The Annals of Applied Probability
2015, Vol. 25, No. 1, 287–323
DOI: 10.1214/13-AAP996

BOOTSTRAP PERCOLATION ON THE HAMMING TORUS

BY JANKO GRAVNER[1], CHRISTOPHER HOFFMAN[2], JAMES PFEIFFER[3] AND DAVID SIVAKOFF[4]

University of California, Davis, University of Washington, University of Washington and Duke University

The Hamming torus of dimension d is the graph with vertices $\{1, \dots, n\}^d$ and an edge between any two vertices that differ in a single coordinate. Bootstrap percolation with threshold θ starts with a random set of open vertices, to which every vertex belongs independently with probability p, and at each time step the open set grows by adjoining every vertex with at least θ open neighbors. We assume that n is large and that p scales as $n^{-\alpha}$ for some $\alpha > 1$, and study the probability that an i-dimensional subgraph ever becomes open. For large θ, we prove that the critical exponent α is about $1 + d/\theta$ for $i = 1$, and about $1 + 2/\theta + \Theta(\theta^{-3/2})$ for $i \geq 2$. Our small θ results are mostly limited to $d = 3$, where we identify the critical α in many cases and, when $\theta = 3$, compute exactly the critical probability that the entire graph is eventually open.

1. Introduction. Bootstrap percolation is a simple growth model, introduced to understand nucleation and metastability in physical processes such as crack formations, clustering and alignment of magnetic spins. It was introduced in 1979 by Chalupa, Leath and Reich [11]. For more applications and background, see surveys by Adler and Lev [1] and Holroyd [15].

Given a graph $G = (V, E)$, *bootstrap percolation* with *threshold* θ is the following discrete-time growth process: given an initial configuration $\omega \in \{0, 1\}^V$, an increasing sequence of configurations $\omega = \omega_0, \omega_1, \dots$ is defined by

$$\omega_{j+1}(v) = \begin{cases} 1, & \text{if } \omega_j(v) = 1 \text{ or } \sum_{w \sim v} \omega_j(w) \geq \theta, \\ 0, & \text{else,} \end{cases}$$

and ω_∞ is the pointwise limit of ω_j as $j \to \infty$. The initial configuration ω is random; $\{\omega(v) : v \in V\}$ is a collection of i.i.d. Bernoulli random variables with parameter p. A natural quantity to study is $\mathbb{P}_p(\omega_\infty \equiv 1)$. Indeed, first results in this area were by van Enter [21] and Schonmann [18], who proved that for the

Received September 2012; revised November 2013.

[1]Supported in part by NSF Grant DMS-02-04376 and the Republic of Slovenia's Ministry of Science program P1-285.

[2]Supported in part by NSF Grant DMS-08-06024 and by an AMS Centennial Fellowship.

[3]Supported in part by NSF Grant DMS-11-15293.

[4]Supported in part by NSF Grant DMS-10-57675 and by SAMSI.

MSC2010 subject classifications. 60K35.

Key words and phrases. Bootstrap percolation, critical exponent, Hamming torus, Poisson convergence.

lattice $\mathbb{Z}^d$ this probability is either 1 or 0 according to whether $\theta \le d$ or $\theta > d$. Following the seminal work of Aizenman and Lebowitz [2], it became clear that this process is even more interesting on large *finite* graphs. For a family of graphs depending on a single parameter n, with the number of vertices going to infinity as n increases, we assume that $p = p(n)$, and study the dependence on n of the critical probability p_c defined by

$$\mathbb{P}_{p_c}(\omega_\infty \equiv 1) = 1/2.$$

We mention only a few prominent results on how p_c scales with n. Let $[n] = \{1, \dots, n\}$. For a large lattice cube $[n]^d \subseteq \mathbb{Z}^d$ (where each point is connected to the nearest $2d$ points), Aizenman and Lebowitz [2] proved that p_c behaves as $(\frac{1}{\log n})^{d-1}$ when $\theta = 2$, and later Cerf and Cirillo [9] and Cerf and Manzo [10] established the scaling $(\log_{\theta-1} n)^{-d+\theta-1}$ for $3 \le \theta \le d$; here, $\log_{\theta-1}$ denotes the $(\theta-1)$st iteration of the logarithm. For the hypercube $\{0, 1\}^n$, Balogh and Bollobás [3] proved that the scaling for p_c is $n^{-2}4^{-\sqrt{n}}$ when $\theta = 2$; by contrast, for the very large threshold $\theta = \lceil n/2 \rceil$, the *majority bootstrap percolation* studied by Balogh, Bollobás and Morris [5], p_c is close to $1/2$.

Such scaling results do not tell the whole story. They suggest the existence of an *order parameter*, a function of p and n whose size determines whether $\mathbb{P}_p(\omega_\infty \equiv 1)$ is small or close to 1, for example, on a lattice square $[n]^2$, such a function is $p \log n$. This leads to two natural questions: Does the probability exhibit a sharp jump from 0 to 1 as the order parameter increases? Does the location of the (purported) sharp jump converge as n increases? (There are good reasons to expect the answer to the first question to be positive in surprising generality [12].)

In a major breakthrough, Holroyd [14] established a positive answer to both questions in the lattice square case, and proved that $p_c \sim \frac{\pi^2}{18 \log n}$. This celebrated theorem contradicted conjectures based on simulations, which is due to the fact that $p_c \log n$ converges to its limit very slowly, as about $1/\sqrt{\log n}$ [13]. For lattice cubes $[n]^d$, $d \ge 3$ and $2 \le \theta \le d$, the sharp transition was established by Balogh, Bollobás, Duminil-Copin and Morris [4, 6].

Besides varying the dimension of the lattice or the threshold, one can also vary the neighborhood of a point. For example, Holroyd, Liggett and Romik [16] consider the lattice square $[n]^2$, with the "cross" neighborhood of a point that consists of $k - 1$ points in each of the 4 axis directions, and $\theta = k$. In this case, $p_c \sim \frac{\pi^2}{3k(k+1) \log n}$.

In this paper, we consider bootstrap percolation on the *Hamming torus* (or Hamming graph), the d-fold product graph $K_n \times \cdots \times K_n$, where K_n is the complete graph with n vertices. This graph has vertex set $V = [n]^d$, and two vertices $v \in V$ and $w \in V$ are adjacent iff $v - w$ has exactly one nonzero coordinate. In $d = 2$, this graph could be interpreted as taking the Holroyd–Liggett–Romik neighborhood [16] with $k = \infty$. For any d, the neighborhood of a point v is the union of all d lines through v parallel to the axes. We emphasize, however, that the threshold θ remains fixed as n increases (although some of our results assume that θ is large). Other models of percolation, including bond percolation [8, 22] and site percolation [19], have been considered on the Hamming torus, and were shown to exhibit interesting behavior due to the large neighborhood sizes relative to nearest-neighbor lattices and hypercubes. For the same reason, we expect qualitatively different transition phenomena in bootstrap percolation on the Hamming torus from those described above. First, the critical probability is much smaller. In fact, our results suggest that p_c is of the order $n^{-\alpha}$, for some critical exponent $\alpha > 1$. We are able to determine α exactly in a few cases, and give estimates otherwise. Moreover, we expect that varying the order parameter $n^\alpha p$ does *not* lead to a sharp jump of $\mathbb{P}_p(\omega_\infty \equiv 1)$ from 0 to 1; instead, this probability gradually approaches 0 (resp., 1) as the order parameter approaches 0 (resp., ∞). When $d = 2$, this is easy to demonstrate for arbitrary θ, but when $d \geq 3$ the combinatorics are quite difficult even when α is known exactly. Nevertheless, we succeeded in analyzing the case $d = \theta = 3$, which has $\alpha = 2$: we give an explicit formula for the limit of $\mathbb{P}_p(\omega_\infty \equiv 1)$ when $pn^2 = a \in (0, \infty)$. See [17], Theorem 3.2, for an analogous result for bootstrap percolation on Erdős–Rènyi random graphs.

Moreover, in dimensions $d \geq 3$, we find two distinct critical exponents. When p is much smaller than $n^{-1-d/\theta}$, the model does not accomplish much; with high probability it does not even fill a single line. When p is much larger than $n^{-1-d/\theta}$, but smaller than $n^{-1-2/\theta-c'/\theta^{3/2}}$, for large enough θ, with high probability some lines become open, but no two-dimensional subgraphs do, and thus $\mathbb{P}_p(\omega_\infty \equiv 1) \to 0$. When $p > n^{-1-2/\theta-c''/\theta^{3/2}}$, and θ is large enough, $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1$. Here, $0 < c'' < c'$ are constants depending on d.

It remains an open question for $\theta > 2$ whether the critical exponents for the appearance of open subspaces with dimension i are distinct for each $2 \leq i \leq d$. However, in subsequent work, Slivken has proven that for $\theta = 2$, there are distinct critical exponents for the appearance of open subspaces with dimension $2i$ for $1 \leq i < \sqrt{d}$ [20].

2. Statement of results.

Let $\mathcal{F}$ be a family of subsets of $[n]^d$. Then

$$\mathbb{P}_p(\exists F \in \mathcal{F} : \omega_\infty|_F \equiv 1)$$

is a nondecreasing function in p. (Observe that here the vertical bar does not denote a conditional probability but a restriction, i.e., $\omega_\infty|_F$ is ω_∞ restricted to the set $F \subseteq [n]^d$.) For $\mathcal{F}_i$, the collection of i-dimensional subgraphs of G, there exists a threshold function $p_c(i, d)$ such that

$$\mathbb{P}_{p_c(i,d)}(\exists F \in \mathcal{F}_i : \omega_\infty|_F \equiv 1) = 0.5.$$

If $\omega_j(v) = 1$, we say v is open at step j, and a set $S \subseteq V$ is open if each $v \in S$ is open, that is, $\omega_j|_S \equiv 1$.

For $i = 0$, we have an additional critical probability $p_c^*(0, d)$. We would like to define it to be the threshold function for the event that $\omega_\infty \not\equiv \omega_0$; unfortunately, this is not an increasing event. (Recall that an event $E \subseteq \{0,1\}^V$ is increasing if $\omega \in E$ and $\omega \le \omega'$ together imply $\omega' \in E$.) Instead, we define the event

$$\text{Above Threshold} = \left\{ \exists v : \sum_{w \sim v} \omega_0(w) \ge \theta \right\}$$

and $p_c^*(0,d)$ to be the p for which $P_p(\text{Above Threshold}) = 0.5$.

We write $f(n) \sim g(n)$ if $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$. We conjecture that for every $\theta, i, d \in \mathbb{N}$ with $i \le d$, there exists $a_c = a_c(\theta, i, d)$ and $\alpha_c = \alpha_c(\theta, i, d)$ such that

$$p_c(i,d) \sim a_c n^{-\alpha_c}.$$

Moreover, there exists a nondecreasing function $G = G(\theta, i, d) : \mathbb{R}^+ \to [0,1]$ such that $G(x) \to 0$ as $x \to 0$, $G(x) \to 1$ as $x \to \infty$, and if $p = an^{-\alpha_c}$ then

$$\mathbb{P}_p(\exists F \in \mathcal{F}_i : \omega_\infty|_F \equiv 1) \sim G(a).$$

We are able to prove that this is the case for $d = 2$.

THEOREM 2.1. *Let $d = 2$, $k = \lceil \theta/2 \rceil > 1$ and $p = an^{-1-1/k}$. Then*

$$\mathbb{P}(\omega_\infty \equiv 1) \to \begin{cases} 1 - e^{-2a^k/k!}, & \text{if } \theta \text{ is odd,} \\ (1 - e^{-a^k/k!})^2, & \text{if } \theta \text{ is even.} \end{cases}$$

Thus,

$$p_c(2,2) = p_c(1,2) = p_c^*(0,2) = n^{-1-2/\theta + o(\theta^{-3/2})}.$$

Furthermore,

$$\mathbb{P}\big(\{\omega_\infty \not\equiv \omega_0\} \setminus \{\omega_\infty \equiv 1\}\big) = o(1).$$

As d increases the problem becomes more intricate. For $d = 3$, we are able to identify the limit under critical scaling when $\theta = 3$.

THEOREM 2.2. *Let $d = 3$, $\theta = 3$ and $p = an^{-2}$ with $a > 0$. Then as $n \to \infty$*

$$\begin{aligned} \mathbb{P}_p(\omega_\infty \equiv 1) &\to 1 - e^{-a^3 - (3/2)a^2(1-e^{-2a})} \\ &\quad \times \left[\frac{3}{2} a^2 \big((e^{-a} + ae^{-3a})^2 - e^{-2a} \big) e^{-a^2 e^{-2a}} + e^{a^3 e^{-3a}} \right]. \end{aligned} \tag{2.1}$$

Other three-dimensional results include determining the critical exponents (α_c) for $d = 3$ and low thresholds, but not the exact constants a_c; see Section 5 for details.

Observe the contrast between Theorem 2.1 and Theorem 2.2 and classical results on percolation on lattice cubes $[n]^d$ [6, 14]: not only is the critical scaling $p = an^{-\alpha}$ much smaller in the present case, but also $\lim_n \mathbb{P}_p(\omega_\infty \equiv 1)$ is not a step function of a. Instead, this limiting probability varies continuously from 0 to 1 as a increases from 0 to ∞.

Many of our results state that

$$p_c(i,d) = n^{-1-c_1(i,d)/\theta - \Theta(\theta^{-3/2})},$$

where $c_1 = c_1(i,d)$ is a constant. This shorthand notation means that, for a large n, we can get a lower bound and upper bound for $p_c(i,d)$ of the stated form, with constants in the correction term $\Theta(\theta^{-3/2})$ depending on i and d.

For general $d \geq 3$, we calculate $p_c^*(0,d)$ and $p_c(1,d)$ for all $d \geq 2$ quite precisely.

THEOREM 2.3. *Let $p = f(n)n^{-1-d/\theta}$ and $d, \theta \geq 3$. If $f(n) \to 0$ then*

$$\mathbb{P}(\text{Above Threshold}) \to 0$$

and if $f(n) \to \infty$ then

$$\mathbb{P}(\exists \text{ a line } \ell \text{ such that } \omega_\infty|_\ell \equiv 1) \to 1.$$

Furthermore, we get good bounds on $p_c(2,d)$, the threshold for existence of two-dimensional subspaces in the final configuration.

THEOREM 2.4. *Fix d and fix θ sufficiently large depending on d. For n sufficiently large*,

$$n^{-1-2/\theta-(4d^2+3)/\theta^{3/2}} \leq p_c(2,d) \leq n^{-1-2/\theta-\sqrt{8(d-2.1)}/\theta^{3/2}}.$$

[We have not attempted to optimize the constants $\sqrt{8(d-2.1)}$ and $4d^2+3$ in the above theorem.] The key arguments in the proof of Theorem 2.4 are Lemmas 8.1 and 5.1.

