for $f, g \in L^2(\pi^{\omega_K})$ by

$$\begin{aligned} \mathcal{E}_{\mathrm{GOOD}_K}(f, g) &= \big(f, (I - P^{\omega_K})g\big)_{\pi^{\omega_K}} \\ &= \frac{1}{2} \sum_{x, y \text{ neighbors in } \omega_K} \big(f(y) - f(x)\big)\big(g(y) - g(x)\big) c^{\omega_K}\big([x, y]\big). \end{aligned}$$

We have:

LEMMA 5.3. *For ω such that $\tilde{B}(L, L^{\alpha}) \cap \omega_K \neq \varnothing$, we have*

$$\Lambda_{\omega_K}\big(\tilde{B}\big(L, L^{\alpha}\big)\big) \geq c(K) L^{-(d+1)}.$$

PROOF. From any vertex $x \in \omega_K$, which is a good point, there exists a directed open path $x = p_x(0), p_x(1), \dots, p_x(l_x)$ in ω_K (which are neighbors in $\mathbb{Z}^d$) such that $p_x(i) \in \tilde{B}(L, L^\alpha)$ for $i < l_x$ and $p_x(l_x) \notin \tilde{B}(L, L^\alpha)$. This allows us to say that

$$\max_{i \le l_x} \frac{\pi_{\omega_K}(x)}{c_{\omega_K}([p_x(i+1), p_x(i)])} \le C \max_{i \le l_x} \frac{\pi_\omega(x)}{\pi_\omega(p_x(i))} \le C, \tag{5.3}$$

where we used Lemma 5.2 and Remark 5.1. Moreover $l_x \le CL$.

We use a classical argument of Saloff-Coste [24], and we write for $\|f\|_{L^2(\pi^{\omega_K})} = 1$

$$\begin{aligned} 1 &= \sum_x f^2(x)\pi_{\omega_K}(x) = \sum_x \Big[\sum_i f\big(p_x(i+1)\big) - f\big(p_x(i)\big)\Big]^2 \pi_{\omega_K}(x) \\ &\le \sum_x l_x \Big[\sum_i \big(f\big(p_x(i+1)\big) - f\big(p_x(i)\big)\big)^2\Big] \pi_{\omega_K}(x). \end{aligned}$$

Now by (5.3), we obtain

$$1 \le C \sum_{x,y \text{ neighbors in } \omega_K} \big(f(z) - f(y)\big)^2 c_{\omega_K}\big([x, y]\big) \times \max_{b \in E(\mathbb{Z}^d)} \sum_{x \in \omega_K \cap \tilde{B}(L, L^\alpha), b \in p_x} l_x,$$

where $b \in p_x$ means that $b = [p_x(i), p_x(i+1)]$ for some i. Using that:

(1) $l_x \le CL$ for any $x \in \omega_K$,

(2) $b = [x, y] \in \omega_K$ can only be crossed by paths "p_z" if $b \in E(\mathbb{Z}^d)$ and $z \in B_{\mathbb{Z}^d}(x, CL)$,

we have

$$\max_b \sum_{x \in \tilde{B}(n, n^\alpha), b \in p_x} l_x \le CL^{d+1}$$

and

$$1 \le CL^{d+1} \sum_{x,y \text{ neighbors in } \omega_K} \big(f(z) - f(y)\big)^2 c_{\omega_K}\big([x, y]\big).$$

Since this is true for every f such that $f_{|(\tilde{B}(L, L^\alpha) \cap \omega_K)^c} = 0$ and $\|f\|_{L^2(\pi(\omega_K))} = 1$, we can use (5.2) to see

$$\Lambda_{\omega_K}\big(\tilde{B}(n, n^\alpha)\big) \ge cL^{-(d+1)}. \qquad \square$$

We explained how to obtain Theorem 5.1 at the beginning of Section 5.3. The proof of Theorem 5.1 is almost completely similar to the end of the proof of Theorem 1.4 in [15]. The reader may read Sections 7.4, 7.5 and 7.6 of [15] for the complete details.

To ease this task, the notations have been chosen so that only two minor changes have to be made: in our case $K_\infty = \mathbb{Z}^d$ and $\mathcal{I} = \Omega$.

The proof in [15] uses some reference to previous results, and for the reader's convenience we specify the correspondence. The following results in [15]: Lemma 7.5, Proposition 7.2, Lemma 7.9 and the second part of Lemma 7.6 correspond, respectively, to Lemma 5.1, Proposition 5.1, Lemma 5.3 and (5.1).

A final remark is that any inequality needed on π^{ω_K} can be found in Remark 5.1.

6. Construction of K-open ladder points. A classical tool for analyzing directional transient RWRE is to use a regeneration structure [28]. We call ladder-point a new maximum of the random walk in the direction $\vec{\ell}$. The standard way of constructing regeneration times is to consider successive ladder points and argue that there is a positive probability of never backtracking again. Such a ladder point creates a separation between the past and the future of the random walk leading to interesting independence properties. We call this point a regeneration time.

A major issue in our case is that we do not have any type of uniform ellipticity. Ladder points are conditioned parts of the environment and, at least intuitively, the edge that led us to a ladder point should have uncharacteristically high conductance. Those high conductances (without uniform ellipticity) may strongly hinder the walk from never backtracking and creating a regeneration time. In order to adapt the classical construction we need, in some sense, to show that the environment seen from the particle at a ladder-point is relatively normal. To address this problem, we will prove that we encounter open ladder-points and find tail estimates on the location of the first open ladder-point.

We define the following random variable:

$$\begin{aligned}\mathcal{M}^{(K)} &= \inf\{i \geq 0, X_i \text{ is } K\text{-open and for } j < i-2, X_j \cdot \vec{\ell} < X_{i-2} \cdot \vec{\ell} \\ &\qquad\qquad \text{and } X_i = X_{i-1} + e_1 = X_{i-2} + 2e_1\} \\ &\leq \infty.\end{aligned}$$

This means at that point we have reached a new maximum of the trajectory (in the direction $\vec{\ell}$), made two steps in the direction e_1 and reaching and open site. This definition is just slightly different than the first open ladder point; it is only for technical reasons that we consider this definition.

Our goal for this section is to obtain properties on this random variable, namely that it is finite and has arbitrarily high polynomial moments.

The dependence on K will be dropped outside of major statements and definitions.

6.1. *Preparatory lemmas.* We need three preparatory lemmas before turning to the study of $\mathcal{M}^{(K)}$. For this, we introduce the inner positive boundary of $B(n, n^\alpha)$

$$\partial_i^+ B(n, n^\alpha) = \{x \in B(n, n^\alpha), \text{ where } x \sim y \text{ with } y \in \partial^+ B(n, n^\alpha)\}$$

and the event

$$A(n) = \{T_{\partial B(n,n^\alpha)} \geq T_{\partial_i^+ B(n,n^\alpha)}\}.$$

It follows from Theorem 5.1 that:

LEMMA 6.1. *We have*

$$\mathbb{P}\big[A(n)^c\big] \leq Ce^{-cn}.$$

We say that a vertex $x \in B(n, n^\alpha)$ is K-n-closed if there exists a nearest neighbor $y \notin \mathcal{H}^+(n)$ of x such that $c^*([x, y]) \notin [1/K, K]$.

Let us denote $\overline{K}_x(n)$ the K-n-closed connected component of x. This allows us to introduce the event

$$B(n) = \big\{\text{for all } x \in \partial_i^+ B\big(n, n^\alpha\big), \text{ we have } \big|\overline{K}_x(n)\big| \leq \ln n\big\}. \tag{6.1}$$

It is convenient to set $\overline{K}_x(n) = \{x\}$ when $\overline{K}_x(n)$ is empty.

LEMMA 6.2. *For any $M < \infty$, we can find K_0 such that for any $K \geq K_0$*

$$\mathbf{P}\big[B(n)^c\big] \leq Cn^{-M}.$$

PROOF. Obviously, for any $x \in \partial^+ B(n, n^\alpha)$

$$\overline{K}_x(n) \subset \mathrm{CLOSED}_K(x),$$

where $\mathrm{CLOSED}_K(x)$ is the connected component of K-closed point containing x.

Using Lemma 5.1 in [20], we may notice that there are at most an exponential number of lattice animals. Hence, for any $x \in \partial^+ B(n, n^\alpha)$

$$\mathbf{P}\big[\big|\mathrm{CLOSED}_K(x)\big| \geq \ln n\big] \leq \sum_{k \geq \ln n} C\big(C_1 \varepsilon(K)\big)^k \leq Cn^{-\xi_2(K)},$$

where $\xi_2(K)$ tends to infinity K goes to infinity. The right-hand side can be made lower than n^{-M} for any M by choosing K large enough. The result follows from a union bound. $\square$

Next, we show a result which, in particular, implies that, if we are on the "positive boundary" (i.e., in the direction of the bias and where the largest conductances are) of a finite connected subset of $\mathbb{Z}^d$ surrounded by normal edges, then we have some lower bound on the probability to exit that set through this positive side.

LEMMA 6.3. *Take $G \neq \varnothing$ to be a finite connected subset of $\mathbb{Z}^d$. Assume that each edge e of $\mathbb{Z}^d$ is assigned a positive conductance $c(e)$ and that there exist $c_1 > 0$, $x \in \partial G$ and $y \in G$ such that $x \sim y$ and $c([x, y]) \geq c_1 c(e)$ for any $e \in \partial_E G$. We have*

$$P_y[T_x \leq T_{\partial G}] \geq \frac{c_1}{4d}|G|^{-1},$$

where P_y is the law of the random walk in $\mathbb{Z}^d$ started at y arising from the conductances $(c(e))_{e\in E(\mathbb{Z}^d)}$.

PROOF. We will be using comparisons to electrical networks, and we refer the reader to Chapter 2 of [23] for further background on this topic.

Let us first notice that a walk started at $y \in G$ will reach ∂G before $\mathbb{Z}^d \setminus (G \cup \partial G)$, so this lemma is actually a result on a finite graph $G \cup \partial G$.

To simplify the proof, we will consider the graph $\tilde{G}$ where all edges emanating from x that are not $[x, y]$ will be assigned conductance 0, which corresponds to reflecting the walk on those edges. It is plain to see that

$$P_y[T_x \le T_{\partial G}] \ge P_y^{\tilde{G}\cup\partial\tilde{G}}[T_x \le T_{\partial\tilde{G}}], \tag{6.2}$$

where $P_y^{\tilde{G}\cup\partial\tilde{G}}$ is the law of the random walk started at y in the conductances of the graph $\tilde{G} \cup \partial\tilde{G}$.

Hence, it is enough to prove our statement in the finite graph $\tilde{G} \cup \partial\tilde{G}$. We may see that

$$P_y^{\tilde{G}\cup\partial\tilde{G}}[T_x \le T_{\partial\tilde{G}}] = u(y), \tag{6.3}$$

where $u(\cdot)$ is the voltage function verifying $u(x) = 1$ and $u(z) = 0$ for $z \in \partial\tilde{G} \setminus \{x\}$. Let us denote $i(\cdot)$ the associate intensity. Since y is the only vertex adjacent to x in $\tilde{G} \cup \partial\tilde{G}$, we know that the current flowing into the circuit at x passes through the edges $[x, y]$, so

$$\frac{1}{R^{\tilde{G}\cup\partial\tilde{G}}(x, \partial\tilde{G} \setminus \{x\})} = i\big([x, y]\big),$$

where $R^{\tilde{G}\cup\partial\tilde{G}}(x, \partial\tilde{G} \setminus \{x\})$ is the effective conductance between x and $\partial\tilde{G} \setminus \{x\}$ in $\tilde{G} \cup \partial\tilde{G}$. By Ohm's law, we may deduce that

$$u(x) - u(y) = r^{\tilde{G}\cup\partial\tilde{G}}\big([x, y]\big) i\big([x, y]\big) = \frac{r^{\tilde{G}\cup\partial\tilde{G}}([x, y])}{R^{\tilde{G}\cup\partial\tilde{G}}(x, \partial\tilde{G} \setminus \{x\})}.$$

Now, since x is the only vertex adjacent to y, we can see by an electrical network reduction of resistances in series that $R^{\tilde{G}\cup\partial\tilde{G}}(x, \partial\tilde{G} \setminus \{x\}) = r^{\tilde{G}\cup\partial\tilde{G}}([x, y]) + R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}(y, \partial\tilde{G} \setminus \{x\})$. This means that

$$\begin{aligned} u(y) &= \frac{R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}(y, \partial\tilde{G} \setminus \{x\})}{R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}(y, \partial\tilde{G} \setminus \{x\}) + r^{\tilde{G}\cup\partial\tilde{G}}([x, y])} \\ &= \left(1 + \frac{r^{\tilde{G}\cup\partial\tilde{G}}([x, y])}{R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}(y, \partial\tilde{G} \setminus \{x\})}\right)^{-1}. \end{aligned}$$

We recall that Rayleigh's monotonocity principle (see [23]) states that increasing any value of the conductance of an edge (in particular merging two vertices)

increases any effective conductances. We consider the graph $\tilde{G} \cup \partial\tilde{G} \setminus \{x\}$ and collapse all vertices $\tilde{G} \cup \partial\tilde{G} \setminus \{x, y\}$ into one vertex δ. This increases all effective conductances. In this new graph, y is connected to δ by at most $|\tilde{G} \cup \partial\tilde{G}|$ edges of conductances at least $(1/c_1)c^{\tilde{G}\cup\partial\tilde{G}}([x, y])$, by our assumptions on the graph. By network reduction of conductances in parallel, this means

$$R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}\big(y, \partial\tilde{G} \setminus \{x\}\big) \geq \frac{c_1}{|\tilde{G} \cup \partial\tilde{G}|} r^{\tilde{G}\cup\partial\tilde{G}}\big([x, y]\big).$$

The two last equations imply, with (6.3), that

$$P_y^{\tilde{G}\cup\partial\tilde{G}}[T_x \leq T_{\partial\tilde{G}}] \geq \frac{c_1}{4d}|G|^{-1},$$

and with (6.2) this concludes the proof. $\square$

6.2. *Successive attempts to find an open ladder point.* Let us denote $\mathcal{B}_n := B(n, n^\alpha)$. We will show that an open ladder point can occur shortly after we exit a box $\mathcal{B}_n$.

Let us denote for $k \leq n$ the events

$$R^{(K)}(nk) = \big\{\mathcal{M}^{(K)} > T_{\partial\mathcal{B}_{nk}} + 2\big\}. \tag{6.4}$$

We have:

LEMMA 6.4. *For any $\varepsilon_1 > 0$ and $M < \infty$, we can find $K_0 = K_0(\varepsilon_1, M)$ large enough such that the following holds: for any $K \geq K_0$ and any $k \in [2, n]$,*

$$\mathbb{P}\big[R^{(K)}(kn)\big] \leq \big(1 - cn^{-\varepsilon_1}\big)\mathbb{P}\big[R^{(K)}\big((k-1)n\big)\big] + Cn^{-M},$$

where the constants depend on K.

Essentially, our goal is to construct an open ladder-point. The idea of the proof roughly goes as follows. If we exited $\mathcal{B}_{(k-1)n}$ without encountering an open ladder point, then:

(1) We are likely to exit $\mathcal{B}_{kn}$ through the positive boundary.

(2) At this point, we look at the K-n-closed component [the corresponding definition is above (6.1)] of that exit point. There is a positive probability that the "positive" boundary of that set is open.

(3) If that is the case, Lemma 6.3 implies that there is a positive chance that we exit through the "positive" side of that set, which is included in $\partial^+\mathcal{B}_{kn}$.

The previous construction implies that we exit $\mathcal{B}_{kn}$ at an open point which is on the positive boundary $\partial^+\mathcal{B}_{kn}$. Hence that point is an open ladder point which can be used to easily construct a point verifying the properties of $\mathcal{M}^{(K)}$.

In the end this means that for each new larger box encountered, $\mathcal{B}_{kn}$, $\mathcal{B}_{(k+1)n}, \dots$, there is a positive chance to encounter an open ladder point through a

procedure which is largely independent of what happened before in smaller boxes. This will allow us to show that eventually we encountered a point with all the properties of $\mathcal{M}^{(K)}$.

PROOF. We introduce the K-n-closed component of the exit point of $\mathcal{B}_n$

$$\mathcal{K}(n) = \overline{K}_{X_{T_{\partial_i^+ \mathcal{B}_n}}}(n) \subseteq \mathcal{B}_n, \tag{6.5}$$

where we recall that the notation $\overline{K}_x(n)$ was defined above (6.1). In case $T_{\partial_i^+ \mathcal{B}_n} = \infty$ (i.e., we never reach the inner positive boundary of $\mathcal{B}_n$) we simply set $\mathcal{K}(n) = \varnothing$ and $\partial\mathcal{K}(n) = \varnothing$. This case typically does not occur.

We introduce the event

$$C(n) = \{\text{for all } x \in \partial\mathcal{K}(n) \cap \mathcal{H}^+(n), \text{ the vertex } x \text{ is open}\}$$

as well as the event

$$D(n) = \{T_{\partial\mathcal{K}(n)} \circ \theta_{T_{\partial_i \mathcal{B}_n}} = T_{\partial\mathcal{K}(n) \cap \mathcal{H}^+(n)} \circ \theta_{T_{\partial_i \mathcal{B}_n}}\}$$

and the event

$$E(n) = \{X_{T_{\partial B(n,n^\alpha)}+2} = X_{T_{\partial B(n,n^\alpha)}+1} + e_1 = X_{T_{\partial B(n,n^\alpha)}} + 2e_1 \text{ and } X_{T_{\partial B(n,n^\alpha)}+2}, X_{T_{\partial B(n,n^\alpha)}+1} \text{ are open}\}.$$

Let us consider an event in $A(n) \cap C(n) \cap D(n) \cap E(n)$. The situation is illustrated in Figure 3. It verifies all the following conditions:

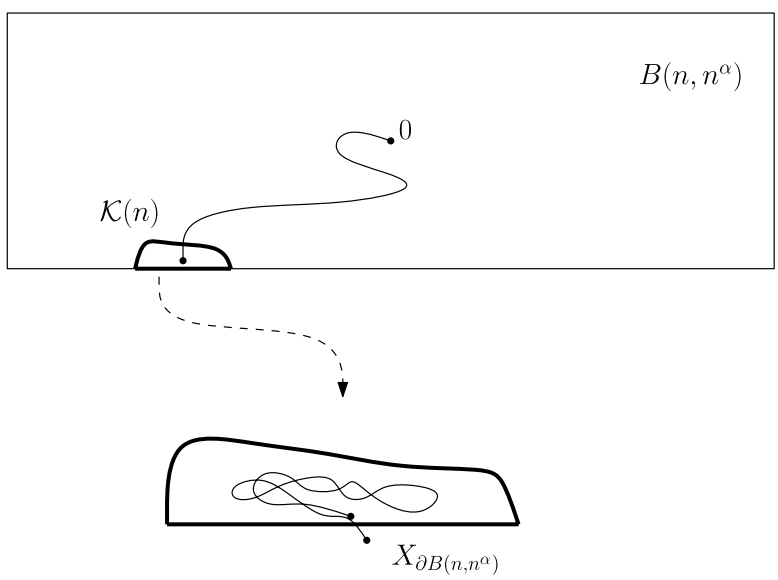

FIG. 3. *A way to find an open ladder point.*

(1) on $A(n)$, we have $T_{\partial_i^+ \mathcal{B}_n} \le T_{\partial \mathcal{B}_n}$, so that

$$T_{\partial B(n,n^\alpha)} \ge T_{\partial \mathcal{K}(n)} \circ \theta_{T_{\partial_i \mathcal{B}_n}};$$

(2) on $D(n)$, by (6.5), we have

$$T_{\partial \mathcal{K}(n)} \circ \theta_{T_{\partial_i \mathcal{B}_n}} = T_{\partial \mathcal{K}(n) \cap \mathcal{H}^+(n)} \circ \theta_{T_{\partial_i \mathcal{B}_n}}$$

and hence

$$T_{\partial B(n,n^\alpha)} = T_{\partial \mathcal{K}(n) \cap \mathcal{H}^+(n)};$$

(3) on $C(n)$, we have $\partial \mathcal{K}(n) \cap \mathcal{H}^+(n)$ is open.

