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LIMIT THEOREMS FOR NONDEGENERATE *U*-STATISTICS OF CONTINUOUS SEMIMARTINGALES

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This paper presents the asymptotic theory for nondegenerate U-statistics of high frequency observations of continuous Itô semimartingales. We prove uniform convergence in probability and show a functional stable central limit theorem for the standardized version of the U-statistic. The limiting process in the central limit theorem turns out to be conditionally Gaussian with mean zero. Finally, we indicate potential statistical applications of our probabilistic results.

1. Introduction. Since the seminal work by Hoeffding [15], U-statistics have been widely investigated by probabilists and statisticians. Nowadays, there exists a vast amount of literature on the asymptotic properties of U-statistics in the case of independent and identically distributed (i.i.d.) random variables or in the framework of weak dependence. We refer to [23] for a comprehensive account of the asymptotic theory in the classical setting. In [4, 5, 11], the authors treat limit theorems for U-statistics under various mixing conditions, while the corresponding theory for long memory processes has been studied, for example, in [9, 14]; see [16] for a recent review of the properties of U-statistics in various settings. The most powerful tools for proving asymptotic results for U-statistics include the classical Hoeffding decomposition (see, e.g., [15]), Hermite expansions (see, e.g., [9, 10]) and the empirical process approach; see, for example, [3]. Despite the activity of this field of research, U-statistics for high frequency observations of a time-continuous process have not been studied in the literature thus far. The notion of high frequency data refers to the sampling scheme in which the time step between two consecutive observations converges to zero while the time span remains fixed. This concept is also known under the name of infill asymptotics. Motivated by the prominent role of semimartingales in mathematical finance, in this paper we present novel asymptotic results for high frequency observations of Itô semimartingales and demonstrate some statistical applications.

The seminal work of Jacod [17] marks the starting point for stable limit theorems for semimartingales. Stimulated by the increasing popularity of semimartingales as natural models for asset pricing, the asymptotic theory for partial sums

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processes of continuous and discontinuous Itô semimartingales has been developed in [2, 18, 22]; see also the recent book [20]. We refer to [25] for a short survey of limit theorems for semimartingales. More recently, asymptotic theory for Itô semimartingales observed with errors has been investigated in [19].

The methodology we employ to derive a limit theory for U-statistics of continuous Itô semimartingales is an intricate combination and extension of some of the techniques developed in the series of papers mentioned in the previous paragraph and the empirical process approach to U-statistics.

In this paper we consider a one-dimensional continuous Itô semimartingale of the form

$$X_t = x + \int_0^t a_s\,ds + \int_0^t \sigma_s\,dW_s, \qquad t \geq 0,$$

defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ (which satisfies the usual assumptions), where $x \in \mathbb{R}$, $(a_s)_{s\geq 0}$, $(\sigma_s)_{s\geq 0}$ are stochastic processes, and W is a standard Brownian motion. The underlying observations of X are

$$X_{i/n}, \qquad i = 0, \dots, [nt],$$

and we are in the framework of infill asymptotics, that is, $n \to \infty$. In order to present our main results, we introduce some notation. We define

$$\mathcal{A}_t^n(d) := \{\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : 1 \leq i_1 < i_2 < \cdots < i_d \leq [nt]\},$$
$$Z_{\mathbf{s}} := (Z_{s_1}, \dots, Z_{s_d}), \qquad \mathbf{s} \in \mathbb{R}^d,$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ is an arbitrary stochastic process. For any continuous function $H : \mathbb{R}^d \to \mathbb{R}$, we define the U-statistic $U(H)_t^n$ of order d as

$$U(H)_t^n = \binom{n}{d}^{-1} \sum_{\mathbf{i}\in\mathcal{A}_t^n(d)} H\big(\sqrt{n}\Delta_{\mathbf{i}}^n X\big) \tag{1}$$

with $\Delta_{\mathbf{i}}^n X = X_{\mathbf{i}/n} - X_{(\mathbf{i}-1)/n}$. For a multi-index $\mathbf{i} \in \mathbb{N}^d$, the vector $\mathbf{i} - 1$ denotes the multi-index obtained by componentwise subtraction of 1 from $\mathbf{i}$. In the following we assume that the function H is symmetric, that is, for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and all permutations π of $\{1, \dots, d\}$, it holds that $H(\pi x) = H(x)$, where $\pi x = (x_{\pi(1)}, \dots, x_{\pi(d)})$.

Our first result determines the asymptotic behavior of $U(H)_t^n$,

$$U(H)_t^n \xrightarrow{\text{u.c.p.}} U(H)_t := \int_{[0,t]^d} \rho_{\sigma_{\mathbf{s}}}(H)\,d\mathbf{s},$$

where $Z^n \xrightarrow{\text{u.c.p.}} Z$ denotes uniform convergence in probability, that is, for any $T > 0$, $\sup_{t\in[0,T]} |Z_t^n - Z_t| \xrightarrow{\mathbb{P}} 0$, and

$$\rho_{\sigma_{\mathbf{s}}}(H) := \int_{\mathbb{R}^d} H(\sigma_{s_1}u_1, \dots, \sigma_{s_d}u_d)\varphi_d(\mathbf{u})\,d\mathbf{u} \tag{2}$$

with φ_d denoting the density of the d-dimensional standard Gaussian law $\mathcal{N}_d(0, \mathbf{I}_d)$. The second result of this paper is the stable functional central limit theorem

$$\sqrt{n}\big(U(H)^n - U(H)\big) \xrightarrow{\text{st}} L,$$

where $\xrightarrow{\text{st}}$ denotes stable convergence in law, and the function H is assumed to be even in each coordinate. The limiting process L lives on an extension of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and it turns out to be Gaussian with mean zero conditionally on the original σ-algebra $\mathcal{F}$. The proofs of the asymptotic results rely upon a combination of recent limit theorems for semimartingales (see, e.g., [17, 20, 22]) and empirical processes techniques.

The paper is organized as follows. In Section 3 we present the law of large numbers for the U-statistic $U(H)^n_t$. The associated functional stable central limit theorem is provided in Section 4. Furthermore, we derive a standard central limit theorem in Section 5. In Section 6 we demonstrate statistical applications of our limit theory including Gini's mean difference, homoscedasticity testing and Wilcoxon statistics for testing of structural breaks. Some technical parts of the proofs are deferred to Section 7.