The higher the dimensions i and d, the more difficult it becomes to calculate $p_c(i,d)$. However, Theorems 2.3 and 2.4 are sufficient for us to get bounds on $p_c(i,d)$ for all $i, d \geq 2$.

THEOREM 2.5. *For all $i \geq 2$ and d, and sufficiently large n*,

$$p_c(i,d) = n^{-1-2/\theta-\Theta(\theta^{-3/2})}.$$

PROOF. It is easy to see that $p_c(i,d)$ is nondecreasing in i and decreasing in d. Also $p_c(d,d)$ is decreasing in d. To see this last inequality note that when $n \geq 3\theta$ and $d = j+1$

$$\mathbb{P}_{p_c(j,j)}(\exists \text{ at least } \theta\ i \text{ such that } \omega_\infty|_{(i,*,*,\dots)} \equiv 1) > 1/2.$$

The event on the left-hand side implies that $\omega_\infty \equiv 1$, and thus

$$p_c(j+1, j+1) \leq p_c(j,j)$$

and inductively

$$p_c(d, d) \leq p_c(3, 3).$$

So

$$p_c(2, d) \leq p_c(i, d) \leq p_c(d, d) \leq p_c(3, 3).$$

By Theorem 2.4,

$$p_c(2, 3) \leq n^{-1-2/\theta-(\sqrt{7.2}+o(1))/\theta^{3/2}}.$$

By coupling it is easy to see that ω chosen when $p = 10\theta p_c(2, 3)$ stochastically dominates the union of 10θ independent ω' chosen with $p = p_c(2, 3)$. Then by the definition of $p_c(2, 3)$

$$\mathbb{P}_{10\theta p_c(2,3)}(\exists \text{ at least } \theta\ i \text{ such that } \omega_\infty|_{(i,*,*)} \equiv 1) > 1/2.$$

The event on the left-hand side implies $\omega|_\infty \equiv 1$, and thus

$$p_c(3, 3) \leq 10\theta p_c(2, 3). \tag{2.2}$$

And putting this all together for all $d \geq 3$ and $2 \leq i \leq d$,

$$\begin{aligned} n^{-1-2/\theta-(4d^2+2+o(1))/\theta^{3/2}} &\leq p_c(2, d) \leq p_c(i, d) \leq p_c(d, d) \leq p_c(3, 3) \\ &\leq 10\theta p_c(2, 3) \leq n^{-1-2/\theta-(\sqrt{7.2}-o(1))/\theta^{3/2}}, \end{aligned}$$

which is the desired result. □

REMARK 2.6. The above results are all asymptotic statements in n. One natural question is whether we can obtain nonasymptotic bounds on the critical parameters. Our arguments do in fact produce bounds on the critical probability for specific values of n. Keeping track of (or even stating) these bounds is quite challenging and we have made no attempt to optimize them. Different results kick in at different values of n, but all of them work if n is at least roughly $e^{\theta^{3/2}}$.

The rest of the paper is organized as follows. In Section 3, we prove the two-dimensional Theorem 2.1. In Section 4, we give a necessary condition for a plane to become open when $d = 3$ and in Section 5 we give a sufficient condition for this event for arbitrary d. Section 5 also features the resulting upper and lower bounds for critical exponents in three dimensions and the proof for the upper bound in Theorem 2.4. Section 6 features the proof of Theorem 2.2, which is, like that of Theorem 2.1, based on Poisson convergence. While the two-dimensional case requires nothing more than Poisson approximation to the binomial, our proof of this three-dimensional result hinges on much more intricate coupling methods introduced in [7]. As some events in question are not positively related, the required couplings need to be explicitly constructed; the details of this construction are deferred to the Appendix. In Section 7, we study when a line is likely to become open and establish Theorem 2.3. In Section 8, we provide a lower bound on the value of p that makes it likely that a plane becomes open; this, together with results in Section 5, will complete the proof of Theorem 2.4. We conclude with a short list of open questions in Section 9.

We end this section with a note on terminology, adopted from [2]. A vertex v (resp., a set $F \subset [n]^d$) is called *open*, or *occupied* at a time $t \in [0, \infty]$ if $\omega_t(v) = 1$ (resp., $\omega_t|_F \equiv 1$). Assume $G \subset [n]^d$ is an arbitrary (deterministic or random) set, and suppose the bootstrap percolation process is run started from the set of open vertices equal to G. Fix also a set $F \subset [n]^d$. We say that G *spans* F if this process makes every vertex in F eventually open. Furthermore, we say that F is *internally spanned* by G if $G \cap F$ spans F. When F is unspecified, it is assumed to be the entire torus $[n]^d$. As throughout this section, the initially open points are by default chosen at random, independently with probability p; if this set spans, we also say that *spanning* occurs. Finally, we denote by $\sigma_\theta(d, p)$ the *spanning probability*, that is, the probability of spanning for the d-dimensional torus with threshold θ and initial occupation density p. (Note that the dependence on n is suppressed in this notation.)

3. Precise two-dimensional results.

In the two-dimensional case, we can describe the limiting behavior exactly as $n \to \infty$. Let $k = \lceil \theta/2 \rceil$ and $p = an^{-1-1/k}$ for some constant a. Also assume $k > 1$; the cases $\theta = 1$ and $\theta = 2$ are easy to work out separately. (For $\theta = 1$, $\omega_\infty \equiv 1$ if and only if $\omega_0 \not\equiv 0$; for $\theta = 2$, $\omega_\infty \equiv 1$ asymptotically if and only if ω_0 contains at least two noncollinear open points.)

LEMMA 3.1. *Let $k = \lceil \theta/2 \rceil$ and $p = an^{-1-1/k}$. With probability going to 1, there are no lines with at least $k + 1$ points initially open.*

PROOF. For a fixed line ℓ, let E_ℓ be the event that ℓ contains $k + 1$ initially open points. For any ℓ,

$$\mathbb{P}_p(E_\ell) \le \binom{n}{k+1} p^{k+1} \le n^{k+1} p^{k+1} \le a^{k+1} n^{-1-1/k},$$

and, as there are $2n$ lines,

$$\mathbb{P}_p\Bigl(\bigcup_\ell E_\ell\Bigr) \le 2n \cdot a^{k+1} n^{-1-1/k} = 2a^{k+1} n^{-1/k} \to 0$$

as $n \to \infty$. □

LEMMA 3.2. *Fix an $\varepsilon > 0$. Let $k = \lceil \theta/2 \rceil$ and $p = \varepsilon n^{-1-1/k}$. Fix constants A, B and choose B fixed vertical* (*resp. horizontal*) *exceptional lines. With probability going to 1, there are at least A horizontal* (*resp., vertical*) *lines, which contain $k - 1$ initially open points none of which are in the union of the exceptional lines.*

PROOF. Each of the n horizontal lines satisfies the condition independently with probability at least

$$\binom{n-B}{k-1} p^{k-1}(1-p)^{n-k+1} = \Theta(n^{-1+1/k}).$$

The probability that there are at least A such lines therefore goes to 1. $\square$

Let E_{horiz} be the event that some horizontal line contains at least k initially open points, E_{vert} the corresponding event for vertical lines, and $E_{\text{horiz}} \circ E_{\text{vert}}$ the event that the two occur disjointly.

LEMMA 3.3. *Let $k = \lceil \theta/2 \rceil$ and $p = an^{-1-1/k}$. We have*

$$\mathbb{P}_p\big((E_{\text{horiz}} \cap E_{\text{vert}}) \setminus (E_{\text{horiz}} \circ E_{\text{vert}})\big) \to 0.$$

Furthermore,

$$\mathbb{P}_p(E_{\text{horiz}} \cap E_{\text{vert}}) \to \big(1 - e^{-a^k/k!}\big)^2$$

and

$$\mathbb{P}_p(E_{\text{horiz}} \cup E_{\text{vert}}) \to 1 - \big(e^{-a^k/k!}\big)^2.$$

PROOF. The event $(E_{\text{horiz}} \cap E_{\text{vert}}) \setminus (E_{\text{horiz}} \circ E_{\text{vert}})$ happens only if some point v is open, and each of the two lines through v contains exactly $k-1$ additional open points. The probability that such a point exists is bounded by

$$n^2 p\big(n^{k-1} p^{k-1}\big)^2 = O\big(n^{-1+1/k}\big) \to 0.$$

This proves the first assertion.

As E_{horiz} and E_{vert} are increasing events, $\mathbb{P}_p(E_{\text{horiz}} \cap E_{\text{vert}}) \geq \mathbb{P}_p(E_{\text{horiz}}) \times \mathbb{P}_p(E_{\text{vert}}) = \mathbb{P}_p(E_{\text{horiz}})^2$ by the FKG inequality. Conversely, the BK inequality gives $\mathbb{P}_p(E_{\text{horiz}})\mathbb{P}_p(E_{\text{vert}}) \geq \mathbb{P}_p(E_{\text{horiz}} \circ E_{\text{vert}})$. Thus, $\mathbb{P}_p(E_{\text{horiz}} \cap E_{\text{vert}}) - \mathbb{P}_p(E_{\text{horiz}})^2 \to 0$. Moreover, the number of horizontal lines with at least k open points is Binomial and converges in distribution to a Poisson random variable with expectation $a^k/k!$. Thus, $\mathbb{P}_p(E_{\text{horiz}}) \to 1 - e^{-a^k/k!}$, which easily ends the proof. $\square$

Let G be the event that the entire graph becomes open, that is, $G = \{\omega_\infty \equiv 1\}$.

LEMMA 3.4. *Let $k = \lceil \theta/2 \rceil$ and $p = an^{-1-1/k}$. If θ is even, $\mathbb{P}_p(G) - P(E_{\text{horiz}} \cap E_{\text{vert}}) \to 0$, while if θ is odd, $\mathbb{P}_p(G) - \mathbb{P}_p(E_{\text{horiz}} \cup E_{\text{vert}}) \to 0$.*

PROOF. If θ is odd, the process adds no new open vertex unless there is some line with at least k vertices initially open. So $G \subseteq E_{\text{horiz}} \cup E_{\text{vert}}$. If θ is even, then by Lemma 3.1, $\mathbb{P}_p(G \setminus (E_{\text{horiz}} \cap E_{\text{vert}})) \to 0$.

Fix an $\varepsilon > 0$ and let ω^*, ω' and ω'' be three independent configurations, the first with $p^* = (1 - 2\varepsilon)n^{-1-1/k}$, and the other two are "sprinkled" with small $p' = \varepsilon n^{-1-1/k}$. Observe that ω_0 (generated with p) stochastically dominates $\omega^* \cup \omega' \cup \omega''$.

Now suppose θ is odd and $E_{\text{horiz}} \cup E_{\text{vert}}$ occurs in ω^*. Then some line ℓ has k points open in ω^*. We now describe the events that occur with probability 1 as $n \to \infty$. By Lemma 3.2, there are θ lines $\{\ell'_i\}$ parallel to ℓ, each with $k - 1$ points open in ω'. Moreover, again by Lemma 3.2, there are θ lines $\{\ell''_j\}$ perpendicular to ℓ, each with $k - 1$ points, which are open in ω'' and avoid ℓ and all ℓ'_i.

Let G^* be the event that the initial configuration $\omega^* \cup \omega' \cup \omega''$ eventually causes every point to be open. We claim that if the events in the above paragraph all happen then G^* happens. First, each point of intersection of ℓ''_j and ℓ becomes open as it sees $k - 1$ open neighbors on ℓ''_j and k on ℓ. Then there are θ open points on ℓ, so ℓ becomes open. Now each point of intersection of ℓ''_j and ℓ'_i becomes open as it sees one open neighbor on ℓ, and $k - 1$ additional open neighbors each on ℓ''_j and ℓ'_i. This results in θ open points on each ℓ''_i and ℓ'_i, so these 2θ lines all become open, and the entire graph becomes open in the next step.

It now follows that $\liminf \mathbb{P}_p(G) \geq \liminf \mathbb{P}_{p^*}(E_{\text{horiz}} \cup E_{\text{vert}})$, and the claim for odd θ follows by continuity (in a) of limits in Lemma 3.3.

Now suppose θ is even. If $E_{\text{horiz}} \cap E_{\text{vert}}$ occurs, then we may assume $E_{\text{horiz}} \circ E_{\text{vert}}$ occurs by Lemma 3.3. That is, there is a horizontal line ℓ_h and a vertical line ℓ_v, each with k points initially open, excluding their point of intersection. This point of intersection becomes open at the first time step.

As in the odd case, we may use sprinkling and Lemma 3.2 to produce θ horizontal lines ℓ'_i and θ vertical ℓ''_j, each with $k - 1$ initially open points that avoid all other lines. Then every point of intersection between ℓ_h and ℓ''_j, and between ℓ_v and ℓ'_i, sees $\theta = (k + 1) + (k - 1)$ open sites, so it becomes open. Then ℓ_h and ℓ_v contain θ open sites, so they become open. Then every point of intersection of an ℓ'_i with an ℓ''_j sees $2 + 2(k - 1) = \theta$ open sites, so becomes open. Now the entire graph becomes open in two additional steps. □

PROOF OF THEOREM 2.1. The claimed convergence follows from Lemmas 3.3 and 3.4. □

4. Upper bound on critical exponent in three dimensions. It is easy to see that with $p = n^{-\alpha}$ for $\alpha > 1 + \frac{d}{\theta}$, with high probability, no points that are not initially open become open. [The expected number of vertices with at least θ open neighbors is at most $Cn^d(np)^\theta = O(n^{d+\theta-\alpha\theta}) = o(1)$.] In this section, we will assume that $d = 3$ and $\theta \geq 3$ and establish a bound on α that ensures that no planes become open (and hence the entire Hamming torus does not become open) with high probability. A similar result is proved for general d in Section 8.

LEMMA 4.1. *Let $d = 3$ and $\theta = 2k - 1 \geq 3$ be odd. Let $p = n^{-\alpha}$ for $\alpha > 1 + \frac{8}{3\theta-1}$. Then $\mathbb{P}_p$(a plane becomes open) $\to 0$. The same holds for $\theta = 2k \geq 4$ when $\alpha > 1 + \frac{8}{3\theta-2}$.*

PROOF. We may assume $\theta \geq 4$, since the $\theta = 3$ bound of $\alpha > 2$ is equivalent to $\alpha > 1 + \frac{d}{\theta}$. We will prove the lemma for θ odd; the even case is similar. Define the following three conditions for a vertex v:

(1) v is initially open,
(2) v is on a line with at least k points initially open,
(3) the neighborhood of v has at least θ points initially open.

