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# Convergence to the Brownian Web for a generalization of the drainage network model

Cristian Coletti[a,1] and Glauco Valle[b,2]

[a] *UFABC - Centro de Matemática, Computação e Cognição, Avenida dos Estados, 5001, Santo André - São Paulo, Brasil.*
*E-mail: cristian.coletti@ufabc.edu.br*
[b] *UFRJ - Departamento de métodos estatísticos do Instituto de Matemática, Caixa Postal 68530, 21945-970, Rio de Janeiro, Brasil.*
*E-mail: glauco.valle@im.ufrj.br*

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**Abstract.** We introduce a system of one-dimensional coalescing nonsimple random walks with long range jumps allowing paths that can cross each other and are dependent even before coalescence. We show that under diffusive scaling this system converges in distribution to the Brownian Web.

**Résumé.** Nous introduisons un système de marches aléatoires coalescentes unidimensionnelles, avec des sauts à longue portée. Ce système autorise des chemins qui se croisent, et qui sont dépendants, même avant leur coalescence. Après une renormalisation diffusive, nous montrons que ce système converge en loi vers le réseau brownien.

*MSC:* 60K35

*Keywords:* Brownian Web; Coalescing Brownian motions; Coalescing random walks; Drainage network; Scaling limit; Invariance principle; Interacting particle systems

## 1. Introduction

The paper is devoted to the analysis of convergence in distribution for a diffusively rescaled system of one-dimensional coalescing random walks starting at each point in the space and time lattice $\mathbb{Z} \times \mathbb{Z}$. To not get repetitive, when we make mention to the convergence of a system of random walks, we always consider one-dimensional random walks and diffusive space–time scaling. In this Introduction, we aim at describing informally the system and explain why its study is relevant.

First, the limit is the Brownian Web (BW) which is a system of coalescing Brownian motions starting at each point in the space and time plane $\mathbb{R} \times \mathbb{R}$. It is the natural scaling limit of a system of simple coalescing random walks. Here, the BW is a proper random element of a metric space whose points are compact sets of paths. This characterization of the BW was given by Fontes, Isopi, Newman and Ravishankar [7], although formal descriptions of systems of coalescing Brownian motions had previously been considered, initially by Arratia [1,2] and then by Tóth and Werner [15]. Following [7], the study of convergence in distribution of systems of random walks to the BW and its variations has become an active field of research, for instance, see [3,4,6,8,9,11,13,14].

In Fontes, Isopi, Newman and Ravishankar [8] a general convergence criteria for the BW is obtained but only verified for a system of simple coalescing random walks that evolve independently up to the time of coalescence (no

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crossing paths). These criteria are used in Newman, Ravishankar and Sun [11] to prove convergence to the Brownian Web of coalescing nonsimple random walks that evolve independently up to the time of coalescence (crossing paths).

Coletti, Fontes and Dias [4] have proved the convergenge of the BW for a system of coalescing Random Walks introduced by Gangopadhyay, Roy and Sarkar [10]. This system is called the drainage network. It evolves in the space–time lattice $\mathbb{Z} \times \mathbb{Z}$ in the following way: Each site of $\mathbb{Z} \times \mathbb{Z}$ is considered open or closed according to an i.i.d. family of Bernoulli Random variables; If at time $z$ a walk is at site $x$ then at time $z+1$ its position is the open site nearest to $x$ (if we have two possible choices for the nearest open site, we choose one of them with probability $1/2$). The Drainage network is a system of coalescing nonsimple random walks with long range jumps where no crossing paths may occur. Furthermore the random walks paths are dependent before coalescence. These properties make the convergence of the drainage network to the BW a relevant and nontrivial example.

We propose here a natural generalization of the drainage network and prove the convergence to the BW. If at time $z$ a walk occupies the site $x$, then it chooses at random $k \in \mathbb{N}$ and jumps to the $k$th open site nearest to $x$, see the next section for the formal description. Now we have a system of coalescing nonsimple random walks with long range jumps and dependence before coalescence where crossing paths may occur. We suggest to the reader the discussion on [11] about the additional difficulties that arise in the study of convergence of systems of coalescing random walks when crossing paths may occur.

We finish this section with a brief description of the BW which follows closely the description given in [13], see also [8] and the appendix in [14]. Consider the extended plane $\bar{\mathbb{R}}^2 = [-\infty, \infty]^2$ as the completion of $\mathbb{R}^2$ under the metric

$$\rho\big((x_1, t_1), (x_2, t_2)\big) = \big|\tanh(t_1) - \tanh(t_2)\big| \vee \left|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\right|,$$

and let $\bar{\rho}$ be the induced metric on $\bar{\mathbb{R}}^2$. In $(\bar{\mathbb{R}}^2, \bar{\rho})$, the lines $[-\infty, \infty] \times \{\infty\}$ and $[-\infty, \infty] \times \{-\infty\}$ correspond respectively to single points $(\star, \infty)$ and $(\star, -\infty)$, see picture 2 in [14]. Denote by $\Pi$ the set of all continuous paths in $(\bar{\mathbb{R}}^2, \bar{\rho})$ of the form $\pi : t \in [\sigma_\pi, \infty] \to (f_\pi(t), t) \in (\bar{\mathbb{R}}^2, \bar{\rho})$ for some $\sigma_\pi \in [-\infty, \infty]$ and $f_\pi : [\sigma, \infty] \to [-\infty, \infty] \cup \{\star\}$. For $\pi_1, \pi_2 \in \Pi$, define $d(\pi_1, \pi_2)$ by

$$\big|\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})\big| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left|\frac{\tanh(f_{\pi_1}(t \vee \sigma_{\pi_1}))}{1+|t|} - \frac{\tanh(f_{\pi_2}(t \vee \sigma_{\pi_2}))}{1+|t|}\right|,$$

we have a metric in $\Pi$ such that $(\Pi, d)$ is a complete separable metric space. Now define $\mathcal{H}$ as the space of compact sets of $(\Pi, d)$ with the topology induced by the Hausdorff metric. Then $\mathcal{H}$ is a complete separable metric space. The Brownian web is a random element $\mathcal{W}$ of $\mathcal{H}$ whose distribution is uniquely characterized by the following three properties (see Theorem 2.1 in [8]):

(a) For any deterministic $z \in \mathbb{R}^2$, almost surely there is a unique path $\pi_z$ of $\mathcal{W}$ that starts at $z$.
(b) For any finite deterministic set of points $z_1, \ldots, z_k$ in $\mathbb{R}^2$, the collection $(\pi_1, \ldots, \pi_n)$ is distributed as coalescing Brownian motions independent up to the time of coalescence.
(c) For any deterministic countable dense subset $\mathcal{D} \subset \mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{\pi_z\colon z \in \mathcal{D}\}$ in $(\Pi, d)$.

This paper is organized as follows: In Section 2 we describe the generalized drainage network model, obtain some basic properties and state the main result of this text which says that it converges to the BW. The convergence is proved in Section 3 where Fontes, Isopi, Newmann and Ravishankar [8] criteria is presented and discussed. As we shall explain, a careful analysis of the proof presented in [11] reduce our work to prove an estimate on the distribution of the coalescing time between two walks in the model, which is carried in Section 3.1, and to prove that a diffusively rescaled system of $n$ walks in the model converge in distribution to $n$ coalescing Brownian motions which are independent up to the coalescence time, this is done in Section 3.2.

## 2. The generalized drainage network model

Let $\mathbb{N}$ the set of positive integers. In order to present our model we need to introduce some notation:

– Let $(\omega(z))_{z\in\mathbb{Z}^2}$ be a family of independent Bernoulli random variables with parameter $p \in (0,1)$ and denote by $\mathrm{P}_p$ the induced probability in $\{0,1\}^{\mathbb{Z}^2}$.
– Let $(\theta(z))_{z\in\mathbb{Z}^2}$ be a family of independent Bernoulli random variables with parameter $1/2$ and denote by $\mathrm{P}_{1/2}$ the induced probability in $\{0,1\}^{\mathbb{Z}^2}$.
– Let $(\zeta(z))_{z\in\mathbb{Z}^2}$ be a family of independent and identically distributed random variables on $\mathbb{N}$ with probability function $q:\mathbb{N}\to[0,1]$ and denote by $\mathrm{P}_{q(\cdot)}$ the induced probability in $\mathbb{N}^{\mathbb{Z}^2}$.

We suppose that the three families above are independent of each other and thus they have joint distribution given by the product probability measure $\mathrm{P}=\mathrm{P}_p\times\mathrm{P}_{1/2}\times\mathrm{P}_{q(\cdot)}$ on the product space $\{0,1\}^{\mathbb{Z}^2}\times\{0,1\}^{\mathbb{Z}^2}\times\mathbb{N}^{\mathbb{Z}^2}$. The expectation induced by P will be denoted by E.

***Remark 2.1.*** 1. *Some coupling arguments used in the proofs ahead will require enlargements of the probability space, we will keep the same notation since the formal specifications of these couplings are straightforward from their descriptions*. 2. *At some points, we will only be interested in the distributions of the sequences* $(\omega((z_1,z_2)))_{z_1\in\mathbb{Z}}$, $(\theta((z_1,z_2)))_{z_1\in\mathbb{Z}}$ *and* $(\zeta((z_1,z_2)))_{z_1\in\mathbb{Z}}$ *which do not depend on* $z_2\in\mathbb{Z}$, *justifying the following abuse of notation* $\omega(z_2)=\omega((1,z_2))$, $\theta(z_2)=\theta((1,z_2))$ *and* $\zeta(z_2):=\zeta((1,z_2))$ *for* $z_2\in\mathbb{Z}$.

For points on the lattice $\mathbb{Z}^2$ we say that $(\tilde z_1,\tilde z_2)$ is above $(z_1,z_2)$ if $z_2<\tilde z_2$, immediately above if $z_2=\tilde z_2-1$, at the right of $(z_1,z_2)$ if $z_1<\tilde z_1$ and at the left if $z_1>\tilde z_1$. Moreover, we say that a site $z\in\mathbb{Z}^2$ is open if $\omega(z)=1$ and closed otherwise.

Let $h:\mathbb{Z}^2\times\mathbb{N}\times\{0,1\}\to\mathbb{Z}^2$ be defined according to the following steps:

(i) Fix $(z,k,i)\in\mathbb{Z}^2\times\mathbb{N}\times\{0,1\}$.
(ii) Suppose that the $k$th closest open site to $z$, which is immediately above it, is uniquely defined. Denote this site by $w$ and set $h(z,k,0)=h(z,k,1)=w$.
(iii) Suppose that the $k$th closest open site to $z$, which is immediately above it, is not uniquely defined. In this case, one of these sites is immediately above and at the left of $z$, denote it $w_1$. The other one is immediately above and at the right of $z$, denote it $w_2$. Set $h(z,k,0)=w_1$ and $h(z,k,1)=w_2$.

Now for every $z\in\mathbb{Z}^2$ define recursively the random path $\Gamma^z(0)=z$ and

$$\Gamma^z(n)=\big(\Gamma_1^z(n),\Gamma_2^z(n)\big)=h\big(\Gamma^z(n-1),\zeta\big(\Gamma^z(n-1)\big),\theta\big(\Gamma^z(n-1)\big)\big)$$

for every $n\ge 1$.

Let $\mathcal{G}=(V,\mathcal{E})$ be the random directed graph with set of vertices $V=\mathbb{Z}^2$ and edges $\mathcal{E}=\{(z,\Gamma^z(1)):\ z\in\mathbb{Z}^2\}$. This generalizes the two-dimensional drainage network model introduced in [10] which corresponds to the case $q(1)=1$. It will be called here the two-dimensional generalized drainage network model (GDNM).

