Central limit theorems in linear dynamics

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Abstract. Given a bounded operator T on a Banach space X, we study the existence of a probability measure μ on X such that, for many functions $f : X \to \mathbb{K}$, the sequence $(f + \cdots + f \circ T^{n-1})/\sqrt{n}$ converges in distribution to a Gaussian random variable.

Résumé. Étant donné un opérateur T agissant sur un espace de Banach X, nous étudions l'existence d'une mesure de probabilité μ sur X telle que, pour de nombreuses fonctions $f : X \to \mathbb{K}$, la suite $(f + \cdots + f \circ T^{n-1})/\sqrt{n}$ converge en loi vers une variable aléatoire gaussienne.

1. Introduction

Linear dynamics (namely the study of the dynamics of linear operators) is a branch of analysis connecting functional analysis and dynamics. Its main topics are detailed in the two books [3] and [12]. As for the classical dynamical systems, one can study the dynamics of linear operators from a topological point of view. Precisely, an operator T defined on a separable Banach space X is called *hypercyclic* provided there exists a vector $x \in X$ such that its orbit $\{T^n x; n \geq 0\}$ under T is dense. In this context, contrary to the general case, there is a very easy criterion to prove that an operator is hypercyclic; it allows to exhibit many hypercyclic operators. Let us recall this criterion.

Theorem 1 (Hypercyclicity criterion). *Let $T \in \mathfrak{L}(X)$. Suppose that there exist a dense subset $\mathcal{D} \subset X$ and a sequence of maps $(S_n)_{n \geq 0}$, $S_n : \mathcal{D} \to X$, such that, for each $x \in \mathcal{D}$,*

(i) $T^n x \to 0$;
(ii) $S_n x \to 0$;
(iii) $T^n S_n x \to x$.

Then T is hypercyclic.

In this paper, we shall concentrate on the other aspect of linear dynamics, that links it with ergodic theory. Let us recall some basic definitions. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measurable map. We set

$$L_0^2(\mu) = \left\{ f \in L^2(\mu); \int_X f \, \mathrm{d}\mu = 0 \right\}.$$

We say that T is a *measure-preserving transformation* (or that μ is T-invariant) if $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{B}$. A measure-preserving transformation $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is *ergodic* (with respect to μ) if

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu\big(A \cap T^{-n}(B)\big) \overset{N \to \infty}{\longrightarrow} \mu(A)\mu(B)$$

for any measurable sets $A, B \subset X$; T is *weakly mixing* (with respect to μ) if

$$\frac{1}{N} \sum_{n=0}^{N-1} \big|\mu\big(A \cap T^{-n}(B)\big) - \mu(A)\mu(B)\big| \overset{N \to \infty}{\longrightarrow} 0$$

for any $A, B \in \mathcal{B}$; and T is *strongly mixing* if

$$\mu\big(A \cap T^{-n}(B)\big) \overset{n \to \infty}{\longrightarrow} \mu(A)\mu(B)$$

for any $A, B \in \mathcal{B}$.

We wonder whether there exists a Borel probability measure μ on the separable Banach space X, which is nondegenerate (namely, $\mu(U) > 0$ for any nonempty and open subset $U \subset X$), such that $T \in \mathcal{L}(X)$ is μ-invariant and T is ergodic (resp. weakly mixing, strongly mixing) with respect to μ. This line of investigation was opened by Flytzanis in [11] and by Rudnicki in [18–20]. It was pursued later by Bayart, Grivaux, Matheron (see [1,2,4]) and also recently by Murillo-Arcilla and Peris (see [16]).

It turns out that, when an operator has many eigenvectors associated to eigenvalues of modulus 1, then it is weakly mixing with respect to a nondegenerate and invariant Gaussian measure on X. Indeed, the following theorem is proved in [4].

Theorem 1.1. *Let $T \in \mathfrak{L}(X)$. Suppose that, for any countable set $D \subset \mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$, the linear span of $\bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(T - \lambda)$ is dense in X. Then there exists a T-invariant weakly mixing Gaussian Borel probability measure μ on X with full support.*

An interesting application of this theorem (strictly speaking, of a variant of this theorem) is that an enhancement of the hypercyclicity criterion leads to a strongly mixing dynamical system.

Theorem 1.2. *Let $T \in \mathfrak{L}(X)$. Suppose that there exist a dense set $\mathcal{D} \subset X$ and a sequence of maps $(S_n)_{n \geq 0}$, $S_n : \mathcal{D} \to X$, such that, for each $x \in \mathcal{D}$,*

  (i) *$\sum_{n \geq 0} T^n x$ converges unconditionally;*
  (ii) *$\sum_{n \geq 0} S_n x$ converges unconditionally;*
 (iii) *$T^n S_n x = x$ for any $n \geq 0$ and $T^m S_n x = S_{n-m} x$ if $n > m$.*

Then there exists a T-invariant strongly mixing (Gaussian) Borel probability measure μ on X with full support.

Theorem 1.2 has also been obtained in [16] in a completely different way. The measure μ constructed in [16] is not a Gaussian measure; in [16], very few properties of this measure are proved. For instance, it is not known whether the norm $\| \cdot \|$ or the linear functionals $\langle x^*, \cdot \rangle$ belong to $L^2(X, \mu)$, even if we can always ensure these properties, see Section 3. On the other hand, the dynamical system (T, μ) is conjugated to an easy strongly mixing dynamical system: a Bernoulli shift.

In this paper, we are interested in properties stronger than just ergodicity or mixing. First we are interested in the speed of mixing. For $f, g \in L^2(X, \mu)$, define the correlation of f and g as

$$\text{Cov}(f, g) = \int_X fg \, d\mu - \int_X f \, d\mu \int_X g \, d\mu$$

and the correlation of order n of f and g (with respect to T) by

$$\mathcal{I}_n(f, g) = \mathrm{Cov}(f \circ T^n, g).$$

T is μ-mixing provided $\mathcal{I}_n(f, g)$ goes to zero for any $f, g \in L^2(X, \mu)$. One may ask if we can estimate the speed of convergence to 0 of $\mathcal{I}_n(f, g)$ for any $f, g \in L^2(X, \mu)$, or at least for a large class of functions.

The second direction we investigate is related to the central limit theorem. Indeed, if T is μ-ergodic, then Birkhoff's theorem says that, for any $f \in L^1(X, \mu)$,

$$\frac{f + \cdots + f \circ T^{n-1}}{n} \to \int f \, d\mu \quad \text{almost surely.}$$

In other words, the sequence $(f \circ T^k)$ satisfies the strong law of large numbers. One can ask whether it also satisfies the central limit theorem, namely if the sequence $(\frac{f + \cdots + f \circ T^{n-1}}{\sqrt{n}})$ converges in distribution to a Gaussian random variable of zero mean and finite variance (we now assume $f \in L_0^2$).

At this point, it is important to notice that we cannot expect results true for all $f \in L_0^2(\mu)$. Indeed, D. Volný has proved in [21] that the Birkhoff means may converge to arbitrary laws on a dense G_δ-set of $L_0^2(\mu)$. More precisely, Volný has shown that, if T is μ-ergodic with $\mu(\{x\}) = 0$ for any $x \in X$, there exists a dense G_δ-set E of $L_0^2(\mu)$ such that, for any $f \in E$ and any probability measure ν on $\mathbb{R}$ satisfying $\int t \, d\nu(t) = 0$ and $\int t^2 \, d\nu(t) = 1$, there exists a sequence (N_k) tending to infinity such that $S_{N_k}/\|S_{N_k}\|_2$ converges in distribution to ν, where $S_N = f + \cdots + f \circ T^{N-1}$. Similarly, the slow decay of correlations is a typical feature of functions in $L_0^2(\mu)$ (see [7]). However, for many concrete dynamical systems, positive results were obtained if we assume some regularity condition on f, like f is Hölder.

Thus, in this paper, we are interested to prove central limit theorems or to estimate the decay of correlation for functions belonging to a large class of $L_0^2(\mu)$, in the context of linear dynamical systems. The first step in that direction was done by V. Devinck in [8]. He started from Theorem 1.1 and he was able to prove that, when the $\mathbb{T}$-eigenvectors of T can be parametrized in a regular way, then $\mathrm{Cov}(f \circ T^n, g)$ decreases fast to zero for f, g belonging to large classes of functions. Let us summarize his main result.

Theorem 1.3. *Let H be a separable Hilbert space and let $T \in \mathcal{L}(H)$. Suppose that there exists $\alpha \in (0, 1]$ and $E : \mathbb{T} \to X$ such that $T E(\lambda) = \lambda E(\lambda)$ for any $\lambda \in \mathbb{T}$ and E is α-Hölderian: there exists $C_E > 0$ such that*

$$\left\| E(e^{i\theta}) - E(e^{i\theta'}) \right\| \le C_E |\theta - \theta'|^\alpha \quad \text{for any } \theta, \theta' \in [0, 2\pi).$$

Suppose moreover that $\mathrm{span}(E(\lambda); \lambda \in \mathbb{T})$ *is dense in H. Then there exist a T-invariant ergodic Gaussian measure μ with full support and two classes of functions $\mathcal{X}, \mathcal{Y}$ such that, for any $(f, g) \in \mathcal{X} \times \mathcal{Y}$,*

$$\left| \mathrm{Cov}(f \circ T^n, g) \right| \le C_{f,g} n^{-\alpha}.$$

We will not describe the two classes $\mathcal{X}$ and $\mathcal{Y}$ in the above theorem. Their definitions involve the Gaussian measure μ and depend on the Hilbertian structure of H. We just mention that a typical function in $\mathcal{X}$ or $\mathcal{Y}$ is an infinitely differentiable function whose sequence of derivatives satisfies some growth condition. For instance, the set of polynomials $\mathcal{P}$ is contained in both $\mathcal{X}$ and $\mathcal{Y}$. We recall that a function $P : X \to \mathbb{R}$ is a homogeneous polynomial of degree d provided there exists a bounded symmetric d-linear form Q on X such that $P(x) = Q(x, \ldots, x)$. A polynomial of degree d is a sum $P = P_0 + \cdots + P_d$, where each P_k is a homogeneous polynomial of degree k.

Theorem 1.3 is really specific to the Hilbert space setting. In this paper, we obtain several results in the Banach space setting regarding the decay of correlations and the validity of the central limit theorem for large classes of functions. We do not start from the Gaussian measure of Theorem 1.1; we rather use the class of measures introduced in [16]. Our theorems will have the following informative form:

Let $T \in \mathcal{L}(X)$ satisfying a strong form of the Hypercyclicity Criterion. Then there exist a T-invariant strongly mixing Borel probability measure μ on X and a "large" subset E of $L^2(\mu)$ such that

- for any $f, g \in E$, the sequence of covariances $(\mathrm{Cov}(f \circ T^n, g))$ converges quickly to zero;
- for any $f \in E$, $(f + \cdots + f \circ T^{n-1})/\sqrt{n}$ converges in distribution to a Gaussian random variable.

Of course, precise statements will be given in Section 2, after we define our large subsets E.

Notations. *Throughout the paper, the letter C will denote an absolute constant whose value may change from line to line. If a constant depends on some parameter x, then we shall denote it by C_x.*

2. Central limit theorems – Results and examples

2.1. *The spaces of functions*

We shall first define the spaces of functions f such that the sequence $(f \circ T^n)$ will satisfy a central limit theorem. There are many differences with the classical cases due to the noncompactness of X. We will require that f is infinitely differentiable; this is stronger than in many situations where a central limit theorem for dynamical systems has been proved. On the contrary, we do not want to restrict ourselves to bounded functions. We expect to apply our results to linear forms for instance.

Let $\omega : \mathbb{N} \to (1, +\infty)$ which goes to infinity. We define E_ω as the set of functions $f : X \to \mathbb{R}$ which are infinitely differentiable at 0, which may be written, for any $x \in X$,

$$f(x) = \sum_{\kappa=0}^{+\infty} \frac{D^\kappa f(0)}{\kappa!} (x, \ldots, x),$$

and whose sequence of derivatives satisfies

$$\|f\|_\omega := \sup_{\kappa \geq 0} \| D^\kappa f(0) \| \omega(\kappa)^\kappa < +\infty.$$

Endowed with the norm $\| \cdot \|_\omega$, E_ω is a Banach space. Clearly, each E_ω contains the polynomials. When ω is well chosen, it also contains other natural classes of functions, as the following proposition indicates.

