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LIMITING GEODESICS FOR FIRST-PASSAGE PERCOLATION ON SUBSETS OF $\mathbb{Z}^2$

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It is an open problem to show that in two-dimensional first-passage percolation, the sequence of finite geodesics from any point to $(n, 0)$ has a limit in n. In this paper, we consider this question for first-passage percolation on a wide class of subgraphs of $\mathbb{Z}^2$: those whose vertex set is infinite and connected with an infinite connected complement. This includes, for instance, slit planes, half-planes and sectors. Writing x_n for the sequence of boundary vertices, we show that the sequence of geodesics from any point to x_n has an almost sure limit assuming only existence of finite geodesics. For all passage-time configurations, we show existence of a limiting Busemann function. Specializing to the case of the half-plane, we prove that the limiting geodesic graph has one topological end; that is, all its infinite geodesics coalesce, and there are no backward infinite paths. To do this, we prove in the Appendix existence of geodesics for all product measures in our domains and remove the moment assumption of the Wehr–Woo theorem on absence of bigeodesics in the half-plane.

1. Introduction. First-passage percolation may be regarded as a family of models, each of which yields a random pseudo-metric on a graph. It was introduced by Hammersley and Welsh [11] as a model for the passage of a fluid through a porous medium and it has provided many interesting problems to the probability and statistical physics community. It also has links to classical physics through disordered Ising models [8, 14] and to mathematical biology through the study of spread of infections and competition models [18].

The main goal is to understand the (properly scaled) random geometry induced by the pseudo-metric. This has been achieved in two (not necessarily unrelated) ways: first, by studying the asymptotics and fluctuations of the distance function between two points of diverging graph distance; second, by understanding the structure of finite or infinite geodesics, length minimizing paths in this pseudo-metric. This paper addresses questions in the latter group.

The study of geodesics in first-passage percolation starts with Newman [16], Licea–Newman [15] and Wehr [19]. It was conjectured that every semi-infinite

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geodesic should have an asymptotic direction and all such geodesics with a given fixed direction should merge. These statements were established in [15, 16] under certain strong assumptions on the limit shape, the $t \to \infty$ scaling limit of the random ball of size t of the origin. Although natural and expected, these assumptions have not been verified.

The analysis of geodesics continues with the work of Häggström–Pemantle [10], Garet–Marchand [9], Hoffman [12, 13], Damron–Hochman [7] and Auffinger–Damron [2]. They establish existence of a wide class of first-passage percolation processes with infinitely many disjoint infinite one-sided geodesics. All these results explored-known properties of the limit shape or a particular choice of passage-time distribution. Under minimal assumptions, however, Damron–Hanson [6] recently proved some forms of Newman's conjectures. They establish almost-sure coalescence of distributional limits of geodesics and nonexistence of certain infinite backward paths. Despite these advances, it is still an open problem to show that in two dimensions, the sequence of finite geodesics from any point to the points $(n, 0)$ has a limit.

In this manuscript, we consider this question on infinite subgraphs of $\mathbb{Z}^2$. Assuming only existence of finite geodesics, we show that sequences of finite geodesics from any point to boundary points have almost sure limits. Our method is motivated by the "paths crossing" trick of Alm and Wierman [1]. In the case of the half-plane, we prove the limiting geodesic graph has one topological end; that is, all its infinite geodesics coalesce and there are no backward infinite paths. To our knowledge, this is the first time that limiting geodesics are shown to exist under minimal assumptions on the passage times.

We close this section by commenting on limitations of our arguments and speculations for further advances. The crucial use of the boundary is to allow the paths crossing argument of Claim 2.2. In the full plane, this is not possible. Even if one leaves the boundary, taking, for example, $(0, n)$ in the upper half-plane, this argument breaks down. Furthermore, the analysis of half-plane geodesics in this paper heavily uses horizontal translation invariance of the passage-time distribution. This is required to apply the ergodic theorem at several points throughout the arguments. So in many other domains (e.g., quarter planes or sectors) we do not know if the geodesics constructed here coalesce, although it is reasonable to expect them to.

1.1. *Outline of the paper.* In the rest of the Introduction, we give the precise definition of the model, and we state the main theorems of the paper. In Section 2 we establish, without any assumption on the passage times, existence of limits for Busemann functions. Under the hypothesis of existence of finite geodesics, in Section 3, we prove existence of the limiting geodesic graph. In Section 4, we show that in the upper half-plane, this limiting graph has one end, establishing coalescence of any pair of its infinite geodesics. We finish the paper with three Appendices. In Appendix A, we give an alternate characterization of our domains. Appendix B proves the existence of finite geodesics for all product measures.

Appendix C extends the Wehr–Woo theorem [20] on absence of doubly infinite geodesics in the half-plane to more general measures.

1.2. *Definitions*. Let $(\mathbb{Z}^2, \mathcal{E}^2)$ denote the square lattice with nearest-neighbor edges. We consider first-passage percolation on particular infinite subsets of this graph. Let $V \subseteq \mathbb{Z}^2$ be a connected [in $(\mathbb{Z}^2, \mathcal{E}^2)$] infinite set whose complement is also connected and infinite. Write E for the set of edges with both endpoints in V. We will need the graph dual to the square lattice, the vertex set of which is $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (1/2, 1/2)$ and the edge set of which is $(\mathcal{E}^2)^* = \mathcal{E}^2 + (1/2, 1/2)$. The edge e^* is said to be dual to $e \in \mathcal{E}^2$ if it bisects e. We prove in Appendix A that there exists some path of dual edges

$$\Upsilon = (e_i^*)_{i \in \mathbb{Z}} \tag{1}$$

which does not (vertex) self-intersect and such that (V, E) is one of the two components of the graph formed from $(\mathbb{Z}^2, \mathcal{E}^2)$ by removing the edges (e_i) dual to those edges (e_i^*).

(2) Let v_i be the endpoint of e_i that lies in V.

Note that while Υ is not self-intersecting, a particular v_i may appear multiple times (at most 3 times).

We do first-passage percolation in (V, E) by setting $\Omega = [0, \infty)^E$ and denoting a typical element of Ω by $\omega = (\omega_e)_{e \in E}$. For $x, y \in V$, a path from x to y in V is a sequence of alternating vertices and edges

$$x = w_0, e_0, w_1, \dots, w_{n-1}, e_{n-1}, w_n = y$$

such that for all i, $e_i = \{w_i, w_{i+1}\} \in E$. Clearly a path is uniquely determined by its sequence of vertices or its sequence of edges, so we will at times refer to it in one of these ways. We will write $\gamma : x \rightsquigarrow y$ to denote that γ is a path from x to y. We will use $\| \cdot \|_1$ to denote the l^1 norm.

The resulting passage time is written τ. That is, $\tau(\gamma) = \sum_{e \in \gamma} \omega_e$ is the passage time of a finite path γ in (V, E) and $\tau(x, y) = \inf_{\gamma : x \rightsquigarrow y} \tau(\gamma)$ is the passage time between x and y in V. As defined, τ is a pseudo-metric. A geodesic from x to y is a path $\gamma : x \rightsquigarrow y$ in (V, E) such that $\tau(\gamma) = \tau(x, y)$. Note that if there exists a geodesic between some pair of points, there is at least one vertex self-avoiding geodesic.

We will define (for x and y elements of V) the Busemann function

$$B_n(x, y) = \tau(x, v_n) - \tau(y, v_n).$$

1.3. *Main results*.

1.3.1. *Arbitrary* (V, E). The first result shows that asymptotic limits of the (B_n) exist under no assumptions on ω. That is, it holds for all passage-time configurations.

THEOREM 1.1. *For any $x, y \in V$ and $\omega \in \Omega$,*

$$B(x, y) := \lim_{n\to\infty} B_n(x, y) \qquad \textit{exists}. \tag{3}$$

REMARK 1.2. We strongly believe that Busemann limits exist in wide generality (in particular, even in the full-plane), but we do not have a proof. That is, we expect that for any $\theta \in [0, 2\pi)$ and any sequence (x_n) of vertices in $\mathbb{Z}^2$ such that $\arg x_n \to \theta$ with $x_n \to \infty$, the limit $\tau(x, x_n) - \tau(y, x_n)$ exists almost surely for $x, y \in \mathbb{Z}^2$.

For the second result we consider a measure $\mathbb{P}$ on Ω (with the product Borel sigma algebra) that admits geodesics; that is,

$$\mathbb{P}(\exists \text{ a geodesic } \gamma : x \rightsquigarrow y) = 1 \qquad \text{for all } x, y \in V.$$

Under this condition we can associate to almost every $\omega \in \Omega$ and each $n \in \mathbb{Z}$ a geodesic graph $\mathbb{G}_n = \mathbb{G}_n(\omega)$. This is a directed graph with vertex set V built from a configuration $\eta_n = \eta_n(\omega)$ from the space $\{0, 1\}^{\vec{E}}$, where $\vec{E}$ is the set of directed edges corresponding to E,

$$\vec{E} = \{(x, y) : \{x, y\} \in E\}.$$

The definition of η_n is as follows. We set $\eta_n((x, y)) = 1$ if $\{x, y\}$ is in a geodesic from some vertex in V to v_n and $\tau(x, v_n) \geq \tau(y, v_n)$. Otherwise we set $\eta_n((x, y)) = 0$. The graph $\mathbb{G}_n$ is then induced by its directed edge set, the set of e such that $\eta_n(e) = 1$.

We say that $\eta_n \to \eta \in \{0, 1\}^{\vec{E}}$ if for each $e \in \vec{E}$, $\eta_n(e) \to \eta(e)$. In this case we write $\mathbb{G}_n \to \mathbb{G}$, where $\mathbb{G}$ is the directed graph corresponding to η.

THEOREM 1.3. *Suppose that $\mathbb{P}$ admits geodesics. Then with probability one, $(\mathbb{G}_n)$ converges to a graph $\mathbb{G}$. Each directed path in $\mathbb{G}$ is a geodesic.*

1.3.2. *On the half-plane* $\mathbb{H}$. Taking the vertex set $V = V_H = \{(x_1, x_2) \in \mathbb{Z}^2 : x_2 \geq 0\}$ and E_H the induced set of edges, we can analyze first-passage percolation more closely on $\mathbb{H} = (V_H, E_H)$, taking advantage of translation invariance of standard measures. The relevant space is $\Omega_H = [0, \infty)^{E_H}$ and we define a family of translation operators $\{T_x : x \in V_H\}$ on Ω_H by

$$(T_x\omega)_e = \omega_{e+x},$$

where if $e = \{v, w\}$ then $e + x = \{v + x, w + x\}$.

For the results in this section we will consider a probability measure $\mathbb{P}$ satisfying one of two assumptions, labeled (A) and (B) below. Assumption (B) includes the upward finite energy property from [6]:

DEFINITION 1.4. Given an edge set E', a Borel probability measure $\mathbb{P}$ on $[0,\infty)^{E'}$ satisfies the upward finite energy property if for each $e \in E'$ and λ such that $\mathbb{P}(\omega_e \geq \lambda) > 0$, we have

$$\mathbb{P}(\omega_e \geq \lambda \mid \check{\omega}) > 0 \qquad \text{almost surely.}$$

In the definition we have used the notation $\omega = (\omega_e, \check{\omega})$, where $\check{\omega} = (\omega_f : f \neq e)$. The assumptions we need are:

(A) $\mathbb{P}$ is a product measure with continuous marginals, or

(B) $\mathbb{P}$ is the restriction to $[0,\infty)^{E_H}$ of a Borel probability measure $\widehat{\mathbb{P}}$ on $[0,\infty)^{\mathcal{E}^2}$ that satisfies the upward finite energy property and the assumptions of Hoffman [13]:

(a) $\widehat{\mathbb{P}}$ is ergodic relative to the translations T_x for $x \in \mathbb{Z}^2$;
(b) $\widehat{\mathbb{P}}$ has all the symmetries of $\mathbb{Z}^2$;
(c) $\int \omega_e^{2+\alpha}\,\mathrm{d}\widehat{\mathbb{P}} < \infty$ for some $\alpha > 0$;
(d) $\widehat{\mathbb{P}}$ has unique passage times: with probability one, no two (edge) nonempty distinct paths have the same passage time and
(e) the limiting shape for $\widehat{\mathbb{P}}$ is bounded.

Under parts (a)–(c) of assumption (B), Kingman's theorem implies that if we write τ' for the passage time in $\mathbb{Z}^2$, then for each $y \in \mathbb{Z}^2$, the limit $g(y) = \lim_{n\to\infty} \tau'(0, ny)/n$ exists almost surely and in L^1. Part (b) is required for the geodesic graph to be a forest. This is used several times in the final arguments. So our arguments do not apply, for instance, to geometric weights. Part (e) of assumption (B) is then the statement that $\inf_{y\neq 0} \frac{g(y)}{\|y\|_1} > 0$.

