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REGULARITY AND STOCHASTIC HOMOGENIZATION OF FULLY NONLINEAR EQUATIONS WITHOUT UNIFORM ELLIPTICITY

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We prove regularity and stochastic homogenization results for certain degenerate elliptic equations in nondivergence form. The equation is required to be strictly elliptic, but the ellipticity may oscillate on the microscopic scale and is only assumed to have a finite dth moment, where d is the dimension. In the general stationary-ergodic framework, we show that the equation homogenizes to a deterministic, uniformly elliptic equation, and we obtain an explicit estimate of the effective ellipticity, which is new even in the uniformly elliptic context. Showing that such an equation behaves like a uniformly elliptic equation requires a novel reworking of the regularity theory. We prove deterministic estimates depending on averaged quantities involving the distribution of the ellipticity, which are controlled in the macroscopic limit by the ergodic theorem. We show that the moment condition is sharp by giving an explicit example of an equation whose ellipticity has a finite pth moment, for every $p < d$, but for which regularity and homogenization break down. In probabilistic terms, the homogenization results correspond to quenched invariance principles for diffusion processes in random media, including linear diffusions as well as diffusions controlled by one controller or two competing players.

1. Introduction. We prove stochastic homogenization and regularity estimates for fully nonlinear elliptic equations in nondivergence form without the assumption of uniform ellipticity. The equations we consider are strictly elliptic but may have ellipticities which are arbitrarily large and oscillating on the microscopic scale. We derive new (deterministic) regularity estimates and show that, under the assumption that the dth moment of the ellipticity is finite, such a degenerate equation homogenizes in the macroscopic limit to an effective equation which is uniformly elliptic. Our analysis yields an explicit estimate for the effective ellipticity which is new, to our knowledge, even in the linear, uniformly elliptic setting. In terms of probability, the main homogenization result, in the special case

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of linear equations, is equivalent to a quenched invariance principle for a diffusion in a random environment. For a nonlinear, positively homogeneous equation, the homogenization result gives similar information about the quenched behavior of controlled diffusions in random environments.

The simplest example of interest is the linear equation

$$-\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\varepsilon},\omega\right)u^{\varepsilon}_{x_i x_j} = f \qquad \text{in } U \subseteq \mathbb{R}^d, d \geq 1. \tag{1.1}$$

The coefficient matrix (a_{ij}), which depends on the random parameter ω, called *the environment*, is assumed to be stationary-ergodic and to satisfy the ellipticity condition

$$\lambda(\omega)|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(0,\omega)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^d, \tag{1.2}$$

where $\Lambda > 0$ is a given constant and $\lambda = \lambda(\omega)$ is a nonnegative random variable. Papanicolaou and Varadhan [27, 28] proved that, if the equation is uniformly elliptic, that is, $\lambda(\omega) \geq \lambda_0 > 0$, then in the almost sure asymptotic limit $\varepsilon \to 0$, equation (1.1) is governed by an effective equation of the form

$$-\sum_{i,j=1}^{d} \bar{a}_{ij} u_{x_i x_j} = f,$$

where the coefficient matrix $(\bar{a}_{ij})$ is uniformly elliptic with ellipticity Λ/λ_0, that is, for every $\xi \in \mathbb{R}^d$,

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{d} \bar{a}_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{1.3}$$

In this paper, we extend this result to the case that $\lambda > 0$ and $\inf \lambda = 0$ under the assumption that $\mathbb{E}[\lambda^{-d}] < \infty$. Due to the presence of small pockets of arbitrarily large ellipticity which become dense for small ε, it is by no means clear at first glance that (1.1) should behave, in the macroscopic limit, like a uniformly elliptic equation. Our result demonstrates that this finite moment condition on λ^{-1} ensures that these tiny regions of very large ellipticity may be effectively controlled and that (1.1) becomes a uniformly elliptic equation in the limit $\varepsilon \to 0$. Furthermore, we show that the moment condition is sharp by exhibiting an explicit example, for each $p < d$ in arbitrary dimension $d \geq 1$, of a nonlinear equation with a finite pth moment and having a finite range of dependence, for which homogenization fails.

Previously, in this linear setting, a result similar to our homogenization result has been proved in a recent paper of Guo and Zeitouni [21] using probabilistic methods. They give a quenched invariance principle for random walks in a random environment, which has an equivalent formulation in terms of stochastic homogenization of a discrete equation (on the lattice $\mathbb{Z}^d$ rather than $\mathbb{R}^d$). They require a slightly stronger moment condition, namely that λ^{-1} have a finite pth moment for some $p > d$. That homogenization occurs in the case $p = d$ is new here.

While our results are therefore of interest in the linear setting, we analyze much more general fully nonlinear equations of the form

$$F\left(D^2 u^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0. \tag{1.4}$$

Such nonlinear equations are more difficult to analyze than (1.1), due to the absence of invariant measures—the key tool used in [27, 28] (and [21]) to overcome the problem of the "lack of compactness." The results here are the first for such equations without a uniform ellipticity assumption. Previously, the homogenization of fully nonlinear equations in stationary-ergodic media was proved in the uniformly elliptic case by Caffarelli, Souganidis and Wang [10]. They introduced a new method based on the obstacle problem, a strategy which we also use in this paper.

The problem one encounters when trying to homogenize (1.4) outside of the uniformly elliptic regime is that most of the regularity theory needed to implement the method of [10] is destroyed by (even tiny) regions of high ellipticity. It therefore seems hopeless, at first glance, to implement the techniques of [10], since they make heavy use of the regularity tools.

To overcome this difficulty, prove new (deterministic) regularity estimates in which the dependence on a uniform upper bound for the ellipticity is replaced by that of its L^d-norm. In particular, we prove a decay of the oscillation lemma at unit scale. This result, and the new arguments we introduce to obtain it, are of independent interest. Indeed, in sharp contrast to the situation for divergence form equations, there are very few results in the literature for equations in nondivergence form, which provide estimates of solutions in terms of averaged quantities.

The estimates refine the classical regularity theory [7] and require several new ideas. One of the basic techniques involves using the area formula to estimate the size of certain "contact sets" between supersolutions and certain families of smooth test functions. This method is a generalization of the classical ABP inequality and was previously used by Cabré [6] to obtain the Harnack inequality on Riemannian manifolds with nonnegative sectional curvature, by Savin in his proof of De Giorgi's conjecture [30] and in his beautiful proof of the Harnack inequality in [29]. In each of these works, supersolutions are touched from below not only by planes, but by translations of balls and paraboloids. In the present paper, one of the key arguments involves touching from below by translations of the *singular function* $|x|^{-\alpha}$, for suitably large $\alpha > 0$.

Since the L^d norm of the ellipticity is controlled on the macroscopic scale almost surely by the ergodic theorem, the regularity results provide effective control on the solutions of (1.4) for small ε. This allows us to homogenize the equation by suitably adapting the arguments of [10]. We expect this two-step approach to homogenization, in which one obtains "effective" regularity and then uses this to homogenize the equation, to be useful in other situations.

As a byproduct of our analysis, we obtain an estimate for the effective ellipticity which is new even in the uniformly elliptic case, which states that (1.3) holds for a $\lambda_0 > 0$ which, in addition to Λ and d, depends only on $\mathbb{E}[\lambda^{-d}]$. It is, to our knowledge, the first such bound for the homogenized coefficients of nondivergence form equations which is nontrivial in the sense that it is given in terms on the averaged microscopic behavior of the equation rather than its uniform properties.

As mentioned above, the homogenization result has an equivalent probabilistic formulation, at least in the linear case, as a quenched invariance principle for the corresponding diffusion in the random environment. It also provides information regarding the recurrence or transience of the diffusion (see [21]). If F is nonlinear but positively homogeneous, the fully nonlinear equation (1.4) is a Bellman–Isaacs equation which arises in the theory of stochastic optimal control and two-player stochastic differential games, and the homogenization result yields similar information about these more general diffusion processes. Although we do not explore this point here, we remark that the recurrence verses transience of such controlled diffusion processes in an isotropic environment was characterized in [3], and this result applied to the effective operator $\overline{F}$, together with its proof, gives information about the corresponding questions for controlled diffusions in random environments.

We now give the precise statement of our results, beginning with the modeling assumptions.

The model. We work in Euclidean space $\mathbb{R}^d$ in dimension $d \geq 1$. The random environment is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an ergodic group $\tau = (\tau_y)_{y \in \mathbb{R}^d}$ of $\mathcal{F}$-measurable, $\mathbb{P}$-preserving transformations on Ω. That is, the action τ of $\mathbb{R}^d$ on Ω satisfies

$$\mathbb{P}[A] = \mathbb{P}[\tau_y A] \qquad \text{for every } y \in \mathbb{R}^d, A \in \mathcal{F} \tag{1.5}$$

and, for every $A \in \mathcal{F}$,

$$\tau_y A = A \qquad \text{for every } y \in \mathbb{R}^d \qquad \text{implies that } \mathbb{P}[A] = 0 \quad \text{or} \quad \mathbb{P}[A] = 1. \tag{1.6}$$

The nonlinear elliptic operator is a map $F : \mathbb{S}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ (here $\mathbb{S}^d$ denotes the space of d-by-d symmetric matrices) which satisfies each of the following four conditions:

(F1) *Stationarity*: for every $M \in \mathbb{S}^d$, $y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$F(M, y, \tau_z \omega) = F(M, y + z, \omega).$$

(F2) *Local uniform ellipticity*: there exists a constant $\Lambda \geq 1$ and a nonnegative random variable $\lambda : \Omega \to [0, \Lambda]$ such that $\mathbb{P}[\lambda > 0] = 1$ and, for every $M, N \in \mathbb{S}^d$, $\omega \in \Omega$ and $y \in B_1$,

$$\mathcal{P}^-_{\lambda(\omega),\Lambda}(M - N) \leq F(M, y, \omega) - F(N, y, \omega) \leq \mathcal{P}^+_{\lambda(\omega),\Lambda}(M - N).$$

(Here, $\mathcal{P}^{\pm}$ are the usual Pucci extremal operators; see the next section.)

(F3) *Uniform continuity and boundedness*: for each $R > 0$,

$$\{F(\cdot, \cdot, \omega) : \omega \in \Omega\} \qquad \text{is uniformly equicontinuous on } B_R \times \mathbb{R}^d$$

and

$$\operatorname*{ess\,sup}_{\omega \in \Omega} |F(0, 0, \omega)| < +\infty.$$

Moreover, there exists a modulus $\rho : [0, \infty) \to [0, \infty)$ and a constant $\sigma > \frac{1}{2}$ such that, for all $(M, p, \omega) \in \mathbb{S}^d \times \mathbb{R}^d \times \Omega$ and $y, z \in \mathbb{R}^d$,

$$|F(M, y, \omega) - F(M, z, \omega)| \leq \rho((1 + |M|)|y - z|^{\sigma}).$$

(F4) *Bounded moment of the ellipticity*: the random variable λ satisfies

$$\mathbb{E}[\lambda^{-d}] < +\infty.$$

The main result. We now present the homogenization result, which for simplicity we state in terms of the Dirichlet problem

$$\begin{cases} F\left(D^2 u^{\varepsilon}, \dfrac{x}{\varepsilon}, \omega\right) = 0, & \text{in } U, \\ u^{\varepsilon} = g, & \text{on } \partial U. \end{cases} \tag{1.7}$$

Here, $U \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain and $g \in C(\partial U)$, and the PDE is to be understood in the viscosity sense (cf. [7, 12]). By modifying our argument in a very minor way (only small changes in part three of Section 4), we may homogenize any other well-posed problem involving F, including parabolic equations like

$$u_t + F\left(D^2 u, \frac{x}{\varepsilon}, \omega\right) = 0$$

with appropriate boundary/initial conditions.

Note that by (F1) and (F2), for each $\varepsilon > 0$, equation (1.7) is uniformly elliptic with probability one. Indeed, if we take $\{B(x_j, 1) : 1 \leq j \leq k\}$ to be a finite covering of $\varepsilon^{-1} U$, then its ellipticity is bounded by the random variable

$$\Lambda \sup_{1 \leq j \leq k} \lambda^{-1}(\tau_{x_j/\varepsilon} \omega),$$

which is almost surely finite by (F2). See (2.1) below. As a consequence, (1.7) is well-posed and has a unique viscosity solution $u^{\varepsilon} = u^{\varepsilon}(x, \omega)$ belonging to $C(\overline{U})$.

The main homogenization result is the following theorem.

THEOREM 1. *Assume* (F1), (F2), (F3) *and* (F4). *Then there exists an event $\Omega_1 \in \mathcal{F}$ of full probability, a positive constant $0 < \lambda_0 < \Lambda$ which depends only on d, Λ and $\mathbb{E}[\lambda^{-d}]$ and a function $\overline{F} : \mathbb{S}^d \to \mathbb{R}$ which satisfies*

$$\mathcal{P}^{-}_{\lambda_0, \Lambda}(M - N) \leq \overline{F}(M) - \overline{F}(N) \leq \mathcal{P}^{+}_{\lambda_0, \Lambda}(M - N)$$

such that, for every $\omega \in \Omega_1$, every bounded Lipschitz domain $U \subseteq \mathbb{R}^d$ and each $g \in C(\partial U)$, the unique solution $u^\varepsilon = u^\varepsilon(x, \omega)$ of the boundary value problem (1.7) satisfies

$$\lim_{\varepsilon \to 0} \sup_{x \in U} \left| u^\varepsilon(x, \omega) - u(x) \right| = 0,$$

where $u \in C(\overline{U})$ is the unique solution of the Dirichlet problem

$$\begin{cases} \overline{F}(D^2 u) = 0, & \text{in } U, \\ u = g, & \text{on } \partial U. \end{cases} \tag{1.8}$$

A brief literature review. The modern regularity theory for elliptic equations in nondivergence form began in the 1980s with the groundbreaking work of Krylov and Safonov, Evans, Caffarelli and others, and we refer to [7] and the references there for more. For degenerate equations, we are unaware of much work that can be compared to ours here. An exception is the *linearized Monge–Ampère equation* which, although degenerate, possesses a special geometric structure allowing for the development of a regularity theory, as discovered by Caffarelli and Gutiérrez [8] (see also Gutiérrez and Nguyen [22] and the references therein). Recently, there has been some progress in obtaining Harnack inequalities and Hölder regularity for certain nonlinear degenerate equations (see, e.g., [16, 23, 24]). In these works, the degeneracy of the equation is typically compensated in some way by dependence on the gradient. A typical model equation considered is

$$|Du|^\gamma F\left(D^2 u\right) = 0, \tag{1.9}$$

where $\gamma > 0$ and F is uniformly elliptic. A solution u of (1.9) may only be irregular if $|Du|$ is small, and this allows to compensate for the degeneracy. This is a very different situation from the "naked" degeneracy of the equations considered here.