2. Preliminaries.

We consider the continuous diffusion model

$$X_t = x + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s, \qquad t \geq 0, \tag{3}$$

where $(a_s)_{s\geq 0}$ is a càglàd process, $(\sigma_s)_{s\geq 0}$ is a càdlàg process, both adapted to the filtration $(\mathcal{F}_s)_{s\geq 0}$. Define the functional class $C^k_p(\mathbb{R}^d)$ via

$$C^k_p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} \,|\, f \in C^k(\mathbb{R}^d) \text{ and all derivatives up to order } k \text{ are of polynomial growth}\}.$$

Note that $H \in C^0_p(\mathbb{R}^d)$ implies that $\rho_{\sigma_{\mathrm{s}}}(H) < \infty$ almost surely. For any vector $y \in \mathbb{R}^d$, we denote by $\|y\|$ its maximum norm; for any function $f : \mathbb{R}^d \to \mathbb{R}$, $\|f\|_\infty$ denotes its supremum norm. Finally, for any $z \neq 0$, Φ_z and φ_z stand for the distribution function and density of the Gaussian law $\mathcal{N}(0, z^2)$, respectively; Φ_0 denotes the Dirac measure at the origin. The bracket $[M, N]$ denotes the covariation process of two local martingales M and N.

3. Law of large numbers.

We start with the law of large numbers, which describes the limit of the U-statistic $U(H)^n_t$ defined at (1). First of all, we remark that the processes $(a_s)_{s\geq 0}$ and $(\sigma_{s-})_{s\geq 0}$ are locally bounded, because they are both càglàd. Since the main results of this subsection (Proposition 3.2 and Theorem 3.3) are *stable under stopping*, we may assume without loss of generality that

(4) The processes a and σ are bounded in (ω, t).

A detailed justification of this statement can be found in [2], Section 3.

We start with the representation of the process $U(H)_t^n$ as an integral with respect to a certain empirical random measure. For this purpose let us introduce the quantity

$$\alpha_j^n := \sqrt{n}\sigma_{(j-1)/n}\Delta_j^n W, \qquad j \in \mathbb{N}, \tag{5}$$

which serves as a first order approximation of the increments $\sqrt{n}\Delta_j^n X$. The empirical distribution function associated with the random variables $(\alpha_j^n)_{1\le j\le [nt]}$ is defined as

$$F_n(t,x) := \frac{1}{n}\sum_{j=1}^{[nt]} \mathbb{1}_{\{\alpha_j^n \le x\}}, \qquad x \in \mathbb{R}, t \ge 0. \tag{6}$$

Notice that, for any fixed $t \ge 0$, $F_n(t,\cdot)$ is a finite random measure. Let $\widetilde{U}(H)_t^n$ be the U-statistic based on α_j^n's, that is,

$$\widetilde{U}(H)_t^n = \binom{n}{d}^{-1} \sum_{\mathbf{i}\in\mathcal{A}_t^n(d)} H(\alpha_{\mathbf{i}}^n). \tag{7}$$

The functional $U_t'^n(H)$ defined as

$$U_t'^n(H) := \int_{\mathbb{R}^d} H(\mathbf{x}) F_n^{\otimes d}(t, d\mathbf{x}), \tag{8}$$

where

$$F_n^{\otimes d}(t, d\mathbf{x}) := F_n(t, dx_1)\cdots F_n(t, dx_d),$$

is closely related to the process $\widetilde{U}(H)_t^n$; in fact, if both are written out as multiple sums over nondecreasing multi-indices, then their summands coincide on the set $\mathcal{A}_t^n(d)$. They differ for multi-indices that have at least two equal components. However, the number of these diagonal multi-indices is of order $O(n^{d-1})$. We start with a simple lemma, which we will often use throughout the paper. We omit a formal proof since it follows by standard arguments.

LEMMA 3.1. *Let $Z_n, Z : [0,T]\times\mathbb{R}^m \to \mathbb{R}$, $n \ge 1$, be random positive functions such that $Z_n(t,\cdot)$ and $Z(t,\cdot)$ are finite random measures on $\mathbb{R}^m$ for any $t \in [0,T]$. Assume that*

$$Z_n(\cdot,\mathbf{x}) \xrightarrow{\text{u.c.p.}} Z(\cdot,\mathbf{x}),$$

for any fixed $\mathbf{x} \in \mathbb{R}^m$, and $\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^m} Z(t,\mathbf{x})$, $\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^m} Z_n(t,\mathbf{x})$, $n \ge 1$, are bounded random variables. Then, for any continuous function $Q : \mathbb{R}^m \to \mathbb{R}$ with compact support, we obtain that

$$\int_{\mathbb{R}^m} Q(\mathbf{x}) Z_n(\cdot, d\mathbf{x}) \xrightarrow{\text{u.c.p.}} \int_{\mathbb{R}^m} Q(\mathbf{x}) Z(\cdot, d\mathbf{x}).$$

The next proposition determines the asymptotic behavior of the empirical distribution function $F_n(t,x)$ defined at (6), and the U-statistic $U_t'^n(H)$ given at (8).

PROPOSITION 3.2. *Assume that $H \in C_p^0(\mathbb{R}^d)$. Then, for any fixed $x \in \mathbb{R}$, it holds that*

$$F_n(t,x) \xrightarrow{\text{u.c.p.}} F(t,x) := \int_0^t \Phi_{\sigma_s}(x)\,ds. \tag{9}$$

Furthermore, we obtain that

$$U_t'^n(H) \xrightarrow{\text{u.c.p.}} U(H)_t := \int_{[0,t]^d} \rho_{\sigma_{\mathbf{s}}}(H)\,ds, \tag{10}$$

where the quantity $\rho_{\sigma_{\mathbf{s}}}(H)$ is defined at (2).

