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Two-parameter non-commutative Central Limit Theorem[1]

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Abstract. In 1992, Speicher showed the fundamental fact that the probability measures playing the role of the classical Gaussian in the various non-commutative probability theories (viz. fermionic probability, Voiculescu's free probability, and q-deformed probability of Bożejko and Speicher) all arise as the limits in a generalized Central Limit Theorem. The latter concerns sequences of non-commutative random variables (elements of a $*$-algebra equipped with a state) drawn from an ensemble of pair-wise commuting or anti-commuting elements, with the respective limiting distributions determined by the average value of the commutation coefficients.

In this paper, we derive a more general form of the Central Limit Theorem in which the pair-wise commutation coefficients are arbitrary real numbers. The classical Gaussian statistics now undergo a second-parameter refinement as a result of controlling for the first *and the second* moments of the commutation coefficients. An application yields the random matrix models for the (q,t)-Gaussian statistics, which were recently shown to have rich connections to operator algebras, special functions, orthogonal polynomials, mathematical physics, and random matrix theory.

Résumé. En 1992, Speicher a montré que les mesures de probabilités jouant le rôle des lois gaussiennes dans les différentes théories des probabilités non-commutatives (probabilités fermioniques, probabilités libres à la Voiculescu, probabilités q-déformées à la Bożejko et Speicher) apparaissent toutes comme limites d'un Théorème de la limite centrale généralisé. Ceci concerne des suites de variables aléatoires non-commutatives (éléments d'une $*$-algèbre munie d'un état) choisies dans un ensemble d'éléments qui commutent ou anti-commutent deux-à-deux, avec les distributions limites respectives déterminées par la valeur moyenne des coefficients de commutation.

Dans ce papier, nous dérivons une forme plus générale du Théorème de la limite centrale où les coefficients de commutation deux-à-deux sont des nombres réels arbitraires. Les statistiques gaussiennes classiques dépendent maintenant d'un second paramètre comme résultat du contrôle du premier *et du second* moment des coefficients de commutation. Une application donne le modèle de matrices aléatoires pour les statistiques (q,t)-gaussiennes, pour lesquelles il a été montré récemment qu'elles ont des profondes connexions avec les algèbres d'opérateurs, les fonctions spéciales, les polynômes orthogonaux, la physique mathématique et la théorie des matrices aléatoires.

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1. Introduction

In *non-commutative probability*, probabilistic interpretations of operator algebraic frameworks give rise to non-commutative analogues of classical results in probability theory. The general setting is that of a non-commutative probability space $(\mathcal{A},\varphi)$, formed by a $*$-algebra $\mathcal{A}$, containing the *non-commutative random variables*, and a posi-

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tive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, playing the role of *expectation*. A particularly rich non-commutative probabilistic framework is Voiculescu's free probability [23], which has been found to both parallel and complement the classical theory (see in-depth treatments in [3,21,24]). Whereas free probability can be seen as characterized by the absence of commutative structure, a parallel – albeit somewhat slower – development has targeted non-commutative settings built around certain types of *commutation relations*.

In [22], Speicher showed a non-commutative version of the classical Central Limit Theorem (CLT) for mixtures of commuting and anti-commuting elements. Speicher's CLT concerns a sequence of elements $b_1, b_2, \ldots \in \mathcal{A}$ whose terms pair-wise satisfy the deformed commutation relation $b_i b_j = s(j,i) b_j b_i$ with $s(j,i) \in \{-1,1\}$. It is not a priori clear that the partial sums

$$S_N := \frac{b_1 + \cdots + b_N}{\sqrt{N}} \tag{1}$$

should converge in some reasonable sense, nor that the limit should turn out to be a natural refinement of the Wick formula for classical Gaussians, but that indeed turns out to be the case. The following theorem is the "almost sure" version of the Central Limit Theorem of Speicher, presented as an amalgamation of Theorem 1 of [22] and Lemma 1 of [22]. Throughout this paper, $\mathscr{P}_2(2n)$ will denote the collection of pair-partitions of $[2n]$, with each $\mathscr{V} \in \mathscr{P}_2(2n)$ uniquely written as $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$ for $w_1 < \cdots < w_n$ and $w_i < z_i$ $(i = 1, \ldots, n)$. For further prerequisite definitions, the reader is referred to Section 2.

Condition 1. *Given a $*$-algebra $\mathscr{A}$ and a state $\varphi : \mathscr{A} \to \mathbb{C}$, consider a sequence $\{b_i\}_{i\in\mathbb{N}}$ of elements of $\mathscr{A}$ satisfying the following*:

1. (*vanishing means*) *for all $i \in \mathbb{N}$, $\varphi(b_i) = \varphi(b_i^*) = 0$;*
2. (*fixed second moments*) *for all for all $i, j \in \mathbb{N}$ with $i < j$ and $\varepsilon, \varepsilon' \in \{1, *\}$, $\varphi(b_i^{\varepsilon} b_i^{\varepsilon'}) = \varphi(b_j^{\varepsilon} b_j^{\varepsilon'})$;*
3. (*uniform moment bounds*) *for all $n \in \mathbb{N}$ and all $j(1), \ldots, j(n) \in \mathbb{N}$, $\varepsilon(1), \ldots, \varepsilon(n) \in \{1, *\}$, $|\varphi(\prod_{i=1}^n b_{j(i)}^{\varepsilon(i)})| \le \alpha_n$ (for $\alpha_n \in \mathbb{R}_+$);*
4. (*"independence"*) *φ factors over the naturally ordered products in $\{b_i\}_{i\in\mathbb{N}}$, in the sense of Definition 2.*

Assume that for all $i \neq j$ and all $\varepsilon, \varepsilon' \in \{1, *\}$, b_i^{ε} and $b_j^{\varepsilon'}$ satisfy the commutation relation

$$b_i^{\varepsilon} b_j^{\varepsilon'} = s(j,i) b_j^{\varepsilon'} b_i^{\varepsilon}, \quad s(j,i) \in \{-1, 1\}. \tag{2}$$

Theorem 1 (Non-commutative CLT [22]). *Consider a non-commutative probability space $(\mathcal{A}, \varphi)$ and a sequence of elements $\{b_i\}_{i\in\mathbb{N}}$ in $\mathcal{A}$ satisfying Condition 1. Fixing $q \in [-1, 1]$, let the commutation signs $\{s(i,j)\}_{1\le i<j}$ be drawn from the collection of independent, identically distributed random variables taking values in $\{-1, 1\}$ with $\mathbb{E}(\mathbf{s}(i,j)) = q$. Then, for almost every sign sequence $\{s(i,j)\}_{1\le i<j}$, the following holds: for every $n \in \mathbb{N}$ and all $\varepsilon(1), \ldots, \varepsilon(2n) \in \{1, *\}$,*

$$\lim_{N\to\infty} \varphi\big(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(2n-1)}\big) = 0, \tag{3}$$

$$\lim_{N\to\infty} \varphi\big(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(2n)}\big) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})} \prod_{i=1}^{n} \varphi\big(b^{\varepsilon(w_i)} b^{\varepsilon(z_i)}\big), \tag{4}$$

with $S_N \in \mathcal{A}$ as given in (1), $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$, and where $\mathrm{cross}(\mathscr{V})$ *denotes the number of crossings in $\mathscr{V}$ (cf. Definition 1).*

In particular, letting $Z_N = S_N + S_N^*$,

$$\lim_{N\to\infty} \varphi\big(Z_N^{2n-1}\big) = 0 \quad \text{and} \quad \lim_{N\to\infty} \varphi\big(Z_N^{2n}\big) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})}, \tag{5}$$

i.e. Z_N converges in moments to the *q-Gaussian* probability measure [6,7] with mean zero and unit variance. The Gaussian (classical), semicircular (free), and Bernoulli (fermionic) measures are recovered for $q = 1, 0, -1$, respectively. Considering more generally the moment expressions in (3) and (4), the stochastic setting with the *average* value of the commutation coefficient set to q turns out to be, from the point of view of limiting distributions, equivalent to the setting of *q-deformed canonical commutation relations*. Namely, given a real, separable Hilbert space $\mathscr{H}$ and two elements $f, g \in \mathscr{H}$, the *creation* and *annihilation* operators on the *q-Fock space* $\mathscr{F}_q(\mathscr{H})$ [7] satisfy the relations:

$$a_q(f)a_q(g)^* - q a_q(g)^* a_q(f) = \langle f, g\rangle_{\mathscr{H}} 1, \tag{6}$$

where $a_q(\cdot)^*$ stands for creation and $a_q(\cdot)$ for annihilation. The mixed moments with respect to the *vacuum expectation state* φ_q of these operators are given by a Wick-type formula which, compared against (3) and (4), yields that for a unit vector e in $\mathscr{H}$, $\lim_{N\to\infty} \varphi(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(n)}) = \varphi_q(a_q(e)^{\varepsilon(1)} \cdots a_q(e)^{\varepsilon(n)})$ for all $n \in \mathbb{N}$ and $\varepsilon(1), \dots, \varepsilon(2n) \in \{1, *\}$. As described in [22], Theorem 1 can be used to provide a general asymptotic model for operators realizing the relation (6), thus providing non-constructive means of settling the question [12] of the positivity of the q-relations.

Finally, any sequence $\{b_i\}_{i\in[n]}$ satisfying Condition 1 has a $*$-representation on $\mathscr{A}_n := \mathcal{M}_2(\mathbb{R})^{\otimes n}$, where $\mathcal{M}_2(\mathbb{R})$ denotes the algebra of 2×2 real matrices. Matricial models for operators satisfying the canonical anti-commutation relation, i.e. the fermionic case corresponding to $q = -1$ in (6), are well known and are provided by the so-called Jordan–Wigner transform (see e.g. [9] for its appearance in a closely-related context). Extending the transform to the setting where there are both commuting and anti-commuting elements and applying Theorem 1 yields random matrix models for operators satisfying the q-commutation relation (6), as remarked in [22] and further developed by Biane in [2]. By replacing 2×2 matrices with 4×4 block-diagonal matrices, Kemp [16] similarly obtained models for the corresponding complex family $(a(f) + \mathrm{i}a(g))/\sqrt{2}$. To describe the extended Jordan–Wigner model, we make the identification $\mathscr{A}_n \cong \mathcal{M}_{2^n}(\mathbb{R})$ and let the $*$ operation be the conjugate transpose on $\mathcal{M}_{2^n}(\mathbb{R})$. Furthermore, let $\varphi_n : \mathcal{M}_{2^n}(\mathbb{R}) \to \mathbb{C}$ be the positive map $a \mapsto \langle ae_0, e_0\rangle_n$, where $\langle \cdot, \cdot\rangle_n$ is the usual inner product on $\mathbb{R}^n$ and $e_0 = (1, 0, \dots, 0)$ an element of the standard basis.