We first prove

$$\text{(4.1)} \quad \mathbb{P}_p\big(\text{there exists a plane each of whose points satisfies one of (1)–(3)}\big) \to 0$$

To prove (4.1), we fix a plane P, which we may assume to be the e_1, e_2-plane, and prove that the probability that all of its points satisfy one of (1)–(3) is exponentially small. Fix an $\varepsilon \in (0, 1/3)$. Consider the lines perpendicular to P, horizontal lines in P, and vertical lines in P, that contain at least one initially open point. Let their respective numbers be S_1, S_2 and S_3, and note that each of these three numbers is Binomially distributed. The probability that a fixed line contains an initially open vertex is at most $np = o(1)$, so $\mathbb{P}_p(S_1 \geq \varepsilon n^2)$, $\mathbb{P}_p(S_2 \geq \varepsilon n)$, and $\mathbb{P}_p(S_3 \geq \varepsilon n)$ are all exponentially small. With probability exponentially close to 1, the number of points in P included in one of the three types of lines is therefore at most $3\varepsilon n^2$, which proves (4.1).

Let E_v be the event that the point v violates all three conditions (1)–(3), but that it becomes open and that no point violating these conditions becomes open earlier. It remains to show that

$$\text{(4.2)} \qquad \mathbb{P}_p(E_v) = o\big(1/n^3\big).$$

We will denote by $\mathcal{N}(v)$ the neighborhood of a point v. If E_v occurs, then $\mathcal{N}(v)$ has m points initially open, for some $0 \leq m \leq \theta - 1$. Then $\mathcal{N}(v)$ contains $\theta - m$ other points $w_1, \dots, w_{\theta-m}$, not initially open, which become open before v. Thus, these w_i must satisfy (2) or (3). Because v violates (2), each w_i shares with v at most $k - 1$ initially open neighbors. Therefore, whether w_i satisfies (2) or (3), $\mathcal{N}_i = \mathcal{N}(w_i) \setminus \mathcal{N}(v)$ must contain k initially open points.

Assume m and w_i are selected. Let N be the number of initially open points in $\mathcal{N}_i \cap \mathcal{N}_j$, for some $i \neq j$. (Note that the intersection of three or more $\mathcal{N}_i$ is empty.) Let H_b^m be the event that $\mathcal{N}(v)$ has m initially open points, $w_1, \dots, w_{\theta-m}$ exist such that $\mathcal{N}_i$ all contain k initially open points *and* that $N = b$. Then

$$\text{(4.3)} \qquad P\big(H_0^m\big) \leq C(np)^m n^{\theta-m}\big((np)^k\big)^{\theta-m}$$

for some constant C. To estimate $P(H_b^m)$, observe that each increase of b by 1 contributes an additional factor of p and removes a factor $(np)^2$ from the right-hand side of (4.3). By monotonicty, we may assume $\alpha \le 2$ so $p \le (np)^2$ [recall $\theta \ge 5$ so $1+8/(3\theta-1) < 2$]; then $P(H_b^m) \le P(H_0^m)$ for all $b \ge 0$ and m. Furthermore, $n^k p^{k-1} = o(1)$ (since $k \ge 2$), thus the upper bound in (4.3) increases with m. It follows that $P(E_v)$ is bounded by the expression in (4.3) with $m = \theta - 1$, which gives

$$n^3 P(E_v) \le C n^{3k+2} p^{3k-2} \to 0,$$

proving (4.2). □

5. Internally spanned planes. In this section, we prove the upper bound in Theorem 2.4 regarding $p_c(2,d)$, the critical probability for the existence of two-dimensional planes in the final configuration. We also introduce a dimension-reduction inequality that allows us to compute lower bounds on the spanning probabilities $\sigma_\theta(\theta, p)$ for arbitrary d and θ. Our first result is a lower bound on $\sigma_\theta(2, p)$, which will allow us to find lower bounds for all d later on.

LEMMA 5.1. *Let $k = \lceil \theta/2 \rceil$ and $\liminf n^\alpha p = b > 0$ with $\alpha > 1 + 1/k$. Then there exists a constant $C > 0$ depending on θ and b such that for all sufficiently large n, $\sigma_\theta(2, p) \ge Cn^{-\beta}$ where*

$$\beta(\alpha) = \begin{cases} \alpha k^2 + a(a+1) - \alpha a(a-1) - (k+1)^2, & \theta \text{ odd}, \\ \alpha k(k+1) + a(a+1) - \alpha a(a-1) - (k+1)(k+2), & \theta \text{ even}, \end{cases} \tag{5.1}$$

and $a = \lfloor \alpha/(\alpha-1) \rfloor$.

REMARK 5.2. If $\alpha = 1 + 1/k$ and $p = b/n^\alpha$ then $\sigma_\theta(2, p) \to c \in (0,1)$ by Theorem 2.1, so $\beta(\alpha) = 0$ for $\alpha \le 1 + 1/k$.

PROOF OF LEMMA 5.1. Observe that the configuration in Figure 1 is sufficient for spanning for odd $\theta = 2k-1$. In the figure, the two-dimensional Hamming graph is first subdivided into nine regions that have dimensions $n/3 \times n/3$. The hashed lines further subdivide some of the regions, and are spaced $\frac{n}{3(k-2)}$ units apart, so each subregion has height and width on the order of n. Each red oval represents the existence of at least one line (in the direction indicated) in that region with the specified number of open vertices. To check that this configuration leads to spanning, observe that the horizontal line containing k open vertices is the first to be spanned: after one step the vertex at the intersection of this line and the vertical line with $k-1$ open vertices becomes open, after two steps the vertex at the intersection of this line and the vertical line with $k-2$ open vertices becomes open, and so on until this line contains $2k-1$ open vertices and the entire line becomes open. As this line is made open, all of the vertical lines each gain one

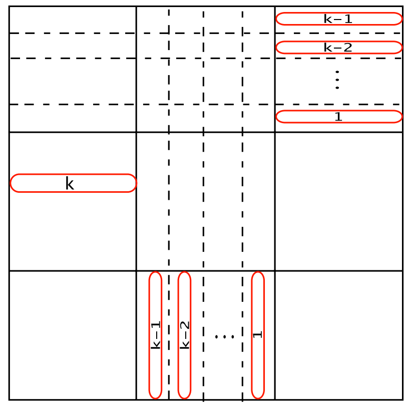


FIG. 1. *This configuration will span the two-dimensional Hamming graph when $\theta = 2k-1$ is odd. Each region bounded by solid lines is approximately $n/3 \times n/3$. The hashed lines are spaced $\frac{n}{3(k-2)}$ units apart, so each subregion has height and width on the order of n. A red oval represents the existence of at least one line (in the direction indicated) in that region with the specified number of open vertices.*

additional open vertex, so the vertical line with $k-1$ initially open vertices is next to be spanned in the same fashion, followed by the horizontal line with $k-1$ open vertices and so on until all $2k-1$ lines with ovals are spanned and cause the rest of the graph to become open. The reason for subdividing the graph into disjoint regions like we have is so that all of the events depicted are independent. Therefore, the spanning probability is bounded below as

$$\begin{aligned} \sigma_{2k-1}(2, p) &\geq \mathbb{P}_p(\text{configuration in Figure 1}) \\ &= \left[1 - \left(1 - \frac{1}{k!}n^k p^k + o\big((np)^k\big)\right)^{n/3}\right] \\ &\quad \times \prod_{\ell=1}^{k-1}\left[1 - \left(1 - \frac{1}{\ell!}(np)^\ell + o\big((np)^\ell\big)\right)^{n/3(k-2)}\right]^2. \end{aligned} \tag{5.2}$$

If $p \asymp n^{-\alpha}$ and $\alpha < 1 + \frac{1}{k}$ then the lower bound in (5.2) tends to 1 as $n \to \infty$, in agreement with Theorem 2.1, so we assume $p \asymp n^{-\alpha}$ and $\alpha > 1 + \frac{1}{k}$. In this case, the terms in the product in the last line of (5.2) for which $\ell \leq 1/(\alpha - 1)$ either tend to 1 or (in the case of equality) are bounded away from 0 as $n \to \infty$. Therefore, by applying the bound $(1-x)^m \leq 1 - mx + m^2x^2$ for $x \in (0, 1)$, we bound (5.2) from below by

$$C\big[n^{k+1}p^k - o\big(n^{k+1}p^k\big)\big] \prod_{\ell=a}^{k-1}\big[n^{\ell+1}p^\ell - o\big(n^{\ell+1}p^\ell\big)\big]^2, \tag{5.3}$$

where $a = \lfloor \alpha/(\alpha - 1) \rfloor$ and the value of C here is not smaller than $(3 \cdot k!)^{-2k}$ for any $\alpha > 1 + 1/k$. We can take $p = (b/2)n^{-\alpha}$ by noting that $\sigma_\theta(2, p)$ is increasing in p, so the constant C appearing in the lemma is not smaller than $(3 \cdot k!)^{-2k}(b/2)^{k(k+1)}$. Computing the exponent of the leading order term in (5.3) when $p = (b/2)n^{-\alpha}$ gives the formula for $\beta(\alpha)$ when θ is odd. A configuration similar to the one in Figure 1, but where there is one additional column with k initially open vertices, provides a sufficient condition for spanning when $\theta = 2k$. This leads to an expression like the one in (5.2), except with the first factor squared, and leads to the formula for $\beta(\alpha)$ when θ is even. $\square$

Our first application of Lemma 5.1 is to prove the upper bound in Theorem 2.4.

THEOREM 5.3. *Fix $d \geq 3$ and fix θ large enough depending on d [$\theta \geq 650(d - 2.1)$ is sufficient]. For all sufficiently large n,*

$$p_c(2, d) \leq n^{-1-2/\theta-\sqrt{8(d-2.1)}/\theta^{3/2}}.$$

To prepare for the proof, we need a bound on the function $\beta(\alpha)$ in Lemma 5.1 that eliminates the use of the floor function. We isolate the reasoning by treating just the terms involving a.

LEMMA 5.4. *If $1 < \alpha \leq 2$ and $a = \lfloor \alpha/(\alpha - 1) \rfloor$ then*

$$a(a+1) - \alpha a(a-1) \leq \frac{1}{\alpha - 1} + 1 + \frac{1}{2}(\alpha - 1). \tag{5.4}$$

PROOF. Let $\varepsilon = \alpha - 1$ and suppose $\frac{1}{\varepsilon} = m + u$ where $m \geq 1$ is an integer and $u \in [0, 1)$. Then we can write (5.4) as

$$a(-\varepsilon a + 2 + \varepsilon) - \frac{1}{\varepsilon} \leq 1 + \frac{1}{2}\varepsilon,$$

so we must prove this inequality. Observe that

$$a = \left\lfloor \frac{1+\varepsilon}{\varepsilon} \right\rfloor = \lfloor m + u + 1 \rfloor = m + 1,$$

so we have

$$\begin{aligned} a(-\varepsilon a + 2 + \varepsilon) - \frac{1}{\varepsilon} &= \frac{-(m+1)^2 + 2(m+u)(m+1) + m + 1 - (m+u)^2}{m+u} \\ &= 1 + \frac{u - u^2}{m+u} \leq 1 + \frac{1}{2}\varepsilon. \end{aligned}$$

$\square$

PROOF OF THEOREM 5.3. We can divide the d-dimensional Hamming torus into n^{d-2} disjoint 2-dimensional planes all parallel to the e_1, e_2-plane. Our goal is to show that at least one of these planes are internally spanned with high probability when $p = n^{-\alpha}$ with $\alpha = 1 + 2/\theta + \sqrt{8(d - 2.1)}/\theta^{3/2}$. The number of these

2-planes that are internally spanned is binomially distributed, so we need only to show that the expected number of internally spanned planes tends to infinity. The expected number of internally spanned planes is

$$n^{d-2}\sigma_\theta(2, n^{-\alpha}) \geq Cn^{d-2-\beta(\alpha)}$$

by Lemma 5.1. By applying Lemma 5.4, we see that when $\theta = 2k-1$ is odd

$$\begin{aligned}
\beta(\alpha) &= \alpha k^2 - (k+1)^2 + a(a+1) - \alpha a(a-1) \\
&\leq \alpha k^2 - (k+1)^2 + \frac{1}{\alpha-1} + 1 + \frac{1}{2}(\alpha-1) \\
&= \left(1 + \frac{2}{\theta} + \frac{\sqrt{8(d-2.1)}}{\theta^{3/2}}\right)\left(\frac{\theta+1}{2}\right)^2 - \left(\frac{\theta+3}{2}\right)^2 + \frac{\theta}{2+\sqrt{8(d-2.1)/\theta}} \\
&\quad + 1 + \frac{1}{\theta} + \frac{\sqrt{8(d-2.1)}}{2\theta^{3/2}} \\
&\leq -\frac{\theta}{2} + \frac{3}{2\theta} + \frac{\sqrt{8(d-2.1)}}{4}(\theta^{1/2} + 2\theta^{-1/2} + \theta^{-3/2}) \\
&\quad + \frac{\theta}{2}\left(1 - \frac{\sqrt{8(d-2.1)}}{2\theta^{1/2}} + \frac{8(d-2.1)}{4\theta}\right) + \frac{\sqrt{8(d-2.1)}}{2\theta^{3/2}} \\
&= d - 2.1 + \frac{3}{2\theta} + \frac{\sqrt{8(d-2.1)}}{4}(2\theta^{-1/2} + 3\theta^{-3/2}) \\
&< d-2,
\end{aligned}$$

where the last inequality holds for θ large relative to d, and in the fourth line we used the inequality $(1+x)^{-1} \leq 1 - x + x^2$ for $x > 0$. This implies that the expected number of internally spanned 2-dimensional planes tends to infinity with n, and completes the proof for odd θ. The proof for even θ is analogous. □

The next theorem is a simple but powerful observation, which we refer to as the dimension reduction inequality.