PROOF. Recall that we always assume (4) without loss of generality. Here and throughout the paper, we denote by C a generic positive constant, which may change from line to line; furthermore, we write C_p if we want to emphasize the dependence of C on an external parameter p. We first show the convergence in (9). Set $\xi_j^n := n^{-1}\mathbb{1}_{\{\alpha_j^n \le x\}}$. It obviously holds that

$$\sum_{j=1}^{[nt]} \mathbb{E}[\xi_j^n|\mathcal{F}_{(j-1)/n}] = \frac{1}{n}\sum_{j=1}^{[nt]} \Phi_{\sigma_{(j-1)/n}}(x) \xrightarrow{\text{u.c.p.}} F(t,x),$$

for any fixed $x \in \mathbb{R}$, due to Riemann integrability of the process Φ_σ. On the other hand, we have for any fixed $x \in \mathbb{R}$,

$$\sum_{j=1}^{[nt]} \mathbb{E}[|\xi_j^n|^2|\mathcal{F}_{(j-1)/n}] = \frac{1}{n^2}\sum_{j=1}^{[nt]} \Phi_{\sigma_{(j-1)/n}}(x) \xrightarrow{\mathbb{P}} 0.$$

This immediately implies the convergence (see [20], Lemma 2.2.11, page 577)

$$F_n(t,x) - \sum_{j=1}^{[nt]} \mathbb{E}[\xi_j^n|\mathcal{F}_{(j-1)/n}] = \sum_{j=1}^{[nt]} (\xi_j^n - \mathbb{E}[\xi_j^n|\mathcal{F}_{(j-1)/n}]) \xrightarrow{\text{u.c.p.}} 0,$$

which completes the proof of (9). If H is compactly supported, then the convergence in (10) follows directly from (9) and Lemma 3.1.

Now, let $H \in C_p^0(\mathbb{R}^d)$ be arbitrary. For any $k \in \mathbb{N}$, let $H_k \in C_p^0(\mathbb{R}^d)$ be a function with $H_k = H$ on $[-k,k]^d$ and $H_k = 0$ on $([-k-1,k+1]^d)^c$. We already know that

$$U'^n(H_k) \xrightarrow{\text{u.c.p.}} U(H_k),$$

for any fixed k, and $U(H_k) \xrightarrow{\text{u.c.p.}} U(H)$ as $k \to \infty$. Since the function H has polynomial growth, that is, $|H(\mathbf{x})| \le C(1+\|\mathbf{x}\|^q)$ for some $q>0$, we obtain for any $p > 0$

$$\mathbb{E}[|H(\alpha_{\mathbf{i}}^n)|^p] \le C_p\mathbb{E}[(1+\|\alpha_{\mathbf{i}}^n\|^{qp})] \le C_p \tag{11}$$

uniformly in $\mathbf{i}$, because the process σ is bounded. Statement (11) also holds for H_k. Recall that the function $H - H_k$ vanishes on $[-k, k]^d$. Hence, we deduce by (11) and Cauchy–Schwarz inequality that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \big|U_t'^n(H - H_k)\big|\Big] \\
&\quad \le C\binom{n}{d}^{-1} \sum_{1\le i_1,\dots,i_d\le[nT]} \big(\mathbb{E}[\mathbb{1}_{\{|\alpha^n_{i_1}|\ge k\}} + \cdots + \mathbb{1}_{\{|\alpha^n_{i_d}|\ge k\}}]\big)^{1/2} \\
&\quad \le C_T \sup_{s\in[0,T]} \big(\mathbb{E}\big[1 - \Phi_{\sigma_s}(k)\big]\big)^{1/2} \to 0
\end{aligned}$$

as $k \to \infty$. This completes the proof of (10). □

Proposition 3.2 implies the main result of this section.

THEOREM 3.3. *Assume that $H \in C_p^0(\mathbb{R}^d)$. Then it holds that*

$$U(H)_t^n \xrightarrow{\text{u.c.p.}} U(H)_t := \int_{[0,t]^d} \rho_{\sigma_{\mathbf{s}}}(H)\, ds, \tag{12}$$

where the quantity $\rho_{\sigma_{\mathbf{s}}}(H)$ is defined at (2).

PROOF. In Section 7 we will show that

$$U(H)^n - \widetilde{U}(H)^n \xrightarrow{\text{u.c.p.}} 0, \tag{13}$$

where the functional $\widetilde{U}(H)_t^n$ is given at (7). In view of Proposition 3.2, it remains to prove that $\widetilde{U}(H)_t^n - U_t'^n(H) \xrightarrow{\text{u.c.p.}} 0$. But due to the symmetry of H and estimation (11), we obviously obtain that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\widetilde{U}(H)_t^n - U_t'^n(H)\big|\Big] \le \frac{C_T}{n} \to 0,$$

since the summands in $\widetilde{U}(H)_t^n$ and $U_t'^n(H)$ are equal except for diagonal multi-indices. □

REMARK 1. The result of Theorem 3.3 can be extended to weighted U-statistics of the type

$$U(H; X)_t^n := \binom{n}{d}^{-1} \sum_{\mathbf{i}\in\mathcal{A}_t^n(d)} H\big(X_{(\mathbf{i}-1)/n}; \sqrt{n}\Delta_{\mathbf{i}}^n X\big). \tag{14}$$

Here, $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be continuous and symmetric in the first and last d arguments. Indeed, similar methods of proof imply the u.c.p. convergence

$$U(H; X)_t^n \xrightarrow{\text{u.c.p.}} U(H; X)_t = \int_{[0,t]^d} \rho_{\sigma_{\mathbf{s}}}(H; X_{\mathbf{s}})\, d\mathbf{s},$$

with

$$\rho_{\sigma_{\mathbf{s}}}(H; X_{\mathbf{s}}) := \int_{\mathbb{R}^d} H(X_{\mathbf{s}}; \sigma_{s_1} u_1, \dots, \sigma_{s_d} u_d) \varphi_d(\mathbf{u})\, d\mathbf{u}.$$

It is not essential that the weight process equals the diffusion process X. Instead, we may consider any k-dimensional $(\mathcal{F}_t)$-adapted Itô semimartingale of type (3). We leave the details to the interested reader.

4. Stable central limit theorem. In this section we present a functional stable central limit theorem associated with the convergence in (12).