Hence, on $A(n) \cap C(n) \cap D(n) \cap E(n)$, we see that $X_{T_{\partial B(n,n^\alpha)}}$ is a new maximum of the trajectory in the direction $\vec{\ell}$ which is open, $X_{T_{\partial B(n,n^\alpha)}+1} = X_{T_{\partial B(n,n^\alpha)}} + e_1$ and $X_{T_{\partial B(n,n^\alpha)}+2} = X_{T_{\partial B(n,n^\alpha)}} + 2e_1$ is a K-open point. This means that

$$A(n) \cap C(n) \cap D(n) \cap E(n) \subset \{\mathcal{M} \le T_{\partial \mathcal{B}_n} + 2\}.$$

We have

$$\begin{aligned}
\mathbb{P}\big[R(kn)\big] &\le \mathbb{P}\big[R\big((k-1)n\big), \big(A(kn) \cap C(kn) \cap D(kn) \cap E(kn)\big)^c\big] \\
&\le \mathbb{P}\big[A(kn)^c\big] + \mathbb{P}\big[B(kn)^c\big] + \cdots \qquad (6.6)\\
&\quad + \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)^c\big] + \cdots \qquad (6.7)\\
&\quad + \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)^c\big] + \cdots \qquad (6.8)\\
&\quad + \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)^c\big], \qquad (6.9)
\end{aligned}$$

which means that

$$\begin{aligned}
&\mathbb{P}\big[R(kn)\big] - \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn)\big] \\
&\quad \le \mathbb{P}\big[A(kn)^c\big] + \mathbb{P}\big[B(kn)^c\big] \qquad (6.10)\\
&\qquad - \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)\big].
\end{aligned}$$

The first term is controlled by Theorem 5.1,

$$\mathbb{P}\big[A(kn)^c\big] \le C \exp(-ckn) \le C \exp(-cn) \qquad (6.11)$$

and for any $M < \infty$, by Lemma 6.2, we can choose K large enough such that

$$\mathbb{P}\big[B(kn)^c\big] \le n^{-M}. \qquad (6.12)$$

To finish the proof, we are going to control the terms in (6.7), (6.8) and (6.9) which will allow us to evaluate (6.10).

Step 1: *Control of the term in* (6.7). We recall that $\mathcal{K}(kn)$ was defined at (6.5). For $k \le n$, on $A(kn) \cap B(kn)$, we see that

$$\big|\mathcal{K}(kn)\big| \le \ln(kn) \le 2 \ln n \qquad (6.13)$$

and in particular,

$$\begin{aligned}
&\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)^c\big] \\
&\quad = \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), \mathcal{K}(kn) = F, C(kn)^c\big] \\
&\quad = \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbf{E}\big[P^\omega\big[R\big((k-1)n\big), A(kn), B(kn), \mathcal{K}(kn) = F\big] \\
&\qquad\qquad\qquad \times \mathbf{1}\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}\big].
\end{aligned}$$

We may now see that:

(1) on the one hand, the random variable $P^\omega[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F]$ is measurable with respect to $\sigma\{c_*([x, y]), \text{ with } x, y \notin \mathcal{H}^+_{nk}\}$;

(2) on the other hand, the event $\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}$ is measurable with respect to $\sigma\{c_*([x, y]), \text{ with } x \in \mathcal{H}^+_{nk}\}$.

Hence, the random variables $P^\omega[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F]$ and $\mathbf{1}\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}$ are $\mathbf{P}$-independent. This yields

$$\begin{aligned}
&\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)^c\big] \\
&\quad = \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), \mathcal{K}(kn) = F\big] \\
&\qquad\qquad \times \mathbf{P}\big[\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\big] \\
&\quad \le \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), \mathcal{K}(kn) = F\big] \\
&\qquad\qquad \times \big(1 - \mathbf{P}\big[\text{all } x \in \partial F \cap \mathcal{H}^+(kn) \text{ are open}\big]\big).
\end{aligned}$$

Now, we know by the Harris inequality [18] that for $F \subset \mathbb{Z}^d$, with $|F| \le 2\ln n$

$$\begin{aligned}
\mathbf{P}\big[\text{all } x \in \partial F \cap \mathcal{H}(kn) \text{ are open}\big] &\ge \mathbf{P}[x \text{ is open}]^{d|F|} \\
&\ge \big(1 - \varepsilon(K)\big)^{2d\ln n} \\
&= n^{2d\ln(1-\varepsilon(K))},
\end{aligned}$$

where we recall that $\varepsilon(K) = P_*[c_* \notin [1/K, K]]$.

For any $\varepsilon_1 > 0$, by choosing K large enough, we can assume that $2d\ln(1 - \varepsilon(K)) \ge -\varepsilon_1$. This means that the two previous equations imply that

$$\begin{aligned}
&\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)^c\big] \\
&\quad \le \big(1 - n^{-\varepsilon_1}\big)\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn)\big],
\end{aligned}$$

which means that for any $\varepsilon_1 > 0$, we have

$$\begin{aligned} &n^{-\varepsilon_1}\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn)\big] \\ &\qquad \le \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)\big]. \end{aligned} \tag{6.14}$$

Step 2: *Control of the term in* (6.8). We want to upper-bound $\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c]$. On $A(kn) \cap B(kn) \cap C(kn)$, we have reached the positive inner border of $\mathcal{B}_{nk}$, and we know that $\mathcal{K}(kn)$ is not too big and its "positive border" is open. By applying Lemma 6.3, we have an estimate for the probability of exiting $\mathcal{K}(kn)$ through the positive border.

To start, we wish to decompose $\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c]$ according to all possible values of $X_{T_{\partial_i \mathcal{B}_{nk}}}$ and $\mathcal{K}(kn)$. For this, we notice that:

(1) on $A(kn)$, we have $X_{T_{\partial_i \mathcal{B}_{nk}}} \in \partial_i^+ \mathcal{B}_{nk}$, and by the definition of $\mathcal{K}(kn)$ [see (6.5)], $X_{T_{\partial_i \mathcal{B}_{nk}}} \in \mathcal{K}(kn)$;

(2) moreover, on $A(kn) \cap B(kn)$, we have (6.13).

Hence,

$$\begin{aligned} &\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)^c\big] \\ &\quad \le \sum_{y,F} \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), \\ &\qquad\qquad X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F, C(kn), D(kn)^c\big], \end{aligned}$$

where $\sum_{y,F}$ stands for $\sum_{y \in \partial_i^+ \mathcal{B}_{nk}} \sum_{F \subset \mathbb{Z}^d, |F| \le 2 \ln n, y \in F}$.

Let us notice that, for a fixed ω, the events $R((k-1)n)$, $A(kn)$, $B(kn)$, $\{X_{T_{\partial_i \mathcal{B}_{nk}}} = y\}$ and $\{\mathcal{K}(kn) = F\}$ are P^ω-measurable with respect to $\{X_i, i \le T_{\partial_i \mathcal{B}_{nk}}\}$. Thus, we may use the Markov property at $T_{\partial_i \mathcal{B}_{nk}}$ to see that

$$\begin{aligned} &\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)^c\big] \\ &\quad \le \sum_{y,F} \mathbf{E}\big[P^\omega\big[R\big((k-1)n\big), A(kn), B(kn), X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F\big] \\ &\qquad\qquad \times P_y^\omega[T_{\partial F} < T_{\partial F \cap \mathcal{H}^+(kn)}]\mathbf{1}\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}\big]. \end{aligned} \tag{6.15}$$

We wish to apply Lemma 6.3, for this we will first prove that on $\{\mathcal{K}(kn) = F\}$ and $\{x$ is open, for $x \in \partial F \cap \mathcal{H}^+(kn)\}$ the set $\partial_E F$ is composed of normal edges. Indeed:

(1) notice that the definition of $\mathcal{K}(kn)$ at (6.5) [which is a K-(kn)-closed component] implies that e is normal for all $e \in \partial_E \mathcal{K}(kn)$ when e has no endpoint in $\mathcal{H}^+(kn)$;

(2) moreover, if for any $x \in \partial F \cap \mathcal{H}^+(kn)$ the vertex x is open, then any edge $e \in \partial_E F$ with one endpoint in $\mathcal{H}^+(kn)$ is normal.

Hence $\partial_E F$ is a set made of normal edges only.

Now, for any $y \in F \cap \partial_i^+ \mathcal{B}_{nk}$, there exists $x \in \mathcal{H}^+(kn)$ adjacent to y. Since $F \subset \mathcal{B}_{nk}$, we can see that for any $z \in \partial F \cup F$ we have $(x - z) \cdot \vec{\ell} \geq -1$ and $(y - z) \cdot \vec{\ell} \geq -2$. Using this, along with the fact that $\partial_E F$ is made of normal edges, we see with (2.1) and Remark 5.1 that

$$\text{for all } e \in \partial_E F \qquad c^\omega(e) \leq K^2 e^{3\lambda} c^\omega([x, y]).$$

We can apply Lemma 6.3 to F, and we see that if $\{\mathcal{K}(kn) = F\}$ and $\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}$, then we obtain

$$\begin{aligned} P_y^\omega[T_{\partial F} < T_{\partial F \cap \mathcal{H}(kn)}] \leq P_y^\omega[T_{\partial F} < T_x] &\leq \big(1 - c|F|^{-1}\big) \\ &\leq \big(1 - c \ln^{-1} n\big), \end{aligned}$$

since $|F| \leq 2 \ln n$.

This turns (6.15) into

$$\begin{aligned} &\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)^c\big] \\ &\quad \leq \sum_{y,F} \mathbf{E}\big[P^\omega\big[R\big((k-1)n\big), A(kn), B(kn), X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F\big] \\ &\qquad\qquad \times \mathbf{1}\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}\big] \\ &\qquad \times \big(1 - c \ln^{-1} n\big) \\ &\quad \leq \big(1 - c \ln n^{-1}\big) \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)\big], \end{aligned}$$

which means that for some $c > 0$, we have

$$\begin{aligned} &c \ln n^{-1} \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn)\big] \\ &\qquad \leq \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)\big]. \end{aligned} \tag{6.16}$$

Step 3: *Control of the term in* (6.9). On $A(kn) \cap B(kn) \cap C(kn) \cap D(kn)$, we know that $X_{T_{\partial \mathcal{B}_{kn}}} \in \partial^+ \mathcal{B}_{kn}$ is an open ladder point. Moreover, it is important to notice that the event $E(kn)$ has a positive probability of happening and does not depend on what happened inside $\mathcal{B}_{kn}$. We introduce

$$R'\big((k-1)n\big) = R\big((k-1)n\big) \cap A(kn) \cap B(kn) \cap C(kn) \cap D(kn).$$

A vertex is said to be x-open if it is open in ω_x coinciding with ω on all edges, but those that are adjacent to x which are normal in ω_x. We see

$$\begin{aligned} &\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)^c\big] \\ &\quad \leq \sum_{x \in \partial^+ \mathcal{B}_{nk}} \mathbb{P}\big[R'\big((k-1)n\big), X_{T_{\partial \mathcal{B}_{nk}}} = x, x \text{ is open}, E(kn)^c\big] \end{aligned}$$

$$\leq \sum_{x\in\partial^+\mathcal{B}_{nk}} \mathbf{E}\big[P^\omega\big[R'\big((k-1)n\big), X_{T_{\partial\mathcal{B}_{nk}}}=x\big]\mathbf{1}\{x \text{ is open}\},$$
$$\big(\mathbf{1}\{x+e_1 \text{ or } x+2e_1 \text{ is not } x\text{-open}\}$$
$$+ P_x^\omega[X_1\neq x+e_1 \text{ or } X_2\neq x+2e_1]$$
$$\times \mathbf{1}\{x+e_1, x+2e_1 \text{ are } x\text{-open}\}\big)\big].$$

On $\{x+e_1, x+2e_1 \text{ are } x\text{-open}\}\cap\{x \text{ is open}\}$, we see that $P_x^\omega[X_1=x+e_1, X_2=x+2e_1]\geq c>0$ by Remark 5.1.

$$\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)^c\big]$$
$$\leq \sum_{x\in\partial^+\mathcal{B}_{nk}} \mathbf{E}\big[P^\omega\big[R'\big((k-1)n\big), X_{T_{\partial\mathcal{B}_{nk}}}=x\big]\mathbf{1}\{x \text{ is open}\},$$
$$\big(\mathbf{1}\{x+e_1 \text{ or } x+2e_1 \text{ is not } x\text{-open}\}$$
$$+(1-c)\mathbf{1}\{x+e_1, x+2e_1 \text{ are } x\text{-open}\}\big)\big].$$

Recalling the definition of ν at the beginning of Section 4, we may also see that $\{R'((k-1)n), X_{T_{\partial\mathcal{B}_{nk}}}=x, x \text{ is open}\}$ is measurable with respect to $\sigma\{c_*(e), e\in E(\mathcal{B}_{nk}) \text{ or } e=[x,x+e']$ with $e'\in\nu\}$, whereas $\{x+e_1, x+2e_1 \text{ are } x\text{-open}\}$ is measurable with respect to $\sigma\{c_*(e), e\notin E(\mathcal{B}_{nk}) \text{ and } e\neq[x,x+e'] \text{ with } e'\in\nu\}$. So these random variables are independent, which yields

$$\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)^c\big]$$
$$\leq \mathbb{P}\big[R'\big((k-1)n\big)\big]\big(\mathbf{P}[x+e_1 \text{ or } x+e_2 \text{ is not } x\text{-open}]$$
$$+(1-c)\mathbf{P}[x+e_1, x+e_2 \text{ are } x\text{-open}]\big)$$
$$\leq (1-c)\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)\big],$$

since $\mathbf{P}[x+e_1, x+e_2 \text{ are } x\text{-open}]>0$. This means that there exists $c>0$ such that

$$\begin{aligned} &c\mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn)\big] \\ &\quad\leq \mathbb{P}\big[R\big((k-1)n\big), A(kn), B(kn), C(kn), D(kn), E(kn)\big]. \end{aligned} \tag{6.17}$$

Step 4: *Conclusion.* For any $\varepsilon_1>0$, we see using (6.11), (6.12), (6.14), (6.16), (6.17) (which are valid K chosen larger than some K_0 depending only on $M<\infty$) and (6.10), that we have for any $k\in[2,n]$

$$\mathbb{P}\big[R(kn)\big]\leq \mathbb{P}\big[R\big((k-1)n\big)\big]\big(1-c\ln n^{-1}n^{-\varepsilon_1}\big)+Cn^{-M},$$

which implies the result. □

We now prove the following:

LEMMA 6.5. *For any M, there exists K_0 such that, for any $K \geq K_0$,*

$$\mathbb{P}[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} \geq n] \leq C(K)n^{-M}.$$

PROOF. For any $M < \infty$, by Lemma 6.4, there exists K_0 such that, for any $K \geq K_0$ such that

$$\mathbb{P}\big[R(nk)\big] \leq \big(1 - cn^{-1/2}\big)\mathbb{P}\big[R\big(n(k-1)\big)\big] + n^{-M}.$$

By a simple induction, this means that for

$$\mathbb{P}\big[R\big(n^2\big)\big] \leq \big(1 - cn^{-1/2}\big)^n + n^{-M+1} \leq 2n^{-M+1}.$$

Recalling the definition of $R(n)$ at (6.4), we see by the Borel–Cantelli lemma that $\mathcal{M}^{(K)} < \infty$.

Also this implies that

$$\mathbb{P}\big[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} > n^2 + 2\big] \leq 2n^{-M+1}$$

and

$$\mathbb{P}[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} > n] \leq Cn^{-(M+1)/2},$$

which proves the lemma, since M is arbitrary. □

6.3. *Consequence of our estimates on $\mathcal{M}$.* A natural consequence of the previous estimate is that the successive open ladder points cannot be too distant, and this is what we aim at showing next in a form that will be useful for us in the sequel. Let us introduce the ladder times

$$W_0 = 0 \quad \text{and} \quad W_{k+1} = \inf\{n \geq 0, X_n \cdot \vec{\ell} > X_{W_k} \cdot \vec{\ell}\}. \tag{6.18}$$

We introduce the event

$$\begin{aligned} M^{(K)}(n) = \big\{&\text{for } k \text{ with } W_k \leq \Delta_n, \\ &\text{we have } X_{\mathcal{M}^{(K)} \circ \theta_{W_k} + W_k} \cdot \vec{\ell} - X_{W_k} \cdot \vec{\ell} \leq n^{1/2}\big\}. \end{aligned} \tag{6.19}$$

LEMMA 6.6. *For any $M < \infty$, there exists K_0 such that, for any $K \geq K_0$ we have*

$$\mathbb{P}\big[M^{(K)}(n)^c\big] \leq Cn^{-M}.$$

PROOF. We introduce the event

$$M_j(n) = \big\{\text{for } k \leq j-1, \text{ we have } X_{\mathcal{M}^{(K)} \circ \theta_{W_k} + W_k} \cdot \vec{\ell} - X_{W_k} \cdot \vec{\ell} \leq n^{1/2}\big\}$$

and the event

$$N_j(n) = \big\{X_{\mathcal{M}^{(K)} \circ \theta_{W_j} + W_j} \cdot \vec{\ell} - X_{W_j} \cdot \vec{\ell} > n^{1/2} \text{ and } X_{W_j} \in B\big(n, n^{\alpha}\big)\big\}.$$

On $M^{(K)}(n)^c \cap \{T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}\}$, since all X_{W_i} are different necessarily, we have $k \le Cn^{2d\alpha}$ for any k such that $W_k \le \Delta_n$. Moreover there exists a $k \le \Delta_n$ such that $N_k(n)$ holds. By decomposing along the smallest such k, we see that

$$\begin{aligned}\mathbb{P}[M(n)^c] \le \sum_{k=1}^{Cn^{2d\alpha}} &\mathbb{P}[M_k(n), N_k(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}] \\ &+ \exp(-cn)\end{aligned} \tag{6.20}$$

by Theorem 5.1.