Proposition 2.1. *There exists a function $\omega : \mathbb{N} \to (1, +\infty)$ tending to infinity such that, for any polynomial $P \in \mathcal{P}$, for any function $\phi : \mathbb{R} \to \mathbb{R}$ which can be written $\phi(x) = \sum_{n \geq 0} \frac{a_n}{(n!)^\sigma} x^n$ with $|a_n| \leq A\tau^n$ for some constants $A, \tau > 0$ and $\sigma > \deg(P)$, $\phi \circ P$ belongs to E_ω.*

Proof. We write $P = P_0 + \cdots + P_d$, where each P_k is homogeneous with degree k. We may assume $\deg(P) > 0$. Let $B > 0$ be such that $|P_k(x)| \leq B\|x\|^k$, for any $k = 0, \ldots, d$. We develop $\phi \circ P$ into

$$\phi \circ P = \sum_{k=0}^{+\infty} \frac{a_k}{(k!)^\sigma} \sum_{j_0 + \cdots + j_d = k} P_0^{j_0} \cdots P_d^{j_d}$$

$$= \sum_{j_0, \ldots, j_d = 0}^{+\infty} P_0^{j_0} \cdots P_d^{j_d} \frac{a_{j_0 + \cdots + j_d}}{[(j_0 + \cdots + j_d)!]^\sigma}$$

$$= \sum_{l=0}^{+\infty} \sum_{\substack{j_0 \geq 0 \\ j_1 + \cdots + d j_d = l}} P_0^{j_0} \cdots P_d^{j_d} \frac{a_{j_0 + \cdots + j_d}}{[(j_0 + \cdots + j_d)!]^\sigma}$$

$$=: \sum_{l=0}^{+\infty} Q_l.$$

Each Q_l is a homogeneous polynomial with degree l. We are looking for a function ω such that the sequence $(\|Q_k\|k!\omega(k)^k)$ is bounded. We observe that

$$\|Q_k\| \leq A \sum_{\substack{j_0 \geq 0 \\ j_1+\cdots+dj_d=k}} \frac{(B\tau)^{j_0+\cdots+j_d}}{((j_0+\cdots+j_d)!)^\sigma}$$

$$\leq A \sum_{j_0 \geq 0}(B\tau)^{j_0} \sum_{j_1+\cdots+dj_d=k} \frac{(B\tau)^{j_1+\cdots+j_d}}{((j_0+\cdots+j_d)!)^\sigma}$$

$$\leq A \sum_{j_0 \geq 0}(B\tau)^{j_0} \sum_{j_1+\cdots+dj_d=k} \frac{(B\tau)^k}{(([k/d]+j_0)!)^\sigma}$$

(we assume, expanding B if necessary, that $B\tau \geq 1$). Now, the number of solutions of the equation $j_1+\cdots+dj_d=k$, $j_i \geq 0$, is less than k^d. We deduce the existence of some positive constant τ_1 (depending on d) such that

$$\|Q_k\| \leq A\tau_1^k \sum_{j_0 \geq 0} \frac{(B\tau)^{j_0}}{(([k/d]+j_0)!)^\sigma}.$$

Now, for k sufficiently large and for any $j_0 \geq 0$,

$$\frac{(B\tau)^{j_0}}{(([k/d]+j_0)!)^\sigma} \leq \left(\frac{1}{2}\right)^{j_0} \times \frac{1}{([k/d]!)^\sigma}.$$

This yields

$$\|Q_k\|k! \leq A'\frac{\tau_1^k k!}{([k/d]!)^\sigma}.$$

We now set $\omega(k) = \log(k+\mathrm{e})$ (which does not depend on anything!) and observe that, by Stirling's formula, the sequence $(\|Q_k\|k!\omega(k)^k)$ is bounded. □

2.2. *Statement of the results and examples*

Having described the spaces of functions where we expect that a central limit theorem holds, we can now give our main statements. We begin with an abstract result.

Theorem 2.2. *Let $T \in \mathfrak{L}(X)$ and let $\omega:\mathbb{N} \to (1, +\infty)$ going to $+\infty$. Suppose that, for any function $\omega_0:[1, +\infty) \to (1, +\infty)$ going to infinity, one can find a sequence $\mathcal{D} = (x_n)_{n \geq 1} \subset X$, a sequence $(\varepsilon_k) \subset \ell^1(\mathbb{Z})$ and a sequence of maps $S_n:\mathcal{D} \to X$, $n \geq 0$, such that*

  (i) *for any $x \in \mathcal{D}$, $\sum_{n \geq 0} T^n x$ converges unconditionally;*
 (ii) *for any $x \in \mathcal{D}$, $\sum_{n \geq 0} S_n x$ converges unconditionally;*
(iii) *for any $x \in \mathcal{D}$, $T^n S_n x = x$ for any $n \geq 0$ and $T^m S_n x = S_{n-m}x$ for any $n > m$;*
(iv) *for any sequence $(n_k) \subset \mathbb{N}^\mathbb{Z}$ such that $\sum_{k \geq 0} T^k x_{n_k}$ and $\sum_{k < 0} S_{-k}x_{n_k}$ are convergent,*

$$\left\|\sum_{k \geq 0} T^k x_{n_k}\right\| + \left\|\sum_{k < 0} S_{-k}x_{n_k}\right\| \leq \sum_{k \in \mathbb{Z}} \omega_0(n_k)\varepsilon_k;$$

  (v) *$\{\sum_{k \in F} T^k x_{n_k} + \sum_{k \in G} S_k x_{n_k}; F, G \subset \mathbb{Z}_+ \text{ finite}, (n_k) \subset \mathbb{N}^{F \cup G}\}$ is dense in X.*

Then there exists a T-invariant strongly mixing Borel probability measure μ on X with full support such that $E_\omega \subset L^2(X, \mathcal{B}, \mu)$.

Suppose moreover that there exists $\alpha > 1/2$ such that, for any $x \in \mathcal{D}$, for any $n, k \geq 0$,

$$\|T^k x_n\| \leq \frac{\omega_0(n)}{(1+k)^\alpha} \quad and \quad \|S_k x_n\| \leq \frac{\omega_0(n)}{(1+k)^\alpha}.$$

Then, for any $f, g \in E_\omega$,

$$\left|\mathrm{Cov}(f \circ T^n, g)\right| \leq C_{f,g,\alpha} \begin{cases} n^{1-2\alpha} & \text{if } \alpha \in (1/2, 1), \\ \frac{\log(n+1)}{n} & \text{if } \alpha = 1, \\ n^{-\alpha} & \text{if } \alpha > 1. \end{cases}$$

If $\alpha > 1$, then for any $f \in E_\omega$ with zero mean, the sequence $\frac{1}{\sqrt{n}}(f + \cdots + f \circ T^{n-1})$ converges in distribution to a Gaussian random variable of zero mean and finite variance.

Of course, this statement does not look very appealing and we shall not prove it immediately. We prefer to give two more readable corollaries. The first one deals with stronger forms of unconditionality.

Theorem 2.3. *Let $T \in \mathfrak{L}(X)$ and let $\omega : \mathbb{N} \to (1, +\infty)$ going to $+\infty$. Suppose that there exist $\alpha > 1$, a dense set $\mathcal{D} \subset X$ and a sequence of maps $S_n : \mathcal{D} \to X$, $n \geq 0$, such that, for any $x \in \mathcal{D}$,*

(i) *$T^n S_n x = x$ for any $n \geq 0$ and $T^m S_n x = S_{n-m} x$ for any $n > m$;*
(ii) *$\|T^n x\| = O(n^{-\alpha})$ and $\|S_n x\| = O(n^{-\alpha})$.*

Then there exists a T-invariant strongly mixing Borel probability measure μ on X with full support with $E_\omega \subset L^2(X, \mathcal{B}, \mu)$ and such that for any $f, g \in E_\omega$,

$$\left|\mathrm{Cov}(f \circ T^n, g)\right| \leq C_{f,g,\alpha} n^{-\alpha}.$$

Moreover, for any $f \in E_\omega$ with zero mean, the sequence $\frac{1}{\sqrt{n}}(f + \cdots + f \circ T^{n-1})$ converges in distribution to a Gaussian random variable of zero mean and finite variance.

Proof. First, the assumptions imply the unconditional convergence of the series $\sum_n T^n x$ and $\sum_n S_n x$ for any $x \in \mathcal{D}$. Moreover, let $\omega_0 : \mathbb{N} \to (1, +\infty)$ going to $+\infty$. Let $(x_n)_{n \geq 1}$ be a dense sequence in $\mathcal{D}$ such that, for any $n \geq 1$, $k \geq 0$,

$$\|T^k x_n\| \leq \frac{\omega_0(n)}{(1+k)^\alpha} \quad and \quad \|S_k x_n\| \leq \frac{\omega_0(n)}{(1+k)^\alpha}.$$

Then for any sequence $(n_k) \subset \mathbb{N}^{\mathbb{Z}}$,

$$\left\|\sum_{k \geq 0} T^k x_{n_k}\right\| + \left\|\sum_{k < 0} S_{-k} x_{n_k}\right\| \leq \sum_{k \in \mathbb{Z}} \frac{\omega_0(n_k)}{(1+|k|)^\alpha},$$

so that the assumptions of Theorem 2.3 are satisfied with $\varepsilon_k = (1+|k|)^{-\alpha}$. $\qquad\square$

The previous theorem may be applied to many examples where we already know that conditions (i) and (ii) hold. For instance, adjoints of multipliers or composition operators associated to hyperbolic automorphisms of the disk, acting on the Hardy space $H^2(\mathbb{D})$ satisfy the assumptions of Theorem 2.3. We refer to [3], Chapter 6, for a description of these examples.

Remark 2.4. *When we work on a Hilbert space, we can compare the speed of convergence to zero of the sequence of correlations given by the measure obtained in Theorem 2.3 with that given by Devinck's theorem. Indeed, the assumptions of Theorem 2.3 imply the existence of a sequence of perfectly spanning $\mathbb{T}$-eigenvectorfields: for any $x \in \mathcal{D}$, put $E(\lambda) = \sum_{n \in \mathbb{Z}} \lambda^n T^n x$. If we assume that $\|T^n x\| = O(n^{-\alpha})$ and $\|S_n x\| = O(n^{-\alpha})$ for any $x \in \mathcal{D}$ and*

some $\alpha \in (1, 2)$, then one can show that E is $(\alpha - 1)$-Hölderian: pick any $\lambda, \mu \in \mathbb{T}$ and let $n \in \mathbb{N}$ be such that $(n + 1)^{-1} < |\lambda - \mu| \leq n^{-1}$. Then

$$\|E(\lambda) - E(\mu)\| \leq C \sum_{k=1}^{n} k|\lambda - \mu|k^{-\alpha} + C \sum_{k>n} k^{-\alpha}$$
$$\leq C|\lambda - \mu|n^{2-\alpha} + Cn^{1-\alpha}$$
$$\leq C|\lambda - \mu|^{\alpha-1}.$$

Thus, with the assumptions of Theorem 2.3*, we may also apply Devinck's theorem to obtain an ergodic measure such that, for any $f, g \in \mathcal{P}$,*

$$\left|\mathrm{Cov}\left(f \circ T^n, g\right)\right| \leq Cn^{-(\alpha-1)}.$$

This is less good than the result we obtain by applying directly Theorem 2.3*.*

  Conversely, let us assume that the assumptions of Devinck's theorem are satisfied. We set

$$\mathcal{D} = \mathrm{span}\left(\int_{\mathbb{T}} \lambda^p E(\lambda) \, \mathrm{d}\lambda; \, p \in \mathbb{Z}\right),$$

which is dense in H. We define S_n on $\mathcal{D}$ by

$$S_n\left(\int_{\mathbb{T}} \lambda^p E(\lambda) \, \mathrm{d}\lambda\right) = \int_{\mathbb{T}} \lambda^{p-n} E(\lambda) \, \mathrm{d}\lambda$$

so that $S_n T^m = S_{n-m}$ on $\mathcal{D}$, for any $n, m \geq 0$. It is not hard to show that, for any $x \in \mathcal{D}$, $\|T^n x\| = \mathrm{O}(n^{-\alpha})$ and $\|S_n x\| = \mathrm{O}(n^{-\alpha})$. Indeed,

$$T^n\left(\int_{\mathbb{T}} \lambda^p E(\lambda) \, \mathrm{d}\lambda\right) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+p)\theta} E\left(\mathrm{e}^{\mathrm{i}\theta}\right)\frac{\mathrm{d}\theta}{2\pi}$$
$$= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+p)\theta}\left(E\left(\mathrm{e}^{\mathrm{i}\theta}\right) - E\left(\mathrm{e}^{\mathrm{i}(\theta+\pi/(p+n))}\right)\right)\frac{\mathrm{d}\theta}{2\pi}$$

by a change of variables. Since E is assumed to be α-Hölderian, this easily implies that $\|T^n x\| = \mathrm{O}(n^{-\alpha})$ for any $x \in \mathcal{D}$. Theorem 2.3 *and Devinck's theorem then give the same decay of correlations. However, we are sure that we can apply Theorem* 2.3 *only if $\alpha > 1$, and we cannot apply it if $\alpha < 1/2$.*

  Thus, in the Hilbert space setting, it is not so easy to decide which measure has better mixing properties. Moreover, we do not know whether we can compare our class E_ω with the classes $\mathcal{X}$ and $\mathcal{Y}$ of Devinck. We also point out that the way we construct E from the assumptions of Theorem 2.3 *or conversely the way we construct $\mathcal{D}$ from the assumptions of Devinck's theorem are often not optimal. In concrete situations, a more natural choice of E (resp. of $\mathcal{D}$) can improve the results we get automatically in this remark. In these concrete situations, our results seem better; see the forthcoming Examples* 2.6 *and* 2.7*.*

2.3. Bilateral weighted shifts

We now apply Theorem 2.2 to backward weighted shift operators on $\ell^p(\mathbb{Z}_+)$, $p \geq 1$. Let $\mathbf{w} = (w_n)_{n \in \mathbb{Z}_+}$ be a bounded sequence in $\mathbb{R}_+$. The bilateral weighted shift $B_{\mathbf{w}}$ on $\ell^p(\mathbb{Z}_+)$ is defined by $B_{\mathbf{w}}((x_n)) = (w_{n+1}x_{n+1})$. We know by [5] that there exists a $B_{\mathbf{w}}$-invariant ergodic Borel probability measure on $\ell^p(\mathbb{Z}_+)$ with full support iff $\sum_{n\geq 1}(w_1 \cdots w_n)^{-p} < +\infty$.

Theorem 2.5. *Let $\omega: \mathbb{N} \to (1, +\infty)$ going to $+\infty$ and let $B_{\mathbf{w}}$ be a bounded backward weighted shift on $\ell^p(\mathbb{Z}_+)$. Suppose moreover that $\sum_{n\geq 1}(w_1 \cdots w_n)^{-p} < +\infty$. Then there exists a $B_{\mathbf{w}}$-invariant strongly mixing Borel probability measure on $\ell^p(\mathbb{Z}_+)$ with full support such that $E_\omega \subset L^2(X, \mathcal{B}, \mu)$.*

Suppose moreover that there exists $C > 0$ and $\alpha > 1/2$ such that, for any $n \geq 1$,

$$w_1 \cdots w_n \geq C n^\alpha.$$

Then for any $f, g \in E_\omega$,

$$\left| \mathrm{Cov}\left(f \circ B_\mathbf{w}^n, g \right) \right| \leq C_{f,g,\alpha} \begin{cases} n^{1-2\alpha} & \text{if } \alpha \in (1/2, 1), \\ \frac{\log(n+1)}{n} & \text{if } \alpha = 1, \\ n^{-\alpha} & \text{if } \alpha > 1. \end{cases}$$

If $\alpha > 1$, then for any $f \in E_\omega$ with zero mean, the sequence $\frac{1}{\sqrt{n}}(f + \cdots + f \circ B_\mathbf{w}^{n-1})$ converges in distribution to a Gaussian random variable of zero mean and finite variance.

