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Local percolative properties of the vacant set of random interlacements with small intensity

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Abstract. Random interlacements at level u is a one parameter family of connected random subsets of $\mathbb{Z}^d$, $d \geq 3$ (*Ann. Math.* **171** (2010) 2039–2087). Its complement, the vacant set at level u, exhibits a non-trivial percolation phase transition in u (*Comm. Pure Appl. Math.* **62** (2009) 831–858; *Ann. Math.* **171** (2010) 2039–2087), and the infinite connected component, when it exists, is almost surely unique (*Ann. Appl. Probab.* **19** (2009) 454–466).

In this paper we study local percolative properties of the vacant set of random interlacements at level u for all dimensions $d \geq 3$ and small intensity parameter $u > 0$. We give a stretched exponential bound on the probability that a large (hyper)cube contains two distinct macroscopic components of the vacant set at level u. In particular, this implies that finite connected components of the vacant set at level u are unlikely to be large. These results are new for $d \in \{3, 4\}$. The case of $d \geq 5$ was treated in (*Probab. Theory Related Fields* **150** (2011) 529–574) by a method that crucially relies on a certain "sausage decomposition" of the trace of a high-dimensional bi-infinite random walk. Our approach is independent from that of (*Probab. Theory Related Fields* **150** (2011) 529–574). It only exploits basic properties of random walks, such as Green function estimates and Markov property, and, as a result, applies also to the more challenging low-dimensional cases. One of the main ingredients in the proof is a certain conditional independence property of the random interlacements, which is interesting in its own right.

Résumé. Un entrelac aléatoire au niveau u est une famille à un paramètre de sous-ensembles connexes aléatoires de $\mathbb{Z}^d$, $d \geq 3$, introduit dans (*Ann. Math.* **171** (2010) 2039–2087). Son complémentaire, l'ensemble vacant au niveau u, possède une transition de percolation non triviale en u, comme il a été montré dans (*Comm. Pure Appl. Math.* **62** (2009) 831–858) et (*Ann. Math.* **171** (2010) 2039–2087). La composante connexe infinie, lorsqu'elle existe, est presque sûrement unique, voir (*Ann. Appl. Probab.* **19** (2009) 454–466).

Dans ce papier, nous étudions les propriétés percolatives locales de l'ensemble vacant au niveau u en toutes dimensions $d \geq 3$ et pour un petit paramètre d'intensité u. Nous donnons une borne exponentielle tendue sur la probabilité qu'un grand (hyper)cube contienne deux composantes macroscopiques distinctes de l'ensemble vacant au niveau u. Nos résultats impliquent qu'il est peu probable que les composantes connexes finies de l'ensemble vacant au niveau u soient grandes. Ces résultats ont été prouvés dans (*Probab. Theory Related Fields* **150** (2011) 529–574) pour $d \geq 5$. Notre approche est différente (de celle de (*Probab. Theory Related Fields* **150** (2011) 529–574)) et est valide pour $d \geq 3$.

L'un des ingrédients principaux de la preuve est une certaine propriété d'indépendance conditionelle des entrelacs aléatoires, qui est intéressante en elle-même.

MSC: 60K35; 82B43

Keywords: Random interlacement; Random walk; Large finite cluster; Supercriticality; Conditional independence

1. Introduction

Random interlacements $\mathcal{I}^u$ at level $u > 0$ on $\mathbb{Z}^d$, $d \geq 3$, is a one parameter family of random connected subsets of $\mathbb{Z}^d$, introduced by Sznitman [11], which arises as the local limit as $N \to \infty$ of the set of sites visited by a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$ when it runs up to time $\lfloor uN^d \rfloor$, see [17]. The law of $\mathcal{I}^u \subseteq \mathbb{Z}^d$ is uniquely characterized by the equations:

$$\mathbb{P}\big[\mathcal{I}^u \cap K = \varnothing\big] = \mathrm{e}^{-u \cdot \mathrm{cap}(K)} \quad \text{for any finite } K \subseteq \mathbb{Z}^d, \tag{1.1}$$

where $\mathrm{cap}(K)$ denotes the discrete capacity of K, defined in (2.6) below. It is proved among other results in [11] that for any $u > 0$, $\mathcal{I}^u$ is almost surely connected, and its law is invariant and ergodic with respect to the lattice shifts. In fact, in [11], a more constructive definition of $\mathcal{I}^u$ is given, which we recall in Section 2.3. Informally, it states that $\mathcal{I}^u$ is the trace of a certain cloud of bi-infinite random walk trajectories in $\mathbb{Z}^d$, with u measuring the density of this cloud.

The vacant set $\mathcal{V}^u$ at level u is the complement of $\mathcal{I}^u$ in $\mathbb{Z}^d$. We view $\mathcal{V}^u$ as a random graph by drawing an edge between any two vertices of the vacant set at L_1-distance 1 from each other. The vacant set exhibits a non-trivial structural phase transition in u, i.e., there exists $u_* \in (0, \infty)$ such that

(i) for any $u > u_*$, almost surely, all connected components of $\mathcal{V}^u$ are finite, and
(ii) for any $u < u_*$, almost surely, $\mathcal{V}^u$ contains an infinite connected component.

In particular, the finiteness of u_* for $d \geq 3$ and the positivity of u_* for $d \geq 7$ were proved in [11], and the latter result was extended to all dimensions $d \geq 3$ in [10]. It is also known that $\mathcal{V}^u$ contains at most one infinite connected component (see [13]); in particular, for any $u < u_*$, the infinite connected component is almost surely unique.

In this paper, we are interested in the local structure of the vacant set in the regime of small u. More specifically, we show that with high probability, the unique infinite connected component of $\mathcal{V}^u$ is "visible" in large hypercubic subsets of $\mathbb{Z}^d$ (as the unique macroscopic connected component in the restriction of $\mathcal{V}^u$ to large hypercubes of $\mathbb{Z}^d$). Our main result is the following theorem.

Theorem 1.1 (Local uniqueness for $\mathcal{V}^u$). *For any $d \geq 3$, there exist $u_1 > 0$, $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $0 \leq u \leq u_1$ and $n \geq 1$, we have*

$$\mathbb{P}\left[\begin{array}{c}\text{the infinite connected component of } \mathcal{V}^u \\ \text{intersects } \mathrm{B}(0,n)\end{array}\right] \geq 1 - C\mathrm{e}^{-n^c} \tag{1.2}$$

and

$$\mathbb{P}\left[\begin{array}{c}\text{any two connected subsets of } \mathcal{V}^u \cap \mathrm{B}(0,n) \text{ with} \\ \text{diameter} \geq n/10 \text{ are connected in } \mathcal{V}^u \cap \mathrm{B}(0,2n)\end{array}\right] \geq 1 - C\mathrm{e}^{-n^c}. \tag{1.3}$$

Statement (1.2) has already been known (it easily follows from [12], Theorem 5.1), but we include it here for completeness. For $d \geq 5$, statement (1.3) follows from the stronger statement of [14], Theorem 3.2. Our contribution to the result of Theorem 1.1 is twofold. Firstly, the result (1.3) is new for $d \in \{3, 4\}$. Secondly, our proof of (1.3) is conceptually different from that of [14], and applies to all dimensions $d \geq 3$. Let us briefly explain the strategy in the proof of [14] and why it cannot be used in low dimensions. The proof in [14] crucially relies on the fact that if $d \geq 5$, the trace of a bi-infinite random walk contains many bilateral cut-points (see [14], (6.1), (6.26)). This gives a decomposition of the random walk trace into a chain of relatively small well-separated "sausages." Heuristically, a chain of sausages cannot separate two macroscopic connected subsets of a box. Random interlacements at level u is the trace of a certain Poisson cloud of doubly infinite random walk trajectories in $\mathbb{Z}^d$, and, therefore, can be viewed as the countable union of doubly infinite chains of "sausages" in $\mathbb{Z}^d$. Thus, in order to show that random interlacements at level u cannot separate two macroscopic connected subsets of a large box, one needs to show that locally it generally looks like the trace of only bounded number of random walks. This is achieved in [14] with a renormalization argument. The sausage decomposition property fails for $d \leq 4$ (see, e.g., [7], Theorem 2.6). In fact, in dimension $d = 3$, even the trace of a single random walk is a "two-dimensional" object, and, therefore, could in principle form a large separating surface in a box. This is not the case, as we discuss in Section 6. Our proof of (1.3) only exploits basic properties of random walks (Green function estimates, Markov property) and works for all dimensions $d \geq 3$.

The results of Theorem 1.1 are in the spirit of the local uniqueness property of supercritical Bernoulli percolation (see, e.g., [4], (7.89)). In fact, the analogues of (1.2) and (1.3) for Bernoulli percolation hold through the whole supercritical phase. We believe that the bounds (1.2) and (1.3) also hold for all $u < u_*$, but with constants $c = c(d, u) > 0$ and $C = C(d, u) < \infty$ depending on u. Our current understanding of the model is not good enough to be able to rigorously justify this belief.

The main technical challenges in the proof of Theorem 1.1 come from the long-range dependence of the random interlacements (see, e.g., [11], Remark 1.6(4)), the lack of the BK inequality (see, e.g., [4], (2.12), and [12], Remark 1.5(3)) and the absence of finite energy property (see, e.g., [11], Remark 2.2(3)).

As an immediate corollary of Theorem 1.1 we obtain that finite connected components of the vacant set at level u are unlikely to be large when u is small enough.

Corollary 1.2. *For any $d \geq 3$, there exist $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $u \leq u_1$ (defined in Theorem 1.1), we have*

$$\mathbb{P}\big[n \leq \operatorname{diam}\big(\mathcal{C}^u(0)\big) < \infty\big] \leq C\mathrm{e}^{-n^c} \tag{1.4}$$

and

$$\mathbb{P}\big[n \leq \big|\mathcal{C}^u(0)\big| < \infty\big] \leq C\mathrm{e}^{-n^c}, \tag{1.5}$$

where $\operatorname{diam}(\mathcal{C}^u(0))$ *and* $|\mathcal{C}^u(0)|$ *denote the diameter and the cardinality of the connected component of the origin in* $\mathcal{V}^u$, *respectively.*

Again, when $d \geq 5$, the result of Corollary 1.2 follows from [14], Theorems 3.5 and 3.6. The analogue of Corollary 1.2 for supercritical Bernoulli percolation is well known, and as Theorem 1.1, it is a property of the whole supercritical phase of Bernoulli percolation (see, e.g., [2,6] and [4], Chapter 8). Moreover, the analogue of (1.4) for Bernoulli percolation holds with exponential decay rate (see, [4], (8.20)), and the analogue of (1.5) holds with stretched exponential decay with the explicit exponent $c = (d-1)/d$ (see, e.g., [4], (8.66)).

Let us now mention some applications of Theorem 1.1. In [9], Theorem 1.1 is used to study the stability of the phase transition of the vacant set under a small quenched noise. The setup is the following. For a positive ε, we allow each vertex of the random interlacement (referred to as occupied) to become vacant, and each vertex of the vacant set to become occupied with probability ε, independently of the randomness of the interlacement, and independently for different vertices. In [9], Theorem 5 it is proved that for any u which satisfies (1.2) and (1.3), the perturbed vacant set at level u still has an infinite connected component if the noise is small enough. In particular, this statement together with Theorem 1.1 imply that the perturbed vacant set at small level u still has an infinite connected component. The use of Theorem 1.1 significantly simplifies the original proof of [9], Theorems 3 and 5, given in the first version of [9].

In [3], Theorem 2.3, we use Theorem 1.1 as an ingredient to prove that the graph distance in the unique infinite connected component of the vacant set at small level u is comparable to the graph distance on $\mathbb{Z}^d$, and establish a shape theorem for balls with respect to graph distance on the infinite connected component.

We believe that the methods of this paper can be applied in order to further explore the fragmentation of the torus $(\mathbb{Z}/N\mathbb{Z})^d$ by the trace of a simple random walk, in a similar fashion to [15], where a strong coupling between the random walk trace on the torus and random interlacements is used to transfer results of [14] to the torus. We further discuss this possibility as well as the analogue of Theorem 1.1 for the set of sites avoided by a simple random walk on $\mathbb{Z}^d$ in Section 6.

We will now briefly sketch the main ideas of the proof of Theorem 1.1. A more detailed description of the main steps of the proof will be given at the beginning of Sections 3, 4, and 5. Before reading those descriptions, we advise the reader to become familiar with basic definitions and results concerning random interlacements in Sections 2.3 and 2.4.

The proof uses coarse graining (see Section 3) and a conditional independence property for random interlacements (see Section 4). The need for coarse graining comes from the fact that the complement of the infinite connected component of the vacant set is almost surely connected, no matter how small the parameter u is. (This is immediate from the fact that $\mathcal{I}^u$ is almost surely connected for any given u, see [11], (2.21).) The reader familiar with Bernoulli percolation may notice that this would not be the case if the vertices were made vacant independently from each other.

In this case, the usual Peierls argument would easily give the analogue of Theorem 1.1 for Bernoulli percolation, when the vacant set has density close to one.

To overcome the problem arising from the connectedness of $\mathcal{I}^u$, we partition $\mathbb{Z}^d$ into L_∞-boxes $(\mathrm{B}(x', R)\colon x' \in (2R+1)\cdot\mathbb{Z}^d)$, with some $R \geq 0$. We use a variant of Sznitman's decoupling inequalities [12] to show that when R is large enough, there is a unique infinite connected subset of good boxes which are "sufficiently vacant." Moreover, the remaining (bad) boxes form only finite connected subsets of $\mathbb{Z}^d$, with stretched exponential decay of the probability that a connected component of bad boxes is large. Our definition of good boxes also assures that the infinite connected component of good boxes contains an infinite connected subset of $\mathcal{V}^u$, which intersects every good box of the above set. For concreteness, in this proof sketch, we call this infinite connected subset of $\mathcal{V}^u$ the "fat" set. As a result, we obtain that with high probability, any nearest-neighbor path of $\mathbb{Z}^d$ with large diameter often intersects the infinite connected component of good boxes, and therefore gets within distance R from the fat set.

However, the possibility of having a long nearest-neighbor path in $\mathcal{V}^u$ which avoids the fat set (but unavoidably, with high probability, gets R-close to it sufficiently often) still remains. We use a conditional independence property of random interlacements (see Section 4) to show that, roughly speaking, conditionally on the fact that a vacant path connects to a good box of the infinite connected set of good boxes and also conditioning on the configuration outside this box, there is still a uniformly positive chance that this vacant path is connected inside the specified good box to the fat set. The difficulty in the proof of this claim comes from the fact that random interlacements do not posess the so-called finite energy property (see, e.g., [11], Remark 2.2(3)). In words, the fact that $\mathcal{I}^u$ is a connected set implies that depending on the realization of $\mathcal{I}^u$ outside a box, not every configuration can be realized by $\mathcal{I}^u$ inside this box. (This is a big constraint, and, for example, causes some difficulties in the proof of the uniqueness of an infinite connected component of $\mathcal{V}^u$, see [13].) Our definition of good boxes is chosen specifically to overcome this problem. Coming back to the proof sketch, since each long path must visit many good boxes in the infinite connected component, we conclude that with high probability each long path in $\mathcal{V}^u$ must be connected to the fat set. This gives us (1.3).

The paper is organized as follows. In Section 2, we define the notation used in the paper, state some basic results about the simple random walk on $\mathbb{Z}^d$, define random interlacements and recall some of its properties, the most important of which is Lemma 2.2. It is based on [12], Corollary 3.5, but formulated more generally (using so-called interlacement local times defined in Section 2.4). Therefore, we give its proof sketch in the Appendix.

In Section 3, we define coarse graining, and prove the existence of a "fat" infinite connected subset of $\mathcal{V}^u$, when u is small enough (see Corollary 3.7).

In Section 4, we prove a conditional independence property of random interlacements (see Lemma 4.4).

In Section 5, we prove Theorem 1.1 using the results of Sections 3 and 4.

Finally, in Section 6, we briefly mention applications of the ideas developed in this paper to the vacant set of a simple random walk on $\mathbb{Z}^d$ and $(\mathbb{Z}/N\mathbb{Z})^d$.

2. Notation, model, preliminaries

2.1. *Basic notation*

We denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of natural numbers, by $\mathbb{Z}$ the set of integers. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of non-negative reals. For $a \in \mathbb{R}$, we write $|a|$ for the absolute value of a, and $\lfloor a \rfloor$ for the integer part of a.

For any $d \geq 1$, we denote by $x = (x_1, \ldots, x_d)$ a generic element of $\mathbb{Z}^d$, also referred to as *vertex* of $\mathbb{Z}^d$. We denote by $|x| = \max_{1\leq i\leq d} |x_i|$ the sup-norm of $x \in \mathbb{Z}^d$ and by $|x|_1 = \sum_{i=1}^d |x_i|$ the L_1-norm of x. For $K \subset \mathbb{Z}^d$, we denote by $|K|$ the cardinality of K. We write $K \subset\subset \mathbb{Z}^d$ when $K \subset \mathbb{Z}^d$ and $|K| < \infty$.

We say that $x, x' \in \mathbb{Z}^d$ are nearest neighbors (respectively, $*$-neighbors) if $|x - x'|_1 = 1$ (respectively, $|x - x'| = 1$). We also denote $|x - x'|_1 = 1$ by $x \sim x'$. We say that $\pi = (z_1, \ldots, z_n)$ is a nearest neighbor path (respectively, $*$-path) if z_i and z_{i+1} are nearest neighbors (respectively, $*$-neighbors) for all $1 \leq i \leq n-1$, and we use the notation $|\pi| = n$ (not to be confused with the cardinality of the set $\{z_1, \ldots, z_n\}$). We say that $V \subseteq \mathbb{Z}^d$ is connected (respectively, $*$-connected) if any pair $x_1, x_2 \in V$ can be connected by a nearest neighbor path (respectively, $*$-path) with vertices in V.

For $x \in \mathbb{Z}^d$ and $R \in \mathbb{N}$ we denote by $\mathrm{B}(x, R) = \{y \in \mathbb{Z}^d\colon |x - y| \leq R\}$ the closed ball of radius R around x with respect to the sup-norm. For any set $V \subseteq \mathbb{Z}^d$, we denote by $V^c = \mathbb{Z}^d \setminus V$.

The interior boundary of $K \subseteq \mathbb{Z}^d$, $\partial_{\mathrm{int}} K$ is the set of vertices of K that have some neighbor in K^c.

The exterior boundary of $K \subseteq \mathbb{Z}^d$, $\partial_{\mathrm{ext}} K$ is the set of vertices of K^c that have some neighbor in K.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $A \in \mathcal{F}$, we denote by $\mathbb{1}_A$ the indicator of the event A. If X is an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, we denote $\mathbb{E}[X; A] = \mathbb{E}[X \cdot \mathbb{1}_A]$.

For $-\infty \le a < b \le +\infty$, we denote by $\mathcal{B}([a, b])$ the Borel σ-algebra on $[a, b]$.

Our agreement about the constants used in the paper is the following. We denote small positive constants by c and large finite constants by C. When needed, we emphasize the dependence of a constant on parameters. If the constant only depends on d, then we sometimes do not mention it at all. The value of a constant may change within the same formula.

2.2. *Simple random walk and potential theory*

The space W_+ stands for the set of infinite nearest-neighbor trajectories, defined for non-negative times and tending to infinity:

$$W_+ = \Big\{ w\colon \mathbb{N} \to \mathbb{Z}^d,\, w(n) \sim w(n+1), n \in \mathbb{N}, \lim_{n\to\infty} \big|w(n)\big| = \infty \Big\}. \tag{2.1}$$

We endow W_+ with the σ-algebra $\mathcal{W}_+$ generated by the canonical coordinate maps X_n, $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define the shift map $\theta_k : W_+ \to W_+$ by $\theta_k(w)(\cdot) = w(\cdot + k)$. For $x \in \mathbb{Z}^d$, let P_x denote the law of simple random walk on $\mathbb{Z}^d$ with starting point x. Simple random walk on $\mathbb{Z}^d$, $d \ge 3$, is transient and the set W_+ has full measure under any P_x. From now on we will view P_x as a measure on $(W_+, \mathcal{W}_+)$, and we write $(X(t)\colon t \in \mathbb{N})$ for a random element of W_+ with distribution P_x.