The homogenization of linear uniformly elliptic equations in random media originated in the work of Papanicolaou and Varadhan [27, 28] and Kozlov [25, 26]. Later, Dal Maso and Modica [14, 15] obtained stochastic homogenization results for nonlinear equations in divergence form and convex variational problems. The homogenization of uniformly elliptic, nonlinear equations in nondivergence form was first considered in the periodic setting by Evans [17] and much later by Caffarelli, Souganidis and Wang [10] in random media. In contrast to the divergence form case (cf. [11]), little seems to be known about the homogenized coefficients for nondivergence form equations, even in the periodic case, other than what is inherited from the uniform properties of the medium. As far as quantitative homogenization results, we mention the work of Yurinskiĭ [31] and Gloria and Otto [19, 20] for linear equations and Caffarelli and Souganidis [9] for fully nonlinear equations.

Outline of the paper. In the next section, we give some preliminary results and notation needed later in the paper and make some comments about our assumptions. In Section 3, we develop the deterministic regularity theory. The proof of Theorem 1 is then given in Section 4. Finally, in Section 5 we construct an explicit example to show that the moment condition (F4) is sharp for general nonlinear equations.

2. Preliminaries. In this section, we present some background results needed in the rest of the paper, including the statements of the ergodic theorems we cite, some remarks about our model and some general remarks concerning viscosity solutions and semiconcave functions. We begin by reviewing the notation.

Notation. The symbols C and c denote positive constants which may vary at each occurrence and which typically depend on known quantities. We work in Euclidean space $\mathbb{R}^d$ for $d \geq 1$. We denote the set of natural numbers by $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{Q}$ is the set of rational numbers. If $r \in \mathbb{R}$, then $\lceil r \rceil$ denotes the smallest positive natural number which is greater than or equal to r, and we write $\lfloor r \rfloor := -\lceil -r \rceil$. The family of bounded Lipschitz subsets of $\mathbb{R}^d$ is denoted by $\mathcal{L}$. The open ball centered at $y \in \mathbb{R}^d$ with radius $r > 0$ is $B_r(y) := \{x \in \mathbb{R}^d : |x - y| < r\}$ and we write $B_r := B_r(0)$. If $E \subseteq \mathbb{R}^d$ is a bounded Borel set, then $\overline{E}$ is its closure and $|E|$ is the Lebesgue measure of E. If $f \in L^1(E)$, then the average of f in E is $\fint_E f(x)\,dx := |E|^{-1} \int_E f(x)\,dx$. If $f : E \to \mathbb{R}$, then we denote $\operatorname{osc}_E f := \sup_E f - \inf_E f$. The characteristic function of a Borel set E is χ_E. We work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as described in the previous section. The indicator random variable of an event $A \in \mathcal{F}$ is written $\mathbb{1}_A$. We say that $A \in \mathcal{F}$ is of *full probability* if $\mathbb{P}[A] = 1$. The space of symmetric d-by-d matrices is $\mathbb{S}^d$. If $M, N \in \mathbb{S}^d$, we write $M \geq N$ if the eigenvalues of $M - N$ are nonnegative. If $x, y \in \mathbb{R}^d$, then $x \otimes y$ denotes the d-by-d matrix with entries $(x_i y_j)$. The trace of $M \in \mathbb{S}^d$ is $\operatorname{tr}(M)$. Recall that any $M \in \mathbb{S}^d$ can be uniquely expressed as a difference $M = M_+ - M_-$ where $M_+ M_- = 0$ and $M_+, M_- \geq 0$. In particular, if $r \in \mathbb{R}$, then we write $r_+ := \max\{0, r\}$ and $r_- := (-r)_+$. The Pucci extremal operators $\mathcal{P}^\pm$ are defined for $0 < \mu \leq \Lambda$ and $M \in \mathbb{S}^d$ by

$$\mathcal{P}^+_{\mu,\Lambda}(M) := -\mu \operatorname{tr}(M_+) + \Lambda \operatorname{tr}(M_-)$$

and

$$\mathcal{P}^-_{\mu,\Lambda}(M) := -\Lambda \operatorname{tr}(M_+) + \mu \operatorname{tr}(M_-).$$

The elementary properties of the Pucci operators can be found in [7]. Here, we remark only that they are uniformly elliptic, $\mathcal{P}^+_{\mu,\Lambda}$ is convex and $\mathcal{P}^-_{\mu,\Lambda}$ is concave. The set of upper and lower semicontinuous functions on $V \subseteq \mathbb{R}^d$ are denoted by $\operatorname{USC}(V)$ and $\operatorname{LSC}(V)$, respectively.

Brief remarks concerning the assumptions. Note that the restriction of (F2) to $y \in B_1$ is merely for convenience, it may be extended to all $y \in \mathbb{R}^d$ by stationarity. Indeed, the combination of (F1) and (F2) yields, for all $y, z \in \mathbb{R}^d$ with $|y - z| < 1$,

$$\mathcal{P}^-_{\lambda(\tau_z\omega),\Lambda}(M - N) \le F(M, y, \omega) - F(N, y, \omega) \le \mathcal{P}^+_{\lambda(\tau_z\omega),\Lambda}(M - N). \tag{2.1}$$

In light of (2.1), it is convenient to abuse notation by writing $\lambda(z, \omega) = \lambda(\tau_z\omega)$. Note also that due to (F3) we may suppose that

$$\{\lambda(\cdot, \omega) : \omega \in \Omega\} \qquad \text{is uniformly equicontinuous on } \mathbb{R}^d. \tag{2.2}$$

Otherwise, we simply redefine λ to be the largest quantity which satisfies (F2), which then satisfies (2.2) by (F3). The operators on the leftmost and rightmost side of (2.1) are the minimal and maximal operators, respectively, which satisfy conditions (F1)–(F3). In particular, since $\lambda(\cdot, \omega) > 0$, our equation is locally uniformly elliptic in the sense that, almost surely, $\inf_V \lambda(\cdot, \omega) > 0$ for each $V \in \mathcal{L}$.

Using ergodicity, we may improve the second part of (F3) to

$$\sup_{y\in\mathbb{R}^d} \operatorname*{ess\,sup}_{\omega\in\Omega} \bigl|F(0, y, \omega)\bigr| < +\infty.$$

Using then the continuity of F and intersecting the event on which the latter holds over all the rational points of $\mathbb{R}^d$, we obtain

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{y\in\mathbb{R}^d} \bigl|F(0, y, \omega)\bigr| < +\infty.$$

Applying also (F2), this yields, for $C_0 := \operatorname{ess\,sup}_{\omega\in\Omega} |F(0, y, \omega)|$ and all $M \in \mathbb{S}^d$,

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{y\in\mathbb{R}^d} \bigl|F(M, y, \omega)\bigr| \le C_0 + \Lambda \operatorname{tr}(M_+ + M_-) \le C\bigl(1 + |M|\bigr). \tag{2.3}$$

The second statement in assumption (F3) is taken in order that the comparison principle hold in each bounded domain for the operator $F(\cdot, \cdot, \omega)$ and for every $\omega \in \Omega$. This is a consequence of the local uniform ellipticity of F and standard comparison results (see [12]).

A brief remark concerning viscosity solutions. All differential inequalities in this paper are to be interpreted in the viscosity sense (cf. [7, 12]). We remark that, while it is not obvious—in fact, it is *equivalent* to the comparison principle—we have transitivity of inequalities in the viscosity sense (see [2], Lemma 3.2). For example, if $V \in \mathcal{L}$ and $u, -v \in \mathrm{USC}(V)$ satisfy

$$F\bigl(D^2u, y, \omega\bigr) \ge 0 \quad \text{and} \quad F\bigl(D^2u, y, \omega\bigr) \le 0 \qquad \text{in } V$$

then formally it follows that for $w := u - v$ we have

$$0 \le F\bigl(D^2u, y, \omega\bigr) - F\bigl(D^2u, y, \omega\bigr) \le \mathcal{P}^+_{\lambda(x,\omega),\Lambda}\bigl(D^2w\bigr). \tag{2.4}$$

We emphasize that we may also deduce $\mathcal{P}^+_{\lambda(x,\omega),\Lambda}(D^2w) \ge 0$ in the viscosity sense, and make other similar formal deductions rigorous, using [2], Lemma 3.2.

Pointwise notions of twice differentiability and $C^{1,1}$. We require the following pointwise regularity notions. We say that $u \in C(B(0,1))$ is *twice differentiable* at $x \in B_1$ if there exist $(X, p) \in \mathbb{S}^d \times \mathbb{R}^d$ such that

$$\limsup_{r \to 0} \sup_{y \in B_r(x)} r^{-2} \left| u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}(y-x) \cdot X(y-x) \right| = 0,$$

in which case we write $D^2 u(x) := X$ and $Du(x) := p$. We also say that u is $C^{1,1}$ *on a set* $E \subseteq B(0,1)$ if u is differentiable at each point of E and

$$\sup_{x \in E} \sup_{y \in B_1} \frac{|u(y) - u(x) - Du(x) \cdot (y-x)|}{|x-y|^2} < +\infty.$$

A function u is *semiconcave* if there exist $k > 0$ such that the map $x \mapsto u(x) - k|x|^2$ is concave. In this paper, we rely many times on the observation that, for any $a > 0$, a semiconcave function is $C^{1,1}$ on the set of points at which it can be touched from below by a C^2 functions with Hessian bounded by a. Moreover, by Rademacher's theorem and the Lebesgue differentiation theorem, any $C^{1,1}$ function on a set E is twice differentiable at (Lebesgue) almost every point of E.

Infimal convolution. We recall a standard tool (cf. [7, 12] for details) in the theory of viscosity solutions. We denote the infimal convolution of $u \in \mathrm{LSC}(B_1)$ by

$$u_\varepsilon(x) := \inf_{y \in B_1} \left(u(y) + \frac{2}{\varepsilon}|x-y|^2 \right). \tag{2.5}$$

The function u_ε is more regular than u and, in particular, is semiconcave. It is a good approximation to u in the sense that $u_\varepsilon \to u$ locally uniformly in B_1 as $\varepsilon \to 0$. Moreover, if $f, \lambda \in C(B_1)$, $\lambda > 0$ and

$$\mathcal{P}^+_{\lambda(x),\Lambda}(D^2 u) \geq f \qquad \text{in } B_1,$$

then there exist sequences of functions $\lambda'_\varepsilon, f'_\varepsilon \in C(B_1)$ which converge locally uniformly to λ and f, respectively, as $\varepsilon \to 0$, such that u_ε satisfies

$$\mathcal{P}^+_{\lambda'_\varepsilon(x),\Lambda}(D^2 u_\varepsilon) \geq f'_\varepsilon \qquad \text{in } B_{1-r_\varepsilon},$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0$. We refer to [7, 12] for details. For us, the principle utility of these approximations is the semiconcavity of u_ε. If u_ε can be touched from below by a smooth function φ at some point $z \in B_1$, then u_ε is $C^{1,1}$ at z, with norm depending only ε and $|D^2\varphi(z)|$. See [7], Theorem 5.1.

Statements of the ergodic theorems. We next recall the two versions of the (multiparameter) ergodic theorem used in this paper. The first is nearly a consequence of the second, but since it is simpler we give it separately. A nice proof can be found in Becker [4].

We emphasize that the assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in particular (1.5) and (1.6), are in force. Recall that $\mathcal{L}$ denotes the set of all bounded Lipschitz subsets of $\mathbb{R}^d$.

PROPOSITION 2.1 (Wiener's ergodic theorem). *Let $f \in L^1(\Omega)$. Then there exists a subset $\Omega_0 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_0$ and $V \in \mathcal{L}$,*

$$\lim_{t\to\infty} \fint_{tV} f(\tau_y\omega)\,dy = \mathbb{E}[f]. \tag{2.6}$$

In particular, the map $y \mapsto f(\tau_y\omega)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for every $\omega \in \Omega_0$.

The version of Proposition 2.1 proved in [4] actually requires V to be star-shaped with respect to the origin. As is well known, this restriction may be removed as follows. First we notice that the conclusion holds for any cube $V = Q$ with sides parallel to the coordinate axes, since any such cube either contains the origin or has the property that, for some larger cube $\widetilde{Q}$, both $\widetilde{Q} \setminus Q$ and $\widetilde{Q}$ are star-shaped with respect to the origin. Since it holds for such cubes, it holds for an arbitrary finite disjoint union of them, and hence any $V \in \mathcal{L}$ by approximation.