The random graph $\mathcal{G}$ may be seen as a set of continuous random paths $X^z=\{X_s^z:\ s\ge z_2\}$. First, define $Z_n^z=\Gamma_1^z(n)$, for every $z\in\mathbb{Z}^2$ and $n\ge 0$. The random path $X^z$ defined by linear interpolation as

$$X_s^z=\big((z_2+n+1)-s\big)Z_n^z+\big(s-(z_2+n)\big)Z_{n+1}^z$$

for every $s\in[z_2+n,z_2+n+1)$ and $n\ge 1$. Put

$$\mathcal{X}=\big\{\big(\sigma^{-1}X_s^z,s\big)_{s\ge z_2}:\ z\in\mathbb{Z}^2\big\},$$

where $\sigma^2$ is the variance of $X_1^{(0,0)}$ (at the end of the section we show that $\sigma^2$ is finite if $q(\cdot)$ has finite second moment). The random set of paths $\mathcal{X}$ is called the generalized drainage network. Let

$$\mathcal{X}_\delta=\big\{\big(\delta x_1,\delta^2x_2\big)\in\mathbb{R}^2:\ (x_1,x_2)\in\mathcal{X}\big\}$$

for $\delta\in(0,1]$, be the diffusively rescaled generalized drainage network. Note that two paths in $\mathcal{X}$ will coalesce when they meet each other at an integer time. Therefore, if $q(1)<1$, we have a system of coalescing random walks that can cross each other. Our aim is to prove that $\mathcal{X}_\delta$ converges in distribution to $\mathcal{W}$. This extends the results in [4] where the case $q(1) = 1$ was considered and no crossing between random walks occurs.

We say that $q(\cdot)$ has finite range if there exists a finite set $F \subset \mathbb{N}$ such that $q(n) \neq 0$ if and only if $n \in F$. Our main result is the following:

**Theorem 2.1.** *If $q(\cdot)$ has finite range, then the diffusively rescaled GDNM, $\mathcal{X}_\delta$, converges in distribution to the $\mathcal{W}$ as $\delta \to 0$.*

***Remark 2.2.*** 1. *Note that the hypothesis that $q(\cdot)$ has finite range does not imply that the random paths in the GDNM have finite range jumps. It should be clear from the definition that any sufficiently large size of jump is allowed with positive probability.* 2. *We only need the hypothesis that $q(\cdot)$ has finite range at a specific point in the proof of Theorem 2.1 (it will be discussed later), other arguments just require that $q(\cdot)$ has at most finite fifth moment. So we conjecture that Theorem 2.1 holds under this condition.*

The next section is devoted to the proof of Theorem 2.1. We finish this section with some considerations about the random paths in the drainage network. The processes $(Z_n^z)_{n\geq 0}$, $z \in \mathbb{Z}^2$, are irreducible aperiodic symmetric random walks identically distributed up to a translation of the starting point. Now consider $X = (X_t)_{\{t\geq 0\}}$ as a random walk with the same distribution as $X^{(0,0)}$. We show that if $q(\cdot)$ has finite absolute $m$th moment then $X_s$ also have this property.

**Proposition 2.2.** *If $q(\cdot)$ has finite $m$th moment then $X_s$ also has finite absolute $m$th moment.*

**Proof.** It is enough to show that the increments of $Z_n := Z_n^{(0,0)}$ have finite absolute $m$th moment. Write

$$\xi_n = Z_n - Z_{n-1}$$

for the increments of $Z$. Then $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with symmetric probability function $\tilde{q}(\cdot)$ such that $\tilde{q}(0) = pq(1)$ and, for $z \geq 1$, $\tilde{q}(z)$ is equal to

$$\sum_{k=1}^{2z} q(k) \mathrm{P}\Bigg( \sum_{j=-z+1}^{z-1} \omega(j) = k-1, \omega(z)\big(1-\omega(-z)\big) = 1 \Bigg)$$
$$+ \frac{1}{2} \sum_{k=1}^{2z} q(k) \mathrm{P}\Bigg( \sum_{j=-z+1}^{z-1} \omega(j) = k-1, \omega(z) = 1 \text{ and } \omega(-z) = 1 \Bigg)$$
$$+ \frac{1}{2} \sum_{k=2}^{2z+1} q(k) \mathrm{P}\Bigg( \sum_{j=-z+1}^{z-1} \omega(j) = k-2, \omega(z) = 1 \text{ and } \omega(-z) = 1 \Bigg)$$

which is equal to

$$p(1-p)^{2z} q(1) + \frac{1}{2} p^{2z+1} q(2z+1) + \frac{1}{2} \sum_{k=1}^{2z} p^{k+1} (1-p)^{2z-k} \binom{2z-1}{k-1} q(k)$$
$$+ \sum_{k=2}^{2z} p^k (1-p)^{2z-k+1} \left[ \binom{2z-1}{k-1} + \frac{1}{2} \binom{2z-1}{k-2} \right] q(k).$$

Therefore, since

$$\binom{2z-1}{k-1} + \frac{1}{2} \binom{2z-1}{k-2} \leq \frac{1}{2} \binom{2z+1}{k},$$

the absolute $m$th moment of $\tilde{q}$ is bounded above by

$$\sum_{z=1}^{+\infty} z^m \sum_{k=1}^{2z+1} q(z)\left[\frac{1}{2p}+\frac{1}{4(1-p)}\right]\binom{2z+1}{k} p^{k+1}(1-p)^{(2z+2)-(k+1)}$$

which is equal to

$$\left[\frac{1}{2p}+\frac{1}{4(1-p)}\right]\sum_{k=1}^{+\infty} q(k) \sum_{z=\lfloor k/2\rfloor}^{+\infty} z^m \binom{2z+1}{k} p^{k+1}(1-p)^{(2z+2)-(k+1)}$$

the sum over $z$ is bounded by one half of the $m$th moment of a negative binomial random variable with parameters $k+1$ and $p$ which is a polynomial of degree $m$ on $k$. Therefore

$$\sum_{z=1}^{+\infty} z^m \tilde{q}(z) \le \sum_{k=1}^{+\infty} r(k) q(k),$$

where $r$ is a polynomial of degree $m$. If $q(\cdot)$ has finite $m$th moment then the right hand side above is finite, which means that $\tilde{q}(\cdot)$ also has finite $m$th moment. □

## 3. Convergence to the Brownian Web

The proof of Theorem 2.1 follows from the verification of four conditions given in [7] for convergence to the Brownian Web of a system of coalescing random paths that can cross each other. These conditions are described below:

(I) Let $\mathcal{D}$ be a countable dense set in $\mathbb{R}^2$. For every $y \in \mathbb{R}^2$, there exist single random paths $\theta_\delta^y \in \mathcal{X}_\delta$, $\delta > 0$, such that for any deterministic $y_1, \dots, y_m \in \mathcal{D}$, $\theta_\delta^{y_1}, \dots, \theta_\delta^{y_n}$ converge in distribution as $\delta \to 0+$ to coalescing Brownian motions, with diffusion constant $\sigma$ depending on $p$, starting at $y_1, \dots, y_m$.

(T) Let $\Gamma_{L,T} = \{z \in \mathbb{Z}^2\colon z_1 \in [-L, L] \text{ and } z_2 \in [-T, T]\}$ and $R_{(x_0,t_0)}(u,t) = \{z \in \mathbb{Z}^2\colon z_1 \in [x_0 - u, x_0 + u] \text{ and } z_2 \in [t_0, t_0 + t]\}$. Define $A(x_0, t_0, u, t)$ as the event that there exists a path starting in the rectangle $R_{(x_0,t_0)}(u,t)$ that exits the larger rectangle $R_{(x_0,t_0)}(Cu, 2t)$ through one of its sides, where $C > 1$ is a fixed constant. Then

$$\limsup_{t\to 0+} \frac{1}{t} \limsup_{\delta\to 0+} \sup_{(x_0,t_0)\in\Gamma_{L,T}} \mathrm{P}\big(A_{\mathcal{X}_\delta}(x_0, t_0; u, t)\big) = 0.$$

($B_1'$) Let $\eta_{\mathcal{V}}(t_0, t; a, b)$, $a < b$, denote the random variable that counts the number of distinct points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths in $\mathcal{V}$ which also touch some point in $[a, b] \times \{t_0\}$. Then, for every $\beta > 0$,

$$\limsup_{\epsilon\to 0+} \limsup_{\delta\to 0+} \sup_{t>\beta} \sup_{(a,t_0)\in\mathbb{R}^2} \mathrm{P}\big(\eta_{\mathcal{X}_\delta}(t_0, t; a, a+\epsilon) \ge 2\big) = 0.$$

($B_2'$) For every $\beta > 0$,

$$\limsup_{\epsilon\to 0+} \frac{1}{\epsilon} \limsup_{\delta\to 0+} \sup_{t>\beta} \sup_{(a,t_0)\in\mathbb{R}^2} \mathrm{P}\big(\eta_{\mathcal{X}_\delta}(t_0, t; a, a+\epsilon) \ge 3\big) = 0$$

for every $\beta > 0$.

In [11] a condition is derived to replace $B_2'$ in the proof of convergence of diffusively rescaled nonnearest neighbour coalescing random walks to the Brownian web. Their condition is based on duality and can be expressed as

(E) Let $\hat{\eta}_{\mathcal{V}}(t_0, t; a, b)$, $a < b$, be the number distinct points in $(a, b) \times \{t_0 + t\}$ that are touched by a path that also touches $\mathbb{R} \times \{t_0\}$. Then for any subsequential limit $\mathcal{X}$ of $\mathcal{X}_\delta$ we have that

$$\mathrm{E}\big[\hat{\eta}_{\mathcal{X}}(t_0, t; a, b)\big] \le \frac{b-a}{\sqrt{\pi t}}.$$

If we have proper estimate on the distributions of the coalescing times and condition I, then the proof of conditions $B_1'$, T and E follows from adaptations of arguments presented in [11], see also [12]. The estimate we need on the distribution of the coalescing time of two random walks in the drainage network starting at time zero is that the probability that they coalesce after time $t$ is of order $1/\sqrt{t}$. This is central in arguments related to estimates on counting variables associated to the density of paths in the system of coalescing random walks under consideration. Let us be more precise. For each $(z,w) \in \mathbb{Z}^2$, let $Y^{z,w}$ be defined as

$$Y_s^{z,w} = X_s^z - X_s^w, \quad s \geq \max(z_2, w_2).$$

The processes $(Y^{z,w})_{(z,w)\in\mathbb{Z}^2}$ are also spatially inhomogeneous random walks with mean zero square integrable increments, identically distributed up to a translation of $z$ and $w$ by the same amount. For $k \in \mathbb{Z}$, let $Y^k = (Y_t^k)_{\{t\geq 0\}}$ be a random walk with the same distribution as $Y^{(0,0),(k,0)}$. Define the random times

$$\tau_k := \min\{t \geq 0, t \in \mathbb{N}\colon\ Y_t^k = 0\}$$

and

$$\nu_k(u) := \min\{t \geq 0, t \in \mathbb{N}\colon\ Y_t^k \geq u\}$$

for every $k \in \mathbb{Z}$ and $u > 0$.