Now, we see that on $\{M_j(n), N_j(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}\}$ we have $X_{W_j} \in B(n, n^\alpha)$, so by a simple union bound argument,

$$\begin{aligned}&\mathbb{P}[M_j(n), N_j(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}] \\ &\quad \le \sum_{x \in B(n,n^\alpha)} \mathbb{P}[X_{W_j} = x, X_{\mathcal{M}^{(K)} \circ \theta_{W_j} + W_j} \cdot \vec{\ell} - x \cdot \vec{\ell} > n^{1/2}] \\ &\quad \le \sum_{x \in B(n,n^\alpha)} \mathbf{E}[P_x^\omega[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} - x \cdot \vec{\ell} \ge n^{1/2}]] \\ &\quad \le |B(n, n^\alpha)| \mathbb{P}[\mathcal{M}^{(K)} \ge n^{1/2}]\end{aligned}$$

by Markov's property at W_j and translation invariance of the environment. Hence, by the two last equations, Lemma 6.5 and using (6.20), we may see that

$$\mathbb{P}[M(n)^c] \le n^{-M}. \qquad \square$$

7. Regeneration times. The aim of this section is to define regeneration times and prove some standard properties on them. These properties are summed up in Section 7.6.

The idea is to find a maximum of the trajectory, in the direction $\vec{\ell}$ from which the random walk will never backtrack, that is, go to a point with lower scalar product with $\vec{\ell}$. Essentially, we would like to call regeneration time the first time that such a situation occurs. For technical reasons it is convenient for us to consider only the maxima which are also K-open points (or more precisely points verifying the properties of $\mathcal{M}$). This is the only difference from the standard definitions of regeneration times.

We define the time it takes for the walk to go back beyond (with respect to the scalar product with $\vec{\ell}$) its starting point

$$D = \inf\{n > 0 \text{ such that } X_n \cdot \vec{\ell} \le X_0 \cdot \vec{\ell}\}.$$

Also we introduce the maximum (in the direction $\vec{\ell}$) of the trajectory before D

$$M = \sup_{n \le D} X_n \cdot \vec{\ell}.$$

We define the configuration dependent stopping times S_k, $k \geq 0$ and the levels M_k, $k \geq 0$,

$$\begin{aligned} &S_0 = 0, \qquad M_0 = X_0 \cdot \vec{\ell} \quad \text{and} \\ &\text{for } k \geq 0 \qquad S_{k+1} = \mathcal{M}^{(K)} \circ \theta_{T_{\mathcal{H}^+(M_k)}} + T_{\mathcal{H}^+(M_k)}, \end{aligned} \tag{7.1}$$

where

$$M_k = \sup\{X_m \cdot \vec{\ell} \text{ with } 0 \leq m \leq R_k\} \tag{7.2}$$

with

$$R_k = D \circ \theta_{S_k} + S_k.$$

In words, R_k is the time it takes to go back beyond X_{S_k}, M_k is the maximum of the trajectory before R_k, and S_{k+1} is the first time we see an open ladder point (more precisely a point verifying the properties of $\mathcal{M}^{(K)}$) after getting past M_k.

These definitions imply that if $S_{i+1} < \infty$, then

$$X_{S_{i+1}} \cdot \vec{\ell} - X_{S_i} \cdot \vec{\ell} \geq 2e_1 \cdot \vec{\ell} \geq \frac{2}{\sqrt{d}}. \tag{7.3}$$

Finally we define the basic regeneration time

$$\tau_1 = S_N \qquad \text{with } N = \inf\{k \geq 1 \text{ with } S_k < \infty \text{ and } M_k = \infty\}. \tag{7.4}$$

Let us give some intuition about those definitions. Assume S_k is constructed; it is, by definition, an open-point ladder point. We will show that at such a point there is a lower bounded chance of never backtracking again. If the walks never backtrack again (then $R_k = \infty$ and thus $M_k = \infty$), we have created a point separating the past and the future of the random walk: a regeneration point called τ_1. This finishes the procedure.

In case a regeneration time is not created, the future of the random walk and the environment ahead of us may be conditioned by the fact that the walk will eventually backtrack: a conditioning limited to the conductances of the edges adjacent to the trajectory of the walk before it backtracks and the trajectory of the walk itself before backtracking.

In our definitions we introduced a random variable M_k chosen large enough so that all the edges we just described have one endpoint in $\mathcal{H}^-(M_k)$. Hence the environment in $\mathcal{H}^+(M_k)$ and the walk after it reaches this set are largely unconditioned. After reaching that set, we have the opportunity to construct another open ladder point (for a walk free of any constraints from its past) by considering the first open ladder point after entering $\mathcal{H}^+(M_k)$. This is how we define S_{k+1} and from there we start over the procedure until we find a regeneration time.

7.1. *Control the variables M.* We want to show that the random variables M_k in (7.2) cannot be too big. For this we prove the following lemma:

LEMMA 7.1. *We have*

$$\mathbb{P}[M \geq n \mid D < \infty] \leq C \exp(-cn).$$

PROOF. We have

$$\begin{aligned}
&\mathbb{P}\big[2^k \leq M < 2^{k+1}\big] \\
&\quad \leq \mathbb{P}[T_{\partial B(2^k, 2^{\alpha k})} \neq T_{\partial^+ B(2^k, 2^{\alpha k})}] \\
&\qquad + \mathbb{P}\big[X_{T_{\partial B(2^k, 2^{\alpha k})}} \in \partial^+ B\big(2^k, 2^{\alpha k}\big), \\
&\qquad\qquad T^+_{\mathcal{H}^-(0)} \circ \theta_{T_{\partial^+ B(2^k, 2^{\alpha k})}} < T_{\mathcal{H}^+(2^{k+1})} \circ \theta_{T_{\partial^+ B(2^k, 2^{\alpha k})}}\big]
\end{aligned}$$

and by a union bound on the $2^{\alpha k}$ possible positions of $X_{T_{\partial^+ B(2^k, 2^{\alpha k})}}$ and using translation invariance arguments,

$$\mathbb{P}\big[2^k \leq M < 2^{k+1}\big] \leq 2^{\alpha k} \mathbb{P}[T_{\partial B(2^{k+1}, 2^{\alpha(k+1)})} \neq T_{\partial^+ B(2^{k+1}, 2^{\alpha(k+1)})}] + e^{-(2^k)}$$

as a consequence of Theorem 5.1.

Hence, using Theorem 5.1, again

$$\mathbb{P}\big[2^k \leq M < 2^{k+1}\big] \leq c 2^{\alpha k} e^{-c(2^k)}$$

and since $M < \infty$ on $D < \infty$, we see that

$$\mathbb{P}[M \geq n \mid D < \infty] \leq \frac{1}{\mathbb{P}[D < \infty]} \sum_{k, 2^k \geq n} \mathbb{P}\big[2^k \leq M < 2^{k+1}\big] \leq C e^{-cn}. \qquad \square$$

Recalling the definition of M_k at (7.2), we introduce the event

$$S(n) = \big\{\text{for } i \text{ with } S_i \leq \Delta_n \text{ and } M_i < \infty,\ M_i - X_{S_i} \cdot \vec{\ell} \leq n^{1/2}\big\}. \tag{7.5}$$

Let us prove the following lemma:

LEMMA 7.2. *We have*

$$\mathbb{P}\big[S(n)^c\big] \leq \exp\big(-n^{1/2}\big).$$

PROOF. By (7.3), we know that $\operatorname{card}\{i; S_i \leq \Delta_n\} \leq n$. So, we see that

$$\begin{aligned}
\mathbb{P}\big[S(n)^c\big] &\leq \mathbb{P}[T_{B(n, n^\alpha)} \neq T_{\partial^+ B(n, n^\alpha)}] \\
&\quad + \sum_{i \leq n} \sum_{x \in B(n, n^\alpha)} \mathbb{P}\big[M_i - X_{S_i} \cdot \vec{\ell} > n^{1/2}, M_i < \infty, X_{S_i} = x\big],
\end{aligned} \tag{7.6}$$

where the first term can be controlled by Theorem 5.1.

Now, we may see that

$$\begin{aligned}\mathbb{P}[M_i - X_{S_i}\cdot\vec{\ell} > n^{1/2}, M_i < \infty, X_{S_i} = x] \\ \le \mathbb{P}\Big[\sup_{n\le D\circ\theta_{S_i}+S_i}(X_n - x)\cdot\vec{\ell} > n^{1/2}, M_i < \infty, X_{S_i} = x\Big].\end{aligned} \tag{7.7}$$

If $M_i < \infty$, we have $D\circ\theta_{S_i} + S_i < \infty$, hence

$$\begin{aligned}\mathbb{P}\Big[\sup_{n\le D\circ\theta_{S_i}+S_i}(X_n - x)\cdot\vec{\ell} \ge n^{1/2}, M_i < \infty, X_{S_i} = x\Big] \\ \le \mathbb{P}\Big[\sup_{n\le D\circ\theta_{S_i}+S_i}(X_n - x)\cdot\vec{\ell} \ge n^{1/2}, D\circ\theta_{S_i} + S_i < \infty, X_{S_i} = x\Big].\end{aligned} \tag{7.8}$$

Recalling that X_{S_i} is a maximum in the direction $\vec{\ell}$ of the past trajectory, we can use Markov's property at the time S_i to see that

$$\begin{aligned}&\mathbb{P}\Big[\sup_{n\le D\circ\theta_{S_i}+S_i}(X_n - x)\cdot\vec{\ell} \ge n^{1/2}, D\circ\theta_{S_i} + S_i < \infty, X_{S_i} = x\Big] \\ &\quad\le \mathbf{E}\Big[P_x^{\omega}\Big[\sup_{n\le D} X_n\cdot\vec{\ell} \ge n^{1/2}, D < \infty\Big]\Big] \\ &\quad\le \mathbb{P}\Big[\sup_{n\le D} X_n\cdot\vec{\ell} \ge n^{1/2} \mid D < \infty\Big] \le C\exp(-cn^{1/2}),\end{aligned} \tag{7.9}$$

where we used translation invariance and Lemma 7.1. The result follows from putting together (7.6), (7.7), (7.8) and (7.9). □

7.2. *Exponential tails for backtracking*. As a result of Theorem 5.1, we know that the walk is exponentially unlikely to backtrack a lot. This can be seen as follows, and given a large n, it is extremely likely to exit $B(2^n, 2^{\alpha n})$ through the positive side. Centering at that exit point a box of size $B(2^{n+1}, 2^{\alpha(n+1)})$, we are again very likely to exit through the positive side. Applying this reasoning recursively, we see that we are unlikely to reach $\mathcal{H}^-(-2^n)$.

LEMMA 7.3. *We have for any n,*

$$\mathbb{P}[T_{\mathcal{H}^-(-n)} < \infty] \le C\exp(-cn).$$

PROOF. Fix $n > 0$. For this proof, we will use the event

$$A(n) = \{T_{\partial B(2^n, 2^{n\alpha})} = T_{\partial^+ B(2^n, 2^{n\alpha})}\}.$$

For any $k \ge n$, let us denote $B_X(2^{k+1}, 2^{(k+1)\alpha}) = \{z \in \mathbb{Z}^d, z = X_{T_{\partial B(2^k, 2^{k\alpha})}} + y$ with $y \in B(2^{k+1}, 2^{(k+1)\alpha})\}$, and we introduce the event

$$C(k) = \{T_{\partial^+ B_X(2^{k+1}, 2^{(k+1)\alpha})}\circ\theta_{T_{\partial B(2^k, 2^{k\alpha})}} = T_{\partial B_X(2^{k+1}, 2^{(k+1)\alpha})}\circ\theta_{T_{\partial B(2^k, 2^{k\alpha})}}\}.$$

A simple induction shows that on $\bigcap_{k\in[n,m]} C(k)\cap A(n)$, we have $\{T_{\mathcal{H}^-(-2^n-1)} \geq T_{B(2^m,m2^{\alpha m})}\}$, and hence we see that

$$\bigcap_{k\geq n} C(k)\cap A(n) \subseteq \{T_{\mathcal{H}^-(-2^n-1)} = \infty\}. \tag{7.10}$$

Denote for $m > n$,

$$D(n,m) = C(k)\cap C(m)^c \cap \Bigg(\bigcap_{n\leq k<m} A(n)\Bigg),$$

so that using (7.10), we see

$$\{T_{\mathcal{H}^-(-2^n)} < \infty\} \subset \Bigg(\bigcap_{k\geq n} C(k)\cap A(n)\Bigg)^c \subset \bigcup_{m\geq n} D(n,m)\cup A(n)^c,$$

which implies

$$\begin{aligned}\mathbb{P}[T_{\mathcal{H}^-(-2^n)} < \infty] &\leq \mathbb{P}\big[A(n)^c\big] + \sum_{m\geq n}\mathbb{P}\big[D(n,m)\big]\\ &\leq \exp\big(-c2^n\big) + \sum_{m\geq n}\mathbb{P}\big[D(n,m)\big]\end{aligned} \tag{7.11}$$

by Theorem 5.1.

We may notice that on $D(n,m)$, we have $\{T_{\partial B(2^m,m2^{m\alpha})} = T_{\partial^+ B(2^m,m2^{m\alpha})}\}$ [note that is different from $A(m)$]. Hence, when using Markov's property at $T_{\partial B(2^m,m2^{m\alpha})}$ the random walk is located in $\partial^+ B(2^m, m2^{m\alpha})$, so

$$\begin{aligned}&\mathbb{P}\big[D(n,m)\big]\\ &\quad\leq \sum_{x\in\partial^+ B(2^m,m2^{m\alpha})} \mathbf{E}\big[P^\omega[X_{T_{\partial B(2^m,m2^{m\alpha})}} = x]\\ &\qquad\qquad \times P_x^\omega[T_{x+\partial B(2^{m+1},2^{(m+1)\alpha})} \neq T_{x+\partial^+ B(2^{m+1},2^{(m+1)\alpha})}]\big]\\ &\quad\leq Cm^d 2^{dm\alpha}\\ &\qquad \times \max_{x\in\partial^+ B(2^m,m2^{m\alpha})} \mathbf{E}\big[P_x^\omega[T_{x+\partial B(2^{m+1},2^{(m+1)\alpha})} \neq T_{x+\partial^+ B(2^{m+1},2^{(m+1)\alpha})}]\big]\\ &\quad\leq Cm^d 2^{dm\alpha}\mathbb{P}\big[A(m+1)^c\big] \leq C\exp\big(-c2^m\big)\end{aligned}$$

by translation invariance and Theorem 5.1.

The lemma follows from the previous and (7.11). □

7.3. *Uniformly bounded chance of never backtracking at open points.* We recall that ν was defined at the beginning of Section 4. We denote $\mathcal{C} = \{x > 0\}^\nu$, which is seen as the configuration, that is, values of conductances, adjacent to a point. For any $a \in \mathcal{C}$, we define the environment ω_x^a on the edges of $\mathbb{Z}^d$ to be the

environment which has the same conductances as in ω except on edges adjacent to x, and on these edges the conductances are given by the configuration a, that is, $c_*^{\omega_x^a}([x, x+e]) = a(e)$ for any $e \in \nu$.

We say that $a \in \mathcal{C}$ is K-open if $a(e) \in [1/K, K]$ for any $e \in \nu$. In the sequel, we will use the notation $\max_{a \in \mathcal{C} \text{ open}}$ to designate the maximum taken over all configurations $a \in \mathcal{C}$ that are open.

LEMMA 7.4. *We have*

$$\mathbf{E}\Big[\max_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[D < \infty] \mid 0 \textit{ is good}\Big] < 1.$$

This result is natural. Indeed, by following the directed open path we can bring the random walk far in the direction of the bias with a positive probability independent of the environment, and after this point it will be unlikely by Lemma 7.3 to backtrack past your starting point. This means that there is always a positive escape probability from a good point.

PROOF. Fix $n > 0$. On the event that $\{0 \text{ is good}\}$, we denote $\mathcal{P}(i)$ a directed path starting at 0 where all points are open. We denote $L_{\partial^+ B(n,n^\alpha)} = \inf\{i, \mathcal{P}(i) \in \partial^+ B(n, n^\alpha)\}$. Now, we see that if the two following conditions are verified:

(1) $X_i = \mathcal{P}(i)$ for $i \le L_{\partial^+ B(n,n^\alpha)}$,
(2) $T_{\mathcal{H}^-(2)} \circ \theta_{T_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}} = \infty$,

then $D = \infty$.

We can see that if $\{0 \text{ is good}\}$, then $L_{\partial^+ B(n,n^\alpha)} \le Cn$, so that

$$\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}\big[X_i = \mathcal{P}(i) \text{ for } i \le L_{\partial^+ B(n,n^\alpha)}\big] \ge \kappa_0^{Cn}$$

by Remark 5.1.

In particular, we have

$$\begin{aligned}
&\mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[D = \infty] \mid 0 \text{ is good}\Big] \\
&\quad \ge \mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}\big[X_i = \mathcal{P}(i) \text{ for } i \le L_{\partial^+ B(n,n^\alpha)}\big] \\
&\qquad\qquad \times P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] \mid 0 \text{ is good}\Big] \\
&\quad \ge \kappa_0^{Cn} \mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] \mid 0 \text{ is good}\Big].
\end{aligned}$$

Moreover, since we will not use the edges adjacent to 0 to know if we hit $\mathcal{H}^-(2)$, we see that

$$P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] = P^{\omega}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty],$$

so that for any n

$$\mathbf{E}\Big[\min_{a\in\mathcal{C}\text{ open}} P^{\omega_0^a}[D=\infty] \mid 0 \text{ is good}\Big]$$
$$\geq \kappa_0^{Cn}\mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)}=\infty] \mid 0 \text{ is good}\big].$$

Now,

$$\mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)}<\infty] \mid 0 \text{ is good}\big]$$
$$\leq \mathbf{P}[0 \text{ is good}]^{-1}\mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)}<\infty]\big]$$
$$\leq C\mathbb{P}[T_{\mathcal{H}^-(-n+2)}<\infty],$$

where we use translation invariance.

Now, by Lemma 7.3, we see that the previous quantity is less than $1/2$ for $n \geq n_0$. Hence combining the last two equations,

$$\mathbf{E}\Big[\min_{a\in\mathcal{C}\text{ open}} P^{\omega_0^a}[D=\infty] \mid 0 \text{ is good}\Big] \geq (1/2)\kappa_0^{Cn_0} > 0,$$

which implies the result. □

7.4. *Number of trials before finding an open ladder point which is a regeneration time.* Let us introduce the collection of edges with maximum scalar product with $\vec{\ell}$

$$\mathcal{E} = \{e \in \nu \text{ such that } e\cdot\vec{\ell} = e_1\cdot\vec{\ell}\}$$

and

$$\mathcal{B}_x = \big\{e \in \mathbb{E}(\mathbb{Z}^d), e = [-e_1, f - e_1] \text{ with } f \text{ any unit vector of } \mathcal{E}\big\}. \tag{7.12}$$

Imagining the bias is oriented to the right, the set of edges to the "left" of x is defined to be

$$\mathcal{L}^x := \big\{[y,z] \in E(\mathbb{Z}^d), y\cdot\ell \leq x\cdot\ell \text{ and } z\cdot\ell \leq x\cdot\ell\big\} \cup \mathcal{B}_x, \tag{7.13}$$

and the edges to the "right" are

$$\mathcal{R}^x := \big\{[y,z] \in E(\mathbb{Z}^d), y\cdot\ell > x\cdot\ell \text{ or } z\cdot\ell > x\cdot\ell\big\} \cup \mathcal{B}_x. \tag{7.14}$$

We recall that N was defined at (7.4). Since each time we arrive at a new S_k, we are at an open-ladder point with unspoiled environment ahead of us, there will be a positive chance of being a good point and never backtracking, which would create a regeneration time. This means that N should be similar to a geometric random variables. Here, we will only prove that N has exponential tails.