We next state the multiparameter subadditive ergodic theorem of Akcoglu and Krengel [1] as modified by Dal Maso and Modica [15], which requires some further notation. We denote by $\mathcal{U}_0$ the family of bounded subsets of $\mathbb{R}^d$. A function $f : \mathcal{U}_0 \to \mathbb{R}$ is *subadditive* if

$$f(A) \le \sum_{j=1}^{k} f(A_j),$$

whenever $k \in \mathbb{N}$ and $A, A_1, \dots, A_k \in \mathcal{U}_0$ are such that $\bigcup_{j=1}^{k} A_j \subseteq A$, the sets $A_1, \dots, A_k$ are pairwise disjoint and $|A \setminus \bigcup_{j=1}^{k} A_j| = 0$. Let $\mathcal{M}$ be the collection of subadditive functions $f : \mathcal{U}_0 \to \mathbb{R}$ which satisfy

$$0 \le f(A) \le |A| \qquad \text{for every } A \in \mathcal{U}_0.$$

A *subadditive process* is a function $f : \Omega \to \mathcal{M}$. It is sometimes convenient to write $f(A, \omega) = f(\omega)(A)$, in which case we have $f(A, \tau_y\omega) = f(y + A, \omega)$.

PROPOSITION 2.2 (Subadditive ergodic theorem). *Let $f : \Omega \to \mathcal{M}$ be a subadditive process. Then there exists an event $\Omega_0 \in \mathcal{F}$ of full probability and a constant $0 \le a \le 1$ such that, for every $\omega \in \Omega_0$ and $V \in \mathcal{L}$,*

$$\lim_{t\to\infty} \frac{f(tV, \omega)}{|tV|} = a. \tag{2.7}$$

This version of the subadditive ergodic theorem is [1], Proposition 1, in the special case that $\mathcal{L}$ is replaced by the family of all cubes, and we recover the general case by an easy approximation argument.

3. Regularity in the macroscopic limit. The classical regularity theory for uniformly elliptic equations (as developed, e.g., in [7]) does not directly help us to homogenize (1.4) because, as ε becomes small, the ergodic theorem guarantees that the set where (1.4) has very high ellipticity becomes dense. What we need are estimates which do not degenerate as $\varepsilon \to 0$, and for this it is necessary to revisit the regularity from the beginning.

What the ergodic theorem ensures is that, almost surely in ω, for every $\mu > 0$,

$$\lim_{\varepsilon\to 0} \fint_{V\cap\{\lambda<\mu\}} \lambda^{-d}\left(\frac{x}{\varepsilon}, \omega\right) dx = \mathbb{E}[\lambda^{-d}\mathbb{1}_{\{\lambda<\mu\}}]. \tag{3.1}$$

In this section we develop a deterministic regularity theory for solutions of (1.4) which will be robust in the almost sure macroscopic limit $\varepsilon \to 0$ by virtue of (3.1).

Since the random environment plays no role here, we drop dependence on ω. Throughout this section, we consider a continuous function $\lambda : \mathbb{R}^d \to (0, \Lambda]$, and we study the regularity of subsolutions and/or supersolutions of the extremal operators $\mathcal{P}^{\pm}_{\lambda(x),\Lambda}$ in bounded Lipschitz domains $V \in \mathcal{L}$. Our estimates must depend only on d, Λ and, for $\mu > 0$, the quantities

$$\int_{V\cap\{\lambda<\mu\}} \lambda^{-d}(x)\, dx.$$

As it is purely deterministic, the regularity developed here is of independent interest.

The primary goal is to obtain an improvement of oscillation result on unit scales, giving us a modulus of continuity [and a Hölder estimate in the macroscopic limit for solutions of (1.4)]. Our arguments are loosely based on the arguments in the classical regularity theory [7], with some nice modifications due to Savin [29], but require several new ideas to overcome the degeneracy of the equation. For instance, "two important tools" are introduced in [7], Section 4.1, which are used repeatedly in what has become the standard proof of the Harnack inequality. In our situation, neither of these tools can be applied in a straightforward way.

First, showing that an appropriate barrier (or "bump") function exists—which is easy in the uniformly elliptic situation (see [7], Lemma 4.1)—is a very nontrivial matter. We construct a barrier by touching a candidate function from below by translations of the singular function $|x|^{-\alpha}$ with $\alpha \gg 1$ and then adapting the proof of the ABP inequality to show that the corresponding contact set would be too large if the function failed to be a barrier. Second, the measure-theoretic argument involving the Calderón–Zygmund cube decomposition must be altered due to the presence of "bad" cubes of high ellipticity, and we use an alternative idea based on the Besicovitch covering theorem.

The development is essentially self-contained and depends also on some novel uses of the area formula for Lipschitz functions, similar to the proof of the ABP inequality, and partially inspired by Savin [29, 30]. The main result of this section is the following proposition.

PROPOSITION 3.1 (Decay of oscillation). *There exists* $\delta > 0$, *depending only on* d *and* Λ, *such that if* $0 < \mu < \frac{1}{2}$ *and*

$$\int_{B_1 \cap \{\lambda < \mu\}} \lambda^{-d}(x)\, dx < \delta,$$

then there exist constants $0 < \tau < 1$ *depending only on* d, Λ *and* μ, *such that for all* $\alpha > 0$ *and* $u \in C(B_1)$ *satisfying*

$$\mathcal{P}^+_{\lambda(x),\Lambda}(D^2 u) \ge -\alpha \quad \text{and} \quad \mathcal{P}^-_{\lambda(x),\Lambda}(D^2 u) \le \alpha \qquad \text{in } B_1,$$

we have

$$\operatorname*{osc}_{B_{1/8}} u \le \tau \operatorname*{osc}_{B_1} u + \alpha.$$

Proceeding with the proof of Proposition 3.1, we begin with three applications of area formula for Lipschitz functions (cf. [18]), which asserts that

$$|f(E)| = \int_E |\det Df(x)|\, dx$$

for all Lebesgue measurable sets $E \subseteq \mathbb{R}^d$ and injective Lipschitz maps $f : E \to \mathbb{R}^d$.

The first is the Alexandroff–Bakelman–Pucci (ABP) inequality. The version we give here is not new: it is actually a corollary to the proof of [7], Theorem 3.2. We include a proof below both for completeness and in order to introduce the style of argument we use below, in a more complicated form, to obtain the barrier function. The argument here is much simpler than the one in [7], which is due to the observation that a semiconcave function is necessarily $C^{1,1}$ on the set where it can be touched from below by a plane.

PROPOSITION 3.2 (ABP inequality). *Let* $f \in C(B_1)$ *and suppose that* $u \in \mathrm{LSC}(\overline{B}_1)$ *satisfies*

$$\begin{cases} \mathcal{P}^+_{\lambda(x),\Lambda}(D^2 u) \ge -f, & \text{in } B_1, \\ u \ge 0, & \text{on } \partial B_1. \end{cases}$$

Then

$$u_-(0) \le \left(\frac{1}{|B_1|} \int_{\{\Gamma_u = u\}} \lambda^{-d}(x) f_+^d(x)\, dx \right)^{1/d},$$

where

$$\Gamma_u(x) := \sup_{p \in \mathbb{R}^d} \inf_{y \in B_1} \big(p \cdot (x - y) - u_-(y) \big)$$

is convex envelope of $-u_- := \min\{0, u\}$.

PROOF. By approximating u by its infimal convolution, we may assume that u is a semiconcave [it is straightforward to see that the limsup of the contact sets for u_ε in (2.5) are contained in the contact set for u].

Let $a := -u(0)$ and assume $a > 0$. Since $u \geq 0$ on ∂B_1, for every $p \in B_a$, there exists $\bar{z}(p) \in B_1$ such that $u(\bar{z}(p)) < 0$ and the map $x \mapsto -u_-(x) - p \cdot x$ attains its infimum over B_1 at $\bar{z}(p)$. Note that we can arrange for $\bar{z} : B_a \to B_1$ to be Lebesgue measurable by choosing $\bar{z}$, say, lexicographically among the minimizers closest to the origin. Since u is semiconcave and can be touched from below by a plane on $A := \bar{z}(B_a)$, it is $C^{1,1}$ on A. In particular, $\bar{z}$ has a Lipschitz inverse $\bar{p} : A \to B_1$ given by $\bar{p}(z) := Du(z)$.

By Rademacher's theorem (cf. [18]) and the Lebesgue differentiation theorem, u is twice differentiable at Lebesgue almost every point of A. At every such $z \in A$, we have that $D^2u(z) \geq 0$, since u can be touched from below by a plane at z, and so the supersolution inequality gives

$$-f(z) \leq \mathcal{P}^+_{\lambda(z),\Lambda}\big(D^2u(z)\big) = -\lambda(z)\operatorname{tr}\big(D^2u(z)\big).$$

Thus, at almost every point $z \in A$,

$$0 \leq D^2u(z) \leq \lambda^{-1}(z) f_+(z) I. \tag{3.2}$$

The area formula yields

$$a^d|B_1| = |B_a| = \int_A \big|\det D\bar{p}(x)\big|\,dx = \int_A \big|\det D^2u(x)\big|\,dx \leq \int_A \lambda^{-d}(x) f_+^d(x)\,dx.$$

Since $A \subseteq \{\Gamma_u = u\}$, we obtain the proposition. □

Using a more sophisticated version of the above argument, we next construct the barrier function, which below plays a critical role in the proof of Lemma 3.6 below, similar to that of the "bump" function in [7]. It is also needed in the next section in proof of Theorem 1 to verify that the limit function satisfies the Dirichlet boundary condition.

LEMMA 3.3. *For each* $0 < r < \frac{1}{2}$, *there exists* $\delta > 0$, *depending only on* d *and* Λ, *such that if* $0 < \mu < \frac{1}{2}$ *and*

$$\int_{B_1 \cap \{\lambda < \mu\}} \lambda^{-d}(x)\,dx < \delta r^d,$$

then there exists a constant $\beta > 0$, *depending only on* d, Λ, r *and* μ, *such that for each* $u \in \operatorname{LSC}(\overline{B}_1 \setminus B_r)$ *satisfying*

$$\begin{cases} \mathcal{P}^+_{\lambda(x),\Lambda}(D^2u) \geq -1, & \text{in } B_1 \setminus \overline{B}_r, \\ u \geq 0, & \text{on } \partial B_1, \\ u \geq \beta, & \text{on } \partial B_r, \end{cases}$$

we have $u > 0$ *on* $B_{1-r} \setminus B_r$.

PROOF. By approximating u by its infimal convolution, we may assume that u is semiconcave. Define

$$\alpha := \frac{2(d-1)\Lambda + 2}{\mu} \quad \text{and} \quad \beta := \left(\frac{4}{r}\right)^{\alpha}.$$

The idea is to show that if u is negative somewhere in $B_{1-r} \setminus B_r$, then it can be touched from below by (too many) small translations of the singular function $\phi(x) := 2^{\alpha}|x|^{-\alpha}$. Let us suppose that $u(x_0) < 0$ for some $x_0 \in B_{1-r} \setminus B_r$. As a consequence we find that, for every $y \in B_{r/2}$, the map $x \mapsto u(x) - \phi(x-y)$ attains its infimum at some point $\bar{z}(y) \in B_1 \setminus B_r$. To verify this we check that, for $y \in B_{r/2}$,

$$\begin{aligned} \inf_{x \in \partial B_1} \big(u(x) - \phi(x-y)\big) &\geq -2^{\alpha}\left|1 - \frac{r}{2}\right|^{-\alpha} \\ &\geq -\phi(x_0 - y) > u(x_0) - \phi(x_0 - y) \end{aligned} \tag{3.3}$$

and, by our choice of β,

$$u(x) \geq \beta \geq \phi(x-y) \qquad \text{for every } x \in \partial B_r. \tag{3.4}$$

It is easy to arrange for the function $\bar{z} : B_{r/2} \to B_1 \setminus B_r$ to be Lebesgue measurable. To obtain the contradiction, we eventually apply the area formula to the inverse of $\bar{z}$. Most of the rest of the argument is concerned with showing that the image of $\bar{z}$ is contained in the region where λ is small, that $\bar{z}$ has an inverse $\bar{y}$ and estimating the determinant of the Jacobian of $\bar{y}$.