Lemma 1 (Extended Jordan–Wigner transform [2,22]). *Fix $q \in [-1, 1]$ and consider a sequence of commutation coefficients $\{s(i, j)\}_{i\le j}$ drawn from $\{-1, 1\}$. Consider the 2×2 matrices $\{\sigma_x\}_{x\in\mathbb{R}}, \gamma$ given as*

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and, for $i = 1, \dots, n$, let the element $b_{n,i} \in \mathcal{M}_2(\mathbb{C})^{\otimes n}$ be given by

$$b_{n,i} = \sigma_{s(1,i)} \otimes \sigma_{s(2,i)} \otimes \cdots \otimes \sigma_{s(i-1,i)} \otimes \gamma \otimes \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{=\sigma_1^{\otimes(n-i)}}. \tag{7}$$

Then, for every $n \in \mathbb{N}$, the non-commutative probablity space $(\mathcal{A}_n, \varphi_n)$ and the elements $b_{n,1}, b_{n,2}, \dots, b_{n,n} \in \mathcal{A}_n$ satisfy Condition 1.

1.1. *Main results*

It is a priori unclear how much of Speicher's elegant theorem hinges on the commutativity/anti-commutativity requirement for the element sequence. More broadly, it is natural to ask whether a limit may be obtained for a more general commutation structure or, conversely, whether the q-Gaussian limit may naturally arise in a larger setting. This article derives a general form of the non-commutative Central Limit Theorem of [22] in which the commutation signs are replaced by real-valued commutation coefficients, both showing that the commutation signs requirement is not essential for the q-Gaussian limit and yielding a broader class of limiting statistics. The setting now concerns a sequence $\{b_i\}_{i\in\mathbb{N}}$ of non-commutative random variables satisfying the commutation relation

$$b_i^{\varepsilon} b_j^{\varepsilon'} = \mu_{\varepsilon',\varepsilon}(j, i) b_j^{\varepsilon'} b_i^{\varepsilon} \quad \text{with } \varepsilon, \varepsilon' \in \{1, *\}, \mu_{\varepsilon',\varepsilon}(i, j) \in \mathbb{R} \tag{8}$$

for $i \ne j$. For $(\mu_{1,1}(i, j))_{i<j}$ an arbitrary real sequence, the consistency of the above commutation relation is ensured by requiring that for all $i < j$ and $\varepsilon, \varepsilon' \in \{1, *\}$

$$\mu_{1,1}(i, j) = \frac{1}{\mu_{*,*}(i, j)}, \qquad \mu_{1,*}(i, j) = \frac{1}{\mu_{*,1}(i, j)}, \tag{9}$$

$$\mu_{*,1}(i,j)=t\mu_{*,*}(i,j), \qquad \mu_{\varepsilon',\varepsilon}(j,i)=\frac{1}{\mu_{\varepsilon,\varepsilon'}(i,j)}, \tag{10}$$

where $t>0$ is a fixed parameter that will appear explicitly in the limits of interest. Relations (9)–(10) arise from the $*$-algebra structure and the positivity of the state φ; for details concerning the appearance of the parameter t, the reader is referred to the Remark 4 of Section 4.

The conditions underlying the generalized non-commutative CLT are the following.

Condition 2. *Given a $*$-algebra $\mathscr{A}$ and a state $\varphi:\mathscr{A}\to\mathbb{C}$, consider a sequence $\{b_i\}_{i\in\mathbb{N}}$ of elements of $\mathscr{A}$ satisfying the following*:

1. (*vanishing means*) *for all $i\in\mathbb{N}$, $\varphi(b_i)=\varphi(b_i^*)=0$;*
2. (*fixed second moments*) *for all $i,j\in\mathbb{N}$, $\varphi(b_ib_i)=\varphi(b_i^*b_i^*)=\varphi(b_i^*b_i)=0$ and $\varphi(b_ib_i^*)=\varphi(b_jb_j^*)$;*
3. (*uniform bounds*) *for all $n\in\mathbb{N}$ and all $j(1),\dots,j(n)\in\mathbb{N}$, $\varepsilon(1),\dots,\varepsilon(n)\in\{1,*\}$, $|\varphi(\prod_{i=1}^n b_{j(i)}^{\varepsilon(i)})|\le\alpha_n$ (for $\alpha_n\in\mathbb{R}_+$);*
4. (*"independence"*) *φ factors over the naturally ordered products in $\{b_i\}_{i\in\mathbb{N}}$, in the sense of Definition 2.*

Assume that for all $i\neq j$ and all $\varepsilon,\varepsilon'\in\{1,*\}$, b_i^ε and $b_j^{\varepsilon'}$ satisfy the commutation relation

$$b_i^\varepsilon b_j^{\varepsilon'}=\mu_{\varepsilon',\varepsilon}(j,i)b_j^{\varepsilon'}b_i^\varepsilon, \quad \mu_{\varepsilon',\varepsilon}(j,i)\in\mathbb{R}. \tag{11}$$

In this generalized setting, the consistency of Condition 2 is not immediately apparent. Indeed, as further discussed in Remark 5 of Section 4, when the commutation coefficients are taken to be real numbers, the moment factoring condition (item 4) hinges on the additional requirement that $\varphi(b_ib_i)=\varphi(b_i^*b_i^*)=\varphi(b_i^*b_i)=0$. (Note that since φ is a state on $\mathscr{A}$, by the Cauchy–Schwarz inequality, the moment-vanishing assumptions can be reduced to $\varphi(b_i^*b_i)=0$.) While this requirement does not explicitly appear in Speicher's setting (Condition 1), it is in fact consistent with the natural choice of matrix models in Lemma 1 and their generalized form discussed shortly. Given Condition 2, the general non-commutative Central Limit Theorem is the following.

Theorem 2 (Generalized non-commutative CLT). *Consider a noncommutative probability space $(\mathcal{A},\varphi)$ and a sequence of elements $\{b_i\}_{i\in\mathbb{N}}$ in $\mathcal{A}$ satisfying Condition 2. Fix $q\in\mathbb{R}$, $t>0$ and let $\{\mu(i,j)\}_{1\le i<j}$ be drawn from a collection of independent, identically distributed, non-vanishing random variables, with*

$$\mathbb{E}\big(\mu(i,j)\big)=qt^{-1}\in\mathbb{R}, \qquad \mathbb{E}\big(\mu(i,j)^2\big)=1. \tag{12}$$

Letting $\mu_{*,*}(i,j)=\mu(i,j)$ for $1\le i<j$, populate the remaining $\mu_{\varepsilon,\varepsilon'}(i,j)$, for $\varepsilon,\varepsilon'\in\{1,*\}$ and $i\neq j$ $(i,j\in\mathbb{N})$, by (9) and (10).

Then, for almost every sequence $\{\mu(i,j)\}_{i\le j}$, the following holds: for every $n\in\mathbb{N}$ and all $\varepsilon(1),\dots,\varepsilon(2n)\in\{1,*\}$,

$$\lim_{N\to\infty}\varphi\big(S_N^{\varepsilon(1)}\cdots S_N^{\varepsilon(2n-1)}\big)=0, \tag{13}$$

$$\lim_{N\to\infty}\varphi\big(S_N^{\varepsilon(1)}\cdots S_N^{\varepsilon(2n)}\big)=\sum_{\mathscr{V}\in\mathscr{P}_2(2n)}q^{\operatorname{cross}(\mathscr{V})}t^{\operatorname{nest}(\mathscr{V})}\prod_{i=1}^n\varphi\big(b^{\varepsilon(w_i)}b^{\varepsilon(z_i)}\big) \tag{14}$$

with $S_N\in\mathcal{A}$ as given in (1), $\mathscr{V}=\{(w_1,z_1),\dots,(w_n,z_n)\}$, and where cross$(\mathscr{V})$ *denotes the number of crossings in $\mathscr{V}$ and* nest$(\mathscr{V})$ *the number of nestings in $\mathscr{V}$ (cf. Definition 1).*

Comparing Theorem 1 to Theorem 2, the unit magnitude requirement for the commutation coefficients is dispelled by Condition 2 and by controlling for the second moments of the commutation coefficients. The resulting theorem is natural at the combinatorial level, at the level of Fock spaces, and in regard to the random matrix models.

Specifically, the generalized commutation structure now generates a second combinatorial statistic in the limiting moments – that of *nestings* in pair partitions, a combinatorial counterpart to crossings (see e.g. [11,15,18]). In particular, letting $Z_N = S_N + S_N^*$ now yields

$$\lim_{N\to\infty} \varphi\big(Z_N^{2n-1}\big) = 0 \quad \text{and} \quad \lim_{N\to\infty} \varphi\big(Z_N^{2n}\big) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})}. \tag{15}$$

These are the moments of the (q,t)-*Gaussian measure* (as referred to in [5]), namely the orthogonalizing measure for the two-parameter deformation of the Hermite orthogonal polynomial sequence $xH_n^{(q,t)}(x) = H_{n+1}^{(q,t)}(x) + [n]_{q,t} H_{n-1}^{(q,t)}(x)$, with $H_0^{(q,t)}(x) = 1$, $H_1(q,t)(x) = x$ and $[n]_{q,t} = (t^n - q^n)/(t-q)$. Specializing to $t = 1$ while imposing no additional constraints on the distribution of $(\mu(i,j))_{i<j}$ now yields a broader class of models for the q-Gaussian statistics.

This second-parameter refinement also extends to the Fock-space level, with the limits (13) and (14) realized as the moments of creation and annihilation operators on the (q,t)-*Fock space* [5] (see also [8]), briefly overviewed in Section 2. Compared to (6), these operators now satisfy the commutation relation

$$a_{q,t}(f)a_{q,t}(g)^* - q a_{q,t}(g)^* a_{q,t}(f) = \langle f, g\rangle_{\mathscr{H}} t^N, \tag{16}$$

where N is the *number operator*. The above (q,t)-commutation relation first appeared in the context of deformed quantum harmonic oscillators, as a defining relation of the *Chakrabarti–Jagannathan* algebra [10] and a generalization of the two types of q-relations frequently appearing in physics [1,4,14,19]. The reader may verify that (12), together with the fact that $t > 0$, recovers the fundamental constraint that $|q| < t$ in order for the (q,t)-Fock space to be a bona fide Hilbert space. Theorem 2 therefore provides independent means of characterizing the parameter range and demonstrating the positivity of the (q,t)-commutation relations.

A natural generalization of the Jordan–Wigner transform of Lemma 1 yields the matrix models satisfying Condition 2. In the resulting construction, the Pauli matrices $\sigma_{\pm 1}$ are deformed into (non-unitary) matrices σ_x for $x \in \mathbb{R}$. The underlying probability space $(\mathcal{A}_n, \varphi_n)$ remains that of the previous section.

Lemma 2 (Two-parameter Jordan–Wigner transform). *Fix $q \in \mathbb{R}$, $t > 0$ and let $\{\mu_{\varepsilon,\varepsilon'}(i,j)\}_{i\neq j,\varepsilon,\varepsilon'\in\{1,*\}}$ be a sequence of commutation coefficients, i.e. a sequence of non-zero real numbers satisfying (9) and (10). Consider the 2×2 matrices $\{\sigma_x\}_{x\in\mathbb{R}}$ and γ given by*

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{t}x \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For $i = 1,\dots,n$, let $\mu(i,j) := \mu_{*,*}(i,j)$ and consider the element $b_{n,i} \in \mathcal{M}_2(\mathbb{R})^{\otimes n}$ given by

$$b_{n,i} = \sigma_{\mu(1,i)} \otimes \sigma_{\mu(2,i)} \otimes \cdots \otimes \sigma_{\mu(i-1,i)} \otimes \gamma \otimes \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{=\sigma_1^{\otimes(n-i)}}. \tag{17}$$

Then, for every $n \in \mathbb{N}$, the non-commutative probablity space $(\mathcal{A}_n, \varphi_n)$ and the elements $b_{n,1}, b_{n,2}, \dots, b_{n,n} \in \mathcal{A}_n$ satisfy Condition 2.