Under either of these assumptions, one can show that $\mathbb{P}$ admits geodesics. Under (A), we show it in Appendix B for general graphs (V, E) considered in this paper. Under (B) it follows from the shape theorem proved by Boivin [3] and boundedness of the limit shape. This means we can use the results from the previous subsection. For the statement of the main theorem, we use the shorthand $x \to y$ for vertices x, y in a directed graph $\vec{G}$ if there is a directed path from x to y in $\vec{G}$.

THEOREM 1.5. *Assume* (A) *or* (B). *Writing* $x_n = (n, 0)$, *the geodesic graphs* $(\mathbb{G}_n)$ *converge almost surely to a directed graph* $\mathbb{G}$ *with the following properties*:

(1) *each vertex in* V_H *has out-degree* 1;
(2) *viewed as an undirected graph*, $\mathbb{G}$ *has no circuits*;
(3) *for each* $x \in V_H$, *the backward cluster* $B_x = \{y \in V_H : y \to x\}$ *is finite*;
(4) *writing* Γ_x *for the unique self-avoiding infinite directed path in* $\mathbb{G}$ *starting from* x, *for all* $x, y \in V_H$, Γ_x *and* Γ_y *coalesce. That is, their edge symmetric difference is finite.*

REMARK 1.6. It is an important problem to show that the geodesics constructed above have direction $\mathbf{e}_1$. We believe this is true; however, we cannot prove it.

2. Existence of Busemann limits.

The main goal of this section is prove Theorem 1.1. We begin with $x, y \in \{v_i\}_{i\in\mathbb{Z}}$, defined in (2).

PROPOSITION 2.1. *For any $x, y \in \{v_i\}_i$ and $\omega \in \Omega$, the limit in* (3) *exists. Moreover, the convergence is monotone.*

PROOF. We assume that $x = v_i$ and $y = v_j$ for $i < j$, and we let $\varepsilon > 0$. Fix any $n_2 > n_1 > j$ such that $v_{n_1} \neq v_{n_2}$. We can now choose vertex self-avoiding paths $\gamma : x \rightsquigarrow v_{n_1}$ and $\gamma' : y \rightsquigarrow v_{n_2}$ to satisfy

$$\tau(\gamma) \leq \tau(x, v_{n_1}) + \varepsilon \quad \text{and} \quad \tau(\gamma') \leq \tau(y, v_{n_2}) + \varepsilon.$$

Form a continuous path β (in $\mathbb{R}^2$) by taking γ, adjoining half of the edge e_{n_1}, adjoining the segment of Υ [recall the definition from (1)] between $e^*_{n_1}$ and e^*_i, and then finally appending half of the edge e_i, to form a continuous circuit based at x. Since this circuit is a Jordan curve, it separates $\mathbb{R}^2$ into an interior and an exterior. See Figure 1 for an illustration of β.

Our first observation is that either $y \in \beta$ or y is in the interior of β (and in fact, $y \in \beta$ only if $y \in \gamma$). The reason is that y is an endpoint of one of the e_i's, which must cross β. Since the other endpoint of this edge is in V^c, it cannot be in the interior of β (or on β). The Jordan curve theorem implies that these endpoints are in different components, and thus if $y \notin \beta$, it must be in the interior of β. We make the following claim:

CLAIM 2.2. *$\gamma' \cap \gamma$ contains a vertex of $\mathbb{Z}^2$.*

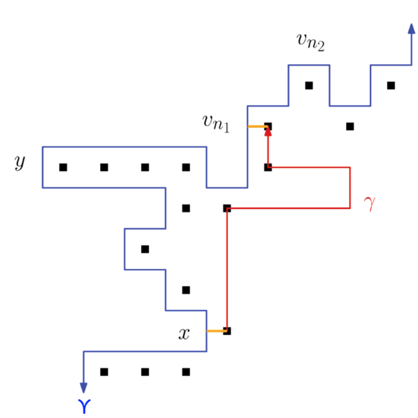

FIG. 1. *Construction of the Jordan curve β. It consists of the right path γ, two half-edges connecting γ to the left path, which is a segment of Υ between v_{n_1} and x.*

To show the claim, we first prove that v_{n_2} is either on β or in the exterior of β. Accordingly, assume v_{n_2} is not on β. Notice that neither endpoint of e_{n_2} can touch β. Furthermore the edge e_{n_2} cannot intersect β because $e^*_{n_2}$ is not contained in β. Therefore both endpoints are in the same component of the complement of β and since the other one is in V^c, they must be in the exterior of β.

Now, considering γ' as a continuous plane curve, we note that γ' must intersect β (since it has to reach v_{n_2}, which is not in the interior of β), but it cannot intersect Υ. Therefore, it must intersect γ; this intersection must happen at a vertex, though it may of course also happen at one or more edges. This proves the claim.

We will complete the existence proof for the limit in (3) by showing that $B_n(x, y)$ is monotone in n for fixed x and y. Let n_1 and n_2 be as above. For any path $\sigma : a \rightsquigarrow b$ and $c \in \sigma$ write $\sigma|_c$ for the segment of σ from the first meeting of c onward and $\sigma|^c$ for the segment of σ to the first meeting of c. Then letting w be a point in $\gamma' \cap \gamma$,

$$\begin{aligned}\tau(x, v_{n_2}) + \tau(y, v_{n_1}) &\le \big[\tau\big(\gamma|^w\big) + \tau\big(\gamma'|_w\big)\big] + \big[\tau\big(\gamma'|^w\big) + \tau(\gamma|_w)\big] \\ &= \big[\tau\big(\gamma|^w\big) + \tau(\gamma|_w)\big] + \big[\tau\big(\gamma'|^w\big) + \tau\big(\gamma'|_w\big)\big] \\ &= \tau(\gamma) + \tau\big(\gamma'\big) \le \tau(x, v_{n_1}) + \tau(y, v_{n_2}) + 2\varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$,

$$\tau(x, v_{n_2}) + \tau(y, v_{n_1}) \le \tau(x, v_{n_1}) + \tau(y, v_{n_2}). \tag{4}$$

We can rearrange the terms in (4) to find that

$$B_{n_2}(x, y) \le B_{n_1}(x, y).$$

Since $B_n(x, y)$ is a sequence bounded below by $-\tau(x, y)$, $\lim B_n(x, y)$ exists. $\square$

We now move on to general $x, y \in V$ and prove the limit in (3) exists. We will need a few geometric notions. Let α denote the vertex set of a finite, connected subgraph of (V, E) which contains some v_i. Denote by (V', E') the graph formed by setting $V' = V \setminus \alpha$ and letting E' be formed from E by removing every edge with an endpoint in α. The graph (V', E') may have multiple components, but the following claim allows us to find a single component defining the Busemann function.

CLAIM 2.3. *There exists a component $(\overline{V}, \overline{E})$ of (V', E') and an $M < \infty$ such that, for all $n > M$, $v_n \in \overline{V}$. Moreover, $(\overline{V}, \overline{E})$ is formed from $(\mathbb{Z}^2, \mathcal{E}^2)$ by the removal of edges dual to a doubly infinite, self-avoiding path $\overline{\Upsilon}$ in the dual lattice.*

PROOF. Note that if $v_n \ne v_{n+1}$, then there exists a path in (V, E) between v_n and v_{n+1} of Euclidean length at most two. Since $\|v_n\|_1 \to \infty$, we can choose M such that

$$\operatorname{dist}\big(\{v_n\}_{n>M}, \alpha\big) \ge 2,$$

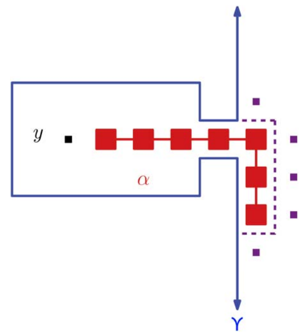

FIG. 2. *Removal of the vertex set α from V. The enlarged squares represent α and the dotted path is the segment of $\overline{\Upsilon}$ that does not lie in Υ. The vertices $\bar{v}_j$ for $j \in J$ are drawn neighboring the dotted path on the right.*

where $\mathrm{dist}(\cdot, \cdot)$ is the (V, E) graph distance. Then $\{v_n\}_{n>M}$ must all lie in one component of (V', E'), which we denote by $(\overline{V}, \overline{E})$.

It remains to show that $(\overline{V}, \overline{E})$ can be formed from $(\mathbb{Z}^2, \mathcal{E}^2)$ by cutting along a doubly infinite, loop-free dual path $\overline{\Upsilon}$. By Proposition A.1 in Appendix A, it suffices to show that both $\overline{V}$ and $\mathbb{Z}^2 \setminus \overline{V}$ are infinite and connected (as subsets of $\mathbb{Z}^2$). Both claims are true for $\overline{V}$. Moreover, $\mathbb{Z}^2 \setminus \overline{V}$ is infinite, since it contains V^c. Because α is connected and contains a point of $\{v_i\}_i$, we see that $\mathbb{Z}^2 \setminus \overline{V}$ is connected; it consists of the union of α, V^c and the sites of V which were only reachable from the large v_n's via sites of α; see Figure 2. Therefore, by the above, the dual edge boundary between $\overline{V}$ and $\mathbb{Z}^2 \setminus \overline{V}$ is a doubly infinite self-avoiding dual path, proving the claim. $\square$

We note that, by Proposition 2.1 and the linearity of the Busemann function, we need only prove the existence of the limit in (3) when $y \notin \{v_i\}_i$ but x is some v_m (which can be chosen as a function of y). Fix y, and denote by α the vertex set of some (vertex self-avoiding, finite) path in (V, E) which starts at a vertex adjacent to y and ends at a vertex $v_m \in \{v_i\}_i$. Form the graph $(\overline{V}, \overline{E})$ as in Claim 2.3; denote by $\overline{\Upsilon}$ the doubly-infinite dual path whose existence is established in the claim, and define $\{\bar{v}_i\}_i$ analogously to $\{v_i\}_i$. We may choose an orientation of $\{\bar{v}_i\}_i$ such that the following holds. There exists $\kappa \in \mathbb{Z}$ such that for all large n, $v_n = \bar{v}_{n+\kappa}$.

If $\bar{\tau}$ and $\overline{B}_n$ are the passage times and Busemann functions in $(\overline{V}, \overline{E})$ (defined in the obvious way), then

$$\overline{B}(\bar{v}_i, \bar{v}_j) = \lim_{n \to \infty} \overline{B}_n(\bar{v}_i, \bar{v}_j) \tag{5}$$

exists for all i and j by Proposition 2.1.

Denote by $J \subseteq \mathbb{Z}$ the finite set of indices such that $\bar{v}_j$ is at Euclidean distance one from α. Note that y is adjacent to some vertex of α; therefore, if $y \in \overline{V}$, then $y = \bar{v}_j$ for some $j \in J$. We will want to apply the following lemma to both $z = y$ and $z = v_m$:

LEMMA 2.4. *Let $z \in V$ be such that either $z \in \{\bar{v}_j : j \in J\}$ or $z \notin \overline{V}$. Then*

$$\tau(z, v_n) = \min_{j \in J}\{\tau(z, \bar{v}_j) + \bar{\tau}(\bar{v}_j, v_n)\}. \tag{6}$$

PROOF. Let $\varepsilon > 0$ and $j \in J$. Then find paths $\gamma : z \rightsquigarrow \bar{v}_j$ in (V, E) and $\bar{\gamma} : \bar{v}_j \rightsquigarrow v_n$ in $(\overline{V}, \overline{E})$ such that $\tau(\gamma) \le \tau(z, \bar{v}_j) + \varepsilon$ and $\bar{\tau}(\bar{\gamma}) \le \bar{\tau}(\bar{v}_j, v_n) + \varepsilon$. Build a path $\sigma : z \rightsquigarrow v_n$ in (V, E) by concatenating γ with $\bar{\gamma}$. Then

$$\tau(z, v_n) \le \tau(\sigma) = \tau(\gamma) + \bar{\tau}(\bar{\gamma}) \le \tau(z, \bar{v}_j) + \bar{\tau}(\bar{v}_j, v_n) + 2\varepsilon.$$

Taking $\varepsilon \to 0$ and a minimum over $j \in J$ gives the inequality $\le$ in (6).