The Hessian of ϕ is given by

$$D^2\phi(x) = \alpha 2^{\alpha}|x|^{-\alpha-2}\left((\alpha+1)\frac{x \otimes x}{|x|^2} - \left(I - \frac{x \otimes x}{|x|^2}\right)\right) \tag{3.5}$$

and thus

$$\text{the eigenvalues of } D^2\phi(x) = \alpha 2^{\alpha}|x|^{-\alpha-2} \cdot \begin{cases} (\alpha+1), & \text{with multiplicity } 1, \\ -1, & \text{with multiplicity } d-1. \end{cases}$$

The differential inequality for u at $z = \bar{z}(y)$ yields

$$\begin{aligned} -1 \leq \mathcal{P}^{+}_{\lambda(z),\Lambda}\big(D^2\phi(z-y)\big) &= \alpha 2^{\alpha}|z-y|^{-\alpha-2}\big((d-1)\Lambda - (\alpha+1)\lambda(z)\big) \\ &\leq \alpha(\alpha+1)2^{\alpha}|z-y|^{-(\alpha+2)}\left(\frac{\mu}{2} - \lambda(z)\right). \end{aligned}$$

Using that $2^{\alpha}|z-y|^{-(\alpha+2)} \geq \frac{1}{4}$ and $\alpha(\alpha+1) \geq 8/\mu$ and rearranging this, we get

$$\lambda(z) < \mu. \tag{3.6}$$

We conclude that

$$A := \bar{z}(B_{r/2}) \subseteq \{x \in B_1 : \lambda(x) < \mu\}. \tag{3.7}$$

Since u is semiconcave and $|D^2\varphi|$ is bounded in $\mathbb{R}^d \setminus B_{r/2}$, we see that u is $C^{1,1}$ on A. In particular, u is differentiable at each point of A and, by Rademacher's theorem and the Lebesgue differentiation theorem, twice differentiable at Lebesgue almost every point of A. For each $y \in B_{r/2}$,

$$Du(\bar{z}(y)) = D\phi(\bar{z}(y) - y) = -\alpha 2^\alpha |\bar{z}(y) - y|^{-(\alpha+2)}(\bar{z}(y) - y).$$

Hence,

$$|Du(\bar{z}(y))| = \alpha 2^\alpha |\bar{z}(y) - y|^{-(\alpha+1)} \tag{3.8}$$

and substituting this into the previous line yields

$$Du(\bar{z}(y)) = -(\alpha 2^\alpha)^{-1/(\alpha+1)} |Du(\bar{z}(y))|^{(\alpha+2)/(\alpha+1)}(\bar{z}(y) - y).$$

Solving this for y, we find that $\bar{z}$ has Lipschitz inverse $\bar{y} : A \to B_{r/2}$ given by

$$\bar{y}(z) := z + (\alpha 2^\alpha)^{1/(\alpha+1)} |Du(z)|^{-(\alpha+2)/(\alpha+1)} Du(z).$$

Since $Du(\bar{z}(y)) = D\phi(\bar{z}(y) - y) \neq 0$ at each $y \in B_{r/2}$ on A, it is clear that $Du \neq 0$ on A and thus $\bar{y}$ is differentiable at each $z \in A$ at which u is twice differentiable; at such $z \in A$, we compute

$$\begin{aligned} D\bar{y}(z) = I &+ (\alpha 2^\alpha)^{1/(\alpha+1)} |Du(z)|^{-(\alpha+2)/(\alpha+1)} \\ &\times \left(D^2u(z)\left(I - \frac{\alpha+2}{\alpha+1} \frac{Du(z)}{|Du(z)|} \otimes \frac{Du(z)}{|Du(z)|} \right)\right). \end{aligned}$$

Using (3.8) we conclude that, at almost every $z \in A$,

$$|D\bar{y}(z)| \le C\left(1 + (\alpha 2^\alpha)^{-1} |\bar{y}(z) - z|^{\alpha+2} |D^2u(z)|\right), \tag{3.9}$$

where $C > 0$ is a constant depending only on d.

It remains to estimate $|D^2u|$ on A. Using (3.5) and that ϕ touches u from below on A, we have, at each $z \in A$ at which u is twice differentiable,

$$D^2u(z) \ge D^2\phi(z - \bar{y}(z)) \ge -\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} I. \tag{3.10}$$

On the other hand, the differential inequality gives

$$-1 \le \mathcal{P}^+_{\lambda(z),\Lambda}(D^2u(z)) = \Lambda \operatorname{tr}(D^2u(z))_- - \lambda(z) \operatorname{tr}(D^2u(z))_+.$$

A rearrangement of the later yields, in light of (3.10),

$$\operatorname{tr}(D^2u(z))_+ \le (d\Lambda\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} + 1)\lambda^{-1}(z)$$

and from this we deduce that

$$D^2u(z) \le C\lambda(z)^{-1}(\alpha 2^\alpha |z - \bar{y}(z)|^{-(\alpha+2)} + 1)I, \tag{3.11}$$

where here and in the rest of the proof $C > 0$ depends only on d and Λ. Combining (3.10) and (3.11), we obtain, at each point $z \in A$ at which u is twice differentiable,

$$|D^2 u(z)| \le C\lambda^{-1}(z)\big(\alpha 2^{\alpha}|z - \bar{y}(z)|^{-(\alpha+2)} + 1\big). \tag{3.12}$$

Inserting this into (3.9) and using that $\alpha \ge 1$, $|z - \bar{y}(z)| \le 2$ and $\lambda^{-1}(z) \ge \mu^{-1} \ge 2$, we at last deduce

$$|D\bar{y}(z)| \le C\big(1 + (\alpha 2^{\alpha})^{-1}|\bar{y}(z) - z|^{\alpha+2}\big)\lambda^{-1}(z) + C \le C\lambda^{-1}(z) \tag{3.13}$$

at Lebesgue almost every point $z \in A$.

We finally apply the area formula, using (3.7), (3.13) and the hypothesis of the lemma to conclude that

$$\begin{aligned}(2^{-d}|B_1|)r^d = |B_{r/2}| &= \int_A |\det D\bar{y}(x)|\,dx \\ &\le C\int_A \lambda^{-d}(x)\,dx \le C\int_{B_1\cap\{\lambda<\mu\}} \lambda^{-d}(x)\,dx \le C\delta r^d.\end{aligned}$$

We get a contradiction if $\delta > 0$ is sufficiently small, depending on d and Λ. □

REMARK 3.4. By an easy modification of the above proof, we also obtain an additional estimate for the barrier. Given $h > 0$, we can modify the choice of $\delta, \beta > 0$ and also select an $r' \in (r, 1)$ depending only on d, Λ, ε, μ, r and h (but not β) such that the supersolution u satisfies $u \ge \beta - h$ on $B_{r'} \setminus B_r$. Later we use this observation to verify the Dirichlet boundary condition for the limit function in the proof of homogenization.

The next lemma, which is inspired by [29], Lemma 2.1, follows from another application of the area formula and a (much easier) variation of the above argument. It asserts that, if a supersolution can be touched from below by sufficiently many translations of a fixed parabola, then the L^d norm of λ^{-1} on the set of points at which the touching occurs cannot be too small.

LEMMA 3.5. *Let $u \in \mathrm{LSC}(B_1)$ satisfy*

$$\mathcal{P}^+_{\lambda(x),\Lambda}(D^2 u) \ge -1 \qquad \text{in } B_1.$$

Suppose that $a \ge 1$ and $V \subseteq \mathbb{R}^d$ such that, for each $y \in V$, the infimum over B_1 of the map $z \mapsto u(z) + \frac{a}{2}|z - y|^2$ is attained. Let $W \subseteq B_1$ denote the union over $y \in V$ of the subset of B_1 at which this map attains its minimum. Then there exists a constant $\delta > 0$, depending only on d and Λ, such that

$$\int_W \lambda^{-d}(x)\,dx \ge \delta|V|.$$

PROOF. By replacing u by $u+\alpha|x|^2$ and letting $\alpha \to 0$, we may suppose that, for some small $\eta > 0$, $W \subseteq B_{1-\eta}$ and, for every $y \in B_1$,

$$\min_{z\in\partial B_{1-\eta}} \left(u(z) + \frac{a}{2}|z-y|^2\right) > \inf_{z\in W}\left(u(z) + \frac{a}{2}|z-y|^2\right). \tag{3.14}$$

As in the proofs of Proposition 3.2 and Lemma 3.3, we may assume that u is semiconcave by infimal convolution approximation. Indeed, due to (3.14), the set V is essentially unchanged by the infimal convolution approximation, while the set W is unchanged or possibly smaller.

Select a Lebesgue-measurable function $\bar{z}: V \to B_1$ such that the map $z \mapsto u(z) + \frac{a}{2}|z-y|^2$ attains its infimum in B_1 at $z = \bar{z}(y)$. The function u is $C^{1,1}$ on $A := \bar{z}(V)$ and $\bar{z}$ has a Lipschitz inverse $\bar{y}$ given by

$$\bar{y}(z) := z + \frac{1}{a} Du(z).$$

By Rademacher's theorem and the Lebesgue differentiation theorem, u is twice differentiable at almost every point of $z \in A$ and, at such z, we have $D^2u(z) \geq -aI$,

$$D\bar{y}(z) = I + \frac{1}{a}D^2u(z) \geq 0 \tag{3.15}$$

as well as

$$\begin{aligned} -\lambda(z)\,\mathrm{tr}\big(D\bar{y}(z)\big) &= \mathcal{P}^+_{\lambda(z),\Lambda}\big(D\bar{y}(z)\big) = \mathcal{P}^+_{\lambda(z),\Lambda}\left(I + \frac{1}{a}D^2u(z)\right) \\ &\geq \frac{1}{a}\mathcal{P}^+_{\lambda(z),\Lambda}\big(D^2u(z)\big) + \mathcal{P}^-_{\lambda(z),\Lambda}(I) \geq -\frac{1}{a} - \Lambda d \end{aligned}$$

and, therefore,

$$0 \leq D\bar{y}(z) \leq \frac{1}{\lambda(z)}\left(\frac{1}{a} + \Lambda d\right). \tag{3.16}$$

An application of the area formula for Lipschitz functions gives

$$|V| = \int_A \big|\det D\bar{y}(x)\big|\,dx \leq \left(\frac{1}{a} + \Lambda d\right)^d \int_A \lambda^{-d}(x)\,dx$$

from which we obtain the lemma, using that $a \geq 1$ and $A \subseteq W$. $\square$

We now give the proof of the decay of oscillation.

PROOF OF PROPOSITION 3.1. Now suppose that $0 < \mu < \frac{1}{2}$ and

$$\int_{B_1\cap\{\lambda<\mu\}} \lambda^{-d}(x)\,dx < \delta := \frac{1}{6N_d}|B_{1/6}|\min\{8^{-d}\delta_1, 32^{-d}\delta_2\}, \tag{3.17}$$

where δ_1 is from Lemma 3.3, δ_2 is from Lemma 3.5 and N_d is the constant from the Besicovitch covering theorem in dimension d.

We first make a reduction by noticing that, to obtain the proposition for $\tau := (1-1/k)$, it suffices to consider $v \in C(B_1)$ which satisfies

$$\mathcal{P}^+_{\lambda(x),\Lambda}(D^2 v) \ge -1 \quad \text{and} \quad \mathcal{P}^-_{\lambda(x),\Lambda}(D^2 v) \le 1 \qquad \text{in } B_1 \tag{3.18}$$

and

$$\operatorname*{osc}_{B_1} v \le 1 + \operatorname*{osc}_{B_{1/8}} v \tag{3.19}$$

and to show that $\operatorname{osc}_{B_1} v < k$. Indeed, suppose u satisfies the hypotheses of the proposition and

$$\operatorname*{osc}_{B_{1/8}} u > (1-k^{-1}) \operatorname*{osc}_{B_1} u + \alpha.$$

Then $\alpha < k^{-1} \operatorname{osc}_{B_1} u$, and so if we set $v := ku/\operatorname{osc}_{B_1} u$, then we see that v satisfies (3.18), (3.19) and $\operatorname{osc}_{B_1} v = k$.

Define, for every $\kappa > 0$,

$$A_\kappa := \left\{ x \in B_1 : \exists y \in B_1, v(x) + \frac{\kappa}{2}|x-y|^2 = \inf_{z \in B_1} \left(v(z) + \frac{\kappa}{2}|z-y|^2 \right) \right\}.$$

In other words, A_κ is the set of points in B_1 at which v can be touched from below by a paraboloid with Hessian $-\kappa I$ and vertex in B_1. To prove the desired estimate on v, it is enough to show that

$$|A_\kappa \cap B_{1/6}| \ge \tfrac{2}{3}|B_{1/6}| \tag{3.20}$$

for some $\kappa > 0$ depending only on d, Λ and μ. Indeed, if we could show this, then using that $-v$ satisfies the same hypotheses as v and applying (3.20) to both functions, we find a point $x \in B_{1/6}$ which can be touched from above and below by parabolas with opening κ. That is, we could conclude that there exist $x \in B_{1/6}$ and $y_1, y_2 \in B_1$ such that, for all $z \in B_1$,

$$v(x) + \frac{\kappa}{2}|x-y_1|^2 - \frac{\kappa}{2}|z-y_1|^2 \le v(z) \le v(x) - \frac{\kappa}{2}|x-y_2|^2 + \frac{\kappa}{2}|z-y_2|^2.$$

This implies that $|v(x) - v(z)| \le 2\kappa$ for all $z \in B_1$, and thus $\operatorname{osc}_{B_1} v \le 4\kappa$, which is the desired estimate.

In order to obtain (3.20) for some $\kappa > 0$ depending on the appropriate quantities, we observe first that (3.19) implies that $A_{576} \cap B_{1/6} \neq \emptyset$. Indeed, $(\frac{1}{6} - \frac{1}{8})^2 = 1/576$ and so v can be touched from below in $B_{1/6}$ by the parabola $-576|x-y|^2$, where $y \in \overline{B}_{1/8}$ is such that $v(y) = \min_{\overline{B}_{1/8}} v$. We then repeatedly apply Lemma 3.6 below to obtain the desired result for $\kappa = 576 \cdot \theta^n$, where $n := \lceil |B_1|/\eta \rceil$ and $\theta, \eta > 0$ are given in the statement of the lemma. The proof of Proposition 3.1 is now complete, pending the verification of Lemma 3.6. □

The following lemma contains the measure theoretic information needed to conclude the proof of Proposition 3.1. In the classical regularity theory, this step traditionally relies on the Calderón–Zygmund cube decomposition (as in the proof of (4.12) in [7]). Since we did not immediately see how to adapt it, and for the sake of variety, we instead use an alternative tool: the Besicovitch covering theorem.[3] The argument also relies in a crucial way on Lemmas 3.3 and 3.5.

LEMMA 3.6. *Let* μ, v *and* A_κ *be as in the proof of Proposition* 3.1. *There exist constant* $\theta > 1$ *and* $\eta > 0$, *depending only on* d, Λ *and* μ, *such that if* $\kappa \geq 1$, $A_\kappa \cap B_{1/6} \neq \varnothing$ *and* $|A_\kappa \cap B_{1/6}| < \frac{2}{3}|B_{1/6}|$, *then* $|A_{\theta\kappa} \cap B_1| \geq |A_\kappa \cap B_1| + \eta$.