For $U \subseteq \mathbb{Z}^d$ and $w \in W_+$, we define

$$H_U(w) = \inf\big\{n \ge 0\colon X_n(w) \in U\big\}, \quad \text{the entrance time in } U, \tag{2.2}$$

$$\widetilde{H}_U(w) = \inf\big\{n \ge 1\colon X_n(w) \in U\big\}, \quad \text{the hitting time of } U, \tag{2.3}$$

$$T_U(w) = \inf\big\{n \ge 0\colon X_n(w) \notin U\big\}, \quad \text{the exit time from } U. \tag{2.4}$$

For $d \ge 3$, the Green function $g : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ of the simple random walk X is defined as

$$g(x, y) = \sum_{t=0}^{\infty} P_x\big[X(t) = y\big], \quad x, y \in \mathbb{Z}^d.$$

Translation invariance yields $g(x, y) = g(0, y - x)$. It follows from [8], Theorem 1.5.4, that for any $d \ge 3$, there exist $c_g = c_g(d) > 0$ and $C_g = C_g(d) < \infty$ such that

$$c_g \cdot \big(|x - y| + 1\big)^{2-d} \le g(x, y) \le C_g \cdot \big(|x - y| + 1\big)^{2-d} \quad \text{for } x, y \in \mathbb{Z}^d. \tag{2.5}$$

The equilibrium measure of $K \subset\subset \mathbb{Z}^d$ is defined by

$$e_K(x) = \begin{cases} P_x[\widetilde{H}_K = \infty], & x \in K, \\ 0, & x \notin K. \end{cases}$$

The capacity of K is the total mass of the equilibrium measure of K:

$$\mathrm{cap}(K) = \sum_x e_K(x). \tag{2.6}$$

Since $\mathbb{Z}^d$ is transient ($d \ge 3$), for any $\varnothing \ne K \subset\subset \mathbb{Z}^d$, the capacity of K is positive. Therefore, we can define for such K the normalized equilibrium measure by

$$\widetilde{e}_K(x) = e_K(x)/\mathrm{cap}(K). \tag{2.7}$$

The following relations for $P_x[H_K < \infty]$ will be useful: for any $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$,

(i) (see, e.g. [11], (1.8))

$$P_x[H_K < \infty] = \sum_{y\in K} g(x,y) e_K(y), \tag{2.8}$$

(ii) (see [11], (1.9))

$$\sum_{y\in K} g(x,y) \Big/ \sup_{z\in K} \sum_{y\in K} g(z,y) \le P_x[H_K < \infty] \le \sum_{y\in K} g(x,y) \Big/ \inf_{z\in K} \sum_{y\in K} g(z,y). \tag{2.9}$$

2.3. *Definition of random interlacements*

Now we recall the definition of the interlacement point process from [11], Section 1. We consider the space of doubly infinite nearest-neighbor trajectories W:

$$W = \Big\{ w\colon \mathbb{Z} \to \mathbb{Z}^d, w(n) \sim w(n+1), n\in\mathbb{Z}, \lim_{n\to\pm\infty} \big|w(n)\big| = \infty \Big\}. \tag{2.10}$$

We endow W with the σ-algebra $\mathcal{W}$ generated by the coordinate maps X_n, $n \in \mathbb{Z}$.

Consider the space W^* of trajectories in W modulo time shift

$$W^* = W/\sim, \text{ where } w \sim w' \quad \Longleftrightarrow \quad w(\cdot) = w'(\cdot + k) \text{ for some } k \in \mathbb{Z}$$

and denote by π^* the canonical projection from W to W^* which assigns to each $w \in W$ the $\sim$-equivalence class $\pi^*(w)$ of w. The map π^* induces a σ-algebra on W^* given by $\mathcal{W}^* = \{A \subset W^*\colon (\pi^*)^{-1}(A) \in \mathcal{W}\}$.

For $K \subset\subset \mathbb{Z}^d$, we denote by W_K the set of trajectories in W that enter the set K, and denote by W_K^* the image of W_K under π^*. Note that $W_K \in \mathcal{W}$ and $W_K^* \in \mathcal{W}^*$.

For any $w^* \in W^*$ and $u \in \mathbb{R}_+$ we call the pair (w^*, u) a labeled trajectory. The space of point measures on which one canonically defines random interlacements is given by

$$\Omega = \Big\{ \omega = \sum_{i\ge 1} \delta_{(w_i^*, u_i)}\colon w_i^* \in W^*, u_i \in \mathbb{R}_+ \text{ and } \forall K \subset\subset \mathbb{Z}^d, u \ge 0\colon \omega\big(W_K^* \times [0,u]\big) < \infty \Big\}. \tag{2.11}$$

The space Ω is endowed with the σ-algebra $\mathcal{F}_\Omega$ generated by the evaluation maps of form $\omega \mapsto \omega(D)$ for $D \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+)$. We recall the definition of the measure Q_K on $(W, \mathcal{W})$ from [11], (1.24): for any $A, B \in \mathcal{W}_+$ and $x \in \mathbb{Z}^d$ let

$$Q_K\big[(X_{-n})_{n\ge 0} \in A, X_0 = x, (X_n)_{n\ge 0} \in B\big] = P_x[A \,|\, \widetilde{H}_K = \infty] \cdot e_K(x) \cdot P_x[B]. \tag{2.12}$$

According to [11], Theorem 1.1, there exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ which satisfies the identity

$$\nu(E) = Q_K\big[\big(\pi^*\big)^{-1}(E)\big] \quad \text{for all } K \subset\subset \mathbb{Z}^d \text{ and } E \in \mathcal{W}^* \text{ with } E \subseteq W_K^*. \tag{2.13}$$

The *interlacement point process* is the Poisson point process on $W^* \times \mathbb{R}_+$ with intensity measure $\nu(\mathrm{d}w^*)\,\mathrm{d}u$, defined on the probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$. Given $\omega = \sum_{i\ge 1} \delta_{(w_i^*, u_i)} \in \Omega$ and $u \ge 0$, the *random interlacement at level u* is the random subset of $\mathbb{Z}^d$ defined by

$$\mathcal{I}^u(\omega) = \bigcup_{i\ge 1, u_i < u} \mathrm{range}\big(w_i^*\big), \tag{2.14}$$

where $\mathrm{range}(w^*) = \{w(n)\colon n \in \mathbb{Z}\}$ for any $w \in \pi^{-1}(w^*)$. The *vacant set at level u* is defined as

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega) \quad \text{for } \omega \in \Omega, u \ge 0.$$

For the sake of consistency, we mention that the law of $\mathcal{I}^u$ is uniquely characterized by (1.1), see [11], Proposition 1.5 and Remark 2.2(2).

2.4. *Discrete interlacement local times*

In this section we define the interlacement local time field $\mathcal{L}^u(\omega)$ at level u, which counts the accumulated number of visits of the interlacement trajectories with label smaller than u to each vertex $x \in \mathbb{Z}^d$, see (2.15). We introduce this notion so that we can control the number of excursions of the interlacement trajectories inside a box in Section 5.

We denote by ℓ a generic element of the product space $\mathbb{N}^{\mathbb{Z}^d}$. For any $x \in \mathbb{Z}^d$, denote by $\Psi_x : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}$ the canonical coordinate function defined by $\Psi_x(\ell) = \ell(x)$. We consider the measurable space $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ where $\mathcal{F}_\ell$ is the σ-algebra generated by the functions Ψ_x, $x \in \mathbb{Z}^d$. For $\ell, \ell' \in \mathbb{N}^{\mathbb{Z}^d}$, we say that $\ell \le \ell'$ if $\ell(x) \le \ell'(x)$ for all $x \in \mathbb{Z}^d$. We say that an event $A \in \mathcal{F}_\ell$ is *increasing* if for any $\ell, \ell' \in \mathbb{N}^{\mathbb{Z}^d}$ the conditions $\ell \in A$ and $\ell \le \ell'$ imply $\ell' \in A$.

Given $\omega = \sum_{i \ge 1} \delta_{(w_i^*, u_i)} \in \Omega$ and $u \ge 0$, we define the discrete interlacement local time profile at level u, $\mathcal{L}^u(\omega) = (\mathcal{L}_x^u(\omega) \colon x \in \mathbb{Z}^d)$ as

$$\mathcal{L}_x^u(\omega) = \sum_{i \ge 1, u_i < u} \sum_{n \in \mathbb{Z}} \mathbb{1}_{\{w_i(n) = x\}}, \quad x \in \mathbb{Z}^d, \tag{2.15}$$

where w_i is any particular element of $\pi^{-1}(w_i^*)$. Note that the function $\mathcal{L}^u : (\Omega, \mathcal{F}_\Omega) \to (\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ is measurable and that $x \in \mathcal{I}^u(\omega)$ if and only if $\mathcal{L}_x^u(\omega) \ge 1$.

Given a measurable function $\mathcal{L} : (\Omega, \mathcal{F}_\Omega) \to (\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ and an event $A \in \mathcal{F}_\ell$, we define

$$A(\mathcal{L}) = \{\omega \in \Omega \colon \mathcal{L}(\omega) \in A\} \quad \text{and} \quad A^u = A(\mathcal{L}^u) \quad \text{for } u \ge 0. \tag{2.16}$$

It follows from (2.15) that for any $0 \le u \le u'$, $\mathbb{P}[\mathcal{L}^u \le \mathcal{L}^{u'}] = 1$. Therefore, for any increasing event $A \in \mathcal{F}_\ell$ and $u \le u'$, we have

$$\mathbb{P}[A^u] \le \mathbb{P}[A^{u'}]. \tag{2.17}$$

Finally, we record that for $x \in \mathbb{Z}^d$ and $u \ge 0$,

$$\mathbb{E}[\mathcal{L}_x^u] = u. \tag{2.18}$$

Indeed, by (2.12) and (2.13), $\mathbb{E}[\mathcal{L}_x^u] = \mathbb{E}[\omega(W_{\{x\}}^* \times [0, u))] \cdot g(x, x) = \mathrm{cap}(\{x\}) \cdot u \cdot g(0, 0) = u$.

2.5. *Cascading events*

In this section we adapt some results of [12] to our setting which involves increasing events of $\mathbb{N}^{\mathbb{Z}^d}$. The result of Lemma 2.2 below is new, but very similar to [12], Corollary 3.5, which is stated for increasing events in $\{0, 1\}^{G \times \mathbb{Z}}$, where G is an infinite, connected, bounded degree weighted graph, satisfying certain regularity conditions (for example, $G = \mathbb{Z}^{d-1}$, with $d \ge 3$). We will use Lemma 2.2 in the proof of Lemma 3.6.

We begin with the definition of uniformly cascading events. We adapt [12], Definition 3.1, to our setting which involves local times.

Definition 2.1. *Let $\lambda > 0$. We say that a family $\mathcal{G} = (G_{x,L,R})_{x \in \mathbb{Z}^d, L \ge 1, R \ge 0}$ of events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ cascades uniformly (in R) with complexity at most $\lambda > 0$ if there exists $C(\lambda) < \infty$ such that*

$$G_{x,L,R} \text{ is } \sigma(\Psi_y, y \in \mathrm{B}(x, 10L))\text{-measurable for each } x \in \mathbb{Z}^d,\ R \ge 0, \text{ and } L \ge 1,$$

and for each l multiple of 100, $x \in \mathbb{Z}^d$, $R \ge 0$, $L \ge 1$, there exists $\Lambda \subseteq \mathbb{Z}^d$ such that

$$\Lambda \subseteq \mathrm{B}(x, 9lL), \tag{2.19}$$

$$|\Lambda| \le C(\lambda) \cdot l^\lambda, \tag{2.20}$$

$$G_{x,lL,R} \subseteq \bigcup_{x_1, x_2 \in \Lambda \colon |x_1 - x_2| \ge (l/100)L} G_{x_1,L,R} \cap G_{x_2,L,R}. \tag{2.21}$$

Lemma 2.2. *Let $\mathcal{G} = (G_{x,L,R})_{x\in\mathbb{Z}^d, L\ge 1, R\ge 0}$ be a family of increasing events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ cascading uniformly (in R) with complexity at most $\lambda > 0$.*

$$\text{Let } L_0 \ge 1,\ l_0 \text{ large enough multiple of } 100, \text{ and } L_n = l_0^n L_0. \tag{2.22}$$

Let $u_{L_0} = {L_0}^{2-d}$, and recall the notation of (2.16). If

$$\inf_{R\ge 0, L_0\ge 1} \sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^{u_{L_0}}_{x,L_0,R}\big] = 0, \tag{2.23}$$

then there exist $l_0 > 1$, $R \ge 0$, $L_0 \ge 1$ and $u > 0$ such that

$$\sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^u_{x,L_n,R}\big] \le 2^{-2^n} \quad \text{for all } n \ge 0. \tag{2.24}$$

The proof of Lemma 2.2 is essentially the same as the proof of [12], Corollary 3.5. For completeness, we include its sketch in the Appendix.

3. Coarse graining of $\mathbb{Z}^d$

In this section we show that when u is small enough, the infinite connected component of $\mathcal{V}^u$ contains a ubiquitous infinite connected subset, which has a well-prescribed structure and useful properties. We do so by partitioning $\mathbb{Z}^d$ into large boxes. We then define a notion of good boxes in Definition 3.3. These boxes are defined to be "sufficiently vacant." In Lemma 3.6, we show that large $*$-connected components of bad boxes are unlikely, where we use Lemma 2.2 to deal with the long-range correlations present in the model. We then combine it with the result of [5], Lemma 2.23, on the connectedness of the exterior $*$-boundary of a $*$-connected finite subset of $\mathbb{Z}^d$ to obtain in Corollary 3.7 that there is a unique infinite connected subset of good boxes (denoted by $\mathcal{G}^\infty$ in Corollary 3.7(2)), and all the remaining bad components are very small. It then follows from the definition of good boxes that the infinite connected component of good boxes contains the desired infinite connected subset of $\mathcal{V}^u$ (see Corollary 3.7(3)). An important consequence of Corollary 3.7, which we will use in the proof of Theorem 1.1 (see (5.2) and (5.5)), is that with high probability, any long nearest-neighbor path in $\mathbb{Z}^d$ will get within distance R from the above defined infinite connected subset of $\mathcal{V}^u$ many times.

3.1. *Setup and auxiliary results*

We consider the hypercubic lattice $\mathbb{Z}^d$ with $d \ge 3$. For an integer $R \ge 0$, let

$$\mathbf{Z} = (2R+1)\cdot\mathbb{Z}^d. \tag{3.1}$$

We say that $x', y' \in \mathbf{Z}$ are (1) nearest-neighbors in $\mathbf{Z}$, if $|x'-y'|_1 = 2R+1$, and (2) $*$-neighbors in $\mathbf{Z}$, if $|x'-y'| = 2R+1$. We denote by $\mathbf{B}(x', N) = \mathrm{B}(x', (2R+1)N) \cap \mathbf{Z}$ the closed ball of radius N in $\mathbf{Z}$. The interior boundary of $\mathbf{K} \subseteq \mathbf{Z}$, denoted by $\partial_{\mathrm{int}}\mathbf{K}$, is the set of vertices of $\mathbf{K}$ that have some nearest neighbor in $\mathbf{Z}\setminus\mathbf{K}$. Note that for $R \ne 0$, the set $\partial_{\mathrm{int}}\mathbf{K}$ is different from $\partial_{\mathrm{int}}\mathbf{K}$, defined in Section 2.1.

With each vertex $x' \in \mathbf{Z}$, we associate the hypercube

$$\mathrm{Q}(x') = \mathrm{B}(x', R) \subset\subset \mathbb{Z}^d. \tag{3.2}$$

This gives us a partition of $\mathbb{Z}^d$ into disjoint hypercubes.

Definition 3.1. *Let $\square$ be the subset of vertices in $\mathrm{Q}(0)$ such that at least two of their coordinates have values in the set $\{-R, -R+1, -R+2, R-2, R-1, R\}$, and let $\square(x') = x' + \square$, for all $x' \in \mathbf{Z}$. We call $\square(x')$ the frame of $\mathrm{Q}(x')$.*

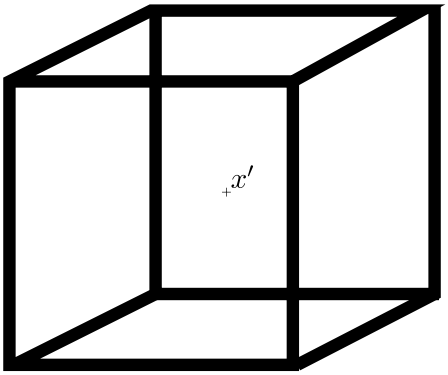

Fig. 1. The frame of $\mathrm{Q}(x')$ in $\mathbb{Z}^3$.

Note that the set $\square$ is connected in $\mathbb{Z}^d$, and for any $x_1', x_2' \in \mathbf{Z}$ nearest-neighbors in $\mathbf{Z}$, the set $\square(x_1') \cup \square(x_2')$ is connected in $\mathbb{Z}^d$.

In the case $d = 3$, the set $\mathrm{Q}(x')$ is the usual cube, and the set $\square(x')$ is just the 2-neighborhood of its edges in the sup-norm, restricted to the vertices inside $\mathrm{Q}(x')$.

Lemma 3.2. *There exists $C = C(d) < \infty$ such that for all $R \geq 2$,*

$$\mathrm{cap}(\square) \leq C R^{d-2} / \log R. \tag{3.3}$$

Proof. The proof easily follows from (2.5), (2.6), (2.8), and (2.9). Let $R \geq 2$. Take $x \in \mathbb{Z}^d$ with $|x| = 2R$. Note that for any $y \in \square$, $R \leq |x - y| \leq 3R$. We have

$$\mathrm{cap}(\square) \overset{(2.6)}{=} \sum_{y \in \square} e_\square(y) \overset{(2.5),(2.8)}{\leq} C R^{d-2} \cdot P_x[H_\square < \infty] \overset{(2.9)}{\leq} C R^{d-2} \cdot \sum_{y \in \square} g(x, y) \Big/ \inf_{z \in \square} \sum_{y \in \square} g(z, y).$$

By (2.5), we get

$$\sum_{y \in \square} g(x, y) \leq C R^{2-d} \cdot |\square| \leq C R^{2-d} \cdot \binom{d}{2} \cdot 6^2 \cdot (2R + 1)^{d-2} \leq C.$$

It remains to show that $\inf_{z \in \square} \sum_{y \in \square} g(z, y) \geq c \cdot \log R$. By the definition of $\square$, for any $z \in \square$ and any integer $1 \leq k \leq R$, we have

$$\left| \left\{ y \in \square : \, |y - z| = k \right\} \right| \geq k^{d-3}.$$

Therefore, uniformly in $z \in \square$, we obtain

$$\sum_{y \in \square} g(z, y) \geq \sum_{k=1}^{R} \sum_{y \in \square: \, |y - z| = k} g(z, y) \overset{(2.5)}{\geq} \sum_{k=1}^{R} c \cdot k^{2-d} \cdot k^{d-3} \geq c \cdot \log R.$$

Putting all the bounds together we get (3.3). $\square$

3.2. *Good vertices*

Definition 3.3. *Let $\ell \in \mathbb{N}^{\mathbb{Z}^d}$. We say that $x' \in \mathbf{Z}$ is R-good for ℓ if*

(1) $\ell(x) = 0$ *for all* $x \in \square(x')$,

(2) $\sum_{x \in \partial_{\mathrm{int}} \mathrm{Q}(x')} \ell(x) \le R^{d-1}$.

If x' is not R-good, then we call it R-bad for ℓ.