**Proposition 3.1.** *If $q(\cdot)$ has finite range, there exists a constant $C > 0$, such that*

$$\mathrm{P}(\tau_k > t) \leq \frac{C|k|}{\sqrt{t}} \quad \text{for every } t > 0 \text{ and } k \in \mathbb{Z}. \tag{3.1}$$

Proposition 3.1 has some consequences that we now state:

**Lemma 3.2.** *If $q(\cdot)$ has finite range, then for every $u > 0$ and $t > 0$, there exists a constant $C = C(t,u) > 0$, such that*

$$\limsup_{\delta\to 0} \frac{1}{\delta} \sup_{k\leq 1} \mathrm{P}\bigl(\nu_k(\delta^{-1}u) < \tau_k \wedge \delta^{-2}t\bigr) < C. \tag{3.2}$$

***Remark 3.1.*** *All we need in order to prove Lemma 3.2 is inequality (3.1) and finite third moment of $q(\cdot)$. In particular, if inequality (3.1) holds under finite third moment of $q(\cdot)$, then the inequality in Lemma 3.2 also holds.*

**Lemma 3.3.** *Let $\mathcal{X}_0 = \{(X_s^{(z_1,0)}, s)\colon\ z_1 \in \mathbb{Z}, s \geq 0\}$ be a system of coalescing random walks starting from every point of $\mathbb{Z}$ at time 0 with increments distributed as the increments of $Z$. Denote by $O_t$ the event that there is a walker seated at the origin by time $t$. If (3.1) holds, then*

$$P(O_t) \leq \frac{C}{\sqrt{t}}$$

*for some positive constant $C$ independent of everything else.*

***Remark 3.2.*** *The Proposition 3.1, Lemma 3.2 and Lemma 3.3 are respectively versions of Lemmas 2.2, 2.4 and 2.7 in Section 2 of [11]. Their results hold for the difference of two independent random walks what is not our case. The Proposition 3.1 is by itself a remarkable result in our case and is going to be proved in Section 3.1. In the proof, we need $q(\cdot)$ with finite range due to an estimate to bound the probability that two walks in the GDNM coalesce given they cross each other (item (iii) of Lemma 3.5). It is the only place where the finite range hypothesis is used. Both lemmas follow from inequality (3.1) in Proposition 3.1 following arguments presented in [11], although we also prove*

*Lemma* 3.2 *in Section* 3.1 *to make clear that it holds in our case. Concerning the proof of Lemma* 3.3, *it is exactly the same as the proof of Lemma* 2.7 *in* [11].

Now let us return to the analysis of conditions $B_1'$, T and E. The idea we carry on here is that if convergence to the BW holds for a system with noncrossing paths then the proof of [11] can be adapted to similar systems allowing crossing paths if we have Proposition 3.1 and condition $I$. In this direction, [11] is extremly useful since it is carefully written in a way that for each result one can clearly identify which hypotheses are needed to generalize its proof to other systems of coalescing random walks.

The proof of $B_1'$ is the same as that of Section 3 in [11] for systems of coalescing random walks independent before coalescence. In order to reproduce the proof given in [11] we need finite second moment of $X_s$ and that the coalescing time of two walkers starting at distance 1 apart is of order $1/\sqrt{t}$ which is the content of Proposition 3.1.

The proof of condition T is also the same as that of Section 4 in [11] for systems of coalescing random walks independent before coalescence. The proof depends only on two facts: first, an upper bound for the probability of the time it takes for two random walks to get far away one from each other before they coalesce, which is the content of Lemma 3.2; second, finite fifth moment of the increments of the random walks, which follows from the finite fifth moment of $q(\cdot)$. See the Appendix for a proof of condition $T$.

Now we consider condition $E$, but first we introduce more notation. For an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathbb{X}$, define $\mathbb{X}^{t^-}$ to be the subset of paths in $\mathbb{X}$ which start before or at time $t$. Also denote by $\mathbb{X}(t) \subset \mathbb{R}$ the set of values at time $t$ of all paths in $\mathbb{X}$. In [11] it is shown that condition $E$ is implied by condition $E'$ stated below:

($E'$) If $\mathcal{Z}_{t_0}$ is any subsequential limit of $\{\mathcal{X}_{\delta_n}^{t_0^-}\}$ for any $t_0 \in \mathbb{R}$, for any nonnegative sequence $(\delta_n)_n$ going to zero as $n$ goes to infinity, then $\forall t, a, b \in \mathbb{R}$ with $t > 0$ and $a < b$, $E[\hat{\eta}_{\mathcal{Z}_{t_0}}(t_0, t; a, b)] \le E[\hat{\eta}_{\mathcal{W}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{\pi t}}$.

The verification of condition $E'$ follows using the same arguments presented in Section 6 of [11]. Denote by $\mathcal{Z}_{t_0}^{(t_0+\epsilon)_T}$ the subset of paths in $\mathcal{Z}_{t_0}$ which start before or at time $t_0$ and truncated before time $(t_0 + \epsilon)$. By Lemma 3.3 we get the locally finiteness of $\mathcal{Z}_{t_0}(t_0 + \epsilon)$ for any $\epsilon > 0$ (for more details see [8,11]). The key point in the verification of condition $E'$ is to note that for any $0 < \epsilon < t$, $E[\hat{\eta}_{\mathcal{Z}_{t_0}}(t_0, t; a, b)] \le E[\hat{\eta}_{\mathcal{Z}_{t_0}^{(t_0+\epsilon)_T}}(t_0 + \epsilon, t - \epsilon; a, b)]$. Then, by the locally finiteness of $\mathcal{Z}_{t_0}(t_0 + \epsilon)$ and condition $I$ we get that $\mathcal{Z}_{t_0}^{(t_0+\epsilon)_T}$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0 + \epsilon)$. We get condition $E'$ from the fact that a family of coalescing Brownian motions starting from a random locally finite subset of the real line is stochastically dominated by the family of paths of coalescing Brownian motions starting at every point in $\mathbb{R}$ at time $t_0 + \epsilon$ and taking limit when $\epsilon$ goes to zero. See Lemma 6.3 in [11] that also holds in our case.

### 3.1. *Estimates on the distributions of the coalescing times*

We start this section proving Lemma 3.2 assuming that Proposition 3.1 holds. We are going to follow closely the proof of Lemma 2.4 in [11] aiming at making clear that their arguments holds in our case. Their first estimate is

$$P\big(\nu_k\big(\delta^{-1}u\big) < \tau_k \wedge \big(\delta^{-2}t\big)\big) < C'(t,u)|k|\delta \tag{3.3}$$

which holds for $\delta$ sufficiently small. It follows from the following estimate based on the strong Markov Property, which in our case also holds by the independence of the increments,

$$P\big(\tau_k > \delta^{-2}t\big) \ge P\big(\nu_k\big(\delta^{-1}u\big) < \tau_k \wedge \big(\delta^{-2}t\big)\big) \inf_{l\in\mathbb{Z}} P\big(Y^l \in \mathrm{B}^l\big(\delta^{-1}u, \delta^{-2}t\big)\big),$$

where $\mathrm{B}^l(x,t)$ is the set of trajectories that remain in the interval $[l-x, l+x]$ during the time interval $[0,t]$. For every $u > 0$ and $t > 0$, write

$$\inf_{l\in\mathbb{Z}} P\big(Y^l \in \mathrm{B}^l\big(\delta^{-1}u, \delta^{-2}t\big)\big) = 1 - \sup_{l\in\mathbb{Z}} P\Big(\sup_{s\le\delta^{-2}t} \big|Y_s^l - l\big| > \delta^{-1}u\Big).$$

Then

$$\begin{aligned}
\limsup_{\delta\to 0}\sup_{l\in\mathbb{Z}} \mathrm{P}\Big(\sup_{s\le\delta^{-2}t}\big|Y_s^l - l\big| > \delta^{-1}u\Big)
&\le \limsup_{\delta\to 0} \mathrm{P}\Big(\sup_{s\le\delta^{-2}t}\big(\big|X_s^0\big| + \big|X_s^l - l\big|\big) > \delta^{-1}u\Big)\\
&\le 2\limsup_{\delta\to 0} \mathrm{P}\Big(\sup_{s\le\delta^{-2}t}\big|X_s^0\big| > \frac{\delta^{-1}u}{2}\Big)\\
&\le 4\mathrm{P}\Big(N > \frac{u}{2\sqrt{t}}\Big) \le 4\mathrm{e}^{-u^2/(8t)},
\end{aligned}$$

where $N$ is a standard normal random variable, the last inequality is a consequence of Donsker Invariance Principle, see Lemma 2.3 in [11]. To obtain (3.2) from (3.3), we only need estimates that do not require independence of the random walks and rely on moment conditions. The computation is the same as that presented in the last paragraph on page 33 in [11] and Lemmas 2.5 and 2.6 in the same paper. Therefore we have that Lemma 3.2 holds.

The remain of the section is entirely devoted to the proof of Proposition 3.1. It suffices to consider the case in which $k = 1$, see the proof of Lemma 2.2 in [11]. Here we will considering that $q(\cdot)$ has finite second moment except near the end of the section, in the proof of condition (iii) in Lemma 3.5, where we need $q(\cdot)$ with finite range. The proof is based on a proper representation of the process $(Y_n^1)_{n\ge 1}$ which has already been used in [4]. However, in our case, the possibility to have $Y_n^1$ assuming negative values, due to crossing paths, requires a new approach. To simplify notation, we write $\tau = \tau_1$.

The process $(Y_n^1)_{n\ge 1}$ has square integrable independent increments with mean zero, thus it is a square integrable martingale with respect to its natural filtration. By Skorohod Embedding Theorem, see Section 7.6 in [5], there exist a standard Brownian Motion $(B(s))_{s\ge 0}$ and stopping times $T_1, T_2, \dots,$ such that $B(T_n)$ has the same distribution as $Y_n^1$, for $n \ge 0$, where $T_0 = 0$. Furthermore, the stopping times $T_1, T_2, \dots,$ have the following representation:

$$T_n = \inf\big\{s \ge T_{n-1}:\ B(s) - B(T_{n-1}) \notin \big(U_n\big(B(T_{n-1})\big), V_n\big(B(T_{n-1})\big)\big)\big\},$$

where $\{(U_n(m), V_n(m)):\ n \ge 1, m \in \mathrm{Z}\}$ is a family of independent random vectors taking values in $\{(0,0)\} \cup \{\dots,-2,-1\} \times \{1,2,\dots\}$. Also, by the definition of $Z$, we have that for each fixed $m \in \mathbb{Z}$, $(U_n(m), V_n(m))$, are identically distributed as random vectors indexed by $n \ge 1$. Indeed, write

$$\xi_n^z = Z_n^z - Z_{n-1}^z$$

for the increments of $Z^z$, then $\xi_n^{(z_1,z_2)} = \xi_1^{(z_1,z_2+(n-1))}$ and the families of identically distributed (but dependent) random variables $(\xi_1^{(z_1,z_2)})_{z_1\in\mathbb{Z}}$, $z_2 \in \mathbb{Z}$, are independent of each other. Moreover

$$\big|U_1(m)\big| \le \big|\xi_1^0\big| + \big|\xi_1^m\big|.$$

Therefore, by Proposition 2.2, $|U_1(m)|$ has finite second moment uniformly bounded in $m \in \mathbb{Z}$. Also note that, by symmetry, for $m > 0$, $|U_1(m)|$ has the same distribution as $V_1(-m)$.

Let us give a brief description of the idea behind the proof of Proposition 3.1. By similar arguments as those of [4] we show that $P(T_t \le \zeta t)$ is of order $1/\sqrt{t}$, for some $\zeta > 0$ sufficiently small. On the other hand if $T_t \ge \zeta t$, then for $t$ large, we show that $\zeta t$ is enough time for the Brownian excursions to guarantee that $(Y_n^1)_{0\le n\le t}$ have sufficient sign changes such that, under these changes, the probability that $(Y_n^1)_{0\le n\le t}$ does not hit 0 is of order $1/\sqrt{t}$.