Proof. Let $\omega_0 : \mathbb{N} \to (1, +\infty)$ going to infinity. Let $(\alpha_n)_{n \geq 1}$ be a dense sequence in $\mathbb{K}$ with $|\alpha_n^p| \leq \omega_0(n)$. We set $\mathcal{D} = (x_n)_{n \geq 1}$, with $x_n = \alpha_n e_0$, where $(e_l)_{l \geq 0}$ is the standard basis of $\ell^p(\mathbb{Z}_+)$. We define S_n on $\mathcal{D}$ by $S_n(e_0) = \frac{1}{w_1 \cdots w_n} e_n$. The unconditional convergence of $\sum_n S_n x$ for any $x \in \mathcal{D}$ follows from the convergence of $\sum_n (w_1 \cdots w_n)^{-p}$ whereas $B_\mathbf{w}^n x = 0$ for any $x \in \mathcal{D}$ and any $n \geq 1$! Moreover, since (α_n) is dense in $\mathbb{K}$, the set of the finite sums $\sum_{k \in F} S_k x_{n_k}$ is dense in X. As observed above, $\sum_{k \geq 0} B_\mathbf{w}^k x_{n_k} = \alpha_{n_0} e_0$. Regarding $\sum_{k < 0} S_{-k} x_{n_k}$, we just write

$$\left\| \sum_{k < 0} S_{-k} x_{n_k} \right\| = \left\| \sum_{k < 0} \alpha_{n_k} e_{-k} \right\|$$

$$= \left(\sum_{k < 0} \frac{|\alpha_{n_k}|^p}{(w_1 \cdots w_{-k})^p} \right)^{1/p}$$

$$\leq \left(\sum_{k < 0} \frac{\omega_0(n_k)}{(w_1 \cdots w_{-k})^p} \right)^{1/p}.$$

Since $\frac{\omega_0(n_{-1})}{w_1} \geq \frac{1}{w_1}$ and $\frac{1}{p} \leq 1$, this in turn yields

$$\left\| \sum_{k < 0} S_{-k} x_{n_k} \right\| \leq C_{\mathbf{w}, p} \sum_{k < 0} \frac{\omega_0(n_k)}{(w_1 \cdots w_{-k})^p}.$$

Thus, the assumptions of Theorem 2.2 are satisfied. Moreover, when $w_1 \cdots w_n \geq C n^\alpha$, it is easy to check that, for any $x \in \mathcal{D}$, $\|T^n x\| = \mathrm{O}(n^{-\alpha})$ and $\|S_n x\| = \mathrm{O}(n^{-\alpha})$, allowing to apply the results regarding the sequence of correlations and the central limit theorem. $\qquad \square$

Example 2.6. *Let us now apply Theorem 2.5 to backward shifts on $\ell^2(\mathbb{Z}_+)$. Let $\alpha \in (0, 1)$ and let $B_\mathbf{w}$ be the weighted backward shift on $\ell^2(\mathbb{Z}_+)$ with weight sequence $w_n = (\frac{n}{n-1})^{\alpha+1/2}$, $n \geq 2$, $w_1 = 1$. An associated perfectly spanning $\mathbb{T}$-eigenvectorfield is*

$$E(\lambda) = \sum_{n \geq 0} \frac{\lambda^n}{w_1 \cdots w_n} e_n,$$

where (e_n) is the standard basis of $\ell^2(\mathbb{Z}_+)$. E is α-Hölder (see [8]). Thus we know that there exists an ergodic Gaussian measure μ on $\ell^2(\mathbb{Z}_+)$ such that, for any $f, g \in \mathcal{P}$,

$$\left| \mathrm{Cov}\left(f \circ B_\mathbf{w}^n, g \right) \right| \leq C_{f,g,\alpha} n^{-\alpha}.$$

On the other hand, we may also apply Theorem 2.5 *to get an ergodic measure* μ *such that, for any* $f, g \in \mathcal{P}$,

$$\left|\mathrm{Cov}\left(f \circ B_{\mathbf{w}}^n, g\right)\right| \leq C_{f,g,\alpha} \begin{cases} n^{-2\alpha} & \text{if } \alpha \in (0, 1/2), \\ \frac{\log(n+1)}{n} & \text{if } \alpha = 1/2, \\ n^{-\alpha-1/2} & \text{if } \alpha > 1/2. \end{cases}$$

In all cases, the speed of mixing is better.

2.4. *Composition operators associated to parabolic automorphisms*

The unpleasant condition (iv) in Theorem 2.2 is useful to handle operators such that $\sum_n T^n x$ and $\sum_n S_n x$ converge unconditionally whereas $\sum_n \|T^n x\| = +\infty$ or $\sum_n \|S_n x\| = +\infty$. This was the case for backward shift operators. Another example is given by composition operators induced by a parabolic automorphism of the disk.

Let ϕ be a parabolic automorphism of the disk, namely ϕ is an automorphism with a unique boundary fixed point. The composition operator C_ϕ defined by $C_\phi(f) = f \circ \phi$ is bounded on the Hardy space $H^2(\mathbb{D})$. Since, for any countable set $D \subset \mathbb{T}$, the linear span of $\bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(C_\phi - \lambda)$ is dense in $H^2(\mathbb{D})$ (see [2]), there exists a C_ϕ-invariant strongly mixing Gaussian measure μ on $H^2(\mathbb{D})$ with full support. Moreover, it is shown in [2], Example 3.9, that C_ϕ admits a $\mathbb{T}$-eigenvector field E which is α-Hölder, for any $\alpha < 1/2$. Devinck's result ensures that, for any $f \in \mathcal{P}$, $\mathrm{Cov}(f \circ T^n, f) = \mathrm{O}(n^{-\alpha})$, $\alpha < 1/2$.

As for backward shifts, we can go further.

Example 2.7. *Let* ϕ *be a parabolic automorphism of the disk, let* $\alpha < 1$ *and let* $\omega : \mathbb{N} \to (1, +\infty)$ *going to infinity. There exists a* C_ϕ-*invariant strongly mixing Borel probability measure* μ *on* $H^2(\mathbb{D})$ *with full support such that* $E_\omega \subset L^2(X, \mathcal{B}, \mu)$ *and, for any* $f, g \in E_\omega$,

$$\mathrm{Cov}\left(f \circ T^n, g\right) = \mathrm{O}\left(n^{1-2\alpha}\right).$$

Proof. Without loss of generality, we may assume $\phi(1) = 1$. It is easier to work with the upper half-plane $\mathbb{P}_+ = \{s \in \mathbb{C}; \Im m(s) > 0\}$ which is biholomorphic to $\mathbb{D}$ via the Cayley map $\sigma(z) = \mathrm{i}(1+z)/(1-z)$. We set $\mathcal{H}^2 := \{f \circ \sigma^{-1}; f \in H^2(\mathbb{D})\}$. The norm on $\mathcal{H}^2$ is given by

$$\|F\|_2^2 = \pi^{-1} \int_{\mathbb{R}} |F(t)|^2 \frac{\mathrm{d}t}{1+t^2}.$$

Moreover, C_ϕ, acting on $H^2(\mathbb{D})$, is similar, via the Cayley map, to a translation operator $\tau_a(F) = F(\cdot + a)$ acting on $\mathcal{H}^2$, $a \in \mathbb{R}^*$. We shall assume $a = 1$.

Let $p \geq 1$ and let $\mathcal{D}$ be the set of all holomorphic polynomials satisfying $P(1) = P'(1) = \cdots = P^{(2p)}(1) = 0$. $\mathcal{D}$ is dense in $H^2(\mathbb{D})$ and any $P \in \mathcal{D}$ satisfies $|P(z)| \leq C_P |z - 1|^{2p}$. Hence $Q = P \circ \sigma^{-1} \in \mathcal{H}^2$ satisfies $|Q(x)| \leq \frac{C_Q}{(1+x^2)^p}$ for any $x \in \mathbb{R}$.

Let now $\omega_0 : \mathbb{N} \to (1, +\infty)$ and let $\theta = \sqrt{\omega_0}$. Let finally $(Q_n)_{n \geq 1}$ be a dense sequence in $\mathcal{H}^2$ such that, for any $n \geq 1$ and any $x \in \mathbb{R}$,

$$\left|Q(x)\right| \leq \frac{\theta(n)}{(1+x^2)^p}.$$

We claim that the assumptions of Theorem 2.2 are satisfied with $T = \tau_1$, $S_n = \tau_{-n}$ and $x_n = Q_n$. The first three points can be found e.g. in [3], Theorem 6.14. Let us prove (iv). Let $(n_k) \subset \mathbb{N}^{\mathbb{Z}}$ and let us majorize

$$\left\|\sum_{k \geq 0} \tau_k Q_{n_k}\right\|_2^2 \leq \int_{\mathbb{R}} \left(\sum_{k \geq 0} \frac{\theta(n_k)}{(1+(t+k)^2)^p}\right)^2 \frac{\mathrm{d}t}{1+t^2}$$

$$\leq 2 \sum_{k \geq j \geq 1} \int_{\mathbb{R}} \frac{\theta(n_k)\theta(n_j)}{(1+(t+k)^2)^p(1+(t+j)^2)^p(1+t^2)} \, \mathrm{d}t.$$

For $k, j \geq 1$, we say that $k \sim j$ provided $[-k - k^{1/4}, -k + k^{1/4}] \cap [-j - j^{1/4}, -j + j^{1/4}] \neq \varnothing$. It is important to notice that

$$k \sim j \quad \Longrightarrow \quad C^{-1} k \leq j \leq Ck. \tag{1}$$

Let us consider $k \geq j \geq 1$ and let us first assume that $k \not\sim j$. We split the integral over $\mathbb{R}$ into three integrals: over $[-k - k^{1/4}, -k + k^{1/4}]$, over $[-j - j^{1/4}, -j + j^{1/4}]$ and outside the two previous intervals. On $[-k - k^{1/4}, -k + k^{1/4}]$, we know that $(t + j)^2 \geq j^{1/2}$ (since $j \not\sim k$), so that

$$\int_{-k-k^{1/4}}^{-k+k^{1/4}} \frac{dt}{(1 + (t + k)^2)^p (1 + (t + j)^2)^p (1 + t^2)} \leq \frac{Ck^{1/4}}{1 \times j^{p/2} \times k^2} \leq \frac{C}{k^{7/4} j^{7/4}}$$

provided $p \geq 7/2$. A similar estimation holds true on $[-j - j^{1/4}, -j + j^{1/4}]$. On the remaining part of $\mathbb{R}$, we have both $(t + j)^2 \geq j^{1/2}$ and $(t + k)^2 \geq k^{1/2}$. Since $\int_{\mathbb{R}} dt/(1 + t^2) < +\infty$, we finally get, provided $k \not\sim j$,

$$\int_{\mathbb{R}} \frac{dt}{(1 + (t + k)^2)^p (1 + (t + j)^2)^p (1 + t^2)} \leq \frac{C}{k^{7/4} j^{7/4}}.$$

Suppose now $k \sim j$. We split the integral over $\mathbb{R}$ into two integrals: over $[-k - k^{1/4}, -j + k^{1/4}]$ (recall that $k \geq j$) and outside this interval. Outside the interval, the estimation of the integral is very similar. Inside the interval, we cannot control both $(t + k)^2$ and $(t + j)^2$, whereas $(1 + t^2) \geq Ck^2$. Observe also that the length of the interval is controlled by $2k^{1/4}$. This leads to

$$\int_{\mathbb{R}} \frac{dt}{(1 + (t + k)^2)^p (1 + (t + j)^2)^p (1 + t^2)} \leq \frac{C}{k^{7/4}}.$$

Summing over all possible values of k and j and using the previous estimates, we find

$$\left\| \sum_{k \geq 1} \tau_k Q_{n_k} \right\|_2^2 \leq C \sum_{k, j \geq 1} \frac{\theta(n_k) \theta(n_j)}{k^{7/4} j^{7/4}} + C \sum_{\substack{k \geq j \geq 1 \\ k \sim j}} \frac{\theta(n_k) \theta(n_j)}{k^{7/4}}$$

$$\leq C \left(\sum_{k \geq 1} \frac{\theta(n_k)}{k^{7/4}} \right)^2 + C \sum_{\substack{k, j \geq 1 \\ k \sim j}} \frac{\theta(n_k)^2}{k^{7/4}} + C \sum_{\substack{k, j \geq 1 \\ k \sim j}} \frac{\theta(n_j)^2}{j^{7/4}},$$

where the last inequality follows from (1). We then observe that, for a fixed $k \geq 1$, there is at most $Ck^{1/4}$ integers j with $j \sim k$. Hence,

$$\left\| \sum_{k \geq 1} \tau_k Q_{n_k} \right\|_2^2 \leq C \left(\sum_{k \geq 1} \frac{\theta(n_k)}{k^{7/4}} \right)^2 + C \sum_{k \geq 1} \frac{\theta(n_k)^2}{k^{3/2}}.$$

We take the square-root and observe that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, $\sqrt{c} \leq c$ if $c \geq 1$. This gives

$$\left\| \sum_{k \geq 1} \tau_k Q_{n_k} \right\|_2 \leq C \sum_{k \geq 1} \frac{\theta(n_k)^2}{k^{3/2}},$$

which proves (iv) with $\varepsilon_k = k^{-3/2}$. We conclude the proof by showing that, for any $k, n \geq 1$,

$$\| \tau_k Q_n \|_2 \leq \frac{C \omega_0(n)}{k^\alpha}.$$

Indeed,

$$\|\tau_k Q_n\|_2^2 \leq \int_{\mathbb{R}} \frac{\theta(n)^2}{(1 + (t+k)^2)^p} \frac{\mathrm{d}t}{1+t^2}.$$

We argue as above. Let $\varepsilon \in (0, 1)$. We split the integral over $\mathbb{R}$ into the integral over $[-k - k^\varepsilon, -k + k^\varepsilon]$ and the integral over $\mathbb{R} \setminus [-k - k^\varepsilon, -k + k^\varepsilon]$. This yields

$$\|\tau_k Q_n\|_2^2 \leq C \frac{\theta(n)^2}{k^{2-\varepsilon}} + C \frac{\theta(n)^2}{k^{2\varepsilon p}},$$

so that

$$\|\tau_k Q_n\|_2 \leq C \omega_0(n) \left(\frac{1}{k^{1-\varepsilon/2}} + \frac{1}{k^{\varepsilon p}} \right) \leq \frac{C \omega_0(n)}{k^\alpha},$$

provided $1 - \varepsilon/2 \geq \alpha$ and $\varepsilon p \geq \alpha$. $\qquad\square$

Remark 2.8. *It is also possible to get a Central Limit Theorem for linear functionals in the context of parabolic composition operators. See Section* 4.