PROOF. Consider the collection $\mathcal{B}$ of balls $B_r(x) \subseteq B_1$ such that $B_{r/2}(x) \subseteq B_1 \setminus A_\kappa$ and $\partial B_{r/2}(x) \cap A_\kappa \neq \varnothing$. Note that since $A_\kappa \cap B_{1/6} \neq \varnothing$ and A_κ is closed, every point of $B_{1/6} \setminus A_\kappa$ is the center of some ball in $\mathcal{B}$. According to the Besicovitch covering theorem, we may select a countable subcollection $\{B_{r_k}(x_k)\}_{k\in\mathbb{N}} \subseteq \mathcal{B}$ that covers $B_{1/6} \setminus A_\kappa$ and such that each point $x \in B_1$ belongs to at most N_d balls.

We say that the ball $B_{r_k}(x_k)$ is *good* if

$$\frac{1}{|B_{r_k}(x)|}\int_{B_{r_k}(x_k)\cap\{\lambda<\mu\}} \lambda^{-d}(x)\,dx < \min\{8^{-d}\delta_1, 32^{-d}\delta_2\}$$

and set $G := \{k \in \mathbb{N} : B_{r_k}(x_k) \text{ is good}\}$, where $\delta_1, \delta_2 > 0$ are as in the proof of Proposition 3.1. We claim that at least half of the Lebesgue measure of $B_{1/6} \setminus A_\kappa$ consists of points which belong to good balls, that is,

$$\left|\bigcup_{k\in G} B_{r_k}(x_k)\right| > \frac{1}{2}|B_{1/6} \setminus A_\kappa| \geq \frac{1}{6}|B_{1/6}|. \tag{3.21}$$

Indeed, if (3.21) were false, then $\sum_{k\notin G} |B_{r_k}| \geq \frac{1}{2}|B_{1/6} \setminus A_\kappa| \geq \frac{1}{6}|B_{1/6}|$ and so

$$\begin{aligned}\int_{B_1\cap\{\lambda<\mu\}} \lambda^{-d}(x)\,dx &\geq \frac{1}{N_d}\sum_{k\notin G}\int_{B_{r_k}(x_k)\cap\{\lambda<\mu\}} \lambda^{-d}(x)\,dx \\ &\geq \frac{1}{N_d}\min\{8^{-d}\delta_1, 32^{-d}\delta_2\}\sum_{k\notin G}|B_{r_k}| \\ &\geq \frac{1}{6N_d}\min|B_{1/6}|\{8^{-d}\delta_1, 32^{-d}\delta_2\},\end{aligned}$$

which contradicts (3.17). Therefore, in light of the Besicovitch covering, it is enough to show that $|B_{r_k/2}(x_k) \cap A_{\theta\kappa}| \geq \eta|B_{r_k}(x_k)|$ for each good ball $B_{r_k}(x_k)$ and some constants $\theta, \eta > 0$ depending only on d, Λ and μ.

[3] Luis Silvestre has since pointed out to us that the Calderón–Zygmund decomposition argument in [7] may indeed be suitably modified to prove Lemma 3.6 and that the best choice is the Vitali covering theorem, which can be used in a similar yet simpler way than the Besicovitch covering theorem.

Fix a good ball $B_r(x) := B_{r_k}(x_k)$ and choose $z_1 \in \partial B_{r/2}(x) \cap A_\kappa$. By the definition of A_κ, we can touch z_1 by a paraboloid of Hessian $-\kappa I$: there exists $y_1 \in B_1$ such that

$$v(z_1) + \frac{\kappa}{2}|z_1 - y_1|^2 = \inf_{z \in B_1}\left(v(z) + \frac{\kappa}{2}|z - y_1|^2\right). \tag{3.22}$$

We argue that, by making this paraboloid steeper and wiggling the vertex, we may touch the function v at a positive proportion of points inside of $B_r(x)$. A key role is played by Lemma 3.3, which keeps the touching points near the center and away from the boundary of $B_r(x)$ as well as by Lemma 3.5, which ensures that we can touch a positive proportion of points by wiggling the vertex of the paraboloid.

Using that $B_r(x)$ is good and applying (a properly scaled) Lemma 3.3, there exists $\beta > 1$, depending only on d, Λ and μ, such that the solution w of the Dirichlet problem

$$\begin{cases} \mathcal{P}^+_{\lambda(x),\Lambda}(D^2 w) = -1, & \text{in } B_r(x) \setminus \overline{B}_{r/8}(x), \\ w = 0, & \text{on } \partial B_r(x), \\ w = \beta r^2, & \text{on } \partial B_{r/8}(x), \end{cases}$$

satisfies $w > 0$ in $\overline{B}_{r/2}(x) \setminus B_{r/8}(x)$. Clearly, $w \le \beta r^2$ in $B_r \setminus B_{r/8}(x)$ by the maximum principle. Observe that the function

$$\varphi(z) := (d\Lambda\kappa + 2)w - \frac{\kappa}{2}|z - y_1|^2,$$

satisfies

$$\mathcal{P}^+_{\lambda(x),\Lambda}(D^2 \varphi) \le -2 \qquad \text{in } B_r(x) \setminus B_{r/8}(x). \tag{3.23}$$

The comparison principle implies that the map $z \mapsto v(z) - \varphi(z)$ attains its infimum in $B_r(x) \setminus B_{r/8}(x)$ at some point $z = z_2 \in \partial B_r(x) \cup \partial B_{r/8}(x)$. Notice, however, that it is impossible that $z_2 \in \partial B_r(x)$, since (3.22), $w \equiv 0$ on $\partial B_r(x)$ and $w(z_1) > 0$ imply that

$$\begin{aligned} v(z_1) &- \varphi(z_1) \\ &= v(z_1) + \frac{\kappa}{2}|z_1 - y_1|^2 - (d\Lambda\kappa + 2)w(z_1) < \inf_{z \in B_1}\left(v(z) + \frac{\kappa}{2}|z - y_1|^2\right) \\ &\le \inf_{z \in \partial B_r(x)}\left(v(z) + \frac{\kappa}{2}|z - y_1|^2\right) = \inf_{z \in \partial B_r(x)}\big(v(z) - \varphi(z)\big). \end{aligned}$$

Hence, $z_2 \in \partial B_{r/8}(x)$ and so, in particular, $\varphi(z_2) = -\frac{\kappa}{2}|z_2 - y_1|^2 + (d\Lambda\kappa + 2)\beta r^2$. Using that $w > 0$ in $B_{r/2}(x) \setminus B_{r/8}(x)$, we obtain that

$$\begin{aligned} &\inf_{z \in B_{r/2}(x) \setminus B_{r/8}(x)}\left(v(z) + \frac{\kappa}{2}|z - y_1|^2\right) \\ &\qquad \ge \inf_{z \in B_{r/2}(x) \setminus B_{r/8}(x)}\big(v(z) - \varphi(z)\big) \\ &\qquad = v(z_2) - \varphi(z_2) = v(z_2) + \frac{\kappa}{2}|z_2 - y_1|^2 - (d\Lambda + 2/\kappa)\kappa\beta r^2. \end{aligned}$$

Using this together with (3.22), $z_1 \in \partial B_{r/2}(x)$ and $\kappa \geq 1$, we obtain

$$\inf_{z \in B_1} \left(v(z) + \frac{\kappa}{2}|z - y_1|^2 \right) \geq v(z_2) + \frac{\kappa}{2}|z_2 - y_1|^2 - (d\Lambda + 2)\kappa\beta r^2. \tag{3.24}$$

It follows that, if we set $\gamma := 16\beta(d\Lambda + 2) + 1$, then for every $y_2 \in B_{r/8}(x)$, the function

$$\psi(z) := v(z) + \frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2$$

satisfies $\psi(z_2) < \min_{B_1 \setminus B_{r/2}(x)} \psi$ and, therefore, must attain its infimum over B_1 somewhere in $B_{r/2}(x)$.

Consider the function $\bar{z} : B_{r/8}(x) \to B_1$ given by $\bar{z}(y) = (y_1 + \gamma y)/(1 + \gamma)$ and observe by completing the square that, for some $a \in \mathbb{R}$,

$$\frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2 = \frac{(\gamma + 1)\kappa}{2}|z - \bar{z}(y_2)|^2 + a \qquad \text{for all } z \in \mathbb{R}^d.$$

It follows that the map $z \mapsto v(z) + \frac{1}{2}(\gamma + 1)\kappa|z - \bar{z}(y_2)|^2$ attains its infimum in B_1 at some point of $B_{r/2}(x)$. Since $\gamma \geq 1$, and thus $\gamma/(\gamma + 1) \geq \frac{1}{2}$, we deduce that

$$\left|\bar{z}(B_{r/8})\right| \geq 2^{-d}\left|B_{r/8}(x)\right|. \tag{3.25}$$

We have succeeded in touching the function v by steepening the paraboloid and wiggling the vertex. Now an application of Lemma 3.5, using (3.25) and that $B_r(x)$ is a good ball, ensures that we have actually touched a positive proportion of points in $B_r(x)$. We obtain

$$\begin{aligned} 2^{-d}\delta_2\left|B_{r/8}(x)\right| &\leq \int_{B_{r/2}(x)\cap A_{(\gamma+1)\kappa}} \lambda^{-d}(x)\,dx \\ &\leq 32^{-d}\delta_2\left|B_r(x)\right| + \mu^{-d}\left|B_{r/2}(x) \cap A_{(\gamma+1)\kappa}\right|, \end{aligned}$$

which implies $|B_{r/2}(x) \cap A_{(\gamma+1)\kappa}| \geq \mu^d 2^{-d}(1 - 2^{-d})\delta_2|B_{r/8}(x)|$, as desired. □

4. Homogenization. The proof of homogenization follows the approach of [10], although we have reorganized the argument for clarity and simplicity as well as to accommodate the modifications required to handle the nonuniform elliptic case. The strategy relies on an application of the subadditive ergodic theorem to a certain quantity involving the obstacle problem. The proof has three steps:

(1) Identifying $\overline{F}$: by applying the subadditive ergodic theorem to the Lebesgue measure of the contact set of a certain obstacle problem, we build the effective operator $\overline{F}$.
(2) Building approximate correctors: with the help of the effective regularity results, we compare the solutions of the obstacle problem to the solution of the Dirichlet problem with zero boundary conditions and show that the latter act as approximate correctors.
(3) Proving convergence: using the approximate correctors, the classical perturbed test function method allows us to conclude.

Step one: *Identifying $\overline{F}$ via the obstacle problem*. Following [10], we introduce, for each bounded Lipschitz domain $V \in \mathcal{L}$, the obstacle problem (with the zero function as the obstacle):

$$\begin{cases} \min\{F(D^2 w, y, \omega), w\} = 0, & \text{in } V, \\ w = 0, & \text{on } \partial V. \end{cases} \tag{4.1}$$

Some important properties of (4.1) are reviewed in Appendix. It is well known that (4.1) has a unique viscosity solution, which we denote by $w = w(y, \omega; V, F)$. We often write $w = w(y, \omega; V)$ or simply $w = w(y, \omega)$ if we do not wish to display the dependence on F or V.

The set $\mathcal{C}(V, \omega) := \{y \in V : w(y, \omega; V) = 0\}$ of points where w touches the obstacle is called the *contact set*. We write $\mathcal{C}(V, \omega; F)$ if we wish to display the dependence on F. The Lebesgue measure of this set is an important quantity, and we denote it by

$$m(V, \omega) := |\mathcal{C}(V, \omega)|. \tag{4.2}$$

We check that m satisfies the hypotheses of the subadditive ergodic theorem (Proposition 2.2). First we observe from the monotonicity of the obstacle problem [see (A.10)], that for all $V, W \in \mathcal{L}$ and $\omega \in \Omega$,

$$V \subseteq W \qquad \text{implies that } \mathcal{C}(W, \omega) \cap V \subseteq \mathcal{C}(V, \omega). \tag{4.3}$$

Immediate from (4.3) is the subadditivity of m. That is, for all $V, V_1, \ldots, V_k \in \mathcal{L}$ such that $\bigcup_{j=1}^k V_j \subseteq V$, the sets $V_1, \ldots, V_k$ are pairwise disjoint and $|V \setminus \bigcup_{j=1}^k V_j| = 0$, we have

$$m(V, \omega) \le \sum_{j=1}^k m(V_j, \omega). \tag{4.4}$$

According to (F1), m is stationary, that is,

$$m(V, \tau_y \omega) = m(y + V, \omega)$$

for every $y \in \mathbb{R}^d$ and $V \in \mathcal{L}$. We may easily extend m to $\mathcal{U}_0$ by defining, for every $A \in \mathcal{U}_0$,

$$\tilde{m}(A, \omega) := \inf\{m(V, \omega) : V \in \mathcal{L} \text{ and } A \subseteq V\}.$$

This extension agrees with m on $\mathcal{L}$ by (4.3) and it is easy to show that the subadditivity and stationarity properties are preserved.

We now obtain the following lemma.

LEMMA 4.1. *There exists an event $\Omega_2 \in \mathcal{F}$ of full probability and a deterministic constant $\bar{m} \in \mathbb{R}$ such that, for every $\omega \in \Omega_2$ and Lipschitz domain $V \subseteq \mathbb{R}^d$*,

$$\lim_{t \to \infty} \frac{1}{t^d} m(tV, \omega) = \bar{m}|V|. \tag{4.5}$$

PROOF. In light of the remarks preceding the statement, the lemma follows from the subadditive ergodic theorem (Proposition 2.2). □

For clarity, we write $\bar{m} = \bar{m}(F)$ to display the dependence of $\bar{m}$ in Lemma 4.1 on the nonlinear operator F.