Remark 3.4. *The choice of R^{d-1} on the right-hand side of (2) is quite arbitrary. Any function $f = f(R)$ which grows faster than linearly would serve our purposes (see the proof of Lemma 3.5). Condition (2) of Definition 3.3 will be important in Section 5, where we use it to give an upper bound on the number of excursions of the interlacement trajectories inside $\partial_{\mathrm{int}} \mathrm{Q}(x')$.*

Note that for any $R \ge 0$ and $x' \in \mathbf{Z}$,

$$\text{the event } \{x' \text{ is } R\text{-good}\} \text{ is decreasing and } \sigma(\Psi_y, y \in \mathrm{B}(x', R))\text{-measurable.} \tag{3.4}$$

Lemma 3.5. *For $R \ge 1$, let $u_R = R^{2-d}$. Then*

$$\mathbb{P}\big[0 \text{ is } R\text{-good for } \mathcal{L}^{u_R}\big] \to 1, \quad \text{as } R \to \infty.$$

Proof. By the definition of R-good vertices, it suffices to prove that

$$\mathbb{P}\big[\Box \subseteq \mathcal{V}^{u_R}\big] \to 1 \quad \text{and} \quad \mathbb{P}\Bigg[\sum_{x \in \partial_{\mathrm{int}} \mathrm{Q}(0)} \mathcal{L}_x^{u_R} \le R^{d-1}\Bigg] \to 1, \quad \text{as } R \to \infty.$$

The first statement follows from Lemma 3.2. Indeed,

$$\mathbb{P}\big[\Box \subseteq \mathcal{V}^{u_R}\big] = \mathrm{e}^{-u_R \cdot \mathrm{cap}(\Box)} \overset{(3.3)}{\ge} \mathrm{e}^{-c/\log R} \to 1.$$

As for the second statement, by the Markov inequality,

$$\mathbb{P}\Bigg[\sum_{x \in \partial_{\mathrm{int}} \mathrm{Q}(0)} \mathcal{L}_x^{u_R} > R^{d-1}\Bigg] \le R^{1-d} \cdot \sum_{x \in \partial_{\mathrm{int}} \mathrm{Q}(0)} \mathbb{E}\big[\mathcal{L}_x^{u_R}\big] = R^{1-d} \cdot \big|\partial_{\mathrm{int}} \mathrm{Q}(0)\big| \cdot \mathbb{E}\big[\mathcal{L}_0^{u_R}\big] \overset{(2.18)}{\le} C \cdot u_R \to 0.$$

This completes the proof of Lemma 3.5. □

For $V_1, V_2 \subseteq \mathbb{Z}^d$ and $\ell \in \mathbb{N}^{\mathbb{Z}^d}$, we write "$V_1 \leftrightarrow V_2$ by a $*$-path in $\mathbf{Z}$ of R-bad vertices for ℓ", if there is a sequence $\pi = (x'_1, \dots, x'_n)$ in $\mathbf{Z}$ of R-bad vertices for ℓ such that

$$x'_1 \in V_1, \qquad x'_n \in V_2, \qquad \forall 1 \le i \le n-1: \; \big|x'_{i+1} - x'_i\big| = 2R + 1. \tag{3.5}$$

The next lemma proves that $*$-connected components of R-bad vertices for $\mathcal{L}^u$ in $\mathbf{Z}$ are small for large enough R and small enough u. Then a standard relation between nearest-neighbor and $*$-connectivities implies the existence of a unique infinite connected component of R-good vertices (see Corollary 3.7).

Lemma 3.6. *There exist $R \ge 0$, $u_1 > 0$, $c = c(d) > 0$ and $C = c(d) < \infty$ such that for all $u \le u_1$ and $N \ge 1$, we have*

$$\mathbb{P}\Big[0 \leftrightarrow \partial_{\mathrm{int}} \mathbf{B}(0, N) \text{ by a } *\text{-path in } \mathbf{Z} \text{ of } R\text{-bad vertices for } \mathcal{L}^u\Big] \le C\mathrm{e}^{-N^c}. \tag{3.6}$$

Proof. First of all, note that the $\mathcal{F}_\ell$-measurable event

$$\Big\{\ell \colon 0 \leftrightarrow \partial_{\mathrm{int}} \mathbf{B}(0, N) \text{ by a } *\text{-path in } \mathbf{Z} \text{ of } R\text{-bad vertices for } \ell\Big\}$$

is increasing. Therefore, it suffices to prove that there exist $R \ge 0$, $u > 0$, $c > 0$ and $C < \infty$ such that for all $N \ge 1$, (3.6) holds. (Then, by (2.17), the result will hold for all u' smaller than u.)

For $x \in \mathbb{Z}^d$ and integers $R \geq 0$, $L \geq 1$, consider the events

$$G_{x,L,R} = \begin{cases} \left\{ \ell \in \mathbb{N}^{\mathbb{Z}^d} : \begin{array}{c} \mathrm{B}(x, L) \leftrightarrow \mathrm{B}(x, 2L)^c \\ \text{by a } *\text{-path in } \mathbf{Z} \text{ of } R\text{-bad vertices for } \ell \end{array} \right\}, & \text{if } L \geq R, \\ \mathbb{N}^{\mathbb{Z}^d}, & \text{if } L < R. \end{cases} \tag{3.7}$$

In order to prove (3.6), it suffices to show that there exist $L_0 \geq 1$, $l_0 > 1$, $R \geq 0$ and $u > 0$ such that

$$\mathbb{P}\big[G_{0,L_n,R}\big(\mathcal{L}^u\big)\big] \leq 2^{-2^n} \quad \text{for all } n \geq 0, \tag{3.8}$$

where L_n are defined in (2.22) (see also the notation in (2.16)). This will immediately follow from Lemma 2.2, as soon as we show that

$$(G_{x,L,R})_{x \in \mathbb{Z}^d, L \geq 1, R \geq 0} \text{ is a family of increasing events cascading uniformly with complexity at most } d, \tag{3.9}$$

and that the family of events $(G_{x,L,R})_{x \in \mathbb{Z}^d, L \geq 1, R \geq 0}$ satisfies (2.23).

We begin with the proof of (3.9). The events $G_{x,L,R}$ are clearly increasing. For $L \geq R$, we have $\ell \in G_{x,L,R}$ if and only if there exists a $*$-path $\pi' = (y_1', \ldots, y_n')$ in $\mathbf{Z}$ of R-bad vertices for ℓ satisfying

$$\big|y_1' - x\big| \leq L, \qquad 2L < \big|y_n' - x\big|, \qquad \forall 1 \leq i \leq n: \big|y_i' - x\big| \leq 2L + 2R + 1. \tag{3.10}$$

Treating the cases $L \geq R$ and $L < R$ separately and using (3.4) and (3.10), one can show that the event $G_{x,L,R}$ is $\sigma(\Psi_y, y \in \mathrm{B}(x, 10L))$-measurable. Let l be a multiple of 100, $x \in \mathbb{Z}^d$, $R \geq 0$, $L \geq 1$. Let

$$\Lambda = L \cdot \mathbb{Z}^d \cap \mathrm{B}(x, 3lL).$$

The set Λ immediately satisfies (2.19) and (2.20) (with $\lambda = d$), so we only need to check that Λ satisfies (2.21). By (3.7), it is enough to consider the non-trivial case $L \geq R$.

If $\ell \in G_{x,lL,R}$, then there exists a $*$-path $\pi' = (y_1', \ldots, y_n')$ in $\mathbf{Z}$ of R-bad vertices for ℓ satisfying $|y_1' - x| \leq lL$ and $2lL < |y_n' - x| \leq 2lL + 2R + 1 \leq 3lL$, so that we can find $x_1, x_2 \in \Lambda$ such that $|y_1' - x_1| \leq L$, $|y_n' - x_2| \leq L$. Note that $|x_1 - x_2| \geq lL - 2L > \frac{l}{100}L$. Moreover, the path π' connects $\mathrm{B}(x_i, L)$ to $\mathrm{B}(x_i, 2L)^c$ for $i \in \{1, 2\}$. Thus $\ell \in G_{x_1,L,R} \cap G_{x_2,L,R}$, which implies (2.21) and hence (3.9).

It remains to prove that $(G_{x,L,R})_{x \in \mathbb{Z}^d, L \geq 1, R \geq 0}$ satisfies (2.23). Let us choose $L_0 = R$. By (3.10) and (3.5) we have

$$G_{x,R,R} \subseteq \bigcup_{x' \in \mathrm{B}(x,R) \cap \mathbf{Z}} \big\{\ell \in \mathbb{N}^{\mathbb{Z}^d} : x' \text{ is } R\text{-bad for } \ell\big\}.$$

Since $|\mathrm{B}(x, R) \cap \mathbf{Z}| = 1$, the condition (2.23) follows from Lemma 3.5. Thus we can apply Lemma 2.2 to infer (3.8), which completes the proof of Lemma 3.6. □

The following result states that there exists a ubiquitous infinite component of good vertices in $\mathbf{Z}$. It is a consequence of Lemma 3.6 and [5], Lemma 2.23, about the connectedness of the exterior $*$-boundary of a $*$-connected subset of $\mathbb{Z}^d$.

Corollary 3.7. *Fix R, u_1, $c = c(d) > 0$, and $C = C(d) < \infty$ as in Lemma 3.6. For all $u \leq u_1$, we have*

(1) *for all $n, N \geq 1$,*

$$\mathbb{P}\left[\begin{array}{c} \mathbf{B}(0, N+n) \setminus \mathbf{B}(0, N) \textit{ contains a set } \mathcal{S} \subset \mathbf{Z} \textit{ such that} \\ \mathcal{S} \textit{ is connected in } \mathbf{Z}, \textit{ each } x \in \mathcal{S} \textit{ is } R\textit{-good for } \mathcal{L}^u, \textit{ and} \\ \textit{every } *\textit{-path in } \mathbf{Z} \textit{ from } \mathbf{B}(0, N+1) \textit{ to } \partial_{\mathrm{int}}\mathbf{B}(0, N+n) \\ \textit{intersects } \mathcal{S} \end{array}\right] \geq 1 - C \cdot \big|\mathbf{B}(0, N+1)\big| \cdot \mathrm{e}^{-n^c}, \tag{3.11}$$

(2) *there exists a unique infinite connected component of R-good vertices for $\mathcal{L}^u$ in $\mathbf{Z}$, which we denote by $\mathcal{G}^\infty$, and for all $n \geq 1$,*

$$\mathbb{P}\big[\mathcal{G}^\infty \text{ contains a vertex in } \mathbf{B}(0,n)\big] \geq 1 - C \cdot \sum_{N \geq n} \mathrm{e}^{-N^c}, \tag{3.12}$$

(3) *the set $\bigcup_{x' \in \mathcal{G}^\infty} \square(x')$ is an infinite connected subset of $\mathcal{V}^u$.*

Proof. (1) Take $n, N \geq 1$. Let

$$\widetilde{\mathcal{S}} = \mathbf{B}(0,N) \cup \big\{x \in \mathbf{B}(0,N+n) \colon x \text{ is connected to } \mathbf{B}(0,N+1) \text{ by a } *\text{-path in } \mathbf{B}(0,N+n) \text{ of } R\text{-bad vertices for } \mathcal{L}^u\big\},$$

and consider the exterior $*$-boundary of $\widetilde{\mathcal{S}}$ in $\mathbf{B}(0,N+n)$:

$$\widehat{\mathcal{S}} = \big\{y \in \mathbf{B}(0,N+n) \setminus \widetilde{\mathcal{S}} \colon y \text{ is a } *\text{-neighbor in } \mathbf{Z} \text{ of some } x \in \widetilde{\mathcal{S}}\big\}.$$

Note that every vertex in $\widehat{\mathcal{S}}$ is R-good. If $\widetilde{\mathcal{S}} \cap \partial_{\mathrm{int}}\mathbf{B}(0,N+n) = \varnothing$, then every $*$-path in $\mathbf{Z}$ from $\mathbf{B}(0,N+1)$ to $\partial_{\mathrm{int}}\mathbf{B}(0,N+n)$ intersects $\widehat{\mathcal{S}}$. Non-trivially, it was proved in [5], Lemma 2.23 (see also a short proof in [16], Theorem 4), that if $\widetilde{\mathcal{S}} \cap \partial_{\mathrm{int}}\mathbf{B}(0,N+n) = \varnothing$, then $\widehat{\mathcal{S}}$ contains a *connected* component $\mathcal{S}$ in $\mathbf{Z}$ such that every $*$-path in $\mathbf{Z}$ from $\mathbf{B}(0,N+1)$ to $\partial_{\mathrm{int}}\mathbf{B}(0,N+n)$ intersects $\mathcal{S}$. By translation invariance of $\mathcal{L}^u$ and (3.6), with $c = c(d) > 0$ and $C = C(d) < \infty$ as in Lemma 3.6, and for all $n, N \geq 1$, we have

$$\mathbb{P}\left[\begin{array}{c}\mathbf{B}(0,N+1) \text{ is connected to } \partial_{\mathrm{int}}\mathbf{B}(0,N+n) \\ \text{by a } *\text{-path in } \mathbf{Z} \text{ of } R\text{-bad vertices for } \mathcal{L}^u\end{array}\right] \leq \big|\mathbf{B}(0,N+1)\big| \cdot C \cdot \mathrm{e}^{-n^c}.$$

Together with the above observations, this implies the first statement of Corollary 3.7.

(2) The existence of $\mathcal{G}^\infty$ as well as (3.12) follow from (3.6) and planar duality (see, e.g., the proof of [9], Theorem 2.1). The uniqueness of $\mathcal{G}^\infty$ follows from (3.11) and the Borel–Cantelli lemma.

(3) The fact that $\bigcup_{x' \in \mathcal{G}^\infty} \square(x')$ is an infinite connected subset of $\mathcal{V}^u$ follows from (2), Definition 3.1 of $\square$, and Definition 3.3 of R-good vertices. □

4. Conditional independence for random interlacements

In this section we prove (in Lemma 4.4) that the behavior of the interlacement trajectories with labels at most u inside a finite set K is independent of their behavior outside of K, given the information about entrance and exit points of all the excursions into K of all the interlacement trajectories with labels at most u. As part of the proof, we will also identify the conditional law of the excursions inside and outside K (see (4.11) and (4.12), respectively).

We begin by introducing notation and recalling some properties of the interlacement point measures, which we will use to identify the above mentioned laws of excursions. We then properly define the excursions (in Section 4.2) and the σ-algebras of events generated by excursions inside, outside, and on the boundary of K (in Section 4.3). Finally, (in Section 4.4) we state and prove the conditional independence of the σ-algebras.

4.1. *More preliminaries about interlacements*

Recall the notation and the definition of the interlacement point process from Section 2.3. Let $\omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)}$ be an interlacement point process on $W^* \times \mathbb{R}_+$. For $K \subset\subset \mathbb{Z}^d$ and $u > 0$, let

$$\omega_{K,u} = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \mathbb{1}_{\{w_i^* \in W_K^*, u_i \leq u\}} \quad \text{and} \quad \omega - \omega_{K,u} = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \mathbb{1}_{\{w_i^* \notin W_K^*\} \cup \{u_i > u\}} \tag{4.1}$$

be the restrictions of ω to the set of pairs (w_i^*, u_i) with, respectively, w_i^* intersecting K and $u_i \leq u$, and either w_i^* not intersecting K or $u_i > u$. By the definition of ω, the point measures $\omega_{K,u}$ and $\omega - \omega_{K,u}$ are independent Poisson point processes. By (2.11), each $\omega_{K,u}$ is a finite point measure. For each $K \subset\subset \mathbb{Z}^d$ and $u > 0$, $\omega_{K,u}$ is a Poisson point process on $W_K^* \times \mathbb{R}_+$ with intensity measure

$$\mathbb{1}_{W_K^* \times [0,u]} \cdot \nu\big(\mathrm{d}w^*\big)\,\mathrm{d}u,$$

where the measure ν is defined in (2.13). In particular, the total mass of $\omega_{K,u}$ has Poisson distribution with parameter $u \cdot \mathrm{cap}(K)$ (this follows from (2.12) and (2.13)), and all the u_i's in the definition of $\omega_{K,u}$ are almost surely different. Therefore, $\omega_{K,u}$ admits the following representation:

$$\omega_{K,u} = \sum_{i=1}^{N_{K,u}} \delta_{(w_i^*, u_i)}, \tag{4.2}$$

where $N_{K,u}$ has Poisson distribution with parameter $u \cdot \mathrm{cap}(K)$, and given $N_{K,u}$, (a) $(u_1, \dots, u_{N_{K,u}})$ and $(w_1^*, \dots, w_{N_{K,u}}^*)$ are independent, (b) $u_1 < \cdots < u_{N_{K,u}}$ are obtained by relabeling independent uniform random variables on $[0, u]$, (c) w_i^* are independent and each distributed according to $\mathbb{1}_{W_K^*} \cdot \nu(\mathrm{d}w^*)/\mathrm{cap}(K)$.

For each w_i^* in (4.2),

$$\text{let } X_i \text{ be the unique trajectory from } (\pi^*)^{-1}(w_i^*) \subset W \text{ parametrized in such a way that } X_i(0) \in K \text{ and } X_i(t) \notin K \text{ for all } t < 0. \tag{4.3}$$

(Here we abuse notation and denote by X_i (bi-infinite) trajectories rather than canonical coordinate maps in W or W_+, see below (2.1).) By (2.12) and (2.13), given $N_{K,u}$ and $(u_i\colon 1 \le i \le N_{K,u})$, the random trajectories $(X_i\colon 1 \le i \le N_{K,u})$ are independent and for all $\mathcal{A}, \mathcal{B} \in \mathcal{W}_+$ (see below (2.1)), $x \in \mathbb{Z}^d$,

$$\mathbb{P}\big[\big(X_i(-t)\colon t \ge 0\big) \in \mathcal{A}, X_i(0) = x, \big(X_i(t)\colon t \ge 0\big) \in \mathcal{B})\big] = P_x^K[\mathcal{A}] \cdot \widetilde{e}_K(x) \cdot P_x[\mathcal{B}], \tag{4.4}$$

where P_x^K is the law of simple random walk started at x and conditioned on $\widetilde{H}_K = \infty$, and $\widetilde{e}_K$ is defined in (2.7).

4.2. *Interlacement excursions*

Definition 4.1. *For $w \in W$, let $R_1(w) = \inf\{n \in \mathbb{Z}\colon w(n) \in K\}$ be the first entrance time of w to K. If $R_1(w) < \infty$, let $D_1(w) = \inf\{n > R_1(w)\colon w(n) \notin K\}$ be the first exit time from K. Similarly, for $k \ge 2$, if $R_{k-1}(w) < \infty$, let*

$$D_{k-1}(w) = \inf\big\{n > R_{k-1}(w)\colon w(n) \notin K\big\} \quad \textit{and} \quad R_k(w) = \inf\big\{n > D_{k-1}(w)\colon w(n) \in K\big\}.$$

For w with $R_1(w) < \infty$, let

$$M(w) = \max\big\{k \ge 1\colon R_k(w) < \infty\big\}.$$

By (2.10), $M(w) < \infty$ for any $w \in W$.

Abusing notation, we extend the above definitions of R_k, D_k and M to trajectories $w_+ \in W_+$ in a natural way, namely, defining $R_1(w_+) = H_K(w_+)$ (see (2.2)), and all the other variables with the same formulas as above.

Given $(X_i\colon 1 \le i \le N_{K,u})$ as in (4.3), for each $1 \le i \le N_{K,u}$, let

$$M_i = M(X_i)$$

be the number of times trajectory X_i revisits K, and for each $1 \le j \le M_i$, let

$$A_{i,j} = R_j(X_i) \quad \text{and} \quad B_{i,j} = D_j(X_i) - 1$$

be the times when jth excursion of X_i inside K begins and ends. Note that $X_i(t) \in K$ if and only if $A_{i,j} \le t \le B_{i,j}$ for some $1 \le j \le M_i$. For $1 \le i \le N_{K,u}$ and $1 \le j \le M_i$, let

$$T_{i,j}^{\mathrm{in}} = B_{i,j} - A_{i,j}, \qquad X_{i,j}^{\mathrm{in}}(t) = X_i(A_{i,j} + t) \quad \text{for } 0 \le t \le T_{i,j}^{\mathrm{in}},$$

and for $1 \le i \le N_{K,u}$, $1 \le j \le M_i - 1$, let

$$T^{\mathrm{out}}_{i,j} = A_{i,j+1} - B_{i,j}, \qquad X^{\mathrm{out}}_{i,j}(t) = X_i(B_{i,j} + t) \quad \text{for } 0 \le t \le T^{\mathrm{out}}_{i,j}.$$

Note that $(X^{\mathrm{in}}_{i,j}\colon 1 \le j \le M_i)$ correspond to the pieces of X_i inside K, and $(X^{\mathrm{out}}_{i,j}\colon 1 \le j \le M_i - 1)$ correspond to the finite pieces of X_i outside K (except for their start and end points). Finally, let

$$X^-_i(t) = X_i(-t) \quad \text{and} \quad X^+_i(t) = X_i(t + B_{i,M_i}) \quad \text{for } t \ge 0,$$

be the (infinite) pieces of trajectory X_i up to the first enter in K and from the last visit to K, respectively.