3. Central limit theorems – The proofs

This section is devoted to the proof of Theorem 2.2.

3.1. *How to prove a central limit theorem*

A central question in this paper is to find ways to prove central limit theorems for linear dynamical systems. This has been already studied in the general context of ergodic theory. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $T : \Omega \to \Omega$ be a bijective bimeasurable transformation preserving the measure μ. We also assume that T is ergodic. Let $f \in L^2(\Omega)$, then $(f \circ T^i)_{i \in \mathbb{Z}}$ is a stationary process. We set $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$ and we say that f satisfies the Central Limit Theorem (in short, CLT) if $\frac{1}{\sqrt{n}} S_n(f)$ converges in distribution to a normal law.

To obtain sufficient conditions on a function f so that the CLT holds, we shall use the *martingale method* which was successfully used recently in various problems (see for instance [6,9,10,14]). This method goes back to Gordin in [13]. The basic idea is to try to approximate a given stationary sequence $f \circ T^n$ by a sequence which is a martingale difference sequence and to deduce the CLT for the given stationary sequence from the result for the martingale. An efficient sufficient condition was obtained by Maxwell and Woodroofe in [17]. Let $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ be a filtration with $\mathcal{F}_i \subset T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1}$. Let $\mathcal{F}_\infty$ be the smallest σ-algebra containing all the $\mathcal{F}_i$ and let $\mathcal{F}_{-\infty} = \bigcap_{i \in \mathbb{Z}} \mathcal{F}_i$. Maxwell and Woodroofe proved that, if $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ is $\mathcal{F}_0$-measurable and

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}(S_n(f)|\mathcal{F}_0)\|_2}{n^{3/2}} < +\infty,$$

then there exists a martingale difference sequence $(m \circ T^i)$ adapted to the filtration $(\mathcal{F}_i)$ such that $\|S_n(f - m \circ T^n)\|_2 = \mathrm{o}(\sqrt{n})$. In particular, f satisfies the CLT.

The monotone filtration $(\mathcal{F}_i)$ must be chosen in accordance with the transformation T; in a given concrete system the construction of this filtration is difficult. In particular, the $\mathcal{F}_0$-measurability of f (meaning that $(\mathcal{F}_i)$ is adapted to the sequence $(f \circ T^i)$) is a too restrictive condition for the applications we have in mind. We will need a nonadapted version of the theorem of Maxwell and Woodroofe. This was done by Volný in [22].

Theorem A. *Let $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ be a filtration with $\mathcal{F}_i \subset T^{-1}(\mathcal{F}_i) = \mathcal{F}_{i+1}$. Let $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ satisfying*

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}(S_n(f)|\mathcal{F}_0)\|_2}{n^{3/2}} < +\infty \quad and \quad \sum_{n=1}^{+\infty} \frac{\|S_n(f) - \mathbb{E}(S_n(f)|\mathcal{F}_n)\|_2}{n^{3/2}} < +\infty.$$

Then f satisfies the CLT.

It should be observed that Theorem A do not exclude that $S_n(f)/\sqrt{n}$ converges to a degenerate normal law, namely to a Dirac mass. If we put stronger assumptions on f (like the convergence of $\sum n|\mathrm{Cov}(f, f \circ T^n)|$, see [15]), then this happens iff f is a coboundary, namely $f = g - g \circ T$ for some $g \in L^2$.

3.2. *The measure on X*

From now on, we fix $\omega : [1, +\infty) \to (1, +\infty)$ going to infinity. Without loss of generality, we may assume that ω is nondecreasing. We consider $\omega_1 : [1, +\infty) \to (1, +\infty)$ tending to infinity and nondecreasing such that

$$\begin{cases} \omega_1(k)^2 \underset{k \to +\infty}{=} \mathrm{o}(\omega(k)), \\ \omega_1(k+1) \leq 2\omega_1(k). \end{cases}$$

We then fix another function $\omega_0 : [1, +\infty) \to (1, +\infty)$ going to $+\infty$, nondecreasing, and such that

$$\forall k, k' \geq 1, \quad \omega_0\big(k + k'\big)^{k+k'} \leq \omega_1(k)^k \omega_1\big(k'\big)^{k'}. \tag{2}$$

For instance, we can set

$$\omega_0(k) = \sqrt{\omega_1\left(\frac{k}{2}\right)}.$$

We claim that ω_0 satisfies (2). Indeed, since ω_1 is nondecreasing,

$$\forall k, k' \geq 1, \quad \omega_1(k)^k \omega_1\big(k'\big)^{k'} \geq \omega_1\left(\frac{k + k'}{2}\right)^{(k+k')/2}.$$

The construction of the measure μ which appears in Theorem 2.2 follows [16]. However, we will need to be more careful during the construction because we want additional properties. For convenience, throughout this section, for $k \in \mathbb{N}$, we shall denote by $T^{-k}x$ the vector $S_k x$, $x \in \mathcal{D}$.

The idea of Murillo-Arcila and Peris is to conjugate T to a Bernoulli shift acting on $\mathbb{N}^{\mathbb{Z}}$ and to transfer the ergodic properties of this shift to T. We start from the sequence (x_n) satisfying the assumptions of Theorem 2.2. We may assume that $x_1 = 0$ and that $S_n x_1 = 0$ for all $n \geq 0$. Let (N_n) be an increasing sequence of positive integers with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ and satisfying, for any $n \in \mathbb{N}$,

$$\left\| \sum_{k > N_n} T^k x_{m_k} \right\| + \left\| \sum_{k < -N_n} T^k x_{m_k} \right\| \leq \frac{1}{2^n} \quad \text{if } m_k \leq l, \text{ for } N_l < |k| \leq N_{l+1}, l \geq n.$$

We define $K = \prod_{k \in \mathbb{Z}} F_k$ where

$$F_k = \{1, \ldots, m\} \quad \text{if } N_m < |k| \leq N_{m+1} \quad \text{and} \quad F_k = \{1\} \quad \text{if } |k| \leq N_1.$$

Let $K(s) := \sigma^{-s}(K)$, $s \in \mathbb{Z}$, where $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ is the forward shift. The intertwining map ϕ is defined on $Z = \bigcup_{s \in \mathbb{Z}} K(s)$ by

$$\phi\big((n_k)\big) = \sum_{k \in \mathbb{Z}} T^k x_{n_k}.$$

ϕ is well defined and continuous, and it satisfies on Z the intertwining relation

$$T \circ \phi = \phi \circ \sigma. \tag{3}$$

Let us now construct on $\mathbb{N}^{\mathbb{Z}}$ a measure $\bar{\mu}$ which is invariant for σ and such that $\bar{\mu}(Z) = 1$. We fix a sequence (p_l) of positive real numbers satisfying $\sum_l p_l = 1$ and such that, setting

$$\beta_l = \left(\sum_{j=1}^{l} p_j\right)^{(N_{l+1} - N_l)} > 0,$$

then $\prod_{l \geq 1} \beta_l^2 > 0$. This condition is satisfied provided (p_l) converges sufficiently fast to zero. Then define $\bar{\mu}_k$ on $\mathbb{N}$ by $\bar{\mu}_k(\{n\}) = p_n$ and $\bar{\mu}$ on $\mathbb{N}^{\mathbb{Z}}$ as $\bar{\mu} = \bigotimes_{k \in \mathbb{Z}} \bar{\mu}_k$. It is shown in [16] that $\bar{\mu}$ is a σ-invariant strongly mixing Borel probability measure on $\mathbb{N}^{\mathbb{Z}}$ satisfying $\bar{\mu}(\mathbb{Z}) = 1$. In particular, ϕ is defined almost everywhere on $\mathbb{N}^{\mathbb{Z}}$ and (3) is a.e. true.

These properties can be transfered to X by setting $\mu(A) = \bar{\mu}(\phi^{-1}(A))$, $A \in \mathcal{B}(X)$. μ is a T-invariant strongly mixing Borel probability measure on X (see [16]). We have just to prove that μ has full support (in [16], this was done under the stronger assumption that $\mathcal{D}$ is dense). Let U be a nonempty open subset of X. Let F be a finite subset of $\mathbb{Z}$ and let $(n_k) \subset \mathbb{N}^F$ be such that

$$y = \sum_{k \in F} T^k x_{n_k} \in U.$$

Let $n \in \mathbb{N}$ be such that $N_n > \max(\max F, -\min F)$ and $y + B(0, 2^{-n}) \subset U$. For $k \in [-n, n] \setminus F$, we set $n_k = 1$. Then U contains

$$\left\{ \sum_{k=-N_n}^{N_n} T^k x_{n_k} + \sum_{|k|>N_n} T^k x_{m_k}; m_k \leq l \text{ for } N_l < |k| \leq N_{l+1}, l \geq n \right\}.$$

Hence,

$$\mu(U) \geq \prod_{k=-N_n}^{N_n} \bar{\mu}_k(\{n_k\}) \prod_{l=n}^{+\infty} \left(\prod_{N_l < |k| \leq N_{l+1}} \bar{\mu}_k(\{1, \ldots, l\}) \right)$$

$$\geq \prod_{k=-N_n}^{N_n} \bar{\mu}_k(\{n_k\}) \prod_{l=n}^{+\infty} \beta_l^2 > 0.$$

To ensure stronger properties than mixing, we will need additional assumptions on the sequence (p_l). We summarize these technical assumptions now, without further comments:

$$\sum_{m>l} p_m = o(p_l), \tag{4}$$

$$\forall l \geq 1, \forall k \geq 1, \quad \sum_{m \geq l} \sqrt{p_m} \omega_0(m)^k \leq C \sqrt{p_l} \max\left(\omega_0(k)^k, \omega_0(l)^k\right), \tag{5}$$

$$\forall l \geq 1, \forall k \geq 1, \quad \sum_{m \geq l} p_m \omega_0(m)^k \leq C p_l \max\left(\omega_0(k)^k, \omega_0(l)^k\right), \tag{6}$$

where C is some absolute constant. These conditions are satisfied if we require that the sequence (p_l) decreases sufficiently fast to 0. This is clear for (4) and also for (5) if we restrict ourselves to $k \leq l$. If we now assume $k > l$, then we can ensure (5) by requiring that, for any $m \geq 1$,

$$\sqrt{p_{m+1}} \left(\omega_0(m+1)\right)^{m+1} \leq \frac{1}{2} \sqrt{p_m} \left(\omega_0(m)\right)^m.$$

Indeed, for any $l \geq 1$ and any $k > l$, one can decompose the sum into

$$\sum_{m \geq l} \sqrt{p_m} \omega_0(m)^k \leq \sum_{m=l}^{k} \sqrt{p_m} \omega_0(m)^k + \sum_{m=k+1}^{+\infty} \sqrt{p_m} \omega_0(m)^m.$$

The first sum is bounded by $(\sum_{m \geq l} \sqrt{p_m}) \omega_0(k)^k$. The second sum is estimated as follows:

$$\sum_{m>k} \sqrt{p_m} \omega_0(m)^m \leq \sqrt{p_k} \omega_0(k)^k \left(\frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$\leq \sqrt{p_l} \omega_0(k)^k.$$

That we may also ensure (6) follows along the same lines.

We now show that our class of functions E_ω is contained in $L^2(X, \mathcal{B}, \mu)$.

Lemma 3.1. (a) *For any $d \geq 1$, $\| \cdot \|^d \in L^2(X, \mathcal{B}, \mu)$;*
(b) $E_\omega \subset L^2(X, \mathcal{B}, \mu)$.