We are now ready to define the effective nonlinearity:

$$\overline{F}(0) := \sup\{\alpha \in \mathbb{R} : \bar{m}(F - \alpha) > 0\}. \tag{4.6}$$

We extend this definition to all symmetric matrices in the obvious way. For each $N \in \mathbb{S}^d$, we denote F_N by

$$F_N(M, y, \omega) := F(M + N, y, \omega)$$

and then we set, for each $M \in \mathbb{S}^d$,

$$\overline{F}(M) := \overline{F_M}(0).$$

To check that $\overline{F}$ is well defined and finite, we first observe that, by (A.8) and (A.12),

$$\inf_{y \in V} F(0, y, \omega) \geq 0 \qquad \text{implies that } \mathcal{C}(V, \omega) = V$$

and

$$\sup_{y \in V} F(0, y, \omega) < 0 \qquad \text{implies that } \mathcal{C}(V, \omega) = \varnothing.$$

Using (F3) and the remarks in Section 2, it follows from these that

$$\operatorname*{ess\,inf}_{\omega \in \Omega} F(M, 0, \omega) \leq \overline{F}(M) \leq \operatorname*{ess\,sup}_{\omega \in \Omega} F(M, 0, \omega). \tag{4.7}$$

The monotonicity of the obstacle problem implies that $\alpha \mapsto \bar{m}(F - \alpha)$ is a decreasing function, and thus $\bar{m}(F - \alpha) > 0$ for $\alpha < \overline{F}(0)$ and $\bar{m}(F - \alpha) = 0$ for $\alpha > \overline{F}(0)$.

It is immediate from the comparison principle for the obstacle problem that, if F_1 and F_2 are two operators satisfying our hypotheses, then

$$\sup_{M \in \mathbb{S}^d} \operatorname*{ess\,sup}_{\omega \in \Omega} \big(F_1(M, 0, \omega) - F_2(M, 0, \omega)\big) \leq 0 \qquad \text{implies } \overline{F_1} \leq \overline{F_2}. \tag{4.8}$$

It is even more obvious that adding constants commutes with the operation $F \mapsto \overline{F}$. From these facts, a number of properties of $\overline{F}$ are immediate, the ones inherited from uniform properties of F. A few of these are summarized in the following lemma.

LEMMA 4.2. *For every $M, N \in \mathbb{S}^d$ such that $M \leq N$, we have*

$$0 \leq \overline{F}(M) - \overline{F}(N) \leq \Lambda \operatorname{tr}(N - M). \tag{4.9}$$

Moreover, if $M \mapsto F(M, 0, \omega)$ is positively homogeneous of order one, odd or linear, then $\overline{F}$ possesses the same property.

PROOF. Each of the properties are proved using the comments before the statement of the proposition. To prove (4.9), we simply observe that, according to (F1), for all $(Y, y, \omega) \in \mathbb{S}^d \times \mathbb{R}^d \times \Omega$,

$$F(M+Y, y, \omega) \le F(N+Y, y, \omega) + \Lambda \operatorname{tr}(N-M) \tag{4.10}$$

and then apply (4.8). It is obvious that $\overline{F}$ inherits the properties of positive homogeneity and oddness from F, and linearity follows from these. □

Observe that (4.9) asserts that $\overline{F}$ is degenerate elliptic. If F were uniformly elliptic, that is, $\lambda^{-1} \in L^\infty(\Omega)$, then it follows from an argument nearly identical to the one for (4.9) that $\overline{F}$ is uniformly elliptic. For more general $\lambda^{-1} \in L^d(\Omega)$, the operator $\overline{F}$ is uniformly elliptic as well, but the proof is more complicated. We postpone it until the next subsection, since it is convenient to deduce it as a consequence of Proposition 4.4, which we prove first.

We next show that, in large domains, the contact set has nearly constant density.

LEMMA 4.3. *For every $\omega \in \Omega_2$ and $V, W \in \mathcal{L}$ with $\overline{W} \subseteq V$,*

$$\lim_{t\to\infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} = \bar{m}. \tag{4.11}$$

PROOF. Let $U := V \setminus W \in \mathcal{L}$ and fix $\omega \in \Omega_2$. Observe that (4.3) gives

$$\limsup_{t\to\infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} \le \lim_{t\to\infty} \frac{|\mathcal{C}(tW, \omega)|}{|tW|} = \bar{m} \tag{4.12}$$

and, by the same argument,

$$\limsup_{t\to\infty} \frac{|\mathcal{C}(tV, \omega) \cap tU|}{|tU|} \le \bar{m}.$$

Therefore,

$$\begin{aligned} \liminf_{t\to\infty} \frac{|\mathcal{C}(tV, \omega) \cap tW|}{|tW|} &= \liminf_{t\to\infty} \frac{|\mathcal{C}(tV, \omega) \cap tV| - |\mathcal{C}(tV, \omega) \cap tU|}{|tW|} \\ &\ge \left(\frac{|V|}{|W|} - \frac{|U|}{|W|}\right)\bar{m} = \bar{m}. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) yields (4.11). □

Step two: *Building approximate correctors*. The next step in the proof of Theorem 1 is to show that, in the macroscopic limit, the obstacle problem controls the solution of the Dirichlet problem

$$\begin{cases} F(D^2 v, y, \omega) = 0, & \text{in } V, \\ v = 0, & \text{on } \partial V. \end{cases} \tag{4.14}$$

As before, $V \in \mathcal{L}$ is a bounded Lipschitz domain and we write $v = v(y, \omega; V, F)$.

The following proposition is the focus of this subsection.

PROPOSITION 4.4. *There exists an event $\Omega_3 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_3$, $M \in \mathbb{S}^d$ and $V \in \mathcal{L}$,*

$$\lim_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} \left| v\big(y, \omega; tV, F_M - \overline{F}(M)\big)\right| = 0. \tag{4.15}$$

Before we give its proof, we remark that Proposition 4.4 is a special case of Theorem 1. We can see this by fixing $U \in \mathcal{L}$, defining

$$v^\varepsilon(x, \omega) := \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon}U, F - \overline{F}(0)\right)$$

and then checking that $v^\varepsilon(\cdot, \omega)$ is the unique solution of the boundary-value problem

$$\begin{cases} F\left(D^2 v^\varepsilon, \dfrac{x}{\varepsilon}, \omega\right) = \overline{F}(0), & \text{in } U, \\ v^\varepsilon = 0, & \text{on } \partial U. \end{cases} \tag{4.16}$$

The conclusion of Proposition 4.4 then asserts that

$$v^\varepsilon \to 0 \qquad \text{uniformly in } U \qquad \text{as } \varepsilon \to 0, \tag{4.17}$$

which is consistent with Theorem 1 since the zero function $v \equiv 0$ is obviously the unique solution

$$\begin{cases} \overline{F}\big(D^2 v\big) = \overline{F}(0), & \text{in } U, \\ v = 0, & \text{on } \partial U. \end{cases} \tag{4.18}$$

As we show in the next subsection, Proposition 4.4 actually implies Theorem 1. This is because, for large $R > 0$, the function $\xi(y) := v(y, \omega; B_R, F_M - \overline{F}(M))$ is an "approximate corrector" in B_R in the sense that it satisfies the equation

$$F\big(M + D^2\xi, y, \omega\big) = \overline{F}(M) \qquad \text{in } B_R \tag{4.19}$$

and is "strictly subquadratic at infinity" [i.e., satisfies (4.15)]. This is precisely what is needed to implement the perturbed test function method.

PROOF OF PROPOSITION 4.4. According to the ergodic theorem, there exists an event $\Omega_4 \in \mathcal{F}$ of full probability such that, for every $\omega \in \Omega_4$, $V \in \mathcal{L}$ and rational $q \in \mathbb{Q}$ with $q > 0$,

$$\lim_{t\to\infty} \fint_{tV} \lambda^{-d}(y, \omega)\, dy = \mathbb{E}\big[\lambda^{-d}\big] \tag{4.20}$$

and

$$\lim_{t\to\infty} \fint_{tV} \lambda^{-d}(y, \omega)\chi_{\{\lambda<q\}}(y)\, dy = \mathbb{E}\big[\lambda^{-d}\mathbb{1}_{\{\lambda<q\}}\big]. \tag{4.21}$$

Note that, according to the ABP inequality (Proposition 3.2, properly scaled), for every $\omega \in \Omega_4$,

$$\lim_{\alpha\to 0} \limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} \frac{1}{R^2} \left| v(y,\omega; tV, F) - v(y,\omega; tV, F+\alpha) \right| = 0. \tag{4.22}$$

We now define $\Omega_3 := \Omega_2 \cap \Omega_4$, where Ω_2 is given in the statement of Lemma 4.1.

We first show that, for all $\omega \in \Omega_3$, $V \in \mathcal{L}$ and $M \in \mathbb{S}^d$

$$\liminf_{t\to\infty} \frac{1}{t^2} \inf_{y\in tV} v\big(y,\omega; tV, F_M - \overline{F}(M)\big) \ge 0. \tag{4.23}$$

We may assume that $M = 0$ by replacing F with F_{-M} and that $\overline{F}(0) = 0$ by replacing F by $F - \overline{F}(0)$. By (4.22), we may also suppose that $\bar{m}(F) = 0$ by considering $F - \alpha$ for $\alpha > 0$ and then sending $\alpha \to 0$. Set

$$K := \operatorname*{ess\,sup}_{\omega\in\Omega} \big(F(0,0,\omega)\big)_+ = \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{y\in\mathbb{R}^d} \big(F(0,y,\omega)\big)_+.$$

According to (4.15) and (A.11), for every $t > 0$, the function $u := w(\cdot,\omega; tV, F) - v(\cdot,\omega; tV, F)$ satisfies

$$\mathcal{P}^-_{\lambda(y,\omega),\Lambda}\big(D^2 u\big) \le K \chi_{\mathcal{C}(tV,\omega)} \qquad \text{in } tV$$

and $u = 0$ on $\partial(tV)$. Using that $w \ge 0$, the ABP inequality (Proposition 3.2, properly scaled) and (4.5), we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} -v(y,\omega tV, F) \\ &\quad\le \limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} u(y) \\ &\quad\le CK \limsup_{t\to\infty} \bigg(\fint_{tV} \lambda^{-d}(y,\omega) \chi_{\mathcal{C}(tV,\omega)}(y)\, dy \bigg)^{1/d}. \end{aligned} \tag{4.24}$$

To estimate the integral on the right, we observe that, for each $k \in \mathbb{N}$,

$$\int_{tV} \lambda^{-d}(y,\omega) \chi_{\mathcal{C}(tV,\omega)}(y)\, dy \le \bigg(k^d \big|m(tV,\omega)\big| + \int_{tV} \lambda^{-d}(y,\omega) \chi_{\{\lambda<1/k\}}(y)\, dy \bigg).$$

Divide this by $|tV|$ and pass to the limit $t \to \infty$ using (4.21) to obtain

$$\limsup_{t\to\infty} \fint_{tV} \lambda^{-d}(y,\omega) \chi_{\mathcal{C}(tV,\omega)}(y)\, dy \le k^d \bar{m}(F) + \mathbb{E}\big[\lambda^{-d} \mathbb{1}_{\{\lambda<1/k\}}\big]. \tag{4.25}$$

Since $\bar{m}(F) = 0$, we may send $k \to \infty$ and combine the resulting expression with (4.24) to obtain (4.23).

To complete the proof, we show that, for every $\omega \in \Omega_3$,

$$\limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} v\big(y,\omega; tV, F_M - \overline{F}(M)\big) \le 0. \tag{4.26}$$

As above, we may suppose that $M = 0$ and $\overline{F}(0) = 0$. We may also assume that $\bar{m}(F) > 0$, by considering $F + \alpha$ for $\alpha > 0$ and then sending $\alpha \to 0$, using (4.22). Since $v \le w$, it suffices for (4.26) to show that

$$\limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tV} w(y, \omega; tV, F) = 0. \tag{4.27}$$

Furthermore, by the monotonicity of the obstacle problem it suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t^2} \sup_{y\in tB_R} w(y, \omega; tB_{2R}, F) = 0, \tag{4.28}$$

where $R > 1$ is large enough that $V \subseteq B_R$. Fix $r > 0$ and observe that, by Lemma 4.3 and an easy covering argument using $\bar{m}(F) > 0$, there exists $T > 0$ sufficiently large such that, for every $t \ge T$ and $x \in B_R$, the function $w(\cdot, \omega; tB_{2R}, F)$ vanishes at some point of $B(tx, tr)$. We therefore have, for every $t \ge T$ and $x \in B_R$,

$$\frac{1}{t^2}\bigl|w(tx, \omega; tB_{2R}; F)\bigr| \le \operatorname*{osc}_{B(tx,tr)} \frac{1}{t^2} w(\cdot, \omega; tB_{2R}; F). \tag{4.29}$$

We prove (4.28) by showing that the lim-sup of the right-hand side of (4.29), as $t \to \infty$, is $o(1)$ as $r \to 0$. For this, we rely on Proposition 3.1.