LEMMA 7.5. *We have*

$$\mathbb{P}[N \geq n] \leq \exp(-cn).$$

PROOF. We introduce the event

$$C(n) := \{\text{for all } k \le n \text{ such that } S_k < \infty, \text{ we have } D \circ S_k + S_k < \infty\},$$

which verifies

$$\{N \ge n\} \subseteq C(n). \tag{7.15}$$

Because of the way our regeneration times are constructed, we can see that $C(n)$ is P^ω-measurable with respect to $\sigma\{X_k \text{ with } k \le S_{n+1}\}$; see the discussion below (7.4). Using Markov's property at S_{n+1},

$$\begin{aligned}\mathbb{P}\big[C(n+1)\big] &\le \sum_{x\in\mathbb{Z}^d} \mathbf{E}\big[P^\omega\big[X_{S_{n+1}} = x, C(n)\big]P_x^\omega[D<\infty]\big] \\ &\le \sum_{x\in\mathbb{Z}^d} \mathbf{E}\Big[P^\omega\big[X_{S_{n+1}} = x, C(n)\big] \max_{a \text{ open}} P_x^{\omega^{x,a}}[D<\infty]\Big],\end{aligned}$$

where we used the fact that $X_{S_{n+1}}$ is open. Furthermore:

(1) $P^\omega[X_{S_{n+1}} = x, C(n)]$ is measurable with respect to $\sigma\{c_*(e) \text{ with } e \in \mathcal{L}_x\}$,

(2) $\max_{a \text{ open}} P_x^{\omega_x^a}[D<\infty]$ is measurable with respect to $\sigma\{c_*(e) \text{ with } e \notin \mathcal{L}_x\}$.

So we have $\mathbf{P}$-independence between the random variables in (1) and in (2). Hence

$$\begin{aligned}&\mathbb{P}\big[C(n+1)\big] \\ &\quad\le \mathbb{P}\big[C(n)\big]\mathbf{P}\Big[\max_{a \text{ open}} P_x^{\omega^{x,a}}[D<\infty]\Big] \\ &\quad\le \mathbb{P}\big[C(n)\big]\Big(\mathbf{P}[x \text{ is not good}] + \mathbf{E}\Big[\mathbf{1}\{x \text{ is good}\} \max_{a\in\mathcal{C} \text{ open}} P^{\omega_x^a}[D<\infty]\Big]\Big) \\ &\quad\le \mathbb{P}\big[C(n)\big]\Big(1 - \mathbf{P}[0 \text{ is good}]\Big(1 - \mathbf{E}\Big[\max_{a\in\mathcal{C} \text{ open}} P^{\omega_0^a}[D<\infty] \mid 0 \text{ is good}\Big]\Big)\Big),\end{aligned}$$

where we used translation invariance. Furthermore, we can use Lemma 7.4 to see that

$$\mathbb{P}\big[C(n+1)\big] \le (1-c)\mathbb{P}\big[C(n)\big] \le \cdots \le (1-c)^n,$$

hence, the result by (7.15). □

7.5. *Tails of regeneration times.* We have all the tools necessary to show that the first regeneration time does not occur too far away from the origin.

THEOREM 7.1. *For any $M < \infty$, there exists K_0 such that, for any $K \ge K_0$ we have $\tau_1^{(K)} < \infty$, $\mathbb{P}$-a.s. and*

$$\mathbb{P}[X_{\tau_1^{(K)}} \cdot \vec{\ell} \ge n] \le C(M)n^{-M}.$$

PROOF. Recalling definitions (7.1) and (6.18), we may see that $\{T_{\mathcal{H}^+(M_k)}, k \geq 0\} \subset \{W_k, k \geq 0\}$. This means that on $M(n)$, defined at (6.19),

$$\text{for } k \text{ such that } S_k \leq \Delta_n \qquad \text{we have } X_{S_{k+1}} \cdot \vec{\ell} - X_{T_{\mathcal{H}^+(M_k)}} \cdot \vec{\ell} \leq n^{1/2}.$$

Moreover, on $S(n)$ [defined at (7.5)], we have

$$\text{for } k \text{ such that } S_k \leq \Delta_n \text{ and } M_k < \infty \qquad \text{we have } M_k - X_{S_k} \cdot \vec{\ell} \leq n^{1/2}.$$

Noticing that $X_{T_{\mathcal{H}^+(M_k)}} \cdot \vec{\ell} \leq M_k + 1$, we may see that, on $S(n) \cap M(n)$

$$X_{S_{k+1}} \cdot \vec{\ell} - X_{S_k} \cdot \vec{\ell} \leq 2n^{1/2} + 1$$

for any k with $S_k < \Delta_n$ and $M_k < \infty$. By induction, this means that if $k \leq n^{1/2}/3$, $S_k < \Delta_n$ and $M_k < \infty$, then

$$X_{S_{k+1}} \cdot \vec{\ell} \leq k(2n^{1/2} + 1) < n \quad \text{and} \quad S_{k+1} \leq \Delta_n,$$

and the second part following from the fact that $X_{S_{k+1}}$ is a new maximum for the random walk in the direction $\vec{\ell}$. In particular, if $N \leq n^{1/2}/3$, then we can apply the previous equation to $k = N - 1$. Recalling (7.4) we see that, if $\{N \leq n^{1/2}/3\}$ and $M(n) \cap S(n)$, then for n large enough,

$$X_{\tau_1} \cdot \vec{\ell} \leq (n^{1/2}/3)(2n^{1/2} + 1) < n.$$

Thus

$$\begin{aligned}\mathbb{P}[X_{\tau_1} \cdot \vec{\ell} \geq n] &\leq \mathbb{P}[N \geq n^{1/2}/3] + \mathbb{P}[M(n)^c] + \mathbb{P}[S(n)^c] \\ &\leq \exp(-cn^{1/2}) + 2n^{-M} \leq 3n^{-M}\end{aligned}$$

by Lemmas 7.2, 6.6 and 7.5. This completes the proof. □

7.6. *Fundamental property of regeneration times.* We are going to define the sequence $\tau_0 := 0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ of successive regeneration times. Using a slight abuse of notation by viewing $\tau_k(\cdot, \cdot)$ as a function of a walk and an environment, we can define previous sequence via the following procedure:

$$\tau_{k+1} = \tau_1 + \tau_k(X_{\tau_1 + \cdot} - X_{\tau_1}, \omega(\cdot + X_{\tau_1})), \qquad k \geq 0, \tag{7.16}$$

meaning that the $k+1$th regeneration time is the kth regeneration time after the first one.

We set

$$\mathcal{G}_k := \sigma\{\tau_1, \dots, \tau_k; (X_{\tau_k \wedge m})_{m \geq 0}; c_*(e) \text{ with } e \in \mathcal{L}^{X_{\tau_k}}\}.$$

Let us introduce for any $x \in \mathbb{Z}^d$,

$$a_x = (c_*([x - e_1, x - e_1 + e]))_{e \in \mathcal{E}} = (c_*(e))_{e \in \mathcal{B}_x} \in [1/K, K]^{\mathcal{E}},$$

recalling notation from (7.12). For $a \in [1/K, K]^{\mathcal{E}}$, we set

$$\mathbf{P}_x^a = \delta_a\big((c_*([x-e_1, x-e_1+e]))_{e\in\mathcal{E}}\big) \otimes \int_{e\in E(\mathbb{Z}^d)\setminus\mathcal{B}_x} \otimes\, d\mathbf{P}\big(c_*(e)\big),$$

where $\otimes$ denotes the product of measures. We introduce the associated annealed measure

$$\mathbb{P}_x^a = \mathbf{P}_x^a \times P_x^\omega.$$

In words, $\mathbb{P}_x^a$ denotes the annealed measure for the walk started at x but where the conductances of the edges in $\mathcal{B}_x$ are fixed and given by a. We will use the notation $\mathbb{P}^a$ (resp., $\mathbf{P}^a$) for $\mathbb{P}_0^a$ (resp., $\mathbf{P}_0^a$).

We may notice that Theorem 7.1, can easily be generalized to become:

THEOREM 7.2. *For any $M < \infty$, there exists K_0 such that, for any $K \geq K_0$, we have $\tau_1^{(K)} < \infty$, $\mathbb{P}_0^a$-a.s. for $a \in [1/K, K]^{\mathcal{E}}$ and*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[X_{\tau_1^{(K)}} \cdot \vec{\ell} \geq n] \leq Cn^{-M}.$$

Similarly, we can turn Theorem 5.1 into:

THEOREM 7.3. *For $\alpha > d+3$,*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[T_{\partial B(L,L^\alpha)} \neq T_{\partial^+ B(L,L^\alpha)}] \leq Ce^{-cL}.$$

The fundamental properties of regeneration times are that:

(1) the past and the future of the random walk that has arrived X_{τ_k} are only linked by the conductances of the edges in $a_{X_{\tau_k}}$;

(2) the law of the future of the random walk has the same law as a random walk under $\mathbb{P}_0^{a_{X_{\tau_k}}}[\cdot \mid D = \infty]$.

We recall that $\mathcal{R}^0$ was defined at (7.14). Let us state a theorem corresponding to the previous heuristic.

THEOREM 7.4. *Let us fix K large enough. Let f, g, h_k be bounded functions which are measurable with respect to $\sigma\{X_n : n \geq 0\}$, $\sigma\{c_*(e), e \in \mathcal{R}^0\}$ and $\mathcal{G}_k$, respectively. Then for $a \in [1/K, K]^{\mathcal{E}}$,*

$$\mathbb{E}^a\big[f(X_{\tau_k+\cdot} - X_{\tau_k})g \circ t_{X_{\tau_k}} h_k\big] = \mathbb{E}^a\big[h_k \mathbb{E}_0^{a_{X_{\tau_k}}}[fg \mid D = \infty]\big].$$

A similar theorem was proved in [25] (as Theorems 3.3 and 3.5). In our context, the random variables studied (τ_1, D etc.) are defined differently from the corresponding ones in [25]. Nevertheless, our notation was chosen so that we may

prove Theorem 7.4 simply by following word for word the proofs of Theorems 3.3 and 3.5 in [25]. The reader should start reading after Remark 3.2 in [25] to have all necessary notation. To avoid any possible confusion we point out that in [25], the measures $\mathbb{P}$, $P_{x,\omega}$ and P correspond, respectively, to the environment, the quenched random walk and the annealed measure and that $\omega(b)$ denotes the conductances of an edge b.

We bring the reader's attention to the fact that we have not proved yet that $\tau_k < \infty$, $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]^{\mathcal{E}}$). We only know this for $k = 1$. This is enough to prove Theorem 7.4 for $k = 1$. Using Theorem 7.4 for $k = 1$ and Theorem 7.2, we may show that $\tau_2 < \infty$, $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]^{\mathcal{E}}$) and thereafter obtain Theorem 7.4 for $k = 2$. Hence, we may proceed by induction to prove Theorem 7.4 alongside the following result.

PROPOSITION 7.1. *Let us fix K large enough. For any $k \geq 1$, we have $\tau_k^{(K)} < \infty$ $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]^{\mathcal{E}}$).*

We see that this implies Proposition 3.1, which states directional transience in the direction $\vec{\ell}$ for the random walk.

As in [25], we may notice that a consequence of Theorem 7.4 is:

PROPOSITION 7.2. *Let*

$$\Gamma := \mathbb{N} \times \mathbb{Z}^d \times [1/K, K]^{\mathcal{E}}$$

with its canonical product σ-algebra, and let $y_i = (j^i, z^i, a^i) \in \Gamma$, $i \geq 0$. For $a \in [1/K, K]^{\mathcal{E}}$ and $G \subset \Gamma$ measurable let also

$$\tilde{R}_K(a; G) := \mathbb{P}_0^a\big[\big(\tau_1^{(K)}, X_{\tau_1^{(K)}}, a_{X_{\tau_1^{(K)}}}\big) \in G \mid D = \infty\big].$$

Then under $\mathbb{P}$ the Γ-valued random variables (with $\tau_0 = 0$),

$$Y_i^K := (J_i, Z_i, A_i) := \big(\tau_{i+1}^{(K)} - \tau_i^{(K)}, X_{\tau_{i+1}^{(K)}} - X_{\tau_i^{(K)}}, a_{\tau_{i+1}^{(K)}}\big), \qquad i \geq 0, \tag{7.17}$$

define a Markov chain on the state space Γ, which has transition kernel

$$\mathbb{P}[Y_{i+1} \in G \mid Y_0 = y_0, \dots, Y_i = y_i] = \tilde{R}_K\big(a^i; G\big)$$

and initial distribution

$$\tilde{\Lambda}_K(G) := \mathbb{P}\big[\big(\tau_1^{(K)}, X_{\tau_1^{(K)}}, a_{X_{\tau_1^{(K)}}}\big) \in G\big].$$

Similarly, on the state space $[1/K, K]^{\mathcal{E}}$, the random variables

$$A_i = a_{X_{\tau_{i+1}^{(K)}}}, \qquad k \geq 0, \tag{7.18}$$

also define a Markov chain under $\mathbb{P}$. *With* $a \in [1/K, K]^{\mathcal{E}}$ *and* $B \subset [1/K, K]^{\mathcal{E}}$ *measurable, its transition kernel is*

$$R_K(a; B) := \mathbb{P}_0^a[a_{X_{\tau_1^{(K)}}} \in B \mid D = \infty] = \sum_{j \in \mathbb{N}, z \in \mathbb{Z}^d} \tilde{R}_K\big(a; (j, z, B)\big)$$

and the initial distribution is

$$\Lambda_K(B) := \mathbb{P}[a_{X_{\tau_1^{(K)}}} \in B] = \tilde{\Lambda}_K\big((j, z, B)\big). \tag{7.19}$$

Now let us quote Lemma 3.7 and Theorem 3.8 from [25], which essentially states that the environment seen at regeneration times converges exponentially fast to some measure ν_K.

THEOREM 7.5. *There exists a unique invariant distribution* ν_K *for the transition kernel* R_K. *It verifies*

$$\sup_{a \in [1/K, K]^{\mathcal{E}}} \big\| R_K^m(a; \cdot) - \nu_K(\cdot) \big\|_{\mathrm{var}} \le C e^{-cm}, \qquad m \ge 0,$$

where $\| \cdot \|_{\mathrm{var}}$ *denotes the total variation distance.*

Further, this probability measure ν_K *is invariant with respect to the transition kernel* R; *that is,* $\nu_K R_K = \nu_K$, *and the Markov chain* $(A_k)_{k \ge 0}$, *defined in* (7.18) *with transition kernel* R_K *and initial distribution* ν_K *on the state space* $[1/K, K]^{\mathcal{E}}$ *is ergodic. Moreover, the initial distribution* $\Lambda_K(\cdot)$ *given in* (7.19) *is absolutely continuous with respect to* $\nu_K(\cdot)$.

THEOREM 7.6. *The distribution* $\tilde{\nu}_K := \nu_K \tilde{R}_K$ *is the unique invariant distribution for the transition kernel* $\tilde{R}_K$. *It verifies*

$$\sup_{a \in [1/M, M]^{\mathcal{E}}} \big\| \tilde{R}_K^m(a; \cdot) - \tilde{\nu}_K(\cdot) \big\|_{\mathrm{var}} \le C e^{-cm}, \qquad m \ge 0.$$

With initial distribution equal $\tilde{\nu}_K$, *the Markov chain* $(Y_k)_{k \ge 0}$ *defined in* (7.17) *is ergodic. Moreover, the law of the Markov chain* $(Y_{k+1})_{k \ge 0}$ *under* $\mathbb{P}$ *is absolutely continuous with respect to the law of the chain with initial distribution* $\tilde{\nu}_K$.

The proofs in [25] carry over to our context simply, once we have shown the following Doeblin condition: there exists $c > 0$ such that for any $a \in [1/K, K]^{\mathcal{E}}$ and $B \subset [1/K, K]^{\mathcal{E}}$, we have

$$R_K(a, B) \ge c \otimes_{\mathcal{E}} \mathbf{P}\big[c_* \in B \mid c_* \in [1/K, K]\big].$$

Before proceeding to the proof of this condition we need to introduce one notation. A vertex $x \in \mathbb{Z}^d$ is called open-good if x is good in a configuration ω_x^a where a is open (the corresponding notations were introduced above Lemma 7.4). Note

that we do not need to specify the value of the conductances of the edges since the event $\{x \text{ is good}\}$ is measurable with respect to the event $\{y \text{ is open}\}$ for $y \in \mathbb{Z}^d$.

To prove this condition, we only need to describe an event where $a_{X_{\tau_1^{(K)}}} \in B$ and $D = \infty$ which has a $\mathbb{P}^a$-probability comparable to $\otimes_{\mathcal{E}} \mathbf{P}[c_* \in B \mid c_* \in [1/K, K]]$. The event we will consider is $X_1 = e_1$, $X_2 = 2e_1$ (with e_1 and $2e_1$ being open) and regenerating at time 2. In terms of equations this translates into

$$\begin{aligned} R_K(a; B) &= \mathbb{P}_0^a[a_{X_{\tau_1}} \in B \mid D = \infty] \\ &= \frac{1}{\mathbb{P}_0^a[D = \infty]} \mathbf{E}^a\big[P_0^\omega[a_{X_{\tau_1}} \in B, D = \infty]\big]. \end{aligned}$$

The previous equation implies, using Remark 5.1,

$$\begin{aligned} R_K(a; B) \geq c\mathbf{E}^a\big[&P_0^\omega[X_1 = e_1, X_2 = 2e_1, D \circ \theta_2 = \infty] \dots \\ &a_{X_2} \in B, 0 \text{ and } e_1 \text{ are open and } 2e_1 \text{ is good}\big] \\ \geq c\kappa_0^2 \mathbf{E}^a\big[&P_{2e_1}^\omega[D = \infty], a_{2e_1} \in B, 0 \text{ and } e_1 \text{ is open and } 2e_1 \text{ is good}\big] \\ \geq c\kappa_0^2 \mathbf{E}^a\Big[&\min_{a \text{ open}} P_{2e_1}^{\omega_0^a}[D = \infty], a_{2e_1} \in B, 0, e_1, 2e_1 \text{ are open} \\ &\text{and } 2e_1 \text{ is open-good}\Big] \end{aligned}$$

and seeing that:

(1) $\min_{a \text{ open}} P_{2e_1}^{\omega_0^a}[D = \infty]$ and $\{2e_1 \text{ is open-good}\}$ are measurable with respect to $\sigma\{c_*([y, z])$ with $(y - 2e_1) \cdot \vec{\ell} > 0$ and $z \neq 2e_1\}$;

(2) $\{a_{2e_1} \in B, 0, e_1, 2e_1 \text{ are open}\}$ is measurable with respect to $\sigma\{c_*([y, z])$ with $(y - e_1) \cdot \vec{\ell} \leq 0$ or $2e_1 = z\}$, and

hence they are $\mathbf{P}$-independent so that

$$\begin{aligned} R_K(a; B) \geq c\mathbf{E}^a\Big[&\min_{a \text{ open}} P_{2e_1}^{\omega_0^a}[D = \infty], 2e_1 \text{ is open-good}\Big] \\ &\times \mathbf{P}_0^a[a_{2e_1} \in B, 0, e_1, 2e_1 \text{ are open}] \\ \geq c\mathbf{P}_0^a&[a_{2e_1} \in B; 0, e_1, 2e_1 \text{ are open}] \end{aligned}$$

by Lemma 7.4 (since $\{2e_1 \text{ is good}\} \subset \{2e_1 \text{ is open-good}\}$). Now by simple combinatorics we see that

$$R_K(a; B) \geq c \otimes_{\mathcal{E}} \mathbf{P}\big[c_* \in B \mid c_* \in [1/K, K]\big],$$

which is the Doeblin condition we were looking for.