Then, analogously to [2], Theorem 2 and Lemma 2 together yield an asymptotic random matrix models for the creation and annihilation operators on the (q,t)-Fock space:

Corollary 1. *Consider a sequence of commutation coefficients drawn according to Theorem 2 and the corresponding matrix construction of Lemma 2. Let*

$$S_{N,k} := \frac{1}{\sqrt{N}} \sum_{i=N(k-1)+1}^{Nk} b_{Nk,i}. \tag{18}$$

Then, for any choice of $k, i(1), \dots, i(k) \in \mathbb{N}$, $\varepsilon(1), \dots, \varepsilon(k) \in \{1, *\}$,

$$\lim_{N\to\infty} \varphi_{Nk}\big(S_{N,i(1)}^{\varepsilon(1)} \cdots S_{N,i(k)}^{\varepsilon(k)}\big) = \varphi_{q,t}\big(a_{q,t}(e_1)^{\varepsilon(1)} \cdots a_{q,t}(e_k)^{\varepsilon(k)}\big), \tag{19}$$

where $\varphi_{q,t}$ *is the vacuum expectation state on the* (q,t)*-Fock space* $\mathscr{F}_{q,t}(\mathscr{H})$ *and* $a_{q,t}(e_i)$ *is the twisted annihilation operator on* $\mathscr{F}_{q,t}(\mathscr{H})$ *associated with the element* e_i *of the orthonormal basis of* $\mathscr{H}$.

Given that the (q,t)-Gaussian statistics arise as the limits in a general Central Limit Theorem, it may not come as a surprise that these also appear in several other contexts [5]. For instance, the $q = 0 < t$ specialization encodes the first-order statistics of the reduced Wigner process, first computed in [17,20], strengthening its interpretation as a natural deformation of free probablity. The same measure is also related to the deformed Airy function of Ismail [13], which plays the role of the classical Airy function in the large-degree Plancherel–Rotach-type asymptotics for the q-polynomials of the Askey scheme.

Apropos, note that Section 3 contains yet a more general result: namely, Theorem 3 considers the existence and the form of the limit for any fixed (non-random) sequence of commutation coefficients. While assuming the correlation coefficients to be independent and identically distributed leads back to Theorem 2, it is presently unclear which types of statistics are achievable by allowing a non-trivial correlation structure on the coefficient sequences (e.g. taking the $(\mu(i,j))_{i<j}$ to form a jointly Gaussian family with a non-trivial covariance matrix). In perspective, this more general framework may lead to interesting new quantum statistics, with the two-parameter Jordan–Wigner tranform of Lemma 2 (a general result valid for any sequence commutation coefficients satisfying the consistency conditions (9) and (10)) providing the corresponding random matrix models.

2. Preliminaries

The present section overviews the key combinatorial constructs used to encode the mechanics of the non-commutative Central Limit Theorem. It also overviews the Hilbert space framework that provides a natural setting in which to realize the limits of the random matrix models of Corollary 1.

2.1. *Partitions*

Denote by $\mathscr{P}(n)$ the collection of partitions of $[n] := \{1, \dots, n\}$. Set partitions will be extensively used to encode equivalence classes of products of random variables, based on the repetition patterns of individual elements. Specifically, any two r-vectors will be declared equivalent if element repetitions occur at same locations in both vectors; i.e. for $(i(1), \dots, i(r)), (j(1), \dots, j(r)) \in [N]^r$,

$$\big(i(1), \dots, i(r)\big) \sim \big(j(1), \dots, j(r)\big) \quad \Longleftrightarrow \quad \begin{aligned} &\text{for all } 1 \le k_1 < k_2 \le r, \\ &i(k_1) = i(k_2) \text{ iff } j(k_1) = j(k_2). \end{aligned} \tag{20}$$

It then immediately follows that the equivalence classes of "$\sim$" can be identified with the set $\mathscr{P}(r)$ of the partitions of $[r]$. An example is shown in Fig. 1. Note that writing "$(i(1), \dots, i(r)) \sim \mathscr{V}$" will indicate that $(i(1), \dots, i(r))$ is in the equivalence class identified with the partition $\mathscr{V} \in \mathscr{P}(r)$.

Particularly relevant is the collection $\mathscr{P}_2(2n)$ of *pair partitions* of $[2n]$, also referred to as *pairings* or *perfect matchings*, which are partitions whose each part contains exactly two elements. A pair partition will be represented as a list of ordered pairs, that is, $\mathscr{P}_2(2n) \ni \mathscr{V} = \{(w_1, z_1), \dots, (w_n, z_n)\}$, where $w_i < z_i$ for $i \in [n]$ and $w_1 < \dots < w_n$.

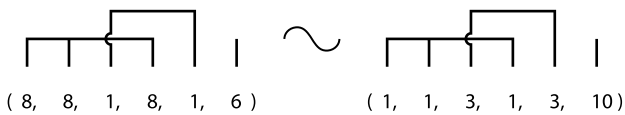

Fig. 1. Two elements of $[N]^r$ (for $N = 10$, $r = 6$) that belong to the same equivalence class, where the latter is represented as the corresponding partition $\mathscr{V} \in \mathscr{P}(r)$ given by $\mathscr{V} = \{(1,2,4), (3,5), (6)\}$.

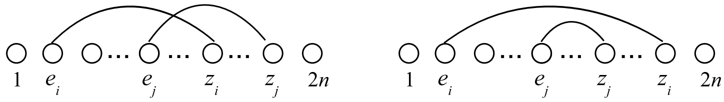

Fig. 2. An example of a crossing [left] and nesting [right] of a pair partition $\mathscr{V} = \{(e_1, z_1), \ldots, (e_n, z_n)\}$ of $[2n]$.

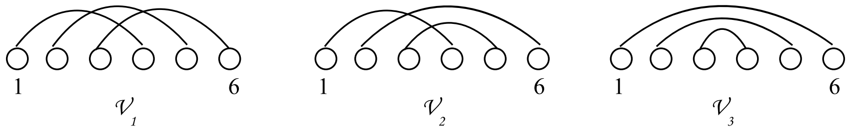

Fig. 3. Example of three pair partitions on $[2n] = \{1, \ldots, 6\}$: $\mathrm{cross}(\mathscr{V}_1) = 3, \mathrm{nest}(\mathscr{V}_1) = 0$ [left], $\mathrm{cross}(\mathscr{V}_2) = 2, \mathrm{nest}(\mathscr{V}_2) = 1$ [middle], $\mathrm{cross}(\mathscr{V}_3) = 0, \mathrm{nest}(\mathscr{V}_3) = 3$ [right].

In the present setting, the pair partitions will typically appear with additional refinements given by the following two statistics on $\mathscr{P}_2(2n)$.

Definition 1 (Crossings and nestings). *For $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\} \in \mathscr{P}_2(2n)$, pairs (w_i, z_i) and (w_j, z_j) are said to* cross *if $w_i < w_j < z_i < z_j$. The corresponding* crossing *is encoded by (w_i, w_j, z_i, z_j) with* $\mathrm{Cross}(\mathscr{V}) := \{(w_i, w_j, z_i, z_j) \mid (w_i, z_i), (w_j, z_j) \in \mathscr{V}$ *with* $w_i < w_j < z_i < z_j\}$ *as the set of all crossings in $\mathscr{V}$ and* $\mathrm{cross}(\mathscr{V}) := |\mathrm{Cross}(\mathscr{V})|$ *counting the crossings in $\mathscr{V}$.*

For $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\} \in \mathscr{P}_2(2n)$, pairs (w_i, z_i) and (w_j, z_j) are said to nest *if $w_i < w_j < z_j < z_i$. The corresponding* nesting *is encoded by (w_i, w_j, z_j, z_i) with* $\mathrm{Nest}(\mathscr{V}) := \{(w_i, w_j, z_j, z_i) \mid (w_i, z_i), (w_j, z_j) \in \mathscr{V}$ *with* $w_i < w_j < z_j < z_i\}$ *as the set of all nestings in $\mathscr{V}$ and* $\mathrm{nest}(\mathscr{V}) := |\mathrm{Nest}(\mathscr{V})|$ *counting the nestings in $\mathscr{V}$.*

The two concepts are illustrated in Figs 2 and 3, by visualizing the pair partitions as collections of disjoint chords with end-points labeled (increasing from left to right) by elements in $[2n]$.

2.2. *Operators on the (q, t)-Fock space*

The (q, t)-Fock space $\mathscr{F}_{q,t}(\mathscr{H})$ [5], for $|q| < t$, is a two-parameter deformation of the classical Bosonic and Fermionic Fock spaces. Consider the tensor algebra on the Hilbert space $\mathscr{H}$ (taken as real and separable) given by $\mathscr{F}(\mathscr{H}) := \bigoplus_{n \geq 0} (\mathbb{C} \otimes \mathscr{H})^{\otimes n}$, with $(\mathbb{C} \otimes \mathscr{H})^0$ defined as a complex vector space spanned by a real unit vector $\Omega \notin \mathscr{H}$. The algebra $\mathscr{F}(\mathscr{H})$ is spanned by the pure tensors $\{h_1 \otimes \cdots \otimes h_n \mid n \in \mathbb{N}, h_1, \ldots, h_n \in \mathscr{H}\} \cup \{\Omega\}$. The completion of $\mathscr{F}(\mathscr{H})$ with respect to the usual inner product, denoted $\langle \cdot, \cdot \rangle_0$ and given by $\langle \Omega, \Omega \rangle_0$ and $\langle f_1 \cdots f_n, h_1 \cdots h_m \rangle_0 = \delta_{n,m} \langle f_1, h_1 \rangle_{\mathscr{H}} \cdots \langle f_n, h_n \rangle_{\mathscr{H}}$, yields the *full (Boltzmann) Fock space*. In the present scenario, it will be more interesting to complete with respect to the "(q, t)-symmetrized" inner product $\langle \cdot, \cdot \rangle_{q,t}$ given by $\langle \Omega, \Omega \rangle_{q,t} = 1$ and

$$\langle f_1 \otimes \cdots \otimes f_n, h_1 \otimes \cdots \otimes h_m \rangle_{q,t} = \begin{cases} 0, & n \neq m, \\ \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{cinv}(\pi)} \langle f_1, h_{\pi(1)} \rangle_{\mathscr{H}} \cdots \langle f_n, h_{\pi(n)} \rangle_{\mathscr{H}}, & n = m, \end{cases} \tag{21}$$

where $\mathrm{inv}(\pi)$ denotes the number of inversions of the permutation $\pi \in S_n$ (viz. all pairs $1 \leq i < j \leq n$ such that $\pi(j) < \pi(i)$) and similarly denotes the number of co-inversions π (viz. all pairs $1 \leq i < j \leq n$ such that $\pi(i) < \pi(j)$). The completion of $\mathscr{F}(\mathscr{H})$ with respect to $\langle \cdot, \cdot \rangle_{q,t}$ yields the *(q, t)-Fock space $\mathscr{F}_{q,t}(\mathscr{H})$* [5], where letting $t = 1$ recovers the q-Fock space $\mathscr{F}_q(\mathscr{H})$ of Bożejko and Speicher [7]. Note by letting $t \mapsto s^2$ and $q \mapsto qs^2$, $\mathscr{F}_{q,t}(\mathscr{H})$ equivalently becomes the (q, s)-Fock space of Bożejko and Yoshida [8], though the parameter changes somewhat obscure the combinatorial statistics that will later appear in the structure of the Central Limit Theorem.