Now define the random discrete times $a_0 = 0$,

$$a_1 = \inf\big\{n \ge 1:\ Y_n^1 \le 0\big\}$$

and for $j \ge 2$

$$a_j = \begin{cases} \inf\big\{n > a_{j-1}:\ Y_n^1 \ge 0\big\}, & j \text{ even},\\ \inf\big\{n > a_{j-1}:\ Y_n^1 \le 0\big\}, & j \text{ odd}.\end{cases}$$

Then $(a_j)_{j\geq 1}$ is an increasing sequence of stopping times for the random walk $(Y_n^1)_{n\geq 1}$. Note that $V_{a_j}(Y^1_{a_{j-1}}) \geq -Y^1_{a_{j-1}}$ if $j$ is even and $U_{a_j}(Y^1_{a_{j-1}}) \leq -Y^1_{a_{j-1}}$ if $j$ is odd. The sequence $(a_j)_{j\geq 1}$ induces the increasing sequence of continuous random times $(T_{a_j})_{j\geq 1}$.

We have the inequality

$$\begin{aligned} \mathrm{P}(\tau > t) &\leq \mathrm{P}(\tau > t, T_t < \zeta t) + \mathrm{P}(\tau > t, T_t \geq \zeta t) \\ &\leq \mathrm{P}(\tau > t, T_t \leq \zeta t) \\ &\quad + \sum_{n=1}^{t} \mathrm{P}(\tau > t, T_t \geq \zeta t, T_{a_{l-1}} < \zeta t, T_{a_l} \geq \zeta t), \end{aligned} \tag{3.4}$$

where $a_0 := 0$. We are going to show that both probabilities in the rightmost side of (3.4) are of order $1/\sqrt{t}$. During the proof we will need some technical lemmas whose proofs are postponed to the end of this section.

We start with $\mathrm{P}(T_t \leq \zeta t)$. Note that we can write

$$T_t = \sum_{i=1}^{t}(T_i - T_{i-1}) = \sum_{i=1}^{t} S_i\big(Y^1_{i-1}\big),$$

where $(S_i(k), i \geq 1, k \in \mathbb{Z})$ are independent random variables. The sequences $(S_i(k), i \geq 1)$, indexed by $k \in \mathbb{Z}$, are not identically distributed. Fix $\lambda > 0$. By the Markov inequality we have

$$\mathrm{P}(\tau > t, T_t \leq \zeta t) = \mathrm{P}\big(\tau > t, \mathrm{e}^{-\lambda T_t} \geq \mathrm{e}^{-\lambda\zeta t}\big) \leq \mathrm{e}^{\lambda\zeta t}\mathrm{E}\big(\mathrm{e}^{-\lambda T_t}\mathbb{I}_{\{\tau > t\}}\big).$$

As in [4], put $S(m) = S_1(m)$, then we have that

$$\mathrm{P}(\tau > t, T_t \leq \zeta t) \leq \Big[\mathrm{e}^{\lambda\zeta} \sup_{m\in\mathbb{Z}-\{0\}} \mathrm{E}\big(\mathrm{e}^{-\lambda S(m)}\big)\Big]^t. \tag{3.5}$$

To estimate the expectation above, we need some uniform estimates on the distribution of $U_1(m)$ and $V_1(m)$.

**Lemma 3.4.** *For every $p < 1$ and every probability function $q : \mathbb{N} \to [0, 1]$, we have that*

$$0 < c_1 := \sup_{m\neq 0} P\big(\big(U_1(m), V_1(m)\big) = (0, 0)\big) < 1. \tag{3.6}$$

Let $S_{-1,1}$ be the exit time of interval $(-1, 1)$ by a standard Brownian motion. We have that

$$\begin{aligned} \mathrm{E}\big(\mathrm{e}^{-\lambda S(m)}\big) &= \mathrm{E}\big(\mathrm{e}^{-\lambda S(m)} | \big(U(m), V(m)\big) \neq (0,0)\big)\mathrm{P}\big(\big(U(m), V(m)\big) \neq (0,0)\big) \\ &\qquad + \mathrm{P}\big(\big(U(m), V(m)\big) = (0,0)\big) \\ &\leq \mathrm{E}\big(\mathrm{e}^{-\lambda S_{-1,1}}\big)\big(1 - \mathrm{P}\big(\big(U(m), V(m)\big) = (0,0)\big)\big) \\ &\qquad + \mathrm{P}\big(\big(U(m), V(m)\big) = (0,0)\big) \\ &= \mathrm{P}\big(\big(U(m), V(m)\big) = (0,0)\big)(1 - c_2) + c_2 \\ &\leq c_1(1 - c_2) + c_2, \end{aligned} \tag{3.7}$$

where $c_1 < 1$ is given by Lemma 3.4 and $c_2 = \mathrm{E}(\mathrm{e}^{-\lambda S_{-1,1}}) < 1$. Now, choose $\zeta$ such that $c_3 = \mathrm{e}^{\lambda\zeta}[c_1(1-c_2)+c_2] < 1$, then from (3.5) and (3.7) we have

$$\mathrm{P}(\tau > t, T_t \leq \zeta t) \leq c_3^t,$$

where $c_3 > 0$. Finally, choose $c_4 > 0$ such that $c_3^t \le \frac{c_4}{\sqrt{t}}$ and

$$\mathrm{P}(\tau > t, T_t \le \zeta t) \le \frac{c_4}{\sqrt{t}} \tag{3.8}$$

for $t > 0$.

It remains to estimate the second term in the rightmost side of (3.4) which is

$$\sum_{l=1}^{t} \mathrm{P}(\tau > t, T_t \ge \zeta t, T_{a_{l-1}} < \zeta t, T_{a_l} \ge \zeta t). \tag{3.9}$$

We need the following result:

**Lemma 3.5.** *There exist independent nonnegative square integrable random variables $\tilde{R}_0$, $R_j$, $j \ge 1$, such that*:

(i) $\{R_j\}_{j=1}^{\infty}$ *are i.i.d. random variables*;
(ii) $R_j|\{R_j \neq 0\}$ *has the same distribution as* $\tilde{R}_0$;
(iii) $c_5 := P(R_1 \neq 0) < 1$;
(iv) $T_{a_j}$ *is stochastically dominated by* $J_j$ *which is defined as* $J_0 = 0$,

$$J_1 = \inf\{s \ge 0\colon\ B(s) - B(0) = (-1)^j (R_1 + \tilde{R}_0)\},$$

*and*

$$J_j = \inf\{s \ge J_{j-1}\colon\ B(s) - B(J_{j-1}) = (-1)^j (R_j + R_{j-1})\}, \quad j \ge 2,$$

*where* $(B(s))_{s\ge 0}$ *is a standard Brownian motion independent from the sequence* $\{R_n\}_{n=1}^{\infty}$;
(v) $Y_{a_j} \neq 0$ *implies* $B(J_j) \neq 0$ *which is equivalent to* $R_j \neq 0$ *given that* $B(0) = \tilde{R}_0$.

Let $(J_j)_{j\ge 1}$ be as in the statement of Lemma 3.5. Since

$$\{\tau > t, T_t \ge \zeta t, T_{a_{l-1}} < \zeta t, T_{a_l} \ge \zeta t\}$$

is a subset of

$$\{Y_{a_j} \neq 0 \text{ for } j = 1, \dots, l-1, T_{a_l} \ge \zeta t\},$$

by Lemma 3.5, we have that (3.9) is bounded above by

$$\begin{aligned}
&\sum_{l=1}^{t} \mathrm{P}\bigl(B(J_j) \neq 0 \text{ for } j = 1, \dots, l-1, J_l \ge \zeta t\bigr)\\
&\quad = \sum_{l=1}^{t} \mathrm{P}(R_j \neq 0 \text{ for } j = 1, \dots, l-1, J_l \ge \zeta t).
\end{aligned} \tag{3.10}$$

For the right hand side above, write

$$\begin{aligned}
&\mathrm{P}(R_j \neq 0 \text{ for } j = 1, \dots, l-1, J_l \ge \zeta t)\\
&\quad = \mathrm{P}(R_j \neq 0 \text{ for } j = 1, \dots, l-1)\mathrm{P}(J_l \ge \zeta t | R_j \neq 0 \text{ for } j = 1, \dots, l-1)\\
&\quad = \mathrm{P}(R_1 \neq 0)^{l-1}\mathrm{P}(J_l \ge \zeta t | R_j \neq 0 \text{ for } j = 1, \dots, l-1)\\
&\quad = c_5^{l-1}\mathrm{P}(J_l \ge \zeta t | R_j \neq 0 \text{ for } j = 1, \dots, l-1),
\end{aligned} \tag{3.11}$$

where the last equality follows from the independence of the $R_j$'s. Now put $\tilde{R}_j = R_j|\{R_j \neq 0\}$ and define $\tilde{J}_0 = 0$ and

$$\tilde{J}_j = \inf\{s \geq \tilde{J}_{j-1}\colon\ B(s) - B(\tilde{J}_{j-1}) = (-1)^j(\tilde{R}_j + \tilde{R}_{j-1})\}.$$

Then $\tilde{R}_j$ is also a sequence of i.i.d. square integrable random variables and, from Lemma 3.5, we get that

$$\mathrm{P}(J_l \geq \zeta t | R_j \neq 0 \text{ for } j = 1, \dots, l-1) = \mathrm{P}(\tilde{J}_l \geq \zeta t). \tag{3.12}$$

To estimate the right hand side in the previous equality, write

$$W_1 = \tilde{J}_1 \quad \text{and} \quad W_j = \tilde{J}_j - \tilde{J}_{j-1} \quad \text{for } j \geq 1.$$

Then it is easy to verify that $W_j$, $j \geq 1$, are identically distributed random variables which are not independent. However $\{W_{2j}\}_{j\geq 1}$ and $\{W_{2j-1}\}_{j\geq 1}$ are families of i.i.d. random variables.

We need the following estimate on the distribution function of $W_j$:

**Lemma 3.6.** *Let $W_1$ be defined as above. Then, there exists a constant $c_6 > 0$ such that for every $x > 0$ we have that*

$$\mathrm{P}(W_1 \geq x) \leq \frac{c_6}{\sqrt{x}}.$$

By Lemma 3.6, it follows that

$$\mathrm{P}(\tilde{J}_l \geq \zeta t) \leq \mathrm{P}\left(\sum_{j=1}^{l} W_j \geq \zeta t\right) \leq l\mathrm{P}\left(W_1 \geq \frac{\zeta t}{l}\right) \leq \frac{c_7 l^{3/2}}{\sqrt{t}}, \tag{3.13}$$

where $c_7 = c_6/\sqrt{\zeta}$. From (3.10), (3.11), (3.12) and (3.13), we have that

$$\sum_{l=1}^{t} \mathrm{P}\big(B(J_j) \neq 0 \text{ for } j = 1, \dots, l-1, J_l \geq \zeta t\big) \leq \frac{c_7}{\sqrt{t}} \sum_{l=1}^{+\infty} c_5^l l^{3/2} \leq \frac{c_8}{\sqrt{t}}. \tag{3.14}$$

Back to (3.4), using (3.8) and (3.14), we have shown that

$$\mathrm{P}(\tau > t) \leq \frac{c_4 + c_8}{\sqrt{t}}$$

finishing the proof of Proposition 3.1.