4.1. *Stable convergence.* The concept of stable convergence of random variables was originally introduced by Renyi [26]. For properties of stable convergence, we refer to [1, 25]. We recall the definition of stable convergence: let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space $(E, \mathcal{E})$. We say that Y_n converges stably with limit Y, written $Y_n \xrightarrow{\text{st}} Y$, where Y is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if and only if for any bounded, continuous function g and any bounded $\mathcal{F}$-measurable random variable Z it holds that

$$\mathbb{E}\big[g(Y_n)Z\big] \to \mathbb{E}'\big[g(Y)Z\big], \qquad n \to \infty. \tag{15}$$

Typically, we will deal with $E = \mathbb{D}([0, T], \mathbb{R})$ equipped with the Skorohod topology, or the uniform topology if the process Y is continuous. Notice that stable convergence is a stronger mode of convergence than weak convergence. In fact, the statement $Y_n \xrightarrow{\text{st}} Y$ is equivalent to the joint weak convergence $(Y_n, Z) \xrightarrow{d} (Y, Z)$ for any $\mathcal{F}$-measurable random variable Z; see, for example, [1].

4.2. *Central limit theorem.* For the stable central limit theorem we require a further structural assumption on the volatility process $(\sigma_s)_{s\geq 0}$. We assume that σ itself is a continuous Itô semimartingale,

$$\sigma_t = \sigma_0 + \int_0^t \tilde{a}_s\, ds + \int_0^t \tilde{\sigma}_s\, dW_s + \int_0^t \tilde{v}_s\, dV_s, \tag{16}$$

where the processes $(\tilde{a}_s)_{s\geq 0}$, $(\tilde{\sigma}_s)_{s\geq 0}$, $(\tilde{v}_s)_{s\geq 0}$ are càdlàg, adapted and V is a Brownian motion independent of W. This type of condition is motivated by potential applications. For instance, when $\sigma_t = f(X_t)$ for a C^2-function f, then the Itô formula implies representation (16) with $\tilde{v} \equiv 0$. In fact, a condition of type (16) is nowadays a standard assumption for proving stable central limit theorems for functionals of high frequency data; see, for example, [2, 18]. Moreover, we assume that the process σ does not vanish, that is,

$$\sigma_s \neq 0 \qquad \text{for all } s \in [0, T]. \tag{17}$$

We believe that this assumption is not essential, but dropping it would make the following proofs considerably more involved and technical. As in the previous subsection, the central limit theorems presented in this paper are stable under stopping. This means, we may assume, without loss of generality, that

(18) The processes $a, \sigma, \sigma^{-1}, \tilde{a}, \tilde{\sigma}$ and $\tilde{v}$ are bounded in (ω, t).

We refer again to [2], Section 3, for a detailed justification of this statement.

We need to introduce some further notation to describe the limiting process. First, we will study the asymptotic properties of the empirical process

$$\mathbb{G}_n(t,x) := \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \big(\mathbb{1}_{\{\alpha_j^n \le x\}} - \Phi_{\sigma_{(j-1)/n}}(x)\big), \tag{19}$$

where α_j^n is defined at (5). This process is of crucial importance for proving the stable central limit theorem for the U-statistic $U(H)_t^n$. We start with the derivation of some useful inequalities for the process $\mathbb{G}_n$.

LEMMA 4.1. *For any even number $p \ge 2$ and $x, y \in \mathbb{R}$, we obtain the inequalities*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbb{G}_n(t,x)\big|^p\Big] \le C_{T,p}\phi(x), \tag{20}$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbb{G}_n(t,x) - \mathbb{G}_n(t,y)\big|^p\Big] \le C_{T,p}|x-y|, \tag{21}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a bounded function (that depends on p and T) with exponential decay at $\pm\infty$.

PROOF. Recall that the processes σ and σ^{-1} are assumed to be bounded. We begin with inequality (20). For any given $x \in \mathbb{R}$, $(\mathbb{G}_n(t,x))_{t\in[0,T]}$ is an $(\mathcal{F}_{[nt]/n})$-martingale. Hence, the discrete Burkhölder inequality implies that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbb{G}_n(t,x)\big|^p\Big] \le C_{T,p}\mathbb{E}\Bigg[\Bigg|\sum_{j=1}^{[nT]} \zeta_j^n\Bigg|^{p/2}\Bigg]$$

with $\zeta_j^n := n^{-1}(\mathbb{1}_{\{\alpha_j^n \le x\}} - \Phi_{\sigma_{(j-1)/n}}(x))^2$. Recalling that $p \ge 2$ is an even number und applying the Hölder inequality, we deduce that

$$\begin{aligned}\Bigg|\sum_{j=1}^{[nT]} \zeta_j^n\Bigg|^{p/2} &\le C_T n^{-1} \sum_{j=1}^{[nT]} \big(\mathbb{1}_{\{\alpha_j^n \le x\}} - \Phi_{\sigma_{(j-1)/n}}(x)\big)^p \\ &= C_T n^{-1} \sum_{j=1}^{[nT]} \sum_{k=0}^{p} \binom{p}{k} (-1)^k \Phi^k_{\sigma_{(j-1)/n}}(x) \mathbb{1}_{\{\alpha_j^n \le x\}}.\end{aligned}$$

Thus we conclude that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{G}_n(t,x)\big|^p\Big] \le C_{T,p}\sup_{s\in[0,T]}\mathbb{E}\big[\Phi_{\sigma_s}(x)\big(1-\Phi_{\sigma_s}(x)\big)^p\big] =: C_{T,p}\phi(x),$$

where the function ϕ obviously satisfies our requirements. This completes the proof of (20). By exactly the same methods we obtain, for any $x \ge y$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{G}_n(t,x)-\mathbb{G}_n(t,y)\big|^p\Big]\\&\quad\le C_{T,p}\sup_{s\in[0,T]}\mathbb{E}\big[\big(\Phi_{\sigma_s}(x)-\Phi_{\sigma_s}(y)\big)\big(1-\big(\Phi_{\sigma_s}(x)-\Phi_{\sigma_s}(y)\big)\big)^p\big].\end{aligned}$$

Since σ and σ^{-1} are both bounded, there exists a constant $M > 0$ such that

$$\sup_{s\in[0,T]}\big|\Phi_{\sigma_s}(x)-\Phi_{\sigma_s}(y)\big| \le |x-y|\sup_{M^{-1}\le z\le M,\, y\le r\le x}\varphi_z(r).$$

This immediately gives (21). $\square$

Our next result presents a functional stable central limit theorem for the process $\mathbb{G}_n$ defined at (19).