THEOREM 5.5. *For any $d \geq 2$, $\theta \geq 2$, and $1 \leq d' \leq d-1$*

$$\sigma_\theta(d, p) \geq \sigma_\theta(d-d', \sigma_\theta(d', p)). \tag{5.5}$$

PROOF. We can subdivide the d-dimensional Hamming torus into $n^{d-d'}$ disjoint sub-Hamming tori of dimension d'. The probability of internally spanning a fixed sub-Hamming torus is $\sigma_\theta(d', p)$, and the initially open sets in the sub-Hamming tori are mutually independent. Therefore, we may identify each d'-dimensional sub-Hamming torus with a single vertex, which is open independently with probability $\sigma_\theta(d', p)$, and the result is a random subset of a $(d-d')$-dimensional Hamming torus that spans with probability $\sigma_\theta(d-d', \sigma_\theta(d', p))$. If

TABLE 1
Upper and lower bounds for the critical exponent when $d = 3$

	θ										
Bound	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Lower	5/2	2	7/4	11/7	3/2	7/5	19/14	17/13	23/18	5/4	27/22
Upper	5/2	2	7/4	11/7	3/2	7/5	15/11	17/13	9/7	5/4	21/17

Note: If $p \asymp n^{-\alpha}$ and α is larger than the upper bound, then spanning will not occur with high probability, while if α is smaller than the lower bound then spanning will occur with high probability.

this procedure spans the $(d - d')$-dimensional Hamming torus, then the original configuration in the d-dimensional graph will span as well. □

Since we can compute bounds for $\sigma_\theta(2, p)$ and $\sigma_\theta(1, p)$ for all θ and p, the dimension reduction inequality yields lower bounds on the critical exponents for all d and θ. In some cases, the lower bounds obtained this way match our upper bounds, so we can precisely compute the critical exponent. For instance, when $d = 3$ and $\theta = 4$ we see that the critical exponent is $\alpha_c = 1 + d/\theta = 7/4$. In this case, if $\alpha = (7 - \varepsilon)/4$ with $0 < \varepsilon < 1$ then Lemma 5.1 with $k = 2$ implies that $\sigma_4(2, n^{-\alpha}) \geq cn^{6-4\alpha} = cn^{-1+\varepsilon}$. Then, since $\sigma_\theta(d, p)$ is increasing in p,

$$\sigma_4(3, n^{-\alpha}) \geq \sigma_4(1, \sigma_4(2, n^{-\alpha})) \geq \sigma_4(1, cn^{-1+\varepsilon}) = P(\mathrm{Bin}(n, cn^{-1+\varepsilon}) \geq 4) \to 1.$$

Theorem 7.6 implies that $1 + d/\theta$ is always an upper bound for the critical exponent, so in the case $d = 3$, $\theta = 4$ the critical exponent is $7/4$.

As a second example of how to apply Lemma 5.1 and Theorem 5.5, consider the case $d = 6$, $\theta = 5$. Applying dimension reduction and Lemma 5.1 twice yields

$$\sigma_5(6, n^{-\alpha}) \geq \sigma_5(4, \sigma_5(2, n^{-\alpha})) \geq \sigma_5(4, Cn^{-\beta(\alpha)}) \geq \sigma_5(2, cn^{-\beta(\beta(\alpha))}).$$

The last term above tends to 1 as $n \to \infty$ if $\beta(\beta(\alpha)) < 4/3$ by Theorem 2.1, so finding the supremum over α satisfying this inequality gives a lower bound on the critical exponent in this case. With a little help from Matlab, we can numerically compute this supremum, and generate lower bounds for other d and θ. See Figure 2 for plots of upper and lower bounds on α_c for $d \in \{2, 3, 4, 5, 6\}$ and $\theta \in \{2, \ldots, 20\}$. Table 1 lists all cases for which our upper and lower bounds match when $d = 3$, and a few cases for which they conspicuously do not ($\theta = 8, 10, 12$). The upper bounds in the table are the smaller of $1 + 3/\theta$ and the bounds from Theorem 4.1—either $1 + 8/(3\theta - 1)$ or $1 + 8/(3\theta - 2)$, depending on whether θ is odd or even.

6. A precise three-dimensional result. In this section, we precisely compute the limiting spanning probability in the case $d = 3$ and $\theta = 3$. As computed in

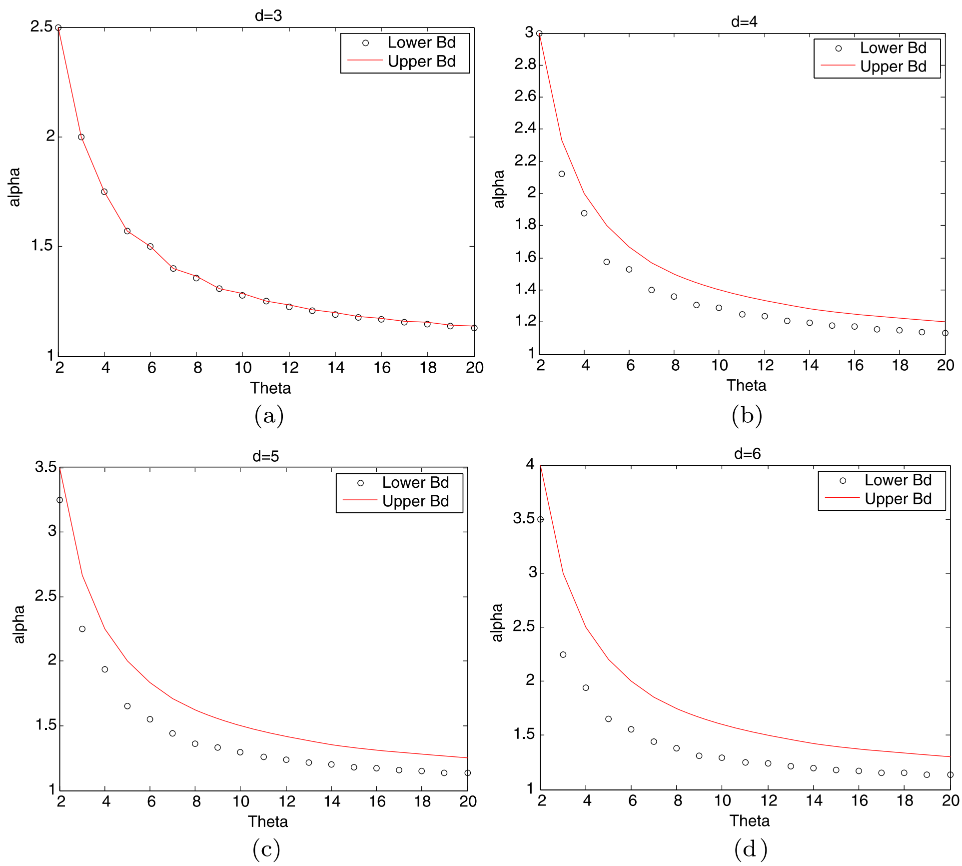
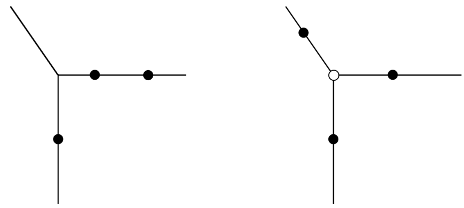


FIG. 2. *Upper and lower bounds for the critical exponent when* $p \asymp n^{-\alpha}$.

Section 5, the critical exponent in this case is $\alpha = 2$ (see Table 1), so we consider the scaling $p = an^{-2}$ when $a > 0$ is a constant.

The resulting limit in Theorem 2.2 is a simplified expression for a probability involving Poisson random variables with means depending on a. Indeed, to compute the spanning probability, we identify the minimal ingredients that lead to spanning, and show that their frequencies of occurrence in ω_0 converge jointly to independent Poisson random variables by using the Chen–Stein method [7]. First, we identify two fundamental configurations, which we will define carefully later: points that see at least one open vertex in each direction [Figure 3(b)] and lines that contain at least two open vertices and at least one more open vertex in the same plane [Figure 3(a)]. At least one of these configurations is necessary (in the limit) for spanning because lines that contain 3 or more open vertices do not appear when $p = an^{-2}$, as the expected number of such lines is $O(n^2(np)^3) = O(n^{-1})$. Note that in the definitions of our configurations we allow for there to be three or more open vertices in a line, even though this is unlikely to occur for large n. This is to



(a) (b)

FIG. 3. *Without one of these configurations appearing somewhere in the graph at time* 0, *nothing will become open at time* 1 *when* $d = \theta = 3$. *The open circle in* (b) *is to emphasize that this "Basic" configuration is with respect to a focal vertex which will become open at time* 1. *The "Line" configuration in* (a) *is indexed with respect to the line which contains two open points, and the single open vertex off of the horizontal line signifies that at least one vertex on one of the two planes containing the focal line must be open.*

maintain some monotonicity of the events, and simplifies the Poisson convergence proofs. Each fundamental configuration also has a corresponding "enhanced" configuration (Figures 4 and 6), which requires additional open vertices in certain planes. Each of these configurations has nonzero probability in the limit, and affects the limiting spanning probability.

We must now determine which combinations of these ingredients are asymptotically necessary and sufficient for spanning. This is summarized as follows:

(1) At least one "basic" configuration like that in Figure 3(b), AND at least one "line" configuration like that in Figure 3(a); OR

(2) At least one "enhanced basic" configuration like that in Figure 4; OR

(3) At least one "line" configuration, AND at least one askew (nonparallel, nonintersecting) line that contains at least two open vertices (see the configuration in Figure 5); OR

(4) At least two "line" configurations like the one in Figure 3(a); OR

(5) At least one "enhanced line" configuration like those in Figure 6.

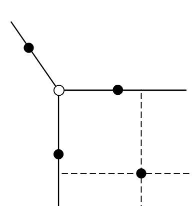


FIG. 4. *"Enhanced Basic": First the two lines containing the open circle in the front plane will be spanned, followed by the two dotted lines then the front plane. Once a plane is spanned, the rest of the graph is likely to be spanned* (*see the last paragraph in the proof of Lemma* 6.1).

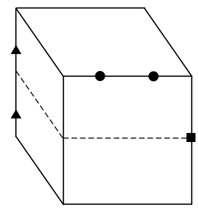


FIG. 5. *This configuration leads to the front plane being spanned, and the graph is likely to be spanned. There is a "line" configuration with respect to the line that contains the two closed circles—the rectangle in the front plane completes the configuration and leads to the spanning of the top line in two steps. After the line with two circles is spanned, the line with two triangles is now in a "line" configuration, and is spanned in two more steps. The vertex at the intersection of the dotted line and the line with the triangles is now open, and leads to the vertex at the intersection of the dotted lines becoming open, which leads to the spanning of the front plane in three more steps. Note that it is crucial for the lines with the circles and triangles to be askew—if these lines were parallel then the front plane would not be spanned without additional help.*

We call ω_0 *good* if it contains at least one of the recipes (1)–(4) described above; a formal definition is given below. The event $\{\omega_0 \text{ is good}\}$ is asymptotically equivalent to the event $\{\omega_0 \text{ spans}\}$ in the sense of the following lemma.

LEMMA 6.1. *If $d = \theta = 3$ and $p = an^{-2}$, then as $n \to \infty$*

$$\mathbb{P}(\omega_0 \textit{ is good}) - \mathbb{P}(\omega_\infty \equiv 1) \to 0.$$

To formally define the event $\{\omega_0 \text{ is good}\}$, and for the proofs that follow, we need to introduce some notation.

Notation. Let e_1, e_2, e_3 denote the standard basis vectors in $\mathbb{R}^3$. For $v, w \in V$ let $d(v, w)$ be the number of nonzero coordinates of $v - w$. Let $\mathcal{N}(v) = \{w \in$

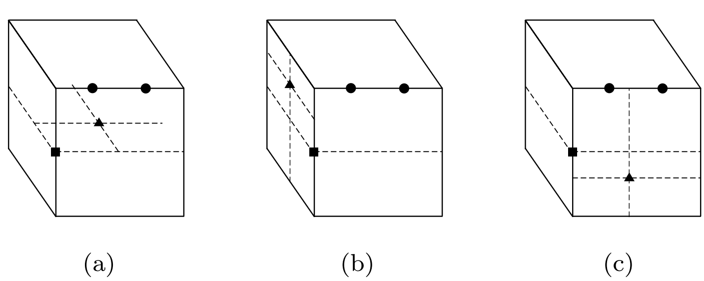


FIG. 6. *"Enhanced Line": These configurations labeled by* (a), (b) *and* (c) (*and any rotations or shifts of them*) *are likely to span. The triangle vertex will cause a second line in the front plane to be spanned, thus the full front plane will be spanned if there is an additional open vertex anywhere in the graph that is not coplanar with this line or the line with two circles. Once a plane is spanned, the rest of the graph is likely to be spanned.*

$V : d(v, w) = 1\}$ denote the neighborhood of v, and for $A \subseteq V$ let $\mathcal{N}(A) = \bigcup_{v \in A} \mathcal{N}(v) \setminus A$.

The basic and enhanced basic configurations will be indexed by vertices, while the line and enhanced line configurations will be indexed by lines. So, we let

$$\mathscr{L} = \{\ell \subseteq V : |\ell| = n \text{ and } \forall v, w \in \ell, d(v, w) \leq 1\}$$

be the set of lines in V. Also, for $i = 1, 2, 3$, let

$$\mathscr{L}_i = \{\ell \in \mathscr{L} : \forall u, v \in \ell, \exists m = m(u, v) \in \mathbb{Z} \text{ s.t. } u = v + m e_i\}$$

denote the collection of lines in V parallel to the coordinate axis in the e_i direction. For the duration of this paper, we will use ℓ to refer to a generic line.

In order to apply the Chen–Stein method, we let `Basic`, `Line`, `Line∅`, `EnhancedBasic`, `EnhancedLine` and `NonEnhancedLine` be the random variables that count the number of occurrences of the corresponding configurations in ω_0, which we now define carefully. The relevant events are a bit difficult to describe, so we refer the reader to Figures 3–6 for guidance.

Define the *basic* event, for $v \in V$, to be

$$G_v^{\mathrm{B}} = \{\exists w_1, w_2, w_3 \in \omega_0 \setminus \{v\} \text{ and } \exists m_1, m_2, m_3 \in \mathbb{Z} \\ \text{ s.t. } v = w_i + m_i e_i \text{ for } i = 1, 2, 3\}.$$

As Figure 3(b) indicates, the basic event occurs at v if v has at least one initially open neighbor in each basis direction. Define the *enhanced basic* event, for $v \in V$, to be

$$G_v^{\mathrm{EB}} = \{\exists w \in \omega_0 \text{ s.t. } d(v, w) = 2, \text{ and } \exists w_1, w_2, w_3 \in \omega_0 \setminus (\mathcal{N}(w) \cup \{v\}) \\ \text{and } \exists m_1, m_2, m_3 \in \mathbb{Z} \text{ s.t. } v = w_i + m_i e_i \text{ for } i = 1, 2, 3\}.$$

As Figure 4 indicates, the enhanced basic event occurs at v if the basic event occurs at v and there is at least one open vertex in one of the planes containing v that is not a neighbor of v. Further, this additional open vertex should not be collinear with the sole open neighbor of v in any direction; if there were two open neighbors of v in a single direction, then we could allow the additional open vertex to be collinear with one of them, but this event is rare. Let I_v^{B} be the indicator random variable for the event G_v^{B}, so $\texttt{Basic} = \sum_v I_v^{\mathrm{B}}$, and let I_v^{EB} be the indicator random variable for the event G_v^{EB}, so $\texttt{EnhancedBasic} = \sum_v I_v^{\mathrm{EB}}$. In general, we will denote by $I_\dagger^*$ the indicator of the event $G_\dagger^*$.