The annihilation operators $\{a_{q,t}(h)\}_{h \in \mathscr{H}}$ on $\mathscr{F}_{q,t}(\mathscr{H})$ and their adjoints (with respect to $\langle \cdot, \cdot \rangle_{q,t}$), the creation operators $\{a_{q,t}(h)^*\}_{h \in \mathscr{H}}$, are densely defined on $\mathscr{F}(\mathscr{H})$ by

$$a_{q,t}(f)^* \Omega = f, \qquad a_{q,t}(f) \Omega = 0, \tag{22}$$

$$a_{q,t}(f)^* h_1 \otimes \cdots \otimes h_n = f \otimes h_1 \otimes \cdots \otimes h_n, \tag{23}$$

$$a_{q,t}(f) h_1 \otimes \cdots \otimes h_n = \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle f, h_k \rangle_{\mathscr{H}} h_1 \otimes \cdots \otimes \check{h}_k \otimes \cdots \otimes h_n, \tag{24}$$

where the superscript $\check{h}_k$ indicates that h_k has been deleted from the product. Letting t^N be the linear operator defined by $t^N \Omega = \Omega$ and $t^N h_1 \otimes \cdots \otimes h_n = t^n h_1 \otimes \cdots \otimes h_n$, the creation and annihilation operators are readily shown to satisfy the (q,t)-commutation relation (16). The two-parameter family of the (self-adjoint) field operators $s_{q,t}(h) := a_{q,t}(h) + a_{q,t}(h)^*$, for $h \in \mathscr{H}$, is referred to as a (q,t)-Gaussian family. The *moments* of the creation, annihilation, and field operators are computed with respect to the vacuum expectation state $\varphi_{q,t} : \mathscr{B}(\mathscr{F}_{q,t}(\mathscr{H})) \to \mathbb{C}$, $\varphi_{q,t}(a) = \langle a\Omega, \Omega \rangle_{q,t}$. In particular, for every $n \in \mathbb{N}$ and all $\varepsilon(1), \ldots, \varepsilon(2n) \in \{1, *\}$,

$$\varphi_{q,t}\big(a_{q,t}(h_1)^{\varepsilon(1)} \cdots a_{q,t}(h_{2n-1})^{\varepsilon(2n-1)}\big) = 0, \tag{25}$$

$$\varphi_{q,t}\big(a_{q,t}(h_1)^{\varepsilon(1)} \cdots a_{q,t}(h_{2n})^{\varepsilon(2n)}\big) = \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} q^{\operatorname{cross}(\mathscr{V})} t^{\operatorname{nest}(\mathscr{V})} \prod_{i=1}^{n} \varphi\big(a_{q,t}(h_{w_i})^{\varepsilon(w_i)} a_{q,t}(h_{z_i})^{\varepsilon(z_i)}\big), \tag{26}$$

$$\varphi_{q,t}\big(s_{q,t}(h)^{2n}\big) = \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} q^{\operatorname{cross}(\mathscr{V})} t^{\operatorname{nest}(\mathscr{V})}, \tag{27}$$

where $\mathscr{P}_2(2n)$ is again the collection of pair partitions of $[2n]$ and each $\mathscr{V} \in \mathscr{P}_2(2n)$ is (uniquely) written as a collection of pairs $\{(w_1, z_1), \ldots, (w_n, z_n)\}$ with $w_1 < \cdots < w_n$ and $w_i < z_i$.

Remark 1. *To elucidate the link between (inversions, co-inversions) of permutations and (crossings, nestings) of pair partitions, consider the mixed moment* $\varphi_{q,t}(a_{q,t}(h_1) \cdots a_{q,t}(h_n) a_{q,t}(h_{n+1})^* \cdots a_{q,t}(h_{2n})^*)$. *Whereas* (26) *yields a moment expression indexed over pair partitions of* $[2n]$, *the fact that*

$$\varphi_{q,t}\big(a_{q,t}(h_1) \cdots a_{q,t}(h_n) a_{q,t}(h_{n+1})^* \cdots a_{q,t}(h_{2n})^*\big) = \langle h_{n+1} \otimes \cdots \otimes h_{2n}, h_n \otimes \cdots \otimes h_1 \rangle_{q,t}$$

yields an equivalent formulation, via (21), *indexed over permutations of* $[n]$. *The reader may verify that for the mixed moment considered presently, the relevant pair partitions of* $[2n]$ *are equinumerous with the permutations of* $[n]$, *and there is a simple correspondence (cf. Remark* 1 *in* [5]*) sending crossings to inversions and nestings to coinversions. Of course, a simple cardinality argument shows that pair-partitions are generally not equivalent to the permutations and no such correspondence can exist more generally.*

3. The extended non-commutative Central Limit Theorem

The goal of this section is to extend the "deterministic formulation" of the non-commutative Central Limit Theorem of Speicher [22]. The deterministic result differs from the previously stated Theorem 1 in that the sequence of commutation signs $(s(i,j))_{i,j}$, taking values in $\{-1, 1\}$ and associated with the commutation relations $b_i^{\varepsilon} b_j^{\varepsilon'} = s(j,i) b_j^{\varepsilon'} b_i^{\varepsilon}$, is now *fixed*. In [22], an analogous Wick-type formula is nevertheless shown to exist, provided the existence of the following limit:

$$\lim_{N \to \infty} \frac{1}{N^{2n}} \sum_{\substack{i(1),\ldots,i(2n) \in [N] \text{ s.t.} \\ (i(1),\ldots,i(2n)) \sim \mathscr{V}}} \prod_{(w_j, w_k, z_j, z_k) \in \operatorname{Cross}(\mathscr{V})} s\big(i(w_j), i(w_k)\big) := \lambda_{\mathscr{V}}$$

for each pair partition $\mathscr{V} \in \mathscr{P}(2n)$.

At present, the focus is on a sequence $(b_i)_{i \in \mathbb{N}}$ of non-commutative random variables satisfying a more general type of commutation relations, where for all $i \neq j$ and $\varepsilon, \varepsilon' \in \{1, *\}$,

$$b_i^{\varepsilon} b_j^{\varepsilon'} = \mu_{\varepsilon',\varepsilon}(j,i) b_j^{\varepsilon'} b_i^{\varepsilon} \quad \text{for some } \mu_{\varepsilon',\varepsilon}(j,i) \in \mathbb{R}. \tag{28}$$

At the outset, the sequence of commutation coefficients $\{\mu_{\varepsilon,\varepsilon'}(i,j)\}_{i\neq j,\varepsilon,\varepsilon'\in\{1,*\}}$ must satisfy certain properties. In particular, interchanging the roles of i and j in the commutation relation implies that

$$\mu_{\varepsilon,\varepsilon'}(i,j)=\frac{1}{\mu_{\varepsilon',\varepsilon}(j,i)}. \tag{A}$$

Similarly, conjugating (via the $*$ operator) both sides of the commutation relation yields

$$\mu_{*,*}(i,j)=\mu_{1,1}(j,i),\qquad \mu_{1,*}(i,j)=\mu_{1,*}(j,i),\qquad \mu_{*,1}(i,j)=\mu_{*,1}(j,i).$$

(For example, $b_ib_j=\mu_{1,1}(j,i)b_jb_i$ and therefore $b_j^*b_i^*=(b_ib_j)^*=\mu_{1,1}(j,i)b_i^*b_j^*$, but also $b_j^*b_i^*=\mu_{*,*}(i,j)b_i^*b_j^*$.) Therefore, by (A),

$$\mu_{*,*}(i,j)=\frac{1}{\mu_{1,1}(i,j)},\qquad \mu_{*,1}(i,j)=\frac{1}{\mu_{1,*}(i,j)}. \tag{B}$$

The second key ingredient in a non-commutative CLT is a moment-factoring assumption. As in [22], the factoring is assumed to follow the underlying partition structure. Drawing on the notation of Section 2, viz. the equivalence relation "$\sim$" on the set $[N]^r$ of r-tuples in $[N]:=\{1,\dots,N\}$, the two relevant ways in which the moments may be assumed to factor are defined as follows.

Definition 2. *Consider a sequence $\{b_i\}_{i\in\mathbb{N}}$ of random variables, elements of some non-commutative probability space $(\mathcal{A},\varphi)$. The element $b_{i(1)}^{\varepsilon(1)}\cdots b_{i(n)}^{\varepsilon(n)}$, for $\varepsilon(1),\dots,\varepsilon(n)\in\{1,*\}$ and $i(1),\dots,i(n)\in\mathbb{N}$, is said to be an* interval-ordered product *if $(i(1),\dots,i(n))\sim\mathscr{V}$ where $\mathscr{V}=\{\{1,\dots,k_1\},\{k_1+1,\dots,k_2\},\dots,\{k_{|\mathscr{V}|-1}+1,\dots,k_{|\mathscr{V}|}\}\}$ is an interval partition of $[n]$. The same element is said to be a* naturally ordered product *if, in addition, $i(1)<i(k_1+1)<\cdots<i(k_{|\mathscr{V}|-1}+1)$.*

The state φ is said to factor over naturally (resp. interval) ordered products *in $\{b_i\}_{i\in\mathbb{N}}$ if*

$$\varphi\big(b_{i(1)}^{\varepsilon(1)}\cdots b_{i(n)}^{\varepsilon(n)}\big)=\varphi\big(b_{i(1)}^{\varepsilon(1)}\cdots b_{i(k_1)}^{\varepsilon(k_i)}\big)\cdots\varphi\big(b_{i(k_{|\mathscr{V}|-1}+1)}^{\varepsilon(k_{|\mathscr{V}|-1}+1)}\cdots b_{i(k_{|\mathscr{V}|})}^{\varepsilon(k_{|\mathscr{V}|})}\big)$$

whenever $b_{i(1)}^{\varepsilon(1)}\cdots b_{i(n)}^{\varepsilon(n)}$ is a naturally (resp. interval) ordered product.

The following remark ensures that the commutation relations (28) are consistent with the moment factoring assumptions.