3.1.1. *Proofs of the technical lemmas of Section 3.1*

Here we prove Lemmas 3.4, 3.5 and 3.6.

**Proof of Lemma 3.4.** Since $(U_1(m), V_1(m))$, $m \geq 1$ takes values in $\{(0,0)\} \cup \{\dots, -2, -1\} \times \{1, 2, \dots\}$, then we have that $(U_1(m), V_1(m)) = (0,0)$ if and only if $Y_1^m = Y_0^m$, which means that $Z_1^m - Z_1^0 = Z_0^m - Z_0^0 = m$.

Fix $k = \inf\{r \geq 1\colon\ q(r) > 0\}$. Then

$$\begin{aligned}
&\mathrm{P}\big(\big(U_1(m), V_1(m)\big) \neq (0,0)\big) \\
&\quad \geq \mathrm{P}\big(Z_1^m < Z_1^0 + m\big) \\
&\quad \geq \mathrm{P}\big(Z_1^0 = k, Z_1^m < m + k\big) \\
&\quad \geq \mathrm{P}\big(\zeta(0) = \zeta(m) = k, \omega(1-j) = 0, \omega(j) = \omega(m+j-1) = 1, j = 1, \dots, k+1\big) \\
&\quad \geq q(k)^2 p^{2k+2}(1-p)^{k+1} > 0.
\end{aligned}$$

Since the last term in the previous inequality does not depend on $m$, we have that

$$\sup_m \mathrm{P}\big(\big(U_1(m), V_1(m)\big) = (0,0)\big) \le 1 - q(k)^2 p^{2k+2}(1-p)^{k+1} < 1.$$

To prove the other inequality we consider two cases. First, suppose that $m < \lfloor k/2 \rfloor$, then

$$\begin{aligned}
\mathrm{P}\big(\big(U_1(m), V_1(m)\big) = (0,0)\big) &\ge \mathrm{P}\big(Z_1^0 = k-1, Z_1^m = m+k-1\big) \\
&\ge \mathrm{P}\big(\zeta(0) = \zeta(m) = k, \theta(0) = 1, \omega(j) = 0, j = m-k+1, \dots, m-1, \\
&\quad \omega(j-k) = 1, j = 1, \dots, m, \omega(m+j) = 1, j = 0, \dots, k\big) \\
&\ge \tfrac{1}{2} q(k)^2 (1-p)^{k-1} p^{m+k} \ge \tfrac{1}{2} q(k)^2 (1-p)^{k-1} p^{(3/2)k}.
\end{aligned}$$

Now we consider the case $m \ge \lfloor k/2 \rfloor$. In this case

$$\begin{aligned}
\mathrm{P}\big(\big(U_1(m), V_1(m)\big) = (0,0)\big) &\ge P\big(Z_1^0 = \lfloor k/2 \rfloor, Z_1^m = m + \lfloor k/2 \rfloor\big) \\
&\ge \mathrm{P}\big(\zeta(0) = \zeta(m) = k, \omega(j) = \omega(m+j) = 1, j = -\lfloor k/2 \rfloor, \dots, \lfloor k/2 \rfloor\big) \\
&\ge \tfrac{1}{4} q(k)^2 p^{2k}.
\end{aligned}$$

Therefore,

$$\sup_m \mathrm{P}\big(\big(U_1(m), V_1(m)\big) = (0,0)\big) \ge \min\left\{\frac{1}{2} q(k)^2 (1-p)^{k-1} p^{(3/2)k}, \frac{1}{4} q(k)^2 p^{2k}\right\} > 0. \qquad \square$$

**Proof of Lemma 3.5.** Consider independent standard Brownian motions $(B(s))_{s\ge 0}$ and $(\mathbb{B}(s))_{s\ge 0}$ such that $(B(s))_{s\ge 0}$ is used in the Skorohod representation of $(Y_n^1)_{n\ge 1}$ and $(\mathbb{B}(s))_{s\ge 0}$ is also independent of $(Y_n^1)_{n\ge 1}$. In the time interval $[0, T_{a_1}]$, $(B(s))_{s\ge 0}$ makes an excursion being able to visit $(-\infty, 0]$ when $U(Y_n^1) \le -Y_n^1$ for some $1 \le n \le a_1$. When this happens two things may occur:

(1) $U(Y_n^1) = -Y_n^1$ which implies $a_1 = n$ and $B(T_{a_1}) = Y_{a_1} = 0$ meaning that $Z^0$ and $Z_1$ have coalesced;
(2) $U(Y_n^1) < -Y_n^1$ which implies that, with probability greater than some strictly positive constant $\beta$, $a_1 = n$ and $(B(s))_{s\ge 0}$ will leave the interval $[U(Y_n^1) + Y_n^1, V(Y_n^1) + Y_n^1]$ by its left side (strong Markov property with the fact that $(B(s))_{s\ge 0}$ visits $(-\infty, 0)$) meaning that $Z^0$ and $Z^1$ have crossed.

Denote $\mathcal{N}_1$ by the random variable that denotes the cardinality of

$$E := \big\{n \in \{1, \dots, a_1\} \colon \big(B(s)\big)_{s\ge 0} \text{ visits } (-\infty, 0] \text{ in time interval } [T_{n-1}, T_n]\big\}.$$

Note that $\mathcal{N}_1 \ge 1$ and, from (2), $\mathcal{N}_1$ is stochastically bounded by a geometric random variable with parameter $\beta$. We will construct below a sequence of i.i.d. nonpositive square integrable random variables $(\mathcal{R}_n)_{n\ge 1}$ such that for each $n \in E$, $\mathcal{R}_n \le U(Y_n^1) + Y_n^1$. Define

$$R_1 = -\sum_{n=1}^{\mathcal{G}_1} \mathcal{R}_n,$$

where $\mathcal{G}_1$ is a geometric random variable with parameter $\beta$ such that $\mathcal{G}_1 \ge \mathcal{N}_1$. It is a simple to show that $R_1$ is square integrable and independent of $(\mathbb{B}(s))_{s\ge 0}$. Clearly from the definitions

$$R_1 \ge -\sum_{n=1}^{\mathcal{N}_1} \mathcal{R}_n \ge \big|Y_{a_1}^n\big| = \big|B(T_{a_1})\big|.$$

So if we take $\tilde{R}_0$ with the same distribution as $R_1 | \{R_1 \ne 0\}$ then the time for $(B(s))_{s\ge 0}$ starting at 1 to hit $-R_1$ is stochastically dominated by the time for $(\mathbb{B}(s))_{s\ge 0}$ starting at $\tilde{R}_0$ to hit $-R_1$ which we denote $J_1$.

From this point, it is straightforward to use an induction argument to build the sequence $\{R_j\}_{j\geq 1}$. At step $j$ in the induction argument, we consider the $(B(s))_{s\geq 0}$ excursion in time interval $[T_{a_{j-1}}, T_{a_j}]$, and since $|Y^n_{a_{j-1}}| \leq R_{j-1}$ we can obtain $R_j$ and define $J_j$ using $(\mathbb{B}(s))_{s\geq 0}$ as before. By the strong Markov property of $(Y^1_n)$, we obtain that the $R_j$'s are independent and properties (i), (ii), (iv) and (v) follows directly from the construction.

It remains to prove (iii) and construct $(\mathcal{R}_n)_{n\geq 1}$.

**Construction of** $(\mathcal{R}_n)_{n\geq 1}$**.** Fix $n \geq 1$. We have to show that there exists a square integrable random variable $\mathcal{R}$ that stochastically dominates $(U_n(m)+m)|\{U_n(m) \leq -m\}$ for every $m > 0$. From this it is straightforward to obtain the sequence $(\mathcal{R}_n)_{n\geq 1}$, by induction and extension of the probability space, in a way that $\mathcal{R}_n$ has the same distribution as $\mathcal{R}$.

Now fix $m > 0$. We will be conditioning of the event $Y^1_n = m$. Recall that $Y^1_n = Z^1_n - Z^0_n$, $n \geq 1$. If $(Z^0_n)_{n\geq 1}$ and $(Z^1_n)_{n\geq 1}$ were independent, $(Y^1_n)_{n\geq 1}$ would be spatially homogeneous and $\mathcal{R}$ could be fixed as the sum of the absolute values of two independent increments of the processes $(Z^i_n)_{n\geq 1}$, $i = 1, 2$. That is not the case.

We define $\mathcal{R} = 0$ if $U_n(m) = -m$. On the event $\{U_n(m) < -m\}$, $\mathcal{R}$ should be smallest value that the environment at time $n+1$ allows for $U_n(m) + m < 0$. So let us regard the environment at time $n+1$. Consider the partition of the sample space on the events $G_{k,m} = \{Z^0_n = k\} \cap \{Z^1_n = m + k\}$, for $m > 0$ and $k \in \mathbb{Z}$. We define random variables $\eta$, $\tilde{\eta}$ and $\hat{\eta}$ in the following way: On $G_{k,m}$

$$\eta := (-k) + \text{ position of the } \zeta\big(Z^0_n\big) \text{ open site at time } n+1 \text{ to the right of position } k,$$

$$\tilde{\eta} := k - \text{ position of the } \zeta\big(Z^0_n\big) \text{ open site at time } n+1 \text{ to the left of position } k,$$

$$\hat{\eta} := k - \tilde{\eta} - \text{ position of the } \zeta\big(Z^1_n\big) \text{ open site at time } n+1 \text{ to the left of position } (k - \tilde{\eta}),$$

see Fig. 1. Clearly, conditioned to each $G_{k,m}$, $\eta$, $\tilde{\eta}$ and $\hat{\eta}$ are i.i.d. whose distribution is characterized as the sum of $N$ geometric random variables, where $N$ is distributed according to the probability function $q(\cdot)$ and is independent of the geometric random variables. In particular, $\eta$, $\tilde{\eta}$ and $\hat{\eta}$ do not depend on $k$ and $m$ and it is straightforward to verify that they have finite second moment if $q(\cdot)$ does. Define $\mathcal{R} = -(\eta + \tilde{\eta} + \hat{\eta})$. On the event $\{U_n(m) < -m\}$, $U_n(m) + m$ cannot be smaller than $\mathcal{R}$. Indeed on $G_{k,m}$, if the random walks $Z^0$ and $Z^1$ cross each other, we have

$$k - \tilde{\eta} - \hat{\eta} \leq Z^1_{a_j} < Z^0_{a_j} \leq k + \eta,$$

thus

$$0 < U_n(m) + m = Y^1_{n+1} = Z^1_{a_j} - Z^0_{a_j} \geq -(\eta + \tilde{\eta} + \hat{\eta}) = \mathcal{R}.$$

***Remark 3.3.*** *In Fig. 1 below the interval with size $\eta$ has exactly $\zeta(Z^0_n)$ open sites, the interval with size $\tilde{\eta}$ has $\zeta(Z^0_n)$ open sites and finally the interval with size $\hat{\eta}$ has exactly $\zeta(Z^1_n)$ open sites.*

**Proof of (iii).** Conditioned on the event $\{\mathcal{G}_1 = 1\}$, which occurs with probability $\beta$, we have that $B(T_{a_1}) \leq 0$ and we can suppose $R_1$ equal in distribution to $\mathcal{R}$ defined above. Thus

$$\begin{aligned} P(R_1 = 0) &= \beta P\big(\mathcal{R} = 0 | B(T_{a_1}) \leq 0\big) \\ &\geq \beta \inf_{m>0} P\big(U_1(m) = -m | U_1(m) \leq -m, B(T_1) - B(0) = U_1(m)\big) \\ &= \beta \inf_{m>0} P\big(Z^m_1 - Z^0_1 = 0 | Z^m_1 - Z^0_1 \leq 0\big), \end{aligned}$$

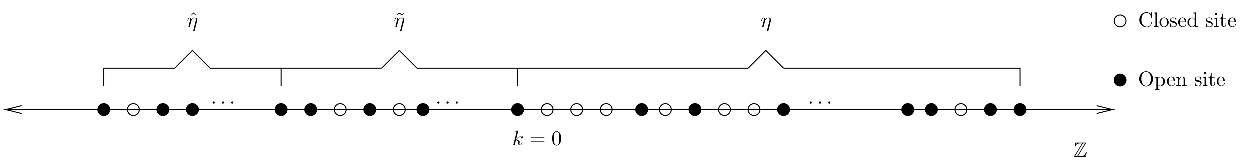


Fig. 1. A realization of the random variables $\eta$, $\tilde{\eta}$, $\hat{\eta}$.

where the first inequality follows from the definition of $\mathcal{R}$. So we have (ii) if

$$\inf_{m>0} P\big(Z_1^m - Z_1^0 = 0 | Z_1^m - Z_1^0 \le 0\big) > 0. \tag{3.15}$$

Note that the probability above, is the probability that two walks in the GDNM at distance $m$ coalesce after their first jumps given that they cross each other.