Proof. Let $d \geq 1$. The construction of the measure $\bar\mu$ ensures that, for almost every $(n_k) \subset \mathbb{N}^{\mathbb{Z}}$, the series $\sum_{k \in \mathbb{Z}} T^k x_{n_k}$ is convergent. By condition (iv),

$$\left\| \phi\big((n_k)\big) \right\| \leq \sum_{k \in \mathbb{Z}} \varepsilon_k \omega_0(n_k).$$

We expand the product and then use Hölder's inequality to get

$$\int_X \|x\|^{2d} \, \mathrm{d}\mu(x) \leq \sum_{k_1,\dots,k_{2d} \in \mathbb{Z}} \int_{\mathbb{N}^{\mathbb{Z}}} \prod_{i=1}^{2d} \varepsilon_{k_i} \omega_0(n_{k_i}) \, \mathrm{d}\bar\mu\big((n_k)\big)$$

$$\leq \sum_{k_1,\dots,k_{2d} \in \mathbb{Z}} \prod_{i=1}^{2d} \left(\int_{\mathbb{N}^{\mathbb{Z}}} \varepsilon_{k_i}^{2d} \omega_0(n_{k_i})^{2d} \, \mathrm{d}\bar\mu\big((n_k)\big) \right)^{1/2d}.$$

Now,

$$\int_{\mathbb{N}^{\mathbb{Z}}} \varepsilon_{k_i}^{2d} \omega_0(n_{k_i})^{2d} \, \mathrm{d}\bar\mu\big((n_k)\big) \leq \varepsilon_{k_i}^{2d} \sum_{l \geq 1} p_l \omega_0(l)^{2d} \leq C \varepsilon_{k_i}^{2d} \omega_0(2d)^{2d},$$

where we have used (6). Coming back to the L^2-norm of $\| \cdot \|^d$, we finally obtain

$$\int_X \|x\|^{2d} \, \mathrm{d}\mu(x) \leq C \omega_0(2d)^{2d} \sum_{k_1,\dots,k_{2d} \in \mathbb{Z}} \prod_{i=1}^{2d} \varepsilon_{k_i}$$

$$\leq C \omega_0(2d)^{2d} \left(\sum_{k \in \mathbb{Z}} \varepsilon_k \right)^{2d}$$

$$\leq C^{2d} \omega_0(2d)^{2d}.$$

To prove (b), we start from a function $f \in E_\omega$ and we observe that

$$\left\| D^\kappa f(0)(x, \dots, x) \right\| \leq \left\| D^\kappa f(0) \right\| \times \|x\|^\kappa$$

so that, from the proof of the first point, we deduce

$$\left\| D^\kappa f(0) \right\|_{L^2} \leq C^\kappa \omega_0(\kappa)^\kappa \left\| D^\kappa f(0) \right\|.$$

Since $\sup_\kappa \| D^\kappa f(0) \| \omega(\kappa)^\kappa < +\infty$ and $\omega_0(\kappa) \underset{+\infty}{=} \mathrm{o}(\omega(\kappa))$, the series $\sum_\kappa \frac{D^\kappa f(0)(x,\dots,x)}{\kappa!}$ is convergent in $L^2(X, \mathcal{B}, \mu)$, showing that $E_\omega \subset L^2(X, \mathcal{B}, \mu)$. $\qquad \square$

Remark 3.2. *If $f : X \to \mathbb{R}$ is infinitely differentiable at 0, satisfies for any $x \in X$*

$$f(x) = \sum_{\kappa=0}^{+\infty} \frac{D^\kappa f(0)}{\kappa!}(x, \dots, x),$$

and verifies moreover that, for any $R > 0$,

$$\sup_{\kappa > 0} \| D^{\kappa} f(0) \| R^{\kappa} < +\infty,$$

then it is not hard to show that one can construct a function $\omega : \mathbb{N} \to (1, +\infty)$ going to $+\infty$ and such that $f \in E_{\omega}$. Under the assumptions of Theorem 2.2, this means that one can define a measure μ_f on X such that the sequence $(f \circ T^n)$ satisfies the central limit theorem in $L^2(X, \mathcal{B}, \mu_f)$. However, to get a single measure μ which works for a large class of functions, we have to fix ω before.

3.3. *An orthonormal basis of $L^2(X, \mathcal{B}, \mu)$*

In this section, we describe an orthonormal basis of $L^2(X, \mathcal{B}, \mu)$ or, equivalently, an orthonormal basis of $L^2(\mathbb{N}^{\mathbb{Z}}, \bar{\mu})$. There is a usual way to do so. Suppose that $(e_l)_{l \geq 0}$ is an orthonormal basis of $\ell^2(\mathbb{N}, (p_n))$ with $e_0 = 1$. Let $\mathbb{N}^{\infty} = \bigcup_{k \geq 1} \mathbb{N}^k$, $\mathbb{Z}^{\infty} = \bigcup_{k \geq 1} \mathbb{Z}^k$ and $\mathbb{Z}_{<}^{\infty} = \{\mathbf{j} = (j_1, \ldots, j_r) \in \mathbb{Z}^{\infty}; j_1 < j_2 < \cdots < j_r\}$. For $\mathbf{j} \in \mathbb{Z}^{\infty}$, denote by $|\mathbf{j}|$ the unique positive integer r such that $\mathbf{j} \in \mathbb{Z}^r$. Then, for $\mathbf{l} = (l_1, \ldots, l_r) \in \mathbb{N}^r$ and $\mathbf{j} = (j_1, \ldots, j_r) \in \mathbb{Z}^r$ with $j_1 < \cdots < j_r$, define

$$e_{\mathbf{l}, \mathbf{j}} = e_{l_1, j_1} \times \cdots \times e_{l_r, j_r},$$

where $e_{l, j}$ is a copy of e_l on the jth coordinate, namely $e_{l, j}((n_k)) = e_l(n_j)$. It is well known that $\{e_{\mathbf{l}, \mathbf{j}}; \mathbf{l} \in \mathbb{N}^{\infty}, \mathbf{j} \in \mathbb{Z}_{<}^{\infty}, |\mathbf{l}| = |\mathbf{j}|\} \cup \{1\}$ is an orthonormal basis of $L^2(\mathbb{N}^{\mathbb{Z}}, \bar{\mu})$. Thus we just need to concentrate on the choice of (e_l). We construct a triangular orthonormal basis.

Lemma 3.3. *There exists an orthonormal basis (e_l) of $\ell^2(\mathbb{N}, (p_n))$ with $e_0 = 1$ and, for $l \geq 1$,*

$$e_l(u) = \begin{cases} 0 & \text{if } u < l, \\ \dfrac{1}{\sqrt{p_l}} \times \dfrac{\sqrt{\sum_{m > l} p_m}}{\sqrt{p_l + \sum_{m > l} p_l}} & \text{if } u = l, \\ -\sqrt{p_l} \times \dfrac{1}{\sqrt{p_l(\sum_{m > l} p_m) + (\sum_{m > l} p_m)^2}} & \text{if } u > l. \end{cases}$$

Proof. Given these formula, we just need to verify that this is an orthonormal basis. We observe that $(e_l)_{l \geq 0}$ is obtained by orthonormalization of the basis $(1, 1, \ldots), (0, 1, 1, \ldots), (0, 0, 1, 1, \ldots), \ldots$ $\qquad \square$

Of course, the exact values of $e_l(u)$ are not very appealing. In the sequel, we will just need the following estimations, which are satisfied thanks to (4):

$$\begin{cases} |e_l(l)| \leq \dfrac{C}{\sqrt{p_l}} \\ |e_l(u)| \leq \dfrac{C}{\sqrt{p_{l+1}}} & \text{if } u > l. \end{cases}$$

Let us point out that this orthonormal basis behaves very well with respect to σ. Indeed, it is easy to check that

$$e_{\mathbf{l}, \mathbf{j}} \circ \sigma = e_{\mathbf{l}, \mathbf{j} - 1}. \tag{7}$$

Let us also mention the following property which is the key for our forthcoming estimations. The $\ell^2(\mathbb{N}, (p_n))$-norm of each e_l is equal to 1. However, the $\ell^1(\mathbb{N}, (p_n))$-norm of e_l goes very quickly to zero: it behaves like $\sqrt{p_l}$.

3.4. *The Fourier coefficients*

In this subsection, we control the Fourier coefficients of a homogeneous polynomial. This will be the key point to control later the behaviour of the sequence of the covariances $\text{Cov}(f \circ T^n, g)$ for $f, g \in E_{\omega}$. We first observe that many Fourier coefficients of a homogeneous polynomial are equal to zero.

Lemma 3.4. *Let* $P(x) = Q(x, \ldots, x)$ *be a homogeneous polynomial of degree* d. *Let* $\mathbf{j} \in \mathbb{Z}^\infty_<$, $\mathbf{l} \in \mathbb{N}^\infty$ *with* $|\mathbf{j}| = |\mathbf{l}| > d$. *Then*

$$\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi \rangle_{L^2(\mathbb{N}^\mathbb{Z}, \bar{\mu})} = 0.$$

Proof. Let us introduce some notations which will also be useful for the next lemma. We write $\mathbf{j} = \{j_1, \ldots, j_r\}$, $\mathbf{l} = \{l_1, \ldots, l_r\}$ and we set

$$\phi_{\mathbf{j}}((n_k)) = \phi((n_k)) - T^{j_1}(x_{n_{j_1}}) - \cdots - T^{j_r}(x_{n_{j_r}}),$$

$$\widehat{\mathbb{N}}_{\mathbf{j}} = \mathbb{N} \setminus \{j_1, \ldots, j_r\},$$

$$\bar{\nu}_{\mathbf{j}} \text{ is the projection of } \bar{\mu} \text{ onto } \widehat{\mathbb{N}}_{\mathbf{j}}.$$

Moreover, for $u_0, \ldots, u_r$ nonnegative integers with $u_0 + \cdots + u_r = d$, we set

$$Q_{u_0, \ldots, u_r}((n_k)) = Q\big(\underbrace{\phi_{\mathbf{j}}((n_k)), \ldots, \phi_{\mathbf{j}}((n_k))}_{u_0 \text{ times}}, \underbrace{T^{j_1}(x_{n_{j_1}}), \ldots, T^{j_1}(x_{n_{j_1}})}_{u_1 \text{ times}}, \ldots,$$

$$\underbrace{T^{j_r}(x_{n_{j_r}}), \ldots, T^{j_r}(x_{n_{j_r}})}_{u_r \text{ times}}\big).$$

The d-linearity of Q yields

$$\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi \rangle = \sum_{\substack{u_0 + \cdots + u_r = d \\ u_i \geq 0}} \binom{d}{u_0, \ldots, u_r} \int_{\mathbb{N}^\mathbb{Z}} e_{\mathbf{l},\mathbf{j}}((n_k)) Q_{u_0, \ldots, u_r}((n_k)) \, d\bar{\mu}((n_k)).$$

Since $r > d$, in each term of the sum, one of the $u_1, \ldots, u_r$, say u_k, is equal to zero. We use Fubini's theorem and we integrate first with respect to the j_kth coordinate. We get zero since

$$\int_{\mathbb{N}} e_{l,j}(n) \, d\bar{\mu}_j(n) = 0 \quad \text{for any } l \in \mathbb{N} \text{ and any } j \in \mathbb{Z}. \qquad \square$$

The nonzero coefficients will be estimated thanks to the following lemma.

Lemma 3.5. *Let* $P(x) = Q(x, \ldots, x)$ *be a homogeneous polynomial of degree* d. *Let* $\mathbf{j} \in \mathbb{Z}^\infty_<$, $\mathbf{l} \in \mathbb{N}^\infty$ *with* $|\mathbf{j}| = |\mathbf{l}| = r \leq d$. *Then*

$$\big|\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi \rangle_{L^2(\mathbb{N}^\mathbb{Z}, \bar{\mu})}\big| \leq \frac{C^d r^d \|Q\| \omega_0(d)^d \sup_i \omega_0(l_i)^d}{(1 + |j_1|)^\alpha \cdots (1 + |j_r|)^\alpha} \sqrt{p_{l_1} \cdots p_{l_r}}.$$

Proof. We keep the same notations so that

$$\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi \rangle = \sum_{\substack{u_0 + \cdots + u_r = d \\ u_i \geq 0}} \binom{d}{u_0, \ldots, u_r} \int_{\mathbb{N}^\mathbb{Z}} e_{\mathbf{l},\mathbf{j}}((n_k)) Q_{u_0, \ldots, u_r}((n_k)) \, d\bar{\nu}_{\mathbf{j}}((n_k)) \, d\bar{\mu}_{j_1}(n_{j_1}) \cdots d\bar{\mu}_{j_r}(n_{j_r}).$$

As in the previous lemma we have just to consider the terms in the sum such that $u_1 \geq 1, \ldots, u_r \geq 1$. Moreover,

$$\big|Q_{u_0, \ldots, u_r}((n_k))\big| \leq \|Q\| \times \big\|\phi_{\mathbf{j}}((n_k))\big\|^{u_0} \prod_{i=1}^r \big\|T^{j_i}(x_{n_{j_i}})\big\|^{u_i}.$$

We get

$$\left|\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi\rangle\right| \leq \|Q\| \sum_{\substack{u_0+\cdots+u_r=d \\ u_i \geq 1}} \binom{d}{u_0, \ldots, u_r} \int_{\widehat{\mathbb{N}_{\mathbf{j}}}} \left\|\phi_{\mathbf{j}}\big((n_k)\big)\right\|^{u_0} \, d\bar{\nu}_{\mathbf{j}}\big((n_k)\big)$$

$$\times \prod_{i=1}^{r} \int_{\mathbb{N}} \left|e_l(n)\right| \left\|T^{j_i}(x_n)\right\|^{u_i} \, d\bar{\mu}_0(n).$$

The first integral may be handled exactly like in Lemma 3.1 and we get

$$\int_{\widehat{\mathbb{N}_{\mathbf{j}}}} \left\|\phi_{\mathbf{j}}\big((n_k)\big)\right\|^{u_0} \, d\bar{\nu}_{\mathbf{j}}\big((n_k)\big) \leq C^{u_0} \omega_0(u_0)^{u_0}.$$

The estimation of the other integrals needs the properties of the sequences (p_l) and (e_l). Indeed, for $u \geq 1$, $j \in \mathbb{Z}$ and $l \in \mathbb{N}$, one can write

$$\int_{\mathbb{N}} \left|e_l(n)\right| \left\|T^j(x_n)\right\|^u \, d\bar{\mu}_0(n) = \sum_{m \geq l} p_m \left\|T^j(x_m)\right\|^u \left|e_l(m)\right|$$

$$\leq \frac{C\sqrt{p_l}}{(1+|j|)^{\alpha u}} \omega_0(l)^u + \frac{C}{\sqrt{p_{l+1}}(1+|j|)^{\alpha u}} \sum_{m \geq l+1} p_m \omega_0(m)^u.$$

We then apply (6) to get

$$\int_{\mathbb{N}} \left|e_l(n)\right| \left\|T^j(x_n)\right\|^u \, d\bar{\mu}_0(n) \leq \frac{C\sqrt{p_l}}{(1+|j|)^{\alpha u}} \omega_0(l)^u + \frac{C}{\sqrt{p_{l+1}}(1+|j|)^{\alpha u}} p_{l+1} \max\big(\omega_0(l+1)^u, \omega_0(u)^u\big)$$

$$\leq \frac{C^u \sqrt{p_l} \max(\omega_0(l)^u, \omega_0(u)^u)}{(1+|j|)^{\alpha}}$$

$$\leq \frac{C^u \omega_0(u)^u \sqrt{p_l} \omega_0(l)^u}{(1+|j|)^{\alpha}}.$$

Coming back to our original Fourier coefficient and using $u_0 + \cdots + u_r = d$, we get

$$\left|\langle e_{\mathbf{l},\mathbf{j}}, P \circ \phi\rangle\right| \leq C^d \|Q\| \omega_0(d)^d \sum_{\substack{u_0+\cdots+u_r=d \\ u_i \geq 1}} \binom{d}{u_0, \ldots, u_r} \frac{\omega_0(l_1)^{u_1} \cdots \omega_0(l_r)^{u_r}}{(1+|j_1|)^{\alpha} \cdots (1+|j_r|)^{\alpha}} \sqrt{p_{l_1} \cdots p_{l_r}}.$$

We conclude by noting that the cardinal number of $\{(u_0, \ldots, u_r) \in \mathbb{N}^{r+1}; u_0 + \cdots + u_r = d, u_1, \ldots, u_r \geq 1\}$ is less than or equal to 2^d and that the multinomial coefficient is less than or equal to $(r+1)^d$. $\qquad\square$

We give a first application of the previous lemmas. It deals with the sum of the Fourier coefficients of a given order for a function in E_ω.