To prove the other inequality, let $\sigma : z \rightsquigarrow v_n$ in (V, E) be a path such that $\tau(\sigma) \le \tau(z, v_n) + \varepsilon$. The path σ must have a terminal segment $\bar{\gamma}$ which lies in $(\overline{V}, \overline{E})$ from some $\bar{v}_{j_0}$ to v_n—this terminal segment may be equal to the singleton $\{v_n\}$. Write γ for the segment of σ from z to the last meeting of $\bar{v}_{j_0}$. Then

$$\begin{aligned}\min_{j \in J}\{\tau(z, \bar{v}_j) + \bar{\tau}(\bar{v}_j, v_n)\} &\le \tau(z, \bar{v}_{j_0}) + \bar{\tau}(\bar{v}_{j_0}, v_n) \\ &\le \tau(\gamma) + \bar{\tau}(\bar{\gamma}) = \tau(\sigma) \le \tau(z, v_n) + \varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ proves (6). □

So, defining

$$\varphi_j(z, n) := \tau(z, \bar{v}_j) + \bar{\tau}(\bar{v}_j, v_n) - \bar{\tau}(\bar{v}_1, v_n),$$

we see that $\tau(z, v_n) = \bar{\tau}(\bar{v}_1, v_n) + \min_{j \in J} \varphi_j(z, n)$. Moreover,

$$\lim_{n \to \infty} \varphi_j(z, n) =: \varphi_j(z)$$

exists by (5), and therefore so does

$$\lim_{n \to \infty}\big[\tau(z, v_n) - \bar{\tau}(\bar{v}_1, v_n)\big]. \tag{7}$$

Finally, we can use the above to show convergence of $B_n(y, v_m)$ as $n \to \infty$. Write

$$\begin{aligned}\lim_{n \to \infty} B_n(y, v_m) &= \lim_{n \to \infty}\big[\tau(y, v_n) - \tau(v_m, v_n)\big] \\ &= \lim_{n \to \infty}\big[\tau(y, v_n) - \bar{\tau}(\bar{v}_1, v_n) + \bar{\tau}(\bar{v}_1, v_n) - \tau(v_m, v_n)\big] \\ &= \lim_{n \to \infty}\big[\tau(y, v_n) - \bar{\tau}(\bar{v}_1, v_n)\big] - \lim_{n \to \infty}\big[\tau(v_m, v_n) - \bar{\tau}(\bar{v}_1, v_n)\big].\end{aligned}$$

Using (7) with $z = y$ and $z = v_m$ completes the proof.

3. Geodesic limits. Our aim in this section is to prove Theorem 1.3. We begin with general properties of geodesic graphs from [6].

3.1. *Geodesic graphs.* We will show that the geodesic graph is in fact a union of geodesics with the appropriate directions. Moreover, under the assumption of unique passage times, it is a directed forest.

PROPOSITION 3.1. *Assume $\mathbb{P}$ admits geodesics.*

(1) *Almost surely, every finite directed path in $\mathbb{G}_n$ is a geodesic. It is a subpath of a geodesic ending in v_n.*

(2) *Assume $\mathbb{P}$ has unique passage times. Then each $x \in V \setminus \{v_n\}$ has out-degree 1 in $\mathbb{G}_n$. Furthermore viewed as an undirected graph, $\mathbb{G}_n$ has no circuits.*

PROOF. Let γ be a directed path in $\mathbb{G}_n$ and write the (directed) edges of γ in order as $e_1, \dots, e_k$. Write $J \subseteq \{1, \dots, k\}$ for the set of j such that the path γ_j induced by $e_1, \dots, e_j$ is a subpath of a geodesic from some vertex to v_n. We will show that $k \in J$. By construction of $\mathbb{G}_n$, the edge e_1 is in a geodesic from some point to v_n. Furthermore, if $e_1 = (x, y)$, then $\tau(x, v_n) \geq \tau(y, v_n)$ because $\eta_n(e_1) = 1$, so if these passage times are not equal, e_1 must be traversed from x to y in this geodesic, giving $1 \in J$. If they are equal, then $\omega_{\{x,y\}} = 0$ and $1 \in J$ as well.

Now suppose that $j \in J$ for some $j < k$; we will show that $j + 1 \in J$. Take σ to be a geodesic from a point z to v_n which contains γ_j as a subpath. Write σ' for the segment of the path from z to the far endpoint w_j of e_j (i.e., we terminate σ directly after traversing the path γ_j for the first time). The edge e_{j+1} is also in $\mathbb{G}_n$ so it is in a geodesic from some point to v_n. If we write $\hat{\sigma}$ for the piece of this geodesic from its first meeting of w_j to v_n, we claim that the concatenation of σ' with $\hat{\sigma}$ is a geodesic from z to v_n. To see this,

$$\tau(z, v_n) = \sum_{e \in \sigma'} \omega_e + \sum_{e \in \sigma \setminus \sigma'} \omega_e = \sum_{e \in \sigma'} \omega_e + \sum_{e \in \hat{\sigma}} \omega_e.$$

The last equality holds since both $\hat{\sigma}$ and the segment of σ from w_j to v_n are geodesics, so they have equal passage time. Hence $j + 1 \in J$, and we are done with the first item.

For the second item, assume that $\mathbb{P}$ has unique passage times so that in particular, almost surely, no edges have passage time 0. Therefore if a directed edge is in a geodesic from a point to v_n, it must be traversed in this direction. Note that from each vertex $v \in V \setminus \{v_n\}$ there is at least one geodesic from v to v_n. The first edge of this geodesic is pointed away from v, so v has an out-degree of at least one. Assuming v has an out-degree of at least two, then we write e_1 and e_2 for two such directed edges. By the first item, there are two geodesics, γ_1 and γ_2, to v_n such that $e_i \in \gamma_i$ for $i = 1, 2$. If either of these paths returned to v, then there would exist a finite path with passage time zero, contradicting unique passage times. So the portions of the γ_i's from v to v_n have distinct edge sets and therefore have different passage times. This contradicts both being geodesics.

We finish by arguing for the absence of circuits. If there is a circuit in the undirected version of $\mathbb{G}_n$, then by virtue of each vertex having out-degree one, this is a directed circuit and thus a geodesic. But then it has passage time zero, a contradiction. □

3.2. *Proof of Theorem* 1.3. The second statement of the theorem follows directly from the previous section: each directed path in $\mathbb{G}_n$ is a geodesic. So we prove the first statement and show that for each directed edge (x, y) in $\vec{E}$, with probability one the value of $\eta_n((x, y))$ is eventually constant. Fix $x \in V$ and choose $m \in \mathbb{N}$ such that, defining [with $d(\cdot,\cdot)$ the graph distance in (V, E)]

$$S_m = \{w \in V : d(x, w) \le m\},$$
$$\partial S_m = \{w \in V : d(x, w) = m + 1\},$$

we have $S_m \cap \{v_i\}_i \neq \varnothing$. Setting $\alpha = S_m$, we may apply Claim 2.3 to find $(\overline{V}, \overline{E})$, a component of the graph generated by removing α from (V, E) containing v_n for all large n. As before, it can be alternatively created by cutting $(\mathbb{Z}^2, \mathcal{E}^2)$ along a doubly infinite self-avoiding dual path $\overline{\Upsilon}$. As in the last section, we will decorate expressions with an overline when they are meant for the model in $(\overline{V}, \overline{E})$ (e.g., $\bar{\tau}$). For the remainder, we also fix $\omega \in \Omega$ such that for each $x, y \in V$, there is a geodesic from x to y.

For each $\zeta \in T_m := \partial S_m \cap \overline{V}$, and n such that $v_n \in \overline{V}$, we define the quantity

$$f_n(\zeta) = \tau(x, \zeta) + \bar{\tau}(\zeta, v_n). \tag{8}$$

Let $\mathfrak{m}_n$ be the set of minimizers of f_n.

LEMMA 3.2. *There exists* $\mathfrak{m} \subset T_m$ *such that* $\mathfrak{m}_n = \mathfrak{m}$ *for all large* n.

PROOF. First, note that $T_m \subset \{\bar{v}_i\}_i$. Therefore by Proposition 2.1, for $\zeta, \zeta' \in T_m$,

$$\begin{aligned} f_n(\zeta) - f_n(\zeta') &= \tau(x, \zeta) + \bar{\tau}(\zeta, v_n) - \tau(x, \zeta') - \bar{\tau}(\zeta', v_n) \\ &= \tau(x, \zeta) - \tau(x, \zeta') + \overline{B}_n(\zeta, \zeta') \end{aligned}$$

is eventually monotone. Suppose that $\zeta \in T_m$ satisfies $\zeta \notin \mathfrak{m}_n$ for infinitely many n. Then we can find ζ' such that $f_n(\zeta) - f_n(\zeta') > 0$ for infinitely many n. By monotonicity this means that actually $f_n(\zeta) - f_n(\zeta') > 0$ for all large n and thus $\zeta \notin \mathfrak{m}_n$ for all large n. This also implies that if $\zeta \in \mathfrak{m}_n$ for infinitely many n, then $\zeta \in \mathfrak{m}_n$ for all large n, completing the proof. □

Given this lemma, the theorem will follow once we show that $\eta_n((x, y)) = 1$ if and only if $\{x, y\}$ is in a geodesic from x to a vertex of $\mathfrak{m}_n$. Note that T_m is equal to the set of vertices in $\overline{V}$ at Euclidean distance one from S_m. Applying Lemma 2.4 with $z = x$, any $\zeta \in T_m$ satisfies

$$\zeta \in \mathfrak{m}_n \text{ if and only if } f_n(\zeta) = \tau(x, v_n).$$

So suppose first that $\eta_n((x, y)) = 1$; then $\{x, y\}$ is in a geodesic γ from x to v_n. γ has a last intersection ζ with T_m. Then the segment $\bar{\gamma}$ of γ from this intersection to v_n has

$$\tau(\zeta, v_n) = \tau(\bar{\gamma}) \geq \bar{\tau}(\zeta, v_n).$$

But $\bar{\tau}(\zeta, v_n) \geq \tau(\zeta, v_n)$, so $\tau(\bar{\gamma}) = \bar{\tau}(\zeta, v_n)$. Therefore

$$\tau(x, v_n) = \tau(\gamma) = \tau(x, \zeta) + \tau(\bar{\gamma}) = \tau(x, \zeta) + \bar{\tau}(\zeta, v_n) = f_n(\zeta),$$

giving $\zeta \in \mathfrak{m}_n$. Furthermore the segment of γ up to the last intersection with ζ is a geodesic from x to ζ that contains $\{x, y\}$.

Conversely, suppose that $\{x, y\}$ is in a geodesic γ_1 from x to a vertex ζ of $\mathfrak{m}_n$; we will show that $\eta_n((x, y)) = 1$. Choose γ_2 as any geodesic from ζ to v_n. Concatenate them to form a path γ from x to v_n. We compute

$$\tau(\gamma) = \tau(\gamma_1) + \tau(\gamma_2) = \tau(x, \zeta) + \tau(\zeta, v_n) \leq \tau(x, \zeta) + \bar{\tau}(\zeta, v_n) = f_n(\zeta).$$

However since $\zeta \in \mathfrak{m}_n$, $f_n(\zeta) = \tau(x, v_n)$, so $\tau(\gamma) \leq \tau(x, v_n)$. The opposite inequality holds because $\gamma : x \rightsquigarrow v_n$, so γ is a geodesic from x to v_n. It remains to show that $\tau(x, v_n) \geq \tau(y, v_n)$. But this holds because y appears in γ after the first appearance of x. Therefore if we write σ for the segment of γ from the first intersection with y to v_n, then

$$\tau(x, v_n) = \tau(\gamma) \geq \tau(\sigma) = \tau(y, v_n).$$

4. Geodesics graphs on $\mathbb{H}$. In this section we prove Theorem 1.5. Because $\mathbb{P}$ admits geodesics, Theorem 1.3 implies that the sequence of graphs $(\mathbb{G}_n)$ converge almost surely to a directed graph $\mathbb{G}$, each of whose directed paths is a geodesic. As $\mathbb{P}$ also has unique passage times, Proposition 3.1 states that each vertex of $\mathbb{G}_n$ has out-degree one and there are no undirected circuits, so these same properties survive in the limit for $\mathbb{G}$. The finiteness of backward clusters is a consequence of nonexistence of bigeodesics in the half-plane, proved by Wehr and Woo [20]. Unfortunately this result was only proved under (A) with the additional assumption $\mathbb{E}\omega_e < \infty$, so we provide a proof in Appendix C under either (A) or (B).