Lets prove (3.15). Suppose that $q(\cdot)$ has finite range (here is the single point in the text where we need $q(\cdot)$ with finite range). Fix $L \ge 1$ such that $q(k) = 0$, if $k \ge L$. Since clearly $P(Z_1^m - Z_1^0 = 0) > 0$ for every $m > 0$, it is enough to show that

$$\inf_{m>4L} P\big(Z_1^m - Z_1^0 = 0 | Z_1^m - Z_1^0 \le 0\big) > 0.$$

Then fix $m > L$ and write

$$\begin{aligned} P\big(Z_1^m < Z_1^0\big) &\le P\big(Z_1^m < Z_1^0, Z_1^0 < m - L\big) + P\big(Z_1^m < Z_1^0, Z_1^m > L\big) \\ &\quad + P\big(Z_1^m < Z_1^0, Z_1^m \le L, Z_1^0 \ge m - L\big) \end{aligned} \tag{3.16}$$

where by symmetry the right hand side is equal to

$$2P\big(Z_1^m < Z_1^0, Z_1^0 < m - L\big) + P\big(Z_1^m < Z_1^0, Z_1^m \le L, Z_1^0 \ge m - L\big). \tag{3.17}$$

We claim that each term in the previous sum is bounded above by $CP(Z_1^m = Z_1^0)$, for some $C > 0$ not depending on $m$, which implies that

$$P\big(Z_1^m - Z_1^0 = 0 | Z_1^m - Z_1^0 \le 0\big) \ge \frac{1}{3C+1}.$$

Let us consider $P(Z_1^m < Z_1^0, Z_1^0 < m - L)$ first. If $Z_1^m < Z_1^0$ and $Z_1^0 < m - L$, we define

$$M = \sum_{j=Z^m}^{Z^0-1} \omega(j).$$

Clearly $M < L$ and $\sum_{j=m-L}^{m+L} \omega(k) < L$ and we can choose $j(1) < \cdots < j(M)) \in \{m - L, \dots, m + L\}$ such that $\omega(j(i)) = 0$ for $i \in \{1, \dots, M\}$ and if $\omega(j) = 0$ than $j > j(E_{k,l})$. By changing the occupancies at sites $\{j(1), \dots, j(M)\}$ we have that the set $\{Z_1^m < Z_1^0, Z_1^0 < m - L\}$ has probability bounded above by

$$\max\left(\left(\frac{1-p}{p}\right)^L, 1\right) P\big(Z_1^m = Z_1^0\big).$$

Therefore we have proved the claim for the first probability in (3.17).

Now we estimate $P(Z_1^m < Z_1^0, Z_1^m \le L, Z_1^0 \ge m - L)$. Since $m > 4L$, if $Z_1^m < Z_1^0$, $Z_1^m \le L$ and $Z_1^0 \ge m - L$, then there exists at least $L$ open sites in the interval $\{L, L + 1, \dots, m - L\}$. By changing the occupancies of at most $L$ of these sites, we get a configuration with $Z_1^m = Z_1^0$. Thus, $P(Z_1^m < Z_1^0, Z_1^m \le L, Z_1^0 \ge m - L)$ is bounded above by

$$\max\left(\left(\frac{1-p}{p}\right)^L, 1\right) P\big(Z_1^m = Z_1^0\big).$$

□

**Proof of Lemma 3.6.** Put $\mu := E[|\tilde{R}_1|]$ and $\mathcal{J}_m := \inf\{t \ge 0\colon\ B(t) = m\}$. We have that

$$\mathrm{P}(W_1 \ge x) = \sum_{k\ge1}\sum_{l\ge0} \mathrm{P}(W_1 \ge x | \tilde{R}_1 = k, \tilde{R}_2 = l)\mathrm{P}(\tilde{R}_1 = k, \tilde{R}_2 = l)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0} \mathrm{P}(W_1 \geq x | \tilde{R}_1 = k, \tilde{R}_2 = l)\mathrm{P}(\tilde{R}_1 = k)\mathrm{P}(\tilde{R}_2 = l)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0} \mathrm{P}^0(\mathcal{J}_{k+l} \geq x)\mathrm{P}(\tilde{R}_1 = k)\mathrm{P}(\tilde{R}_1 = l)$$

$$= \sum_{k\geq 1}\sum_{l\geq 0} \int_x^\infty \frac{k+l}{\sqrt{2\pi y^3}} \mathrm{e}^{-(k+l)/(2y)}\,\mathrm{d}y \mathrm{P}(\tilde{R}_1 = k)\mathrm{P}(\tilde{R}_1 = l)$$

$$\leq 2\mu \int_x^\infty \frac{1}{\sqrt{2\pi y^3}} \mathrm{e}^{-1/(2y)}\,\mathrm{d}y \leq \frac{c_6}{\sqrt{x}}. \qquad \square$$

### 3.2. *Verification of condition I*

Condition $I$ clearly follows from the next result (see Theorem 4 in [4] for the equivalent result in the case $q(1) = 1$):

**Proposition 3.7.** *Suppose that inequality (3.1) holds and $q(\cdot)$ has finite third moment. Let $(x_0, s_0)$, $(x_1, s_1), \ldots,$ $(x_m, s_m)$ be $m+1$ distinct points in $\mathbb{R}^2$ such that $s_0 \leq s_1 \leq \cdots \leq s_m$, and if $s_{i-1} = s_i$ for some $i$, $i = 1, \ldots, m$, then $x_{i-1} < x_i$. Then*

$$\left(\left(\frac{X^{x_0 n}_{sn^2}}{n}\right)_{s\geq s_0}, \ldots, \left(\frac{X^{x_m n}_{sn^2}}{n}\right)_{s\geq s_m}\right)_{s\geq 0} \Longrightarrow^D \left(\left(B_0^{x_0}(s)\right)_{s\geq s_0}, \ldots, \left(B_m^{x_m}(s)\right)_{s\geq s_m}\right),$$

*where $(B_0^{x_0}(s))_{s\geq s_0}, \ldots, (B_m^{x_m}(s))_{s\geq s_m}$ are coalescing Brownian Motions with constant diffusion coefficient $\sigma$ starting at $((x_0, s_0), \ldots, (x_m, s_m))$.*

To prove Proposition 3.7, we first remark that the same proof of part III in the proof of Theorem 4 of [4] holds in our case. So it is enough to consider the case $s_0 = s_1 = \cdots = s_k = 0$.

Let $(B_0^{x_0}(s), B_1^{x_1}(s), \ldots, B_m^{x_m}(s))_{s\geq 0}$ be a vector of coalescing Brownian Motions starting at $(x_0, \ldots, x_m)$ with diffusion coefficient $\sigma$. Under the assumption that $q$ is finite range, we show in this section that

$$\left(\frac{X^{x_0 n}_{sn^2}}{n}, \ldots, \frac{X^{x_m n}_{sn^2}}{n}\right)_{s\geq 0} \Longrightarrow^D \left(B_0^{x_0}(s), \ldots, B_m^{x_m}(s)\right)_{s\geq 0}. \tag{3.18}$$

We have that the convergence of $(X^{xn}_{sn^2}/n)_{s\geq 0}$ to a Brownian Motion starting at $x$ is a direct consequence of Donsker invariance principle. For the case $q(1) = 1$, in the proof presented in [4], it is shown by induction that for $j = 1, \ldots, m$

$$\left(\frac{X^{x_k n}_{sn^2}}{n}\right)_{s\geq 0} \Bigg| \left(\frac{X^{x_0 n}_{sn^2}}{n}, \ldots, \frac{X^{x_{k-1} n}_{sn^2}}{n}\right)_{s\geq 0} \Longrightarrow^D \left(B_k^{x_k}(s)\right)_{s\geq 0} \big| \left(B_0^{x_0}(s), \ldots, B_{k-1}^{x_{k-1}}(s)\right)_{s\geq 0}.$$

It relies on the fact that no crossing can occur which allows the use of a natural order among trajectories of the random walks which we do not have.

We will take a slight different approach here. We make the proof in the case $x_0 = 0$, $x_1 = 1, \ldots, x_m = m$, the other cases can be carried out in an analogous way. To simplify the notation write

$$\frac{X^0_{sn^2}}{n} = \Delta_n^0(s), \qquad \frac{X^n_{sn^2}}{n} = \Delta_n^1(s), \qquad \ldots, \qquad \frac{X^{mn}_{sn^2}}{n} = \Delta_n^m(s).$$

Let us fix a uniformly continuous bounded function $H : D([0, s])^{m+1} \to \mathbb{R}$. We show that

$$\lim_{n\to\infty} \left|\mathrm{E}\left[H\left(\Delta_n^0, \ldots, \Delta_n^m\right)\right] - \mathrm{E}\left[H\left(B_0^0, \ldots, B_m^m\right)\right]\right| = 0. \tag{3.19}$$

As mentioned above $\Delta_n^0$ converges in distribution to $B_0^0$. Now, we suppose that $(\Delta_n^0, \dots, \Delta_n^{m-1})$ converges in distribution to $(B_0^0, \dots, B_{m-1}^{m-1})$ and we are going to show that (3.19) holds. By induction and the definition of convergence in distribution we obtain (3.18).

We start defining modifications of the random walks $X^0, \dots, X^{(m-1)n}, X^{mn}$ in order that the modification of $X^{mn}$ is independent of the modifications of $X^0, \dots, X^{(m-1)n}$, until the time of coalescence with one of them. We achieve this through a coupling which is constructed using a suitable change of the environment.