4.3. *Interior, exterior, and boundary σ-algebras*

Let $\mathcal{F}^{\mathrm{in}}_{K,u}$ be the σ-algebra generated by the random variables

$$N_{K,u}, \qquad (u_i\colon 1 \le i \le N_{K,u}), \qquad (M_i\colon 1 \le i \le N_{K,u}), \qquad \bigl(X^{\mathrm{in}}_{i,j}\colon 1 \le i \le N_{K,u}, 1 \le j \le M_i\bigr),$$

i.e., $\mathcal{F}^{\mathrm{in}}_{K,u}$ is generated by the excursions of the interlacement trajectories with labels at most u inside K.

Let $\mathcal{F}^{\mathrm{out}}_{K,u}$ be the σ-algebra generated by

$$\begin{aligned} &\omega - \omega_{K,u}, \qquad N_{K,u}, \qquad (u_i\colon 1 \le i \le N_{K,u}), \qquad (M_i\colon 1 \le i \le N_{K,u}), \\ &\bigl(X^-_i\colon 1 \le i \le N_{K,u}\bigr), \qquad \bigl(X^{\mathrm{out}}_{i,j}\colon 1 \le i \le N_{K,u}, 1 \le j \le M_i - 1\bigr), \qquad \bigl(X^+_i\colon 1 \le i \le N_{K,u}\bigr) \end{aligned}$$

i.e., $\mathcal{F}^{\mathrm{out}}_{K,u}$ is generated by the excursions of the interlacement trajectories with labels at most u outside K and $\omega - \omega_{K,u}$ (see (4.1)).

Let $\mathcal{F}^{AB}_{K,u}$ be the σ-algebra generated by

$$\begin{aligned} &N_{K,u}, (u_i\colon 1 \le i \le N_{K,u}), \qquad (M_i\colon 1 \le i \le N_{K,u}), \\ &\bigl(\bigl(X_i(A_{i,j}), X_i(B_{i,j})\bigr)\colon 1 \le i \le N_{K,u}, 1 \le j \le M_i\bigr), \end{aligned}$$

i.e., $\mathcal{F}^{AB}_{K,u}$ is generated by the entrance and exit points of the interlacement trajectories with labels at most u to K.

The following properties are immediate from the definitions.

Claim 4.2. *For any $K \subset\subset \mathbb{Z}^d$,*

(1) $\mathcal{F}^{AB}_{K,u} \subset \mathcal{F}^{\mathrm{in}}_{K,u}$ *and* $\mathcal{F}^{AB}_{K,u} \subset \mathcal{F}^{\mathrm{out}}_{K,u}$,
(2) $\sigma(\mathcal{F}^{\mathrm{in}}_{K,u}, \mathcal{F}^{\mathrm{out}}_{K,u}) = \mathcal{F}_\Omega$ (*see below* (2.11)),
(3) $(\mathcal{L}^u_x\colon x \in K)$ *is* $\mathcal{F}^{\mathrm{in}}_{K,u}$*-measurable, and* $(\mathcal{L}^u_x\colon x \in \mathbb{Z}^d \setminus K)$ *is* $\mathcal{F}^{\mathrm{out}}_{K,u}$*-measurable* (*recall the definition of* $\mathcal{L}^u$ *in* (2.15)).

4.4. *Conditional independence*

In this section we prove the main result of Section 4, which states that the σ-algebras $\mathcal{F}^{\mathrm{in}}_{K,u}$ (generated by the excursions of the interlacement trajectories inside K) and $\mathcal{F}^{\mathrm{out}}_{K,u}$ (generated by the excursions outside K and $\omega - \omega_{K,u}$ (see (4.1))) are conditionally independent, given $\mathcal{F}^{AB}_{K,u}$ (generated by the entrance and exit points of the interlacement trajectories to K). In the proof of (1.3), we will only use Lemma 4.4(a) and (4.11) (see the proofs of Lemmas 5.11 and 5.13, respectively). We begin with a definition.

Definition 4.3. *For integers $n \ge 1$, $1 \le i \le n$, $\mathcal{U}_i \in \mathcal{B}([0,u])$, $\mathcal{A}_i, \mathcal{B}_i \in \mathcal{W}_+$, integers $m_i \ge 1$, $1 \le j \le m_i$, $x_{i,j}, y_{i,j} \in \partial_{\mathrm{int}} K$, finite nearest-neighbor trajectories $\tau^{\mathrm{in}}_{i,j}$ from $x_{i,j}$ to $y_{i,j}$ in K, and for $1 \le j' \le m_i - 1$, finite nearest-neighbor* *trajectories $\tau^{\text{out}}_{i,j'}$ from $y_{i,j'}$ to $x_{i,j'+1}$ outside K except for the start and end points, consider the events*

$$\mathcal{E}^{AB}_{K,u} = \left\{ \begin{array}{c} N_{K,u} = n, u_i \in \mathcal{U}_i, M_i = m_i, X_i(A_{i,j}) = x_{i,j}, X_i(B_{i,j}) = y_{i,j}, \\ \text{for all } 1 \le i \le n, 1 \le j \le m_i \end{array} \right\} \in \mathcal{F}^{AB}_{K,u}, \tag{4.5}$$

$$\mathcal{E}^{\text{in}}_{K,u} = \left\{ \begin{array}{c} N_{K,u} = n, u_i \in \mathcal{U}_i, M_i = m_i, X_i(A_{i,j}) = x_{i,j}, X_i(B_{i,j}) = y_{i,j}, \\ X^{\text{in}}_{i,j} = \tau^{\text{in}}_{i,j}, \text{for all } 1 \le i \le n, 1 \le j \le m_i \end{array} \right\} \in \mathcal{F}^{\text{in}}_{K,u}, \tag{4.6}$$

$$\mathcal{E}^{\text{out}}_{K,u} = \left\{ \begin{array}{c} N_{K,u} = n, u_i \in \mathcal{U}_i, M_i = m_i, X_i(A_{i,j}) = x_{i,j}, X_i(B_{i,j}) = y_{i,j}, \\ X^{\text{out}}_{i,j'} = \tau^{\text{out}}_{i,j'}, X^-_i \in \mathcal{A}_i, X^+_i \in \mathcal{B}_i, \\ \text{for all } 1 \le i \le n, 1 \le j \le m_i, 1 \le j' \le m_i - 1 \end{array} \right\} \in \mathcal{F}^{\text{out}}_{K,u}. \tag{4.7}$$

Note that

the σ-algebras $\mathcal{F}^{\text{in}}_{K,u}$ and $\mathcal{F}^{AB}_{K,u}$ are respectively generated by events of form (4.6) and (4.5), and $\mathcal{F}^{\text{out}}_{K,u}$ is generated by the events $\mathcal{E}^{\text{out}}_{K,u} \cap \{\omega - \omega_{K,u} \in \mathcal{E}\}$, with $\mathcal{E} \in \mathcal{F}_\Omega$ (see below (2.11)). (4.8)

Lemma 4.4. *For any $K \subset\subset \mathbb{Z}^d$, $u > 0$,*

(a) *$\mathcal{F}^{\text{in}}_{K,u}$ and $\mathcal{F}^{\text{out}}_{K,u}$ are conditionally independent, given $\mathcal{F}^{AB}_{K,u}$, and*
(b) *for any choice of the parameters in Definition 4.3, we have*

$$\mathbb{P}\big[\mathcal{E}^{\text{in}}_{K,u} \cap \mathcal{E}^{\text{out}}_{K,u}\big] = \mathbb{P}\big[\mathcal{E}^{AB}_{K,u}\big] \cdot \mathbb{P}\big[\mathcal{E}^{\text{in}}_{K,u} | \mathcal{E}^{AB}_{K,u}\big] \cdot \mathbb{P}\big[\mathcal{E}^{\text{out}}_{K,u} | \mathcal{E}^{AB}_{K,u}\big], \tag{4.9}$$

and

$$\begin{aligned} \mathbb{P}\big[\mathcal{E}^{AB}_{K,u}\big] = {} & \mathbb{P}[N_{K,u} = n] \cdot \mathbb{P}[u_i \in \mathcal{U}_i \colon 1 \le i \le n] \\ & \cdot \prod_{i=1}^n \widetilde{e}_K(x_{i,1}) \cdot P_{x_{i,1}} \left[\begin{array}{c} M(X) = m_i, X(R_j) = x_{i,j}, X(D_j - 1) = y_{i,j} \\ \text{for all } 1 \le j \le m_i \end{array} \right], \end{aligned} \tag{4.10}$$

$$\mathbb{P}\big[\mathcal{E}^{\text{in}}_{K,u} | \mathcal{E}^{AB}_{K,u}\big] = \prod_{i=1}^n \prod_{j=1}^{m_i} \frac{P_{x_{i,j}}[(X(t) \colon 0 \le t \le T_K - 1) = \tau^{\text{in}}_{i,j}]}{P_{x_{i,j}}[X(T_K - 1) = y_{i,j}]}, \tag{4.11}$$

$$\mathbb{P}\big[\mathcal{E}^{\text{out}}_{K,u} | \mathcal{E}^{AB}_{K,u}\big] = \prod_{i=1}^n P^K_{x_{i,1}}[\mathcal{A}_i] \cdot P^K_{y_{i,m_i}}[\mathcal{B}_i] \cdot \prod_{j'=1}^{m_i - 1} \frac{P_{y_{i,j'}}[(X(t) \colon 0 \le t \le \widetilde{H}_K) = \tau^{\text{out}}_{i,j'}, \widetilde{H}_K < \infty]}{P_{y_{i,j'}}[X(\widetilde{H}_K) = x_{i,j'+1}, \widetilde{H}_K < \infty]}, \tag{4.12}$$

where T_K and $\widetilde{H}_K$ are defined in (2.4) and (2.3), respectively.

Proof. Statement (a) immediately follows from (4.9), the fact that point processes $\omega_{K,u}$ and $\omega - \omega_{K,u}$ are independent, the inclusion $\mathcal{E}^{\text{in}}_{K,u}, \mathcal{E}^{\text{out}}_{K,u} \subseteq \mathcal{E}^{AB}_{K,u}$, and (4.8).

To prove (b), we first observe that the expressions in (4.10), (4.11), and (4.12) indeed give rise to probability distributions.

We rewrite the left-hand side of (4.9) using the definition (4.2) of $\omega_{K,u}$ and (4.4) as

$$\begin{aligned} \mathbb{P}\big[\mathcal{E}^{\text{in}}_{K,u} \cap \mathcal{E}^{\text{out}}_{K,u}\big] = {} & \mathbb{P}[N_{K,u} = n] \cdot \mathbb{P}[u_i \in \mathcal{U}_i \colon 1 \le i \le n] \cdot \prod_{i=1}^n P^K_{x_{i,1}}[\mathcal{A}_i] \cdot \widetilde{e}_K(x_{i,1}) \\ & \cdot \prod_{i=1}^n P_{x_{i,1}} \left[\begin{array}{c} M(X) = m_i, X(R_j) = x_{i,j}, X(D_j - 1) = y_{i,j}, \\ (X(t) \colon R_j \le t \le D_j - 1) = \tau^{\text{in}}_{i,j}, (X(t) \colon D_{j'} - 1 \le t \le R_{j'+1}) = \tau^{\text{out}}_{i,j'}, \\ (X(t + D_{m_i} - 1) \colon t \ge 0) \in \mathcal{B}_i \text{ for all } 1 \le j \le m_i, 1 \le j' \le m_i - 1 \end{array} \right]. \end{aligned}$$

Note that this equality immediately implies (4.10) by taking all $\mathcal{A}_i$ and $\mathcal{B}_i$ equal to W_+ and summing over all possible paths $\tau^{\text{in}}_{i,j}$ and $\tau^{\text{out}}_{i,j'}$.

Consecutive applications of the Markov property for simple random walk imply that the above expression equals

$$\begin{aligned}
&\mathbb{P}[N_{K,u}=n]\cdot\mathbb{P}[u_i\in\mathcal{U}_i\colon\ 1\le i\le n]\cdot\prod_{i=1}^{n}P^K_{x_{i,1}}[\mathcal{A}_i]\cdot\widetilde{e}_K(x_{i,1})\\
&\quad\cdot\prod_{i=1}^{n}\prod_{j=1}^{m_i}P_{x_{i,j}}\bigl[X(t)=\tau^{\mathrm{in}}_{i,j}(t)\colon\ 0\le t\le\bigl|\tau^{\mathrm{in}}_{i,j}\bigr|-1\bigr]\\
&\quad\cdot\prod_{i=1}^{n}\prod_{j'=1}^{m_i-1}P_{y_{i,j'}}\bigl[X(t)=\tau^{\mathrm{out}}_{i,j'}(t)\colon\ 0\le t\le\bigl|\tau^{\mathrm{out}}_{i,j'}\bigr|-1\bigr]\\
&\quad\cdot\prod_{i=1}^{n}P_{y_{i,m_i}}[\mathcal{B}_i,\widetilde{H}_K=\infty].
\end{aligned}\tag{4.13}$$

We will now rearrange the terms in (4.13) to obtain (4.9), (4.11), and (4.12). We begin with a few observations. Note that

$$P_{y_{i,j'}}\bigl[X(t)=\tau^{\mathrm{out}}_{i,j'}(t),0\le t\le\bigl|\tau^{\mathrm{out}}_{i,j'}\bigr|-1\bigr]=P_{y_{i,j'}}\bigl[\bigl(X(t)\colon\ 0\le t\le\widetilde{H}_K\bigr)=\tau^{\mathrm{out}}_{i,j'}\bigr],\tag{4.14}$$

and

$$P_{y_{i,m_i}}[\mathcal{B}_i,\widetilde{H}_K=\infty]=e_K(y_{i,m_i})\cdot P^K_{y_{i,m_i}}[\mathcal{B}_i].\tag{4.15}$$

Also note that by the Markov property at time $|\tau^{\mathrm{in}}_{i,j}|-1$, we have

$$\begin{aligned}
&P_{x_{i,j}}\bigl[X(t)=\tau^{\mathrm{in}}_{i,j}(t),0\le t\le\bigl|\tau^{\mathrm{in}}_{i,j}\bigr|-1\bigr]\\
&\quad=\frac{P_{x_{i,j}}[X(t)=\tau^{\mathrm{in}}_{i,j}(t),0\le t\le|\tau^{\mathrm{in}}_{i,j}|-1,X(|\tau^{\mathrm{in}}_{i,j}|)\notin K]}{P_{y_{i,j}}[X(1)\notin K]}\\
&\quad=\frac{P_{x_{i,j}}[(X(t)\colon\ 0\le t\le T_K-1)=\tau^{\mathrm{in}}_{i,j}]}{P_{y_{i,j}}[X(1)\notin K]}.
\end{aligned}\tag{4.16}$$

We now plug in the expressions (4.14), (4.15), and (4.16) into (4.13) to get that $\mathbb{P}[\mathcal{E}^{\mathrm{in}}_{K,u}\cap\mathcal{E}^{\mathrm{out}}_{K,u}]$ equals

$$\begin{aligned}
&\mathbb{P}[N_{K,u}=n]\cdot\mathbb{P}[u_i\in\mathcal{U}_i\colon\ 1\le i\le n]\cdot\prod_{i=1}^{n}P^K_{x_{i,1}}[\mathcal{A}_i]\cdot\widetilde{e}_K(x_{i,1})\cdot e_K(y_{i,m_i})\cdot P^K_{y_{i,m_i}}[\mathcal{B}_i]\\
&\quad\cdot\prod_{i=1}^{n}\prod_{j=1}^{m_i}\frac{P_{x_{i,j}}[(X(t)\colon\ 0\le t\le T_K-1)=\tau^{\mathrm{in}}_{i,j}]}{P_{y_{i,j}}[X(1)\notin K]}\cdot\prod_{j'=1}^{m_i-1}P_{y_{i,j'}}\bigl[\bigl(X(t)\colon\ 0\le t\le\widetilde{H}_K\bigr)=\tau^{\mathrm{out}}_{i,j'}\bigr].
\end{aligned}\tag{4.17}$$

By taking $\mathcal{A}_i=\mathcal{B}_i=W_+$ in (4.17) and summing over all $\tau^{\mathrm{in}}_{i,j}$ and $\tau^{\mathrm{out}}_{i,j'}$, we obtain that

$$\begin{aligned}
\mathbb{P}\bigl[\mathcal{E}^{AB}_{K,u}\bigr]&=\mathbb{P}[N_{K,u}=n]\cdot\mathbb{P}[u_i\in\mathcal{U}_i\colon\ 1\le i\le n]\\
&\quad\cdot\prod_{i=1}^{n}\widetilde{e}_K(x_{i,1})\cdot e_K(y_{i,m_i})\cdot\prod_{j=1}^{m_i}\frac{P_{x_{i,j}}[X(T_K-1)=y_{i,j}]}{P_{y_{i,j}}[X(1)\notin K]}\cdot\prod_{j'=1}^{m_i-1}P_{y_{i,j'}}\bigl[X(\widetilde{H}_K)=x_{i,j'+1}\bigr].
\end{aligned}\tag{4.18}$$

The expression (4.11) follows from (4.17) by taking all $\mathcal{A}_i=\mathcal{B}_i=W_+$ in (4.17), summing over all $\tau^{\mathrm{out}}_{i,j'}$, and dividing by (4.18). Similarly, the expression (4.12) follows from (4.17) by summing (4.17) over all $\tau^{\mathrm{in}}_{i,j}$ and dividing by (4.18).

Finally, to obtain (4.9), we observe that the product of the right-hand sides of (4.11), (4.12), and (4.18) equals (4.17). The proof of Lemma 4.4 is complete. □

5. Proof of Theorem 1.1

Statement (1.2) of Theorem 1.1 follows from Corollary 3.7(2) and (3). Statement (1.3) is proved in Section 5.3. We will deduce it there from Claim 5.2 and Lemma 5.4, which we state in Section 5.1.

We begin with a general overview of the proof of (1.3). As we already know from Corollary 3.7, we can choose R and u such that $\mathcal{V}^u$ contains an infinite connected subset $\bigcup_{x'\in\mathcal{G}^\infty}\Box(x')$, where $\mathcal{G}^\infty$ is the unique infinite connected component of R-good vertices in $\mathbf{Z}$ for $\mathcal{L}^u$. The goal is to show that if a vertex of $\mathbb{Z}^d$ is in a large connected component of $\mathcal{V}^u$, then, with high probability, it must be (locally) connected to $\bigcup_{x'\in\mathcal{G}^\infty}\Box(x')$. This is realized in Lemma 5.4. The crucial observation is that by Corollary 3.7, with high probability, any long nearest-neighbor path in $\mathbb{Z}^d$ will often intersect $\bigcup_{x'\in\mathcal{G}^\infty}\mathrm{B}(x',R)$ (see (5.2) and (5.5)).

The proof of Lemma 5.4 proceeds by exploring the connected component of a vertex in $\mathcal{V}^u$, and showing that every visit to a new box of $\bigcup_{x'\in\mathcal{G}^\infty}\mathrm{B}(x',R)$ gives a fresh, uniformly positive chance for the (already explored) vacant set to merge with $\bigcup_{x'\in\mathcal{G}^\infty}\Box(x')$ (see Lemmas 5.5 and 5.10). The key observation in proving that the history of this exploration does not have a negative effect on the success probability of the next merger comes from Lemma 4.4: if we consider a box of radius R, the events which depend on the behavior of the interlacement trajectories outside this box are conditionally independent of what they do inside the box, given the collection of entrance and exit points of the excursions inside the box. As we already pointed out earlier, some care is still needed, since random interlacements do not posess the finite energy property. Our definition of good vertices (more precisely, property (1) of Definition 3.3) allows to overcome this difficulty (see the proof of Lemma 5.10). In order to get a uniform lower bound in (5.31) of Lemma 5.10, we use the fact that the number of excursions of the interlacement trajectories inside good boxes (corresponding to good vertices) is bounded (see property (2) of Definition 3.3).

We now proceed with the proof of (1.3).

$$\text{From now on we fix } R \text{ and } u_1 \text{ that satisfy (3.6), and consider } u \le u_1. \tag{5.1}$$

Since R is now fixed, we will call R-good/R-bad vertices (see Definition 3.3) simply good/bad.