This section is devoted to showing coalescence of directed paths in $\mathbb{G}$. Because each vertex in $\mathbb{G}_H$ has an out-degree of one, it suffices to show that each Γ_v and Γ_w (defined in the statement of Theorem 1.5) share a vertex. The main difficulty will be proving this statement for all v, w on the first coordinate axis; that is, the set L_0, where

$$\text{for } k \in \mathbb{N} \cup \{0\}, \qquad L_k := \{(x, k) : x \in \mathbb{Z}\}.$$

To see why this implies coalescence for all paths, assume we have proved this statement, and note that it suffices then to show that for all $v, w \in V_H$ with $w \in L_0$, the geodesics Γ_v and Γ_w coalesce. Write $v = (v_1, v_2)$ and consider the set

$$\widetilde{L}_v = \{(v_1, y) \in V_H : 0 \le y \le v_2\}.$$

With probability one, for each $v' \in \widetilde{L}_v$, the backward cluster $B_{v'}$ is finite. Thus we can find $m, n \in \mathbb{Z}$ with $m < v_1 < n$ such that for all $v' \in \widetilde{L}_v$, both points $(m, 0)$ and $(n, 0)$ are not in $B_{v'}$. This means in particular that $\Gamma_{(m,0)}$ and $\Gamma_{(n,0)}$ cannot intersect $\widetilde{L}_v$ and, since they coalesce, they must meet "above" v. In other words, v is in the bounded component of $V_H \setminus (\Gamma_{(m,0)} \cup \Gamma_{(n,0)})$ (viewing these paths only as their vertex sets). By planarity, Γ_v must intersect $\Gamma_{(m,0)}$. Because $\Gamma_{(m,0)}$ coalesces with Γ_w, this completes the proof.

So we move to proving coalescence starting from the first coordinate axis. We will prove by contradiction, so assume either (A) or (B) but that

(9) with positive probability, there are vertices $v, w \in L_0$ with $\Gamma_v \cap \Gamma_w = \varnothing$.

4.1. *Estimates on density of disjoint geodesics.*

4.1.1. *Definitions.* For each $k \in \mathbb{N} \cup \{0\}$ and $m, n \in \mathbb{Z}$ with $m < n$ define $N_{m,n}^{(k)}$ as the largest number N such that we can find vertices $v_1, \ldots, v_N \in [m, n] \times \{k\}$ such that:

(a) $\Gamma_{v_1}, \ldots, \Gamma_{v_N}$ are pairwise disjoint, and
(b) for all i, $\Gamma_{v_i} \cap [L_0 \cup \cdots \cup L_k] = \{v_i\}$.

Similarly, for $k \in \mathbb{N}$ let $M_{m,n}^{(k)}$ be the largest M such that we can find $v_1, \ldots, v_M \in [m, n] \times \{k\}$ such that (a) and (b) above hold but also (c) for all $i = 1, \ldots, M$, every $v \in L_0$ has $\Gamma_v \cap \Gamma_{v_i} = \varnothing$. See Figure 3 for an illustration of these definitions.

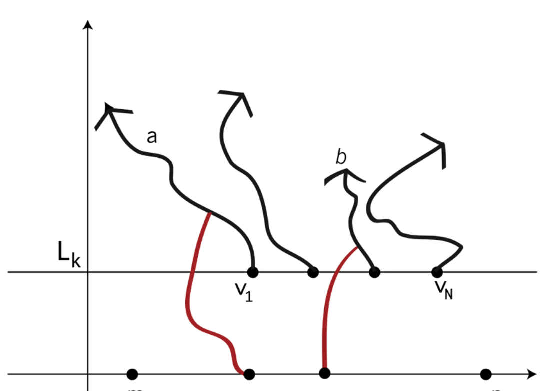

FIG. 3. *In this example $N_{m,n}^{(k)}$ is at least 4. The arrowed paths are geodesics emanating from vertices on the line L_k. They do not intersect each other, and they intersect L_k only at their initial points. The nonarrowed paths are segments of geodesics starting from L_0. Note that the initial points of a and b do not contribute to the random variable $M_{m,n}^{(k)}$.*

LEMMA 4.1. *For each $k_1 \in \mathbb{N} \cup \{0\}$ and $k_2 \in \mathbb{N}$, there exist deterministic $\alpha_{k_1}, \beta_{k_2} \geq 0$ such that*

$$\lim_{n\to\infty} \frac{N_{0,n}^{(k_1)}}{n} = \alpha_{k_1} \quad \textit{and} \quad \lim_{n\to\infty} \frac{M_{0,n}^{(k_2)}}{n} = \beta_{k_2} \qquad \textit{almost surely and in } L^1(\mathbb{P}).$$

We have the characterization

$$\alpha_{k_1} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}N_{0,n}^{(k_1)}}{n} \quad \textit{and} \quad \beta_{k_2} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}M_{0,n}^{(k_2)}}{n}.$$

Furthermore, assuming (9), $\alpha_0 > 0$.

PROOF. Note that for all $m < n < p$ in $\mathbb{Z}$ and $k_1 \in \mathbb{N} \cup \{0\}$, $k_2 \in \mathbb{N}$, we have

$$N_{m,p}^{(k_1)} \leq N_{m,n}^{(k_1)} + N_{n,p}^{(k_1)} \quad \text{and} \quad M_{m,p}^{(k_2)} \leq M_{m,n}^{(k_2)} + M_{n,p}^{(k_2)}.$$

Further $\max\{N_{m,n}^{(k_1)}, M_{m,n}^{(k_2)}\} \leq n - m + 1$ surely, so they have finite mean, and $(N_{m,n}^{(k_1)}, M_{m,n}^{(k_2)})$ has the same distribution as $(N_{0,n-m}^{(k_1)}, M_{0,n-m}^{(k_2)})$. Therefore we can apply Kingman's subadditive ergodic theorem to find deterministic $\alpha_{k_1}, \beta_{k_2} \geq 0$ such that

$$\frac{1}{n} N_{0,n}^{(k_1)} \to \alpha_{k_1} \quad \text{and} \quad \frac{1}{n} M_{0,n}^{(k_2)} \to \beta_{k_2} \qquad \text{almost surely and in } L^1(\mathbb{P}).$$

Furthermore, $\alpha_{k_1} = \inf_{n\in\mathbb{N}} \mathbb{E}N_{0,n}^{(k_1)}/n$ and $\beta_{k_2} = \inf_{n\in\mathbb{N}} \mathbb{E}M_{0,n}^{(k_2)}/n$.

We claim now that under assumption (9), $\alpha_0 > 0$. By countability and invariance of $\mathbb{P}$ under $T_{(1,0)}$, we can find $i_0 \in \mathbb{N}$ such that $\mathbb{P}(A(1, i_0)) > 0$, where $A(1, i_0)$ is the event that $\Gamma_{(1,0)}$ and $\Gamma_{(i_0,0)}$ do not intersect. Note that if $i_1 < i_2 < i_3 < i_4$ are integers such that $\Gamma_{(i_l,0)}$ and $\Gamma_{(i_{l+1},0)}$ are disjoint for $l = 1, 3$, then by planarity, at least three of them must be disjoint. So the ergodic theorem implies that with probability one, $A(1, i_0) \circ T_{(j,0)}$ occurs for infinitely many j and therefore we can find 4 geodesics starting from L_0 that are all disjoint. The middle two of these must intersect L_0 only finitely often. This implies that for some $j_0 \in \mathbb{N}$, $\mathbb{P}(B(1, j_0)) > 0$, where $B(1, j_0)$ is the event that $\Gamma_{(1,0)}$ and $\Gamma_{(j_0,0)}$ do not intersect and only touch L_0 at their initial points.

Again, by the ergodic theorem,

$$\frac{1}{N} \sum_{l=0}^{N} T_{(j_0,0)}^{l} 1_{B(1,j_0)} \to \mathbb{P}\big(B(1, j_0)\big) \qquad \text{almost surely and in } L^1(\mathbb{P}).$$

The reasoning given above, but applied to sets $\{j_1, j_2, \ldots\}$ of size bigger than 4, implies that for $n \in \mathbb{N}$,

$$N_{0,j_0 n}^{(0)} - 1 \geq \sum_{l=0}^{n} T_{(j_0,0)}^{l} 1_{B(1,j_0)}.$$

Dividing by $j_0 n$ and taking $n \to \infty$, we find $\alpha_0 \geq \mathbb{P}(B(1, j_0))/j_0 > 0$. $\square$

4.1.2. *Lower bound on α_k.*

PROPOSITION 4.2. *For each $k \in \mathbb{N}$, $\alpha_k \geq \beta_k + \alpha_0$.*

PROOF. For the proof we need a lemma stating that any geodesic starting at L_0 intersects L_k only finitely often.

LEMMA 4.3. *Assume (9). For each $v \in L_0$ and $k \in \mathbb{N}$, with probability one, the set $\Gamma_v \cap L_k$ is finite.*

PROOF. Assume that there exists $k \in \mathbb{N}$ such that with positive probability, there exists $v \in L_0$ with $\Gamma_v \cap L_k$ infinite. By countability and invariance of $\mathbb{P}$ under $T_{(1,0)}$,

$$\mathbb{P}(B) > 0 \qquad \text{where } B = \{\#(\Gamma_{(0,0)} \cap L_k) = \infty\}.$$

By Lemma 4.1, we can find $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}(N^{(0)}_{1,N_0+1} > k+2) > 1 - \mathbb{P}(B)/2$$

and then by translation invariance, with positive $\mathbb{P}$-probability, the event $B \cap \{N^{(0)}_{1,N_0+1} > k+2\} \cap \{N^{(0)}_{-1-N_0,-1} > k+2\}$ occurs. However any outcome in this event must have contradictory properties, as we now explain. Since B occurs, $\Gamma_{(0,0)}$ must intersect infinitely many vertices of either $L_k \cap \{(x,y) : x \geq 0\}$ or $L_k \cap \{(x,y) : x \leq 0\}$. Let us assume the first; the subsequent argument is similar in the other case. Then $\Gamma_{(0,0)}$ must be disjoint from at least $k+1$ different geodesics $\Gamma_{v_1}, \ldots, \Gamma_{v_{k+1}}$ with $v_i \in L_0 \cap [1, N_0+1]$ for all i, but it must intersect some vertex (x,k) for $x > N_0$. By planarity, the geodesics Γ_{v_i} must all intersect the set $\{(x,j) : 0 \leq j \leq k\}$, but then they cannot be disjoint. This is a contradiction. $\square$

Returning to the proof of Proposition 4.2, fix $k \in \mathbb{N}$. For each $m \in \mathbb{Z}$, define $d_k(m)$ as the first coordinate of the last vertex (by the natural ordering) on $\Gamma_{(m,0)}$ in the line L_k. This quantity exists almost surely by Lemma 4.3. For any $a, b \in \mathbb{Z}$ with $a < b$, define the set

$$X_{a,b} = \{j \in \mathbb{Z} : d_k(j) \in [a,b]\}.$$

We claim that for some fixed $N_0 \in \mathbb{N}$,

$$\mathbb{P}(X_{-N_0,n+N_0} \text{ contains } [0,n] \text{ for infinitely many } n \in \mathbb{N}) \geq 1/2. \tag{10}$$

To show this, first choose $N_0 \in \mathbb{N}$ such that $\mathbb{P}(|d_k(0)| \leq N_0) \geq 3/4$. Next note that by invariance of $\mathbb{P}$ under $T_{(1,0)}$, $\mathbb{P}(d_k(n) \leq n + N_0) \geq 3/4$ for all $n \in \mathbb{N}$. These two events occur simultaneously with probability at least $1/2$, so

$$\mathbb{P}(d_k(0) \geq -N_0 \text{ and } d_k(n) \leq n + N_0 \text{ for infinitely many } n \in \mathbb{N}) \geq 1/2.$$

Last, observe that by planarity and the fact that if two Γ's touch, they must merge, the function $m \mapsto d_k(m)$ is monotonic. This implies that if $d_k(0) \geq -N_0$ and $d_k(n) \leq n + N_0$, then the set $X_{-N_0,n+N_0}$ contains $[0, n]$.