8. Positive speed regime. Our aim for this section is to show:

THEOREM 8.1. *If $E_*[c_*] < \infty$, we have*

$$\max_{a \in [1/K, K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0 \in \mathrm{GOOD}_K\}\big] \le C(K)n$$

for any $K \ge K_0$ for some K_0.

Used in combination with the existence of a law of large numbers provided by the existence of a regeneration structure, this will allow us to prove the positivity of the speed if $E_*[c_*] < \infty$. Let us enumerate the key points for proving the previous result:

(1) The number of visits to a good point is bounded; see Lemma 8.1. This limits the expected number of entries in a trap to, roughly, the size of its border.

(2) The time spent during one visit to a trap is linked to the size of the trap and the conductances in that trap; see Lemma 8.3. It is already known by Lemma 5.1 that the size of traps is extremely small, so we will be able to neglect the effect from the size of traps.

(3) The conductances in traps are, relatively, similar to usual conductances. In particular, they do not have infinite expectation and cannot force zero-speed; see Lemma 8.4.

This reasoning allows us to say that, essentially, Δ_n should be of the same order as the number of sites visited before Δ_n (i.e., of order n), since there is no local trapping. More precisely, we get an upper-bound of $\mathbb{E}[\Delta_n]$ in terms of the number of the probability of reaching a point during the first regeneration time; see Lemma 8.6. The last step of the proof is to estimate the probability that we reach x during the first regeneration time; see Lemma 8.7.

We proceed to give the details associated with the previous outline. First, we notice:

LEMMA 8.1. *For any $x \in \mathrm{GOOD}_K(\omega)$, we have*

$$E_x^\omega\left[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\right] \le C(K) < \infty.$$

PROOF. We see that

$$E_x^\omega\left[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\right] = \frac{1}{P_x^\omega[T_x^+ = \infty]} = \frac{\pi^\omega(x)}{C^\omega(x \leftrightarrow \infty)},$$

where $C^\omega(x \leftrightarrow \infty)$ is the effective conductance between x and infinity in ω. Since $x \in \mathrm{GOOD}_K$, we can upper-bound $\pi^\omega(x)$ using Remark 5.1, and we may use

Rayleigh's monotonicity principle (see [23]) to see that

$$C^{\omega}(x \leftrightarrow \infty) \geq \frac{c}{K} \sum_{i \geq 0} c^{\omega}(p_i) \geq c \exp(2\lambda x \cdot \vec{\ell}),$$

where $(p_i)_{i \geq 0}$ is a directed path of open points starting at x. This yields the result. □

8.1. *Time spent in traps.* For $x \in \partial \mathrm{BAD}(\omega)$, we define $\mathrm{BAD}_x^{\mathrm{s}}(K) = \{x\} \cup \bigcup_{y \sim x} \mathrm{BAD}_K(y)$ the union of all bad areas adjacent to x. Following Lemma 5.1 is:

LEMMA 8.2. *For $x \in \partial \mathrm{BAD}(\omega)$, we have that $\mathrm{BAD}_x^{\mathrm{s}}(K)$ is finite $\mathbf{P}$-a.s. for K large enough and*

$$\mathbf{P}_p\big[W\big(\mathrm{BAD}_x^{\mathrm{s}}(K)\big) \geq n\big] \leq C \exp\big(-\xi_1(K)n\big),$$

where $\xi_1(K) \to \infty$ as K tends to infinity.

By a slightly subtle use of the mean return time formula we are able to obtain the following:

LEMMA 8.3. *For any $x \in \partial \mathrm{BAD}(\omega)$ we have that*

$$E_x^{\omega}\big[T^+_{\mathrm{GOOD}(\omega)}\big] \leq C(K) \exp\big(3\lambda \big|\partial \mathrm{BAD}_x^{\mathrm{s}}(\omega)\big|\big)\Big(1 + \sum_{e \in E(\mathrm{BAD}_x^{\mathrm{s}})} c_*^{\omega}(e)\Big).$$

PROOF. The first remark to be made is that since $x \in \partial \mathrm{BAD}(\omega) \subset \mathrm{GOOD}(\omega)$, all $y \sim x$, then $c_*([x, y]) \in [1/K, K]$.

We introduce the notation $\mathrm{BAD}_x^{\mathrm{ss}}(K) = \mathrm{BAD}_x^{\mathrm{s}}(K) \setminus \{x\}$.

Let us consider the finite network obtained by taking $\mathrm{BAD}_x^{\mathrm{ss}}(\omega) \cup \partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)$ and merging all points of $\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)$ (which contains x) to one point δ. We denote ω_δ the resulting graph which is obviously finite by Lemma 8.2 and connected, since the different connected components $\mathrm{BAD}_K(y)$ ($x \sim y$) are connected through x. We may apply the mean return formula at δ (see [23] exercise 2.33) to obtain that

$$E_\delta^{\omega_\delta}\big[T_\delta^+\big] = 2\frac{\sum_{e \in E(\omega_\delta)} c(e)}{\pi^{\omega_\delta}(\delta)} = 2\frac{\sum_{e \in E(\mathrm{BAD}_x^{\mathrm{ss}})} c(e) + \pi^{\omega_\delta}(\delta)}{\pi^{\omega_\delta}(\delta)},$$

where $\pi^{\omega_\delta}(\delta) = \sum_{e \in \partial_E \mathrm{BAD}_x^{\mathrm{ss}}(K)} c^{\omega}(e)$.

We know that δ was formed by merging only good (hence open) points. Hence, for y a neighbor of δ in ω_δ, we have, by Remark 5.1,

$$c \exp\Big(2\lambda \min_{y \in \partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y \cdot \vec{\ell}\Big) \leq c^{\omega_\delta}\big([\delta, y]\big) \leq C \exp\Big(2\lambda \max_{y \in \partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y \cdot \vec{\ell}\Big), \tag{8.1}$$

so that we know that

$$
\begin{aligned}
&c\exp\Big(2\lambda \max_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot\vec{\ell}\Big)\\
&\quad\le \pi^{\omega_\delta}(\delta) \le C\big|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)\big|\exp\Big(2\lambda \max_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot\vec{\ell}\Big).
\end{aligned}
\tag{8.2}
$$

Using (8.2), we get

$$
\text{for } e\in E\big(\mathrm{BAD}_x^{\mathrm{ss}}\big) \qquad \frac{c(e)}{\pi^{\omega_\delta}(\delta)} \le Cc_*(e),
$$

which means that

$$
E_\delta^{\omega_\delta}\big[T_\delta^+\big] \le C\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{ss}})} c_*(e) + C. \tag{8.3}
$$

The transition probabilities of the random walk in ω_δ at any point different from δ are the same as that of the walk in ω. This implies that

$$
\begin{aligned}
E_\delta^{\omega_\delta}\big[T_\delta^+\big] &= \sum_{y\sim\delta} P_\delta^{\omega_\delta}[X_1=y]E_y^{\omega}[T_{\partial\mathrm{BAD}_x^{\mathrm{ss}}}]\\
&= \sum_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} P_\delta^{\omega_\delta}[X_1=y]E_y^{\omega}[T_{\partial\mathrm{BAD}_x^{\mathrm{ss}}}]\\
&\ge \max_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} P_\delta^{\omega_\delta}[X_1=y]E_y^{\omega}[T_{\partial\mathrm{BAD}_x^{\mathrm{ss}}}]\\
&\ge \min_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} P_\delta^{\omega_\delta}[X_1=y] \max_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} E_y^{\omega}[T_{\partial\mathrm{BAD}_x^{\mathrm{ss}}}].
\end{aligned}
\tag{8.4}
$$

Moreover by (8.1) and (8.2), we have

$$
P_\delta^{\omega_\delta}[X_1=y] = \frac{c^{\omega_\delta}([\delta,y])}{\pi^{\omega_\delta}(\delta)} \ge c\frac{\exp(2\lambda\min_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell})}{|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)|\exp(2\lambda\max_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell})},
$$

and, since $\mathrm{BAD}_x^{\mathrm{ss}}\cup\partial\mathrm{BAD}_x^{\mathrm{ss}}$ is connected, we have

$$
\max_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot\vec{\ell} - \min_{y\in\partial\mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell} \le \big|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)\big|,
$$

so that

$$
\min_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} P_\delta^{\omega_\delta}[X_1=y] \ge c\big|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)\big|^{-1}\exp\big(-2\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)\big|\big).
$$

This, with (8.3) and (8.4), and considering the fact that $\partial\mathrm{BAD}_x^{\mathrm{ss}}\subset\mathrm{GOOD}$ yields

$$
\max_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\, y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} E_y^{\omega}[T_{\mathrm{GOOD}(\omega)}] \le C\exp\big(3\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)\big|\big)\Big(1+\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{ss}})} c_*(e)\Big).
$$

This implies that

$$
E_x^{\omega}\big[T_{\mathrm{GOOD}(\omega)}^+\big] \le 1 + C\exp\big(3\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)\big|\big)\Big(1+\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}})} c_*(e)\Big).
$$

□

8.2. *Conductances in traps*. Let us understand, partially, how the conductances in traps are conditioned.

LEMMA 8.4. *Fix $a \in [1/K, K]^{\mathcal{E}}$. Take $n \geq 0$ and $K \geq 1$, $F \subset E(\mathbb{Z}^d)$ such that $0 \notin V(F)$, and $e \in F$. If $E_*[c_*] < \infty$, then*

$$\begin{aligned}\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}P^\omega[T_{V(F)} \leq \Delta_n]c_*(e)\big]\\ \leq C(K)\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big].\end{aligned}$$

If $\lim \frac{\ln P_*[c_* > n]}{\ln n} = -\gamma$ with $\gamma < 1$, then for any $\varepsilon > 0$ we have

$$\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}c_*(e)^{\gamma-\varepsilon}\big] \leq C(K)\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}\big].$$

Let us recall that abnormal edges were defined in the beginning of Section 5.1. We introduce the notation $\omega^{e\mathrm{an}}$ to signify that for $e' \in E(\mathbb{Z}^d) \setminus \{e\}$,

$$\mathbf{1}\{e' \text{ is abnormal}\}(\omega^{e\mathrm{an}}) = \mathbf{1}\{e' \text{ is abnormal}\}(\omega)$$

and

$$\mathbf{1}\{e \text{ is abnormal}\}(\omega^{e\mathrm{an}}) = 1.$$

PROOF. First, let us notice that if there exists M such that $\mathbf{P}^a[c_* < M] = 1$, then we may obtain the first part of the lemma with $C = M$. We will now assume that $\mathbf{P}^a[c_* > M] > 0$ for any M.

Take $F \subset E(\mathbb{Z}^d)$ such that $x \in F$ and $0 \notin F$. For any $e \in F$, let us notice that on the event $\{c_*(e) > K\}$, e is abnormal which means that $E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))$. This means that

$$\begin{aligned}&\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \leq \Delta_n]c_*(e)\big]\\ &\quad\leq K\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\qquad+\mathbf{E}^a\big[c_*(e)\mathbf{1}\{c_*(e) > K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\quad= K\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\qquad+\mathbf{E}^a\big[c_*(e)\mathbf{1}\{c_*(e) > K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\quad\leq K\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\qquad+\mathbf{E}^a\big[c_*(e)\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big].\end{aligned} \tag{8.5}$$

Using the fact that $c_*(e)$ is independent of $\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}$ and $P^\omega[T_{V(F)} \leq \Delta_n]$ since $0 \notin V(F)$ and $e \in F$. Hence

$$\begin{aligned}&\mathbf{E}^a\big[c_*(e)\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big]\\ &\quad\leq \mathbf{E}^a\big[c_*(e)\big]\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \leq \Delta_n]\big].\end{aligned} \tag{8.6}$$

We can use this same independence property again to write

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big] \\
&\quad \le \frac{1}{\mathbf{P}^a\big[c_*(e) > K\big]} \\
&\quad\quad \times \mathbf{E}^a\big[\mathbf{1}\{c_*(e) > K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\ \mathrm{an}})) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big] \qquad (8.7)\\
&\quad \le \frac{1}{\mathbf{P}^a\big[c_*(e) > K\big]}\mathbf{E}^a\big[\mathbf{1}\{c_*(e) > K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big] \\
&\quad \le \frac{1}{\mathbf{P}^a\big[c_*(e) > K\big]}\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big].
\end{aligned}$$

Putting together (8.5), (8.6) and (8.7) proves the first part of the result. The second part can be handled using exactly the same techniques. □

8.3. *Correlations of hitting probabilities of traps and their shape.* We have:

LEMMA 8.5. *Fix $a \in [1/K, K]^{\mathcal{E}}$ and $x \in \mathbb{Z}^d$. Take $n \ge 0$ and $F \subset E(\mathbb{Z}^d)$, for K large enough, we have*

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big] \\
&\quad \le C\exp\big(\lambda\big|\partial V(F)\big|\big)\mathbf{P}^a\big[E(\mathrm{BAD}_x^{\mathrm{s}}) = F\big]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n],
\end{aligned}$$

where $B^\infty_{\mathbb{Z}^d}(x,r)$ is the ball of center x and radius r in $\mathbb{Z}^d$ for the infinity norm $\|\cdot\|_\infty$.

PROOF. First, we introduce the set of vertices

$$\partial^{\mathrm{good}} F = \{z \notin F \text{ where } z = y + e_1 \pm e_i \text{ with } y \in F \text{ and } i \le d\},$$

which implies that $\partial^{\mathrm{good}}\mathrm{BAD}_x^{\mathrm{s}}(K)$ is composed of good points.

Then we set $\partial^{\mathrm{good}}_{\mathrm{edge}} F$ to be the set of edges $\{[z, z+e_1], [z+e_1, z+e_1+e_i]$ with $z+e_1+e_i \in \partial^{\mathrm{good}} F$ and $i \le d\}$. Those definitions ensure that $\{E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F\}$ is measurable with respect to the conductances of the edges adjacent to F, the edges in $\partial^{\mathrm{good}}_{\mathrm{edge}} F$ and the events $\{y$ is good$\}$ for $y \in \partial^{\mathrm{good}} F$. These notations are illustrated in Figure 4.

We extend the notion of x-open in the following manner: for any set A of edges, we say that a point y is A-open in ω, if y is open in the environment coinciding with ω on all edges of A and where all other edges are open.

Let us notice that if $E(\mathrm{BAD}_x^{\mathrm{s}}(K)) = F$, and then we have the two following events:

(1) $E(\mathrm{BAD}_x^{\mathrm{s}}(K))$ contains at least F, and this can be determined by the conductances of the edges adjacent to $y \in F$ and the edges in $\partial^{\mathrm{good}}_{\mathrm{edge}} F$;

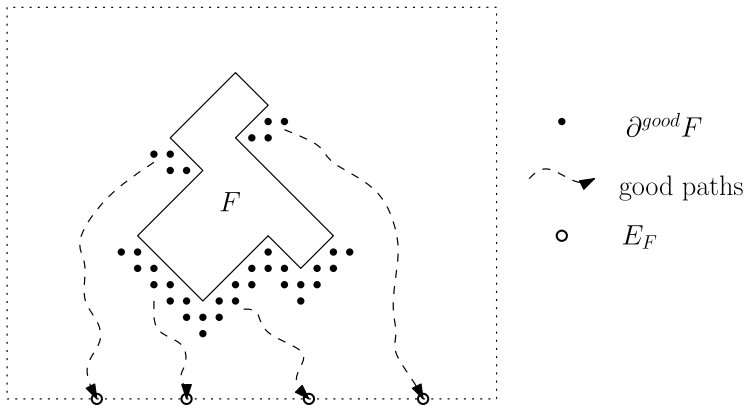

FIG. 4. *The event C_F depends only on what happens inside the dotted box $E(B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|))$. In this picture the direction of e_1 is down.*

(2) all vertices of $\partial^{\text{good}} F$ are connected with an $E(B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|))$-open directed path with even length to $\partial^+ B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|) := \{z \in B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|)$, with $(z - x) \cdot e_1 = 3|\partial V(F)|\}$. We denote by E_F the endpoints of these paths in $\partial^+ B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|)$.

We denote by C_F the event which is the intersection of the two previous events. The key properties of this definition are that:

(1) C_F is measurable with respect to the conductances of the edges in $E(B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|))$, which meant that it is $\mathbf{P}$-independent of

$$P^\omega[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n];$$

(2) on C_F, if the points in E_F are $E(B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|))^c$-good, then all points in $\partial^{\text{good}} F$ are good, and since $E(\text{BAD}^{\text{s}}_x(K))$ contains at least F, we have $E(\text{BAD}^{\text{s}}_x(K)) = F$.