PROPOSITION 4.2. *We obtain the stable convergence*

$$\mathbb{G}_n(t,x) \xrightarrow{\text{st}} \mathbb{G}(t,x)$$

on $\mathbb{D}([0,T])$ *equipped with the uniform topology, where the convergence is functional in* $t \in [0,T]$ *and in finite distribution sense in* $x \in \mathbb{R}$. *The limiting process* $\mathbb{G}$ *is defined on an extension* $(\Omega',\mathcal{F}',\mathbb{P}')$ *of the original probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and it is Gaussian conditionally on* $\mathcal{F}$. *Its conditional drift and covariance kernel are given by*

$$\begin{aligned}&\mathbb{E}'\big[\mathbb{G}(t,x)|\mathcal{F}\big] = \int_0^t \overline{\Phi}_{\sigma_s}(x)\,dW_s,\\&\mathbb{E}'\big[\mathbb{G}(t_1,x_1)\mathbb{G}(t_2,x_2)|\mathcal{F}\big]-\mathbb{E}'\big[\mathbb{G}(t_1,x_1)|\mathcal{F}\big]\mathbb{E}'\big[\mathbb{G}(t_2,x_2)|\mathcal{F}\big]\\&\qquad = \int_0^{t_1\wedge t_2}\Phi_{\sigma_s}(x_1\wedge x_2)-\Phi_{\sigma_s}(x_1)\Phi_{\sigma_s}(x_2)-\overline{\Phi}_{\sigma_s}(x_1)\overline{\Phi}_{\sigma_s}(x_2)\,ds,\end{aligned}$$

where $\overline{\Phi}_z(x) = \mathbb{E}[V\mathbb{1}_{\{zV\le x\}}]$ *with* $V \sim \mathcal{N}(0,1)$.

PROOF. Recall that due to (18) the process σ is bounded in (ω,t). [However, note that we do not require the condition (16) to hold.] For any given $x_1,\dots,x_k \in \mathbb{R}$, we need to prove the functional stable convergence

$$\big(\mathbb{G}_n(\cdot,x_1),\dots,\mathbb{G}_n(\cdot,x_k)\big) \xrightarrow{\text{st}} \big(\mathbb{G}(\cdot,x_1),\dots,\mathbb{G}(\cdot,x_k)\big).$$

We write $\mathbb{G}_n(t,x_l)=\sum_{j=1}^{[nt]}\chi_{j,l}^n$ with

$$\chi_{j,l}^n:=\frac{1}{\sqrt{n}}\big(\mathbb{1}_{\{\alpha_j^n\le x_l\}}-\Phi_{\sigma_{(j-1)/n}}(x_l)\big),\qquad 1\le l\le k.$$

According to [21], Theorem IX.7.28, we need to show that

$$\sum_{j=1}^{[nt]}\mathbb{E}\big[\chi_{j,r}^n\chi_{j,l}^n|\mathcal{F}_{(j-1)/n}\big] \xrightarrow{\mathbb{P}} \int_0^t\big(\Phi_{\sigma_s}(x_r\wedge x_l)-\Phi_{\sigma_s}(x_r)\Phi_{\sigma_s}(x_l)\big)\,ds, \tag{22}$$

$$\sum_{j=1}^{[nt]}\mathbb{E}\big[\chi_{j,l}^n\Delta_j^nW|\mathcal{F}_{(j-1)/n}\big]\xrightarrow{\mathbb{P}}\int_0^t\overline{\Phi}_{\sigma_s}(x_l)\,ds, \tag{23}$$

$$\sum_{j=1}^{[nt]}\mathbb{E}\big[|\chi_{j,l}^n|^2 1_{\{|\chi_{j,l}^n|>\varepsilon\}}|\mathcal{F}_{(j-1)/n}\big]\xrightarrow{\mathbb{P}}0\qquad\text{for all }\varepsilon>0, \tag{24}$$

$$\sum_{j=1}^{[nt]}\mathbb{E}\big[\chi_{j,l}^n\Delta_j^nN|\mathcal{F}_{(j-1)/n}\big]\xrightarrow{\mathbb{P}}0, \tag{25}$$

where $1\le r,l\le d$ and the last condition must hold for all bounded continuous martingales N with $[W,N]=0$. The convergence in (22) and (23) is obvious, since Δ_j^nW is independent of $\sigma_{(j-1)/n}$. We also have that

$$\sum_{j=1}^{[nt]}\mathbb{E}\big[|\chi_{j,l}^n|^2 1_{\{|\chi_{j,l}^n|>\varepsilon\}}|\mathcal{F}_{(j-1)/n}\big]\le\varepsilon^{-2}\sum_{j=1}^{[nt]}\mathbb{E}\big[|\chi_{j,l}^n|^4|\mathcal{F}_{(j-1)/n}\big]\le Cn^{-1},$$

which implies (24). Finally, let us prove (25). We fix l and define $M_u:=\mathbb{E}[\chi_{j,l}^n|\mathcal{F}_u]$ for $u\ge(j-1)/n$. By the martingale representation theorem we deduce the identity

$$M_u=M_{(j-1)/n}+\int_{(j-1)/n}^u\eta_s\,dW_s$$

for a suitable predictable process η. By the Itô isometry we conclude that

$$\mathbb{E}\big[\chi_{j,l}^n\Delta_j^nN|\mathcal{F}_{(j-1)/n}\big]=\mathbb{E}\big[M_{j/n}\Delta_j^nN|\mathcal{F}_{(j-1)/n}\big]=\mathbb{E}\big[\Delta_j^nM\Delta_j^nN|\mathcal{F}_{(j-1)/n}\big]=0.$$

This completes the proof of Proposition 4.2. $\square$

We suspect that the stable convergence in Proposition 4.2 also holds in the functional sense in the x variable. However, proving tightness (even on compact sets) turns out to be a difficult task. In particular, inequality (21) is not sufficient for showing tightness.

REMARK 2. We highlight some probabilistic properties of the limiting process $\mathbb{G}$ defined in Proposition 4.2.