Remark 2. *In assuming φ factors over naturally ordered products, one must be able to bring a moment $\varphi(b_i^{\varepsilon_i}b_i^{\varepsilon_i'}b_j^{\varepsilon_j}b_j^{\varepsilon_j'})$ for $i>j$ into naturally-ordered form. Alternatively, should it be further assumed that φ factors over interval-ordered products of the sequence $\{b_i\}_{i\in\mathbb{N}}$, one must allow that $\varphi(b_i^{\varepsilon_i}b_i^{\varepsilon_i'}b_j^{\varepsilon_j}b_j^{\varepsilon_j'})=\varphi(b_j^{\varepsilon_j}b_j^{\varepsilon_j'}b_i^{\varepsilon_i}b_i^{\varepsilon_i'})$ for all i,j and $\varepsilon,\varepsilon'\in\{1,*\}$. When commutation coefficients are constrained to take values in $\{-1,1\}$, it is in fact the case that $b_i^{\varepsilon_i}b_i^{\varepsilon_i'}$ commutes with $b_j^{\varepsilon_j}b_j^{\varepsilon_j'}$, and the moment-factoring assumptions are consistent with the commutativity structure. However, this need not be the case for the general setting. In particular,*

$$\varphi\big(b_i^{\varepsilon_i}b_i^{\varepsilon_i'}b_j^{\varepsilon_j}b_j^{\varepsilon_j'}\big)=\mu_{\varepsilon_i,\varepsilon_j'}(j,i)\mu_{\varepsilon_i,\varepsilon_j}(j,i)\mu_{\varepsilon_i',\varepsilon_j'}(j,i)\mu_{\varepsilon_i',\varepsilon_j}(j,i)\varphi\big(b_j^{\varepsilon_j}b_j^{\varepsilon_j'}b_i^{\varepsilon_i}b_i^{\varepsilon_i'}\big).$$

The reader may verify that any sequence of real-valued commutation coefficients for which the above product evaluates to unity regardless of the choice of ε,ε' must in fact take values in $\{-1,1\}$.

Instead, rather than imposing additional restrictions on the sign sequence, the alternative approach is that of restricting the range of φ when applied to the sequence $\{b_i\}$. In particular, by (A)–(B),

$$\mu_{\varepsilon_i,\varepsilon_j'}(j,i)\mu_{\varepsilon_i,\varepsilon_j}(j,i)\mu_{\varepsilon_i',\varepsilon_j'}(j,i)\mu_{\varepsilon_i',\varepsilon_j}(j,i)=1$$

whenever $\varepsilon_i \neq \varepsilon'_i$ *and* $\varepsilon_j \neq \varepsilon'_j$. *Thus, by imposing that* $\varphi(b_i^* b_i^*) = \varphi(b_i b_i) = 0$ *for all* $i \in \mathbb{N}$, *the assumption on the factoring of naturally-ordered second moments conveniently becomes equivalent to factoring of interval-ordered second moments. Note that factoring an interval-ordered product containing higher moments generally still incurs a product of commutation coefficients. However, as will become apparent shortly, the contribution of such expressions vanishes in the limits of interest.*

The stage is now set for the main result of this section.

Theorem 3 (Extended non-commutative CLT). *Consider a noncommutative probability space* $(\mathcal{A}, \varphi)$ *and a sequence* $\{b_i\}_{i\in\mathbb{N}}$ *of elements of* $\mathcal{A}$ *satisfying Condition 2, with the real-valued commutation coefficients* $\{\mu_{\varepsilon',\varepsilon}(i,j)\}$ *satisfying the consistency conditions (A)–(B). For* $n \in \mathbb{N}$, *fix* $\varepsilon(1), \dots, \varepsilon(2n) \in \{1, *\}$ *and, letting* $\mathscr{P}_2(2n)$ *denote the collection of pair partitions of* $[2n]$, *assume that for all* $\mathscr{V} = \{(w_1, z_1), \dots, (w_n, z_n)\} \in \mathscr{P}_2(2n)$ *the following limit exists*:

$$\lambda_{\mathscr{V},\varepsilon(1),\dots,\varepsilon(2n)} := \lim_{N\to\infty} N^{-n} \sum_{\substack{i(1),\dots,i(2n)\in[N]\ s.t.\\ (i(1),\dots,i(2n))\sim\mathscr{V}}} \Biggl(\prod_{\substack{(w_j,w_k,z_j,z_k)\\ \in\mathrm{Cross}(\mathscr{V})}} \mu_{\varepsilon(z_j),\varepsilon(w_k)}\bigl(i(z_j), i(w_k)\bigr) \times \prod_{\substack{(w_j,w_m,z_m,z_j)\\ \in\mathrm{Nest}(\mathscr{V})}} \mu_{\varepsilon(z_j),\varepsilon(z_m)}\bigl(i(z_j), i(z_m)\bigr)\mu_{\varepsilon(z_j),\varepsilon(w_m)}\bigl(i(z_j), i(w_m)\bigr)\Biggr), \tag{29}$$

where $\mathrm{Cross}(\mathscr{V})$ *and* $\mathrm{Nest}(\mathscr{V})$ *denote, respectively, the sets of crossings and nestings in* $\mathscr{V}$ *(cf. Definition 1) and where the equivalence relation* $\sim$ *is given by (20).*

Then, for every $n \in \mathbb{N}$ *and all* $\varepsilon(1), \dots, \varepsilon(2n) \in \{1, *\}$,

$$\lim_{N\to\infty} \varphi\bigl(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(2n-1)}\bigr) = 0, \tag{30}$$

$$\lim_{N\to\infty} \varphi\bigl(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(2n)}\bigr) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} \lambda_{\mathscr{V},\varepsilon(1),\dots,\varepsilon(2n)} \prod_{i=1}^{n} \varphi\bigl(b^{\varepsilon(w_i)} b^{\varepsilon(z_i)}\bigr) \tag{31}$$

for $S_N \in \mathcal{A}$ *as given in (1) and with each* $\mathscr{V} \in \mathscr{P}_2(2n)$ *written as* $\mathscr{V} = \{(w_1, z_1), \dots, (w_n, z_n)\}$ *for* $w_1 < \cdots < w_n$ *and* $w_i < z_i$ $(i = 1, \dots, n)$.

Proof. The notation and the development follow closely those of [22].

Fix $r \in \mathbb{N}$ and $\varepsilon(1), \dots, \varepsilon(r) \in \{1, *\}$ and recall that the focus is the $N \to \infty$ limit of the corresponding mixed moment of S_N. Namely, let

$$M_N := \varphi\bigl(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(r)}\bigr) = \frac{1}{N^{r/2}} \sum_{i(1),\dots,i(r)\in[N]} \varphi\bigl(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\bigr).$$

Making use of the previously-defined equivalence relation, M_N can be rewritten as

$$M_N = \sum_{\mathscr{V}\in\mathscr{P}(r)} \frac{1}{N^{r/2}} \sum_{\substack{i(1),\dots,i(r)\in[N]\ \text{s.t.}\\ (i(1),\dots,i(r))\sim\mathscr{V}}} \varphi\bigl(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\bigr) = \sum_{\mathscr{V}\in\mathscr{P}(r)} \frac{1}{N^{r/2}} M_N^{\mathscr{V}},$$

where

$$M_N^{\mathscr{V}} := \sum_{\substack{i(1),\dots,i(r)\in[N]\ \text{s.t.}\\ (i(1),\dots,i(r))\sim\mathscr{V}}} \varphi\bigl(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\bigr).$$

Focusing on $M_N^{\mathscr{V}}$, suppose first that $\mathscr{V}$ contains a singleton, i.e. a single-element part $\{k\} \in \mathscr{V}$ for some $k \in [r]$. Via the commutation relation (11), $b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}$ can be brought into a naturally ordered form (incurring, in the process, a multiplying factor given by the corresponding product of the commutation coefficients). In turn, by the assumption on the factoring of the naturally ordered products (cf. Definition 2), $\varphi(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)})$ factors according to the blocks in $\mathscr{V}$. Since $\varphi(b_k) = \varphi(b_k^*) = 0$, it follows that for all $N \in \mathbb{N}$, $M_N^{\mathscr{V}} = 0$ for all partitions $\mathscr{V}$ containing a singleton block.

Focus next on partitions of $[r]$ containing blocks with two or more elements or, equivalently, partitions $\mathscr{V} \in \mathscr{P}(r)$ with $|\mathscr{V}| \le \lfloor r/2 \rfloor$, where $|\mathscr{V}|$ denotes the number of blocks in $\mathscr{V}$. Recalling that, by the assumption on the existence of uniform bounds on the moments, we have that for all $\mathscr{V} \in \mathscr{P}(r)$,

$$\left|\varphi\left(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\right)\right| \le \alpha_{\mathscr{V}}$$

for some $\alpha_{\mathscr{V}} \in \mathbb{R}$. Thus, for a partition $\mathscr{V}$ with ℓ blocks, summing over all $i(1), \dots, i(r) \in [N]$ with $(i(1), \dots, i(r)) \sim \mathscr{V}$ yields

$$\left|M_N^{\mathscr{V}}\right| \le \binom{N}{\ell} \ell! \alpha_{\mathscr{V}},$$

and therefore

$$|M_N| \le \sum_{\mathscr{V} \in \mathscr{P}(r)} \frac{\binom{N}{\ell}\ell!}{N^{r/2}} \alpha_{\mathscr{V}}.$$

Noting that (1) the above sum is taken over a fixed (finite) index r, (2) that the only N-dependent term in the above expression is the ratio $\binom{N}{\ell}/N^{r/2}$ and (3) that $\binom{N}{\ell}/N^{r/2} \to 0$ as $N \to \infty$ for $\ell < \lceil r/2 \rceil$, it follows that only those partitions $\mathscr{V}$ with $|\mathscr{V}| \ge \lceil r/2 \rceil$ contribute to the $N \to \infty$ limit of M_N. But, since $|\mathscr{V}| \le \lfloor r/2 \rfloor$, it follows that the only non-vanishing contributions are obtained for r even and partitions with exactly $r/2$ blocks – i.e. pair-partitions, $\mathscr{V} \in \mathscr{P}_2(r)$. Therefore, for r odd,

$$\lim_{N\to\infty} \varphi\left(S_N^{\varepsilon(1)} \cdots S_N^{\varepsilon(r)}\right) = 0$$

and, otherwise,

$$\lim_{N\to\infty} |M_N| = \sum_{\mathscr{V} \in \mathscr{P}_2(r)} \lim_{N\to\infty} N^{-r/2} \sum_{\substack{i(1),\dots,i(r)\in[N] \text{ s.t.} \\ (i(1),\dots,i(r))\sim\mathscr{V}}} \varphi\left(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\right).$$

Next, fixing $i(1), \dots, i(r) \in [N]$ and recalling that $\mathscr{V}$ is a pair-partition of $[r]$, consider the following algorithm for transforming $b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}$, via the commutation relation (11), into an interval-ordered product. Starting with $i(1)$ and recalling that $\mathscr{V}$ is a pair-partition of $[r]$, let $1 < k_1 \le r$ denote the unique index for which $i(k_1) = i(1)$. Consider element $b_{i(1)}$ to be already in place and commute $b_{i(k_1)}$ with the elements to its left until $b_{i(k_1)}$ is immediately to the right of $b_{i(1)}$, recording all the while the commutation coefficients incurred in each transposition. The next iteration, proceeding in the analogous manner, is carried out on the string of length $r-2$ given by $i(2), \dots, i(\check{k}_1), \dots, i(r)$, where $i(\check{k}_1)$ indicates that $i(k_1)$ has been suppressed from the string. Continuing in this manner, the algorithm terminates when the remaining string is the empty string. The resulting moment is of the form

$$\varphi\left(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\right) = \beta_{i(1),\dots,i(r)}^{\varepsilon(1),\dots,\varepsilon(r)} \varphi\left(b_{i(w_1)}^{\varepsilon(w_1)} b_{i(z_1)}^{\varepsilon(z_1)} \cdots b_{i(w_{r/2})}^{\varepsilon(w_{r/2})} b_{i(z_{r/2})}^{\varepsilon(z_{r/2})}\right),$$

where $\mathscr{V} = \{(w_1, z_1), \dots, (w_{r/2}, z_{r/2})\}$ with $w_1 < \cdots < w_{r/2}$ is the underlying pair-partition and $\beta_{i(1),\dots,i(r)}^{\varepsilon(1),\dots,\varepsilon(r)}$ denotes the product of the commutation coefficients incurred in this transformation. Note that though $i(w_j) = i(z_j)$ for all $j = 1, \dots, r/2$, in general $\varepsilon(w_j) \ne \varepsilon(z_j)$ and the above expression therefore also (artificially) distinguishes between $i(w_j)$ and $i(z_j)$.