Hence

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{E\big(\text{BAD}^{\text{s}}_x(K)\big) = F\}P^\omega[T_{V(F)} \le \Delta_n]\big] \\
&\quad \le \mathbf{E}^a\big[\mathbf{1}\{C_F\}P^\omega[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n\big] \\
&\quad = \mathbf{P}^a[C_F]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n].
\end{aligned}$$

Moreover, since E_F can be injected into $\partial^{\text{good}} F$, we see that $|E_F| \le |\partial^{\text{good}} F| \le C(d)|\partial V(F)|$. Now, using the FKG inequality (see [18]) in the first line and the fact that $\{E_F = E\}$ and {the points in E are $E(B^\infty_{\mathbb{Z}^d}(x, 3|\partial V(F)|))^c$-good} are $\mathbf{P}^a$-

independent in the second, we obtain

$$\begin{aligned}
&\mathbf{P}[0 \text{ is good}]^{C(d)|\partial V(F)|}\mathbf{P}^a[C_F] \\
&\quad \le \sum_{E\subset\mathbb{Z}^d, |E|\le C(d)|\partial V(F)|} \mathbf{P}^a[E_F = E, C_F] \\
&\qquad\qquad \times \mathbf{P}^a\big[\text{points in } E \text{ are } E\big(B^\infty_{\mathbb{Z}^d}\big(x, 3\big|\partial V(F)\big|\big)\big)^c\text{-good}\big] \\
&\quad \le \sum_{E\subset\mathbb{Z}^d, |E|\le C(d)|\partial V(F)|} \mathbf{P}^a\big[E_F = E, C_F, \text{points in } E \\
&\qquad\qquad \text{are } E\big(B^\infty_{\mathbb{Z}^d}\big(x, 3\big|\partial V(F)\big|\big)\big)^c\text{-good}\big] \\
&\quad \le \sum_{E\subset\mathbb{Z}^d, |E|\le C(d)|\partial V(F)|} \mathbf{P}^a\big[E_F = E, E\big(\mathrm{BAD}^{\mathrm{s}}_x(K)\big) = F\big] \\
&\quad = \mathbf{P}^a\big[E\big(\mathrm{BAD}^{\mathrm{s}}_x(K)\big) = F\big].
\end{aligned}$$

For K large enough, we have $\mathbf{P}[0 \text{ is } K\text{-good}]^{C(d)|\partial V(F)|} \ge \exp(-\lambda|\partial V(F)|)$, and hence using two previous equation, we see that

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\big\{E\big(\mathrm{BAD}^{\mathrm{s}}_x(K)\big) = F\big\}P^\omega[T_{V(F)} \le \Delta_n]\big] \\
&\qquad \le C\exp\big(\lambda\big|\partial V(F)\big|\big)\mathbf{P}^a\big[E\big(\mathrm{BAD}^{\mathrm{s}}_x\big) = F\big]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n]. \qquad \square
\end{aligned}$$

8.4. *Proof of Theorem* 8.1. We are now getting to the pivotal point in the proof where we control Δ_n in terms of the expected number of sites encountered before Δ_n.

LEMMA 8.6. *There exists K_0 such that, for any $K \ge K_0$, we have*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0 \in \mathrm{GOOD}_K\}\big] \le C(K) \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[N_n],$$

where $N_n = |\{x \in \mathbb{Z}^d \text{ such that } T_x \le \Delta_n\}|$ is the number of sites reached before Δ_n.

PROOF. *Step* 1: *Decompose Δ_n into the time spent in traps and outside of traps.* If $0 \in \mathrm{GOOD}(\omega)$, then a walk started at 0 can only be in a vertex of $\mathrm{BAD}(\omega)$ between visits to $\partial\mathrm{BAD}(\omega)$. Hence, on $\{0 \in \mathrm{GOOD}(\omega)\}$,

$$\sum_{x\in\mathrm{BAD}(\omega)} \sum_{i=0}^{\Delta_n} \mathbf{1}\{X_i = x\} \le \sum_{x\in\partial\mathrm{BAD}(\omega)} \sum_{i=0}^{\Delta_n} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i.$$

Hence, on $\{0 \in \mathrm{GOOD}(\omega)\}$, since $\mathbb{Z}^d$ is partitioned into two parts $\mathrm{GOOD}(\omega)$ and $\mathrm{BAD}(\omega)$ we have

$$\begin{aligned}
\Delta_n &\le \sum_{x\in\mathbb{Z}^d}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i = x\} \\
&\le \sum_{x\in\mathrm{GOOD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i = x\} + \sum_{x\in\mathrm{BAD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i = x\} \qquad (8.8)\\
&\le \sum_{x\in\mathrm{GOOD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i = x\} + \sum_{x\in\partial\mathrm{BAD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i = x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i.
\end{aligned}$$

Hence, on $\{0 \in \mathrm{GOOD}(\omega)\}$,

$$\begin{aligned}
\Delta_n \le{}& \sum_{x\in\mathrm{GOOD}(\omega)}\mathbf{1}\{T_x \le \Delta_n\}\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\} \\
&+ \sum_{x\in\partial\mathrm{BAD}(\omega)}\mathbf{1}\{T_x \le \Delta_n\}\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i. \qquad (8.9)
\end{aligned}$$

Step 2: *Reduce the problem to hitting probabilities and time of excursions in traps.* We can use Markov's property to say that for any $x \in \mathbb{Z}^d$, on $\{0 \in \mathrm{GOOD}(\omega)\}$

$$E^\omega\left[\mathbf{1}\{T_x \le \Delta_n\}\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}\right] = P^\omega[T_x \le \Delta_n]E^\omega_x\left[\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}\right]$$

and

$$\begin{aligned}
&E^\omega\left[\mathbf{1}\{T_x \le \Delta_n\}\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\right] \\
&\qquad = P^\omega[T_x \le \Delta_n]E^\omega_x\left[\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\right].
\end{aligned}$$

This implies, using (8.9), that on $\{0 \in \mathrm{GOOD}(\omega)\}$

$$\begin{aligned}
E^\omega[\Delta_n] \le{}& \sum_{x\in\mathrm{GOOD}(\omega)} P^\omega[T_x \le \Delta_n]E^\omega_x\left[\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}\right] \\
&+ \sum_{x\in\partial\mathrm{BAD}(\omega)} P^\omega[T_x \le \Delta_n]E^\omega_x\left[\sum_{i=0}^{\infty}\mathbf{1}\{X_i = x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\right].
\end{aligned}$$

Now we have

$$\begin{aligned}
&\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0\in \mathrm{GOOD}_K\}\big] \\
&\quad \le C \max_{a\in[1/K,K]^{\mathcal{E}}} \Bigg[\mathbf{E}^a\Bigg[\sum_{x\in \mathrm{GOOD}(\omega)} P^\omega[T_x\le \Delta_n] E_x^\omega\Bigg[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\}\Bigg]\Bigg]\Bigg] \\
&\qquad + \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{E}^a\Bigg[\mathbf{1}\{0\notin \mathrm{BAD}\} \\
&\qquad\qquad \times \sum_{x\in\partial\mathrm{BAD}(\omega)} P^\omega[T_x\le\Delta_n] \\
&\qquad\qquad\qquad \times E_x^\omega\Bigg[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\} T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Bigg]\Bigg]\Bigg].
\end{aligned} \tag{8.10}$$

So that, using Lemma 8.1,

$$\begin{aligned}
&\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0\in \mathrm{GOOD}_K\}\big] \\
&\quad \le C\Bigg[\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a\Bigg[\sum_{x\in\mathbb{Z}^d} \mathbf{1}\{T_x\le\Delta_n\}\Bigg] \\
&\qquad + \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{E}^a\Bigg[\sum_{x\in\partial\mathrm{BAD}(\omega)} P^\omega[T_x\le\Delta_n] \\
&\qquad\qquad \times E_x^\omega\Bigg[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\} T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Bigg]\Bigg]\Bigg].
\end{aligned}$$

Let us focus, for now, on the second term which is the more difficult to upper-bound. It corresponds to the time spent in the traps we encounter in the first regeneration time. By Markov's property at $x\in\partial\mathrm{BAD}(\omega)$,

$$\begin{aligned}
&E_x^\omega\Bigg[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\} T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Bigg] \\
&\quad = \sum_{i=0}^\infty E_x^\omega\big[\mathbf{1}\{X_i=x\}\big] E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big] \\
&\quad = \Bigg(\sum_{i=0}^\infty E_x^\omega\big[\mathbf{1}\{X_i=x\}\big]\Bigg) E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big] \\
&\quad \le C E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big],
\end{aligned} \tag{8.11}$$

where we used Lemma 8.1. Hence

$$\begin{aligned}
&\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{E}^a\Bigg[\mathbf{1}\{0\notin \mathrm{BAD}\}\sum_{x\in\partial\mathrm{BAD}(\omega)} P^\omega[T_x\le\Delta_n] \\
&\qquad\qquad\qquad\qquad \times E_x^\omega\Bigg[\sum_{i=0}^{\infty}\mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Bigg]\Bigg] \\
&\le C\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{E}^a\Bigg[\mathbf{1}\{0\notin \mathrm{BAD}\}\sum_{x\in\partial\mathrm{BAD}(\omega)} P^\omega[T_x\le\Delta_n]E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big]\Bigg] \\
&\le C\max_{a\in[1/K,K]^{\mathcal{E}}} \sum_{x\in\mathbb{Z}^d}\mathbf{E}^a\big[\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}\mathbf{1}\{0\notin \mathrm{BAD}\} \\
&\qquad\qquad\qquad\qquad \times P^\omega[T_x\le\Delta_n]E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big]\big] \qquad (8.12)\\
&\le C\max_{a\in[1/K,K]^{\mathcal{E}}} \sum_{x\in\mathbb{Z}^d}\mathbf{E}^a\big[\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}\mathbf{1}\{0\notin \mathrm{BAD}\} \\
&\qquad\qquad\qquad\qquad \times P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}}\le\Delta_n]E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big]\big],
\end{aligned}$$

where we used that for $x\in\partial\mathrm{BAD}(\omega)$, we have $x\in\mathrm{BAD}_x^{\mathrm{s}}$ by definition.

Step 3: *Estimate the time spent during an excursion in a trap by its size.* Using Lemma 8.3,

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{0\notin \mathrm{BAD}\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}}\le\Delta_n]E_x^\omega\big[T^+_{\mathrm{GOOD}(\omega)}\big]\big] \\
&\le C\mathbf{E}^a\Bigg[\mathbf{1}\{0\notin \mathrm{BAD}\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\} \\
&\qquad\qquad \times P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}}\le\Delta_n]\exp\big(3\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)\big|\big) \\
&\qquad\qquad\qquad\qquad \times\Bigg(1+\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}})}c_*(e)\Bigg)\Bigg] \qquad (8.13)\\
&\le \sum_{\substack{F\subset E(\mathbb{Z}^d)\\ x\in V(F),0\notin V(F)}} C\exp\big(3\lambda\big|\partial V(F)\big|\big) \\
&\qquad\qquad \times\sum_{e\in F}\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\} \\
&\qquad\qquad\qquad\qquad \times P^\omega[T_{V(F)}\le\Delta_n]\big(1+c_*(e)\big)\big].
\end{aligned}$$

Now by Lemma 8.4 (applied in the case $E_*[c_*]<\infty$) and Lemma 8.5,

$$\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\le\Delta_n]c_*(e)\big] \\
&\le C\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\le\Delta_n]\big] \qquad (8.14)\\
&\le C\exp\big(\lambda\big|\partial V(F)\big|\big)\mathbf{P}^a\big[E(\mathrm{BAD}_x^{\mathrm{s}})=F\big]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)}<\Delta_n],
\end{aligned}$$

so using (8.13), we have

$$
\begin{aligned}
&\mathbf{E}^a\big[\mathbf{1}\{0 \notin \mathrm{BAD}\}\mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\}P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}} \le \Delta_n]E_x^\omega[T^+_{\mathrm{GOOD}(\omega)}]\big] \\
&\quad \le C \sum_{\substack{F \subset E(\mathbb{Z}^d) \\ x \in V(F), 0 \notin V(F)}} \exp\big(4\lambda|\partial V(F)|\big)\mathbf{P}^a\big[E\big(\mathrm{BAD}_x^{\mathrm{s}}\big) = F\big] \\
&\qquad\qquad \times \mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3|\partial V(F)|)} < \Delta_n] \\
&\quad = C \sum_{k \ge 0} \exp(4\lambda k)\mathbf{P}^a\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}| = k\big]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3k)} < \Delta_n],
\end{aligned}
\tag{8.15}
$$

where we sum over the sets F of same size in the last line.

Step 4: *Conclusion.* When $x \notin \partial\mathrm{BAD}$ we use the notation $\mathrm{BAD}_x^{\mathrm{s}} = \partial\mathrm{BAD}_x^{\mathrm{s}} = \{x\}$. Using (8.12), (8.15) and (8.10) we obtain

$$
\begin{aligned}
&\max_{a \in [1/K,K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0 \in \mathrm{GOOD}_K\}\big] \\
&\quad \le C \max_{a \in [1/K,K]^{\mathcal{E}}} \Big[\sum_{x \in \mathbb{Z}^d} \mathbb{P}^a[T_{\mathrm{BAD}_x^{\mathrm{s}}} \le \Delta_n] \\
&\qquad\qquad + \sum_{x \in \mathbb{Z}^d} \mathbf{E}^a\big[\exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}|\big)P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}} \le \Delta_n]\big]\Big] \\
&\quad \le C \max_{a \in [1/K,K]^{\mathcal{E}}} \sum_{x \in \mathbb{Z}^d} \mathbf{E}^a\big[\exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}|\big)P^\omega[T_{\mathrm{BAD}_x^{\mathrm{s}}} \le \Delta_n]\big] \\
&\quad \le C \max_{a \in [1/K,K]^{\mathcal{E}}} \sum_{x \in \mathbb{Z}^d} \sum_{k \ge 0} \exp(4\lambda k)\mathbf{P}^a\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}| = k\big]\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3k)} < \Delta_n] \\
&\quad \le C \max_{a \in [1/K,K]^{\mathcal{E}}} \max_{x \in \mathbb{Z}^d} \Big[\sum_{k \ge 0} \Big[\exp(4\lambda k)\mathbf{P}^a\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}| = k\big] \\
&\qquad\qquad \times \Big(\sum_{x \in \mathbb{Z}^d} \mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3k)} < \Delta_n]\Big)\Big]\Big].
\end{aligned}
\tag{8.16}
$$

Now, we can use the fact that $\mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3k)} < \Delta_n] \le \sum_{y \in B^\infty_{\mathbb{Z}^d}(x,3k)} \mathbb{P}^a[T_y < \Delta_n]$ so that $\sum_{x \in \mathbb{Z}^d} \mathbb{P}^a[T_{B^\infty_{\mathbb{Z}^d}(x,3k)} < \Delta_n] \le Ck^d \sum_{x \in \mathbb{Z}^d} \mathbb{P}^a[T_x < \Delta_n] \le Ck^d \mathbb{E}^a[N_n]$. This means that

$$
\begin{aligned}
&\max_{a \in [1/K,K]^{\mathcal{E}}} \mathbb{E}^a\big[\Delta_n \mathbf{1}\{0 \in \mathrm{GOOD}_K\}\big] \\
&\quad \le C \max_{a \in [1/K,K]^{\mathcal{E}}} \max_{x \in \mathbb{Z}^d} \Big[\sum_{k \ge 0} k^d \exp(4\lambda k)\mathbf{P}^a\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}| = k\big]\Big]
\end{aligned}
\tag{8.17}
$$

$$\times \Big(\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[N_n]\Big)$$
$$\le C \max_{a\in[1/K,K]^{\mathcal{E}}} \max_{x\in\mathbb{Z}^d} \mathbf{E}^a\big[\exp\big(5\lambda\big|\partial \mathrm{BAD}_x^{\mathrm{s}}\big|\big)\big] \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[N_n].$$

The random variables $\mathrm{BAD}_x^{\mathrm{s}}$ are measurable with respect to the events $\{y \text{isopen}\}$ for $y \in \mathbb{Z}^d$. The expectation is that $\mathbf{E}^a$ conditions only one such random variable, so

$$\max_{x\in\mathbb{Z}^d} \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{E}^a\big[\exp\big(5\lambda\big|\partial \mathrm{BAD}_x^{\mathrm{s}}\big|\big)\big] \le \frac{1}{\mathbf{P}[0 \text{ is open}]} \max_{x\in\mathbb{Z}^d} \mathbf{E}\big[\exp\big(5\lambda\big|\partial \mathrm{BAD}_x^{\mathrm{s}}\big|\big)\big]$$

by translation invariance, and using Lemma 8.2 the right-hand side is finite for K large enough. This and the two last equations complete the proof. $\square$

Let us estimate $\mathbb{P}^a[T_x \le \tau_1^{(K)}]$.

LEMMA 8.7. *Take $x \in \mathbb{Z}^d$, and then for any $M < \infty$, there exists K_0 such that for any $K \ge K_0$,*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a\big[T_x \le \tau_1^{(K)}\big] \le C\|x\|_\infty^{-M}.$$

PROOF. Denote by χ the smallest integer so that $\{X_i, i \in [0, \tau_1]\} \subset B(\chi, \chi^\alpha)$. First let us notice that

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[\chi \ge k]$$
$$\le \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[X_{\tau_1}\cdot\vec{\ell} \ge k] \tag{8.18}$$
$$+ \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a\Big[X_{\tau_1}\cdot\vec{\ell} < k, \max_{0\le i,j\le\tau_1} \max_{l\in[2,d]} \big|(X_j - X_i)\cdot f_l\big| \ge k^\alpha\Big].$$

We can upper-bound the first term as follows:

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[X_{\tau_1^{(K)}}\cdot\vec{\ell} \ge k] \le Ck^{-M}$$

for any M by choosing K large enough by Theorem 7.2.

The second term can be upper-bounded with the following reasoning: on the event that $X_{\tau_1}\cdot\vec{\ell} \le k$ and $\max_{j\ne1}\max_{0\le j_1,j_2\le\tau_1}|(X_{j_1} - X_{j_2})\cdot f_j| \ge k^\alpha$, X_n does not exit the box $B(k, k^\alpha)$ through $\partial^+B(k, k^\alpha)$, this means

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a\Big[X_{\tau_1}\cdot\vec{\ell} < k, \max_{0\le i,j\le\tau_1} \max_{l\in[2,d]} \big|(X_j - X_i)\cdot f_l \ge k^\alpha\big|\Big]$$
$$\le \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_{\partial B(k,k^\alpha)} \ne T_{\partial^+B(k,k^\alpha)}] \le ce^{-ck}$$

by Theorem 7.3.

This turns (8.18) into

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[\chi \geq k] \leq Ck^{-M} \tag{8.19}$$

for any M for K large enough.

Now assume that $\{T_x \leq \tau_1\}$ then $\chi \geq \|x\|_\infty^{1/\alpha}$. Hence, by the previous equation

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_x \leq \tau_1] \leq C\|x\|_\infty^{-M/\alpha},$$

which proves the lemma, since α is fixed and M is arbitrary. □

We can now prove Theorem 8.1.

PROOF. Let us introduce $\tilde{N}_{\tau_i}$ the number of different sites seen between τ_{i-1} and τ_i for any $i \geq 1$. By Theorem 7.4 and choosing $K \geq K_0$ so that we may apply Lemma 8.7 with $M > 2d$, we can see that for $i \geq 1$

$$\begin{aligned}\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\tilde{N}_{\tau_i}] &\leq \sum_{x\in\mathbb{Z}^d} \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_x < \tau_1 \mid D = \infty] \\ &\leq \sum_{x\in\mathbb{Z}^d} C\|x\|_\infty^{-M/2} < C,\end{aligned}$$

where C does not depend on $i \geq 1$. A similar inequality holds for $i = 0$.