(i) Proposition 4.2 can be reformulated as follows. Let $x_1, \dots, x_k \in \mathbb{R}$ be arbitrary real numbers. Then it holds that

$$\big(\mathbb{G}_n(\cdot, x_1), \dots, \mathbb{G}_n(\cdot, x_k)\big) \xrightarrow{\text{st}} \int_0^{\cdot} v_s\, dW_s + \int_0^{\cdot} w_s^{1/2}\, dW_s',$$

where W' is a k-dimensional Brownian motion independent of $\mathcal{F}$, and v and w are $\mathbb{R}^k$-valued and $\mathbb{R}^{k\times k}$-valued processes, respectively, with coordinates

$$\begin{aligned} v_s^r &= \overline{\Phi}_{\sigma_s}(x_r), \\ w_s^{rl} &= \Phi_{\sigma_s}(x_r \wedge x_l) - \Phi_{\sigma_s}(x_r)\Phi_{\sigma_s}(x_l) - \overline{\Phi}_{\sigma_s}(x_r)\overline{\Phi}_{\sigma_s}(x_l), \end{aligned}$$

for $1 \le r, l \le k$. This type of formulation appears in [21], Theorem IX.7.28. In particular, $(\mathbb{G}(\cdot, x_l))_{1\le l\le k}$ is a k-dimensional martingale.

(ii) It is obvious from (i) that $\mathbb{G}$ is continuous in t. Moreover, $\mathbb{G}$ is also continuous in x. This follows from Kolmogorov's criterion and the inequality $(y \le x)$

$$\begin{aligned} &\mathbb{E}'\big[\big|\mathbb{G}(t,x) - \mathbb{G}(t,y)\big|^p\big] \\ &\quad \le C_p \mathbb{E}\bigg[\bigg(\int_0^t \big\{\Phi_{\sigma_s}(x) - \Phi_{\sigma_s}(y) - \big(\Phi_{\sigma_s}(x) - \Phi_{\sigma_s}(y)\big)^2\big\}\, ds\bigg)^{p/2}\bigg] \\ &\quad \le C_p (x-y)^{p/2}, \end{aligned}$$

for any $p > 0$, which follows by the Burkhölder inequality. In particular, $\mathbb{G}(t, \cdot)$ has Hölder continuous paths of order $1/2 - \varepsilon$, for any $\varepsilon \in (0, 1/2)$.

(iii) A straightforward computation [cf. (20)] shows that the function $\mathbb{E}[\sup_{t\in[0,T]} \mathbb{G}(t,x)^2]$ has exponential decay as $x \to \pm\infty$. Hence, for any function $f \in C_p^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(x)\mathbb{G}(t, dx) < \infty, \qquad \text{a.s.}$$

If f is an even function, we also have that

$$\int_{\mathbb{R}} f(x)\mathbb{G}(t, dx) = \int_{\mathbb{R}} f(x)\big(\mathbb{G}(t, dx) - \mathbb{E}'\big[\mathbb{G}(t, dx)|\mathcal{F}\big]\big),$$

since

$$\int_{\mathbb{R}} f(x)\mathbb{E}'\big[\mathbb{G}(t, dx)|\mathcal{F}\big] = \int_0^t \bigg(\int_{\mathbb{R}} f(x)\overline{\Phi}_{\sigma_s}(dx)\bigg)\, dW_s,$$

and, for any $z > 0$,

$$\int_{\mathbb{R}} f(x)\overline{\Phi}_z(dx) = \int_{\mathbb{R}} x f(x)\varphi_z(x)\, dx = 0,$$

because $f\varphi_z$ is an even function. The same argument applies for $z < 0$. Furthermore, the integration by parts formula and the aforementioned argument imply the identity

$$\mathbb{E}'\left[\left|\int_{\mathbb{R}} f(x)\mathbb{G}(t,dx)\right|^2\Big|\mathcal{F}\right]$$
$$= \int_0^t \left(\int_{\mathbb{R}^2} f'(x)f'(y)\big(\Phi_{\sigma_s}(x\wedge y) - \Phi_{\sigma_s}(x)\Phi_{\sigma_s}(y)\big)\,dx\,dy\right)ds.$$

We remark that, for any $z \neq 0$, we have

$$\mathrm{var}[f(V)] = \int_{\mathbb{R}^2} f'(x)f'(y)\big(\Phi_z(x\wedge y) - \Phi_z(x)\Phi_z(y)\big)\,dx\,dy$$

with $V \sim \mathcal{N}(0, z^2)$.

Now, we present a functional stable central limit theorem of the U-statistic $U_t'^n(H)$ given at (8), which is based on the approximative quantities $(\alpha_j^n)_{1\le j\le[nt]}$ defined at (5).

PROPOSITION 4.3. *Assume that conditions* (16), (17) *and* (18) *hold. Let* $H \in C_p^1(\mathbb{R}^d)$ *be a symmetric function that is even in each* (*or, equivalently, in one*) *argument. Then we obtain the functional stable convergence*

$$\sqrt{n}\big(U'^n(H) - U(H)\big) \xrightarrow{\mathrm{st}} L, \tag{26}$$

where

$$L_t = d\int_{\mathbb{R}^d} H(x_1,\dots,x_d)\mathbb{G}(t,dx_1)F(t,dx_2)\cdots F(t,dx_d). \tag{27}$$

The convergence takes place in $\mathbb{D}([0,T])$ *equipped with the uniform topology. Furthermore,* $\mathbb{G}$ *can be replaced by* $\mathbb{G} - \mathbb{E}'[\mathbb{G}|\mathcal{F}]$ *without changing the limit and, consequently,* L *is a centered Gaussian process, conditionally on* $\mathcal{F}$.