The second step is to prove that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{N_{0,n}^{(k)} - M_{0,n}^{(k)}}{n} \geq \alpha_0\right) \geq 1/4. \tag{11}$$

Because $(N_{0,n}^{(k)} - M_{0,n}^{(k)})/n$ converges almost surely to $\alpha_k - \beta_k$, this suffices to complete the proof of the proposition. First, given $\varepsilon > 0$, by Lemma 4.1, pick N_1 such that

$$\mathbb{P}\big(N_{0,n}^{(0)}/n \geq \alpha_0 - \varepsilon \text{ for all } n \geq N_1\big) \geq 3/4.$$

On this event, for $n \geq N_1$, setting $a_n = \lfloor n(\alpha_0 - \varepsilon)\rfloor$, we may find $x_1^{(n)}, \ldots, x_{a_n}^{(n)}$ in $[0, n]$ such that the geodesics $\Gamma_{(x_1^{(n)},0)}, \ldots, \Gamma_{(x_{a_n}^{(n)},0)}$ are pairwise disjoint. If, in addition, the event in (10) occurs, then for infinitely many n, all of $d_k(x_1^{(n)}), \ldots, d_k(x_{a_n}^{(n)})$ are in $[-N_0, n + N_0]$. Note that the geodesics emanating from each of the points $(d_k(x_i^{(n)}), k)$ are disjoint and do not intersect $L_0 \cup \cdots \cup L_k$ except for their initial vertices. Next, choose a maximal set $\widehat{\Gamma}_1^{(n)}, \ldots, \widehat{\Gamma}_{M_{-N_0,n+N_0}^{(k)}}^{(n)}$ of geodesics starting in $[-N_0, n + N_0] \times \{k\}$ which are disjoint and intersect $L_0 \cup \cdots \cup L_k$ only at their initial vertices, and such that no $v \in L_0$ has $\Gamma_v \cap \widehat{\Gamma}_i^{(n)} \neq \varnothing$ for $i = 1, \ldots, M_{-N_0,n+N_0}^{(k)}$. Note that these $\widehat{\Gamma}$'s are disjoint from the geodesics starting from the points $(d_k(x_i^{(n)}), k)$. Therefore for each $n \geq N_1$, with probability at least $1/4$ we have $N_{-N_0,n+N_0}^{(k)} \geq a_n + M_{-N_0,n+N_0}^{(k)}$. Thus

$$\mathbb{P}\big(N_{-N_0,n+N_0}^{(k)} \geq a_n + M_{-N_0,n+N_0}^{(k)} \text{ for infinitely many } n\big) \geq 1/4.$$

By invariance of $\mathbb{P}$ under $T_{(1,0)}$,

$$\mathbb{P}\big(N_{0,n+2N_0}^{(k)} - M_{0,n+2N_0}^{(k)} \geq a_n \text{ for infinitely many } n\big) \geq 1/4.$$

Finally, as $(n + 2N_0)/n \to 1$ as $n \to \infty$ and ε is arbitrary, (11) holds. □

4.1.3. *Upper bound on α_k.* In this section we combine the lower bound from last section with an upper bound to conclude that $\beta_k = 0$. In what follows, we will denote by $G(x, y)$ the unique geodesic between x and y.

PROPOSITION 4.4. *For $k \in \mathbb{N}$, $\alpha_k \leq \alpha_0$. Therefore by Proposition 4.2, $\beta_k = 0$.*

We will couple together the upper half-plane with shifted half-planes. For any $k \in \mathbb{N}$ we consider the shifted configuration $T_{(0,k)}\omega$ and the unique geodesics

$G(v, (n, 0))$ in this configuration. Specifically, for any $\omega \in \Omega_H$ and $v \in V_H^k = \{(x, y) \in V_H : y \geq k\}$, we set

$$G_n^{(k)}(v) = T_{(0,-k)}\big[G\big(v - (0, k), (n, 0)\big)(T_{(0,k)}\omega)\big], \tag{12}$$

where for a path γ in $\mathbb{H}$ we denote by $T_{(0,-k)}\gamma$ the path γ shifted up by k units. By Theorem 1.3, there is an almost sure limit $G^{(k)}(v) = \lim_{n\to\infty} G_n^{(k)}(v)$.

LEMMA 4.5. *Let $k \in \mathbb{N}$. With probability one, for all $v \in L_k$, if $\Gamma_v \cap [L_0 \cup \cdots \cup L_{k-1}] = \varnothing$, then*

$$\Gamma_v = G^{(k)}(v).$$

PROOF. Let $v \in L_k$ such that $\Gamma_v \cap [L_0 \cup \cdots \cup L_{k-1}] = \varnothing$ and write it as $v = (v_1, v_2)$. Let σ be the nonself intersecting continuous curve obtained by concatenating (a) the edges of Γ_v, (b) the vertical line segment connecting $(v_1, -1/2)$ and v and (c) the ray $\{(x, -1/2) \in \mathbb{R}^2 : x \geq v_1\}$. One component of the complement of σ contains all vertices of L_{k-1} to the right of $v - (0, 1)$, and the other contains all vertices of L_{k-1} to the left of $v - (0, 1)$; call the first C_1 and the second C_2. Because the sequence $G(v, (n, 0))$ converges to Γ_v as $n \to \infty$, there exists N_0 such that if $n \geq N_0$, then $G(v, (n, 0))$ does not contain any vertices of the form (v_1, y) for $y < v_2$. For $n \geq N_0$ the geodesic $G(v, (n, 0))$ cannot contain any vertices in C_2. For if it did, it would start at v, go through a vertex in C_2, and then touch $(n, 0)$, a vertex in C_1. Because this geodesic cannot cross $\{(v_1, y) : y < v_2\}$, it must cross Γ_v and violate unique passage times.

For $n \geq N_0$, let w_n denote the first intersection of $G(v, (n, 0))$ with L_{k-1}. The vertex v_n directly before this must be in L_k, and the segment γ_n of $G(v, (n, 0))$ from v to v_n has all vertices in V_H^k. Therefore writing $v_n = (a_n, k)$, we have $\gamma_n = G_{a_n}^{(k)}(v)$. Because Γ_v does not intersect $L_0 \cup \cdots \cup L_{k-1}$, $\|w_n\|_1 \to \infty$. However w_n is in C_1, so $a_n \to +\infty$. Taking n to infinity, these segments converge to $G^{(k)}(v)$. However they converge to Γ_v. $\square$

For $n \in \mathbb{N}$, choose $r = N_{0,n}^{(k)}$ pairwise disjoint geodesics $\Gamma_{v_1}, \ldots, \Gamma_{v_r}$ for $v_1, \ldots, v_r \in [0, n] \times \{k\}$ such that for each $i = 1, \ldots, r$, $\Gamma_{v_i} \cap [L_0 \cup \cdots \cup L_k] = \{v_i\}$. By Lemma 4.5, $r \leq N_{0,n}^{(0)}(T_{(0,k)}(\omega))$. Therefore

$$\frac{N_{0,n}^{(k)}(\omega)}{n} \leq \frac{N_{0,n}^{(0)}(T_{(0,k)}(\omega))}{n} \qquad \text{for all } n \in \mathbb{N}.$$

Taking $n \to \infty$ and using invariance of $\mathbb{P}$ under $T_{(0,k)}$, we find $\alpha_k \leq \alpha_0$.

4.2. *Deriving a contradiction.* In this section we will show that assuming (9), there exists $k \geq 1$ such that $\beta_k > 0$. This will contradict Proposition 4.4 and complete the proof of coalescence starting from the first-coordinate axis.

4.2.1. *Lemmas for edge modification.* The first lemma will let us apply an edge modification argument. For a typical element ω and edge $e \in E_H$ we write $\omega = (\omega_e, \check{\omega})$. We say an event $A \subset \Omega_H$ is *e-increasing* if, for all $(\omega_e, \check{\omega}) \in A$ and $r > 0$, $(\omega_e + r, \check{\omega}) \in A$. The following is a weaker version of [6], Lemma 6.6, and uses the upward finite energy property.

LEMMA 4.6. *Let $\lambda > 0$ be such that $\mathbb{P}(\omega_e \geq \lambda) > 0$. If $A \subset \Omega_H$ is e-increasing with $\mathbb{P}(A) > 0$, then*

$$\mathbb{P}(A, \omega_e \geq \lambda) > 0.$$

PROOF. We estimate

$$\begin{aligned}\mathbb{P}(A, \omega_e \geq \lambda) &= \mathbb{E}\big[\mathbb{E}\big[1_A(\omega_e, \check{\omega})1_{\{\omega_e \geq \lambda\}}|\check{\omega}\big]\big]\\ &\geq \mathbb{E}\big[1_A(\lambda, \check{\omega})\mathbb{P}(\omega_e \geq \lambda|\check{\omega})\big].\end{aligned}$$

Because A is e-increasing, the variable $1_A 1_{\{\omega_e \leq \lambda\}}$ is less than or equal to the random variable $1_A(\lambda, \check{\omega})$. Therefore if the statement of the lemma is false, then $1_A(\lambda, \check{\omega})$ is positive on a set of positive probability. By the upward finite energy property, $\mathbb{P}(\omega_e \geq \lambda|\check{\omega})$ is positive almost surely, so the above estimates give $\mathbb{P}(A, \omega_e \geq \lambda) > 0$, a contradiction. $\square$

The second lemma is a shape theorem-type upper bound. For it, we define

$$\lambda_0^+ = \sup\{\lambda \geq 0 : \mathbb{P}(\omega_e \geq \lambda) > 0\}. \tag{13}$$

LEMMA 4.7. *Suppose that $\lambda_0^+ < \infty$. There exists $c^+ < \lambda_0^+$ such that*

$$\mathbb{P}\big(\tau(0, x) \leq c^+\|x\|_1 \textit{ for all but finitely many } x \in V_H\big) = 1.$$

PROOF. Because $\mathbb{P}$ has unique passage times, the marginal of ω_e is not concentrated at a point and therefore $\mathbb{E}\omega_e < \lambda_0^+$. For any $x \in V_H$ choose a deterministic path $\gamma_x : 0 \rightsquigarrow x$ in $\mathbb{H}$ with $\|x\|_1$ number of edges. Then

$$\mathbb{E}\tau(0, x) \leq \mathbb{E}\tau(\gamma_x) = \|x\|_1 \mathbb{E}\omega_e.$$

We now set $c^+ = \frac{\mathbb{E}\omega_e + \lambda_0^+}{2}$ and argue that this value satisfies the condition of the lemma. The argument will be similar to the proof of the shape theorem in the full space.

For any $z \in \mathbb{Q}^2$ with second coordinate nonnegative, let N be any natural number such that $Nz \in V_H$. Then for $n \in \mathbb{N}$, write $n = \lfloor \frac{n}{N} \rfloor + r$, where $0 \leq r < N$ and estimate

$$\tau(0, nz) \leq N\lambda_0^+\|z\|_1 + \sum_{i=0}^{\lfloor n/N \rfloor - 1} \tau(0, Nz)\big(T_{Nz}^i \omega\big).$$

Divide by n and use the ergodic theorem to find

$$\limsup_{n\to\infty} \frac{\tau(0, nz)}{n} \le \frac{\mathbb{E}\tau(0, Nz)}{N} \le \|z\|_1 \mathbb{E}\omega_e. \tag{14}$$

Let Ω'_H be the full-probability event on which (14) holds for all $z \in \mathbb{Q}^2$ with second coordinate nonnegative. Assume by way of contradiction that on some positive probability event A, the lemma does not hold for the c^+ fixed above. Then we can find $\omega \in A \cap \Omega'_H$; we will show that this ω has contradictory properties.

Let (z_n) be a sequence of vertices in V_H such that $\|z_n\|_1 \to \infty$ and

$$\tau(0, z_n) > c^+ \|z_n\|_1 \qquad \text{for all } n \in \mathbb{N}.$$

By compactness (and by restricting to a subsequence), given a positive a such that $a\lambda_0^+ < c^+ - \mathbb{E}\omega_e$, we can find some $z \in \mathbb{Q}^2$ with second coordinate nonnegative and

$$\|z\|_1 = 1 \text{ such that } \left\| \frac{z_n}{\|z_n\|_1} - z \right\|_1 < a \qquad \text{for all } n \in \mathbb{N}.$$

Then we can estimate

$$\tau(0, z_n) \le \tau\big(0, \|z_n\|_1 z\big) + \tau\big(\|z_n\|_1 z, z_n\big) \le \tau\big(0, \|z_n\|_1 z\big) + \big\| \|z_n\|_1 z - z_n \big\|_1 \lambda_0^+.$$

Therefore

$$c^+ < \frac{\tau(0, z_n)}{\|z_n\|_1} \le \frac{\tau(0, \|z_n\|_1 z)}{\|z_n\|_1} + \left\| z - \frac{z_n}{\|z_n\|_1} \right\|_1 \lambda_0^+.$$

Taking limsup on the right-hand side gives $c^+ \le \mathbb{E}\omega_e + a\lambda_0^+$, a contradiction. □

The final lemma deals with spatial concentration of geodesics emanating from the first coordinate axis. For $v_1, v_2, v_3 \in L_0$ let $B(v_1, v_2, v_3)$ be the event that:

(1) the geodesics $\Gamma_{v_1}, \Gamma_{v_2}$ and Γ_{v_3} are disjoint;
(2) they intersect L_0 only at their initial points;
(3) their intersection with each L_k is finite.

We will also need a subevent of $B(v_1, v_2, v_3)$. Let

$$B^G(v_1, v_2, v_3) = \left\{ \begin{array}{c} B(v_1, v_2, v_3) \text{ occurs and for each } \varepsilon > 0, \\ \text{there are infinitely many } k \in \mathbb{N} \text{ such that} \\ \text{the last intersections } \zeta_k \text{ and } \zeta'_k \text{ of} \\ \Gamma_{v_1} \text{ and } \Gamma_{v_3} \text{ with } L_k \text{ have } \|\zeta_k - \zeta'_k\|_1 < \varepsilon k \end{array} \right\}.$$

LEMMA 4.8. *Suppose* $v_1 = (x_1, 0)$, $v_2 = (x_2, 0)$ *and* $v_3 = (x_3, 0)$ *with* $x_1 < x_2 < x_3$. *Then* $\mathbb{P}(B^G(v_1, v_2, v_3) | B(v_1, v_2, v_3)) = 1$.