5.1. *Large cluster in $\mathcal{V}^u$ is likely to be ubiquitous*

The main result of this section is Lemma 5.4. We begin with definitions and preliminary observations. Recall the definitions of the coarse grained lattice $\mathbf{Z}$ from (3.1) and the ball $\mathbf{B}(x',N)$ in $\mathbf{Z}$ from below (3.1).

For $N\ge 1$, let

$$k_N = \lfloor\sqrt{N}\rfloor \quad \text{and} \quad K_{N,k} = N + k_N\cdot k \quad \text{for } 0\le k\le k_N.$$

Now we define an event that a large hypercube $\mathbf{B}(0,2N)$ in $\mathbf{Z}$ contains a (large) connected component of good vertices in $\mathbf{Z}$ which contains separating shells in each of k_N concentric annuli $\mathbf{B}(0,K_{N,k})\setminus\mathbf{B}(0,K_{N,k-1})$, $1\le k\le k_N$.

Definition 5.1. *For $N\ge 1$, let $\mathcal{H}_N$ be the event that*

1. $\mathbf{B}(0,N)$ *is connected to* $\partial_{\mathrm{int}}\mathbf{B}(0,2N)$ *by a nearest-neighbor path of good vertices for* $\mathcal{L}^u$ *in* $\mathbf{Z}$,
2. *for all* $1\le k\le k_N$, $\mathbf{B}(0,K_{N,k})\setminus\mathbf{B}(0,K_{N,k-1})$ *contains a set* $\mathcal{S}_k\subset\mathbf{Z}$ (*which we call a shell in* $\mathbf{B}(0,K_{N,k})$ *around* $\mathbf{B}(0,K_{N,k-1})$) *such that*
   (a) $\mathcal{S}_k$ *is connected in* $\mathbf{Z}$,
   (b) *each* $x\in\mathcal{S}_k$ *is good for* $\mathcal{L}^u$, *and*
   (c) *every* $*$*-path in* $\mathbf{Z}$ *from* $\mathbf{B}(0,K_{N,k-1})$ *to* $\partial_{\mathrm{int}}\mathbf{B}(0,K_{N,k})$ *intersects* $\mathcal{S}_k$.

Claim 5.2. *It follows from Corollary 3.7 that for R and $u\le u_1$ as in (5.1), there exist constants $c=c(d)>0$ and $C=C(d)<\infty$ (possibly different from the ones in Lemma 3.6) such that*

$$\mathbb{P}[\mathcal{H}_N]\ge 1-C\mathrm{e}^{-N^c}. \tag{5.2}$$

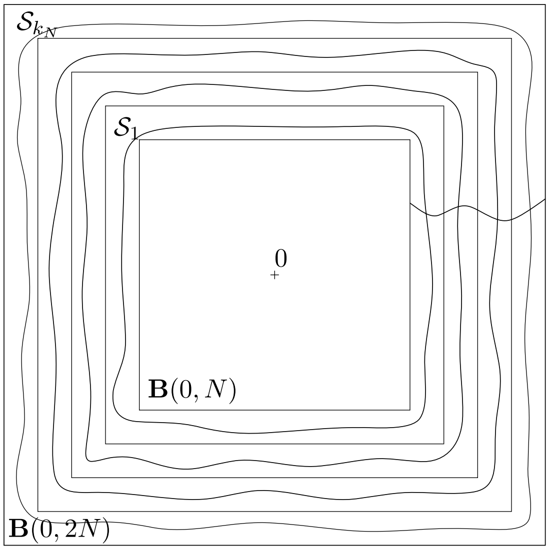

Fig. 2. The event $\mathcal{H}_N$. In each of the k_N concentric annuli $\mathbf{B}(0, K_{N,k}) \setminus \mathbf{B}(0, K_{N,k-1})$, $1 \le k \le k_N$, there exists a connected component $\mathcal{S}_k$ in $\mathbf{Z}$ of good vertices (which we call a shell) separating $\mathbf{B}(0, K_{N,k-1})$ from $\partial_{\text{int}}\mathbf{B}(0, K_{N,k})$, and all the $\mathcal{S}_k$ are (disjoint) parts of the same connected component of good vertices in $\mathbf{B}(0, 2N)$.

Note that if $\mathcal{H}_N$ occurs, then for each $1 \le k \le k_N$,

$$\mathcal{S}_k \text{ can be defined as the } \textit{unique} \text{ connected component of good vertices in } \mathbf{B}(0, K_{N,k}) \setminus \mathbf{B}(0, K_{N,k-1}) \text{ such that every } *\text{-path in } \mathbf{Z} \text{ from } \mathbf{B}(0, K_{N,k-1}) \text{ to } \partial_{\text{int}}\mathbf{B}(0, K_{N,k}) \text{ intersects } \mathcal{S}_k. \tag{5.3}$$

We will use this definition of $\mathcal{S}_k$ here. If $\mathcal{H}_N$ does not occur, we set $\mathcal{S}_k = \varnothing$ for all k. Note that by Definition 5.1 the sets $\mathcal{S}_k$ are disjoint subsets of $\mathbf{Z}$, and for each $1 \le k \le k_N$,

$$\mathcal{S}_1, \dots, \mathcal{S}_k \text{ are in the same connected component of good vertices in } \mathbf{B}(0, K_{N,k}). \tag{5.4}$$

In terms of connectivities in $\mathbb{Z}^d$, the key property of $\mathcal{S}_k$ can be stated as follows: if the event $\mathcal{H}_N$ occurs, then for each $1 \le k \le k_N$,

$$\text{every nearest-neighbor path in } \mathbb{Z}^d \text{ from } \mathrm{B}(0, (2R+1)K_{N,k-1}) \text{ to } \partial_{\text{int}}\mathrm{B}(0, (2R+1)K_{N,k}) \text{ intersects the set } \bigcup_{x' \in \mathcal{S}_k} \mathrm{B}(x', R). \tag{5.5}$$

By (5.3), (5.4), Definition 3.1 and Definition 3.3, if $\mathcal{H}_N$ occurs, then for each $1 \le k \le k_N$,

$$\text{the sets } \bigcup_{x'_1 \in \mathcal{S}_1} \square(x'_1), \dots, \bigcup_{x'_k \in \mathcal{S}_k} \square(x'_k) \text{ are in the same connected component of } \mathcal{V}^u \cap \mathrm{B}(0, (2R+1)K_{N,k} + R), \text{ which we denote by } \mathcal{C}_k. \tag{5.6}$$

If $\mathcal{H}_N$ does not occur, we define $\mathcal{C}_k = \varnothing$. By (5.6),

$$\mathcal{C}_k \subseteq \mathcal{C}_{k+1} \quad \text{for all } 1 \le k \le k_N - 1. \tag{5.7}$$

As we will see in Section 5.3, in order to prove (1.3), it suffices to show that, with high probability, $\mathcal{C}_{k_N}$ is the only connected component of $\mathcal{V}^u \cap \mathrm{B}(0, (2R+1) \cdot 2N + R)$ that intersects $\mathrm{B}(0, (2R+1) \cdot N)$ and $\partial_{\text{int}}\mathrm{B}(0, (2R+1) \cdot 2N)$. To prove the latter statement, we need a more general definition.

Definition 5.3. *For $z \in \mathrm{B}(0, (2R+1) \cdot N)$ and $1 \le k \le k_N$, let $\mathcal{A}_{z,k}$ be the event that*

1. *$\mathcal{H}_N$ occurs,*

2. *z is connected to $\partial_{\mathrm{int}}\mathrm{B}(0,(2R+1)\cdot K_{N,k})$ by a nearest-neighbor path in $\mathcal{V}^u$*,
3. *$z \notin \mathcal{C}_k$*,

and let $\mathcal{A}_{z,0} = \mathcal{H}_N$.

The main result of this section is the following lemma.

Lemma 5.4. *For R and u as in (5.1), there exists $\gamma = \gamma(d,R) > 0$ such that*

$$\mathbb{P}[\mathcal{A}_{z,k_N}] \le (1-\gamma)^{k_N} \quad \textit{for all } N \ge 1 \textit{ and } z \in \mathrm{B}\big(0,(2R+1)\cdot N\big). \tag{5.8}$$

Proof. Fix $z \in \mathrm{B}(0,(2R+1)\cdot N)$. Without loss of generality we may assume that $\mathbb{P}[\mathcal{A}_{z,k_N}] \neq 0$. By (5.7), we have the inclusion

$$\mathcal{A}_{z,k} \subseteq \mathcal{A}_{z,k-1} \quad \text{for all } 1 \le k \le k_N. \tag{5.9}$$

Using (5.9), we obtain

$$\mathbb{P}[\mathcal{A}_{z,k_N}] = \mathbb{P}[\mathcal{H}_N] \cdot \prod_{k=1}^{k_N} \mathbb{P}[\mathcal{A}_{z,k} | \mathcal{A}_{z,k-1}].$$

To complete the proof of (5.8) it suffices to show that for all $z \in \mathrm{B}(0,(2R+1)\cdot N)$, $1 \le k \le k_N$ and some $\gamma = \gamma(d,R) > 0$,

$$\mathbb{P}[\mathcal{A}_{z,k} | \mathcal{A}_{z,k-1}] \le 1-\gamma. \tag{5.10}$$

This follows from the more general Lemma 5.5 below. Before we state the lemma, we need some notation.

Define the random variable $\Sigma_{\mathcal{G},N} : \Omega \to \{0,1\}^{\mathbf{B}(0,2N)}$ which keeps track of good and bad vertices in $\mathbf{B}(0,2N)$ as

$$\Sigma_{\mathcal{G},N} = \big(\mathbb{1}_{\{x' \text{ is good for } \mathcal{L}^u\}} \colon x' \in \mathbf{B}(0,2N)\big). \tag{5.11}$$

Note that

$$\mathcal{H}_N \in \sigma(\Sigma_{\mathcal{G},N}) \quad \text{for all } N, \tag{5.12}$$

and, in particular,

$$\text{for all } 1 \le k \le k_N, \text{ the set } \mathcal{S}_k \text{ is measurable with respect to } \sigma(\Sigma_{\mathcal{G},N}). \tag{5.13}$$

For $1 \le k \le k_N$, if $\mathcal{H}_N$ occurs,

$$\text{let } \mathcal{D}_k \text{ be the (unique) connected component of } \mathbb{Z}^d \setminus \textstyle\bigcup_{x' \in \mathcal{S}_k} \mathrm{B}(x',R) \text{ which contains the origin,} \tag{5.14}$$

and let $\mathcal{D}_k = \mathrm{B}(0,(2R+1)\cdot K_{N,k} - R)$ otherwise. By (5.13),

$$\mathcal{D}_k \text{ is measurable with respect to } \sigma(\Sigma_{\mathcal{G},N}), \text{ for all } 1 \le k \le k_N. \tag{5.15}$$

By (5.5),

$$\mathrm{B}\big(0,(2R+1)\cdot K_{N,k-1} + R\big) \subseteq \mathcal{D}_k \subseteq \mathrm{B}\big(0,(2R+1)\cdot K_{N,k} - R\big). \tag{5.16}$$

Define the random variables $\Sigma_k : \Omega \to \{0,1\}^{\mathrm{B}(0,(2R+1)\cdot K_{N,k}-R)}$ which keep track of the interlacement configuration inside $\mathcal{D}_k$ as

$$\Sigma_k = \big(\mathbb{1}_{\{x \in \mathcal{I}^u \cap \mathcal{D}_k\}} \colon x \in \mathrm{B}\big(0,(2R+1)\cdot K_{N,k} - R\big)\big), \quad 1 \le k \le k_N. \tag{5.17}$$

The following lemma implies (5.10), as we show in (5.20).

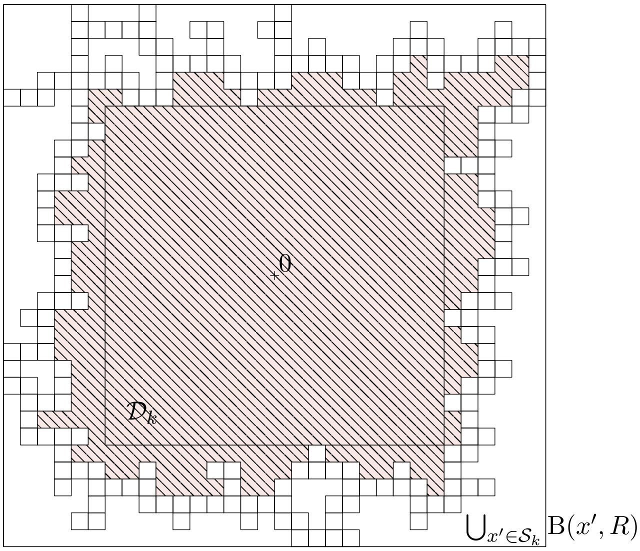

Fig. 3. The inner and outer boxes are $\mathrm{B}(0, (2R+1) \cdot K_{N,k-1} + R)$ and $\mathrm{B}(0, (2R+1) \cdot K_{N,k} + R)$, respectively. The set $\mathcal{D}_k \subseteq \mathbb{Z}^d$ is the unique connected component of $\mathbb{Z}^d \setminus \bigcup_{x' \in \mathcal{S}_k} \mathrm{B}(x', R)$, which contains the origin.

Lemma 5.5. *There exists $\gamma = \gamma(d, R) > 0$ such that for all $z \in \mathrm{B}(0, (2R+1) \cdot N)$ and $1 \le k \le k_N$,*

$$\mathbb{1}_{\mathcal{H}_N} \cdot \mathbb{P}[\mathcal{A}_{z,k} | \Sigma_{\mathcal{G},N}, \Sigma_k] \le 1 - \gamma, \quad \mathbb{P}\text{-a.s.} \tag{5.18}$$

We postpone the proof of Lemma 5.5 until Section 5.2, and now complete the proof of Lemma 5.4 by showing how Lemma 5.5 implies (5.10).

By (5.6), Definition 5.3, (5.12), (5.15), and (5.16), we have

$$\mathcal{A}_{z,k-1} \in \sigma(\Sigma_{\mathcal{G},N}, \Sigma_k) \quad \text{for each } 1 \le k \le k_N. \tag{5.19}$$

Therefore, for each $1 \le k \le k_N$,

$$\mathbb{P}[\mathcal{A}_{z,k}] \overset{(5.9),(5.19)}{=} \mathbb{E}\big[\mathbb{1}_{\mathcal{A}_{z,k-1}} \cdot \mathbb{P}[\mathcal{A}_{z,k} | \Sigma_{\mathcal{G},N}, \Sigma_k]\big] \overset{(5.18)}{\le} (1-\gamma) \cdot \mathbb{P}[\mathcal{A}_{z,k-1}]. \tag{5.20}$$

This implies (5.10) and completes the proof of Lemma 5.4 subject to Lemma 5.5, which will be proved in Section 5.2. □

5.2. *Proof of Lemma 5.5*

In this section we prove Lemma 5.5. Recall the definitions of the configuration $\Sigma_{\mathcal{G},N}$ of good and bad vertices of $\mathbf{B}(0, 2N)$ (see (5.11)), the event $\mathcal{H}_N$ (see Definition 5.1) guaranteeing the presence of $k_N = \lfloor \sqrt{N} \rfloor$ connected shells $\mathcal{S}_k$, $1 \le k \le k_N$ of good boxes (see (5.3)), the domain $\mathcal{D}_k \subseteq \mathbb{Z}^d$ surrounded by $\bigcup_{x' \in \mathcal{S}_k} \mathrm{B}(x', R)$ (see (5.14)), and the configuration Σ_k of occupied/vacant vertices of $\mathcal{D}_k$ (see (5.17)).

The occurrence of event $\mathcal{A}_{z,k}$ guarantees the existence of a vacant path in $\mathcal{D}_k$ from z to $\partial_{\mathrm{int}} \mathcal{D}_k$ with certain restrictions on the location of the end point of this path on $\partial_{\mathrm{int}} \mathcal{D}_k$. These properties are reflected in the following definition.

Definition 5.6. *For $z \in \mathrm{B}(0, (2R+1) \cdot N)$, and $1 \le k \le k_N$, let $\widetilde{\mathcal{A}}_{z,k}$ be the event that (a) $\mathcal{H}_N$ occurs, and (b) there exists a nearest-neighbor path π_k in $\mathcal{D}_k$ from z to a vertex $x_k \in \partial_{\mathrm{int}} \mathcal{D}_k \setminus \partial_{\mathrm{ext}} \bigcup_{x' \in \mathcal{S}_k} \square(x')$ such that every vertex x along this path (including x_k) satisfies $\Sigma_k(x) = 0$ (i.e. $x \in \mathcal{V}^u$, cf. (5.17)). If there are several such paths, we pick one in a predetermined, non-random fashion.*

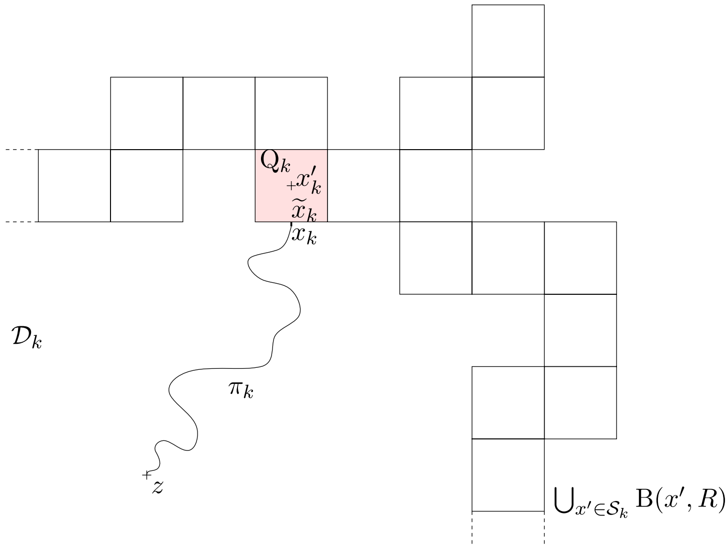

Fig. 4. If the event $\widetilde{\mathcal{A}}_{z,k}$ occurs, there exists a vacant path π_k from z to $x_k \in \partial_{\mathrm{int}}\mathcal{D}_k$ in $\mathcal{D}_k$ such that $x_k \notin \partial_{\mathrm{ext}} \bigcup_{x'\in\mathcal{S}_k} \Box(x')$. There exists a unique $x'_k \in \mathcal{S}_k$ such that $x_k \in \partial_{\mathrm{ext}}\mathrm{Q}(x'_k)$ (and $x_k \notin \partial_{\mathrm{ext}}\Box(x'_k)$). The cube $\mathrm{Q}(x'_k)$ is denoted by Q_k. The unique neighbor of x_k in $\partial_{\mathrm{int}}\mathrm{Q}_k$ is denoted by $\widetilde{x}_k$.

The properties of $\widetilde{\mathcal{A}}_{z,k}$ that are useful to us are the following:

$$\widetilde{\mathcal{A}}_{z,k} \text{ (and hence } \pi_k \text{ and } x_k) \text{ is measurable with respect to } \sigma(\Sigma_{\mathcal{G},N}, \Sigma_k), \tag{5.21}$$

and, by Definition 5.3,

$$\mathcal{A}_{z,k} \subseteq \widetilde{\mathcal{A}}_{z,k}. \tag{5.22}$$

Indeed, (5.21) is immediate from Definition 5.6. To see that (5.22) holds, note that if $\mathcal{A}_{z,k}$ occurs, then by (5.16) z is connected to $\partial_{\mathrm{int}}\mathcal{D}_k$ by a nearest-neighbor path of vertices x with $\Sigma_k(x) = 0$. However, by (5.6), $\bigcup_{x'\in\mathcal{S}_k} \Box(x') \subseteq \mathcal{C}_k$ and, by Definition 5.3, $z \notin \mathcal{C}_k$, therefore any such path must avoid $\partial_{\mathrm{ext}} \bigcup_{x'\in\mathcal{S}_k} \Box(x')$. This implies (5.22).