While it need not be the case that $i(w_1) < \cdots < i(w_{r/2})$, and the moment

$$\varphi\big(b_{i(w_1)}^{\varepsilon(w_1)} b_{i(z_1)}^{\varepsilon(z_1)} \cdots b_{i(w_{r/2})}^{\varepsilon(w_{r/2})} b_{i(z_{r/2})}^{\varepsilon(z_{r/2})}\big)$$

therefore need not be naturally ordered, φ nevertheless factors over the pairs. Specifically, as $\varphi(b_j b_j) = \varphi(b_j^* b_j^*) = 0$, it can be assumed that $\varepsilon(w_j) \neq \varepsilon(z_j)$ for $j = 1, \ldots, r/2$. By Remark 2, it then follows that

$$\varphi\big(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(r)}^{\varepsilon(r)}\big) = \beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)} \varphi\big(b_{i(w_1)}^{\varepsilon(w_1)} b_{i(z_1)}^{\varepsilon(z_1)}\big) \cdots \varphi\big(b_{i(w_{r/2})}^{\varepsilon(w_{r/2})} b_{i(z_{r/2})}^{\varepsilon(w_{r/2})}\big).$$

Next, $\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}$ can be expressed combinatorially as follows. Fixing some $j \in [r/2]$ and considering the corresponding pair $(w_j, z_j) \in \mathscr{V}$ (where $w_j < z_j$), note that for every $k \in [r/2]$ for which $w_j < w_k < z_j < z_k$, the above algorithm commutes z_j and w_k. Additionally note that this commutation is performed exactly once, on the jth iteration, as the process does not revisit pairs that were brought into the desired form in one of the previous steps. The corresponding contribution to $\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}$ is therefore given by $\mu_{\varepsilon(z_j),\varepsilon(w_k)}(i(z_j), i(w_k))$. Similarly, for every $m \in [r/2]$ for which $w_j < w_m < z_m < z_j$, the above algorithm commutes z_j and z_m *as well as* z_j and w_m, and both commutations occur exactly once. The corresponding contribution to $\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}$ is therefore given by $\mu_{\varepsilon(z_j),\varepsilon(z_m)}(i(z_j), i(z_m))\mu_{\varepsilon(z_j),\varepsilon(w_m)}(i(z_j), i(w_m))$. Recall now (cf. Definition 1) that the 4-tuple given by $w_j < w_k < z_j < z_k$ is what is referred to as a *crossing* in $\mathscr{V}$ and encoded by $(w_j, w_k, z_j, z_k) \in \mathrm{Cross}(\mathscr{V})$, whereas the 4-tuple $w_j < w_m < z_m < z_j$ is referred to as a *nesting* in $\mathscr{V}$ and is encoded as $(w_j, w_m, z_m, z_j) \in \mathrm{Nest}(\mathscr{V})$. Finally, realizing that the algorithm performs no other commutations than the two types described, it follows that

$$\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)} = \prod_{\substack{(w_j,w_k,z_j,z_k)\\ \in \mathrm{Cross}(\mathscr{V})}} \mu_{\varepsilon(z_j),\varepsilon(w_k)}\big(i(z_j), i(w_k)\big) \times \prod_{\substack{(w_j,w_m,z_m,z_j)\\ \in \mathrm{Nest}(\mathscr{V})}} \mu_{\varepsilon(z_j),\varepsilon(z_m)}\big(i(z_j), i(z_m)\big)\mu_{\varepsilon(z_j),\varepsilon(w_m)}\big(i(z_j), i(w_m)\big).$$

The encoding of $\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}$ through nestings and crossings of $\mathscr{V}$ is illustrated in Figs 4 and 5.

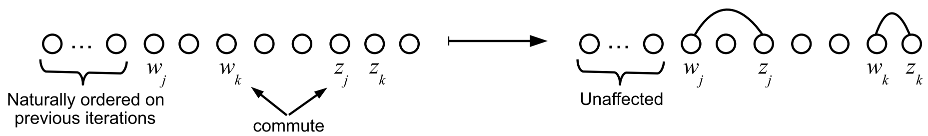

Fig. 4. The process of bringing a mixed moment into a naturally-ordered form involves commuting all the inversions and all the nestings in each of the underlying pair partitions. In commuting a crossing (w_j, w_k, z_j, z_k), as depicted, the corresponding moment incurs a factor $\mu_{\varepsilon(z_j),\varepsilon(w_k)}(i(z_j), i(w_k))$.

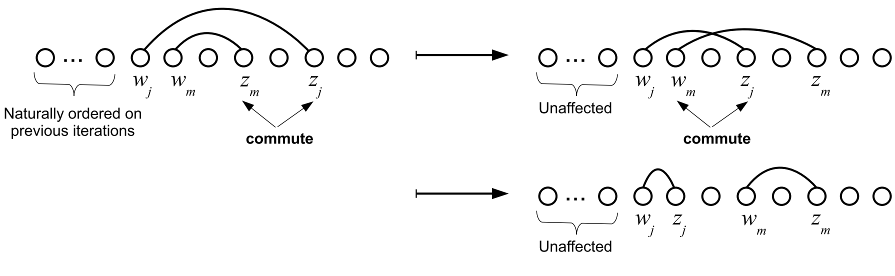

Fig. 5. The process of bringing a mixed moment into a naturally-ordered form involves commuting all the inversions and all the nestings in each of the underlying pair partitions. In commuting a nesting (w_j, w_m, z_m, z_j), as depicted, the corresponding moment incurs a factor $\mu_{\varepsilon(z_j),\varepsilon(z_m)}(i(z_j), i(z_m))\mu_{\varepsilon(z_j),\varepsilon(w_m)}(i(z_j), i(w_m))$.

Putting it all together,

$$\lim_{N\to\infty}\varphi\big(S_N^{\varepsilon(1)}\cdots S_N^{\varepsilon(2n)}\big)$$
$$=\sum_{\mathscr{V}\in\mathscr{P}_2(2n)}\lim_{N\to\infty}N^{-r/2}\sum_{\substack{i(1),\ldots,i(r)\in[N]\text{ s.t.}\\(i(1),\ldots,i(r))\sim\mathscr{V}}}\big(\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}\varphi\big(b_{i(w_1)}^{\varepsilon(w_1)}b_{i(z_1)}^{\varepsilon(z_1)}\big)\cdots\varphi\big(b_{i(w_n)}^{\varepsilon(w_n)}b_{i(z_n)}^{\varepsilon(z_n)}\big)\big).\tag{32}$$

By the assumption on the covariances of the b_i's and the existence of the limit in (29),

$$\lim_{N\to\infty}\varphi\big(S_N^{\varepsilon(1)}\cdots S_N^{\varepsilon(2n)}\big)$$
$$=\sum_{\mathscr{V}\in\mathscr{P}_2(2n)}\Bigg(\varphi\big(b^{\varepsilon(w_1)}b^{\varepsilon(z_1)}\big)\cdots\varphi\big(b^{\varepsilon(w_n)}b^{\varepsilon(z_n)}\big)\lim_{N\to\infty}N^{-r/2}\sum_{\substack{i(1),\ldots,i(r)\in[N]\text{ s.t.}\\(i(1),\ldots,i(r))\sim\mathscr{V}}}\beta_{i(1),\ldots,i(r)}^{\varepsilon(1),\ldots,\varepsilon(r)}\Bigg),$$

which yields (31) and completes the proof. □

Remark 3. *The assumption of Theorem 3 that the covariances of the b_i's are independent of i, namely, that $\varphi(b_i^{\varepsilon_1}b_i^{\varepsilon_2})=\varphi(b_j^{\varepsilon_1}b_j^{\varepsilon_2})$ for all $i,j\in\mathbb{N}$ and $\varepsilon_1,\varepsilon_2\in\{1,*\}$, was not used in obtaining (30) and (32). Provided the existence of the limit in (32), the additional assumption is solely used for the purpose of simplifying (32) as (31).*

The above Theorem 3 differs from Theorem 1 of [22] in the following ways:

1. The more general commutation relation $b_i^{\varepsilon}b_j^{\varepsilon'}=\mu_{\varepsilon',\varepsilon}(j,i)b_j^{\varepsilon'}b_i^{\varepsilon}$ with $\mu_{\varepsilon,\varepsilon'}(i,j)\in\mathbb{R}$ now replaces the commutation relation $b_i^{\varepsilon}b_j^{\varepsilon'}=s(i,j)b_j^{\varepsilon'}b_i^{\varepsilon}$ with spins $s(i,j)\in\{-1,1\}$.
2. For the purpose of factoring naturally ordered second moments as interval-ordered second moments, it is presently additionally assumed that $\varphi(b_i^*b_i^*)=\varphi(b_ib_i)=0$. (Cf. Remark 2.)
3. The convergence of the moments now hinges on the existence of a more complicated limit, which is not only a function of the commutation coefficients and of the underlying partition, as was the case in [22], but also on the pattern of adjoints in the mixed moment of interest (i.e. on the string $\varepsilon(1),\ldots,\varepsilon(n)$).

Note that the assumption on the uniform bounds on the moments is not new, but is instead implicit in [22].

4. Stochastic interpolation

Recall that, analogously to Theorem 1 in [22], the "deterministic version" of the non-commutative CLT hinges on an existence of the limit (29), which is determined by the sequence of commutation coefficients $\{\mu_{\varepsilon,\varepsilon'}(i,j)\}$. Rather than providing more explicit conditions for the existence of the above limit, this section follows the philosophy of [22] and instead considers the scenario where the coefficients "may have been chosen at random." The outcome will be that, starting with a probability law for a single coefficient and extending it to a product measure on the entire coefficient sequence, almost any choice of commutation coefficients will yield a finite and easily describable limit. For this, it is first necessary to define a suitable product measure on the coefficient sequence that is consistent with the dependency structure given by (A)–(B), which is accomplished in Remark 4. In turn, Remark 5 considers the effect on the limit achieved by imposing the vanishing of certain second moments. Finally, Lemma 3 is the remaining ingredient in the "almost sure version" of the non-commutative CLT (viz. the present Theorem 2).