For each line $\ell \in \mathscr{L}$, we define the *line* event

$$G_\ell^{\mathrm{L}} = \{|\ell \cap \omega_0| = 2, |\mathcal{N}(\ell) \cap \omega_0 \setminus \mathcal{N}(\ell \cap \omega_0)| \geq 1\} \\ \cup \{|\ell \cap \omega_0| \geq 3, |\mathcal{N}(\ell) \cap \omega_0| \geq 1\}.$$

As Figure 3(a) suggests, the line event occurs at ℓ if ℓ contains at least two initially open vertices, and there is at least one additional open vertex in the same plane as ℓ.

This additional open vertex should not be in the neighborhood of the two open vertices in ℓ, though if there are three or more open vertices in ℓ then the location of the additional vertex does not matter. We now define $\mathtt{Line} = \sum_{\ell\in\mathscr{L}} I_\ell^{\mathrm{L}}$, and because we will also need to count the number of line events in a particular direction [for case (3) in the recipe for spanning], for $i = 1, 2, 3$ we let $\mathtt{Line}_i = \sum_{\ell\in\mathscr{L}_i} I_\ell^{\mathrm{L}}$. For each $\ell \in \mathscr{L}$, we define the *∅-line* event

$$G_\ell^{\emptyset\mathrm{L}} = \{|\ell \cap \omega_0| \geq 2\} \setminus G_\ell^{\mathrm{L}},$$

and let $I_\ell^{\emptyset\mathrm{L}}$ be the corresponding indicator random variable so $\mathtt{Line}\emptyset = \sum_{\ell\in\mathscr{L}} I_\ell^{\emptyset\mathrm{L}}$ and for $i = 1, 2, 3$, $\mathtt{Line}\emptyset_i = \sum_{\ell\in\mathscr{L}_i} I_\ell^{\emptyset\mathrm{L}}$. The ∅-line event occurs at ℓ if ℓ contains at least two initially open vertices, and there are no other open vertices in the same plane as ℓ (except possibly those that are collinear with one of the two open vertices in ℓ).

For each line $\ell \in \mathscr{L}$, we define the *enhanced line* event

$$\begin{aligned} G_\ell^{\mathrm{EL}} = \{&|\ell \cap \omega_0| = 2 \text{ and } \exists v \in \mathcal{N}(\ell) \cap \omega_0 \setminus \mathcal{N}(\ell \cap \omega_0) \\ &\text{s.t. } |\mathcal{N}(\mathcal{N}(v)) \cap \omega_0 \setminus \mathcal{N}(\ell \cap \mathcal{N}(v))| \geq 1\} \\ &\cup \{|\ell \cap \omega_0| \geq 3, \exists v \in \mathcal{N}(\ell) \cap \omega_0 \text{ s.t. } |\mathcal{N}(\mathcal{N}(v)) \cap \omega_0 \setminus \mathcal{N}(\ell \cap \mathcal{N}(v))| \geq 1\} \end{aligned}$$

and let I_ℓ^{EL} be the corresponding indicator random variable so $\mathtt{EnhancedLine} = \sum_{\ell\in\mathscr{L}} I_\ell^{\mathrm{EL}}$ and for $i = 1, 2, 3$, $\mathtt{EnhancedLine}_i = \sum_{\ell\in\mathscr{L}_i} I_\ell^{\mathrm{EL}}$. For the enhanced line event to occur at ℓ, a line configuration must appear in ω_0 at ℓ and there must be at least one additional open vertex. This additional open vertex is coplanar with the open vertex in $\mathcal{N}(\ell)$ from the line configuration (there may be more than one), but is not counted if it is collinear with this vertex or on the other plane containing ℓ. Finally, define the nonenhanced line event

$$G_\ell^{\mathrm{NEL}} = G_\ell^{\mathrm{L}} \setminus G_\ell^{\mathrm{EL}}$$

and its corresponding indicator I_ℓ^{NEL}, so that $I_\ell^{\mathrm{NEL}} = I_\ell^{\mathrm{L}} - I_\ell^{\mathrm{EL}}$ for every $\ell \in \mathscr{L}$, $\mathtt{NonEnhancedLine} = \mathtt{Line} - \mathtt{EnhancedLine}$ and for $i = 1, 2, 3$, $\mathtt{NonEnhancedLine}_i = \mathtt{Line}_i - \mathtt{EnhancedLine}_i$.

Now we define the event that ω_0 is good by

$$\begin{aligned} \{\omega_0 \text{ is good}\} = {}&\{\mathtt{Basic} \geq 1, \mathtt{Line} \geq 1\} \cup \{\mathtt{EnhancedBasic} \geq 1\} \\ &\cup \bigcup_{i=1}^{3} \Big\{\mathtt{Line}_i \geq 1, \sum_{j\neq i} \mathtt{Line}\emptyset_j \geq 1\Big\} \\ &\cup \{\mathtt{Line} \geq 2\} \cup \{\mathtt{EnhancedLine} \geq 1\}. \end{aligned}$$

The third term above covers the scenario in Figure 5 when $\mathtt{Line} \leq 1$, which is otherwise covered by the event $\{\mathtt{Line} \geq 2\}$. Using inclusion–exclusion, exploiting

TABLE 2
Means of the random variables appearing in (6.1)

Random variable	**Mean**
$\mathtt{Basic}$	a^3
$\mathtt{EnhancedBasic}$	$a^3(1-e^{-3a})$
$\mathtt{Line}$	$\frac{3}{2}a^2(1-e^{-2a})$
$\mathtt{Line\emptyset}_i$	$\frac{1}{2}a^2e^{-2a}$
$\mathtt{NonEnhancedLine}_i$	$\frac{1}{2}a^2[(e^{-a}+ae^{-3a})^2-e^{-2a}]$
$\mathtt{EnhancedLine}$	$\frac{3}{2}a^2[1-(e^{-a}+ae^{-3a})^2]$

obvious symmetries of the graph, and combining like terms:

$$\begin{aligned}
&\mathbb{P}(\omega_0 \text{ is good})\\
&\quad = \mathbb{P}(\mathtt{Basic} \geq 1, \mathtt{Line} = 1) + \mathbb{P}(\mathtt{EnhancedBasic} \geq 1, \mathtt{Line} = 0)\\
&\qquad + \mathbb{P}(\mathtt{Line} \geq 2)\\
&\qquad + \mathbb{P}(\mathtt{Basic} = 0, \mathtt{EnhancedLine} = 1, \mathtt{NonEnhancedLine} = 0) \qquad (6.1)\\
&\qquad + 3\mathbb{P}(\mathtt{Basic} = 0, \mathtt{NonEnhancedLine}_1 = 1,\\
&\qquad\qquad \mathtt{NonEnhancedLine}_2 + \mathtt{NonEnhancedLine}_3 = 0,\\
&\qquad\qquad\qquad \mathtt{EnhancedLine} = 0, \mathtt{Line\emptyset}_2 + \mathtt{Line\emptyset}_3 \geq 1).
\end{aligned}$$

Therefore, once we compute the probabilities in (6.1), Lemma 6.1 implies Theorem 2.2. Lemma 6.2 allows us to do just this, and is followed by the proof of Lemma 6.1. The proof of Lemma 6.2 uses the Chen–Stein method, and is outlined in the Appendix.

LEMMA 6.2. *If $p = an^{-2}$, then as $n \to \infty$ Table 2 gives the means of the random variables appearing in* (6.1). *Furthermore, the two random variables* $\mathtt{EnhancedBasic}$ *and* $\mathtt{Line}$ *converge jointly in distribution to independent Poisson random variables with the above means, as do the eight random variables* $\mathtt{Basic}$, $\mathtt{EnhancedLine}$, *and for* $i = 1, 2, 3$, $\mathtt{NonEnhancedLine}_i$ *and* $\mathtt{Line\emptyset}_i$.

REMARK 6.3. Lemma 6.2 allows us to compute the limiting probability in (6.1) by treating all of the random variables that appear as independent Poisson random variables with the means given by the table. The means that appear in the limit are straightforward to compute. For example, to compute the expected number of basic events, the probability that a fixed vertex has at least one initially open neighbor in each direction is $\sim (np)^3 = a^3/n^3$, and there are n^3 vertices at which a basic configuration can be centered. To obtain the expected number of enhanced basic configurations, observe that a fixed vertex must first see a basic configuration, then independently at least one of the $3(n-2)^2$ coplanar but not collinear vertices must be present. This has probability $1-(1-p)^{3(n-2)^2} \sim 1 - e^{-3a}$ of occurring.

PROOF OF LEMMA 6.1. We will first show that spanning does not occur with high probability when ω_0 is not good. The expected number of lines that contain at least three initially open vertices is $\sim 3n^2\binom{n}{3}p^3 = O(n^{-1})$, so at least one line configuration or basic configuration is necessary for any vertices to become open after one step.

Any vertex that becomes open in the second step must be neighbors with at least one vertex that becomes open in the first step, that is, with a vertex in $\omega_1 \setminus \omega_0$. If $\texttt{Line} = 0$ and $\texttt{EnhancedBasic} = 0$ then any two basic events located at vertices v and w cannot be coplanar unless $\mathcal{N}(v) \cap \mathcal{N}(w) \subseteq \omega_0$, otherwise a line or an enhanced basic configuration would exist. The probability that there exist two vertices, v and w, with $I_v^{\mathrm{B}} I_w^{\mathrm{B}} = 1$, $d(v,w) = 2$ and $\mathcal{N}(v) \cap \mathcal{N}(w) \subseteq \omega_0$ is at most $3n\binom{n^2}{2}(np)^2p^2 = O(n^{-1})$, so with high probability there are no coplanar basic events. Therefore, no pair of vertices in $\omega_1 \setminus \omega_0$ have a common neighbor, and no vertex in $\mathcal{N}(\omega_1 \setminus \omega_0) \setminus \omega_0$ has more than one neighbor in ω_0 (or else a line or enhanced basic configuration would have existed in ω_0). This implies that no vertices can become open in the second step, so spanning cannot occur with high probability when $\texttt{Line} = 0$ and $\texttt{EnhancedBasic} = 0$.

Also, if simultaneously $\texttt{NonEnhancedLine}_1 = 1$, $\texttt{NonEnhancedLine}_2 + \texttt{NonEnhancedLine}_3 = 0$, $\texttt{Basic} = 0$, $\texttt{EnhancedLine} = 0$ and $\texttt{Line}\emptyset_2 + \texttt{Line}\emptyset_3 = 0$ then spanning is unlikely to occur. The sole line configuration will span the focal line, ℓ, after two steps. There may be parallel lines that contain two occupied vertices, but they cannot be coplanar with ℓ or else the line configuration would be enhanced. These parallel lines will not span the cube as their neighborhoods do not intersect ℓ, so no other vertices will become open after two steps. Therefore, $\mathbb{P}(\{\omega_\infty \equiv 1\} \setminus \{\omega_0 \text{ is good}\}) \to 0$.

The probability of ω_0 containing a basic configuration and a line configuration that share a plane [i.e., there exist v and ℓ so that $I_v^{\mathrm{B}} I_\ell^{\mathrm{L}} = 1$ and $v \in \mathcal{N}(\ell) \cup \ell$] is at most $Cn^3(n)(np)^3(np)^2 = O(n^{-1})$. Similarly, the probability of having two or more coplanar line configurations is $O(n^{-1})$. Conditional on the complements of these last two events, observe that a line configuration will cause a basic configuration to become an enhanced basic configuration in two steps. Likewise, a line configuration will cause a second line configuration to become an enhanced line configuration in two steps; and similarly a line configuration will with high probability cause an askew line with two initially open vertices to become a line configuration (and subsequently an enhanced line configuration).

Both the enhanced basic and enhanced line configurations lead to a plane becoming open. Once a plane is open, two nonneighboring, coplanar open vertices will cause another plane to become open, then one more open vertex elsewhere will cause the rest of the graph to become open. With probability exponentially close to 1, there are at least $n^{1/2}$ planes with at least two nonneighboring open vertices in ω_0. Therefore, $\mathbb{P}(\{\omega_0 \text{ is good}\} \setminus \{\omega_\infty \equiv 1\}) = O(n^{-1})$, and the two events are asymptotically equivalent. □

7. Open one-dimensional subgraphs. In this section, we obtain an upper bound on the threshold probability for lines, $p_c(1, d)$. The main idea is the following. Assume that the line ℓ contains $r \le \theta$ initially open vertices, that it intersects one line with $\theta - r$ initially open sites (not on ℓ), and that it intersects θ other lines, each with $\theta - r - 1$ sites (not on ℓ) initially open. Then after one step, ℓ has $r + 1$ points open, and after two steps, θ points open. After three steps, ℓ is completely open. See Figure 7 for an illustration.