Since $\Delta_n \leq \tau_n = \sum_{i=1}^n (\tau_i - \tau_{i-1})$, we see that $N_n \leq \sum_{i=1}^n \tilde{N}_{\tau_i}$ which means that

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[N_n] \leq \sum_{i=1}^n \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\tilde{N}_{\tau_i}] \leq Cn.$$

Then, using Lemma 8.6, we can prove Theorem 8.1. □

8.5. *Law of large numbers*. We can use exactly the same type of proof as in [25] to obtain:

PROPOSITION 8.1. *If $E_*[c_*] < \infty$, then there exists K_0 such that for any $K \geq K_0$ we have*

$$\frac{X_n}{n} \to v = \frac{\mathbb{E}^{\Pi_K}[X_{\tau_1^{(K)}}]}{\mathbb{E}^{\Pi_K}[\tau_1^{(K)}]}, \qquad \mathbb{P}\text{-a.s. with } v \cdot \vec{\ell} > 0,$$

where

$$\Pi_K[\cdot] = \int \nu_K(da)\mathbb{P}_0^a[\cdot \mid D = \infty] \quad \text{and} \quad \mathbb{E}^{\Pi_K}[\cdot] := \int \nu_K(da)\mathbb{E}_0^a[\cdot \mid D = \infty],$$

where ν_K is the unique invariant distribution on $[1/K, K]^{\mathcal{E}}$ given in Theorem 7.5.

Let us explain quickly where this result comes from. The regeneration structure will allow us to obtain a law of large numbers for $X_n/n \to v$ and $\Delta_n/n \to C_\infty$. By a standard inversion argument we will see that $v \cdot \vec{\ell} = 1/C_\infty$. Then Theorem 8.1 will imply that C_∞ cannot be infinite which means that X_n/n cannot go to 0.

PROOF. We will now follow the strategy of proof of Theorem 5.1 in [25] to obtain the result.

First, we notice that by (8.19), we have $\mathbb{E}^\Pi[|X_{\tau_1}|] < \infty$.

Recalling the notations of Proposition 7.2, we know that Y_i^K with initial distribution $\tilde{\nu}$ is stationary and ergodic and that the law of $(Y_i^K)_{i\geq 1}$ under $\mathbb{P}$ is absolutely continuous with respect to the law with initial distribution $\tilde{\nu}$. Therefore using Birkhoff's ergodic theorem (page 341 of [12]) we obtain that for any $f \in L^1(\Gamma, \tilde{\nu})$,

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) \to_{n\to\infty} \int f d\tilde{\nu}, \qquad \mathbb{P}\text{-a.s.}$$

Applying the previous formula to the functions $f(y) = j$ and $f(y) = x$ where $y = (j, z, a) \in \Gamma$, we obtain $\mathbb{P}$-a.s.

$$\frac{1}{n-1}\big(\tau_n^{(K)} - \tau_1^{(K)}\big) \to_{n\to\infty} \int \mathbb{E}^a\big[\tau_1^{(K)} \mid D = \infty\big]\nu(da) = \mathbb{E}^{\Pi_K}\big[\tau_1^{(K)}\big]$$

and

$$\frac{1}{n-1}(X_{\tau_n^{(K)}} - X_{\tau_1^{(K)}}) \to_{n\to\infty} \int \mathbb{E}^a[X_{\tau_1^{(K)}} \mid D = \infty]\nu(da) = \mathbb{E}^{\Pi_K}[X_{\tau_1^{(K)}}].$$

Moreover recalling that, by Proposition 7.1, we have $\tau_1^{(K)} < \infty$, $\mathbb{P}$-a.s. we actually see that $\mathbb{P}$-a.s.

$$\frac{1}{n}\tau_n^{(K)} \to_{n\to\infty} \mathbb{E}^{\Pi_K}\big[\tau_1^{(K)}\big] \quad \text{and} \quad \frac{1}{n}X_{\tau_n^{(K)}} \to_{n\to\infty} \mathbb{E}^{\Pi_K}[X_{\tau_1^{(K)}}].$$

Now, we may introduce k_n, a nondecreasing sequence going to infinity such that $\tau_{k_n} \leq n < \tau_{k_{n+1}}$. By dividing this equation by k_n we see that the previous estimate implies that $\mathbb{P}$-a.s.

$$\frac{k_n}{n} \to \frac{1}{\mathbb{E}^{\Pi_K}[\tau_1^{(K)}]}.$$

Furthermore, we observe that $\mathbb{P}$-a.s.

$$\frac{X_n}{n} = \frac{X_{\tau_{k_n}}}{n} + \frac{X_n - X_{\tau_{k_n}}}{n},$$

which in view of the two previous equations implies that $\mathbb{P}$-a.s.

$$\frac{X_{\tau_{k_n}}}{n} = \frac{X_{\tau_{k_n}}}{k_n}\frac{k_n}{n} \to_{n\to\infty} \frac{\mathbb{E}^{\Pi_K}[X_{\tau_1^{(K)}}]}{\mathbb{E}^{\Pi_K}[\tau_1^{(K)}]},$$

where the right-hand side is always well defined (even if $\mathbb{E}^{\Pi}[\tau_1] = \infty$), since $\mathbb{E}^{\Pi_K}[|X_{\tau_1^{(K)}}|] \in (0, \infty)$. The last assertion follows from Theorem 7.2 and the fact that $X_{\tau_1} \cdot \vec{\ell} > 2/\sqrt{d}$.

Moreover we have $\mathbb{P}$-a.s.

$$\frac{|X_n - X_{\tau_{k_n}}|}{n} \le \frac{\tau_{k_n+1} - \tau_{k_n}}{n} = \frac{\tau_{k_n+1}}{k_n+1}\frac{k_n+1}{n} - \frac{\tau_{k_n}}{k_n}\frac{k_n}{n} \to_{n\to\infty} 0.$$

The three last equations imply that $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{X_n}{n} = v = \frac{\mathbb{E}^{\Pi}[X_{\tau_1}]}{\mathbb{E}^{\Pi}[\tau_1]}, \tag{8.20}$$

even in the case where $\mathbb{E}^{\Pi}[\tau_1] = \infty$.

We introduce k_n', a nondecreasing sequence going to infinity such that $\tau_{k_n'} \le \Delta_n < \tau_{k_n'+1}$. This means that

$$X_{\tau_{k_n'}} \cdot \vec{\ell} \le X_{\Delta_n} \cdot \vec{\ell} < X_{\tau_{k_n'+1}} \cdot \vec{\ell}$$

and in particular,

$$X_{\tau_{k_n'}} \cdot \vec{\ell} \le n+1 \quad \text{and} \quad X_{\tau_{k_n'+1}} \cdot \vec{\ell} \ge n.$$

Dividing the left equation by $\tau_{k_n'}$ and the right one by $\tau_{k_n'+1}$ we can take the limit as n goes to infinity using (8.20), which yields $\mathbb{P}$-a.s.

$$\limsup_{n\to\infty} \frac{n}{\tau_{k_n'+1}} \le \frac{\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]}{\mathbb{E}^{\Pi}[\tau_1]} \quad \text{and} \quad \liminf_{n\to\infty} \frac{n}{\tau_{k_n'}} \ge \frac{\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]}{\mathbb{E}^{\Pi}[\tau_1]}.$$

Now, we see that

$$\begin{aligned}
\frac{\tau_{k_{n+1}'} - \tau_{k_n'}}{n} &= \frac{\tau_{k_n'+1}}{k_n'+1}\frac{k_n'+1}{n} - \frac{\tau_{k_n'}}{k_n'}\frac{k_n'}{n} \\
&= \big(\mathbb{E}^{\Pi}[\tau_1] + o(1)\big)\left(\frac{k_n'}{n} + o(1)\right) - \big(\mathbb{E}^{\Pi}[\tau_1] + o(1)\big)\frac{k_n'}{n} \\
&= \left(\frac{k_n'}{n} + o(1)\right) o(1),
\end{aligned}$$

where the right-hand side goes to zero because $k_n' \le n$ (since $\Delta_n < \tau_n$), so that the two previous equations imply that

$$\lim_{n\to\infty} \frac{n}{\tau_{k_n'}} = \frac{\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]}{\mathbb{E}^{\Pi}[\tau_1]}.$$

Recalling the definition of the sequence k_n', this implies $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\Delta_n}{n} = \frac{\mathbb{E}^{\Pi}[\tau_1]}{\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]}, \tag{8.21}$$

even in the case where $\mathbb{E}^{\Pi}[\tau_1] = \infty$.

If $E_*[c_*] < \infty$, then Theorem 8.1 (and the fact that $\{0 \in \mathrm{GOOD}_K\}$ has positive probability) imply that the almost sure limit in (8.21) cannot be infinite. This means that $\mathbb{E}^{\Pi}[\tau_1] < \infty$. Since $X_{\tau_1} \cdot \vec{\ell} > 2/\sqrt{d}$, this means that (8.20) implies that $v \cdot \vec{\ell} > 0$. $\square$

REMARK 8.1. Interestingly, we do not know of any direct way of showing that $\mathbb{E}^{\Pi}[\tau_1] < \infty$.

9. Zero-speed regime.

9.1. *Characterization of the zero-speed regime.* We set $\mathcal{A}$ to be the set of vertices: 0, e_1, $e_1 \pm e_i$, $2e_1 \pm e_i$, $2e_1 \pm 2e_i$, $3e_1 \pm 2e_i$, $3e_1 \pm e_i$, $4e_1 \pm e_i$, for all $i \in [2, d]$, $5e_1$, $6e_1$ and the events

$$A = \{\text{any } x \in \mathcal{A} \text{ is } 6e_1\text{-open}\} \quad \text{and} \quad B = \{6e_1 \text{ is good}\},$$

where we recall that a vertex is called x-open if it is open in ω_x coinciding with ω on all edges, but those that are adjacent to x which are normal in ω_x.

Note that A and B are independent and independent of $c_*([2e_1, 3e_1])$.

LEMMA 9.1. *If* $E_*[c_*] = \infty$, *then* $\min_{a \in [1/K, K]^{\mathcal{E}}} E^a[\tau_1 \mid D = \infty] = \infty$.

The typical configuration that will slow the walk down is depicted in Figure 5: the walk is likely to reach the edge $[2e_1, 3e_1]$ and then stay there for a long time. More precisely, we will consider the following chain of events on $A \cap B$:

(1) $X_1 = e_1$, $X_2 = 2e_1$, $X_3 = e_1$, $X_4 = 2e_1$, which forces τ_1 to be after time 4.

(2) From there we go back and forth on $[2e_1, 3e_1]$, the quenched expected time spent on this edge is of order $c_*([2e_1, 3e_1])$. With the previous point, this implies that τ_1 is larger than the time spent on $[2e_1, 3e_1]$ and so has, in some sense, an infinite annealed expectation.

(3) After leaving the edge $[2e_1, 3e_1]$ the walk goes to $6e_1$ before reaching any point of ∂A and from there never backtracks.

In the series of events described above, we have that $D = \infty$, and in some sense, we have τ_1 has infinite expectation. Moreover, under the quenched measure all the three events described are independent.

PROOF. We have

$$\mathbb{E}^a[\tau_1 \mid D = \infty] \geq \mathbb{E}^a\big[\mathbf{1}\{D = \infty\}\tau_1\big] \geq \mathbb{E}^a\big[\mathbf{1}\{A, B\}\mathbf{1}\{D = \infty\}\tau_1\big]. \tag{9.1}$$

We may notice that on $A \cap B$, if:

(1) $X_1 = e_1$, $X_2 = 2e_1$, $X_3 = e_1$, $X_4 = 2e_1$ (hence $\tau_1 \geq 4$),

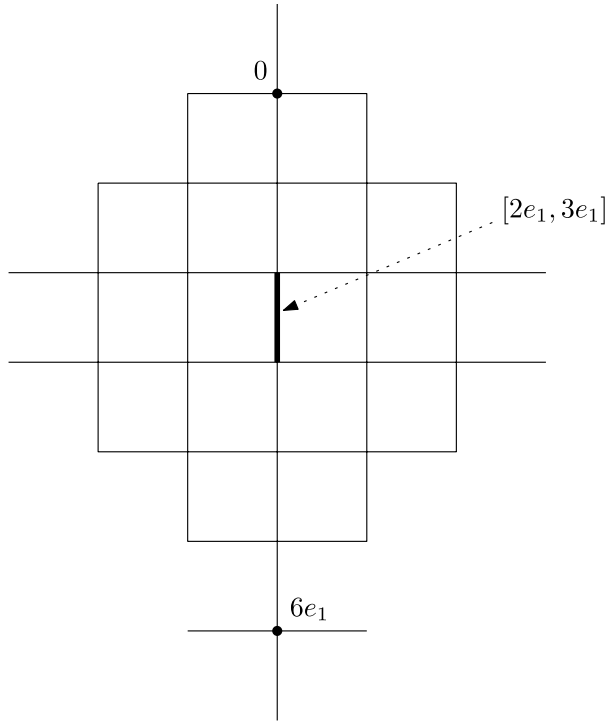

FIG. 5. *The typical configurations slowing the walk down. All the edges depicted are normal and the bold one has unusually high conductance.*

(2) $T_{6e_1} \circ \theta_{T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4} \leq T_{\partial(\mathcal{A} \setminus \{0\})} \circ \theta_{T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4}$,
(3) $D \circ \theta_{T_{6e_1}} = \infty$,

then we have $D = \infty$ and $\tau_1 \geq T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4$. Now, we aim at estimating the three different events and $T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4$, which will allow us to give a lower-bound in (9.1).

On $A \cap B$, we see, by Remark 5.1, that we have

$$P^{\omega}[X_1 = e_1, X_2 = 2e_1, X_3 = e_1, X_4 = 2e_1] \geq \kappa_0^4 \tag{9.2}$$

and moreover on $A \cap B$

$$P^{\omega}_{2e_1}[T_{6e_1} \circ \theta_{T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4} \leq T_{\partial(\mathcal{A} \setminus \{0\})} \circ \theta_{T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \circ \theta_4}] \geq \kappa_0^7, \tag{9.3}$$

which follows from Remark 5.1 and the fact that, on $A \cap B$, for any neighbor of $2e_1$ or $3e_1$, there exists an open nearest-neighbor path of length at most 7 in $\mathcal{A} \setminus \{0\}$ to $6e_1$.

Using Remark 5.1 again, we may see that on $A \cap B$, we have $P^{\omega}_{2e_1}[X_1 \neq 3e_1] \leq C/c_*([2e_1, 3e_1])$ and $P_{3e_1}[X_1 \neq 2e_1] \leq C/c_*([2e_1, 3e_1])$. This implies that

$$P^{\omega}_{2e_1}[T_{\mathbb{Z}^d \setminus \{2e_1, 3e_1\}} \geq n] \geq \big(1 - C/c_*([2e_1, 3e_1])\big)^n, \tag{9.4}$$

so

$$E^{\omega}_{2e_1}[T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}] \geq cc_*\big([2e_1, 3e_1]\big). \tag{9.5}$$

Using the series of events below (9.1) and Markov's property (at times 4, $T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}} \circ \theta_4$ and T_{6e_1}) along with (9.5), (9.2) and (9.3) we may see

$$\begin{aligned}
&\mathbb{E}^a\big[\mathbf{1}\{A, B\}\mathbf{1}\{D=\infty\}\tau_1\big] \\
&\quad \geq c\mathbf{E}^a\big[\mathbf{1}\{A, B\}E^{\omega}_{2e_1}[T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}]P^{\omega}_{6e_1}[D=\infty]\big] \\
&\quad \geq c\mathbf{E}^a\big[\mathbf{1}\{A, B\}c_*\big([2e_1, 3e_1]\big)P^{\omega}_{6e_1}[D=\infty]\big],
\end{aligned} \tag{9.6}$$

which is a consequence of the estimates of the events defined after (9.1).

We may now notice that $\mathbf{1}\{A\}$, $c_*([2e_1, 3e_1])$ and $\mathbf{1}\{B\}P^{\omega}_{6e_1}[D=\infty]$ are $\mathbf{P}^a$-independent, so that

$$\mathbb{E}^a\big[\mathbf{1}\{A, B\}\mathbf{1}\{D=\infty\}\tau_1\big] \geq \mathbf{P}^a[A]\mathbf{E}^a\big[c_*\big([2e_1, 3e_1]\big)\big]\mathbf{E}^a\big[\mathbf{1}\{B\}P^{\omega}_{6e_1}[D=\infty]\big].$$

We have $\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{P}^a[A] \geq c > 0$, $\mathbf{E}^a[c_*([2e_1, 3e_1])] = \mathbf{E}[c_*([2e_1, 3e_1])] = \infty$, and by translation invariance

$$\mathbf{E}^a\big[\mathbf{1}\{B\}P^{\omega}_{6e_1}[D=\infty]\big] = \mathbf{E}\big[\mathbf{1}\{0 \text{ is good}\}P^{\omega}[D=\infty]\big] > 0$$

by Lemma 7.4 and the fact that $\mathbf{P}[0 \text{ is good}] > 0$. This means that

$$\mathbf{E}^a\big[\mathbf{1}\{A, B\}c_*\big([2e_1, 3e_1]\big)P^{\omega}_{6e_1}[D=\infty]\big] = \infty.$$

Hence, by (9.1) and (9.6), we have

$$\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\tau_1 \mid D=\infty] = \infty. \qquad \square$$

REMARK 9.1. Using a reasoning similar to the previous proof but using a normal edge surrounded by edges with small conductances (see Figure 1), we may show that if $P_*[1/c_* \geq x] \geq c\ln(x)^{-\varepsilon}$ for any $\varepsilon > 0$, then

$$\mathbb{E}_0\big[\tau_1 \ln(\tau_1)^{\varepsilon} \mid D=\infty\big] = \infty$$

for any $\varepsilon > 0$. Essentially, without any assumption on the tail of c_* at 0, we cannot expect any stronger integrability of regeneration times than the first moment being finite.

REMARK 9.2. Using a reasoning similar to the previous proof and a modified version using a normal edge surrounded by edges with $4d-2$ small conductances (see Figure 1), we may show that

$$\text{if } P_*[1/c_* \geq x] \geq x^{-1/(4d-2)+\varepsilon} \text{ or } P_*[c_* \geq x] \geq x^{-2+\varepsilon} \qquad \text{for some } \varepsilon > 0,$$

then

$$\mathbb{E}_0\big[\tau_1^2 \mid D=\infty\big] = \infty,$$

which is a strong indication that the annealed central limit theorem does not hold.

The previous lemma implies:

PROPOSITION 9.1. *If* $E_*[c_*] = \infty$, *then* $\lim X_n/n = \vec{0}$ $\mathbb{P}$-*a.s.*

PROOF. By Lemma 9.1, we see that

$$\mathbb{E}^{\Pi}[\tau_1] = \infty,$$

and then (8.20) implies that

$$\lim_{n\to\infty} \frac{X_n}{n} = \vec{0}. \qquad \square$$

9.2. *Lower-bound on the polynomial order of* Δ_n. The trap described in Figure 5 is the most efficient when conductances have heavy tails. Actually, before Δ_n, the walk will typically have encountered such a trap associated with a high conductances of roughly $n^{1/\gamma}$ which, just on it's own, will be responsible for a sharp lower bound on Δ_n.