Lemma 3.6. *Let* $f \in E_\omega$, $F = f \circ \phi = \sum_{\mathbf{l},\mathbf{j}} a_{\mathbf{l},\mathbf{j}} e_{\mathbf{l},\mathbf{j}}$. *Then, for any* $M > 1$, *there exists* $C_{\omega_0, \omega, M} > 0$ *such that, for any* $r \geq 1$ *and any* $\mathbf{j} \in \mathbb{Z}_<^\infty$ *with* $|\mathbf{j}| = r$,

$$\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}}| \leq \frac{C_{\omega_0, \omega, M} \|f\|_\omega}{M^r (1+|j_1|)^{\alpha} \cdots (1+|j_r|)^{\alpha}}.$$

Proof. By the above lemmas,

$$\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}}| \le \sum_{|\mathbf{l}|=r} \sum_{\kappa=0}^{+\infty} \frac{|\langle e_{\mathbf{l},\mathbf{j}}, D^\kappa f(0) \circ \phi \rangle|}{\kappa!}$$

$$\le \sum_{\kappa=r}^{+\infty} C^\kappa r^\kappa \frac{\|D^\kappa f(0)\| \omega_0(\kappa)^\kappa}{\kappa!(1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha} \sum_{|\mathbf{l}|=r} \sup_i \omega_0(l_i)^\kappa \sqrt{p_{l_1} \cdots p_{l_r}}.$$

By Stirling's formula, $\frac{r^\kappa}{\kappa!} \le C^\kappa$ for any $\kappa \ge r$, so that

$$\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}}| \le \sum_{\kappa=r}^{+\infty} \frac{C^\kappa \|D^\kappa f(0)\| \omega_0(\kappa)^\kappa}{(1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha} \sum_{|\mathbf{l}|=r} \sup_i \omega_0(l_i)^\kappa \sqrt{p_{l_1} \cdots p_{l_r}}.$$

Now, for $\kappa \ge r$,

$$\sum_{|\mathbf{l}|=r} \sup_i \omega_0(l_i)^\kappa \sqrt{p_{l_1} \cdots p_{l_r}} \le \sum_{i=1}^r \sum_{|\mathbf{l}|=r} \omega_0(l_i)^\kappa \sqrt{p_{l_1} \cdots p_{l_r}}$$

$$\le \sum_{i=1}^r \sum_{l_i=1}^{+\infty} \omega_0(l_i)^\kappa \sqrt{p_{l_i}} \prod_{\substack{j=1 \\ j \ne i}}^r \sum_{l=1}^{+\infty} \sqrt{p_l}$$

$$\le C^r \sum_{l \ge 1} \sqrt{p_l} \omega_0(l)^\kappa$$

$$\le C^\kappa \omega_0(\kappa)^\kappa,$$

where we have used (5).

To conclude, observe that, given any absolute constant C, there exists some constant $C_{\omega_0,\omega,M}$ such that, for any κ, $C^\kappa \omega_0(\kappa)^{2\kappa} \le C_{\omega_0,\omega,M} M^{-\kappa} \omega(\kappa)^\kappa$. This yields

$$\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}}| \le \frac{C_{\omega_0,\omega,M}}{(1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha} \sum_{\kappa=r}^{+\infty} \frac{\|D^\kappa f(0)\| \omega(\kappa)^\kappa}{M^\kappa}$$

$$\le \frac{C_{\omega_0,\omega,M} \|f\|_\omega}{M^r (1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha}. \qquad \square$$

3.5. *The sequence of correlations*

In this subsection, we prove the first part of Theorem 2.2, that devoted to the behaviour of the sequence of correlations. We begin with a first lemma which is an easy consequence of (7).

Lemma 3.7. *Let* $F = \sum_{\mathbf{j},\mathbf{l}} a_{\mathbf{l},\mathbf{j}} e_{\mathbf{l},\mathbf{j}}$, $G = \sum_{\mathbf{j},\mathbf{l}} b_{\mathbf{l},\mathbf{j}} e_{\mathbf{l},\mathbf{j}} \in L_0^2(\mathbb{N}^\mathbb{Z}, \bar{\mu})$. *Then*

$$\mathrm{Cov}(F \circ \sigma^p, G) = \sum_{\mathbf{j},\mathbf{l}} a_{\mathbf{l},\mathbf{j}} b_{\mathbf{l},\mathbf{j}-p}.$$

Proof. By linearity and by (7),

$$\mathrm{Cov}(F \circ \sigma^p, G) = \sum_{\mathbf{j},\mathbf{k},\mathbf{l},\mathbf{m}} a_{\mathbf{l},\mathbf{j}} b_{\mathbf{m},\mathbf{k}} \mathrm{Cov}(e_{\mathbf{l},\mathbf{j}-p}, e_{\mathbf{m},\mathbf{k}}).$$

Now, $e_{\mathbf{l},\mathbf{j}-p}$ and $e_{\mathbf{m},\mathbf{k}}$ are orthogonal, unless $\mathbf{l} = \mathbf{m}$ and $\mathbf{k} = \mathbf{j} - p$. This gives the lemma. $\qquad \square$

Let us now start with $f, g \in E_\omega$. Without loss of generality, we may assume that they have zero mean. We set $F = f \circ \phi$, $G = g \circ \phi$ so that $\mathrm{Cov}(f \circ T^p, g) = \mathrm{Cov}(F \circ \sigma^p, G)$. Let us write

$$F = \sum_{r=1}^{+\infty} \sum_{|\mathbf{l}|=|\mathbf{j}|=r} a_{\mathbf{l},\mathbf{j}} e_{\mathbf{l},\mathbf{j}} \quad \text{and} \quad G = \sum_{r=1}^{+\infty} \sum_{|\mathbf{l}|=|\mathbf{j}|=r} b_{\mathbf{l},\mathbf{j}} e_{\mathbf{l},\mathbf{j}},$$

and let us compute $\mathrm{Cov}(F \circ \sigma^p, G)$ using Lemma 3.7. We apply the estimations of the Fourier coefficients given by Lemma 3.4 and Lemma 3.5. Using Stirling's formula, we find, for a given $\mathbf{j} \in \mathbb{Z}_<^\infty$ with $|\mathbf{j}| = r$,

$$\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}} b_{\mathbf{l},\mathbf{j}-p}| \leq \sum_{|\mathbf{l}|=r} \sum_{\kappa,\kappa'=r}^{+\infty} \frac{\|D^\kappa f(0)\| \|D^{\kappa'} f(0)\| C^{\kappa+\kappa'} \omega_0(\kappa)^\kappa \omega_0(\kappa')^{\kappa'} \sup_i \omega_0(l_i)^{\kappa+\kappa'} p_{l_1} \cdots p_{l_r}}{(1+|j_1-p|)^\alpha \cdots (1+|j_r-p|)^\alpha (1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha}.$$

Now, for $\kappa, \kappa' \geq r$,

$$\begin{aligned}
\sum_{|\mathbf{l}|=r} \sup_i \omega_0(l_i)^{\kappa+\kappa'} p_{l_1} \cdots p_{l_r} &\leq \sum_{i=1}^{r} \sum_{|\mathbf{l}|=r} \omega_0(l_i)^{\kappa+\kappa'} p_{l_1} \cdots p_{l_r} \\
&\leq \sum_{i=1}^{r} \left(\sum_{l_i=1}^{+\infty} p_{l_i} \omega_0(l_i)^{\kappa+\kappa'} \right) \times \left(\sum_{l \geq 1} p_l \right)^{r-1} \\
&\leq C r \omega_0 (\kappa + \kappa')^{\kappa+\kappa'} \\
&\leq C^{\kappa+\kappa'} \omega_1(\kappa)^\kappa \omega_1(\kappa')
\end{aligned}$$

(at this stage, we use the strange relation (2) satisfied by ω_0 and also (6)). Hence,

$$\begin{aligned}
\sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}-p} b_{\mathbf{l},\mathbf{j}}| &\leq \left(\sum_{\kappa=r}^{+\infty} C^\kappa \|D^\kappa f(0)\| \omega_1(\kappa)^{2\kappa} \right) \times \left(\sum_{\kappa=r}^{+\infty} C^\kappa \|D^\kappa g(0)\| \omega_1(\kappa)^{2\kappa} \right) \\
&\quad \times \frac{1}{(1+|j_1-p|)^\alpha \cdots (1+|j_r-p|)^\alpha (1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha} \\
&\leq \frac{C \|f\|_\omega \|g\|_\omega}{(1+|j_1-p|)^\alpha \cdots (1+|j_r-p|)^\alpha (1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha}.
\end{aligned}$$

We now sum over $\mathbf{j}$ with $|\mathbf{j}| = r$:

$$\begin{aligned}
\sum_{|\mathbf{j}|=|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}-p} b_{\mathbf{l},\mathbf{j}}| &\leq \sum_{j_1 < j_2 < \cdots < j_r} \frac{C \|f\|_\omega \|g\|_\omega}{(1+|j_1-p|)^\alpha \cdots (1+|j_r-p|)^\alpha (1+|j_1|)^\alpha \cdots (1+|j_r|)^\alpha} \\
&\leq C \|f\|_\omega \|g\|_\omega \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1+|j-p|)^\alpha (1+|j|)^\alpha} \right)^r.
\end{aligned}$$

We split this last sum into three sums: $\sum_{j \leq -1}$, $\sum_{j=0}^{p}$ and $\sum_{j=p+1}^{+\infty}$ and we observe that the first sum and the last sum are equal. We first consider these sums. We get different estimates following the value of α. When $\alpha > 1$, it is easy to check that, for $j \geq p+1$,

$$\frac{1}{(1+(j-p))^\alpha} - \frac{1}{(1+j)^\alpha} = \frac{(1+j)^\alpha - (1+j-p)^\alpha}{(1+j)^\alpha(1+(j-p))^\alpha} \geq \frac{p^\alpha}{(1+j)^\alpha(1+(j-p))^\alpha}.$$

Thus,

$$\sum_{j \geq p+1} \frac{1}{(1+j)^\alpha (1+(j-p))^\alpha} \leq p^{-\alpha} \sum_{j \geq p+1} \left(\frac{1}{(1+(j-p))^\alpha} - \frac{1}{(1+j)^\alpha} \right)$$

$$\leq p^{-\alpha} \left(\frac{1}{2^\alpha} + \cdots + \frac{1}{(p+1)^\alpha} \right) \leq C_\alpha p^{-\alpha}.$$

On the other hand,

$$\sum_{j=0}^{p} \frac{1}{(1+j)^\alpha (1+(p-j))^\alpha} \leq C_\alpha p^{-\alpha} \sum_{j=0}^{[p/2]} \frac{1}{(1+j)^\alpha} \leq C_\alpha p^{-\alpha}.$$

We finally find

$$\sum_{|\mathbf{j}|=|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}-p} b_{\mathbf{l},\mathbf{j}}| \leq C_\alpha \|f\|_\omega \|g\|_\omega \left(\frac{C_\alpha}{p^\alpha} \right)^r.$$

Summing this over r, we get

$$\left| \mathrm{Cov}\left(F \circ \sigma^p, G \right) \right| \leq \frac{C_\alpha}{p^\alpha} \|f\|_\omega \|g\|_\omega$$

provided $p^\alpha \geq 2C_\alpha$.

The case $\alpha = 1$ is the easiest one. Indeed, in that case, for $j \geq p+1$ one can write

$$\frac{1}{1+(j-p)} - \frac{1}{1+j} = \frac{p}{(1+j)(1+j-p)}.$$

We then argue exactly as before. Suppose now that $\alpha \in (1/2, 1)$. On the one hand

$$\sum_{j \geq p+1} \frac{1}{(1+j)^\alpha (1+(j-p))^\alpha} \leq \sum_{j=1}^{p} \frac{1}{j^\alpha (j+p)^\alpha} + \sum_{j=p+1}^{+\infty} \frac{1}{j^\alpha (j+p)^\alpha}.$$

Now,

$$\sum_{j=1}^{p} \frac{1}{j^\alpha (j+p)^\alpha} \leq \frac{1}{p^\alpha} \sum_{j=1}^{p} \frac{1}{j^\alpha} \leq C_\alpha p^{1-2\alpha},$$

whereas

$$\sum_{j=p+1}^{+\infty} \frac{1}{j^\alpha (j+p)^\alpha} \leq \sum_{j=p+1}^{+\infty} \frac{1}{j^{2\alpha}} \leq C_\alpha p^{1-2\alpha}.$$

On the other hand,

$$\sum_{j=0}^{p} \frac{1}{(1+j)^\alpha (1+(p-j))^\alpha} \leq \frac{2}{p^\alpha} + \sum_{j=1}^{p-1} \frac{1}{j^\alpha (p-j)^\alpha}$$

$$\leq \frac{2}{p^\alpha} + p^{1-2\alpha} \times \frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(j/p)^\alpha (1-j/p)^\alpha}.$$

We recognize a Riemann sum of the function $x \mapsto x^\alpha (1 - x)^\alpha$, so that

$$\sum_{j=0}^{p} \frac{1}{(1 + j)^\alpha (1 + (p - j))^\alpha} \leq C_\alpha p^{1 - 2\alpha}.$$

We conclude exactly like for the other cases.