For a set $S \subseteq V$ and $x \in \mathbb{N}$, let $\mathrm{Initial}(S, \ge x)$ be the event that the set S has at least x points initially open, that is,

$$\mathrm{Initial}(S, \ge x) = \left\{\sum_{v \in S} \omega_0(v) \ge x\right\}.$$

For a point $v \in V$, let $P_{1,2}(v)$ be the e_1, e_2-parallel plane through v:

$$P_{1,2}(v) = \{(a_1, a_2, v_3, v_4, \dots, v_d) : a_1, a_2 \in [n]\}.$$

Let $\ell_2(v)$ be the e_2-parallel line through v:

$$\ell_2(v) = \{(v_1, a_2, v_3, v_4, \dots, v_d) : a_2 \in [n]\}.$$

For any e_1-parallel line ℓ, define

$$\ell_l = \{w \in \ell, w_1 < n/3\}, \qquad \ell_m = \{w \in \ell, n/3 \le w_1 \le 2n/3\},$$

and

$$\ell_r = \{w \in \ell, w_1 > 2n/3\}$$

to be the left, middle and right thirds of ℓ. Define

$$\text{Cross Lines}_m(\ell) = \left\{\sum_{v \in \ell_m} \mathbf{1}_{\mathrm{Initial}(\ell_2(v), \ge \theta - r)} \ge 1\right\}$$

$$\text{Cross Lines}_r(\ell) = \left\{\sum_{v \in \ell_r} \mathbf{1}_{\mathrm{Initial}(\ell_2(v), \ge \theta - r - 1)} \ge \theta\right\}$$

and

$$F_\ell = \mathrm{Initial}(\ell_l, \ge r) \cap \text{Cross Lines}_m(\ell) \cap \text{Cross Lines}_r(\ell).$$

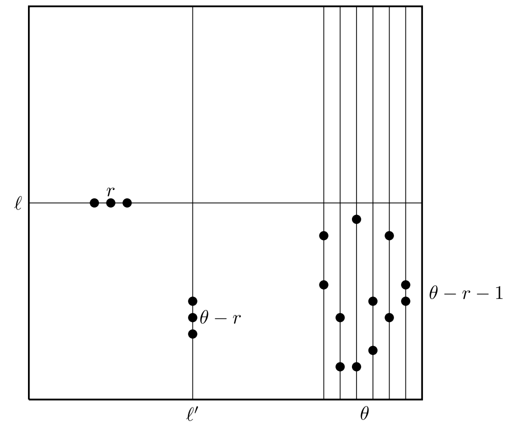


FIG. 7. *An instance of the event F_ℓ. Here, $\theta = 6$, $r = 3$. After one step, the intersection of lines ℓ and ℓ' becomes open so ℓ has $r + 1$ vertices open. At step 2, the θ intersections with ℓ and the other θ vertical lines become open. At step 3, all of ℓ becomes open.*

Notice that the event F_ℓ depends only on the sites in $P_{1,2}(v)$ for any $v \in \ell$. Also note that

$$\text{Cross Lines}_m(\ell) = \text{Cross Lines}_m(\ell') \quad \text{and} \quad \text{Cross Lines}_r(\ell) = \text{Cross Lines}_r(\ell')$$

for any e_1-parallel lines $\ell \neq \ell'$ that lie in a common e_1, e_2-parallel plane. Finally, note that $\text{Initial}(\ell_l, \geq r)$, $\text{Cross Lines}_m(\ell)$, and $\text{Cross Lines}_r(\ell)$ are independent, and $\text{Initial}(\ell_l, \geq r)$ and $\text{Initial}(\ell'_l, \geq r)$ are independent.

We exhibit the role of F_ℓ (see Figure 7) in the following lemma.

LEMMA 7.1. *If ℓ is a line parallel to the e_1 axis and F_ℓ occurs, then the entire line ℓ is open after three steps.*

REMARK 7.2. Computation of $P(F_\ell)$ is facilitated by independence of the three events. A more natural definition would not restrict the orientations of the lines, or demand that the event happen in the left, middle or right sections thereof, and would increase the probability by a constant factor, independent of n.

We set $r = \lceil \frac{(d-1)\theta}{d} \rceil - 1$ and $p = n^{-1-d/\theta} f(n)$, where $f(n)$ is any function such that $f(n) \to \infty$. We will show that in this regime some line becomes open asymptotically almost surely. We will use the following elementary fact about the binomial distribution.

LEMMA 7.3. *Assume that S is* Binomial(n, p), *with large n and $p = p(n)$, and that k does not depend on n. If $np = O(1)$, then $P(S \geq k) \geq c(np)^k$ for some constant c dependent on k. If $np \to \infty$, then $P(S \geq k) \to 1$.*

LEMMA 7.4. *Fix $v \in V$ and $\theta, d \geq 3$. Let $p = n^{-1-d/\theta} f(n)$ where $f(n) \to \infty$. Then for any $c > 0$, the probability that there exists an e_1-parallel line ℓ in $P_{1,2}(v)$ such that F_ℓ occurs is at least cn^{2-d} for n sufficiently large.*

PROOF. As the event in the statement is increasing, its probability is monotone in p. Thus, we may assume that $f(n)$ grows to ∞ as slowly as we need in the proof.

Note that when $\theta, d \geq 3$ then $rd/\theta \geq 1$ as

$$rd/\theta \geq \left(\frac{(d-1)\theta}{d} - 1\right)\frac{d}{\theta} = d - 1 - d/\theta.$$

The right-hand side is strictly greater than 1 except if $d = \theta = 3$. We assume that at least one of d and θ is at least 4, and leave the exceptional case to the reader.

The three events that define F_ℓ depend on disjoint sets of sites, so they are independent and we compute their probabilities separately. Furthermore, for the set of lines ℓ we consider, the events Cross $\text{Lines}_m(\ell)$ and Cross $\text{Lines}_r(\ell)$ do not depend on ℓ, which will thus be dropped from the notation. For any ℓ, by Lemma 7.3

$$\begin{aligned}\mathbb{P}\big(\text{Initial}(\ell_l, \geq r)\big) &\geq c_1(np)^r \\ &\geq c_1\big(f(n)n^{-d/\theta}\big)^r.\end{aligned}$$

As this is $o(1/n)$, we can use Lemma 7.3 again to get that

$$\mathbb{P}\big(\exists \ell \text{ such that } \text{Initial}(\ell_l, \geq r) \text{ occurs}\big) \geq c_2 n\big(f(n)n^{-d/\theta}\big)^r.$$

To estimate the second probability, observe that

$$\mathbb{P}\big(\text{Initial}\big(\ell_2(v), \geq \theta - r\big)\big) \geq c_3(np)^{\theta - r},$$

which is $o(1/n)$, as $r < (d-1)\theta/d$. Thus,

$$\begin{aligned}\mathbb{P}(\text{Cross Lines}_m) &\geq c_4 n(np)^{\theta-r} \\ &\geq c_4 n\big(f(n)n^{-d/\theta}\big)^{\theta - r}.\end{aligned}$$

For the third probability,

$$\mathbb{P}\big(\text{Initial}\big(\ell_2(v), \geq \theta - r\big)\big) \geq c_5(np)^{\theta-r-1},$$

and

$$n \cdot (np)^{\theta-r-1} \geq f(n)^{\theta-r-1} n^{1-d+(r+1)d/\theta} \to \infty$$

as $n \to \infty$, so Lemma 7.3 implies that

$$\mathbb{P}(\text{Cross Lines}_r) \to 1,$$

and for large n the probability is bounded below by a constant $c_6 > 0$. Multiplying together the probabilities, we have that for any c and all sufficiently large n

$$\begin{aligned}
&\mathbb{P}\big(\exists \ell \text{ in } P_{1,2}(v) \text{ such that that } F_\ell \text{ occurs}\big)\\
&\quad = \mathbb{P}\big(\exists \ell \text{ such that } \operatorname{Initial}(\ell_l, \geq r)\big)\mathbb{P}(\text{Cross Lines}_m)\mathbb{P}(\text{Cross Lines}_r)\\
&\quad \geq c_2 n\big(f(n)n^{-d/\theta}\big)^r c_4 n\big(f(n)n^{-d/\theta}\big)^{\theta-r} c_6\\
&\quad = c_7 f(n)^\theta n^{2-d}\\
&\quad > cn^{2-d},
\end{aligned}$$

ending the proof. □

THEOREM 7.5. *Suppose that* $p = n^{-1-d/\theta} f(n)$ *with* $f(n) \to \infty$. *Then* $\mathbb{P}(\bigcup_\ell F_\ell) \to 1$ *as* $n \to \infty$, *where the union is taken over all* e_1*-parallel lines. Thus, with probability going to* 1, *some line becomes open after three steps.*

PROOF. We can choose n^{d-2} distinct vertices v_i such that $P_{1,2}(v_i)$ are disjoint. Then the events that there exist ℓ in $P_{1,2}(v_i)$ where F_ℓ occurs are independent. Moreover,

$$n^{d-2}\mathbb{P}\big(\exists \ell \text{ in } P_{1,2}(v_i) \text{ such that } F_\ell \text{ occurs}\big) \geq n^{d-2} c n^{2-d} = c$$

for any fixed c. Thus, $\mathbb{P}(\bigcup_\ell F_\ell) \to 1$ by Lemma 7.3. □

THEOREM 7.6. *Assume that* $p = n^{-1-d/\theta} f(n)$, *with* $f(n) \to 0$, *then* $\mathbb{P}(\text{Above Threshold}) \to 0$.

PROOF. Using the union bound,

$$\begin{aligned}
\mathbb{P}(\text{Above Threshold}) &\leq \sum_{v \in V} \mathbb{P}\Big(\sum_{w \sim v} \omega_0(w) \geq \theta\Big)\\
&= n^d \mathbb{P}\Big(\sum_{w \sim v} \omega_0(w) \geq \theta\Big)\\
&\leq n^d \binom{n}{\theta} p^\theta\\
&\leq f(n)^\theta
\end{aligned}$$

which approaches 0 as $n \to \infty$. □

PROOF OF THEOREM 2.3. Combining Theorems 7.5 and 7.6 proves the result. □

8. Open two-dimensional subgraphs. In previous sections, we have encountered several possibilities for a vertex v to become open:

- v is initially open;
- the neighborhood of v has at least θ vertices initially open, causing v to become open by time 1; and
- a line containing v has at least $\theta(d-1)/d$ vertices initially open, with some additional open sites "nearby" (see Section 7).

Let Plane Active be the event that some plane eventually becomes open. In this section, we show that if p is sufficiently small then with high probability all of the vertices that are eventually open satisfy a condition like one of the three above. By doing this, we prove an upper bound on the probability of Plane Active and consequently a lower bound on the threshold probability $p_c(2, d)$.

Let A be some integer, $1 \le A \le \theta$, which we will specify later. Let E be the event that there exists a vertex v such that:

(1) v is initially not open;
(2) the neighborhood of v has at most A vertices initially open;
(3) each line containing v has at most $A/2$ vertices initially open; and
(4) v becomes open.

Our strategy to demonstrate that $\mathbb{P}(\text{Plane Active})$ is small for sufficiently small p is to show that $\mathbb{P}(E)$ and $\mathbb{P}(\text{Plane Active} \setminus E)$ are both small.

For each vertex v, let E_v be the event that v satisfies (1)–(4), and none among such vertices becomes open earlier. If the event E occurs, then there must be a first time a vertex satisfying (1)–(4) exists, thus $E \subseteq \bigcup_v E_v$, and consequently, $\mathbb{P}(E) \le n^d \mathbb{P}(E_v)$.

LEMMA 8.1. *Suppose $p = o(n^{-1-\beta})$ with $\beta > (\frac{2d^2}{\theta - A} + 1)\frac{2}{A}$. Fix a line ℓ. The probability that ℓ contains at least $\frac{\theta - A}{2d}$ vertices v that have at least $A/2$ initially open points in $\mathcal{N}(v) \setminus \ell$ is*

$$o\big(n^{(\theta - A)/(2d)(1 - \beta A/2)}\big).$$

PROOF. The reduced neighborhoods $\mathcal{N}(v) \setminus \ell$, $v \in \ell$, are pairwise disjoint, and in each the number of initially open vertices is a Binomial$((d-1)(n-1), p)$ random variable. The probability that such a random variable is at least $A/2$ is bounded by a constant times $(np)^{A/2} = o(n^{-\beta A/2})$. These random variables are independent, thus the probability that at least $\frac{\theta - A}{2d}$ of them are at least $A/2$ is $o((n \cdot n^{-\beta A/2})^{(\theta - A)/(2d)})$. □

LEMMA 8.2. *Assume p satisfies the same bound as in Lemma 8.1. Fix a line ℓ. The probability that ℓ has at least $\frac{\theta - A}{2d}$ vertices w, for which there exists a line $\ell' \ne \ell$ through w such that $\ell' \setminus \{w\}$ contains at least $A/2$ initially open points is*

$$o\big(n^{(\theta - A)/(2d)(1 - \beta A/2)}\big).$$

PROOF. We need to bound the probability of at least $\frac{\theta-A}{2d}$ successes in $n(d-1)$ independent trials, each of which is a success with the probability that a given line has at least $A/2$ points initially open. Same estimates as in the proof of Lemma 8.1 apply. □

LEMMA 8.3. *Assume p satisfies the same bound as in Lemma 8.1. Then $\mathbb{P}(E) \to 0$ as $n \to \infty$.*

PROOF. As we have already observed, $\mathbb{P}(E) \le n^d \mathbb{P}(E_v)$. Now, if E_v occurs, by (2) at least $\theta - A$ vertices in the neighborhood of v must be initially closed but become open strictly before v; therefore, they violate at least one of (1)–(4). But since they are not open initially and become open, they must violate one of (2) or (3). By the pigeonhole principle, of the d lines through v, at least one must either contain $\frac{\theta-A}{2d}$ vertices which violate (2), or $\frac{\theta-A}{2d}$ vertices which violate (3).

By Lemmas 8.1 and 8.2, each of these happens with probability

$$o\big(n^{(\theta-A)/(2d)(1-\beta A/2)}\big).$$

Rearranging using the inequality $\beta > (\frac{2d^2}{\theta-A}+1)\frac{2}{A}$, we see that $\mathbb{P}(E_v) = o(n^{-d})$, as claimed. □

LEMMA 8.4. *Let $p = n^{-1-\beta}$, with $\beta > 0$, and assume $A \ge 4$. Then $\mathbb{P}$(Plane Active $\setminus$ E) $\to 0$ as $n \to \infty$.*

PROOF. There are $\binom{d}{2}n^{d-2}$ planes, P, and Plane Active $= \bigcup_P\{P$ becomes open$\}$, so we have

$$\mathbb{P}(\text{Plane Active} \setminus E) \le \binom{d}{2} n^{d-2}\mathbb{P}\big(\{P \text{ becomes open}\} \setminus E\big).$$

Now if P becomes open but E does not occur, then since each point in P becomes open, they must all violate one of (1), (2) or (3). By the pigeonhole principle, at least $n^2/3$ of these points must together violate a single condition. We will check that the probabilities of these three cases are $o(n^{-(d-2)})$. In fact, we will see that they are exponentially small by reducing each case to a large deviation probability involving a Binomial random variable with a small chance of success. We will use the fact that neighborhoods of two points in P do not intersect outside P.

- $\mathbb{P}(n^2/3$ vertices in P are initially open) is exponentially small in n^2, as $p = o(1)$.
- $\mathbb{P}(n^2/3$ vertices in P are each on a line with $A/2$ points initially open) is exponentially small in n.

  As every line covers at most n points in P, this event implies that there are at least $n/(3d)$ parallel lines, in some direction e_i, each with at least $A/2$ points initially open. The probability that a given line has at least $A/2$ points initially open is $O((np)^{\lfloor A/2 \rfloor}) = o(1)$, thus the probability that $n/(3d)$ lines in a given direction e_i satisfy this is exponentially small in n.