Notice that (A.11), (4.20) and the ABP inequality (Proposition 3.2) yield, for $t > 0$ sufficiently large, the bound

$$\frac{1}{t^2} \sup_{tB_{2R}} \bigl|w(\cdot, \omega; tB_{2R}, F)\bigr| \le CKR^2, \tag{4.30}$$

where C depends only on d, Λ and $\mathbb{E}[\lambda^{-d}]$. Select $\mu > 0$ such that

$$\mathbb{E}\bigl[\lambda^{-d} \mathbb{1}_{\{\lambda<\mu\}}\bigr] < 4^{-d}\delta,$$

where $\delta > 0$, $\delta \in \mathbb{Q}$ is as in Proposition 3.1. By (4.21), and making $T > 0$ larger, if necessary, we have that for all $t \ge T$, $r < r' < R$ and $x \in B_R$,

$$\fint_{B_{tr'}(tx)} \lambda^{-d}(y, \omega)\chi_{\{\lambda<\mu\}}\, dy < \delta. \tag{4.31}$$

To see this, consider a finite covering $\{B_s(x_i)\}$ of B_R by balls of radius $s = 2^{-k}R$ for some $k \in \mathbb{N}$. According to (4.21), for sufficiently large t, the average of $\lambda^{-d}(y, \omega)\chi_{\{\lambda<\mu\}}$ in each of the balls $B_{2s}(x_i)$ will be less than $4^{-d}\delta$. But every ball $B_{r'}(x)$, with $s/2 \le r' \le s$ and $x \in B_R$, is contained in one of the balls $B_{2s}(x_i)$. Since $4r' \ge s$, this yields

$$\fint_{B_{tr'}(tx)} \lambda^{-d}(y, \omega)\chi_{\{\lambda<\mu\}}\, dy \le 4^d \fint_{B_{2s}(x_i)} \lambda^{-d}(y, \omega)\chi_{\{\lambda<\mu\}}\, dy < \delta.$$

Repeating this covering argument for $k = 0, 1, 2, \ldots, \lceil \log_2(R/r) \rceil$ and making $T > 0$ larger, if necessary, we obtain (4.31) for every $t \ge T$, $r \le r' \le R$ and $x \in B_R$.

Iterating Proposition 3.1, using (4.30), (4.31) as well as (A.9) and (A.11), we obtain, for every $x \in B_R$ and $t \ge T$,

$$\operatorname*{osc}_{B(tx,tr)} \frac{1}{t^2} w(\cdot, \omega; tB_{2R}; F) \le C r^{\gamma} \tag{4.32}$$

for some constants $\gamma > 0$ and $C > 0$ which may depend on d, Λ, $\mathbb{E}[\lambda^{-d}]$, μ, K and R, but do not depend on r or T. Combining this with (4.29) and sending $t \to \infty$ and then $r \to 0$, we obtain (4.27), and thus the proposition. □

We conclude the second part by showing that $\overline{F}$ is uniformly elliptic and giving an estimate of its ellipticity. The proof is based on Lemma 3.5 and Proposition 4.4.

PROPOSITION 4.5. *There exists $c > 0$, depending only on d and Λ, such that $\overline{F}$ is uniformly elliptic with constants $\lambda_0 := c\mathbb{E}[\lambda^{-d}]^{-1}$ and Λ, that is, for all $M, N \in \mathbb{S}^d$,*

$$\mathcal{P}^-_{\lambda_0,\Lambda}(M - N) \le \overline{F}(M) - \overline{F}(N) \le \mathcal{P}^+_{\lambda_0,\Lambda}(M - N). \tag{4.33}$$

PROOF. Select $M, N \in \mathbb{S}^d$ such that $M \ge N$. Fix $\omega \in \Omega_3$ and define, for each $\varepsilon > 0$,

$$\begin{aligned} V_\varepsilon(x) := {}& \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon} B_1, F_M - \overline{F}(M)\right) - \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega; \frac{1}{\varepsilon} B_1, F_N - \overline{F}(N)\right) \\ & + \frac{1}{2} x \cdot (M - N)x. \end{aligned}$$

It is easy to check that V satisfies the inequality

$$\mathcal{P}^+_{\lambda(x/\varepsilon,\omega),\Lambda}(D^2 V_\varepsilon) \ge \overline{F}(M) - \overline{F}(N) \qquad \text{in } B_1. \tag{4.34}$$

According to Proposition 4.4,

$$V_\varepsilon(x) \to \tfrac{1}{2} x \cdot (M - N)x \qquad \text{as } \varepsilon \to 0 \qquad \text{uniformly in } B_1. \tag{4.35}$$

Suppose that $M - N$ has a largest eigenvalue $a > 0$ with corresponding normalized eigenvector $\xi \in \mathbb{R}^d$, $|\xi| = 1$ so that

$$a\xi \otimes \xi \le M - N \le aI. \tag{4.36}$$

Fix $\beta > 0$ and, for each $y \in \mathbb{R}^d$, denote by $\bar{z}(y) \in \mathbb{R}^d$ the (unique) point at which the map $x \mapsto \Phi(x, y) := \frac{1}{2} x \cdot (M - N)x + \beta |x - y|^2$ attains its (strict) global minimum on $\mathbb{R}^d$. Note that

$$\bar{z}(y) = (M - N + 2\beta I)^{-1} 2\beta y.$$

In particular, $|\bar{z}(y)| \le |y|$ and

$$\big|\xi \cdot \bar{z}(y)\big| = (a+2\beta)^{-1}\big|\xi \cdot (2\beta y)\big| \le \frac{2\beta}{2\beta + a}|y|.$$

Applying (4.35), we deduce that, for sufficiently small $\varepsilon > 0$ and every $y \in B_{1/3}$, the infimum in B_1 of the map $x \mapsto V_\varepsilon(x) + \beta|x-y|^2$ is attained in $B_{1/2}$ and any point z at which the minimum is attained satisfies

$$|z \cdot \xi| \le \frac{2\beta}{2\beta + a}\frac{1}{3} < \frac{\beta}{a}. \tag{4.37}$$

Let $A := \{x \in B_{1/2} : |x \cdot \xi| < \beta/a\}$ and note that $|A| \le \beta/a$. In the case that $\overline{F}(M) - \overline{F}(N) \ge -2\beta$, we may apply Lemma 3.5, using (4.20), to obtain

$$c \le |B_{1/2}| \le C \limsup_{\varepsilon \to 0} \int_A \lambda\left(\frac{x}{\varepsilon}, \omega\right)^{-d} dx = C|A|\mathbb{E}[\lambda^{-d}] \le C\frac{\beta}{a}\mathbb{E}[\lambda^{-d}].$$

This is impossible if $a \ge C\beta\mathbb{E}[\lambda^{-d}]$. Here, $C > 0$ depends only on d and Λ.

We conclude that $a/\beta \ge \widetilde{C} := C\mathbb{E}[\lambda^{-d}]$ implies that $\overline{F}(M) - \overline{F}(N) < -2\beta$. Define $\lambda_0 := 2/d\widetilde{C}$ and deduce that, for all $M \ge N$,

$$\overline{F}(M) - \overline{F}(N) \le -2a/\widetilde{C} = -\lambda_0 a d = \mathcal{P}^+_{\lambda_0,\Lambda}(aI) \le \mathcal{P}^+_{\lambda_0,\Lambda}(M-N).$$

Recalling (4.9), we also have, for every $M \ge N$,

$$\overline{F}(M) - \overline{F}(N) \ge -\Lambda\,\mathrm{tr}(N-M) = \mathcal{P}^-_{\lambda_0,\Lambda}(M-N).$$

We have verified (4.33) for all $M, N \in \mathbb{S}^d$ with $M \ge N$.

To remove the latter restriction, fix any $M, N \in \mathbb{S}^d$ and write

$$\begin{aligned}\overline{F}(M) - \overline{F}(N) &= \overline{F}(M) - \overline{F}\big(M - (N-M)_-\big) \\ &\quad + \overline{F}\big(M-(N-M)_-\big) - \overline{F}\big(M - (N-M)_- + (N-M)_+\big)\end{aligned}$$

and observe by what we have shown above that

$$\overline{F}(M) - \overline{F}(N) \le \mathcal{P}^+_{\lambda_0,\Lambda}\big((N-M)_-\big) - \mathcal{P}^-_{\lambda_0,\Lambda}\big((N-M)_+\big) = \mathcal{P}^+_{\lambda_0,\Lambda}(M-N).$$

This yields the second inequality of (4.33) and arguing again after interchanging M and N yields the first inequality. □

Step three: *Concluding by the perturbed test function method*. By adapting the classical perturbed test function method, first introduced in the context of periodic homogenization by Evans [17], we now complete the proof of Theorem 1. The test functions are perturbed by the approximate correctors constructed in Proposition 4.4. The argument we present here is similar in spirit to the one given in Section 4 of [10], although a bit less complicated.

PROOF OF THEOREM 1. Fix a bounded Lipschitz domain $U \in \mathcal{L}$, $g \in C(\partial U)$ and an environment $\omega_0 \in \Omega_3$, where the event $\Omega_3 \in \mathcal{F}$ is given in the statement of Proposition 4.4.

We first argue that, for every $x \in U$,

$$\widetilde{u}(x) := \limsup_{\varepsilon \to 0} u^\varepsilon(x, \omega_0) \le u(x). \tag{4.38}$$

To show (4.38), we begin by checking that $\widetilde{u}(x) \le g$ on ∂U. By approximation, we may assume that $g \equiv 0$ and that U is smooth (and in particular has the exterior ball condition). By dilation, we may also assume that $F(0, \cdot, \omega) \le 1$ and that $U \subseteq B_{R/2}(0)$. Given $y \in \partial U$, we may select $B_r(x) \subseteq \mathbb{R}^d \setminus U$ such that $\overline{B}_r(x) \cap \overline{U} = \{y\}$. Given $h > 0$, we apply Lemma 3.3 with the modification in Remark 3.4. Using $\omega_0 \in \Omega_3$, we may select $\beta > 0$ and $r' \in (r, R - r)$ such that the solution $\varphi^\varepsilon \in C(\overline{B}_R \setminus B_r)$ of

$$\begin{cases} \mathcal{P}^-_{\lambda(x/\varepsilon,\omega_0),\Lambda}(D^2\varphi^\varepsilon) = 1, & \text{in } B_R \setminus B_r, \\ \varphi^\varepsilon = \beta, & \text{on } \partial B_R, \\ \varphi^\varepsilon = 0, & \text{on } \partial B_r, \end{cases}$$

satisfies $\limsup_{\varepsilon \to 0} \varphi^\varepsilon \le h$ in $V \cap B_{r'}(x)$. Since $U \subseteq B_R(x)$ and $u^\varepsilon \le 0$ on ∂U, the comparison principle implies that $u^\varepsilon \le \varphi^\varepsilon$. It follows that

$$\limsup_{\varepsilon \to 0} \sup_{V \cap B_{r'-r}(y)} u^\varepsilon(\cdot, \omega_0) \le h.$$

Since $h > 0$ was arbitrary, we conclude that $\widetilde{u} \le g$ on ∂U.

By the comparison principle, to prove (4.38) it suffices to check that the function $\widetilde{u}(x) := \limsup_{\varepsilon \to 0} u^\varepsilon(x, \omega_0)$ satisfies, in the viscosity sense,

$$F\bigl(D^2\widetilde{u}\bigr) \le 0 \qquad \text{in } U. \tag{4.39}$$

To verify (4.39), we select a smooth test function $\phi \in C^2(U)$ and a point $x_0 \in U$ such that

$$x \mapsto (\widetilde{u} - \phi)(x) \qquad \text{has a strict local maximum at } x = x_0.$$

We must show that $\overline{F}(D^2\phi(x_0)) \le 0$. Set $M := D^2\phi(x_0)$ and suppose on the contrary that $\theta := \overline{F}(M) > 0$.

Since the local maximum of $\widetilde{u} - \phi$ at x_0 is strict, there exists $r_0 > 0$ such that $B_{r_0}(x_0) \subseteq U$ and, for every $0 < r \le r_0$,

$$(\widetilde{u} - \phi)(x_0) > \sup_{\partial B_r(x_0)} (\widetilde{u} - \phi). \tag{4.40}$$

We next introduce the perturbed test function

$$\phi^\varepsilon(x) := \phi(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}, \omega_0; \frac{1}{\varepsilon} B_{r_0}(x_0), F_M - \overline{F}(M)\right).$$

We claim that, in some neighborhood of x_0, ϕ^ε is a strict supersolution of the oscillatory equation at microscopic scale ε. More precisely, we will argue that, for some suitably small $0 < s < r_0$ to be selected below (and which may depend on ϕ),

$$F\left(D^2\phi^\varepsilon, \frac{x}{\varepsilon}, \omega_0\right) \geq \frac{1}{2}\theta \qquad \text{in } B_s(x_0). \tag{4.41}$$

To check (4.41), we select a smooth test function $\psi \in C^2(B_s(x_0))$ and a point $x_1 \in B_s(x_0)$ such that

$$x \mapsto (\phi^\varepsilon - \psi)(x) \qquad \text{has a local minimum at } x = x_1.$$

Using the definition of ϕ^ε and rescaling, we have

$$y \mapsto v\left(y, \omega_0; \frac{1}{\varepsilon}B_{r_0}(x_0), F_M - \overline{F}(M)\right) - \frac{1}{\varepsilon^2}\big(\psi(\varepsilon y) - \phi(\varepsilon y)\big)$$
$$\text{has a local minimum at } y = \frac{x_1}{\varepsilon}.$$

Using the equation for v, we obtain

$$F\left(M + D^2\psi(x_1) - D^2\phi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) - \overline{F}(M) \geq 0.$$

Since $\phi \in C^2$, we may make $|M - D^2\phi(x_1)| = |D^2\phi(x_0) - D^2\phi(x_1)|$ as small as we like by taking $s > 0$ small enough. Thus, in light of (F2), we may fix $s > 0$ so that

$$\left|F\left(M + D^2\psi(x_1) - D^2\phi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) - F\left(D^2\psi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right)\right| \leq \frac{1}{2}\theta.$$

The previous two inset inequalities and $\theta = \overline{F}(M)$ yield

$$F\left(D^2\psi(x_1), \frac{x_1}{\varepsilon}, \omega_0\right) \geq \frac{1}{2}\theta. \tag{4.42}$$

This completes the proof of (4.41).