3.6. *Central limit theorem*

We now prove that the central limit theorem holds for $f \in E_\omega$ with zero mean. Throughout the proof, we assume $\alpha > 1$. We set $F = f \circ \phi = \sum_{\mathbf{l}, \mathbf{j}} a_{\mathbf{l}, \mathbf{j}} e_{\mathbf{l}, \mathbf{j}}$. We will apply Theorem A with the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ defined by $\mathcal{F}_i = \sigma^{-i}(\mathcal{F}_0)$ and

$$\mathcal{F}_0 = \cdots \times \Omega \times \cdots \times \Omega \times \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \times \cdots,$$

where $\Omega = \{\varnothing, \mathbb{N}\}$ and the first $\mathcal{P}(\mathbb{N})$ is at the 0th position. The filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ is increasing, with $\mathcal{F}_\infty = \mathcal{P}(\mathbb{N})^{\mathbb{Z}}$ and $\mathcal{F}_{-\infty} = \{\varnothing, \mathbb{N}^{\mathbb{Z}}\}$. Thus, $F \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$. By (7),

$$F \circ \sigma^p = \sum_{\mathbf{l}, \mathbf{j}} a_{\mathbf{l}, \mathbf{j}} e_{\mathbf{l}, \mathbf{j} - p} = \sum_{\mathbf{l}, \mathbf{j}} a_{\mathbf{l}, \mathbf{j} + p} e_{\mathbf{l}, \mathbf{j}}$$

so that

$$S_n(F) = \sum_{\mathbf{l}, \mathbf{j}} \sum_{p=0}^{n-1} a_{\mathbf{l}, \mathbf{j} + p} e_{\mathbf{l}, \mathbf{j}}.$$

If we take the conditional expectation, then we find

$$\mathbb{E}(S_n(F) | \mathcal{F}_0) = \sum_{\substack{\mathbf{l}, \mathbf{j} \\ j_1 \geq 0}} \sum_{p=0}^{n-1} a_{\mathbf{l}, \mathbf{j} + p} e_{\mathbf{l}, \mathbf{j}},$$

$$S_n(F) - \mathbb{E}(S_n(F) | \mathcal{F}_n) = \sum_{\substack{\mathbf{l}, \mathbf{j} \\ j_1 < -n}} \sum_{p=0}^{n-1} a_{\mathbf{l}, \mathbf{j} + p} e_{\mathbf{l}, \mathbf{j}}.$$

During the proof, we will need the two following technical facts.

Fact 1. *For any $n \geq 1$,*

$$\sum_{j \geq 0} \left(\sum_{p=0}^{n-1} \frac{1}{(1 + j + p)^\alpha} \right)^2 \leq C_\alpha \max(n^{3 - 2\alpha}, \log(n + 1)).$$

Fact 2. *For any $n \geq 1$ and any $r \geq 1$,*

$$\sum_{\substack{j_1 < -n \\ j_2, \ldots, j_r \in \mathbb{Z}}} \left| \sum_{p=0}^{n-1} \frac{1}{(1 + |j_1 + p|)^\alpha \cdots (1 + |j_r + p|)^\alpha} \right|^2 \leq \sum_{j_1 < -n} \left(\sum_{p=0}^{n-1} \frac{1}{(1 + |j_1 + p|)^\alpha} \right)^2$$

$$\times \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |j|)^\alpha} \right)^{r-1}.$$

We postpone the proof of these two facts and we show that the conditions of Theorem A are satisfied. First,

$$\left\| \mathbb{E}\big(S_n(F)|\mathcal{F}_0\big)\right\|_2^2 = \sum_{\substack{r \geq 1 \\ }} \sum_{\substack{|\mathbf{j}|=r \\ j_1 \geq 0}} \sum_{|\mathbf{l}|=r} \left| \sum_{p=0}^{n-1} a_{\mathbf{l},\mathbf{j}+p} \right|^2$$

$$\leq \sum_{\substack{r \geq 1 \\ }} \sum_{\substack{|\mathbf{j}|=r \\ j_1 \geq 0}} \left(\sum_{p=0}^{n-1} \sum_{|\mathbf{l}|=r} |a_{\mathbf{l},\mathbf{j}+p}| \right)^2.$$

We apply Lemma 3.6 with $M^2 \geq 2\sum_{j \geq 1} \frac{1}{(1+j)^{2\alpha}}$. This yields

$$\left\| \mathbb{E}\big(S_n(F)|\mathcal{F}_0\big)\right\|_2^2 \leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{M^{2r}} \sum_{0 \leq j_1 < \cdots < j_r} \left(\sum_{p=0}^{n-1} \frac{1}{(1+j_1+p)^\alpha \cdots (1+j_r+p)^\alpha} \right)^2$$

$$\leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{M^{2r}} \sum_{j_2,\ldots,j_r \geq 1} \frac{1}{(1+j_2)^{2\alpha} \cdots (1+j_r)^{2\alpha}}$$

$$\times \sum_{j_1 \geq 0} \left(\sum_{p=0}^{n-1} \frac{1}{(1+j_1+p)^\alpha} \right)^2.$$

We now apply Fact 1 to get

$$\left\| \mathbb{E}\big(S_n(F)|\mathcal{F}_0\big)\right\|_2^2 \leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{2^r} \times \max\big(n^{3-2\alpha}, \log(n+1)\big)$$

$$\leq C_{\omega_0,\omega,\alpha} \|f\|_\omega^2 \max\big(n^{3-2\alpha}, \log(n+1)\big).$$

Since $(3-2\alpha)/2 < 1/2$, this yields the convergence of $\sum_{n \geq 1} \frac{\|\mathbb{E}(S_n(F)|\mathcal{F}_0)\|_2}{n^{3/2}}$.

We now turn to the second sum. The beginning of the estimation is completely similar, except that we now apply Lemma 3.6 with $M^2 = 2\sum_{j \in \mathbb{Z}} \frac{1}{(1+|j|)^\alpha}$. We thus obtain

$$\left\| S_n(F) - \mathbb{E}\big(S_n(F)|\mathcal{F}_n\big)\right\|_2^2 \leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{M^{2r}} \sum_{\substack{|\mathbf{j}|=r \\ j_1 < -n}} \left(\sum_{p=0}^{n-1} \frac{1}{(1+|j_1+p|)^\alpha \cdots (1+|j_r+p|)^\alpha} \right)^2.$$

At this stage, we can no longer majorize $\frac{1}{(1+|j_k+p|)^\alpha}$ by $\frac{1}{(1+|j_k|)^\alpha}$ for $k \geq 2$ since it is possible that $j_k \leq 0$. We use Fact 2 instead. It yields

$$\left\| S_n(F) - \mathbb{E}\big(S_n(F)|\mathcal{F}_n\big)\right\|_2^2 \leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{M^{2r}} \sum_{j_1 < -n} \left| \sum_{p=0}^{n-1} \frac{1}{(1+|j_1+p|)^\alpha} \right|^2 \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1+|j|)^\alpha} \right)^{r-1}.$$

The definition of M and the changes of variables $j = -j_1 - n$, $q = n - 1 - p$ imply

$$\left\| S_n(F) - \mathbb{E}\big(S_n(F)|\mathcal{F}_n\big)\right\|_2^2 \leq \sum_{r \geq 1} \frac{C_{\omega_0,\omega,\alpha} \|f\|_\omega^2}{2^r} \sum_{j > 0} \left| \sum_{q=0}^{n-1} \frac{1}{(1+j+q)^\alpha} \right|^2$$

$$\leq C_{\omega_0,\omega,\alpha} \|f\|_\omega^2 \max\big(n^{3-2\alpha}, \log(n+1)\big),$$

where the last inequality follows from Fact 1. As before,

$$\sum_{n\geq 1} \frac{\|S_n(F) - \mathbb{E}(S_n(F)|\mathcal{F}_n)\|_2}{n^{3/2}} < +\infty.$$

Hence, F satisfies the CLT.

Proof of Fact 1. We first observe that there exists $C_\alpha > 0$ such that, for any $j \geq 0$,

$$\sum_{p=0}^{n-1} \frac{1}{(1+j+p)^\alpha} \leq C_\alpha \frac{1}{(1+j)^{\alpha-1}}.$$

Hence,

$$\sum_{j=0}^{n} \left(\sum_{p=0}^{n-1} \frac{1}{(1+j+p)^\alpha} \right)^2 \leq C_\alpha \sum_{j=0}^{n} \frac{1}{(1+j)^{2\alpha-2}}$$

$$\leq C_\alpha \begin{cases} 1 & \text{provided } \alpha > 3/2, \\ \log(n+1) & \text{provided } \alpha = 3/2, \\ n^{3-2\alpha} & \text{provided } \alpha \in (1, 3/2). \end{cases}$$

For the remaining part of the sum, we just write

$$\sum_{p=0}^{n-1} \frac{1}{(1+j+p)^\alpha} \leq \frac{n}{j^\alpha}$$

so that

$$\sum_{j>n} \left(\sum_{p=0}^{n-1} \frac{1}{(1+j+p)^\alpha} \right)^2 \leq n^2 \sum_{j>n} \frac{1}{j^{2\alpha}} \leq C_\alpha n^{3-2\alpha}. \qquad \square$$

Proof of Fact 2. Let

$$S = \sum_{\substack{j_1 < -n \\ j_2,\ldots,j_r \in \mathbb{Z}}} \left(\sum_{p=0}^{n-1} \frac{1}{(1+|j_1+p|)^\alpha \cdots (1+|j_r+p|)^\alpha} \right)^2.$$

We expand the square to get

$$S = \sum_{\substack{j_1 < -n \\ j_2,\ldots,j_r \in \mathbb{Z}}} \left(\sum_{p=0}^{n-1} \frac{1}{(1+|j_1+p|)^{2\alpha} \cdots (1+|j_r+p|)^{2\alpha}} + 2 \sum_{0 \leq p < q \leq n-1} \frac{1}{(1+|j_1+p|)^\alpha \cdots (1+|j_r+q|)^\alpha} \right).$$

We put the sum over $j_2, \ldots, j_r$ inside and we observe that, for a fixed $p \in \mathbb{Z}$,

$$\sum_{j_2,\ldots,j_r \in \mathbb{Z}} \frac{1}{(1+|j_2+p|)^{2\alpha} \cdots (1+|j_r+p|)^{2\alpha}} = \sum_{j_2,\ldots,j_r \in \mathbb{Z}} \frac{1}{(1+|j_2|)^{2\alpha} \cdots (1+|j_r|)^{2\alpha}}$$

$$= \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1+|j|)^{2\alpha}} \right)^{r-1}.$$

Similarly, for a fixed $(p, q) \in \mathbb{Z}^2$, since $(1 + |j_k + q|) \geq 1$, we get

$$
\sum_{j_2,\ldots,j_r \in \mathbb{Z}} \frac{1}{(1 + |j_2 + p|)^\alpha \cdots (1 + |j_r + q|)^\alpha} \leq \sum_{j_2,\ldots,j_r \in \mathbb{Z}} \frac{1}{(1 + |j_2 + p|)^\alpha \cdots (1 + |j_r + p|)^\alpha}
$$

$$
\leq \sum_{j_2,\ldots,j_r \in \mathbb{Z}} \frac{1}{(1 + |j_2|)^\alpha \cdots (1 + |j_r|)^\alpha}
$$

$$
= \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |j|)^\alpha} \right)^{r-1}.
$$

Hence,

$$
S \leq \sum_{j_1 < -n} \left(\sum_{p=0}^{n-1} \frac{1}{(1 + |j_1 + p|)^{2\alpha}} + 2 \sum_{0 \leq p < q \leq n-1} \frac{1}{(1 + |j_1 + p|)^\alpha (1 + |j_1 + q|)^\alpha} \right)
$$

$$
\times \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |j|)^\alpha} \right)^{r-1}
$$

$$
\leq \sum_{j_1 < -n} \left(\sum_{p=0}^{n-1} \frac{1}{(1 + |j_1 + p|)^\alpha} \right)^2 \times \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |j|)^\alpha} \right)^{r-1}. \qquad \square
$$

4. Further remarks

4.1. *Unconditional convergence*

In the statement of Theorem 2.2, condition (iv) is not very pleasant. During the proof it is used at two places: in Lemma 3.1 and in Lemma 3.5. We can delete this assumption if we accept to work only with polynomials instead of functions in E_ω.

Theorem 4.1. *Let $T \in \mathfrak{L}(X)$. Suppose that there exist a dense set $\mathcal{D} \subset X$ and a sequence of maps $S_n : \mathcal{D} \to X$, $n \geq 0$, such that, for any $x \in X$,*

  (i) $\sum_{n \geq 0} T^n x$ *converges unconditionally;*
  (ii) $\sum_{n \geq 0} S_n x$ *converges unconditionally;*
  (iii) $T^n S_n x = x$ *for any $n \geq 0$ and $T^m S_n x = S_{n-m} x$ for any $n > m$.*

Then there exists a T-invariant strongly mixing Borel probability measure μ on X with full support such that $\mathcal{P} \subset L^2(X, \mathcal{B}, \mu)$.

   Suppose moreover that there exists $\alpha > 1/2$ such that, for any $x \in \mathcal{D}$, $\|T^n x\| = \mathrm{O}(n^{-\alpha})$ and $\|S_n x\| = \mathrm{O}(n^{-\alpha})$. Then, for any $f, g \in \mathcal{P}$,

$$
\left| \mathrm{Cov}(f \circ T^n, g) \right| \leq C_{f,g,\alpha} \begin{cases} n^{1-2\alpha} & \text{if } \alpha \in (1/2, 1), \\ \frac{\log(n+1)}{n} & \text{if } \alpha = 1, \\ n^{-\alpha} & \text{if } \alpha > 1. \end{cases}
$$

If $\alpha > 1$, then for any $f \in \mathcal{P}$ with zero mean, the sequence $\frac{1}{\sqrt{n}}(f + \cdots + f \circ T^{n-1})$ converges in distribution to a Gaussian random variable of zero mean and finite variance.