- $\mathbb{P}(n^2/3$ vertices in P each have at least A initially open vertices in their neighborhoods) is exponentially small in n.

  If a vertex w has at least A initially open vertices in its neighborhood then either one of the two lines through w in P contain at least $A/4$ initially open vertices or the $d-2$ lines through w not in P together contain at least $A/2$ initially open vertices. This implies that either (a) there are at least $n/12$ parallel lines in P with at least $A/4$ vertices initially open, or (b) there are at least $n^2/6$ vertices with at least $A/2$ vertices in their neighborhoods outside of P.

  The probability of (a) is exponentially small by the same argument as in the previous case. For a fixed w, the probability that $(d-2)(n-1)$ sites in $\mathcal{N}(w) \setminus P$ contain at least $A/2$ initially open sites is again $O(np) = o(1)$. Thus, the probability of (b) is exponentially small in n^2.

Therefore, $\mathbb{P}(\text{Plane Active} \setminus E)$ goes to 0 exponentially fast. □

PROOF OF THEOREM 2.4. To get the lower bound set, $A = \lfloor \theta - \sqrt{\theta} \rfloor$. Then Lemmas 8.1–8.3 are (for large enough θ) satisfied with

$$\beta = \frac{2}{\theta} + \frac{4d^2+3}{\theta^{3/2}}.$$

The upper bound was proved in Theorem 5.3. □

9. Further questions and conjectures. We begin with a general form of threshold probabilities; we believe that the answer to the question below is positive.

QUESTION 9.1. Do there exist positive constants $c_1 = c_1(i,d)$ and $c_{3/2} = c_{3/2}(i,d)$, so that, for all i and d, a lower bound and an upper bound for $p_c(i,d)$ are both of the form

$$n^{-1-c_1/\theta - c_{3/2}/\theta^{3/2} + o(\theta^{-3/2})}$$

for large enough n?

We next ask whether it is possible that generation of open planes does not likely lead to spanning of the entire graph when $d \geq 4$.

QUESTION 9.2. Can one find d and $\theta > 2$ such that $\log_n(p_c(2,d)) - \log_n(p_c(d,d))$ is bounded away from 0 as $n \to \infty$, that is, $p_c(2,d) \approx n^{-\zeta}$ and $p_c(d,d) \approx n^{-\xi}$ with $\zeta > \xi$? Does this hold for all θ and $d \geq 4$? Note that it does not hold for $d = 3$ by (2.2).

It would be desirable to have a general method to determine the critical exponent for any given (small) d and θ; here we merely recall the simplest unsolved instances.

QUESTION 9.3. When $d = 3$, we know the critical exponents for $\theta = 2, 3, 4, 5, 6, 7, 9, 11$; what are the correct exponents for $\theta = 8, 10$ and $\theta \geq 12$?

APPENDIX: POISSON CONVERGENCE FOR $d = \theta = 3$

In this section, we outline the proof of Lemma 6.2 regarding Poisson convergence of the random variables that count the configurations that lead to spanning when $d = \theta = 3$ and $p = an^{-2}$. Our approach is to apply the Chen–Stein method [7], and to do so we need to introduce some notation.

We want to show that a collection of random variables, which are sums of indicator random variables, converge to independent Poisson random variables in the limit. That is, suppose we have disjoint sets of indices, $\Gamma_1, \Gamma_2, \dots, \Gamma_\ell$, let $\Gamma = \bigcup_{i=1}^{\ell} \Gamma_i$, and for each $\gamma \in \Gamma$ suppose I_γ is an indicator random variable. For $i = 1, \dots, \ell$ let $W_i = \sum_{\gamma \in \Gamma_i} I_\gamma$ and suppose that $EW_i = \lambda_i$ and $EI_\gamma = p_\gamma$. In our application, the index sets are going to be V for the indicators of the basic and enhanced basic events, and $\mathscr{L}$ for the indicators of the line, Ø-line, enhanced line and nonenhanced line events.

To apply the Chen–Stein method in many cases, we need to construct a coupling for every fixed $\gamma \in \Gamma$ between I_η and $J_{\eta\gamma}$ so that

$$(J_{\eta\gamma})_{\eta \neq \gamma} \stackrel{d}{=} (I_\eta | I_\gamma = 1)_{\eta \neq \gamma}. \tag{A.1}$$

Many of the indicators that we have constructed are increasing functions of ω_0, which makes those sets of indicators positively related ([7], Section 2.1). However, the Ø-line and nonenhanced line indicators, $I_\ell^{\text{ØL}}$ and I_ℓ^{NEL}, are not increasing functions of ω_0, so whenever these appear we are unable to use the simpler form of the Poisson convergence theorem. Instead, we will explicitly define the couplings below, and use Theorem 10.J of [7], which we state below as Lemma A.1.

Suppose X and Y are two $\mathbb{Z}^m$-valued random variables with laws μ_X and μ_Y, and recall that the total variation distance between μ_X and μ_Y (or with an abuse of notation, between X and Y or X and μ_Y) is

$$d_{\text{TV}}(X, Y) = d_{\text{TV}}(\mu_X, \mu_Y) := \sup_{A \subseteq \mathbb{Z}^m} |\mu_X(A) - \mu_Y(A)| = \frac{1}{2} \sum_{k \in \mathbb{Z}^m} |\mu_X(k) - \mu_Y(k)|.$$

Let P_λ denote the law of a Poisson(λ) random variable (taking values in $\mathbb{Z}_+$). The Chen–Stein method gives us the following bound on the total variation distance between the joint law of $(W_1, W_2, \dots, W_m)$ and $\prod_{i=1}^{m} P_{\lambda_i}$.

LEMMA A.1 ([7], Theorem 10.J and Corollary 10.J.1). *If W_i are defined as above with $\lambda_i = EW_i$ for $i = 1, \dots, \ell$, with $EI_\gamma = p_\gamma$, then*

$$d_{\mathrm{TV}}\left((W_1, \dots, W_m), \prod_{i=1}^m P_{\lambda_i}\right) \le \sum_{\gamma \in \Gamma} p_\gamma^2 + \sum_{\substack{\gamma, \eta \in \Gamma \\ \gamma \neq \eta}} p_\gamma \mathbb{E}|J_{\eta\gamma} - I_\eta|. \tag{A.2}$$

If $\{I_\gamma\}_{\gamma \in \Gamma}$ are positively related then

$$d_{\mathrm{TV}}\left((W_1, \dots, W_m), \prod_{i=1}^m P_{\lambda_i}\right) \le \sum_{\gamma \in \Gamma} p_\gamma^2 + \sum_{\substack{\gamma, \eta \in \Gamma \\ \gamma \neq \eta}} \mathrm{Cov}(I_\gamma, I_\eta). \tag{A.3}$$

REMARK A.2. In all of our applications of Lemma A.1, the first sum on the right-hand side is easy to control, since it merely requires that p_γ are uniformly small. In the case of events indexed by $\mathscr{L}$ this sum is $O(n^{-2})$, since there are $O(n^2)$ summands and the probability of a line configuration is $O(n^2 p^2) = O(n^{-2})$. Similarly, in the case of basic or enhanced basic events this sum is $O(n^{-3})$. The important part of the right-hand side is the term $\mathbb{E}|J_{\eta\gamma} - I_\eta| = \mathbb{P}(J_{\eta\gamma} \neq I_\eta)$, which requires bounding the probability that our coupling destroys or creates the event indicated by I_η. In the case of positively related indicators, no explicit coupling is needed, and we must merely bound the covariances between the relevant indicators.

Construction of couplings. Observe that in equation (6.1), the last term involves random variables that are sums of indicators that are not positively related. So, for each of the indicators I_v^{B}, $I_\ell^{\emptyset\mathrm{L}}$, I_ℓ^{EL}, I_ℓ^{NEL} and every $v \in V$ and $\ell \in \mathscr{L}$, we must construct a suitable coupling between all of the remaining indicators and their conditioned versions as in (A.1). As in (A.1), we will use the letter J for coupled indicator random variables.

Once we show that these random variables appearing in the last term of (6.1) converge jointly to independent Poissons, we will be able to compute the limiting probabilities for all of the terms except the second, which involves the `EnhancedBasic` and `Line` random variables. We will treat this term separately using the simpler form of Lemma A.1, since the enhanced basic and line indicators are positively related.

Our goal is to show that the second summation in (A.2) is $O(n^{-1})$ under the couplings that we construct. We will need to construct four couplings, one for each type of indicator, and for each coupling we have four comparisons (to each of the four types of indicators) that need to be made. Furthermore, for each comparison, there are several cases that need to be checked depending on the relative positions of the vertices and lines that index each event. There are many cases that need to be verified, but the arguments quickly become repetitive, thus we merely outline the proof and give complete details in two typical cases (see proofs of Lemmas A.6 and A.7).

We begin with the simplest case, the *basic coupling* for conditioning on $I_v^{\mathrm{B}} = 1$ for a fixed $v \in V$. In this case, we merely need each of the three lines containing v to contain at least one open vertex. To achieve this, we extend the probability space by possibly resampling the vertices in each of the three lines until this condition is met. That is, if a line through v already contains an open vertex, nothing is resampled for that line, and the original configuration is kept, otherwise it is repeatedly replaced with an independent configuration until it does contain an open vertex. Also, it is important to note that none of the other vertices in the initial configuration, ω_0, are altered. Then J_{wv}^{B}, $J_{\ell v}^{\emptyset\mathrm{L}}$, $J_{\ell v}^{\mathrm{EL}}$, $J_{\ell v}^{\mathrm{NEL}}$ are the indicator random variables of the corresponding events after the local resampling is completed. Since v is fixed and the Hamming torus is transitive, we will drop the index v in the conditioning on $I_v^{\mathrm{B}} = 1$.

LEMMA A.3. *Under the basic coupling, the following sums are all $O(n^{-1})$*:

$$\sum_{v\in V}\sum_{\substack{w\in V\\ w\neq v}} EI_v^{\mathrm{B}}\mathbb{P}(I_w^{\mathrm{B}} \neq J_w^{\mathrm{B}}), \qquad \sum_{v\in V}\sum_{\ell\in\mathscr{L}} EI_v^{\mathrm{B}}\mathbb{P}(I_\ell^{\emptyset\mathrm{L}} \neq J_\ell^{\emptyset\mathrm{L}}),$$

$$\sum_{v\in V}\sum_{\ell\in\mathscr{L}} EI_v^{\mathrm{B}}\mathbb{P}(I_\ell^{\mathrm{NEL}} \neq J_\ell^{\mathrm{NEL}}), \qquad \sum_{v\in V}\sum_{\ell\in\mathscr{L}} EI_v^{\mathrm{B}}\mathbb{P}(I_\ell^{\mathrm{EL}} \neq J_\ell^{\mathrm{EL}}).$$

The next simplest coupling is the *∅-line coupling* for the conditioning on $I_\ell^{\emptyset\mathrm{L}} = 1$ for a fixed $\ell \in \mathscr{L}$. For this coupling, we need the line ℓ to contain at least two initially open vertices, so we first resample the vertices in ℓ if necessary until this condition is met. Given the locations of the open vertices in ℓ, we need the two planes containing ℓ to have no open vertices that are not neighbors of the open vertices in ℓ. To achieve this, we simply remove any violating vertices from ω_0. In the next three lemmas, we use indicators J, with proper subscripts and superscripts, in an analogous fashion as in Lemma A.3.

LEMMA A.4. *Under the ∅-line coupling, the following sums are $O(n^{-1})$*

$$\sum_{\ell\in\mathscr{L}}\sum_{w\in V} EI_\ell^{\emptyset\mathrm{L}}\mathbb{P}(I_w^{\mathrm{B}} \neq J_w^{\mathrm{B}}), \qquad \sum_{\ell\in\mathscr{L}}\sum_{\substack{\ell'\in\mathscr{L}\\ \ell'\neq\ell}} EI_\ell^{\emptyset\mathrm{L}}\mathbb{P}(I_{\ell'}^{\emptyset\mathrm{L}} \neq J_{\ell'}^{\emptyset\mathrm{L}}),$$

$$\sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\emptyset\mathrm{L}}\mathbb{P}(I_{\ell'}^{\mathrm{NEL}} \neq J_{\ell'}^{\mathrm{NEL}}), \qquad \sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\emptyset\mathrm{L}}\mathbb{P}(I_{\ell'}^{\mathrm{EL}} \neq J_{\ell'}^{\mathrm{EL}}).$$

Next, we construct the *enhanced line coupling* for the conditioning on $I_\ell^{\mathrm{EL}} = 1$ for a fixed $\ell \in \mathscr{L}$. To achieve this, we will need the line ℓ to contain at least two open vertices, so we first resample the vertices in ℓ if necessary until this condition is met. Next, given the locations of the open vertices in ℓ, we need that at least one of the two planes containing ℓ has at least one open vertex that is not collinear with an open vertex in ℓ. Again, if necessary, we resample these two planes (excepting the vertices in ℓ) simultaneously until this condition is satisfied. At this point, if one of the two planes containing ℓ has at least two nonneighboring open vertices, then the coupling is completed. Otherwise, conditional on the location of the open vertex (or vertices) in $\mathcal{N}(\ell)$, we need there to be at least one open vertex in the same plane as this vertex (or vertices) but not in the same line. If one does not exist, then we resample the two (or four) planes containing the open vertex (or vertices) in $\mathcal{N}(\ell)$ but not containing ℓ until there is at least one open vertex in any of these planes [we do not resample the vertices in ℓ, $\mathcal{N}(\ell)$, or the neighborhood of the open vertices in $\mathcal{N}(\ell)$].

LEMMA A.5. *Under the enhanced line coupling, the following sums are* $O(n^{-1})$:

$$\sum_{\ell\in\mathscr{L}}\sum_{w\in V} EI_\ell^{\mathrm{EL}}\mathbb{P}(I_w^{\mathrm{B}}\neq J_w^{\mathrm{B}}),\qquad \sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\mathrm{EL}}\mathbb{P}(I_{\ell'}^{\emptyset\mathrm{L}}\neq J_{\ell'}^{\emptyset\mathrm{L}}),$$

$$\sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\mathrm{EL}}\mathbb{P}(I_{\ell'}^{\mathrm{NEL}}\neq J_{\ell'}^{\mathrm{NEL}}),\qquad \sum_{\ell\in\mathscr{L}}\sum_{\substack{\ell'\in\mathscr{L}\\ \ell'\neq\ell}} EI_\ell^{\mathrm{EL}}\mathbb{P}(I_{\ell'}^{\mathrm{EL}}\neq J_{\ell'}^{\mathrm{EL}}).$$

Finally, we construct the *nonenhanced line coupling* for the conditioning on $I_\ell^{\mathrm{NEL}}=1$ for a fixed $\ell\in\mathscr{L}$. To achieve this, we will need the line ℓ to contain at least two open vertices. So, first we resample the vertices in ℓ if necessary until this condition is met. Next, given the locations of the open vertices in ℓ, we need: (1) that at least one of the two planes containing ℓ has at least one open vertex that is not collinear with an open vertex in ℓ, and (2) that neither plane containing ℓ has more than one noncollinear open vertex. Again, if necessary, we resample these two planes simultaneously until these conditions are met (here we do not resample ℓ). Now, conditional on the locations of the open points in $\mathcal{N}(\ell)$, we must guarantee that there are no other points outside of ℓ that are coplanar but not collinear with these points. For this part of the coupling, we simply remove any violating points from ω_0.