By the definition of $\widetilde{\mathcal{A}}_{z,k}$, $x_k \in \partial_{\mathrm{int}}\mathcal{D}_k \setminus \partial_{\mathrm{ext}} \bigcup_{x'\in\mathcal{S}_k} \Box(x')$. Therefore, there exists a unique

$$x'_k \in \mathcal{S}_k \text{ such that } x_k \text{ belongs to the exterior boundary of } \mathrm{Q}_k = \mathrm{Q}(x'_k) \text{ (see (3.2)) and is not adjacent to any of the vertices in } \Box(x'_k). \tag{5.23}$$

Also there exists a (unique) $\widetilde{x}_k \in \mathrm{Q}(x'_k) \setminus \Box(x'_k)$ such that $x_k \sim \widetilde{x}_k$. Moreover, since $\widetilde{x}_k \notin \Box(x'_k)$,

$$x_k \text{ is the only nearest-neighbor of } \widetilde{x}_k \text{ which is outside } \mathrm{Q}(x'_k). \tag{5.24}$$

The key step in the proof of Lemma 5.5 is Lemma 5.10, in which we show that given the configurations $\Sigma_{\mathcal{G},N}$ of good and bad vertices of $\mathbf{B}(0,2N)$ and Σ_k of occupied/vacant vertices of $\mathcal{D}_k$ satisfying the event $\widetilde{\mathcal{A}}_{z,k}$, and given the σ-algebra generated by the interlacement excursions outside $\mathrm{Q}(x'_k)$, with uniformly positive probability there is a realization of the interlacement excursions inside $\mathrm{Q}(x'_k)$ such that x'_k is good, and x_k is connected to $\Box(x'_k)$ in $\mathcal{V}^u \cap (\mathrm{Q}(x'_k) \cup \{x_k\})$. Once this is done, Lemma 5.5 immediately follows, as we show after the statement of Lemma 5.10. To state Lemma 5.10, we need some notation.

Definition 5.7. *Let $K \subset\subset \mathbb{Z}^d$. In the notation of Section 4.2, let X_K^{in} be the (random) vector*

$$X_K^{\mathrm{in}} = \big(X_{i,j}^{\mathrm{in}} \colon 1 \le i \le N_{K,u}, 1 \le j \le M_i\big)$$

of the excursions inside K of the interlacement trajectories from the support of $\omega_{K,u}$ (numbered in order of increase of their labels), and X_K^{AB} the vector

$$X_K^{AB} = \big(\big(X_i(A_{i,j}), X_i(B_{i,j})\big) \colon 1 \le i \le N_{K,u}, 1 \le j \le M_i\big)$$

of start and end points of all these excursions. Note that

$$X_K^{\mathrm{in}} \text{ is measurable with respect to } \mathcal{F}_{K,u}^{\mathrm{in}}, \text{ and } X_K^{AB} \text{ with respect to } \mathcal{F}_{K,u}^{AB}, \tag{5.25}$$

with $\mathcal{F}_{K,u}^{\mathrm{in}}$ and $\mathcal{F}_{K,u}^{AB}$ defined in Section 4.3.

Definition 5.8. *For $x' \in \mathbf{Z}$, let $\mathcal{T}_{x'}^{\mathrm{in}} = \mathcal{T}_{x'}^{\mathrm{in}}(X_{\mathrm{Q}(x')}^{AB})$ be the set of all vectors*

$$\tau^{\mathrm{in}} = \big(\tau_{i,j}^{\mathrm{in}} \colon 1 \le i \le N_{\mathrm{Q}(x'),u}, 1 \le j \le M_i\big)$$

of finite nearest-neighbor trajectories from $X_i(A_{i,j})$ to $X_i(B_{i,j})$ inside $\mathrm{Q}(x')$ such that

(a) *all the $\tau_{i,j}^{\mathrm{in}}$ avoid $\square(x')$, and*
(b) *the total number of visits to $\partial_{\mathrm{int}}\mathrm{Q}(x')$ of all the $\tau_{i,j}^{\mathrm{in}}$ is at most R^{d-1}.*

Note that

$$\mathcal{T}_{x'}^{\mathrm{in}} \text{ is measurable with respect to } \mathcal{F}_{\mathrm{Q}(x'),u}^{AB} \tag{5.26}$$

(see Section 4.3), and by Definition 3.3, for any $x' \in \mathbf{Z}$,

$$\big\{X_{\mathrm{Q}(x')}^{\mathrm{in}} \in \mathcal{T}_{x'}^{\mathrm{in}}\big\} = \big\{x' \text{ is good for } \mathcal{L}^u\big\}. \tag{5.27}$$

Claim 5.9. *Recall the definition of x_k' and Q_k from (5.23).*

(1) *If $\widetilde{\mathcal{A}}_{z,k}$ occurs, then for all $1 \le i \le N_{\mathrm{Q}_k,u}$, $1 \le j \le M_i$, and for any element $(X_i(A_{i,j}), X_i(B_{i,j}))$ of $X_{\mathrm{Q}_k}^{AB}$, we have*

$$X_i(A_{i,j}), X_i(B_{i,j}) \in \partial_{\mathrm{int}}\mathrm{Q}_k \setminus \big(\square\big(x_k'\big) \cup \{\widetilde{x}_k\}\big). \tag{5.28}$$

Indeed, if $\widetilde{\mathcal{A}}_{z,k}$ occurs, then x_k' is good for $\mathcal{L}^u$ and, by Definition 5.6, $x_k \in \mathcal{V}^u$. Together with (5.24), this implies (5.28).

(2) *If $\widetilde{\mathcal{A}}_{z,k}$ occurs, then*

$$X_{\mathrm{Q}_k}^{\mathrm{in}} \in \mathcal{T}_{x_k'}^{\mathrm{in}}. \tag{5.29}$$

Indeed, (5.29) follows from (5.27) and the fact that the vertex x_k' is good for $\mathcal{L}^u$ when $\widetilde{\mathcal{A}}_{z,k}$ occurs.

Lemma 5.5 follows from the next lemma. Recall Definition 5.3 of the event $\mathcal{A}_{z,k}$, the definition of x_k' and Q_k from (5.23), and the notion of the σ-algebra $\mathcal{F}_{K,u}^{\mathrm{out}}$ generated by the interlacement excursions outside of $K \subset\subset \mathbb{Z}^d$ and $\omega - \omega_{K,u}$ from Section 4.3.

Lemma 5.10. *There exists $\gamma = \gamma(d, R) > 0$ such that for any $z \in \mathrm{B}(0, (2R+1)\cdot N)$ and $1 \le k \le k_N$, $\mathbb{P}$-almost surely, for each realization of $\Sigma_{\mathcal{G},N}$, Σ_k, and $X_{\mathrm{Q}_k}^{AB}$ satisfying $\widetilde{\mathcal{A}}_{z,k}$, there exists*

$$\rho^{\mathrm{in}} = \rho^{\mathrm{in}}\big(\Sigma_{\mathcal{G},N}, \Sigma_k, X_{\mathrm{Q}_k}^{AB}\big) \in \mathcal{T}_{x_k'}^{\mathrm{in}} \tag{5.30}$$

such that for all $x' \in \mathbf{Z}$,

$$\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k} \cap \{x_k' = x'\}} \cdot \mathbb{P}\big[X_{\mathrm{Q}_k}^{\mathrm{in}} = \rho^{\mathrm{in}} | \sigma\big(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}_{\mathrm{Q}(x'),u}^{\mathrm{out}}\big)\big] \ge \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k} \cap \{x_k' = x'\}} \cdot \gamma \tag{5.31}$$

and

$$\widetilde{\mathcal{A}}_{z,k} \cap \{X^{\mathrm{in}}_{\mathrm{Q}_k} = \rho^{\mathrm{in}}\} \subseteq \widetilde{\mathcal{A}}_{z,k} \setminus \mathcal{A}_{z,k}. \tag{5.32}$$

Before we prove Lemma 5.10, we use it to finish the proof of Lemma 5.5. We have

$$\begin{aligned}\mathbb{1}_{\mathcal{H}_N} \cdot \mathbb{P}[\mathcal{A}_{z,k} | \Sigma_{\mathcal{G},N}, \Sigma_k] &\overset{(5.21),(5.22)}{=} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}} \cdot \mathbb{P}[\mathcal{A}_{z,k} | \Sigma_{\mathcal{G},N}, \Sigma_k] \\ &\overset{(5.32)}{\le} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}} \cdot \mathbb{P}\big[X^{\mathrm{in}}_{\mathrm{Q}_k} \neq \rho^{\mathrm{in}} | \Sigma_{\mathcal{G},N}, \Sigma_k\big] \\ &\overset{(5.31)}{\le} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}} \cdot (1-\gamma).\end{aligned}$$

This finishes the proof of Lemma 5.5, subject to Lemma 5.10.

It remains to prove Lemma 5.10. We begin with some preliminary results. Recall the notion of the σ-algebras $\mathcal{F}^{\mathrm{in}}_{K,u}$, $\mathcal{F}^{\mathrm{out}}_{K,u}$, and $\mathcal{F}^{AB}_{K,u}$ from Section 4.3.

Lemma 5.11. *For any $x' \in \mathbf{Z}$ and $\mathcal{E}^{\mathrm{in}} \in \mathcal{F}^{\mathrm{in}}_{\mathrm{Q}(x'),u}$, we have, $\mathbb{P}$-almost surely, that*

$$\begin{aligned}&\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k} \cap \{x'_k = x'\}} \cdot \mathbb{P}\big[\mathcal{E}^{\mathrm{in}} | \sigma\big(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\mathrm{out}}_{\mathrm{Q}(x'),u}\big)\big] \\ &\quad = \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k} \cap \{x'_k = x'\}} \cdot \frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}} \cap \{x' \textit{ is good for } \mathcal{L}^u\} | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}]}{\mathbb{P}[\{x' \textit{ is good for } \mathcal{L}^u\} | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}]}.\end{aligned} \tag{5.33}$$

Remark 5.12. *Note that $\widetilde{\mathcal{A}}_{z,k} \cap \{x'_k = x'\} \subseteq \{x' \text{ is good for } \mathcal{L}^u\}$. Therefore,*

$$\mathbb{P}\big[\widetilde{\mathcal{A}}_{z,k} \cap \{x'_k = x'\} \cap \{\omega\colon \mathbb{P}\big[\{x' \textit{ is good for } \mathcal{L}^u\} | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}\big] = 0\}\big] = 0.$$

Proof of Lemma 5.11. Let $z \in \mathrm{B}(0, (2R+1) \cdot N)$. Let

$$\begin{aligned}&\sigma_{\mathcal{G},N} \in \{0,1\}^{\mathbf{B}(0,2N)}, \qquad \sigma_k \in \{0,1\}^{\mathrm{B}(0,(2R+1)\cdot K_{N,k}-R)}, \\ &\partial_k \subseteq \mathrm{B}\big(0, (2R+1) \cdot K_{N,k} - R\big) \quad \text{and} \quad x' \in \mathbf{B}(0,2N)\end{aligned}$$

be such that

$$\{\Sigma_{\mathcal{G},N} = \sigma_{\mathcal{G},N}, \Sigma_k = \sigma_k\} \subseteq \widetilde{\mathcal{A}}_{z,k} \cap \{x'_k = x'\} \cap \{\mathcal{D}_k = \partial_k\}, \tag{5.34}$$

where $\mathcal{D}_k$ is defined in (5.14). Let

$$K = \mathrm{Q}(x').$$

In order to prove (5.33), it suffices to show that for any events $\mathcal{E}^{\mathrm{in}} \in \mathcal{F}^{\mathrm{in}}_{K,u}$ and $\mathcal{E}^{\mathrm{out}} \in \mathcal{F}^{\mathrm{out}}_{K,u}$, we have

$$\begin{aligned}&\mathbb{P}\big[\mathcal{E}^{\mathrm{in}} \cap \mathcal{E}^{\mathrm{out}} \cap \{\Sigma_{\mathcal{G},N} = \sigma_{\mathcal{G},N}, \Sigma_k = \sigma_k\}\big] \\ &\quad = \mathbb{E}\left[\frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}} \cap \{x' \text{ is good for } \mathcal{L}^u\} | \mathcal{F}^{AB}_{K,u}]}{\mathbb{P}[\{x' \text{ is good for } \mathcal{L}^u\} | \mathcal{F}^{AB}_{K,u}]}; \mathcal{E}^{\mathrm{out}} \cap \{\Sigma_{\mathcal{G},N} = \sigma_{\mathcal{G},N}, \Sigma_k = \sigma_k\}\right].\end{aligned} \tag{5.35}$$

Let

$$\Sigma^{\mathrm{out}}_{\mathcal{G},N} = \big(\mathbb{1}_{\{\widetilde{x}' \text{ is good for } \mathcal{L}^u\}}\colon \widetilde{x}' \in \mathbf{B}(0,2N) \setminus \{x'\}\big)$$

be the restriction of $\Sigma_{\mathcal{G},N}$ to $\mathbf{B}(0,2N)\setminus\{x'\}$, and let $\sigma^{\mathrm{out}}_{\mathcal{G},N}\in\{0,1\}^{\mathbf{B}(0,2N)\setminus\{x'\}}$ be the restriction of $\sigma_{\mathcal{G},N}$ to $\mathbf{B}(0,2N)\setminus\{x'\}$. Consider the events

$$\widetilde{\mathcal{E}}^{\mathrm{in}}=\{x' \text{ is good for } \mathcal{L}^u\},$$
$$\widetilde{\mathcal{E}}^{\mathrm{out}}=\{\Sigma^{\mathrm{out}}_{\mathcal{G},N}=\sigma^{\mathrm{out}}_{\mathcal{G},N},\ (\mathbb{1}_{\{x\in\mathcal{I}^u\cap\partial_k\}}\colon x\in \mathrm{B}(0,(2R+1)\cdot K_{N,k}-R))=\sigma_k\}.$$

Note that by Claim 4.2(3) and Definition 3.3, we have

$$\widetilde{\mathcal{E}}^{\mathrm{in}}\in\mathcal{F}^{\mathrm{in}}_{K,u},\tag{5.36}$$

by Claim 4.2(3) and the fact that $\partial_k\cap K=\varnothing$,

$$\widetilde{\mathcal{E}}^{\mathrm{out}}\in\mathcal{F}^{\mathrm{out}}_{K,u},\tag{5.37}$$

and by (5.34),

$$\{\Sigma_{\mathcal{G},N}=\sigma_{\mathcal{G},N},\ \Sigma_k=\sigma_k\}=\widetilde{\mathcal{E}}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{out}}.\tag{5.38}$$

Using these observations and Lemma 4.4(a), we rewrite the left-hand side of (5.35) as

$$\begin{aligned}
&\mathbb{P}\big[\mathcal{E}^{\mathrm{in}}\cap\mathcal{E}^{\mathrm{out}}\cap\{\Sigma_{\mathcal{G},N}=\sigma_{\mathcal{G},N},\ \Sigma_k=\sigma_k\}\big]\\
&\quad\stackrel{(5.38)}{=}\mathbb{P}\big[(\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}})\cap(\mathcal{E}^{\mathrm{out}}\cap\widetilde{\mathcal{E}}^{\mathrm{out}})\big]\\
&\quad\stackrel{\text{Lemma 4.4(a)},(5.36),(5.37)}{=}\mathbb{E}\big[\mathbb{P}\big[\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}\big]\cdot\mathbb{P}\big[\mathcal{E}^{\mathrm{out}}\cap\widetilde{\mathcal{E}}^{\mathrm{out}}|\mathcal{F}^{AB}_{K,u}\big]\big]\\
&\quad=\mathbb{E}\bigg[\frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}{\mathbb{P}[\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}\cdot\mathbb{P}\big[\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}\big]\cdot\mathbb{P}\big[\mathcal{E}^{\mathrm{out}}\cap\widetilde{\mathcal{E}}^{\mathrm{out}}|\mathcal{F}^{AB}_{K,u}\big]\bigg]\\
&\quad\stackrel{\text{Lemma 4.4(a)},(5.36),(5.37)}{=}\mathbb{E}\bigg[\frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}{\mathbb{P}[\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}\cdot\mathbb{P}\big[\widetilde{\mathcal{E}}^{\mathrm{in}}\cap\mathcal{E}^{\mathrm{out}}\cap\widetilde{\mathcal{E}}^{\mathrm{out}}|\mathcal{F}^{AB}_{K,u}\big]\bigg]\\
&\quad\stackrel{(5.38)}{=}\mathbb{E}\bigg[\frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}{\mathbb{P}[\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}\cdot\mathbb{P}\big[\mathcal{E}^{\mathrm{out}}\cap\{\Sigma_{\mathcal{G},N}=\sigma_{\mathcal{G},N},\ \Sigma_k=\sigma_k\}|\mathcal{F}^{AB}_{K,u}\big]\bigg]\\
&\quad=\mathbb{E}\bigg[\frac{\mathbb{P}[\mathcal{E}^{\mathrm{in}}\cap\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]}{\mathbb{P}[\widetilde{\mathcal{E}}^{\mathrm{in}}|\mathcal{F}^{AB}_{K,u}]};\mathcal{E}^{\mathrm{out}}\cap\{\Sigma_{\mathcal{G},N}=\sigma_{\mathcal{G},N},\ \Sigma_k=\sigma_k\}\bigg].
\end{aligned}$$

This is precisely (5.35). The proof of Lemma 5.11 is complete. □

Lemma 5.13. *For $z\in\mathrm{B}(0,(2R+1)\cdot N)$, $x'\in\mathbf{Z}$, non-negative integers n and $(m_i\colon 1\le i\le n)$, vector $\tau^{\mathrm{in}}=(\tau^{\mathrm{in}}_{i,j}\colon 1\le i\le n, 1\le j\le m_i)$ of finite nearest-neighbor trajectories $\tau^{\mathrm{in}}_{i,j}$ in $\mathrm{Q}(x')$ from $x_{i,j}\in\partial_{\mathrm{int}}\mathrm{Q}(x')$ to $y_{i,j}\in\partial_{\mathrm{int}}\mathrm{Q}(x')$, and $1\le k\le k_N$, we have $\mathbb{P}$-almost surely, that*

$$\begin{aligned}
&\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}}\cdot\mathbb{P}\big[X^{\mathrm{in}}_{\mathrm{Q}_k}=\tau^{\mathrm{in}}|\sigma(\Sigma_{\mathcal{G},N},\Sigma_k,\mathcal{F}^{\mathrm{out}}_{\mathrm{Q}(x'),u})\big]\\
&\quad\ge\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}\cap\{\tau^{\mathrm{in}}\in\mathcal{T}^{\mathrm{in}}_{x'}\}}\cdot\prod_{i=1}^{n}\prod_{j=1}^{m_i}(1/2d)^{|\tau^{\mathrm{in}}_{i,j}|}.
\end{aligned}\tag{5.39}$$

Remark 5.14. *Note that by (5.29), we have*

$$\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}\cap\{X^{\mathrm{in}}_{\mathrm{Q}_k}=\tau^{\mathrm{in}}\}\subseteq\{\tau^{\mathrm{in}}\in\mathcal{T}^{\mathrm{in}}_{x'}\},$$

and by (5.26) *and Claim* 4.2(1),

$$\{\tau^{\text{in}} \in \mathcal{T}^{\text{in}}_{x'}\} \in \mathcal{F}^{AB}_{\mathrm{Q}(x'),u} \subset \mathcal{F}^{\text{out}}_{\mathrm{Q}(x'),u}.$$

In particular, the right-hand side of (5.39) *is measurable with respect to* $\sigma(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{AB}_{\mathrm{Q}(x'),u})$.