LEMMA 9.2. *If* $-\lim \frac{\ln P_*[c_*>n]}{\ln n} = \gamma < 1$, *we have*

$$\liminf \frac{\ln \Delta_n}{\ln n} \geq 1/\gamma, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Using Theorem 7.2,

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[X_{\tau_1^{(K)}} \cdot \vec{\ell}] < \infty,$$

which implies

$$\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}] < \infty. \tag{9.7}$$

Because of Theorem 7.6 and (9.7), we may Birkhoff's ergodic theorem (page 341 in [12]) to see that

$$\frac{(X_{\tau_n} - X_{\tau_1}) \cdot \vec{\ell}}{n} \to_{n\to\infty} \mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}] < \infty.$$

This implies that with probability going to 1, we have $X_{\tau_n} \cdot \vec{\ell} < cn$ for $c > \mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]$. Hence, for $c < 1/\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}]$, we have $X_{\tau_{cn}} \cdot \vec{\ell} < n$ with probability going to 1, which means that for a small $c > 0$,

$$\mathbb{P}[\tau_{cn} \leq \Delta_n] \to 1.$$

Using (9.4) and a reasoning similar to the proof of Lemma 9.1,

$$\begin{aligned}\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[\tau_1 \geq n \mid D = \infty] &\geq c\mathbf{E}\big[\big(1 - C/c_*([2e_1, 3e_1])\big)^n\big] \\ &\geq cP_*\big[c_* \geq n^{1+2\varepsilon_1}\big] \geq cn^{-(\gamma(1+\varepsilon_1))}\end{aligned}$$

for any $\varepsilon_1 > 0$.

Notice that if $\tau_{cn} = \sum_{i=1}^{cn-1}(\tau_{i+1} - \tau_i) \le n^{1/\gamma-\varepsilon}$, then $\tau_{i+1} - \tau_i \le n^{1/\gamma-\varepsilon}$ for any $i \le cn - 1$. This and the previous two equations imply that for any $\varepsilon > 0$ we have

$$\begin{aligned}\mathbb{P}\big[\Delta_n \le n^{1/\gamma-\varepsilon}\big] &\le \mathbb{P}\left[\sum_{i=1}^{cn-1}(\tau_{i+1} - \tau_i) \le n^{1/\gamma-\varepsilon}\right] + o(1)\\ &\le o(1) + \prod_{i=2}^{cn-1} \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a\big[\tau_1 < cn^{1/\gamma-\varepsilon} \mid D = \infty\big]\\ &\le o(1) + \big(1 - cn^{-(1/\gamma-\varepsilon)\gamma(1+\varepsilon_1)}\big)^{cn-2}\end{aligned}$$

by Theorem 7.4 and Lemma 7.4, for any $\varepsilon_1 > 0$.

Then taking $\varepsilon_1 > 0$ small enough such that $(\frac{1}{\gamma} - \varepsilon)\gamma(1 + \varepsilon_1) < 1$, we see that

$$\mathbb{P}\big[\Delta_n \le n^{1/\gamma-\varepsilon}\big] \to 0.$$

This being true for all $\varepsilon > 0$, we have the lemma. $\square$

9.3. *Upper-bound on the polynomial order of* Δ_n. Given a realization of an environment ω and a walk, $\widetilde{\mathrm{GOOD}}_K^i$ [resp., $\widetilde{\mathrm{BAD}}^i(x)$] denotes the set of good vertices (resp., the bad cluster of x) in the configuration ω^i where all edges of $\mathcal{L}^{X_{\tau_i}}$ [defined at (7.13)] are considered to be normal, and all other have the same state as in ω.

Let us introduce, for $i \ge 0$, $\tilde{\tau}_i^{(K)} := \{j \ge T_{\widetilde{\mathrm{GOOD}}_K^i} \circ \tau_i$ with $j \le \tau_{i+1}$ or $X_j \in \widetilde{\mathrm{BAD}}^i(X_{\tau_{i+1}})\}$. Let us explain rapidly why we introduce the definition $\tilde{\tau}_i$. These random variables will approximate $\tau_{i+1} - \tau_i$, but they also have the following properties:

(1) we can estimate the time of an excursion in a trap, but not the time to exit a trap. With the previous definition of $\tilde{\tau}_i$, we will only need to consider excursions in traps.

(2) A simple inductance can be used to see that we can upper-bound τ_n by $(T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) + \tilde{\tau}_1 + \cdot + \tilde{\tau}_n$. This is ultimately what allows us to upper-bound Δ_n.

In exchange for these advantages, we lose any independence properties between all $\tilde{\tau}_i$. This will not be a major issue, as we will see Proposition 9.2. Even though this may seem surprising at first sight, we recall that we are working with $\gamma < 1$, which means that we are working in the heavy tailed regime. Hence, the limiting behavior is determined by what happens in only one trap and thus, at least heuristically, correlations should not be important.

On the bright side, the random variables $\tilde{\tau}_i$ are measurable with respect to $\sigma\{X_{n+\tau_i} - X_{\tau_i} : n \geq 0\}$ and $\sigma\{c_*(e), e \in \mathcal{R}^{X_{\tau_i}}\}$, which is well suited for an application of Theorem 7.4. This theorem, will allow us, loosely speaking, to say that $\tilde{\tau}_i$ (for $i \geq 1$) is distributed like $\tilde{\tau}_0$ under the law $\mathbb{P}^{a_{X_{\tau_i}}}$. This is why we seek to prove the following.

LEMMA 9.3. *Assume that* $-\lim \frac{\ln P_*[c_* > n]}{\ln n} = \gamma < 1$. *For any* $\varepsilon > 0$, *there exists* K_0 *such that, for any* $K \geq K_0$,

$$\max_{a \in [1/K, K]^{\mathcal{E}}} \mathbb{P}^a\big[\tilde{\tau}_0^{(K)} > n \mid D = \infty\big] \leq C(K) n^{-(\gamma - \varepsilon)}.$$

We will follow the ideas of Lemma 8.6 to prove it.

PROOF. To simplify the notations, we will do the proof for $\mathbb{P}$. In a similar fashion, we could do it for any $\mathbb{P}^a$ for $a \in [1/K, K]^{\mathcal{E}}$. We may also notice that on the event $\{D = \infty\}$, we have that:

(1) $T_{\widetilde{\mathrm{GOOD}}_K^0} = T_{\mathrm{GOOD}_K}$,
(2) $\widetilde{\mathrm{BAD}}^i(X_{\tau_1}) \subset \mathrm{BAD}(X_{\tau_1})$,

which means that $\tilde{\tau}_0$ is smaller than the random variable $\overline{\tau}_0 := \{j \geq T_{\mathrm{GOOD}_K}$ with $j \leq \tau_1$ or $X_j \in \mathrm{BAD}(X_{\tau_1})\}$. Hence, to prove our theorem it will be enough to prove that

$$\max_{a \in [1/K, K]^{\mathcal{E}}} \mathbb{P}^a\big[\overline{\tau}_0^{(K)} > n\big] \leq C(K) n^{-(\gamma - \varepsilon)}.$$

In this proof, we will point out the K dependence of constants, since the proof requires us to be careful with this dependence.

Fix $\varepsilon > 0$. We have

$$\overline{\tau}_0 \leq \sum_{x \in \mathrm{GOOD}(\omega)} \mathbf{1}\{T_x < \tau_1\} + \sum_{x \in \partial \mathrm{BAD}(\omega)} \sum_{i=1}^{\infty} \mathbf{1}\{T_x < \tau_1\} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i.$$

Recalling the definition of χ at the beginning of the proof of Lemma 8.7, we see that

$$\begin{aligned} &E^\omega\big[\mathbf{1}\{\chi \leq n^\varepsilon\} \overline{\tau}_0\big] \\ &\quad \leq \sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} \Bigg[\mathbf{1}\{x \in \mathrm{GOOD}(\omega)\} \\ &\qquad\qquad + \mathbf{1}\{x \in \partial \mathrm{BAD}(\omega)\} E^\omega\Bigg[\sum_{i=1}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i\Bigg]\Bigg]. \end{aligned}$$

Using Markov's property and Lemma 8.1 [just as in (8.11)], we obtain

$$\begin{aligned}
&E^{\omega}\big[\mathbf{1}\{\chi \le n^{\varepsilon}\}\overline{\tau}_0\big] \\
&\quad \le C(K) \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \big[\mathbf{1}\{x \in \mathrm{GOOD}(\omega)\} \\
&\qquad\qquad + \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} E_x^{\omega}\big[T^{+}_{\mathrm{GOOD}(\omega)}\big]\big],
\end{aligned}$$

and now, since $\gamma < 1$, we have

$$\begin{aligned}
&E^{\omega}\big[\mathbf{1}\{\chi \le n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}\big] \\
&\quad \le E^{\omega}\big[\mathbf{1}\{\chi \le n^{\varepsilon}\}\overline{\tau}_0\big]^{\gamma-\varepsilon} \\
&\quad \le C(K)\Bigg[\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \big[\mathbf{1}\{x \in \mathrm{GOOD}(\omega)\} \\
&\qquad\qquad + \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} E_x^{\omega}\big[T^{+}_{\mathrm{GOOD}(\omega)}\big]\big]\Bigg]^{\gamma-\varepsilon}.
\end{aligned}$$

We may now apply Lemma 8.3,

$$\begin{aligned}
&E^{\omega}\big[\mathbf{1}\{\chi \le n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}\big] \\
&\quad \le C(K)\Bigg[\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \mathrm{GOOD}(\omega)\} \\
&\qquad\qquad + \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} E_x^{\omega}\big[T^{+}_{\mathrm{GOOD}(\omega)}\big]\Bigg]^{\gamma-\varepsilon} \\
&\quad \le C(K)\Bigg[\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \mathrm{GOOD}(\omega)\} \\
&\qquad\qquad + \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} \\
&\qquad\qquad\qquad \times \exp\big(3\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)\big|\big) \\
&\qquad\qquad\qquad \times \Bigg(1 + \sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))} c_*(e)\Bigg)\Bigg]^{\gamma-\varepsilon} \\
&\quad \le C(K) n^{C\varepsilon} \max_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\}\big|E\big(\mathrm{BAD}_x^{\mathrm{s}}(\omega)\big)\big|^{\gamma} \\
&\qquad\qquad \times \exp\big(3\gamma\lambda\big|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)\big|\big) \\
&\qquad\qquad \times \Bigg(1 + \max_{e\in E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))} c_*(e)^{\gamma-\varepsilon}\Bigg)
\end{aligned}$$

$$\leq C(K)n^{C\varepsilon} \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} \exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|\big)$$
$$\times \Big(1 + \sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))} c_*(e)^{\gamma-\varepsilon}\Big),$$

where we used that $|E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))| \leq C|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|^d$.

Now, averaging over the environment, we get

$$\mathbb{E}[\mathbf{1}\{\chi \leq n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}]$$
$$\leq C(K)n^{C\varepsilon}\mathbf{E}\Big[\mathbf{1}\{0 \in \mathrm{GOOD}(\omega)\}$$
$$\times \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x \in \partial\mathrm{BAD}(\omega)\} \exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|\big)$$
$$\times \Big(1 + \sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))} c_*(e)^{\gamma-\varepsilon}\Big)\Big],$$

so using a reasoning similar to (8.13) (based on Lemma 8.4) yields

$$\mathbb{E}[\mathbf{1}\{\chi \leq n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}]/(C(K)n^{C\varepsilon})$$
$$\leq \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{E}\Big[\exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|\big)\Big(1 + \sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))} c_*(e)^{\gamma-\varepsilon}\Big)\Big]$$
$$\leq C(K) \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{E}\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|^d \exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|\big)\big],$$

where we used that $|E(\mathrm{BAD}_x^{\mathrm{s}}(\omega))| \leq C|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|^d$.

We may see that Lemma 8.2 implies that for K large enough,

$$\mathbf{E}\big[|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|^d \exp\big(4\lambda|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|\big)\big] < C(K) < \infty,$$

which means that for any $\varepsilon > 0$,

$$\mathbb{E}[\mathbf{1}\{\chi \leq n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}] \leq C(K)n^{C\varepsilon}.$$

From this, using Chebyshev's inequality, we get

$$\mathbb{P}[\chi \leq n^{\varepsilon}, \overline{\tau}_0 > n] = \mathbb{P}[\mathbf{1}\{\chi \leq n^{\varepsilon}\}\overline{\tau}_0 > n]$$
$$\leq n^{-(\gamma-\varepsilon)}\mathbb{E}[\mathbf{1}\{\chi \leq n^{\varepsilon}\}\overline{\tau}_0^{\gamma-\varepsilon}]$$
$$\leq C(K)n^{-(\gamma-\varepsilon)}n^{C_1\varepsilon}.$$

For any $\varepsilon_1 > 0$, we may apply the previous equality for a small ε (which depends only on γ and C_1 but not on K) to obtain

$$\mathbb{P}[\chi \leq n^{\varepsilon}, \overline{\tau}_0 > n] \leq C(K)n^{-(\gamma-\varepsilon_1)}.$$

Now, using (8.19), we can choose K large enough such that

$$\mathbb{P}[\chi > n^{\varepsilon}] \le Cn^{-1},$$

and the previous two equations imply that for any $\varepsilon_1 > 0$ there exists K large enough, we obtain

$$\mathbb{P}[\overline{\tau}_0^{(K)} > n] \le \mathbb{P}[\chi > n^{\varepsilon}] + \mathbb{P}[\chi \le n^{\varepsilon}, \overline{\tau}_0^{(K)} > n] \le C(K)n^{-(\gamma-\varepsilon_1)},$$

which proves the lemma. □

The random variables $\tilde{\tau}_i$ verify

$$\Delta_n \le \tau_n \le (T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) + \sum_{i=1}^{n} \tilde{\tau}_i, \tag{9.8}$$

since, by (7.3), we necessarily have $X_{\tau_n^{(K)}} \cdot e_1 \ge n$.

This means that the previous lemma will allow us to give a sharp upper bound on Δ_n.

PROPOSITION 9.2. *If* $\lim \frac{\ln P_*[c_* > n]}{\ln n} = -\gamma$ *with* $\gamma < 1$, *then*

$$\limsup \frac{\ln \Delta_n}{\ln n} \le \frac{1}{\gamma}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Fix $M \ge 1$. We know, by Lemma 9.3 and Theorem 7.4, that there exists K large enough such that for $i \le M+1$,

$$\begin{aligned}
&\mathbb{E}[\mathrm{card}\{1 \le j \le n, \tilde{\tau}_j^{(K)} \ge n^{i/(M\gamma)}\}] \\
&\quad = \mathbb{E}\Big[\sum_{1 \le j \le n} 1\{\tilde{\tau}_j^{(K)} \ge n^{i/(M\gamma)}\}\Big] \\
&\quad = \sum_{1 \le j \le n} \mathbb{E}[\mathbb{P}^{a_{X_{\tau_j}}}[\tilde{\tau}_1^{(K)} \ge n^{i/(M\gamma)} \mid D = \infty]] \\
&\quad \le n \max_{a \in [1/K, K]^{\mathcal{E}}} \mathbb{P}^a[\tilde{\tau}_0 \ge n^{i/(M\gamma)} \mid D = \infty] \\
&\quad \le Cnn^{-(i/M)(1-1/M)} = Cn^{(1-i/M)+2/M},
\end{aligned}$$

where we used Lemma 7.4. Hence by Markov's inequality for any L,

$$\begin{aligned}
&\mathbb{P}\Big[\mathrm{card}\{1 \le j \le n, \tilde{\tau}_j^{(K)} \ge n^{i/(M\gamma)}\} \ge \frac{1}{2M} n^{1/\gamma + L/M - (i+1)/(M\gamma)}\Big] \\
&\quad \le C(M)n^{(1-1/\gamma)(1-i/M)+(1/\gamma+2-L)/M}.
\end{aligned}$$

Fix $L \geq 1/\gamma + 3$ (which does not depend on M). We denote the event

$$B(n,i,M) = \Big\{\mathrm{card}\big\{1 \leq j \leq n, \tilde{\tau}_j^{(K)} \in \big(n^{i/(M\gamma)}, n^{(i+1)/(M\gamma)}\big]\big\} \geq \frac{1}{2M} n^{1/\gamma + L/M - (i+1)/(M\gamma)}\Big\},$$

and we get that for any fixed M and $i \leq M$

$$\mathbb{P}\big[B(n,i,M)\big] \leq C n^{-1/M} = o(1), \tag{9.9}$$

since $\gamma < 1$.

In the same way, by Lemma 9.3 and Theorem 7.4, we get that

$$B(n, M+1, M) = \big\{\mathrm{card}\big\{1 \leq j \leq n, \tilde{\tau}_j^{(K)} \geq n^{(M+1)/(M\gamma)}\big\} \geq 1\big\},$$

verifies

$$\mathbb{P}\big[B(n, M+1, M)\big] \leq n^{-\varepsilon} = o(1).$$

This means that, denoting $B(n,M) = \bigcup_{j=0}^{M+1} B(n,i,M)$, we have

$$\mathbb{P}\big[B(n,M)\big] = o(1).$$

Now, on $B(n,M)^c$, we can give an upper bound for Δ_n by using (9.8)

$$\begin{aligned}\Delta_n &\leq (T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) + \sum_{i=0}^{M} n^{(i+1)/(M\gamma)} \left(\frac{1}{2M} n^{1/\gamma + L/M - (i+1)/(M\gamma)}\right) \\ &= (T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) + \frac{M+1}{2M} n^{1/\gamma + L/M}.\end{aligned}$$

It follows that $\Delta_n \leq (T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) + n^{1/\gamma + L/M}$ on $B(n,M)^c$, and hence, since $(T_{\mathrm{GOOD}} \circ \tau_1 + \tau_1) < \infty$, we have

$$\text{on } B(n,M)^c \qquad \frac{\ln \Delta_n}{\ln n} \leq 1/\gamma + L/M.$$

Hence we have proved that for any $M \geq 1$, by (9.9)

$$\limsup \frac{\ln \Delta_n}{\ln n} \leq 1/\gamma + L/M, \qquad \mathbb{P}\text{-a.s.},$$

and letting M go to infinity, we get the result (we recall that L does not depend on M). $\square$

9.4. *The polynomial order of the distance of the random walk from the origin.* By Proposition 9.2 and Lemma 9.2 we see that if $-\lim \frac{\ln P_*[c_* > n]}{\ln n} = \gamma < 1$, we have

$$\lim \frac{\ln \Delta_n}{\ln n} = 1/\gamma, \qquad \mathbb{P}\text{-a.s.},$$

which implies, using a classical inversion argument similar to the one used in the proof of Theorem 1.3 in Section 5 of [15], we see that

$$\lim \frac{\ln X_n \cdot \vec{\ell}}{\ln n} = \gamma, \qquad \mathbb{P}\text{-a.s.}$$

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