PROOF. First of all, we remark that

$$\int_{\mathbb{R}} H(x_1,\dots,x_d)\mathbb{E}'\big[\mathbb{G}(t,dx_1)|\mathcal{F}\big] = 0$$

follows from Remark 2(iii). The main part of the proof is divided into five steps:

(i) In Section 7.3 we will show that under condition (16) we have

$$\sqrt{n}\left(U(H)_t - \int_{\mathbb{R}^d} H(\mathbf{x})\overline{F}_n^{\otimes d}(t,d\mathbf{x})\right) \xrightarrow{\text{u.c.p.}} 0 \tag{28}$$

with

$$\overline{F}_n(t,x) := \frac{1}{n}\sum_{j=1}^{[nt]} \Phi_{\sigma_{(j-1)/n}}(x).$$

Thus, we need to prove the stable convergence $L^n \xrightarrow{\text{st}} L$ for

$$L_t^n := \sqrt{n}\left(U_t'^n(H) - \int_{\mathbb{R}^d} H(\mathbf{x})\overline{F}_n^{\otimes d}(t, d\mathbf{x})\right). \tag{29}$$

Assume that the function $H \in C^1(\mathbb{R}^d)$ has compact support. Recalling definition (19) of the empirical process $\mathbb{G}_n$, we obtain the identity

$$L_t^n = \sum_{l=1}^d \int_{\mathbb{R}^d} H(\mathbf{x})\mathbb{G}_n(t, dx_l) \prod_{m=1}^{l-1} F_n(t, dx_m) \prod_{m=l+1}^{d} \overline{F}_n(t, dx_m).$$

In step (iv) we will show that both $F_n(t, dx_m)$ and $\overline{F}_n(t, dx_m)$ can be replaced by $F(t, dx_m)$ without affecting the limit. In other words, $L^n - L'^n \xrightarrow{\text{u.c.p.}} 0$ with

$$L_t'^n := \sum_{l=1}^d \int_{\mathbb{R}^d} H(\mathbf{x})\mathbb{G}_n(t, dx_l) \prod_{m \neq l} F(t, dx_m).$$

But, since H is symmetric, we readily deduce that

$$L_t'^n = d \int_{\mathbb{R}^d} H(\mathbf{x})\mathbb{G}_n(t, dx_1) \prod_{m=2}^{d} F(t, dx_m).$$

The random measure $F(t, x)$ has a Lebesgue density in x due to assumption (17), which we denote by $F'(t, x)$. The integration by parts formula implies that

$$L_t'^n = -d \int_{\mathbb{R}^d} \partial_1 H(\mathbf{x})\mathbb{G}_n(t, x_1) \prod_{m=2}^{d} F'(t, x_m)\, d\mathbf{x},$$

where $\partial_l H$ denotes the partial derivative of H with respect to x_l. This identity completes step (i).

(ii) In this step we will start proving the stable convergence $L'^n \xrightarrow{\text{st}} L$ [the function $H \in C^1(\mathbb{R}^d)$ is still assumed to have compact support]. Since the stable convergence $\mathbb{G}_n \xrightarrow{\text{st}} \mathbb{G}$ does not hold in the functional sense in the x variable, we need to overcome this problem by a Riemann sum approximation. Let the support of H be contained in $[-k, k]^d$. Let $-k = z_0 < \cdots < z_l = k$ be the equidistant partition of the interval $[-k, k]$. We set

$$Q(t, x_1) := \int_{\mathbb{R}^{d-1}} \partial_1 H(x_1, \ldots, x_d) \prod_{m=2}^{d} F'(t, x_m)\, dx_2 \cdots dx_d,$$

and define the approximation of $L_t'^n$ via

$$L_t'^n(l) = -\frac{2dk}{l} \sum_{j=0}^{l} Q(t, z_j)\mathbb{G}_n(t, z_j).$$

Proposition 4.2 and the properties of stable convergence imply that

$$\big(Q(\cdot, z_j), \mathbb{G}_n(\cdot, z_j)\big)_{0\le j\le l} \xrightarrow{\text{st}} \big(Q(\cdot, z_j), \mathbb{G}(\cdot, z_j)\big)_{0\le j\le l}.$$

Hence, we deduce the stable convergence

$$L'^n_\cdot(l) \xrightarrow{\text{st}} L_\cdot(l) := -\frac{2dk}{l}\sum_{j=0}^{l} Q(\cdot, z_j)\mathbb{G}(\cdot, z_j)$$

as $n \to \infty$, for any fixed l. Furthermore, we obtain the convergence

$$L(l) \xrightarrow{\text{u.c.p.}} L$$

as $l \to \infty$, where we reversed all above transformations. This convergence completes step (ii).

(iii) To complete the proof of the stable convergence $L'^n \xrightarrow{\text{st}} L$, we need to show that

$$\lim_{l\to\infty}\limsup_{n\to\infty}\sup_{t\in[0,T]}\big|L'^n_t(l) - L'^n_t\big| = 0,$$

where the limits are taken in probability. With $h = l/2k$ we obtain that

$$\big|L'^n_t(l) - L'^n_t\big| = d\left|\int_{\mathbb{R}}\big\{Q\big(t, [xh]/h\big)\mathbb{G}_n\big(t, [xh]/h\big) - Q(t, x)\mathbb{G}_n(t, x)\big\}\,dx\right|.$$

Observe that

$$\sup_{t\in[0,T]}\big|F'(t, x_m)\big| = \int_0^T \varphi_{\sigma_s}(x_m)\,ds \le T \sup_{M^{-1}\le z\le M}\varphi_z(x_m), \tag{30}$$

where M is a positive constant with $M^{-1} \le |\sigma| \le M$. Recalling the definition of $Q(t, x)$ we obtain that

$$\begin{aligned} &\sup_{t\in[0,T]}\big|Q(t, x)\big| \le C_T, \\ &\sup_{t\in[0,T]}\big|Q(t, x) - Q\big(t, [xh]/h\big)\big| \le C_T\eta\big(h^{-1}\big), \end{aligned} \tag{31}$$

where $\eta(\varepsilon) := \sup\{|\partial_1 H(\mathbf{y}_1) - \partial_1 H(\mathbf{y}_2)| : \|\mathbf{y}_1 - \mathbf{y}_2\| \le \varepsilon, \mathbf{y}_1, \mathbf{y}_2 \in [-k, k]^d\}$ denotes the modulus of continuity of the function $\partial_1 H$. We also deduce by Lemma 4.1 that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{G}_n(t, x)\big|^p\Big] \le C_T, \tag{32}$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{G}_n(t, x) - \mathbb{G}_n\big(t, [xh]/h\big)\big|^p\Big] \le C_T h^{-1}, \tag{33}$$

for any even number $p \geq 2$. Combining inequalities (31), (32) and (33), we deduce the convergence

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|L_t'^n(l) - L_t'^n\big|\Big] = 0$$

using that $Q(t,\cdot)$ has compact support contained in $[-k,k]$. Hence, $L'^n \xrightarrow{\text{st}} L$, and we are done.