PROOF. For $z \in L_0$ and $k \in \mathbb{N}$, denote by $\zeta_k(z)$ the last point of intersection of Γ_z with L_k, which exists almost surely by Lemma 4.3. Take $v = v_3 - v_1$ and consider

$$C_k(v) = \{\|\zeta_k(v) - \zeta_k(0)\|_1 \geq \varepsilon k\}.$$

For $k, n \in \mathbb{N}$, define $X_n^{(k)} = \sum_{j=0}^{n-1} 1_{C_k(v)}(T_{(jd,0)}(\omega))$, where $d = \|v\|_1 + 1$. By the ergodic theorem, putting $p_k = \mathbb{P}(C_k(v))$,

$$X_n^{(k)}/n \to p_k \qquad \text{almost surely.} \tag{15}$$

As previously stated in the paper, for $l \in \mathbb{Z}$ and $k \in \mathbb{N}$, define $d_k(l)$ as the first coordinate of $\zeta_k(l)$, and note that by planarity, $d_k(l)$ is monotone in l. Therefore for $n \in \mathbb{N}$, the difference $d_k(nd) - d_k(0)$ is at least equal to $\varepsilon k X_n^{(k)}$, so

$$\frac{d_k(nd) - nd - d_k(0)}{n} \geq \frac{\varepsilon k X_n^{(k)} - nd}{n} = \varepsilon k X_n^{(k)}/n - d.$$

Combining with (15), almost surely,

$$\liminf_{n\to\infty} \frac{d_k(nd) - nd - d_k(0)}{n} \geq \varepsilon k p_k - d.$$

Because $d_k(nd) - nd$ and $d_k(0)$ have the same distribution, $(d_k(nd) - nd - d_k(0))/n \to 0$ in probability. Therefore

$$p_k \leq d/(\varepsilon k),$$

giving $p_k \to 0$. In particular, with probability one, $C_k(v)^c$ occurs for infinitely many k. □

4.2.2. *Main argument*. We will first assume that $\lambda_0^+ < \infty$ and that (9) holds. By Proposition 4.2, $\alpha_0 > 0$ and so we can find v_1, v_2, v_3 and $p > 0$ such that $\mathbb{P}(B(v_1, v_2, v_3)) \geq p$, where this event was defined before Lemma 4.8. Fix any positive

$$\varepsilon < \frac{\lambda_0^+ - c^+}{8\lambda_0^+}. \tag{16}$$

We first define a modified event which combines conditions from the previous section. Specifically, for $k \in \mathbb{N}$ we set $B'(k) = B'(v_1, v_3; k)$ as the event that:

(1) the geodesics Γ_{v_1} and Γ_{v_3} are disjoint and intersect L_j in a finite set for all $j \in \mathbb{N} \cup \{0\}$;

(2) writing $w_1 = w_1(k)$ and $w_3 = w_3(k)$ for the last intersections of Γ_{v_1} and Γ_{v_3} with L_k, there is a vertex x^* in L_k between w_1 and w_3 such that Γ_{x^*} is disjoint from Γ_{v_1} and Γ_{v_3}, and Γ_{x^*} intersects L_k only at x^*;

(3) the finite geodesics $r_1(k)$ and $r_3(k)$, defined as the segments of $\Gamma_{v_1}, \Gamma_{v_3}$ from L_0 to each of w_1 and w_3 satisfy $\tau(r_i(k)) \leq c^+\|v_i - w_i\|_1$ for $i = 1, 3$;

(4) $\|w_1 - w_3\|_1 < \varepsilon k$. (See Figure 4.)

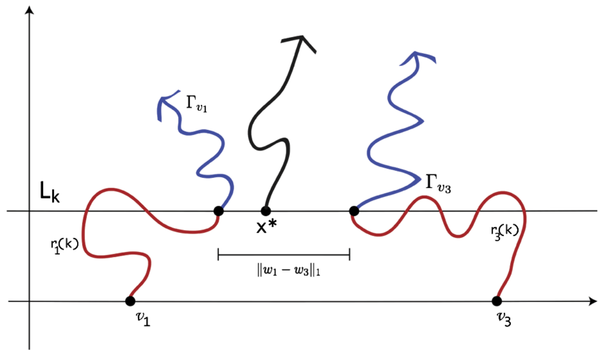

FIG. 4. *The event $B'(k)$. The geodesics $\Gamma_{v_i}, i = 1, 3$, are the left and right paths. The central geodesic Γ_{x^*} does not intersect either Γ_{v_1} or Γ_{v_3} and intersects L_k only at x^*. The initial segments of Γ_{v_1} and Γ_{v_3} satisfy $\tau(r_i(k)) \le c^+\|v_i - w_i\|_1$ while $\|w_1 - w_3\|_1 < \varepsilon k$.*

The first two conditions hold together for all k simultaneously with probability at least p. This is because whenever $B(v_1, v_2, v_3)$ occurs, almost surely each Γ_{v_i} intersects each L_k in a finite set, so we can let x^* be the last intersection point of Γ_{v_2} with L_k. Next, by Lemma 4.7 we can find k_0 such that

$$\mathbb{P}\left(\tau(v_i, w) \le c^+\|v_i - w\|_1 \text{ for all } i = 1, 3 \text{ and } w \in \bigcup_{k=k_0}^{\infty} L_k\right) > 1 - p/2.$$

This implies that the first three conditions hold for all $k \ge k_0$ with probability at least $p/2$. Using Lemma 4.8,

$$\mathbb{P}\big(B'(k)\big) > 0 \qquad \text{for infinitely many } k \ge k_0. \tag{17}$$

We then fix any such $k \ge k_0$ with

$$4\|v_3 - v_1\|_1 \lambda_0^+ < \frac{\lambda_0^+ - c^+}{2} k. \tag{18}$$

Next we modify the edge-weights for a set of edges between the geodesics Γ_{v_1} and Γ_{v_3}. For any configuration ω in $B'(k)$ write X_1 for the closed subset of $\mathbb{R}^2$ with boundary curves Γ_{v_1}, Γ_{v_3} and the segment of the first coordinate axis between v_1 and v_3. Let X_2 be the component of $X_1 \cap \{(x, y) \in \mathbb{R}^2 : 0 \le y \le k\}$ containing v_1. Last, define the set $X \subset E_H$ consisting of all edges not in Γ_{v_1} or Γ_{v_3} but such that both endpoints are in X_2. Because there are only countably many choices, (17) implies there is a deterministic choice X' and a vertex $y \in L_k$ such that

$$\mathbb{P}\big(B'(k), X = X', x^* = y\big) > 0. \tag{19}$$

Here the notation $x^* = y$ means that the (deterministic) vertex y satisfies condition (2) of the definition of $B'(k)$.

We next show that

$$\mathbb{P}\biggl(B'(k), X = X', x^* = y, \bigcap_{e \in X'} \biggl\{\omega_e \geq \frac{c^+ + \lambda_0^+}{2}\biggr\}\biggr) > 0. \tag{20}$$

To prove this we enumerate the edges $e_1, \dots, e_r$ of X' and repeatedly apply Lemma 4.6. By (19), we simply need to verify that for all $j = 2, \dots, r$,

$$B'(k) \cap \{X = X', x^* = y\} \cap \bigcap_{i=1}^{j-1} \biggl\{\omega_{e_i} \geq \frac{c^+ + \lambda_0^+}{2}\biggr\} \qquad \text{is } e_j\text{-increasing.}$$

So take ω in the event on the left for some $j = 2, \dots, r$ with ω' such that $\omega'_f = \omega_f$ for $f \neq e_j$ and $\omega'_{e_j} \geq \omega_{e_j}$. First we claim that Γ_{v_1}, Γ_y and Γ_{v_3} are unchanged from ω to ω'. To see this, note that since e_j is not in Γ_{v_1}, Γ_y or Γ_{v_3} we can find $n_1 = n_1(\omega)$ such that if $n \geq n_1$ then e_j is also not in any of the geodesics $G(v_1, (n, 0))$, $G(y, (n, 0))$ or $G(v_3, (n, 0))$ in ω. Therefore these remain geodesics in ω'; taking the limit as $n \to \infty$ proves the claim. Now it is clear that $X = X'$ in ω' and conditions (1)–(4) of $B'(k)$ hold in ω'. Obviously if $\omega_{e_i} \geq (1/2)(c^+ + \lambda_0^+)$ for $i = 1, \dots, j-1$ in ω, then this is still true in ω'. This proves (20).

On the event in (20), no point $v \in L_0$ can have $\Gamma_v \cap \Gamma_y \neq \varnothing$. We will now argue for this fact and explain why it leads to a contradiction. If such a v exists, it must be on the segment of L_0 strictly between v_1 and v_3; this is a direct consequence of planarity and the fact that each vertex in $\mathbb{G}_H$ has out degree one. Therefore Γ_v must start at L_0 and use only edges in X' until its exit from $L_0 \cup \cdots \cup L_k$. Writing w for the first vertex of Γ_v in L_k, we must then have

$$\tau(v, w) \geq \frac{c^+ + \lambda_0^+}{2} \|v - w\|_1. \tag{21}$$

On the other hand, we can give an upper bound for the passage time from v to w by taking the path obtained by concatenating (a) the segment of L_0 from v to v_1, (b) the geodesic r_1 and (c) the segment of L_k from w_1 to w. We get the bound

$$\begin{aligned} \tau(v, w) &\leq [\|v_3 - v_1\|_1 + \varepsilon k]\lambda_0^+ + c^+\|v_1 - w_1\|_1 \\ &\leq 2[\|v_3 - v_1\|_1 + \varepsilon k]\lambda_0^+ + c^+\|v - w\|_1. \end{aligned}$$

Combining this with (21), we find

$$(\lambda_0^+ - c^+)k \leq 4[\|v_3 - v_1\|_1 + \varepsilon k]\lambda_0^+.$$

This contradicts (16) and (18).

To summarize, we have now shown that for some fixed $w_1, w_2, w_3 \in L_k$ such that the segment of L_k between w_1 and w_3 contains w_2, $C = C(w_1, w_2, w_3)$ has positive probability, where this event is defined by the conditions:

(1) $\Gamma_{w_1}, \Gamma_{w_2}$ and Γ_{w_3} are disjoint and intersect $L_0 \cup \cdots \cup L_k$ only in w_1, w_2 and w_3, respectively, and

(2) no $v \in L_0$ has $\Gamma_{w_2} \cap \Gamma_v \neq \varnothing$.

Fix any $m, n \in \mathbb{Z}$ with $m < n$ and $w_1, w_3 \in [m, n] \times \{k\}$. Let $l \in \mathbb{N}$ be bigger than $\|w_3 - w_1\|_1$, and recall the notation $M^{(k)}_{m,n}$ from Section 4.1.1. Note that if $C \cap T_{(l,0)}C$ occurs, then $M^{(k)}_{m,n+l} \geq 2$. Iterating this reasoning, for any $j \in \mathbb{N}$,

$$M^{(k)}_{m,n+jl}(\omega) \geq \sum_{i=0}^{j-1} 1_C\big(T^i_{(l,0)}\omega\big).$$

Diving by j and using the ergodic theorem gives $\beta_k > 0$, a contradiction. This proves that assumption (9) is false in the case $\lambda_0^+ < \infty$ and thus all geodesics starting from L_0 coalesce.

In the case that $\lambda_0^+ = \infty$, the argument is much easier, and we will just explain the idea. If (9) holds, then we still find v_1, v_2, v_3 in L_0 with v_2 in the segment of L_0 between v_1 and v_3 and such that the Γ_{v_i}'s are disjoint and intersect L_0 in only v_1, v_2 and v_3. Again pick y as the last intersection point of Γ_{v_2} with L_1. Letting S be the set of edges touching any vertex of L_0 between v_1 and v_3 (and therefore not in Γ_{v_1} or Γ_{v_3}), we then modify the edge-weights for edges in S to be larger than some $C_{\text{big}} > 0$. Using Lemma 4.6 we can find C_{big} large enough so that on this event, no vertex v of L_0 can have $\Gamma_v \cap \Gamma_y \neq \varnothing$. As before, this implies $\beta_1 > 0$, a contradiction.

APPENDIX A: DUAL EDGE BOUNDARY OF V

For any set $V_1 \subseteq \mathbb{Z}^2$, let F be the edge boundary of V_1,

$$F = F(V_1) = \big\{\{x, y\} : x \in V_1, y \in V_1^c\big\}.$$

PROPOSITION A.1. *Let $V_1 \subseteq \mathbb{Z}^2$ be infinite, connected and such that V_1^c is infinite and connected. The dual edge set F^* consists of a single doubly infinite dual path which is nonself intersecting. That is, it is connected and infinite, and each dual vertex v^* in W^*, the set of endpoints of dual edges in F^*, has degree exactly 2 in the connected infinite graph $G^* = (W^*, F^*)$.*

PROOF. Assume first that G^* has a cycle. We can then extract from this cycle a self-avoiding one, whose parametrization yields a Jordan curve. This curve must contain a vertex of $\mathbb{Z}^2$ in its interior, showing that either V_1 or V_1^c must be finite, a contradiction.