So let $(\bar{\omega}^0(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ and $(\bar{\omega}^1(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ be two independent families of independent Bernoulli random variables with parameter $p \in (0,1)$ which are also independent of $(\omega(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$, $(\theta(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ and $(\zeta(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$. Considering the processes $Z^0, \dots, Z^{(m-1)n}$ we define the environment $(\tilde{\omega}(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ by

$$\tilde{\omega}(z,t) = \begin{cases} \omega(z,t), & \text{if } t \geq 1, z \leq \max_{0\leq k\leq m-1} Z_{t-1}^{kn} + n^{3/4}, \\ \bar{\omega}^0(z,t), & \text{otherwise}. \end{cases}$$

The processes $\tilde{Z}^0, \dots, \tilde{Z}^{(m-1)n}$ are defined in the following way: For every $k = 0, \dots, m-1$, $\tilde{Z}^{kn}$ starts at $kn$ and evolves in the same way as $Z^{kn}$ except that $\tilde{Z}^{kn}$ sees a different environment. Both $Z^{kn}$ and $\tilde{Z}^{kn}$ processes use the families of random variables $(\theta(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ and $(\zeta(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$, but $\tilde{Z}^{kn}$ jumps according to $(\tilde{\omega}(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$ which is equal to $(\omega(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$ until the random time $t$ such that

$$\max_{0\leq k\leq m-1} \tilde{Z}_t^{kn} \geq \max_{0\leq k\leq m-1} \tilde{Z}_{t-1}^{kn} + n^{3/4},$$

after that time $\tilde{Z}^{kn}$ jumps according to $(\bar{\omega}^0(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$. From $\tilde{Z}^{kn}$ we define $\tilde{X}^{kn}$ and $\tilde{\Delta}_n^k$ as before.

Considering the process $Z^{mn}$ we define the environment $(\hat{\omega}(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ by

$$\hat{\omega}(z,t) = \begin{cases} \omega(z,t), & \text{if } t \geq 1, Z_{t-1}^{mn} - n^{3/4} \leq z \leq Z_{t-1}^{mn} + n^{3/4}, \\ \bar{\omega}^1(z,t), & \text{otherwise}. \end{cases}$$

The process $\hat{Z}^{mn}$ is defined in the following way: $\hat{Z}^{mn}$ starts at $mn$ and evolves in the same way as $Z^{mn}$ except that $\hat{Z}^{mn}$ sees a different environment. Both $Z^{mn}$ and $\hat{Z}^{mn}$ use the families of random variables $(\theta(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ and $(\zeta(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$, but $\hat{Z}^{mn}$ jumps according to $(\tilde{\omega}(z,t))_{z\in\mathbb{Z},t\in\mathbb{Z}_+}$ until the random time $t$ such that

$$\hat{Z}^{mn} \neq \left[\hat{Z}_{t-1}^{mn} - n^{3/4}, \hat{Z}_{t-1}^{mn} + n^{3/4}\right],$$

after that time $\hat{Z}^{mn}$ jumps according to $(\bar{\omega}^1(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$. From $\hat{Z}^{mn}$ we define $\hat{X}^{mn}$ and $\hat{\Delta}_n^m$ as before.

Define the event

$$\mathcal{B}_{n,s} = \bigcap_{k=0}^{m} \left\{\Delta_n^k \text{ does not make a jump of size greater than } n^{-1/4} \text{ in time interval } 0 \leq t \leq s\right\}.$$

Note three facts:

- $\hat{Z}^{mn}$ is independent of $(\tilde{Z}^0, \dots, \tilde{Z}^{(m-1)n})$ when conditioned to the event that $\tilde{Z}^{mn}$ do not get to a distance smaller than $2n^{3/4}$ of some $\tilde{Z}^{jn}$, $0 \leq j < m$.
- For every $k = 0, \dots, m-1$, $Z^{kn}$ and $\tilde{Z}^{kn}$ are equal at least until the first jump by one of them of size greater than $n^{3/4}$, thus they are equal when restricted to the event $\mathcal{B}_{n,s}$. The same is true about $Z^{mn}$ and $\hat{Z}^{mn}$.
- $(\tilde{\Delta}_n^0, \dots, \tilde{\Delta}_n^{m-1})$ and $(\Delta_n^0, \dots, \Delta_n^{m-1})$ have the same distribution.

We have the following lemma:

**Lemma 3.8.** *If $q$ has finite third moment, then*

$$\mathrm{P}\left(\mathcal{B}_{n,s}^c\right) \leq Cn^{-1/4}.$$

**Proof.** By definition, we have

$$\mathrm{P}(\mathcal{B}^c_{n,s}) \le (m+1)\mathrm{P}\big((Z^0_j)_{j=1}^{sn^2} \text{ makes a jump of size greater than } n^{3/4}\big).$$

It is simple to see that the probability in the right hand side is bounded above by

$$sn^2\mathrm{P}\big(|Z^0_1 - Z^0_0| \ge n^{3/4}\big) \le sn^2 n^{-9/4}\mathrm{E}\big[|Z^0_1 - Z^0_0|^3\big] \le Cn^{-1/4}. \qquad \square$$

We still need to replace the process $\hat{Z}^{mn}$ by a process $\bar{Z}^{mn}$ in such a way that $\hat{Z}^{mn}$ is independent of $(\tilde{Z}^0, Z^n, \dots, \tilde{Z}^{(m-1)n})$ until the time $\bar{Z}^{mn}$ coalesce with one of them. In order to do this we define $\bar{Z}^{mn}$ in the following way: Let

$$\nu = \inf\big\{1 \le j \le sn^2 \colon \big|\tilde{Z}^{kn}_j - \hat{Z}^{mn}_j\big| \le n^{3/4} \text{ for some } 0 \le k \le m-1\big\},$$

then $\bar{Z}^{mn}_j = \hat{Z}^{mn}_j$, for $0 \le j < \nu$, otherwise $\bar{Z}^{mn}$ jumps according to $(\bar{\omega}^1(z,t))_{z\in\mathbb{Z},t\in\mathbb{N}}$ or coalesce with one of $\tilde{Z}^0, \dots, \tilde{Z}^{(m-1)n}$, if they meet each other at an integer time. From $\bar{Z}^{mn}$ we define $\bar{X}^{mn}$ and $\bar{\Delta}^m_n$ as before. Define the event

$$\mathcal{A}_{n,s} = \Big\{\inf_{0\le k<m}\inf_{0\le t\le s}\big\|\tilde{\Delta}^k_n(t) - \hat{\Delta}^m_n(t)\big\| \ge 2n^{-1/4}\Big\}$$

and note two facts:

- $\bar{Z}^{mn}$ is independent of $(\tilde{Z}^0, \dots, \tilde{Z}^{(m-1)n})$.
- $\hat{Z}^{mn}$ and $\bar{Z}^{mn}$ are equal when restricted to the event $\mathcal{A}_{n,s}$.

We are now ready to prove (3.19). Write

$$\begin{aligned}
&\big|\mathrm{E}\big[H(\Delta^0_n, \dots, \Delta^m_n)\big] - \mathrm{E}\big[H(B^0_0, \dots, B^m_m)\big]\big| \\
&\quad \le \big|\mathrm{E}\big[H(\Delta^0_n, \dots, \Delta^m_n)\big] - \mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big]\big| \\
&\qquad + \big|\mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big] - \mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big]\big| \\
&\qquad + \big|\mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big] - \mathrm{E}\big[H(B^0_0, \dots, B^m_m)\big]\big|.
\end{aligned} \tag{3.20}$$

By Donsker's Invariance Principle and the induction hypothesis, we have that

$$\lim_{n\to\infty}\big|\mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big] - \mathrm{E}\big[H(B^0_0, \dots, B^m_m)\big]\big| = 0.$$

So we only have to deal with the first and second term in (3.20). For the first term in (3.20) we have that

$$\begin{aligned}
&\big|\mathrm{E}\big[H(\Delta^0_n, \dots, \Delta^m_n)\big] - \mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big]\big| \\
&\quad \le \big|\mathrm{E}\big[\big(H(\Delta^0_n, \dots, \Delta^m_n) - H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big)\mathbb{I}_{\mathcal{B}^c_{n,s}}\big]\big| \\
&\quad \le C\|H\|_\infty n^{-1/4}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\big|\mathrm{E}\big[H(\Delta^0_n, \dots, \Delta^m_n)\big] - \mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big]\big| = 0.$$

It remains to prove that the second term in (3.20) converges to zero as $n$ goes to $+\infty$. Note that

$$\begin{aligned}
&\big|\mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n)\big] - \mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big]\big| \\
&\quad = \big|\mathrm{E}\big[H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n) - H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big]\big| \\
&\quad = \big|\mathrm{E}\big[\big(H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \hat{\Delta}^m_n) - H(\tilde{\Delta}^0_n, \dots, \tilde{\Delta}^{m-1}_n, \bar{\Delta}^m_n)\big)\mathbb{I}_{\mathcal{A}^c_{n,s}}\big]\big|.
\end{aligned}$$

The rightmost expression in the previous equality is bounded above by $C\|H\|_\infty n^{-1/4}$ plus

$$\big|\mathrm{E}\big[\big(H\big(\tilde{\Delta}_n^0,\dots,\tilde{\Delta}_n^{m-1},\hat{\Delta}_n^m\big)-H\big(\tilde{\Delta}_n^0,\dots,\tilde{\Delta}_n^{m-1},\bar{\Delta}_n^m\big)\big)\mathbb{I}_{\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}}\big]\big|$$

which is equal to

$$\big|\mathrm{E}\big[\big(H\big(\Delta_n^0,\dots,\Delta_n^m\big)-H\big(\Delta_n^0,\dots,\Delta_n^{m-1},\bar{\Delta}_n^m\big)\big)\mathbb{I}_{\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}}\big]\big|. \tag{3.21}$$

To deal with the last expectation, define the coalescing times

$$\tau_k=\inf\big\{j\ge 1:\ Z_j^{kn}=Z_j^{mn}\big\}\quad\text{and}\quad\bar{\tau}_k=\inf\big\{j\ge 1:\ Z_j^{kn}=\hat{Z}_j^{mn}\big\}$$

for every $k\in 0,\dots,m-1$. The times $\tau_k$ and $\bar{\tau}_k$ have the tail of their distributions of $\mathrm{O}(t^{-1/2})$. Also define

$$\nu_k=\inf\big\{1\le j\le sn^2:\ \big|\tilde{Z}_j^{mn}-\tilde{Z}_j^{kn}\big|\le n^{3/4}\big\}.$$

Note that on $\mathcal{A}_{n,s}^c$ we have $\nu_k=\nu\le sn^2$, for some $k\in\{0,\dots,m-1\}$. Furthermore, on $\mathcal{B}_{n,s}$ and up to time $\nu$, we have $(\Delta_n^0,\dots,\Delta_n^m)$ equal to $(\Delta_n^0,\dots,\Delta_n^{m-1},\bar{\Delta}_n^m)$. We have that

$$\mathrm{P}\Big(\Big\{\sup_{0\le t\le s}\big|\Delta_n^m(t)-\bar{\Delta}_n^m(t)\big|\ge\log(n)n^{-1/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\Big)$$

is equal to

$$\begin{aligned}&\mathrm{P}\Big(\Big\{\sup_{0\le j\le sn^2}\big|Z_j^{mn}-\bar{Z}_j^{mn}\big|\ge\log(n)n^{7/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\Big)\\&\quad=\mathrm{P}\Big(\Big\{\sup_{\nu\le j\le sn^2}\big|Z_j^{mn}-\bar{Z}_j^{mn}\big|\ge\log(n)n^{7/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\Big)\\&\quad\le\sum_{k=0}^{m-1}\mathrm{P}\Big(\Big\{\sup_{\nu_k\le j\le sn^2}\big|Z_j^{mn}-\bar{Z}_j^{mn}\big|\ge\log(n)n^{7/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\cap\{\nu=\nu_k\}\Big).\end{aligned}$$