Remark 4. *Defining a measure on the sequence of commutation coefficients by focusing on the triangular sequences $\{\mu_{*,*}(i,j)\}_{1\le i<j}$ and attempting to fix the remaining coefficients via (A)–(B) still leaves one degree of freedom. Namely, $\mu_{*,*}(i,j)$ was not until now explicitly related to $\mu_{*,1}(i,j)$. The need for a third relation governing the sign sequence comes into play when considering positivity requirements. Generally, φ is assumed to be* positive, *that is, if*

$\varphi(aa^*) \geq 0$ for all $a \in \mathscr{A}$. Then, $\varphi(b_i b_i^*) \geq 0$ and $\varphi(b_i b_j b_j^* b_i^*) \geq 0$ for all $i, j \in \mathbb{N}$. But, by the commutation relations and the factoring of naturally ordered moments,

$$\varphi\big(b_i b_j b_j^* b_i^*\big) = \mu_{*,1}(i,j)\mu_{*,*}(i,j)\varphi\big(b_i b_i^*\big)\varphi\big(b_j b_j^*\big).$$

If the sequence $b_1, b_2, \ldots$ is such that $\varphi(b_i b_i^*) > 0$ for all i, the commutation signs must therefore also satisfy the following, third, requirement:

$$\frac{\mu_{*,1}(i,j)}{|\,\mu_{*,1}(i,j)\,|} = \frac{\mu_{*,*}(i,j)}{|\,\mu_{*,*}(i,j)\,|}. \tag{C}$$

In the random setting, (C) translates to $\mu_{*,1}(i,j) = \gamma(i,j)\mu_{*,*}(i,j)$ for some random sequence $\{\gamma(i,j)\}$ supported on $(0,\infty)$.

In assuming $\{\gamma(i,j)\}$ to be independent of $\mu_{*,*}(i,j)$ in line with the general philosophy of this section, the reader may soon verify that only the expectation of $\gamma(i,j)$ will matter from the perspective of Lemma 3. Furthermore, since the expectations of $\mu_{*,*}(i,j)$ and $\mu_{*,1}(i,j)$ will be taken to not depend on the index (i,j), one is free to set $t := \mathbb{E}(\gamma(i,j))$. Then, for $i < j$, (C) becomes:

$$\mu_{*,1}(i,j) = t\mu_{*,*}(i,j), \qquad t > 0. \tag{C$'$}$$

Remark 5. *Beyond the existence of the limit (29), the goal of the present section is to develop a probabilistic framework in which this limit takes on a particularly natural form. For this purpose, the basic setting of Theorem 3 will need to fulfill an additional requirement. Specifically, by the assumption of factoring of naturally-ordered moments, $\varphi(b_i b_j b_i^* b_j^*)$ and $\varphi(b_i b_j^* b_i^* b_j)$ for $i < j$ are both brought into their naturally ordered form by performing a single commutation. In the former case, the commutation incurs a factor $\mu_{*,1}(i,j)$ and, in the latter, the factor $\mu_{*,*}(i,j)$. Yet, in the combinatorial formulation, both products are in the equivalence class (in the sense of "$\sim$") of the pair partition $\pi =$ └┴┘ and both are brought into their naturally ordered form by commuting the single crossing in π. Thus, in order for the combinatorial invariant to be preserved, either:*

- *the expected values of $\mu_{*,1}(i,j)$ and $\mu_{*,*}(i,j)$ must be the same, or,*
- *one of the two mixed moments vanishes, i.e. $\varphi(b_i b_i^*) = 0$ or $\varphi(b_i^* b_i) = 0$ for all $i \in \mathbb{N}$.*

By Remark 4, one may without loss of generality let $\mu_{*,1}(i,j) = t\mu_{*,*}(i,j)$. Thus, as the reader may soon be able to verify, opting to make equal the means of $\mu_{*,1}(i,j)$ and $\mu_{*,*}(i,j)$ by letting $t = 1$ reduces the statistics of the desired limit to those of crossings and the outcome is the same as in the case of randomly chosen commutation signs in [22]. The formulation of Lemma 3 instead opts for the second alternative, and the introduction of the second parameter t will give rise to the appearance of a second combinatorial statistic, that of nestings.

Note that while φ is assumed to be positive, it is not assumed to be faithful, and there is no contradiction in assuming that $\varphi(b_i^* b_i) = 0$ while $\varphi(b_i b_i^*) \neq 0$. As further discussed in the following section, letting $\varphi(b_i^* b_i) = 0$ and $\varphi(b_i b_i^*) = 1$ will provide an asymptotic model for a family of "twisted" annihilation operators, whereas making the opposite choice would yield the corresponding analogue for the creation operators.

Lemma 3. *Fix $0 \leq |q| < t$ and let $\{\mu(i,j)\}_{1 \leq i < j}$ be a collection of independent, identically distributed non-vanishing random variables, with*

$$\mathbb{E}\big(\mu(i,j)\big) = qt^{-1} \in \mathbb{R}, \qquad \mathbb{E}\big(\mu(i,j)^2\big) = 1.$$

Letting $\mu_{*,*}(i,j) = \mu(i,j)$ for $1 \leq i < j$, populate the remaining $\mu_{\varepsilon,\varepsilon'}(i,j)$, for $\varepsilon, \varepsilon' \in \{1, *\}$ and $i \neq j$ $(i, j \in \mathbb{N})$, by

$$\mu_{1,1}(i,j) = \frac{1}{\mu_{*,*}(i,j)}, \qquad \mu_{1,*}(i,j) = \frac{1}{\mu_{*,1}(i,j)},$$

$$\mu_{*,1}(i,j) = t\mu_{*,*}(i,j), \qquad \mu_{\varepsilon',\varepsilon}(j,i) = \frac{1}{\mu_{\varepsilon,\varepsilon'}(i,j)},$$

Then, for any $\mathscr{V} \in \mathscr{P}_2(2n)$ *and* $\varepsilon(1), \ldots, \varepsilon(2n) \in \{1, *\}$, *the limit* (29) *exists a.s. Moreover, if* $\mathscr{V}$ *is such as to satisfy* $(\varepsilon(w), \varepsilon(z)) = (1, *)$ *for all blocks* $(w, z) \in \mathscr{V}$, *the corresponding limit is given by*

$$\lambda_{\mathscr{V},\varepsilon(1),\ldots,\varepsilon(2n)} = q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})} \quad a.s.,$$

where $\mathrm{cross}(\mathscr{V}) = |\mathrm{Cross}(\mathscr{V})|$ *and* $\mathrm{nest}(\mathscr{V}) = |\mathrm{Nest}(\mathscr{V})|$ *denote, respectively, the numbers of crossings and nestings in* $\mathscr{V}$ (*cf. Definition* 1).

Proof. Fix $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$ and, recalling that $i(w_m) = i(z_m)$ for all $m \in [n]$, consider the (classical) random variable X_N given by

$$X_N := N^{-n} \sum_{\substack{i(1),\ldots,i(2n)\in[N] \text{ s.t.}\\ (i(1),\ldots,i(2n))\sim\mathscr{V}}} \Bigg(\prod_{\substack{(w_j,w_k,z_j,z_k)\\ \in\mathrm{Cross}(\mathscr{V})}} \mu_{\varepsilon(z_j),\varepsilon(w_k)}\big(i(z_j), i(w_k)\big) \\ \times \prod_{\substack{(w_\ell,w_m,z_m,z_\ell)\\ \in\mathrm{Nest}(\mathscr{V})}} \mu_{\varepsilon(z_\ell),\varepsilon(z_m)}\big(i(z_\ell), i(z_m)\big)\mu_{\varepsilon(z_\ell),\varepsilon(w_m)}\big(i(z_\ell), i(w_m)\big)\Bigg), \tag{33}$$

where the sequence of random variables $\{\mu_{\varepsilon,\varepsilon'}(i,j)\}_{\varepsilon,\varepsilon'\in\{1,*\},i,j\in\mathbb{N},i\neq j}$ is obtained by letting $\mu_{*,*}(i,j) = \mu(i,j)$ for $i < j$ and fixing the remaining coefficients as prescribed by (9)–(10). The first goal is to compute $\mathbb{E}(X_N)$. By the independence assumption, since the overall product includes no repeated terms, the expectation factors over the products. It therefore suffices to evaluate $\mathbb{E}(\mu_{\varepsilon(z_j),\varepsilon(w_k)}(i(w_j), i(w_k)))$ for each crossing (w_j, w_k, z_j, z_k) and

$$\mathbb{E}(\mu_{\varepsilon(z_\ell),\varepsilon(z_m)}\big(i(w_\ell), i(w_m)\big)\mu_{\varepsilon(z_\ell),\varepsilon(w_m)}\big(i(w_\ell), i(w_m)\big)$$

for each nesting $(w_\ell, w_m, z_m, z_\ell)$ of a given pair-partition. At the outset, recall that every pair-partition $\mathscr{V}$ contributing to X_N is such that $(\varepsilon(w), \varepsilon(z)) = (1, *)$. Then, starting with the crossings and assuming that $i(w_j) = i(z_j) < i(w_k) = i(z_k)$, the corresponding commutation coefficient in (33) and its expectation are given as

$$\mu_{*,1}\big(i(z_j), i(w_k)\big) = t\mu_{*,*}\big(i(z_j), i(w_k)\big) = t\mu\big(i(z_j), i(w_k)\big) \overset{\mathbb{E}}{\longmapsto} t(q/t) = q. \tag{34}$$

When it is instead the case that $i(w_j) = i(z_j) > i(w_k) = i(z_k)$, it suffices to notice that by (A)–(B), $\mu_{*,1}(i,j) = \mu_{*,1}(j,i)$. The same conclusion then holds and each crossing therefore contributes a factor of q on average. Moving on to nestings, let $(w_\ell, w_m, z_m, z_\ell)$ be a nesting. If $i(w_\ell) = z_\ell < i(w_m) = i(z_m)$, the corresponding commutation coefficient in (33) and its expectation are given as

$$\begin{aligned}\mu_{*,*}\big(i(z_\ell), i(z_m)\big)\mu_{*,1}\big(i(z_\ell), i(w_m)\big) &= \mu_{*,*}\big(i(z_\ell), i(z_m)\big)t\mu_{*,*}\big(i(z_\ell), i(w_m)\big) \\ &= t\big(\mu\big(i(w_\ell), i(w_m)\big)\big)^2 \overset{\mathbb{E}}{\longmapsto} t.\end{aligned} \tag{35}$$

If on the other hand $i(w_\ell) = i(z_\ell) > i(w_m) = i(z_m)$, by (A)–(B) the commutation coefficient and its expectation become

$$\begin{aligned}\mu_{*,*}\big(i(z_\ell), i(z_m)\big)\mu_{*,1}\big(i(z_\ell), i(w_m)\big) &= \big(\mu_{*,*}\big(i(z_m), i(z_\ell)\big)\big)^{-1}\mu_{*,1}\big(i(z_\ell), i(w_m)\big) \\ &= \big(\mu\big(i(z_m), i(z_\ell)\big)\big)^{-1}t\mu\big(i(z_m), i(z_\ell)\big) = t \overset{\mathbb{E}}{\longmapsto} t.\end{aligned} \tag{36}$$

Thus, each nesting also contributes a factor of t. It follows that $\mathbb{E}(X_N)$ is given by

$$\mathbb{E}(X_N) = N^{-n} \sum_{\substack{i(1),\ldots,i(2n)\in[N] \text{ s.t.}\\ (i(1),\ldots,i(2n))\sim\mathscr{V}}} q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})} = q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})} N^{-n} \binom{N}{n} n!. \tag{37}$$