Proof. We just point out the main differences with the proof of Theorem 2.2. Let $\omega_0 : \mathbb{N} \to (1, +\infty)$ be any nondecreasing function going to $+\infty$. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in $\mathcal{D}$ with $x_1 = 0$ and $\|T^k x_n\| \leq \omega_0(n)$ for any $n \geq 1$ and any $k \in \mathbb{Z}$. We may also ask, if we want to prove the second part of Theorem 2.3, that for any $n \geq 1$ and any $k \in \mathbb{Z}$,

$$\|T^k x_n\| \leq \frac{\omega_0(n)}{(1 + |k|)^\alpha}.$$

We construct the measure exactly like in Section 3.2, except that we require that the sequence (p_l) also satisfies

$$\forall d \geq 1, \forall l \geq 2d, \quad \omega_0(l)^{2d} \leq p_l^{-1/2},$$

$$\forall d \geq 1, \quad \sum_{l \geq 1}(N_{l+1} - N_l)p_l^{1/4d} < +\infty.$$

To prove that $\mathcal{P} \subset L^2(X, \mathcal{B}, \mu)$, it suffices to show that, for any $d \geq 1$, $\|\cdot\|^d \in L^2(X, \mathcal{B}, \mu)$. Now, by the triangle inequality,

$$\int_X \|x\|^{2d} = \int_{\mathbb{N}^{\mathbb{Z}}} \left\| \sum_{l \geq 1} \sum_{|k|=N_l}^{N_{l+1}} T^k x_{n_k} \right\|^{2d} \mathrm{d}\bar{\mu}\big((n_k)\big)$$

$$\leq \sum_{l_1,\ldots,l_{2d} \geq 1} \int_{\mathbb{N}^{\mathbb{Z}}} \prod_{i=1}^{2d} \left\| \sum_{|k|=N_{l_i}}^{N_{l_i+1}} T^k x_{n_k} \right\| \mathrm{d}\bar{\mu}\big((n_k)\big).$$

We then apply Hölder's inequality to get

$$\int_X \|x\|^{2d} \leq \sum_{l_1,\ldots,l_{2d} \geq 1} \prod_{i=1}^{2d} \left(\int_{\mathbb{N}^{\mathbb{Z}}} \left\| \sum_{|k|=N_{l_i}}^{N_{l_i+1}} T^k x_{n_k} \right\|^{2d} \mathrm{d}\bar{\mu}\big((n_k)\big) \right)^{1/2d}.$$

We fix some $l \geq 1$ and we want to estimate $\int_{\mathbb{N}^{\mathbb{Z}}} \|\sum_{|k|=N_l}^{N_{l+1}} T^k x_{n_k}\|^{2d} \mathrm{d}\bar{\mu}((n_k))$. Let $(n_k) \subset \mathbb{N}^{\mathbb{Z}}$ and let us write

$$\left\| \sum_{|k|=N_l}^{N_{l+1}} T^k x_{n_k} \right\|^{2d} \leq 2^{2d} \left(\left\| \sum_{\substack{|k|=N_l \\ n_k \leq l}}^{N_{l+1}} T^k x_{n_k} \right\|^{2d} + \left\| \sum_{\substack{|k|=N_l \\ n_k > l}}^{N_{l+1}} T^k x_{n_k} \right\|^{2d} \right)$$

$$\leq \frac{2^{2d}}{2^{ld}} + 2^{4d-1}(N_{l+1} - N_l)^{2d-1} \sum_{|k|=N_l}^{N_{l+1}} \|T^k x_{n_k}\|^{2d} \times \mathbf{1}_{\{n_k > l\}}.$$

We integrate this inequality over $\mathbb{N}^{\mathbb{Z}}$ to find

$$\int_{\mathbb{N}^{\mathbb{Z}}} \left\| \sum_{|k|=N_l}^{N_{l+1}} T^k x_{n_k} \right\|^{2d} \mathrm{d}\bar{\mu}\big((n_k)\big) \leq \frac{2^{2d}}{2^{ld}} + 2^{4d-1}(N_{l+1} - N_l)^{2d-1} \sum_{|k|=N_l}^{N_{l+1}} \sum_{m>l} p_m \|T^k x_m\|^{2d}$$

$$\leq \frac{2^{2d}}{2^{ld}} + 2^{4d}(N_{l+1} - N_l)^{2d} \sum_{m>l} p_m \omega_0(m)^{2d}$$

$$\leq \frac{2^{2d}}{2^{ld}} + 2^{4d}(N_{l+1} - N_l)^{2d} p_l \max\big(\omega_0(l), \omega_0(2d)\big)^{2d}$$

$$\leq \frac{2^{2d}}{2^{ld}} + 2^{4d}(N_{l+1} - N_l)^{2d} p_l^{1/2} \omega_0(2d)^{2d},$$

since we assumed $\omega_0(l)^{2d} p_l \leq p_l^{1/2}$. We take the power $1/2d$ and we sum the inequalities to get

$$\int_X \|x\|^{2d} \, d\mu(x) \leq C \left(\sum_{l \geq 1} \left(\frac{2}{2^{l/2}} + 4(N_{l+1} - N_l) p_l^{1/4d} \right) \right)^{2d} \omega_0(2d)^{2d}$$

$$\leq C_d.$$

Thus, $\| \cdot \|^d$ belongs to $L^2(X, \mathcal{B}, \mu)$. Contrary to what happens in Lemma 3.1, we cannot control its norm by $\omega_0(2d)^d$, but only by some constant C_d which can be much larger. This also affects Lemma 3.5, where we have to replace $C^d r^d \omega_0(d)^d$ by some constant C_d depending on d. However, when we want to study the sequence of covariances and the validity of the Central Limit Theorem for the sequence $(f \circ T^n)$, with f a polynomial, this is unimportant. Indeed, the sum over r which appears in both proofs is now a finite sum. $\qquad \square$

4.2. *Central limit theorems and the Gaussian measure*

When T satisfies the assumptions of Theorem 1.2, we have two ways to define a T-invariant Gaussian measure on X with full support: the Gaussian measure of [4] and the measure constructed in [16] or in Section 3. The Gaussian measure is probably simpler. However, it is easier to understand why T is ergodic with respect to the measure of Section 3: it behaves like a Bernoulli shift. This was very useful to apply the martingale method, in particular to have a "canonical" choice of the filtration $(\mathcal{F}_i)$ to apply Theorem A.

   If we want to apply Theorem A with the Gaussian ergodic measure, it is not clear which filtration could be convenient. A particular case is that of backward shifts: in that case, T is already a shift! Hence, on $\ell^2(\mathbb{Z}_+)$, we can prove a statement similar to Theorem 2.5 with a Gaussian measure and replacing E_ω by some subspace of L^2 similar to those of Devinck.

Question 4.2. *Let $T \in \mathfrak{L}(X)$ satisfying the assumptions of Theorem 1.1 and let μ be a T-invariant and ergodic Gaussian measure on X with full support. Does there exist a big subspace $E \subset L^2(\mu)$ such that any $f \in E$ satisfies the CLT?*

4.3. *Central limit theorem for linear forms*

It is a little bit deceiving in the statement of Theorem 2.2 that the sequence (ε_k) is not directly related to the decay needed on $\|T^k x\|$ to obtain a CLT. It turns out that this is the case if we restrict ourselves to linear forms.

Theorem 4.3. *Let $T \in \mathfrak{L}(X)$, let $\alpha > 1$ and let $\omega : \mathbb{N} \to (1, +\infty)$ going to $+\infty$. Suppose that, for any function $\omega_0 : [1, +\infty) \to (1, +\infty)$ going to infinity, one can find a sequence $\mathcal{D} = (x_n)_{n \geq 1} \subset X$ and a sequence of maps $S_n : \mathcal{D} \to X$, $n \geq 0$, such that*

   (i) *for any $x \in \mathcal{D}$, $\sum_{n \geq 0} T^n x$ converges unconditionally;*
  (ii) *for any $x \in \mathcal{D}$, $\sum_{n \geq 0} S_n x$ converges unconditionally;*
 (iii) *for any $x \in \mathcal{D}$, $T^n S_n x = x$ for any $n \geq 0$ and $T^m S_n x = S_{n-m} x$ for any $n > m$;*
 (iv) *for any sequence $(n_k) \subset \mathbb{N}^{\mathbb{Z}}$ such that $\sum_{k \geq 0} T^k x_{n_k}$ and $\sum_{k < 0} S_{-k} x_{n_k}$ are convergent,*

$$\left\| \sum_{k \geq 0} T^k x_{n_k} \right\| + \left\| \sum_{k < 0} S_{-k} x_{n_k} \right\| \leq \sum_{k \in \mathbb{Z}} \omega_0(n_k) k^{-\alpha};$$

   (v) *$\{ \sum_{k \in F} T^k x_{n_k} + \sum_{k \in G} S_k x_{n_k}; F, G \subset \mathbb{Z}_+ \text{ finite}, (n_k) \subset \mathbb{N}^{F \cup G} \}$ is dense in X.*

Then there exists a T-invariant strongly mixing Borel probability measure μ on X with full support such that $E_\omega \subset L^2(X, \mathcal{B}, \mu)$. Moreover, for any $x^* \in X^*$, the sequence $\frac{1}{\sqrt{n}}(x^* + \cdots + x^* \circ T^{n-1})$ converges in distribution to a Gaussian random variable of finite variance.

Proof. Let $x^* \in X^*$ and let us show that x^* satisfies the CLT. Let $F = f \circ \phi = \sum_{j \in \mathbb{Z}, l \in \mathbb{N}} a_{l,j} e_{l,j}$. A look at the proof of Theorem 2.2 shows that we only need to prove

$$\sum_{j \geq 0} \sum_{l \geq 1} \left| \sum_{p=0}^{n-1} a_{l,j+p} \right|^2 = O(n^{1-\varepsilon}), \tag{8}$$

$$\sum_{j < -n} \sum_{l \geq 1} \left| \sum_{p=0}^{n-1} a_{l,j+p} \right|^2 = O(n^{1-\varepsilon}) \tag{9}$$

for some $\varepsilon > 0$. Let $j \geq 0$ and $l \geq 1$. Like in Lemma 3.5,

$$a_{l,j} = \int_{\mathbb{N}} e_{l,j}\big((n_k)\big)\big\langle x^*, \phi\big((n_k)\big)\big\rangle \mathrm{d}\bar{\mu}\big((n_k)\big)$$

$$= \sum_{m \geq l} p_m e_l(m)\langle x^*, T^j x_m \rangle.$$

By linearity of x^*,

$$\sum_{p=0}^{n-1} a_{l,j+p} = \sum_{m \geq l} p_m e_l(m)\left\langle x^*, \sum_{p=0}^{n-1} T^{j+p} x_m \right\rangle.$$

We can now use assumption (iv) and argue as in Lemma 3.5:

$$\left| \sum_{p=0}^{n-1} a_{l,j+p} \right| \leq C \sum_{m \geq l} p_m \big|e_l(m)\big|\omega_0(m)\|x^*\| \sum_{p=0}^{n-1} \frac{1}{(1 + j + p)^\alpha}$$

$$\leq C \sqrt{p_l}\,\omega_0(l)\|x^*\| \sum_{p=0}^{n-1} \frac{1}{(1 + j + p)^\alpha}.$$

This implies, arguing as in Lemma 3.6,

$$\sum_{j \geq 0} \sum_{l \geq 1} \left| \sum_{p=0}^{n-1} a_{l,j+p} \right|^2 \leq C\|x^*\|^2 \sum_{j \geq 0} \left| \sum_{p=0}^{n-1} \frac{1}{(1 + j + p)^\alpha} \right|^2 \leq C\|x^*\|^2 n^{3-2\alpha}.$$

Since $3 - 2\alpha < 1$, this shows (8). The proof of (9) is completely similar and omitted. $\qquad\square$

This theorem is interesting for operators such that $\sum_n T^n x$ converges unconditionnaly whereas $\sum_n \|T^n x\| = +\infty$. For instance, we get the following corollary for backward shifts, where the value $1/p$ is optimal.

Corollary 4.4. *Let $\omega : \mathbb{N} \to (1, +\infty)$ going to $+\infty$ and let $B_{\mathbf{w}}$ be a bounded backward weighted shift on $\ell^p(\mathbb{Z}_+)$. Suppose that there exists $\alpha > 1/p$ such that, for any $n \geq 1$, $w_1 \cdots w_n \geq Cn^\alpha$. Then there exists a $B_{\mathbf{w}}$-invariant strongly mixing Borel probability measure μ on $\ell^p(\mathbb{Z}_+)$ with full support such that $E_\omega \subset L^2(X, \mathcal{B}, \mu)$. Moreover, for any $x^* \in \ell^q(\mathbb{Z}_+)$, $\frac{1}{p} + \frac{1}{q} = 1$, the sequence $\frac{1}{\sqrt{n}}(x^* + \cdots + x^* \circ T^{n-1})$ converges in distribution to a Gaussian random variable of finite variance.*

More surprisingly, we obtain that a central limit theorem holds in the context of parabolic composition operators. This was unavailable with Theorem 2.2.

Corollary 4.5. *Let ϕ be a parabolic automorphism of the disk. There exists a C_ϕ-invariant strongly mixing Borel probability measure μ on $H^2(\mathbb{D})$ such that $(H^2)^* \subset L^2(\mu)$ and any x^* in $(H^2)^*$ satisfies the CLT.*

Question 4.6. *Do the above corollaries remain true if we consider polynomials instead of linear forms?*

4.4. *Fréchet spaces*

Theorem 1.2 was proved in [16] for an *F*-space. For convenience, we restrict ourselves to Banach spaces; however, there are interesting examples beside this context, especially in the Fréchet space setting.

Question 4.7. *Can we extend Theorem 2.2 to Fréchet spaces?*

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