An application of the comparison principle now yields

$$\begin{aligned} u^\varepsilon(x_0, \omega_0) - \phi^\varepsilon(x_0) &\leq \sup_{B_s(x_0)} \big(u^\varepsilon(\cdot, \omega_0) - \phi^\varepsilon\big) \\ &= \sup_{\partial B_s(x_0)} \big(u^\varepsilon(\cdot, \omega_0) - \phi^\varepsilon\big). \end{aligned} \tag{4.43}$$

Taking the limsup of both sides of (4.43) as $\varepsilon \to 0$ and applying Proposition 4.4, we obtain

$$\widetilde{u}(x_0) - \phi(x_0) \leq \sup_{\partial B_s(x_0)} (\widetilde{u} - \phi).$$

This contradicts (4.40) and completes the proof that $\overline{F}(M) \leq 0$, and hence of (4.39), and hence of (4.38).

It remains to show that, for every $x \in U$,

$$\liminf_{\varepsilon \to 0} u^\varepsilon(x, \omega_0) \geq u(x).$$

This is obtained by mimicking the argument above with very obvious modifications. We omit the details. □

5. Breakdown of homogenization and regularity for $p < d$. In this section, we show that the condition that the dth moment of λ^{-1} is finite is sharp for both the homogenization and regularity results. We remark that the example we construct shows that the exponent $p = d$ is sharp with respect to the general class of (fully nonlinear) operators, but not with respect to the subclass of linear operators. One interpretation of the reason for this difference is that some nonlinear equations correspond to stochastic optimal control problems, and the controller is under no obligation to select a stationary control. A variant of our construction leads to a linear counterexample for all $p < 1$, which was already discovered in [21] (see also [5]) using a similar trap model. The range $1 \leq p < d$ thus remains open in the linear case; we believe that $p = 1$ is the critical exponent.

For each $p < d$, we construct a stationary-ergodic random environment $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ and stationary random field $\lambda : \mathbb{R}^d \times \Omega \to (0, 1]$ such that

$$\mathbb{E}[\lambda^{-p}] < +\infty, \tag{5.1}$$

but for which homogenization fails for the equation

$$\mathcal{P}^-_{\lambda(x/\varepsilon,\omega),1}(D^2 u^\varepsilon) = 1. \tag{5.2}$$

To show the breakdown of homogenization, we check that the solution u^ε of the Dirichlet problem

$$\begin{cases} \mathcal{P}^-_{\lambda(x/,\omega),1}(D^2 u^\varepsilon) = 1, & \text{in } B_1, \\ u^\varepsilon = 0, & \text{on } \partial B_1, \end{cases} \tag{5.3}$$

satisfies $\lim_{\varepsilon \to 0} u^\varepsilon(0, \omega) = +\infty$ almost surely. We conclude that there is no "effective" ABP inequality, in the limit $\varepsilon \to 0$, and hence no effective regularity or effective equation. The random field $\lambda : \mathbb{R}^d \times \Omega \to (0, 1]$ we construct has a finite range of dependence, so even this strongest possible mixing assumption cannot save homogenization for a general nonlinear operator without a bounded dth moment of ellipticity.

The idea underlying the construction of λ is to build spatial "traps" where, from the probabilistic perspective, the corresponding controlled diffusion process becomes stuck for long periods of time, resulting in subdiffusive behavior on large scales. We fix $0 < \alpha < 1$ small, take $0 < \lambda_* < 1/2d$ to be selected below and choose, for each $k \in \mathbb{N}$, a random arrangement $P_k(\omega) \subseteq \mathbb{R}^d$ of points (also specified below). We construct the random field λ in such a way that $0 < \lambda \leq \lambda_*$ almost surely and $\lambda(y, \omega) \leq \lambda_k := 1/(k^{1+\alpha} \log^3(2 + k))$ in each ball of radius 1 with center in $P_k(\omega)$. To be more precise, for each $k \in \mathbb{N}$ we select a (deterministic) continuous function θ_k on $\mathbb{R}^d$ which is at most λ_k in B_1, takes the value λ_* in $\mathbb{R}^d \setminus B_2$ and satisfies $\lambda_k \le \theta_k(y)$ and $\theta_k(y) \le \lambda_*$. We then set

$$\lambda(y,\omega) := \inf_{k\in\mathbb{N}} \inf_{x\in P_k(\omega)} \theta_k(y-x).$$

We may also easily arrange that the family $\{\theta_k\}_{k\in\mathbb{N}}$ is equicontinuous.

We take the point configurations P_k to be independent Poisson point processes (cf. [13]), with intensities depending on k such that the expected number of points of $P_k \cap V$ is equal to $a|V|k^{-1-d}$, where $a > 0$ is a parameter independent of k which we also choose below. Since the series $\sum_{k=1}^{\infty} k^{-1-d}$ converges, it follows that the number of points of $\bigcup_{k=1}^{\infty} P_k$ is almost surely locally finite by the Borel–Cantelli lemma, and this implies that $\lambda(0,\omega) > 0$ almost surely. In fact, for all $p < d/(1+\alpha)$,

$$\begin{aligned}\mathbb{E}\bigl[\lambda^{-p}\bigr] &\le 1 + Ca\sum_{k=1}^{\infty} \lambda_k^{-(1+\alpha)p} k^{-1-d} \\ &= 1 + Ca\sum_{k=1}^{\infty} k^{-1-d+(1+\alpha)p}\log^{3p}(2+k) < \infty.\end{aligned}$$

The stationarity of the Poisson point processes implies that λ is a stationary function, and it is clear that $\lambda(\cdot,\omega)$ is uniformly continuous (almost surely in ω) since the family $\{\theta_k\}_{k\in\mathbb{N}}$ is equicontinuous.

Let us see how we can increase the frequency of "traps" (i.e., regions in which λ is small) by taking $a > 0$ large. For each fixed $t > 1 + a\log t|V|$, we see that

$$\begin{aligned}&\mathbb{P}\bigl[P_k \cap t(\log t)^{1/d} V \ne \varnothing \text{ for some } k \ge t\bigr] \\ &\quad \ge 1 - \prod_{k=\lceil t\rceil}^{\infty} \bigl(1 - at^d \log t|V|k^{-1-d}\bigr) \ge 1 - \exp\bigl(-a\log t|V|\bigr).\end{aligned} \tag{5.4}$$

Choose the constant $a > 0$ large enough that $a|B_1| > d+1$, so that

$$\mathbb{P}\bigl[P_k \cap t(\log t)^{1/d} V \ne \varnothing \text{ for some } k \ge t\bigr] \ge 1 - t^{-(d+1)}.$$

By covering B_{t^2} with Ct^d balls of radius $\frac{1}{6}t(\log t)^{1/d}$, we deduce that

$$\mathbb{P}\bigl[\text{for all } x \in B_{t^2}, \operatorname{dist}(x, P_k) < \tfrac{1}{3}t(\log t)^{1/d} \text{ for some } k \ge t\bigr] \ge 1 - Ct^{-1}.$$

By using Borel–Cantelli along the sequence $t_j = 2^j$, it follows that

$$\mathbb{P}\bigl[\exists s > 1 \text{ s.t. } \forall t > s, x \in B_{t^2}, \exists k \ge t \text{ s.t. } \operatorname{dist}(x, P_k) < t(\log t)^{1/d}\bigr] = 1. \tag{5.5}$$

The previous line says that we will have sufficiently many traps to work with. We next measure the local effect of one trap.

LEMMA 5.1. *Fix $\omega \in \Omega$, and suppose that $k \geq t > 10$, $\operatorname{dist}(0, P_k(\omega)) < t(\log t)^{1/d}$, $R(t) := 10t(\log t)^{1/d}$ and $v \in C(B_R)$ satisfies*

$$\begin{cases} \mathcal{P}^-_{\lambda(x,\omega),1}(D^2 v) \geq 1, & \text{in } B_{R(t)}, \\ v \geq \ell, & \text{on } \partial B_{R(t)}, \end{cases}$$

where ℓ is an affine function. If $0 < \lambda_* < 1/2d$ is chosen small enough (depending on α), then there exists $c, q > 0$ depending on d and λ_*, but not on t, such that

$$v(0) \geq \ell(0) + cR(t)^2 (\log(t))^q \tag{5.6}$$

for a constant $c > 0$ depending only on d and a lower bound for α.

PROOF. We may assume with loss of generality that $\ell = 0$. The goal is to find an explicit subsolution, taking advantage of the trap near the origin and the fact that the ellipticity is larger than λ_*^{-1}. Set $\beta := 1 - 2d\lambda_* > 0$, $a = (1-\beta)/2 = d\lambda_* > 0$ and

$$\phi(x) := -(a + |x|^2)^{\beta/2}.$$

The Hessian of ϕ is given by

$$D^2\phi(x) = \beta(a + |x|^2)^{\beta/2-2}\left(((1-\beta)|x|^2 - a)\frac{x \otimes x}{|x|^2} - (a + |x|^2)\left(I - \frac{x \otimes x}{|x|^2}\right)\right).$$

For $\mu \leq \lambda_*$, we find that

$$\mathcal{P}^-_{\mu,1}(D^2\phi(x)) \leq \beta(a + |x|^2)^{\beta/2-2}((\beta - 1 + a) + (a+1)\mu(d-1)) \leq 0 \quad \text{in } \mathbb{R}^d \setminus B_1$$

and, for a constant $C > 0$ depending only on d,

$$\mathcal{P}^-_{\mu,1}(D^2\phi(x)) \leq C\mu(a + |x|^2)^{\beta/2-2} \leq C\mu\lambda_*^{\beta/2-2} \qquad \text{in } B_1.$$

Now suppose $x_0 \in P_k(\omega)$ such that $|x_0| < t(\log t)^{1/d}$. Then for a small constant $c > 0$ depending only on dimension, the function

$$\psi(x) := ct^{1+\alpha}(\log t)^3 \lambda_*^{2-\beta/2}\phi(x - x_0)$$

satisfies

$$\mathcal{P}^+_{\lambda(x,\omega),1}(D^2\psi) \leq 1 \qquad \text{in } \mathbb{R}^d.$$

By the comparison principle,

$$\psi(0) - v(0) \leq \max_{\partial B_R}(\psi - v) = \max_{\partial B_R} \psi,$$

which yields, for $T := t(\log t)^{1/d}$,

$$\begin{aligned} v(0) &\geq \psi(0) - \max_{\partial B_R} \psi \\ &\geq ct^{1+\alpha}(\log t)^3 \lambda_*^{2-\beta/2}\big(-(a+T^2)^{\beta/2} + (a+(R-1)^2)^{\beta/2}\big) \\ &\geq ct^{1+\alpha}(\log t)^3 \lambda_*^{2-\beta/2} T^\beta \qquad \text{(since } R > 10T > 100a) \\ &= \big(c\lambda_*^{2-\beta/2}\big) T^{1+\alpha+\beta}(\log t)^q, \end{aligned}$$

where $q := 3 - (1+\alpha)/2 > 0$. By taking $\lambda_* := \alpha/2d$ so that $\alpha + \beta + 1 = 2$, and noting that $R(t) = 10T$, we obtain (5.6). □

The above lemma, after rescaling and in light of (5.5), implies that the difference of u^ε and the paraboloid $c(\log t)^q(1-|x|^2)$, with $\varepsilon = t^{-2}$, cannot achieve its maximum on B_1 except in a boundary strip of ∂B_1 of width at most $t^{-1} = \sqrt{\varepsilon}$. This easily gives that $u^\varepsilon \to +\infty$ locally uniformly in B_1 with at least rate $|\log \varepsilon|^q$.

APPENDIX: ELEMENTARY PROPERTIES OF THE OBSTACLE PROBLEM

For the convenience of the reader, we briefly review (and sketch the proofs of) some well-known properties of the obstacle problem

$$\text{(A.7)} \qquad \begin{cases} \min\{F(D^2 w, y), w\} = 0, & \text{in } V, \\ w = 0, & \text{on } \partial V. \end{cases}$$

We have dropped the dependence of F on ω since the random environment plays no role here, and we furthermore assume that F is uniformly elliptic by the remarks preceding Theorem 1.

First of all, problem (A.7) has a unique solution $w \in C(\overline{V})$, which may be expressed as the least nonnegative supersolution of $F = 0$ in V:

$$\text{(A.8)} \qquad w(x; V) = \inf\{u(x) : u \geq 0 \text{ in } V \text{ and } F(D^2 u, y) \geq 0 \text{ in } V\}.$$

This from the Perron method and the fact that the obstacle problem has a comparison principle. Immediate from this expression is that w is a global subsolution:

$$\text{(A.9)} \qquad F(D^2 w, y) \geq 0 \qquad \text{in } V$$

as well as the monotonicity property:

$$\text{(A.10)} \qquad V \subseteq W \qquad \text{implies that } w(\cdot; V) \leq w(\cdot; W) \qquad \text{on } \overline{V}.$$

In order to use some regularity theory, we also need the fact that w satisfies

$$\text{(A.11)} \quad F(D^2 w, y) \leq k\chi_{\{w=0\}} \qquad \text{in } V \qquad \text{where } k := \sup_{y \in V}\big(F(0, y)\big)_+.$$

This is typically handled by considering an approximate equation with a penalty term (the Levy–Stampacchia penalization method) whose solutions satisfy (A.11) and showing that their uniform limit is w. We instead opt for a more natural and simpler proof by showing that w is also given by the formula

$$w(x;V) = \sup\{u(x) : u \le 0 \text{ on } \partial V \text{ and } F(D^2u, y) \le k\chi_{\{u\le 0\}} \text{ in } V\}. \tag{A.12}$$

It is clear that (A.12) implies (A.11). To check the former, we let $\hat{w}$ denote the expression on the right-hand side and observe that, since the zero function belongs to the admissible class by the definition of k, we have $\hat{w} \ge 0$. The Perron method implies that $\hat{w}$ satisfies $F(D^2\hat{w}, y) = 0$ in $\{\hat{w} > 0\}$. Therefore, $\hat{w}$ is a solution of (4.1), and by uniqueness we deduce $w = \hat{w}$.

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