Proof of Lemma 5.13. By (5.25), $\{X^{\text{in}}_{\mathrm{Q}(x')} = \tau^{\text{in}}\} \in \mathcal{F}^{\text{in}}_{\mathrm{Q}(x'),u}$. Using Lemma 5.11, we obtain

$$\begin{aligned}
&\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}} \cdot \mathbb{P}\big[X^{\text{in}}_{\mathrm{Q}_k} = \tau^{\text{in}} | \sigma\big(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\text{out}}_{\mathrm{Q}(x'),u}\big)\big] \\
&\quad= \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}} \cdot \mathbb{P}\big[X^{\text{in}}_{\mathrm{Q}(x')} = \tau^{\text{in}} | \sigma\big(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\text{out}}_{\mathrm{Q}(x'),u}\big)\big] \\
&\quad\overset{(5.33)}{=} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}} \cdot \frac{\mathbb{P}[X^{\text{in}}_{\mathrm{Q}(x')} = \tau^{\text{in}}, x' \text{ is good for } \mathcal{L}^u | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}]}{\mathbb{P}[x' \text{ is good for } \mathcal{L}^u | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}]} \\
&\quad\overset{(5.27)}{\geq} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x'_k=x'\}\cap\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \mathbb{P}\big[X^{\text{in}}_{\mathrm{Q}(x')} = \tau^{\text{in}} | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}\big].
\end{aligned} \tag{5.40}$$

Using (4.11), we get

$$\begin{aligned}
&\mathbb{1}_{\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \mathbb{P}\big[X^{\text{in}}_{\mathrm{Q}(x')} = \tau^{\text{in}} | \mathcal{F}^{AB}_{\mathrm{Q}(x'),u}\big] \\
&\quad= \mathbb{1}_{\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m_i} \frac{P_{x_{i,j}}[(X(t)\colon 0 \leq t \leq T_{\mathrm{Q}(x')} - 1) = \tau^{\text{in}}_{i,j}]}{P_{x_{i,j}}[X(T_{\mathrm{Q}(x')} - 1) = y_{i,j}]} \\
&\quad\geq \mathbb{1}_{\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m_i} P_{x_{i,j}}\big[\big(X(t)\colon 0 \leq t \leq T_{\mathrm{Q}(x')} - 1\big) = \tau^{\text{in}}_{i,j}\big] \\
&\quad= \mathbb{1}_{\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m_i} P_{x_{i,j}}\big[\big(X(t)\colon 0 \leq t \leq \big|\tau^{\text{in}}_{i,j}\big| - 1\big) = \tau^{\text{in}}_{i,j}, X\big(\big|\tau^{\text{in}}_{i,j}\big|\big) \notin \mathrm{Q}\big(x'\big)\big] \\
&\quad\geq \mathbb{1}_{\{\tau^{\text{in}}\in\mathcal{T}^{\text{in}}_{x'}\}} \cdot \prod_{i=1}^{n}\prod_{j=1}^{m_i} (1/2d)^{|\tau^{\text{in}}_{i,j}|}.
\end{aligned}$$

Together with (5.40), this implies (5.39) and finishes the proof of Lemma 5.13. □

Proof of Lemma 5.10. Fix $z \in \mathrm{B}(0, (2R+1)\cdot N)$, $1 \leq k \leq k_N$, and a realization of $\Sigma_{\mathcal{G},N}$, Σ_k, and $X^{AB}_{\mathrm{Q}_k}$ satisfying $\widetilde{\mathcal{A}}_{z,k}$. Our aim is to construct $\rho^{\text{in}} = \rho^{\text{in}}(\Sigma_{\mathcal{G},N}, \Sigma_k, X^{AB}_{\mathrm{Q}_k})$ satisfying (5.30), (5.31), and (5.32).

We begin by defining a "tunnel" from $\widetilde{x}_k$ to $\square(x'_k)$ inside Q_k, which we will later force to be vacant. Recall that $\widetilde{x}_k \in \partial_{\text{int}}\mathrm{Q}_k \setminus \square(x'_k)$. By Definition 3.1, precisely one of the coordinates of the vector $\widetilde{x}_k - x'_k$ is $-R$ or R, and the values of all the remaining coordinates are between $-R+3$ and $R-3$. Let i be this unique coordinate, and let j be the first among the remaining $(d-1)$ coordinates which is not i. For $1 \leq s \leq d$, let e_s be the sth unit vector. We define the subset T_k of Q_k to be

$$\{\widetilde{x}_k, \widetilde{x}_k + e_i, \widetilde{x}_k + 2e_i\} \cup \big(\{\widetilde{x}_k + 2e_i + te_j\colon t \geq 0\} \cap \mathrm{Q}_k\big)$$

if the value of the ith coordinate of $\widetilde{x}_k - x'_k$ is $-R$, or

$$\{\widetilde{x}_k, \widetilde{x}_k - e_i, \widetilde{x}_k - 2e_i\} \cup \big(\{\widetilde{x}_k - 2e_i + te_j\colon t \geq 0\} \cap \mathrm{Q}_k\big)$$

if the value of the ith coordinate of $\widetilde{x}_k - x'_k$ is R. Note that for $R \geq 4$,

(1) $T_k \cap \square(x_k') \neq \varnothing$,
(2) $\mathrm{Q}_k \setminus (\partial_{\mathrm{int}}\mathrm{Q}_k \cup \square(x_k') \cup T_k)$ is a connected subset of Q_k, and
(3) every $x \in \partial_{\mathrm{int}}\mathrm{Q}_k \setminus (\square(x_k') \cup \{\widetilde{x}_k\})$ has a neighbor in $\mathrm{Q}_k \setminus (\partial_{\mathrm{int}}\mathrm{Q}_k \cup \square(x_k') \cup T_k)$.

In particular, (2) and (3) imply that any two points $a, b \in \partial_{\mathrm{int}}\mathrm{Q}_k \setminus (\square(x_k') \cup \{\widetilde{x}_k\})$ are connected by a self-avoiding path in $\{a, b\} \cup (\mathrm{Q}_k \setminus (\partial_{\mathrm{int}}\mathrm{Q}_k \cup \square(x_k') \cup T_k))$.

Taking into account (5.28), the above mentioned properties of T_k imply that for each element $(X_i(A_{i,j}), X_i(B_{i,j}))$ of $X^{AB}_{\mathrm{Q}_k}$, there exist self-avoiding paths $\rho^{\mathrm{in}}_{i,j}$ which connect $X_i(A_{i,j})$ to $X_i(B_{i,j})$ and are entirely contained in $\mathrm{Q}_k \setminus (\partial_{\mathrm{int}}\mathrm{Q}_k \cup \square(x_k') \cup T_k)$ except for their start and end points, $X_i(A_{i,j})$ and $X_i(B_{i,j})$, which are in $\partial_{\mathrm{int}}\mathrm{Q}_k \setminus (\square(x_k') \cup \{\widetilde{x}_k\})$. (Note that if $X_i(A_{i,j}) = X_i(B_{i,j})$, then $\rho^{\mathrm{in}}_{i,j} = \{X_i(A_{i,j})\}$ is the unique self-avoiding path from $X_i(A_{i,j})$ to $X_i(B_{i,j})$.) We choose one of such collections of self-avoiding paths $\rho^{\mathrm{in}} = \rho^{\mathrm{in}}(\Sigma_{\mathcal{G},N}, \Sigma_k, X^{AB}_{\mathrm{Q}_k})$ in a predetermined, non-random way.

We will now show that ρ^{in} satisfies the requirements of Lemma 5.10. First we show (5.30). Recall Definition 5.8 of $\mathcal{T}^{\mathrm{in}}_{x'}$. By construction, the total number of visits of all the $\rho^{\mathrm{in}}_{i,j}$ to $\partial_{\mathrm{int}}\mathrm{Q}_k$ is the smallest one among all the possible collections of paths $\tau^{\mathrm{in}} = (\tau^{\mathrm{in}}_{i,j}\colon 1 \le i \le N_{\mathrm{Q}_k,u}, 1 \le j \le M_i)$ inside Q_k from $X_i(A_{i,j})$ to $X_i(B_{i,j})$. In particular, it is almost surely smaller or equal to the total number of visits to $\partial_{\mathrm{int}}\mathrm{Q}_k$ by the trajectories in $X^{\mathrm{in}}_{\mathrm{Q}_k}$, which is at most R^{d-1} by (5.29). Thus, ρ^{in} satisfies (5.30).

Now we show that ρ^{in} satisfies (5.32). If $X^{\mathrm{in}}_{\mathrm{Q}_k} = \rho^{\mathrm{in}}$, then $T_k \subset \mathcal{V}^u$. In particular, since $\widetilde{x}_k$ is connected to $\square(x_k')$ by T_k, we obtain that $\widetilde{x}_k$ is connected to $\square(x_k')$ in $\mathcal{V}^u \cap \mathrm{Q}_k$. Recall that $\widetilde{x}_k \sim x_k$ and, by Definition 5.6, x_k is connected to z in $\mathcal{V}^u \cap \mathcal{D}_k$. Therefore, z is connected to $\square(x_k') \subset \mathcal{C}_k$ (recall (5.6) and Definition 5.3) in $\mathcal{V}^u \cap (\mathcal{D}_k \cup \mathrm{Q}_k)$, and the event $\mathcal{A}_{z,k}$ does not occur. In other words, ρ^{in} satisfies (5.32).

It remains to show that ρ^{in} satisfies (5.31). Remember that the total number of visits of all the $\rho^{\mathrm{in}}_{i,j}$ to $\partial_{\mathrm{int}}\mathrm{Q}_k$ is at most R^{d-1}. In particular, the total number of trajectories $\rho^{\mathrm{in}}_{i,j}$ in ρ^{in} is at most R^{d-1}, namely

$$\sum_{i=1}^{N_{\mathrm{Q}_k,u}} M_i \le R^{d-1}. \tag{5.41}$$

Since each $\rho^{\mathrm{in}}_{i,j}$ is a self-avoiding path in Q_k,

$$\left|\rho^{\mathrm{in}}_{i,j}\right| \le |\mathrm{Q}_k| \le (2R+1)^d. \tag{5.42}$$

Finally, observe that for any $x' \in \mathbf{Z}$ and vector $\tau^{\mathrm{in}} \in \mathcal{T}^{\mathrm{in}}_{x'}$,

$$\widetilde{\mathcal{A}}_{z,k} \cap \left\{x_k' = x'\right\} \cap \left\{\rho^{\mathrm{in}} = \tau^{\mathrm{in}}\right\} \in \sigma\left(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\mathrm{out}}_{\mathrm{Q}(x'),u}\right). \tag{5.43}$$

We get

$$\begin{aligned}
&\mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x_k'=x'\}} \cdot \mathbb{P}\left[X^{\mathrm{in}}_{\mathrm{Q}_k} = \rho^{\mathrm{in}} | \sigma\left(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\mathrm{out}}_{\mathrm{Q}(x'),u}\right)\right] \\
&\quad\overset{(5.26),(5.30),(5.43)}{=} \sum_{\tau^{\mathrm{in}}\in\mathcal{T}^{\mathrm{in}}_{x'}} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x_k'=x'\}\cap\{\rho^{\mathrm{in}}=\tau^{\mathrm{in}}\}} \cdot \mathbb{P}\left[X^{\mathrm{in}}_{\mathrm{Q}_k} = \tau^{\mathrm{in}} | \sigma\left(\Sigma_{\mathcal{G},N}, \Sigma_k, \mathcal{F}^{\mathrm{out}}_{\mathrm{Q}(x'),u}\right)\right] \\
&\quad\overset{(5.39),(5.30),(5.41),(5.42)}{\ge} \mathbb{1}_{\widetilde{\mathcal{A}}_{z,k}\cap\{x_k'=x'\}} \cdot (1/2d)^{R^{d-1}\cdot(2R+1)^d}.
\end{aligned}$$

This proves that ρ^{in} satisfies (5.31) with $\gamma = (1/2d)^{R^{d-1}\cdot(2R+1)^d}$. The proof of Lemma 5.10 is complete. □

5.3. *Proof of* (1.3)

In this section we complete the proof of Theorem 1.1 by showing how to deduce (1.3) from (5.2) and (5.8). We begin with the following lemma.

Lemma 5.15. *For R and u_1 as in (5.1), there exist constants $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $u < u_1$ and $N \ge 1$,*

$$\mathbb{P}\left[\begin{array}{c}\mathrm{B}(0, 2(2R+1)N) \cap \mathcal{V}^u \textit{ contains two nearest-neighbor paths} \\ \textit{from } \mathrm{B}(0, (2R+1)N) \textit{ to } \partial_{\mathrm{int}}\mathrm{B}(0, 2(2R+1)N) \textit{ which are} \\ \textit{in different connected components of } \mathrm{B}(0, 2(2R+1)N) \cap \mathcal{V}^u\end{array}\right] \le C \cdot \mathrm{e}^{-N^c}. \tag{5.44}$$

Proof. Recall Definition 5.1 of $\mathcal{H}_N$ and Definition 5.3 of $\mathcal{A}_{z,k}$. Note that when $\mathcal{H}_N$ occurs, the event in (5.44) implies that $\mathcal{A}_{z,k_N}$ occurs for some $z \in \mathrm{B}(0, (2R+1)N)$. Therefore, we can bound the probability in (5.44) from above by

$$\mathbb{P}\big[\mathcal{H}_N^c\big] + \sum_{z \in \mathrm{B}(0,(2R+1)N)} \mathbb{P}[\mathcal{A}_{z,k_N}].$$

The result now follows from (5.2) and (5.8). □

As an immediate corollary to Lemma 5.15, we obtain that for u_1 as in (5.1) there exist constants $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $u \le u_1$ and $n \ge 1$,

$$\mathbb{P}\left[\begin{array}{c}\mathrm{B}(0, 3n) \cap \mathcal{V}^u \text{ contains two paths from } \mathrm{B}(0, n) \text{ to } \partial_{\mathrm{int}}\mathrm{B}(0, 3n) \\ \text{which are in different connected components of } \mathrm{B}(0, 3n) \cap \mathcal{V}^u\end{array}\right] \le C \cdot \mathrm{e}^{-n^c}. \tag{5.45}$$

We are now ready to prove (1.3). Take u_1 as in (5.1) and $u \le u_1$. It suffices to consider $n \ge 100$. Let $k = \lfloor n/100 \rfloor$. Note that if $\mathrm{B}(0, n) \cap \mathcal{V}^u$ contains at least 2 different connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ with diameter $\ge n/10$, then there exist two vertices $x_1, x_2 \in \mathrm{B}(0, n)$ (possibly equal) such that $\mathcal{C}_i \cap \mathrm{B}(x_i, k) \ne \varnothing$ and $\mathcal{C}_i \setminus \mathrm{B}(x_i, 7k) \ne \varnothing$, for $i \in \{1, 2\}$.

For $x \in \mathrm{B}(0, n)$, let $\mathcal{A}_x$ be the event that

(a) $\mathrm{B}(x, k)$ is connected to $\partial_{\mathrm{int}}\mathrm{B}(x, 7k)$ in $\mathcal{V}^u$, and
(b) every two nearest-neighbor paths from $\mathrm{B}(x, 2k)$ to $\partial_{\mathrm{int}}\mathrm{B}(x, 6k)$ in $\mathcal{V}^u$ are in the same connected component of $\mathcal{V}^u \cap \mathrm{B}(x, 6k)$.

Let $\mathcal{A} = \bigcap_{x \in \mathrm{B}(0,n)} \mathcal{A}_x$. By (1.2) and (5.45), we have

$$\mathbb{P}[\mathcal{A}] \ge 1 - C \cdot \mathrm{e}^{-n^c}.$$

However, if the event $\mathcal{A}$ occurs, then $\mathcal{C}_1$ and $\mathcal{C}_2$, defined earlier, cannot exist. Indeed, take a nearest-neighbor path $\pi = (z_1, \dots, z_t)$ in $\mathrm{B}(0, n)$ from x_1 to x_2. For each $1 \le i \le t-1$, the occurrence of the events $\mathcal{A}_{z_i}$ and $\mathcal{A}_{z_{i+1}}$ implies that (a) there exist nearest-neighbor paths π_1 and π_2 in $\mathcal{V}^u$, π_1 from $\mathrm{B}(z_i, k)$ to $\partial_{\mathrm{int}}\mathrm{B}(z_i, 7k)$, and π_2 from $\mathrm{B}(z_{i+1}, k)$ to $\partial_{\mathrm{int}}\mathrm{B}(z_{i+1}, 7k)$, and (b) any two such paths are in the same connected component of $\mathcal{V}^u \cap \mathrm{B}(0, 2n)$. This implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ must be connected in $\mathcal{V}^u \cap \mathrm{B}(0, 2n)$. As a result, we have

$$\mathbb{P}\left[\begin{array}{c}\text{any two connected subsets of } \mathcal{V}^u \cap \mathrm{B}(0, n) \text{ with} \\ \text{diameter} \ge n/10 \text{ are connected in } \mathcal{V}^u \cap \mathrm{B}(0, 2n)\end{array}\right] \ge \mathbb{P}[\mathcal{A}] \ge 1 - C \cdot \mathrm{e}^{-n^c}.$$

This implies (1.3). The proof of Theorem 1.1 is completed.

6. Extensions to other models

6.1. *Random walk on $\mathbb{Z}^d$*

Consider a simple random walk on $\mathbb{Z}^d$, $d \ge 3$, started at $x \in \mathbb{Z}^d$. The random walk is transient, and the probability that $y \in \mathbb{Z}^d \setminus \{x\}$ is ever visited by the random walk is comparable to $|x - y|^{2-d}$.

Let $\mathcal{V}$ be the set of vertices which are never visited by the random walk. The approach that we develop in this paper also applies to the study of the local connectivity properties of $\mathcal{V}$. Similarly to the proof of Theorem 1.1, one can show that the set $\mathcal{V}$, viewed as a random subgraph of $\mathbb{Z}^d$, contains a unique infinite connected component, which is also locally unique. Namely, the statements (1.2) and (1.3) hold with $\mathcal{V}^u$ replaced by $\mathcal{V}$, and the law $\mathbb{P}$ of random interlacements replaced by the law of a simple random walk started from $x \in \mathbb{Z}^d$.

6.2. *Random walk on* $(\mathbb{Z}/N\mathbb{Z})^d$

Consider a simple discrete time random walk on a d-dimensional torus $(\mathbb{Z}/N\mathbb{Z})^d$, with $d \geq 3$. The vacant set at time t is the set of vertices which have not been visited by the random walk up to time t. We view the vacant set as a (random) graph by drawing an edge between any two vertices of the vacant set at L_1-distance 1 from each other. The study of percolative properties of the vacant set was initiated in [1] and recently significantly boosted in [15]. It was proved in [15], Theorems 1.2 and 1.3, that the vacant set at time $\lfloor uN^d \rfloor$ exhibits different connectivity properties for small and large u:

(i) if u is large, there exists $\lambda = \lambda(u) < \infty$, such that the largest connected component of the vacant set at time $\lfloor uN^d \rfloor$ is smaller than $(\log N)^\lambda$ asymptotically almost surely, and
(ii) if $u > 0$ is small, there exists $\delta = \delta(u) > 0$, such that the largest connected component of the vacant set at time $\lfloor uN^d \rfloor$ is larger than δN^d asymptotically almost surely,

where "asymptotically almost surely" means "with probability going to 1 as $N \to \infty$." Moreover, it is proved in [15], Theorem 1.4, that when $d \geq 5$ and u is small enough, with high probability, the vacant set on the torus at time $\lfloor uN^d \rfloor$ has the following properties:

(a) the largest connected component has an asymptotic density, and
(b) the size of the second largest connected component is at most $(\log N)^\kappa$, for some $\kappa > 0$.

The proof of [15], Theorem 1.4, relies on a strong coupling between random interlacements and the random walk trace (see [15], Theorem 1.1) and the existence of *strongly supercritical* values of u for $d \geq 5$ (see [15], Definition 2.4 and Remark 2.5). We believe that the ideas used in the proof of Theorem 1.1 can be applied in order to yield an extension of [15], Theorem 1.4, for all $d \geq 3$ and small enough u, despite the fact that Theorem 1.1 does not imply the existence of strongly supercritical values of u.

Appendix: Decoupling inequalities for interlacement local times

In this appendix we prove Lemma 2.2. The proof is essentially the same as the proof of [12], Corollary 3.5. We sketch the main ideas here and refer the reader to corresponding formulas in [12] for details.

A.1. *Notation from* [12], *Section* 1

For $K \subset\subset \mathbb{Z}^d$, we denote by $s_K : W_K^* \to W_K$ the map which associates with each element $w^* \in W_K^*$ the unique element $w^0 = s_K(w^*) \in W_K$ such that (a) $\pi^*(w^0) = w^*$ and (b) $w^0(0) \in K$, $w^0(t) \notin K$ for all $t < 0$. For $w \in W$, we denote by w_+ the element in W_+ (see (2.1)) such that $w_+(n) = w(n)$, for $n \geq 0$.

For a finite measure ρ on $\mathbb{Z}^d$, we denote by P_ρ the measure $\sum_{x \in \mathbb{Z}^d} \rho(x) P_x$ on $(W_+, \mathcal{W}_+)$.