Next we prove that each dual vertex $v^* \in W^*$ has degree 2 in G^*. If v^* has degree 1, then it has one incident dual edge $e^* \in F^*$, and this is dual to an edge $e \in F$. One endpoint of e is in V_1 and one is in V_1^c, but they can be connected outside of F using the 3 other edges dual to those which have v^* as an endpoint, a contradiction. This means each $v^* \in W^*$ has degree at least 2 in G^*. However if v^* has degree at least 3 in G^*, then three such dual edges e_1^*, e_2^* and e_3^* incident to v^* are the first edges of disjoint self-avoiding infinite dual paths P_1, P_2, P_3. These paths split $\mathbb{Z}^2$ into at least 3 components, violating the fact that $(\mathbb{Z}^2, \mathcal{E}^2) \setminus F$ has two components.

Last we must show that G^* is connected. Indeed, if G^* were not connected, it would have two components G_1^*, G_2^* (and possibly others). Since each dual vertex of G_i^* must have degree two, and since there can be no cycles, G_1^* and G_2^* must be disjoint, self-avoiding, doubly infinite dual paths. But this breaks $\mathbb{Z}^2$ into at least three components, a contradiction. □

APPENDIX B: EXISTENCE OF GEODESICS

In this section, we prove that if $\mathbb{P}$ is a product measure and x and y are arbitrary vertices of V, then there almost surely exists a (finite) geodesic between x and y. For $V = \mathbb{Z}^2$ this was proved by Wierman and Reh [21]; for general d, this appears to be open; see the remark under Theorem 8.1.8 in [22]. The proof will rely on the following "partial shape theorem."

LEMMA B.1. *Assume that* $\mathbb{P}(\omega_e = 0) < 1/2$. *Then, with probability one,*

$$\liminf_{\|x\|_1 \to \infty} \frac{\tau(0, x)}{\|x\|_1} > 0.$$

PROOF. Because (V, E) is a subgraph of $(\mathbb{Z}^2, \mathcal{E}^2)$, it suffices to show the lemma in the first-passage model on $\mathbb{Z}^2$. So let (ω_e) be a passage time realization on $\mathcal{E}^2$, and define the truncated $\hat{\omega}_e = \min\{\omega_e, 1\}$, with $\hat{\tau}$ the passage time in the environment $(\hat{\omega}_e)$. Then by the shape theorem (see [17], Theorem 1, and the references therein), the lemma holds for $\hat{\tau}$. However, $\tau \geq \hat{\tau}$, so we are done. □

THEOREM B.2. *Let x and y be elements of V. Then, almost surely, there exists a geodesic* $\gamma : x \rightsquigarrow y$.

PROOF. The proof will be broken up into two cases, depending on the probability that $\omega_e = 0$. In both cases, we will show that if we write for $N \in \mathbb{N}$,

$$\tau_N(x, y) = \min_{\substack{\gamma : x \rightsquigarrow y \\ \gamma \subseteq (x + [-N, N]^2) \cap V}} \tau(\gamma),$$

then

$$\mathbb{P}\bigl(\tau_N(x, y) = \tau(x, y) \text{ for all large } N\bigr) = 1. \tag{22}$$

This suffices to prove the theorem, as a function on a finite set attains its minimum.

Case I: $\mathbb{P}(\omega_e = 0) < 1/2$. In this case, we fix some deterministic path γ_0 in V connecting x and y and define $N = N(\tau(\gamma_0))$ to be the smallest number such that

$$\min_{z \in V \setminus (x + [-N, N]^2)} \tau(x, z) > \tau(\gamma_0).$$

Note that N is almost surely finite by Lemma B.1. Then no path containing a vertex of $V \setminus (x + [-N, N]^2)$ can have passage time less than or equal to $\tau(x, y)$. In particular, (22) holds.

Case II: $\mathbb{P}(\omega_e = 0) \geq 1/2$. Choose a deterministic $N_0 > 1$ such that there exists a path connecting x and y lying entirely in $[-N_0, N_0]^2 \cap V$. We will consider $\mathbb{P}$ to actually be defined on $\mathbb{R}^{\mathcal{E}^2}$, though of course the weights of edges outside of E will have no bearing on the first-passage model in (V, E).

Consider a sequence of annuli $A_n \subseteq \mathbb{R}^2$ of the form

$$A_n = [-N_0^{n+1}, N_0^{n+1}]^2 \setminus (-N_0^n, N_0^n)^2;$$

denote by G_n the event that there is a (vertex) self-avoiding circuit α in A_n of edges e such that $\omega_e = 0$. By the RSW theorem for independent percolation (see [4], Section 3.1), we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} G_n\right) = 1.$$

For any $N \in \mathbb{N}$ write $L_N = N_0^{N+1}$. For a given ω such that G_N occurs, choose α as above, and consider it as a continuous plane curve. Further, let γ be any vertex self-avoiding path in (V, E) from x to y. We will show that there exists another path γ' in $[-L_N, L_N]^2$ from x to y such that $\tau(\gamma') \leq \tau(\gamma)$. This suffices to complete the proof. To do so, we use the following construction. Let β be any path from x to y in (V, E) lying entirely in $[-N_0, N_0]^2$. Since γ intersects β at x and y we may list their common vertices in order (along γ) as $x = x_1, \ldots, x_k = y$. We proceed along γ from each x_i to x_{i+1}, calling this subpath γ_i. If γ_i is not just one edge of β, we create a Jordan curve C by concatenating the portion of β from x_i to x_{i+1} with γ_i. If α intersects the interior of C, then we choose any common point p and proceed in both directions along α from it. In each direction we must meet C again; otherwise α was in the interior of C, which is false. Furthermore we meet C before we meet Υ, since Υ is in the exterior of C. Therefore the component of $\alpha \cap \operatorname{int} C$ containing p is a segment of α from some vertex a to another b. Since a and b are in C, they must be in γ_i, and we can replace the segment of γ_i from a to b with this segment of α. In this way we obtain a new path we call $\tilde{\gamma}_i$ and corresponding Jordan curve $\widetilde{C}$. Note that $\tau(\tilde{\gamma}_i) \leq \tau(\gamma_i)$. See Figure 5 for a depiction of this procedure.

It remains to show that the procedure defined above eventually terminates in some path $\hat{\gamma}_i$ and Jordan curve $\widehat{C}$. At this point α will not intersect the interior of $\widehat{C}$, implying that $\hat{\gamma}_i$ does not leave $[-L_N, L_N]^2$. To prove this, assume that $p \in \alpha \cap \operatorname{int} C$ and define a and b as above. Let σ_i be the segment of γ_i from a to b. If σ_i does not leave α, then it must be the complementary segment of α from a to b, implying that $\alpha \subset (C \cup \operatorname{int} C)$. Then $\operatorname{int} \alpha \subset \operatorname{int} C$, a contradiction, since β is in the interior of α. Therefore we can find some edge adjacent to α in σ_i. When we construct $\hat{\gamma}_i$, we remove this edge from γ_i and only add edges of α. Since there are only finitely many edges adjacent to α, the process terminates. $\square$

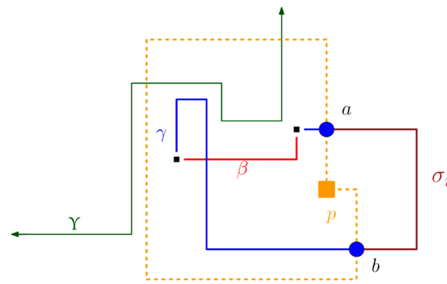

FIG. 5. *Modifying the path γ by replacing a segment σ_i of γ with a segment of α. In the figure, α is the dotted path and p is a point on α in the interior of C, the Jordan curve formed by the union of γ_i with β.*

APPENDIX C: ABSENCE OF BIGEODESICS IN $\mathbb{H}$

In this section we outline the modifications needed to carry over the proof of the main theorem of [20] to our setting. An infinite geodesic indexed by $\mathbb{Z}$ is called a bigeodesic. When we assume unique passage times, such a path is (vertex) self-avoiding.

C.1. Lemmas from Wehr–Woo. Assume either (A) or (B), and let K^* be the event

$$K^* = \{\text{there exists a bigeodesic}\}.$$

Note that for all x, $\mathbb{P}(\#B_x = \infty, (K^*)^c) = 0$, where B_x was defined in Theorem 1.5. By horizontal translation ergodicity, $\mathbb{P}(K^*)$ is zero or one; let us assume for a contradiction that $\mathbb{P}(K^*) = 1$.

Any bigeodesic γ divides $\mathbb{R}^2 \setminus \gamma$ into two components, say $R^+ = R^+(\gamma)$ and $R^- = R^-(\gamma)$; that is,

$$\begin{aligned} R^+(\gamma) \cap R^-(\gamma) &= \varnothing, \\ R^+(\gamma) \cup R^-(\gamma) &= \mathbb{R}^2 \setminus \gamma, \\ \partial R^+ = \partial R^- &= \gamma, \end{aligned}$$

where R^- is a region that contains $(0, -1)$ and where ∂A denotes the usual boundary of a set $A \subset \mathbb{R}^2$. Hence by unique passage times, for any points $x, y \in R^-(\gamma)$, no bond b belonging to the finite geodesic $G(x, y)$ can be an element of $R^+(\gamma)$. The following is [20], Proposition 4.

PROPOSITION C.1. *Consider the sequence $G((-n, 0), (n, 0))$ for $n \in \mathbb{N}$. With probability* 1, *this sequence has a limit*

$$\gamma_0 = \lim_{n\to\infty} G\big((-n, 0), (n, 0)\big).$$

Moreover, γ_0 is a bigeodesic, and for any bigeodesic γ,

$$\gamma_0 \subset [R^-(\gamma) \cup \gamma].$$

PROOF. The same proof as in [20] works here. The only assumption needed is that of unique passage times. □

The next is [20], Lemma 5.

LEMMA C.2. *Let $n \in \mathbb{N}$ and $\mathbb{H}' = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le n\}$. With probability 1, for any bigeodesic γ intersecting $z = (z_1, z_2)$ with $z_2 < n$,*

$$\mathbb{H}' \cap R^+(\gamma) \neq \varnothing \text{ and all its components are bounded.}$$

The boundary of each component is a self-avoiding loop, which is a bond-disjoint union of segments of γ and segments of the boundary of $\mathbb{H}'$.

PROOF. Because we do not assume independence of the variables (ω_e), we must modify the proof of [20], replacing independence with the upward finite energy property.

In order to prove the boundedness of each component of $\mathbb{H}' \cap R^+(\gamma)$, it is sufficient to prove that

$$\mathbb{P}(\text{there is a bigeodesic with an infinite connected part in } \mathbb{H}') = 0. \tag{23}$$

For each $k \in \mathbb{Z}$ consider a rectangular box

$$C_k = C_k(m, n) = \{(x_1, x_2) : 2km \le x_1 \le (2k+1)m, 0 \le x_2 \le n\}.$$

Let T_k be the minimum passage time of all paths in C_k which start at a vertex in the left boundary of C_k and end at a vertex in the right boundary of C_k, without intersecting the top boundary. Let $\widehat{C}_k$ for the set of edges in ∂C_k that do not lie on the first coordinate axis; then set

$$E_k = \left\{ \sum_{e \in \widehat{C}_k} \tau_e < T_k \right\}.$$

We claim that for some m large enough, $\mathbb{P}(E_k) > 0$ for all k. To prove this, we consider two cases. Assume first that λ_0^+, defined in (13), is finite. Then by the ergodic theorem, writing $e_k = \{(k, 0), (k+1, 0)\}$, $(1/m) \sum_{k=0}^{m-1} \omega_{e_k} \to \mathbb{E}\omega_e$. Therefore, using the bound $\omega_e \le \lambda_0^+$,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{e \in \widehat{C}_0} \omega_e = \mathbb{E}\omega_e.$$

As $\mathbb{P}$ has unique passage times, $\mathbb{E}\omega_e < \lambda_0^+$, so choose m such that

$$\mathbb{P}\biggl(\sum_{e \in \widehat{C}_0} \omega_e < \frac{\mathbb{E}\omega_e + \lambda_0^+}{2} m\biggr) > 0.$$

Writing C_k^0 for the set of edges with an endpoint in $C_k \setminus \widehat{C}_k$, we see that the above event is e-increasing for all $e \in C_0^0$. So by Lemma 4.6,

$$\mathbb{P}\biggl(\sum_{e \in \widehat{C}_0} \omega_e < \frac{\mathbb{E}\omega_e + \lambda_0^+}{2} m, \omega_f \geq \frac{\mathbb{E}\omega_e + \lambda_0^+}{2} \text{ for all } f \in C_0^0\biggr) > 0.$$

On this event, each path which passes from the left to the right-hand side of C_0, taking only edges in C_0^0, must have passage time at least $\frac{\mathbb{E}\omega_e + \lambda_0^+}{2} m$. So for such m, horizontal translation invariance gives $\mathbb{P}(E_k) > 0$.