Thus,

$$\lim_{N\to\infty} \mathbb{E}(X_N) = q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})}. \tag{38}$$

It now remains to show that $\lim_{N\to\infty} X_N = \lim_{N\to\infty} \mathbb{E}(X_N)$ a.s., that is, that for every $\eta > 0$,

$$\mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{M\geq N} \big\{\big|X_M - \mathbb{E}(X_M)\big| \geq \eta\big\}\Big) = 0.$$

The calculation is analogous to that in [22]. By the subadditivity of $\mathbb{P}$ and a standard application of Markov inequality,

$$\mathbb{P}\Big(\bigcup_{M\geq N} \big\{\big|X_M - \mathbb{E}(X_M)\big| \geq \eta\big\}\Big) \leq \sum_{M\geq N} \mathbb{P}\big(\big|X_M - \mathbb{E}(X_M)\big| \geq \eta\big) \leq \frac{1}{\eta^2} \sum_{M\geq N} \mathbb{E}\big(\big|X_M - \mathbb{E}(X_M)\big|^2\big). \tag{39}$$

In turn, $\mathbb{E}(|X_M - \mathbb{E}(X_M)|^2) = \mathbb{E}(X_M^2) - \mathbb{E}(X_M)^2$, with

$$\mathbb{E}(X_M)^2 = q^{2\,\mathrm{cross}(\mathscr{V})} t^{2\,\mathrm{nest}(\mathscr{V})}$$

and

$$\begin{aligned}\mathbb{E}\big(X_M^2\big) = M^{-2n} &\sum_{\substack{i(1),\dots,i(2n)\in[N] \text{ s.t.}\\ (i(1),\dots,i(2n))\sim\mathscr{V},\\ j(1),\dots,j(2n)\in[N] \text{ s.t.}\\ (j(1),\dots,j(2n))\sim\mathscr{V}}} \mathbb{E}\Big[\prod_{(w_k,w_\ell)\in\mathrm{Cross}(\mathscr{V})} \big(\mu_{*,1}\big(i(w_k), i(w_\ell)\big)\big) \\ &\times \big(\mu_{*,1}\big(j(w_k), j(w_\ell)\big)\big) \prod_{(w_m,w_m)\in\mathrm{Nest}(\mathscr{V})} \big(\mu_{*,*}\big(i(w_m), i(w_n)\big)\mu_{*,1}\big(i(w_m), i(w_n)\big)\big) \\ &\times \big(\mu_{*,*}\big(j(w_m), j(w_n)\big)\mu_{*,1}\big(j(w_m), j(w_n)\big)\big)\Big],\end{aligned} \tag{40}$$

where, for convenience of notation, each crossing $(w_k, w_\ell, z_k, z_\ell)$ was abbreviated as (w_k, w_ℓ), and similarly for the nestings. Now suppose that for two choices of indices and the corresponding sets (not multisets) $\{i(1), \dots, i(2n)\}$ and $\{j(1), \dots, j(2n)\}$, there is at most one index in common, i.e. suppose that $\{i(1), \dots, i(2n)\} \cap \{j(1), \dots, j(2n)\} \leq 1$. In that case, $(i(k), i(k')) \neq (j(m), j(m'))$ for all $k, k', m, m' \in [2n]$ with $k \neq k', m \neq m'$. By the independence assumption, the above expectation factors over the product (up to the parenthesized terms) and the contribution of each such $\{i(1), \dots, i(2n)\}$, $\{j(1), \dots, j(2n)\}$ is simply $q^{2\,\mathrm{cross}(\mathscr{V})} t^{2\,\mathrm{nest}(\mathscr{V})}$. Thus, the choices of indices with $\{i(1), \dots, i(2n)\} \cap \{j(1), \dots, j(2n)\} \leq 1$ do not contribute to the variance $\mathbb{E}(|X_M - \mathbb{E}(X_M)|^2)$. It now remains to consider the $\Theta(M^{2n-2})$ remaining terms of the sum (40).

By the Cauchy–Schwarz inequality, the expectation of the product is bounded, and thus

$$\mathbb{E}\big(\big|X_M - \mathbb{E}(X_M)\big|^2\big) \leq M^{-2n} M^{2n-2} C = \frac{C}{M^2},$$

where C does not depend on M. Since $\sum_{M\geq 0} M^{-2}$ converges,

$$\lim_{N\to\infty} \sum_{M\geq N} \mathbb{E}\big(\big|X_M - \mathbb{E}(X_M)\big|^2\big) \to 0,$$

and therefore by (39),

$$\mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{M\geq N} \big\{\big|X_M - \mathbb{E}(X_M)\big| \geq \eta\big\}\Big) = \lim_{N\to\infty} \mathbb{P}\Big(\bigcup_{M\geq N} \big\{\big|X_M - \mathbb{E}(X_M)\big| \geq \eta\big\}\Big) = 0.$$

This completes the proof. □

5. Random matrix models

Considering some prescribed sequence $\{\mu_{\varepsilon,\varepsilon'}(i,j)\}_{\varepsilon,\varepsilon'\in\{1,*\},i,j\in\mathbb{N},i\neq j}$ of real-valued commmutation coefficients satisfying (A)–(B), Lemma 1 exhibits a set of elements of a matrix algebra that satisfy the corresponding commutativity structure. The construction is analogous to the one given in Lemma 1 and the latter is in fact stated in a form that renders the present generalization natural.

Proof of Lemma 2. To show that $b_i^{\varepsilon} b_j^{\varepsilon'} = \mu_{\varepsilon',\varepsilon}(j,i) b_j^{\varepsilon'} b_i^{\varepsilon}$, it suffices to consider the definitions in (17) and commute 2×2 matrices. Specifically, let $i<j$ and, by elementary manipulations on tensor products, write

$$\begin{aligned} b_i b_j = {}& (\sigma_{\mu(1,i)}\sigma_{\mu(1,j)}) \otimes \cdots \otimes (\sigma_{\mu(i-1,i)}\sigma_{\mu(i-1,j)}) \otimes (\gamma\sigma_{\mu(i,j)}) \otimes (\sigma_1\sigma_{\mu(i+1,j)}) \otimes \cdots \\ & \otimes (\sigma_1\sigma_{\mu(j-1,j)}) \otimes (\sigma_1\gamma) \otimes (\sigma_1\sigma_1)^{\otimes(n-j)}. \end{aligned} \tag{41}$$

Now note that $\sigma_x\sigma_y = \sigma_y\sigma_x$ for all $x, y \in \mathbb{R}$. Moreover, $\gamma\sigma_x = \sqrt{t}x\sigma_x\gamma$. Thus,

$$\gamma\sigma_{\mu(i,j)} = \sqrt{t}\mu(i,j)\sigma_{\mu(i,j)}\gamma \quad \text{and} \quad \sigma_1\gamma = (\sqrt{t})^{-1}\gamma\sigma_1,$$

and, therefore,

$$b_i b_j = \frac{\sqrt{t}\mu(i,j)}{\sqrt{t}} b_j b_i = \mu_{*,*}(i,j) b_j b_i = \mu_{1,1}(j,i) b_j b_i.$$

Next, in commuting b_i^* with b_j, the only non-trivial commutations are that of γ^* with $\sigma_{\mu(i,j)}$ and $\sigma_1^* = \sigma_1$ with γ. Since $\gamma^*\sigma_x = (\sqrt{t}x)^{-1}\sigma_x\gamma^*$, it follows that

$$b_i^* b_j = \frac{1}{\sqrt{t}}\frac{1}{\sqrt{t}\mu(i,j)} b_j b_i^* = \frac{1}{t\mu(i,j)} b_j b_i^* = \frac{1}{t\mu_{*,*}(i,j)} b_j b_i^* = \frac{1}{\mu_{*,1}(i,j)} b_j b_i^* = \mu_{1,*}(j,i) b_j b_i^*.$$

The remaining relations now follow by taking adjoints, and the result is that $b_i^{\varepsilon} b_j^{\varepsilon'} = \mu_{\varepsilon',\varepsilon}(j,i) b_j^{\varepsilon'} b_i^{\varepsilon}$.

It remains to show that, in addition to the commutation relation, the resulting matrix sequences also satisfy the assumptions (1)–(4) of Theorem 2. Start by noting that for $a_1,\dots,a_k \in \mathscr{M}_2$, $\varphi(a_1\otimes\cdots\otimes a_k) = (a_1)_{11}\cdots(a_k)_{11}$, where $(a)_{11} := \langle e_1 a, e_1\rangle_2$. It therefore immediately follows that for all $i\in\mathbb{N}$, $\varphi(b_i) = \varphi(b_i^*) = 0$. By the same token, it is also clear that for all $i, j \in \mathbb{N}$, $\varphi(b_i b_i^*) = \varphi(b_j b_j^*) = 1$ and $\varphi(b_i^{\varepsilon} b_i^{\varepsilon'}) = \varphi(b_i^{\varepsilon} b_i^{\varepsilon'}) = 0$ for $\varepsilon, \varepsilon' \in \{1,*\}$ with $(\varepsilon,\varepsilon') \neq (1,*)$, and, furthermore, $|\varphi(\prod_{i=1}^n b_{j(i)}^{\varepsilon(i)})| \leq 1$ for all n and all choices of exponents and indices. The factoring over naturally ordered products also follows immediately, completing the proof. □

Combining Theorem 2 with Lemma 2, and comparing the resulting moments with those given in Section 2.2, immediately yields the desired asymptotic models for the creation, annihilation, and field operators on the (q,t)-Fock space. For instance, the mixed moments of S_N converge to those of the annihilation operator $a(e_1)$, where e_1 is an element of the orthonormal basis of $\mathscr{H}$. More generally, the expressions of Section 2.2 in fact consider *systems* of operators, e.g. they specify the *joint mixed moments* of annihilation operators $a(e_1),\dots,a(e_n)$ associated with basis elements $e_1,\dots,e_n$. In order to asymptotically realize the joint moments of $a(e_1),\dots,a(e_n)$ rather than the moments of $a(e_1)$ alone, it suffices to consider a sequence $S_{N,1},\dots,S_{N,n}$ of partial sums built from non-intersecting subsets of $\{b_i\}_{i\in\mathbb{N}}$. For instance, the fact that e_i and e_j are orthogonal for $i\neq j$ and that the moment $\varphi_{q,t}(a(e_i)a(e_j))$ vanishes follows (in this asymptotic setting) from the fact that $\varphi(b_i b_j) = 0$ for $i \neq j$. The general formulation is found in Corollary 1.

Finally, it should be emphasized that the two-parameter Jordan–Wigner transform produces the desired random matrix models for *any* coefficient sequence satisfying the consistency conditions (A) and (B). Whereas the coefficient sequences drawn as described in Theorem 2 almost surely give rise to the (q,t)-Gaussian random matrix models, coefficient sequences drawn from a different random process may also lead to a finite – albeit different – limit in Theorem 3. In that case, the two-parameter Jordan–Wigner transform will provide a new class of random matrix models. Given the appearance of the (q,t)-Gaussian statistics in several areas of mathematics and in physics [5], Theorem 3 together with Lemma 2 may provide a framework for discovering and realizing further relevant classes of non-commutative Gaussian statistics.

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