Let $\omega = \sum_{i \geq 1} \delta_{(w_i^*, u_i)}$ be the interlacement point process on $W^* \times \mathbb{R}_+$ defined on the canonical probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$. For $K \subset\subset \mathbb{Z}^d$, and $0 \leq u' < u$, we define on $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ the Poisson point processes on the space W_+ denoted by $\mu_{K,u}$ and $\mu_{K,u',u}$ in the following way:

$$\begin{aligned} \mu_{K,u',u} &= \sum_{i \geq 1} \mathbb{1}_{\{w_i^* \in W_K^*, u' \leq u_i < u\}} \delta_{s_K(w_i^*)_+} \\ \mu_{K,u} &= \mu_{K,0,u} \end{aligned} \tag{A.1}$$

With these definitions, we have (analogously to [12], (1.27), (1.28)): for $K \subset\subset \mathbb{Z}^d$ and $0 \leq u' < u$,

(i) $\mu_{K,u',u}$ and $\mu_{K,u'}$ are independent with respective intensity measures $(u - u')P_{e_K}$ and $u' P_{e_K}$,
(ii) $\mu_{K,u} = \mu_{K,u'} + \mu_{K,u',u}$.

Let I denote a finite or countable set. If $\mu = \sum_{i \in I} \delta_{w_i}$ is a point measure on W_+, we define (by slightly abusing the notation of (2.15)) the local time of μ at $x \in \mathbb{Z}^d$ to be

$$\mathcal{L}_x(\mu) = \sum_{i \in I} \sum_{n \in \mathbb{N}} \mathbb{1}_{\{w_i(n) = x\}}. \tag{A.2}$$

Using (2.15), (A.1) and (A.2), we obtain that for any $\omega \in \Omega$, $K \subseteq K' \subset\subset \mathbb{Z}^d$, and $u \geq 0$,

$$\mathcal{L}_x^u(\omega) = \mathcal{L}_x(\mu_{K',u}) \quad \text{for } x \in K. \tag{A.3}$$

A.2. Decoupling inequalities for the interlacement local times

In this section we extend the results of [12], Section 2, 3, about certain decoupling inequalities for increasing events in $\{0,1\}^{G\times\mathbb{Z}}$ to increasing events in $\mathbb{N}^{\mathbb{Z}^d}$. The graphs G considered in [12] are infinite, connected, bounded degree weighted graphs, satisfying certain regularity conditions, and in particular, include the case of $\mathbb{Z}^{d-1}$, with $d \geq 3$.

Since our current aim is to prove Lemma 2.2 on $\mathbb{Z}^d$, the notation of [12] become slightly simpler. When $G = \mathbb{Z}^{d-1}$, the volume growth exponent of G is $\alpha = d-1$, the diffusivity exponent of the random walk on G is $\beta = 2$, thus $\nu = \alpha - \frac{\beta}{2} = d-2$ is the ususal exponent of the Green function on $\mathbb{Z}^d$, cf. (2.5) and [12], (0.2). Moreover, in [12], (0.3), a special metric $d(\cdot,\cdot)$ on $G\times\mathbb{Z}$ is introduced, but in our special case $\mathbb{Z}^d = G\times\mathbb{Z}$, the results of [12] remain valid if we replace the distance $d(x,x')$ by the usual sup-norm distance $|x-x'|$, cf. the first paragraph of [12], Section 2.

Remark A.1. *The definition* (2.15) *carries over to the more general setting which involves local times of the interlacement point process on $G\times\mathbb{Z}$* (*where G satisfies the conditions described in* [12], *Section* 1), *and in fact all the results and proofs of* [12], *Sections* 2, 3, *have their analogous*, *more general counterparts which involve $\mathcal{L}^u$ rather than $\mathcal{I}^u$. To simplify the notation*, *we only consider the special case of $G = \mathbb{Z}^{d-1}$ here.*

Now we recall some notation from [12], Section 2, which we adapt to our setting.

Our definition of the length scales $L_n = l_0^n L_0$ in (2.22) is the same as [12], (2.1).

For $n \geq 0$, we denote the dyadic tree of depth n by $T_n = \bigcup_{0\leq k\leq n}\{1,2\}^k$ and the set of vertices of the tree at depth k by $T_{(k)} = \{1,2\}^k$. We call $\varnothing \in T_{(0)}$ the root of T_n and $1,2 \in T_{(1)}$ the children of the root. Given a mapping $\mathcal{T}: T_n \to \mathbb{Z}^d$, we define

$$x_{m,\mathcal{T}} = \mathcal{T}(m), \qquad \widetilde{C}_{m,\mathcal{T}} = \mathrm{B}(x_{m,\mathcal{T}}, 10L_{n-k}) \quad \text{for } m \in T_{(k)}, 0 \leq k \leq n.$$

For any $0 \leq k < n$, $m \in T_{(k)}$, we say that m_1, m_2 are the two descendants of m in $T_{(k+1)}$ if they are obtained by respectively concatenating 1 and 2 to m. We say that $\mathcal{T}$ is an admissible embedding if for any $0 \leq k < n$ and $m \in T_{(k)}$,

$$\widetilde{C}_{m_1,\mathcal{T}} \cup \widetilde{C}_{m_2,\mathcal{T}} \subseteq \widetilde{C}_{m,\mathcal{T}}, \qquad |x_{m_1,\mathcal{T}} - x_{m_2,\mathcal{T}}| \geq \frac{1}{100}L_{n-k}.$$

For any $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we denote by $\Lambda_{x,n}$ the set of admissible embeddings of T_n in $\mathbb{Z}^d$ with $\mathcal{T}(\varnothing) = x$, and let $\Lambda_n = \bigcup_{x\in\mathbb{Z}^d}\Lambda_{x,n}$.

Recall the definition of the space $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ and the coordinate maps Ψ_x, $x \in \mathbb{Z}^d$ from Section 2.4. Given $n \geq 0$ and $\mathcal{T} \in \Lambda_n$, we say that a collection $(B_m\colon m \in T_{(n)})$ of $\mathcal{F}_\ell$-measurable subsets of $\mathbb{N}^{\mathbb{Z}^d}$ is $\mathcal{T}$-adapted if

$$B_m \text{ is } \sigma(\Psi_x, x \in \widetilde{C}_{m,\mathcal{T}})\text{-measurable for each } m \in T_{(n)}. \tag{A.4}$$

Recall that given $u \geq 0$, the collection of $\mathcal{F}_\Omega$-measurable events $(B_m^u\colon m \in T_{(n)})$ is defined by (2.16).

For $n \geq 0$ and $\mathcal{T} \in \Lambda_{n+1}$, we denote by $\mathcal{T}_1 \in \Lambda_n$ the embedding of T_n corresponding to the restriction of $\mathcal{T}$ to the descendants of $1 \in T_{(1)}$ in T_{n+1}. We define $\mathcal{T}_2$ similarly using $2 \in T_{(1)}$. Given a $\mathcal{T}$-adapted collection $(B_m\colon m \in T_{(n+1)})$, we then define the $\mathcal{T}_1$-adapted collection $(B_{m,1}\colon m \in T_{(n)})$ and the $\mathcal{T}_2$-adapted collection $(B_{m,2}\colon m \in T_{(n)})$ in a natural way.

We can now restate and adapt [12], Theorem 2.1, to fit our setting related to $\mathcal{L}^u$ on $\mathbb{Z}^d$.

Theorem A.2. *There exist $c = c(d) > 0$ and $c_1 = c_1(d) > 0$ such that for all $l_0 \geq c$, $n \geq 0$, $\mathcal{T} \in \Lambda_{n+1}$, any $\mathcal{T}$-adapted collection $(B_m\colon m \in T_{(n+1)})$ of increasing events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$, and any $0 < u' < u$ satisfying*

$$u \geq \bigl(1 + c_1(n+1)^{-3/2} l_0^{-(d-2)/4}\bigr)u',$$

we have

$$\mathbb{P}\Big[\bigcap_{m\in T_{(n+1)}} B_m^{u'}\Big] \le \mathbb{P}\Big[\bigcap_{\overline{m}_1\in T_{(n)}} B^u_{\overline{m}_1,1}\Big]\mathbb{P}\Big[\bigcap_{\overline{m}_2\in T_{(n)}} B^u_{\overline{m}_2,2}\Big] + 2\exp\Big(-2u'\frac{2}{(n+1)^3}L_n^{d-2}l_0^{(d-2)/2}\Big). \tag{A.5}$$

Proof. The proof is analogous to that of [12], Theorem 2.1. We only need to mechanically replace events defined in terms of $\mathcal{I}^u$ (see (2.14)) by events defined in terms of $\mathcal{L}^u$ (see (2.15)).

When we adapt [12], Theorem 2.1, to suit our purposes, we make the following choices: $\mathbb{Z}^d = G\times\mathbb{Z}$, $G=\mathbb{Z}^{d-1}$, $\alpha = d-1$, $\beta = 2$, $\nu = d-2$, we use the sup-norm distance $|x-x'|$ on $\mathbb{Z}^d$ (cf. the first paragraph of [12], Section 2), moreover we choose $K=2$ and $\nu' = \frac{d-2}{2}$ (where the latter parameters appear in the statement of [12], Theorem 2.1).

From [12], (2.11), to [12], (2.59), we do not need to modify the proof at all, but we recall some further notation before we state the key domination result (A.10).

Given $n\ge 0$ and $\mathcal{T}\in\Lambda_{n+1}$, we define, as in [12], (2.11) and (2.13),

$$\widehat{C}_i = \bigcup_{m\in T_{(n)}} \widetilde{C}_{m,\mathcal{T}_i} \quad \text{for } i\in\{1,2\}, \quad \text{and} \quad V = \widehat{C}_1\cup\widehat{C}_2,$$

and

$$U_i = \mathrm{B}\Big(x_{i,\mathcal{T}}, \frac{L_{n+1}}{1000}\Big) \quad \text{for } i\in\{1,2\}, \quad \text{and} \quad U = U_1\cup U_2.$$

Finally, we take a set $\mathrm{W}\subset\mathbb{Z}^d$ such that $V\subseteq \mathrm{W}\subseteq U$. Recall the notation (2.2) and (2.4). For a trajectory in W_+ (see (2.1)), we define the sequence of successive returns to W and departures from U:

$$\begin{aligned} &R_1 = H_{\mathrm{W}},\ D_1 = T_U\circ\theta_{R_1} + R_1, \quad \text{and by induction} \\ &R_{k+1} = R_1\circ\theta_{D_k} + D_k,\ D_{k+1} = D_1\circ\theta_{D_k} + D_k \quad \text{for } k\ge 1, \end{aligned} \tag{A.6}$$

where it is understood that if $R_k=\infty$ for some $k\ge 1$, then $D_k = R_{k+1} = \infty$. Let $0\le u' < u$. Recalling (A.1), we introduce, similarly to [12], (2.17), the Poisson point processes on W_+,

$$\begin{aligned} &\zeta'_l = \mathbb{1}_{\{R_l<\infty = R_{l+1}\}}\mu_{\mathrm{W},u'} \quad \text{for } l\ge 1, \\ &\zeta^*_l = \mathbb{1}_{\{R_l<\infty = R_{l+1}\}}\mu_{\mathrm{W},u',u} \quad \text{for } l\ge 1. \end{aligned}$$

Both ζ'_l and ζ^*_l are supported on the subspace of W_+ which consists of trajectories that perform exactly l returns to W in the sense of (A.6). By the properties of $\mu_{\mathrm{W},u'}$ and $\mu_{\mathrm{W},u',u}$,

$$\zeta'_l,\ l\ge 1, \text{ and } \zeta^*_1 \text{ are independent Poisson point processes on } W_+. \tag{A.7}$$

Recalling (A.2), we define the local times

$$\begin{aligned} &\mathcal{L}'_{l,x} = \mathcal{L}_x(\zeta'_l), \qquad \mathcal{L}'_l = (\mathcal{L}'_{l,x}\colon x\in V) \quad \text{for } l\ge 1, \\ &\mathcal{L}^*_{1,x} = \mathcal{L}_x(\zeta^*_1), \qquad \mathcal{L}^*_1 = (\mathcal{L}^*_{1,x}\colon x\in V). \end{aligned}$$

These definitions are counterparts of [12], (2.60) and (2.61). It follows from (A.7) and (A.3) that

the random variables $\mathcal{L}'_l$, $l\ge 1$, and $\mathcal{L}^*_1$ are independent, $\mathcal{L}^{u'}_x = \sum_{l\ge 1}\mathcal{L}'_{l,x}$, and $\mathcal{L}^u_x \ge \mathcal{L}^*_{1,x} + \mathcal{L}'_{1,x}$, for all $x\in V$. (A.8)

This is analogous to [12], (2.62). Moreover, similarly to [12], (2.64), we have that

$$(\mathcal{L}^*_{1,x} + \mathcal{L}'_{1,x}\colon x\in\widehat{C}_1) \text{ and } (\mathcal{L}^*_{1,x} + \mathcal{L}'_{1,x}\colon x\in\widehat{C}_2) \text{ are independent.} \tag{A.9}$$

The main ingredients in the proof of [12], Theorem 2.1, are [12], Lemma 2.4 and (2.59). We will only use a weaker result that immediately follows from [12], Lemma 2.4 and (2.59): for a specific choice of W (see [12], (2.15) and (2.58)), there exists a coupling $(\overline{\mathcal{L}}', \overline{\mathcal{L}}^*)$ on $(\overline{\Omega}, \overline{\mathcal{F}_\Omega}, \overline{\mathbb{P}})$ of $\sum_{l\geq 2}\mathcal{L}'_l$ and $\mathcal{L}^*_1$ such that

$$\text{if } l_0 \geq c(d) \text{ and } u \geq \left(1 + c_1(n+1)^{-3/2} l_0^{-(d-2)/4}\right)u', \text{ then } \overline{\mathbb{P}}[\overline{\mathcal{L}}' \leq \overline{\mathcal{L}}^*] \geq 1 - 2\exp\left(-u'\frac{4}{(n+1)^3} L_n^{d-2} l_0^{(d-2)/2}\right). \tag{A.10}$$

Informally, (A.10) states that with high probability, the local times in V of the collection of interlacement trajectories which have labels less than u' and reenter W after leaving U are dominated by the local times in V of the collection of interlacement trajectories with labels between u' and u that never reenter W after leaving U.

We now prove (A.5) by mimicking [12], (2.68). We recall the notation from (2.16). Let l_0, u and u' satisfy (A.10). Since the B_m, $m \in T_{(n+1)}$, are increasing and $\mathcal{T}$-adapted, cf. (A.4), we see that

$$\begin{aligned}
\mathbb{P}\Bigg[\bigcap_{m\in T_{(n+1)}} B_m^{u'}\Bigg] &\overset{(A.8)}{=} \mathbb{P}\Bigg[\bigcap_{m\in T_{(n+1)}} B_m\Bigg(\sum_{l=1}^{\infty}\mathcal{L}'_l\Bigg)\Bigg] \\
&\overset{(A.8),(A.10)}{\leq} \mathbb{P}\Bigg[\bigcap_{m\in T_{(n+1)}} B_m\big(\mathcal{L}^*_1 + \mathcal{L}'_1\big)\Bigg] + 2\exp\Bigg(-u'\frac{4}{(n+1)^3} L_n^{d-2} l_0^{(d-2)/2}\Bigg) \\
&\overset{(A.8),(A.9)}{\leq} \mathbb{P}\Bigg[\bigcap_{\overline{m}_1\in T_{(n)}} B^u_{\overline{m}_1,1}\Bigg]\mathbb{P}\Bigg[\bigcap_{\overline{m}_2\in T_{(n)}} B^u_{\overline{m}_2,2}\Bigg] + 2\exp\Bigg(-u'\frac{4}{(n+1)^3} L_n^{d-2} l_0^{(d-2)/2}\Bigg).
\end{aligned}$$

This is precisely (A.5). □

Now we derive the decoupling inequalities of [12], Theorem 2.6, adapted to our setting which involves local times. Given c_1 and $l_0 \geq c$ as in Theorem A.2, for any $u_0 > 0$ we define (analogously to [12], (2.70))

$$u^-_\infty = u_0 \cdot \prod_{k=0}^{\infty}\left(1 + \frac{c_1}{(k+1)^{3/2}} l_0^{-(d-2)/4}\right)^{-1}. \tag{A.11}$$

Note that $u^-_\infty > 0$ and $u^-_\infty \to u_0$ as $l_0 \to \infty$.

Theorem A.3 (Decoupling inequalities). *For any $L_0 \geq 1$, $l_0 \geq c(d)$, $u_0 > 0$, $n \geq 0$, $\mathcal{T} \in \Lambda_n$, and all $\mathcal{T}$-adapted collections $(B_m\colon m \in T_{(n)})$ of increasing events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$, one has*

$$\mathbb{P}\Bigg[\bigcap_{m\in T_{(n)}} B_m^{u^-_\infty}\Bigg] \leq \prod_{m\in T_{(n)}} \big(\mathbb{P}\big[B_m^{u_0}\big] + \varepsilon\big(u^-_\infty, l_0, L_0\big)\big),$$

where

$$\varepsilon(u, l_0, L_0) = f\big(2uL_0^{d-2} l_0^{(d-2)/2}\big), \quad \textit{with } f(v) = \frac{2\cdot e^{-v}}{1 - e^{-v}}. \tag{A.12}$$

Proof. The proof of Theorem A.3 is identical to that of [12], Theorem 2.6. We only need to make the particular choices $K = 2$, $\nu = d-2$ and $\nu' = \frac{d-2}{2}$, and replace references to [12], Theorem 2.1 by references to Theorem A.2. We omit the details. □

Recall the definition of uniformly cascading events from Definition 2.1. We now restate [12], Theorem 3.4, adapted to our setting, which involves local times.

Lemma A.4. *Consider the collection $\mathcal{G} = (G_{x,L,R})_{x\in\mathbb{Z}^d, L\geq 1, R\geq 0}$ of increasing events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$, cascading uniformly (in R) with complexity at most λ. Then for any $l_0 \geq c(d)$, $L_0 \geq 1$, $n \geq 0$, $u_0 > 0$ and $R \geq 0$, we have*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^{u_\infty^-}_{x,L_n,R}\big] \leq \big(C(\lambda)^2 \cdot l_0^{2\lambda}\big)^{2^n-1}\Big(\sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^{u_0}_{x,L_0,R}\big] + \varepsilon\big(u_\infty^-, l_0, L_0\big)\Big)^{2^n}, \tag{A.13}$$

where the constant $C(\lambda)$ was defined in (2.20).

Proof. The proof of Lemma A.4 is identical to that of [12], Theorem 3.4. We only need to make the particular choices $K = 2$, $\nu = d - 2$ and $\nu' = \frac{d-2}{2}$, and note that the inequality (A.13) holds uniformly in R because the bound of (2.20) holds uniformly in R. We omit the details. □

A.3. Proof of Lemma 2.2

We are now ready to prove Lemma 2.2, using Lemma A.4. This is similar to the proof of [12], Corollary 3.5.

Let $\mathcal{G} = (G_{x,L,R})_{x\in\mathbb{Z}^d, L\geq 1, R\geq 0}$ be a family of increasing events on $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{F}_\ell)$ cascading uniformly in R with complexity at most $\lambda > 0$. Recall the notation from (A.11) and (A.12). We will choose $l_0 \geq c(d)$, $L_0 \geq 1$, $u_0 > 0$, and $R \geq 0$ so that

$$C(\lambda)^2 \cdot l_0^{2\lambda} \cdot \Big(\sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^{u_0}_{x,L_0,R}\big] + \varepsilon\big(u_\infty^-, l_0, L_0\big)\Big) \leq \frac{1}{2}. \tag{A.14}$$

Once we do so, (2.24) will immediately follow from Lemma A.4 with l_0, L_0, and $R \geq 0$ as in (A.14) and $u = u_\infty^-$.

Let $u_0 = u_{L_0} = {L_0}^{2-d}$. By (A.11) and (A.12), for all large enough $l_0 \geq c(d)$, we have

$$\sup_{L_0\geq 1} C(\lambda)^2 \cdot l_0^{2\lambda} \cdot \varepsilon\big(u_\infty^-, l_0, L_0\big) \leq \frac{1}{4}. \tag{A.15}$$

We fix l_0 satisfying (A.15). Now we use our assumption (2.23) to choose $L_0 \geq 1$ and $R \geq 0$ such that

$$C(\lambda)^2 \cdot l_0^{2\lambda} \cdot \sup_{x\in\mathbb{Z}^d} \mathbb{P}\big[G^{u_0}_{x,L_0,R}\big] \leq \frac{1}{4}. \tag{A.16}$$

The combination of (A.15) and (A.16) gives (A.14) and finishes the proof of Lemma 2.2. □

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