In the case that $\lambda_0^+ = \infty$, the proof of $\mathbb{P}(E_k) > 0$ is easier. We simply modify the edge-weights for edges in C_0^0 to be larger than the sum of the boundary edge-weights with positive probability. In either case, the ergodic theorem shows that

$$\mathbb{P}(E_k \text{ occurs for infinitely many } k > 0 \text{ and } k < 0) = 1.$$

For any k such that E_k occurs, no geodesic can pass from the left-hand to the right-hand side of C_k taking only edges in C_k^0, because we can replace the segment between the left-hand and right-hand sides by a portion of the boundary ∂C_0. This shows (23). The rest of the lemma follows immediately. $\square$

We now move to [20], Proposition 6, the main observation showing that unique passage times implies that γ_0 must intersect any large box with probability bounded below uniformly of the position of the box. For $l \in \mathbb{N}$, let us write $B = B(l) = [-l, l] \times [0, 2l]$, and let K be the event that at least one bigeodesic intersects B. Define for $L \in \mathbb{N}$, translations of B by

$$B_{i,j} = B_{i,j}(l, L) = B + (iL, jL) \qquad \text{for } (i, j) \in V_H.$$

For $L > 2l$, the $B_{i,j}$ are mutually disjoint.

PROPOSITION C.3. *Let $\delta = 1 - \mathbb{P}(K)$. Then*

$$\mathbb{P}\bigl(B_{i,j} \subset R^+(\gamma_0)\bigr) \leq \delta,$$
$$\mathbb{P}\bigl(B_{i,j} \subset R^-(\gamma_0)\bigr) \leq \delta.$$

PROOF. The proof is the same as that in [20]. $\square$

C.2. Main modifications. From this point on we must obtain a contradiction in a different manner than what was used in [20]; this is because the large deviation estimate [20], Lemma 9, does not necessarily hold in our setting.

A consequence of Proposition C.3 is that for any i, j, $\mathbb{P}(B_{i,j} \cap \gamma_0 \neq \varnothing) > 1-2\delta$. So using $\mathbb{P}(K^*) = 1$, choose l large enough that $1 - 2\delta > 0$ and fix $L = 2l + 1$. For any $n \in \mathbb{N}$ let N_n be the number of boxes $B_{i,j}$ contained in $R_n := [-l, nL + l] \times [0, nL + 2l]$ such that $B_{i,j} \cap \gamma_0 \neq \varnothing$ (the maximum number is n^2). The choice of l ensures that there is a constant c_1 with $0 < c_1 \leq 1$ such that

$$\mathbb{E}N_n \geq c_1 n^2 \qquad \text{for all } n \in \mathbb{N}.$$

Therefore writing $\mathcal{E}_n$ for the set of edges with both endpoints in R_n, for some $c_2 > 0$,

$$\mathbb{E}\#\gamma_0 \cap \mathcal{E}_n \geq c_2 n^2 \qquad \text{for all } n \in \mathbb{N}. \tag{24}$$

We can then argue the following.

LEMMA C.4. *Assuming $\mathbb{P}(K^*) = 1$, there exists $c_3 > 0$ such that with positive probability, for an infinite number of $n \in \mathbb{N}$, there are vertices $v_1, v_2 \in \partial R_n$ such that the geodesic $G(v_1, v_2)$ contains at least $c_3 n^2$ edges in $\mathcal{E}_n$ with weight at least c_3.*

PROOF. Let $a > 0$ and choose $C > a$ such that $\#\mathcal{E}_n \leq Cn^2$ for all $n \in \mathbb{N}$. Use (24) to estimate

$$c_2 n^2 \leq an^2 + (Cn^2 - an^2)\mathbb{P}(\#\gamma_0 \cap \mathcal{E}_n \geq an^2),$$

giving

$$\mathbb{P}(\#\gamma_0 \cap \mathcal{E}_n \geq an^2) \geq \frac{c_2 - a}{C - a}. \tag{25}$$

Furthermore for $b > 0$, writing $p_b = \mathbb{P}(\omega_e < b)$, and $N_n' = \#\{e \in \mathcal{E}_n : \omega_e < b\}$,

$$\#\mathcal{E}_n p_b = \mathbb{E}N_n' \geq \sqrt{p_b} n^2 \mathbb{P}(N_n' \geq \sqrt{p_b} n^2),$$

so $\mathbb{P}(N_n' \geq \sqrt{p_b} n^2) \leq \frac{\#\mathcal{E}_n \sqrt{p_b}}{n^2}$.

Because $\mathbb{P}$ has unique passage times, $\mathbb{P}(\omega_e = 0) = 0$ and so $p_b \to 0$ as $b \to 0$. Thus $\mathbb{P}(N_n' \geq \sqrt{p_b} n^2) \to 0$ uniformly in n as $b \to 0$. Combining this with (25), choosing a and b small enough,

$$\mathbb{P}(N_n' < an^2/2 \text{ and } \#\gamma_0 \cap \mathcal{E}_n \geq an^2) > \frac{c_2}{2C} \qquad \text{for all } n \in \mathbb{N}.$$

With probability at least $c_2/(2C)$, this event occurs for infinitely many n and gives at least $an^2/2$ edges in $\gamma_0 \cap \mathcal{E}_n$ with weight at least b, so set $c_3 < \min\{c_2/(2C), a/2\}$. For such an n, we take v_1 and v_2 to be the first and last vertices that γ_0 touches in R_n. $\square$

To contradict Lemma C.4, we will need to handle assumptions (A) and (B) differently.

C.2.1. *Contradiction under* (A). Because assumption (A) does not include a moment condition on the variable ω_e, we will need to define modified passage times similarly to [5]. Choose any $D > 0$ such that

$$\mathbb{P}(\omega_e > D) \leq 1/5$$

and define a percolation process by setting $\eta_D = \eta_D(\omega) \in \{0, 1\}^{V_H}$ to be

$$\eta_D(e) = \begin{cases} 0, & \text{if } \omega_e > D, \\ 1, & \text{if } \omega_e \leq D. \end{cases}$$

Because the weights (ω_e) are i.i.d., so are the variables $(\eta_D(e))$. Because the critical value for bond percolation on $\mathbb{Z}^2$ is $1/2$, this is a supercritical percolation process. The following lemma holds for any D such that $(\eta_D(e))$ is supercritical, but we will give a simple proof for D as above. For the statement, we define an open half-circuit to be a path in $\mathbb{H}$ whose initial and final endpoints are on the first coordinate axis and all of whose edges e have $\eta_D(e) = 1$.

LEMMA C.5. *Define B_n as the box $[-n, n] \times [0, 2n]$ and A_n as the half-annulus $A_n = B_n \setminus B_{n-\sqrt{n}}$. Then*

$$\sum_n \mathbb{P}\big(\textit{there is no open half-circuit of edges in } A_n \textit{ enclosing } (0,0)\big) < \infty.$$

PROOF. We will consider the dual half-plane lattice $\mathbb{H}^*$, whose vertex set is $V_H^* = V_H - (1/2, 1/2)$ and whose edge set is $E_H^* = [E_H \setminus X] - (1/2, 1/2)$, where X is the set of edges joining vertices on the first coordinate axis. The configuration η_D induces one on the dual lattice η_D^*, where we set $\eta_D^*(e^*) = 1$ if $\eta_D(e) = 1$ and 0 otherwise. Here e^* is the edge dual to $e \in E_H$; that is, the unique dual edge which bisects e. Note that η_D^* has a product distribution with $\mathbb{P}(\eta_D^*(e^*) = 1) = \mathbb{P}(\omega_e \leq D)$.

For $v \in V_H^*$ and $n \in \mathbb{N}$, let $F_n(v)$ be the event that there is a dual path of n dual edges e^* starting at v satisfying $\eta_D^*(e^*) = 0$ for all e^*. Then

$$\mathbb{P}\big(F_n(v)\big) \leq \sum_{|P|=n} \mathbb{P}(\omega_e > D)^n \leq \big(4\mathbb{P}(\omega_e > D)\big)^n \leq (4/5)^n,$$

where the sum is over all dual paths P starting at v with length n. Therefore, letting ∂_n^* be the set of dual vertices in B_n within Euclidean distance 1 of ∂B_n,

$$\sum_n \sum_{v \in \partial_n^*} \mathbb{P}\big(F_{\sqrt{n}}(v)\big) < \infty.$$

But if there is no open half-circuit of edges in A_n enclosing $(0, 0)$, then there is a dual path with all dual edges e^* satisfying $\eta_D^*(e^*) = 0$ starting at a dual vertex in ∂_n^* and ending in $B_{n-\sqrt{n}}$. $\square$

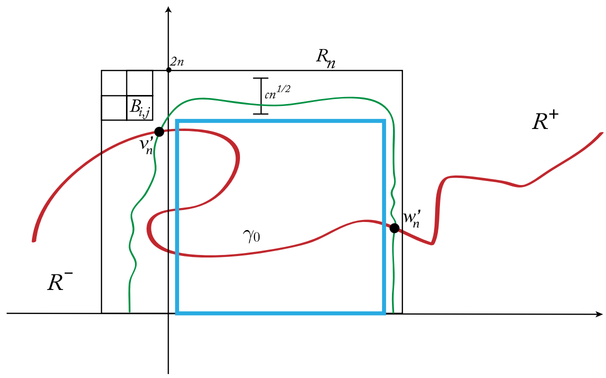

FIG. 6. *The annulus $R_n \setminus [B_{nL/2+l-\sqrt{nL/2+l}} + (nL/2+l, 0)]$. The open half-circuit $\mathcal{C}_n$ is between the two half-boxes, and the geodesic γ_0 is the bold path entering and leaving the large box. The first intersection of γ_0 with $\mathcal{C}_n$ is v_n' and the last intersection is w_n'. Because γ_0 intersects order n^2 number of edges in the inner half-box with weight at least c_3, $\tau(v_n', w_n')$ is at least order n^2.*

Note that there is some $C > 0$ such that, if v, w are vertices in such an open half-circuit mentioned in the previous lemma, then

$$\tau(v, w) \le CDn^{3/2}. \tag{26}$$

Combining this with Lemma C.4, we see that with positive probability, for infinitely many n, both of the following occur:

(1) there exist $v_n, w_n \in \partial R_n$ such that the geodesic $G(v_n, w_n)$ contains at least $c_3 n^2$ edges in $\mathcal{E}_n$ with edge-weight at least c_3, and

(2) the annulus $R_n \setminus [B_{nL/2+l-\sqrt{nL/2+l}} + (nL/2+l, 0)]$ contains an open half-circuit $\mathcal{C}_n$ of edges enclosing $(nL/2+l, 0)$. (See Figure 6.)

Note that the above annulus contains only order $n^{3/2}$ edges total. Therefore when these two conditions hold for large n, the geodesic $G(v_n, w_n)$ must contain at least $c_3 n^2/2$ edges in $B_{nL/2+l-\sqrt{nL/2+l}} + (nL/2+l, 0)$ with weight at least c_3. This means that this geodesic must intersect $\mathcal{C}_n$ and contain at least $c_3 n^2/2$ edges with weight at least c_3 between two intersections with $\mathcal{C}_n$. Consequently, there exist vertices v_n' and w_n' on $\mathcal{C}_n$ such that $\tau(v_n', w_n') \ge c_3^2 n^2/2$. This contradicts (26) for large n.

C.2.2. *Contradiction under* (B). Lemma C.4 implies that with positive probability, for infinitely many n, there are two vertices v, w in ∂R_n such that $\tau(v, w) \ge c_3^2 n^2$. But this passage time is bounded above by the sum of edge weights for edges in ∂R_n, and we find

$$\mathbb{P}\big(\tau(v,w) \geq c_3^2 n^2 \text{ for some } v, w \in \partial R_n\big) \leq \mathbb{P}\bigg(\sum_{e \in \partial R_n} \omega_e \geq c_3^2 n^2\bigg)$$
$$\leq \frac{1}{c_3^4 n^4} \mathbb{E}\bigg(\sum_{e \in \partial R_n} \omega_e\bigg)^2 = O(n^{-2})$$

as $n \to \infty$. Borel–Cantelli then contradicts Lemma C.4.

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