LEMMA A.6. *Under the nonenhanced line coupling, the following sums are* $O(n^{-1})$:

$$\sum_{\ell\in\mathscr{L}}\sum_{w\in V} EI_\ell^{\mathrm{NEL}}\mathbb{P}(I_w^{\mathrm{B}}\neq J_w^{\mathrm{B}}),\qquad \sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\mathrm{NEL}}\mathbb{P}(I_{\ell'}^{\emptyset\mathrm{L}}\neq J_{\ell'}^{\emptyset\mathrm{L}}),$$

$$\sum_{\ell\in\mathscr{L}}\sum_{\substack{\ell'\in\mathscr{L}\\ \ell'\neq\ell}} EI_\ell^{\mathrm{NEL}}\mathbb{P}(I_{\ell'}^{\mathrm{NEL}}\neq J_{\ell'}^{\mathrm{NEL}}),\qquad \sum_{\ell\in\mathscr{L}}\sum_{\ell'\in\mathscr{L}} EI_\ell^{\mathrm{NEL}}\mathbb{P}(I_{\ell'}^{\mathrm{EL}}\neq J_{\ell'}^{\mathrm{EL}}).$$

PROOF. We now outline the proof by bounding the first summation above. There are three cases.

Case 1: $w \in \ell$. This term appears in the sum $O(n^3)$ times, and $EI_\ell^{\mathrm{NEL}} = O(n^{-2})$, so we must show that $\mathbb{P}(I_w^{\mathrm{B}} \neq J_w^{\mathrm{B}}) = O(n^{-2})$. Now there are two subcases, *destruction* and *creation,* respectively: $\mathbb{P}(I_w^{\mathrm{B}} = 1, J_w^{\mathrm{B}} = 0)$ and $\mathbb{P}(I_w^{\mathrm{B}} = 0, J_w^{\mathrm{B}} = 1)$. Clearly, $\mathbb{P}(I_w^{\mathrm{B}} = 1, J_w^{\mathrm{B}} = 0) \leq \mathbb{P}(I_w^{\mathrm{B}} = 1) = O(n^{-3})$. Next, in order for the creation event to occur, the resampling procedure must have generated at least one open vertex in both planes containing ℓ, and both of these points must lie in the neighborhood of w. The probability of this is $O(n^{-2})$, since we require an open vertex in each of two fixed lines.

Case 2: $w \in \mathcal{N}(\ell)$. This term appears in the sum $O(n^4)$ times, and $EI_\ell^{\mathrm{NEL}} = O(n^{-2})$, so we must show that $\mathbb{P}(I_w^{\mathrm{B}} \neq J_w^{\mathrm{B}}) = O(n^{-3})$. Once again, there are two subcases as above. The creation event cannot occur in this case because an open vertex in $\mathcal{N}(\ell)$ that is collinear with w must not see any coplanar open vertices (off of ℓ), which includes a line in the neighborhood of w, so w can no longer see an open vertex in each direction. The probability of the destruction event can be trivially bounded by $O(n^{-3})$ as in Case 1.

Case 3: $w \notin \mathcal{N}(\ell) \cup \ell$. This term appears in the sum $O(n^5)$ times, and $EI_\ell^{\mathrm{NEL}} = O(n^{-2})$, so we must show that $\mathbb{P}(I_w^{\mathrm{B}} \neq J_w^{\mathrm{B}}) = O(n^{-4})$. Once again, the creation event cannot occur for the same reason as cited in Case 2. The destruction event can only occur if one of the initially open points in the neighborhood of w is in one of the resampled planes. At most six planes are affected with probability $1 - O(n^{-1})$, and with the same probability none of the resampled planes contain a line in the neighborhood of w. The probability of the destruction event is at most $O(n^{-4})$, since w must first have three open neighbors initially [an event with probability $O(n^{-3})$], and at least one must coincide with one of the resampled planes [an event with probability $O(n^{-1})$]. □

Positively related case. Since $\{I_v^{\mathrm{EB}}\}_{v \in V}$ and $\{I_\ell^{\mathrm{L}}\}_{\ell \in \mathscr{L}}$ are all increasing functions of ω_0, these collections of indicators are positively related so we may apply the simpler form of Lemma A.1 by bounding the covariances.

LEMMA A.7. *The collections of indicators* $\{I_v^{\mathrm{EB}}\}_{v \in V}$ *and* $\{I_\ell^{\mathrm{L}}\}_{\ell \in \mathscr{L}}$ *are positively related and the following sums are* $O(n^{-1})$:

$$\sum_{v \in V} \sum_{\substack{w \in V \\ w \neq v}} \mathrm{Cov}(I_v^{\mathrm{EB}}, I_w^{\mathrm{EB}}), \qquad \sum_{v \in V} \sum_{\ell \in \mathscr{L}} \mathrm{Cov}(I_v^{\mathrm{EB}}, I_\ell^{\mathrm{L}}), \qquad \sum_{\ell \in \mathscr{L}} \sum_{\substack{\ell' \in \mathscr{L} \\ \ell' \neq \ell}} \mathrm{Cov}(I_\ell^{\mathrm{L}}, I_{\ell'}^{\mathrm{L}}).$$

Note that the bound on the last sum, which involves only indicators of line events, is implied by combining the results for the enhanced line and nonenhanced line couplings in Lemmas A.5 and A.6 by writing $I_\ell^{\mathrm{L}} = I_\ell^{\mathrm{EL}} + I_\ell^{\mathrm{NEL}}$.

PROOF. We will explain the proof of the bound on the first sum, as the second sum is evaluated in a similar fashion and the third is implied by previous lemmas.

We break up the sum into three cases depending on the Hamming distance between v and w.

Case 1: $d(v,w)=1$. There are $O(n^4)$ such terms in the sum, so we need to show that the covariance is $O(n^{-5})$. In this case it suffices to use the trivial bound $\mathrm{Cov}(I_v^{\mathrm{EB}}, I_w^{\mathrm{EB}}) \le \mathbb{E} I_v^{\mathrm{EB}} I_w^{\mathrm{EB}} = \mathbb{P}(G_v^{\mathrm{EB}} \cap G_w^{\mathrm{EB}})$, which is the probability that an enhanced basic configuration appears at v and at w. For this event to occur, v must have one open neighbor in each direction, one of which is shared with w, so w needs only one open neighbor in each direction orthogonal to $w-v$. This is a total of at least five open points on five fixed lines, which has probability $O((np)^5) = O(n^{-5})$ as desired.

Case 2: $d(v,w)=2$. There are $O(n^5)$ such terms in the sum, so we need to show that the covariance is $O(n^{-6})$. Again, it suffices to use the bound $\mathrm{Cov}(I_v^{\mathrm{EB}}, I_w^{\mathrm{EB}}) \le \mathbb{E} I_v^{\mathrm{EB}} I_w^{\mathrm{EB}}$. In this case, the vertices v and w have exactly two common neighbors, so there are three cases: zero, one, or two of these common neighbors are initially open. If neither common neighbor is initially open, then v and w each independently need one open neighbor in each direction—a total of six open vertices in six fixed lines, which has probability $O(n^{-6})$. If one of the common neighbors is open, an event with probability $O(p) = O(n^{-2})$, then v and w each need an open neighbor in two other directions—a total of four open vertices in four fixed lines which has probability $O(n^{-4})$. This gives a probability of $O(n^{-6})$ to the case where one common neighbor is open. The event that both common neighbors are open has probability $p^2 = O(n^{-4})$, and v and w each require one more occupied neighbor in one direction, which has probability $O(n^{-2})$ for a total probability of $O(n^{-6})$.

Case 3: $d(v,w)=3$. There are $O(n^6)$ such terms in the sum, so we need to show that the covariance is $O(n^{-7})$, and the trivial upper bound on the covariance will not suffice. Observe that the planes containing v and the planes containing w intersect only along 6 lines, and conditional on the event that none of the points on these lines are initially open, I_v^{EB} and I_w^{EB} are independent. Call this event E_{empty}, then since I_v^{EB} and I_w^{EB} are increasing functions of ω_0, the covariance is bounded by

$$\mathrm{Cov}(I_v^{\mathrm{EB}}, I_w^{\mathrm{EB}}) \le \mathbb{P}(I_v^{\mathrm{EB}} I_w^{\mathrm{EB}} = 1, E_{\mathrm{empty}}^c).$$

We now divide the event E_{empty}^c into subcases according to which vertices in the intersection are open. There are two types of vertices in the intersection—those which are neighbors to either v or w, and those which are only in the same plane as each vertex. There are exactly 6 vertices in the former category and $6(n-2)$ in the latter. The probability that j of the 6 vertices in $[\mathcal{N}(v) \cap \mathcal{N}(\mathcal{N}(w))] \cup [\mathcal{N}(w) \cap \mathcal{N}(\mathcal{N}(v))]$ are initially open is $O(p^j) = O(n^{-2j})$. Conditional on this, v and w collectively require an initially open vertex in each of the remaining $6-j$ lines in their neighborhoods, which has probability $O(n^{-6+j})$, giving a total probability of $O(n^{-6-j})$ to the event that there are j of these 6 vertices initially open and both enhanced basic events occur. Therefore, if $j \geq 1$ we are done, otherwise we must consider the case where $j = 0$ and then E^c_{empty} requires that at least one vertex among the $6(n-2)$ vertices in $\mathcal{N}(\mathcal{N}(v)) \cap \mathcal{N}(\mathcal{N}(w))$ are initially open. This event has probability $O(np) = O(n^{-1})$, and when $j = 0$, v and w still need one open vertex in each line of their neighborhoods, which has probability $O(n^{-6})$, giving a total probability of $O(n^{-7})$. $\square$

PROOF OF LEMMA 6.2. The limiting means are straightforward to calculate, as outlined in Remark 6.3. It is also not difficult to show that $d_{\text{TV}}(P_{\lambda_n}, P_\lambda) \leq |\lambda_n - \lambda|$ so if $\lambda_n \to \lambda$ then P_{λ_n} converges to P_λ. Therefore, applying Lemma A.1 and using Lemmas A.3–A.6 to bound the second summation in (A.2) implies that the random variables Basic, $\texttt{Line}\emptyset_i$, $\texttt{NonEnhancedLine}_i$ and EnhancedLine (where $i = 1, 2, 3$, so there are a total of 8 random variables) converge jointly to independent Poisson random variables with the appropriate limiting means. Similarly, applying Lemma A.1 and using Lemma A.7 to bound the second summation in (A.3) implies that the random variables EnhancedBasic and Line converge jointly to independent Poisson random variables with the appropriate limiting means. $\square$

REFERENCES

[1] ADLER, J. and LEV, U. (2003). Bootstrap percolation: Visualizations and applications. *Braz. J. Phys.* **33** 641–644.

[2] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

[3] BALOGH, J. and BOLLOBÁS, B. (2006). Bootstrap percolation on the hypercube. *Probab. Theory Related Fields* **134** 624–648. MR2214907

[4] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Bootstrap percolation in three dimensions. *Ann. Probab.* **37** 1329–1380. MR2546747

[5] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Majority bootstrap percolation on the hypercube. *Combin. Probab. Comput.* **18** 17–51.

[6] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. MR2888224

[7] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation. Oxford Studies in Probability* **2**. Oxford Univ. Press, New York. MR1163825

[8] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2005). Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. *Ann. Probab.* **33** 1886–1944. MR2165583

[9] CERF, R. and CIRILLO, E. N. M. (1999). Finite size scaling in three-dimensional bootstrap percolation. *Ann. Probab.* **27** 1837–1850. MR1742890

[10] CERF, R. and MANZO, F. (2002). The threshold regime of finite volume bootstrap percolation. *Stochastic Process. Appl.* **101** 69–82. MR1921442

[11] CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C* **12** L31–L35.

[12] FRIEDGUT, E. and KALAI, G. (1996). Every monotone graph property has a sharp threshold. *Proc. Amer. Math. Soc.* **124** 2993–3002. MR1371123

[13] GRAVNER, J., HOLROYD, A. E. and MORRIS, R. (2012). A sharper threshold for bootstrap percolation in two dimensions. *Probab. Theory Related Fields* **153** 1–23. MR2925568

[14] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342

[15] HOLROYD, A. E. (2007). Astonishing cellular automata. *Bulletin du Centre de Recherches Mathematiques* **13** 10–13.

[16] HOLROYD, A. E., LIGGETT, T. M. and ROMIK, D. (2004). Integrals, partitions, and cellular automata. *Trans. Amer. Math. Soc.* **356** 3349–3368 (electronic). MR2052953

[17] JANSON, S., ŁUCZAK, T., TUROVA, T. and VALLIER, T. (2012). Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.* **22** 1989–2047. MR3025687

[18] SCHONMANN, R. H. (1992). On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20** 174–193. MR1143417

[19] SIVAKOFF, D. (2014). Site percolation on the d-dimensional Hamming torus. *Combin. Probab. Comput.* **23** 290–315.

[20] SLIVKEN, E. Bootstrap percolation on the hamming torus with threshold 2. Unpublished manuscript.

[21] VAN ENTER, A. C. D. (1987). Proof of Straley's argument for bootstrap percolation. *J. Stat. Phys.* **48** 943–945. MR0914911

[22] VAN DER HOFSTAD, R. and LUCZAK, M. J. (2010). Random subgraphs of the 2D Hamming graph: The supercritical phase. *Probab. Theory Related Fields* **147** 1–41. MR2594346

J. GRAVNER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616
USA
E-MAIL: gravner@math.ucdavis.edu

C. HOFFMAN
J. PFEIFFER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195
USA
E-MAIL: hoffman@math.washington.edu
jpfeiff@math.washington.edu

D. SIVAKOFF
MATHEMATICS DEPARTMENT
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27708
USA
E-MAIL: djsivy@math.duke.edu