For each $k\in\{0,\dots,m-1\}$, the respective term in the previous sum is bounded above by

$$\begin{aligned}&\mathrm{P}\Big(\Big\{\sup_{\nu\le j\le sn^2}\big|Z_j^{mn}-\bar{Z}_j^{mn}\big|\ge\log(n)n^{7/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\cap\big\{\tau,\bar{\tau}\in\big[\nu,\nu+n^{7/4}\big]\big\}\Big),\\&\mathrm{P}\big(\big\{\tau_k>\nu_k+n^{7/4}\big\}\cup\big\{\bar{\tau}_k>\nu_k+n^{7/4}\big\}\big).\end{aligned} \tag{3.22}$$

The second term in (3.22) is bounded above by $2\frac{n^{3/4}}{n^{7/8}}=2n^{-1/8}$ and the first by

$$\begin{aligned}&\mathrm{P}\Big(\Big\{\sup_{\nu\le j\le(\nu+n^{7/4})\wedge sn^2}\big|Z_j^{mn}-\bar{Z}_j^{mn}\big|\ge\log(n)n^{7/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}\Big)\\&\quad\le 2\mathrm{P}\Big(\sup_{0\le j\le(\nu+n^{7/4})}\frac{|Z_j^0|}{n^{7/8}}\ge\frac{\log(n)}{2}\Big),\end{aligned}$$

which by Donsker invariance principle goes to zero as $n$ goes to infinity. Finally we have that (3.21) is bounded above by a term that converges to zero as $n\to\infty$ plus

$$\big|\mathrm{E}\big[\big(H\big(\Delta_n^0,\dots,\Delta_n^m\big)-H\big(\Delta_n^0,\dots,\Delta_n^{m-1},\bar{\Delta}_n^m\big)\big)\mathbb{I}_{\mathcal{E}_{n,s}}\big]\big|,$$

where

$$\mathcal{E}_{n,s}=\Big\{\sup_{0\le t\le s}\big|\Delta_n^m(t)-\bar{\Delta}_n^m(t)\big|\le\log(n)n^{-1/8}\Big\}\cap\mathcal{A}_{n,s}^c\cap\mathcal{B}_{n,s}.$$

By the uniform continuity of $H$ the rightmost expectation in the previous expression converges to zero as $n$ goes to $+\infty$.

## Appendix

The verification of condition $T$ follows in the same lines of the analogous result proved by Newman et al. [11] in the context of coalescing nonsimple random walks which evolve independently before coalescence. We include here a sketch of the proof only for the sake of completeness. Our aim is just to point out that the proof in [11] works in our case, so we try to keep the notation as close as possible to that of [11].

By the translation invariance property of the system, we only need to consider $(x_0, t_0) = (0, 0)$ in the statement of condition $T$. Denote by $A^+_{\chi_\delta}(u, t)$ (respectively $A^-_{\chi_\delta}(u, t)$) the event that there exists a path in $\mathcal{X}_\delta$ crossing the rectangle $R_{(0,0)}(u, t) := R(u, t)$ and at a later time touching the right (respectively, left) boundary of the bigger rectangle $R(Cu, 2t)$ (for simplicity we take $C = 17$). Thus, $A_{\chi_\delta}(0, 0; u, t) = A^+_{\chi_\delta}(u, t) \cup A^-_{\chi_\delta}(u, t)$ and to verify condition $T$ it is enough to show that

$$\limsup_{t\to 0^+} \frac{1}{t} \limsup_{\delta\to 0^+} \mathrm{P}\big(A^+_{\chi_\delta}(u, t)\big) = 0, \tag{A.1}$$

and

$$\limsup_{t\to 0^+} \frac{1}{t} \limsup_{\delta\to 0^+} \mathrm{P}\big(A^-_{\chi_\delta}(u, t)\big) = 0. \tag{A.2}$$

Verification of (A.2) is analogous to the proof of (A.1).

Now we proceed to show (A.1). We start introducing some notation. Denote $\tilde{u} = \sigma\delta^{-1}u$ and $\tilde{t} = \delta^{-2}t$. For $(x, m) \in \mathbb{Z} \times \mathbb{Z}$, let $\pi^{x,m}$ be the random walk in the GDNM starting at $(x, m)$. For $i = 1, 2, 3, 4$, let $\pi_i$ be the random walk in the GDNM starting at time 0 from $x_i = \lfloor (3 + (i - 1)4)\tilde{u} \rfloor$. Denote by $B_i, i = 1, 2, 3, 4$, the event that $\pi_i$ remains confined in a region of size $\tilde{u}$ centered at its starting position $x_i$ during the time interval $0 \le t \le 2\tilde{t}$, i.e., $|\pi(t) - x_i| < \tilde{u}$ for all $0 \le t \le 2\tilde{t}$. Denote by $J(\tilde{u}, \tilde{t})$ the event that some walk in the GDNM, starting at the left of $R(\tilde{u}, \tilde{t})$, crosses this rectangle in one unit of time. Without loss of generality, we will be assuming that $\tilde{t} \in \mathbb{Z}$.

The idea behind the proof of (A.1) is the following:

(i) By the arguments presented in the proof of Lemma (3.2), we have that $\limsup_{\delta\to 0^+} \mathrm{P}(B_i^c) \le 4\mathrm{e}^{-u^2/(8t)}$. Therefore

$$\limsup_{t\to 0^+} \frac{1}{t} \limsup_{\delta\to 0^+} \mathrm{P}\left(\bigcup_{i=1}^{4} B_i^c\right) = 0.$$

This is an estimate for single trajectories, and independence is not required.

(ii) Denote by $J(\tilde{u}, n)$ the event that some walk located to the left of $-\tilde{u}$ at time $n$ gives a jump to the right of $\tilde{u}$ in one unit of time. Denote by $\xi$ a random variable with the same distribution of the increments of the process $Z$. Then

$$\begin{aligned}\mathrm{P}\big(J(\tilde{u}, n)\big) &\le \sum_{x=0}^{\infty} \mathrm{P}\big(|\xi| \ge 2\tilde{u} + x\big) \le \sum_{x=0}^{\infty} \frac{\mathrm{E}[|\xi|^k, |\xi| \ge 2\tilde{u} + x]}{(2\tilde{u} + x)^k} \\ &\le C\frac{\mathbb{E}[\|\xi\|^k]}{\tilde{u}^{k-1}}.\end{aligned}$$

Thus, if for some $\epsilon > 0$, $\xi$ has finite moment of order $3 + \epsilon$, then,

$$\mathrm{P}\big(J(\tilde{u}, \tilde{t})\big) \le \sum_{n=0}^{\tilde{t}-1} \mathrm{P}\big(J(\tilde{u}, n)\big) \le C\frac{\tilde{t}}{\tilde{u}^{2+\epsilon}} \mathrm{E}\big[|\xi|^{3+\epsilon}\big] = C'(u, t)\delta^{\epsilon}. \tag{A.3}$$

Hence

$$\limsup_{\delta\to 0^+} \mathrm{P}\big(J(\tilde{u},\tilde{t})\big)=0.$$

We have used additive arguments and estimates for single trajectories, and independence was not required.

(iii) On the event $\bigcap_{i=1}^4 B_i \cap J(\tilde{u},\tilde{t})^c$, if $A^+_{\chi_\delta}(u,t)$ holds then none of $\pi^{x,m}$, $(x,m)\in R(\tilde{u},\tilde{t})$, coalesce with any of the random walks $\pi_i$, $i=1,\dots,4$, see Fig. 1 in [11]. We claim that this event has probability of order $o(t)$ as $\delta\to 0$. The rest of the proof is devoted to show this claim.

For $(x,m)\in R(\tilde{u},\tilde{t})$, denote by $\tau_i^{x,m}$ the integer stopping time that a random walk starting from $(x,m)$ first exceeds $(1+4k)\tilde{u}$, $k=1,2,3,4$. Also, set $\tau_0^{x,m}=m$ and $\tau_5^{x,m}=2\tilde{t}$. Finally, denote by $C_i(x,m)$ the event that $|\pi^{x,m}(t)-\pi_i(t)|>0$ for all $0\le t\le 2\tilde{t}$. We have to estimate

$$\mathrm{P}\left(\bigcap_{i=1}^4 B_i;\exists(x,m)\in R\big(\delta^{-1}u,\delta^{-2}t\big)\text{ s.t. }\bigcap_{i=1}^4 C_i(x,m),\tau_4^{x,m}<2\tilde{t}\right). \tag{A.4}$$

This probability is bounded above by

$$\sum_{x\in[-\tilde{u},\tilde{u}]}\sum_{m\in[0,\tilde{t}]\cap\mathbb{Z}}\mathrm{P}\left(\bigcap_{i=1}^4 B_i,\bigcap_{i=1}^4 C_i(x,m),\tau_4^{x,m}<2\tilde{t}\right)$$

$$\le 2\tilde{u}\tilde{t}\sup_{(x,m)\in R(\tilde{u},\tilde{t})}\mathrm{P}\left(\bigcap_{i=1}^4 B_i,\bigcap_{i=1}^4 C_i(x,m),\tau_4^{x,m}<2\tilde{t}\right).$$

To estimate the probabilities in the right hand side of the previous inequality, we fix $(x,m)\in R(\tilde{u},\tilde{t})$ and we suppress $(x,m)$ in $\pi^{x,m}$, $C_i(x,m)$ and $\tau_i(x,m)$. We have that $\mathrm{P}(\bigcap_{i=1}^4 B_i,\bigcap_{i=1}^4 C_i,\tau_4<2\tilde{t})$ is bounded above by the sum of

$$\mathrm{P}\big(\pi(\tau_1)>\big(5+\tfrac{1}{2}\big)\tilde{u}\text{ or }\pi(\tau_2)>\big(9+\tfrac{1}{2}\big)\tilde{u}\text{ or }\pi(\tau_3)>\big(13+\tfrac{1}{2}\big)\tilde{u}\big) \tag{A.5}$$

and $\mathrm{P}(H)$, where

$$H=\left\{\pi(\tau_1)\le\left(5+\frac{1}{2}\right)\tilde{u},\pi(\tau_2)\le\left(9+\frac{1}{2}\right)\tilde{u},\pi(\tau_3)\le\left(13+\frac{1}{2}\right)\tilde{u},\tau_4<2\tilde{t},\bigcap_{i=1}^4 B_i,\bigcap_{i=1}^4 C_i\right\}.$$

The probability in (A.5) concerns a single trajectory. Thus, analogously to [11] (see inequality (4.3) in that paper), under finite fifth moment condition on the increments of the GDNM, (A.5) times $\tilde{u}\tilde{t}$ goes to zero as $\delta\to 0^+$.

It remains to estimate $\mathrm{P}(H)$. For $s>0$, define $G_s$ as the event that none of the conditions in the definition of H are violated by time $s$. Note that

$$\mathrm{P}(H)=\mathrm{P}(G_{2\tilde{t}})\le\prod_{k=1}^4\mathrm{P}(G_{\tau_k}|G_{\tau_{k-1}}),$$

since $G_{\tau_k}\subset G_{\tau_{k-1}}$. Exactly as in [11], the strong Markov property and Lemma 3.2 imply that for $\delta$ sufficiently small there exists $C(t,u)$ such that $\mathbb{P}(G_{\tau_k}|G_{\tau_{k-1}})\le C(t,u)\delta$. Therefore, for $\delta$ sufficiently small,

$$\mathrm{P}(G_{2\tilde{t}})\le C^4(t,u)\delta^4.$$

Since $\tilde{u}\tilde{t}=\frac{\sigma ut}{\delta^3}$, we have that

$$2\tilde{u}\tilde{t}\mathrm{P}(H)$$

goes to zero as $\delta$ goes to zero from the right. Then, the claim in (iii) has been proved and condition $T$ follows from (i), (ii) and (iii).

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