(iv) In this step we will prove the convergence

$$L^n - L'^n \xrightarrow{\text{u.c.p.}} 0.$$

This difference can be decomposed into several terms; in the following we will treat a typical representative (all other terms are treated in exactly the same manner). For $l < d$ define

$$\begin{aligned} R_t^n(l) := \int_{\mathbb{R}^d} H(\mathbf{x}) \mathbb{G}_n(t, dx_l) \prod_{m=1}^{l-1} F_n(t, dx_m) \\ \times \prod_{m=l+1}^{d-1} \overline{F}_n(t, dx_m) \big[\overline{F}_n(t, dx_d) - F(t, dx_d)\big]. \end{aligned}$$

Now, we use the integration by parts formula to obtain that

$$R_t^n(l) = \int_{\mathbb{R}} N_n(t, x_l) \mathbb{G}_n(t, x_l)\, dx_l,$$

where

$$\begin{aligned} N_n(t, x_l) = \int_{\mathbb{R}^{d-1}} \partial_l H(\mathbf{x}) \prod_{m=1}^{l-1} F_n(t, dx_m) \\ \times \prod_{m=l+1}^{d-1} \overline{F}_n(t, dx_m) \big[\overline{F}_n(t, dx_d) - F(t, dx_d)\big]. \end{aligned}$$

As in step (iii) we deduce for any even $p \geq 2$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbb{G}_n(t, x_l)\big|^p\Big] \leq C_p,$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|N_n(t, x_l)\big|^p\Big] \leq C_p.$$

Recalling that the function H has compact support and applying the dominated convergence theorem, it is sufficient to show that

$$N_n(\cdot, x_l) \xrightarrow{\text{u.c.p.}} 0,$$

for any fixed x_l. But this follows immediately from Lemma 3.1, since

$$F_n(\cdot, x) \xrightarrow{\text{u.c.p.}} F(\cdot, x), \qquad \overline{F}_n(\cdot, x) \xrightarrow{\text{u.c.p.}} F(\cdot, x),$$

for any fixed $x \in \mathbb{R}$, and $\partial_l H$ is a continuous function with compact support. This finishes the proof of step (iv).

(v) Finally, let $H \in C_p^1(\mathbb{R}^d)$ be arbitrary. For any $k \in \mathbb{N}$, let $H_k \in C_p^1(\mathbb{R}^d)$ be a function with $H_k = H$ on $[-k, k]^d$ and $H_k = 0$ on $([-k-1, k+1]^d)^c$. Let us denote by $L_t^n(H)$ and $L_t(H)$ the processes defined by (29) and (27), respectively, that are associated with a given function H. We know from the previous steps that

$$L^n(H_k) \xrightarrow{\text{st}} L(H_k)$$

as $n \to \infty$, and $L(H_k) \xrightarrow{\text{u.c.p.}} L(H)$ as $k \to \infty$. So, we are left to proving that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{t\in[0,T]} \left|L_t^n(H_k) - L_t^n(H)\right| = 0,$$

where the limits are taken in probability. As in steps (ii) and (iii) we obtain the identity

$$\begin{aligned}
&L_t^n(H_k) - L_t^n(H) \\
&\quad = \sum_{l=1}^d \int_{\mathbb{R}^d} \partial_l (H - H_k)(\mathbf{x}) \mathbb{G}_n(t, x_l)\, dx_l \prod_{m=1}^{l-1} F_n(t, dx_m) \prod_{m=l+1}^{d} \overline{F}_n(t, dx_m) \\
&\quad =: \sum_{l=1}^d Q^l(k)_t^n.
\end{aligned}$$

We deduce the inequality

$$\begin{aligned}
\left|Q^l(k)_t^n\right| \le n^{-(l-1)} \sum_{i_1,\dots,i_{l-1}=1}^{[nt]} \int_{\mathbb{R}^{d-l+1}} &\left|\partial_l (H - H_k)\left(\alpha_{i_1}^n, \dots, \alpha_{i_{l-1}}^n, x_l, \dots, x_d\right)\right| \\
&\times \left|\mathbb{G}_n(t, x_l)\right| \prod_{m=l+1}^d \overline{F}_n'(t, x_m)\, dx_l \cdots dx_d.
\end{aligned}$$

We remark that $\partial_l (H_k - H)$ vanishes if all arguments lie in the interval $[-k, k]$. Hence

$$\begin{aligned}
\left|Q^l(k)_t^n\right| \le n^{-(l-1)} \sum_{i_1,\dots,i_{l-1}=1}^{[nt]} \int_{\mathbb{R}^{d-l+1}} &\left|\partial_l (H - H_k)\left(\alpha_{i_1}^n, \dots, \alpha_{i_{l-1}}^n, x_l, \dots, x_d\right)\right| \\
&\times \left(\sum_{m=1}^{l-1} \mathbb{1}_{\{|\alpha_{i_m}^n| > k\}} + \sum_{m=l}^d \mathbb{1}_{\{|x_m| > k\}} \right) \\
&\times \left|\mathbb{G}_n(t, x_l)\right| \prod_{m=l+1}^d \overline{F}_n'(t, x_m)\, dx_l \cdots dx_d.
\end{aligned}$$

Now, applying Lemma 4.1, (11), (30) and the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\big|Q^l(k)_t^n\big|\Big] \\
&\quad\le C_T\int_{\mathbb{R}^{d-l+1}}\Bigg((l-1)\sup_{M^{-1}\le z\le M}\big(1-\Phi_z(k)\big)+\sum_{m=l}^d \mathbb{1}_{\{|x_m|>k\}}\Bigg)^{1/2} \\
&\qquad\qquad\times \psi(x_l,\dots,x_d)\phi(x_l)\prod_{m=l+1}^d \sup_{M^{-1}\le z\le M}\varphi_z(x_m)\,dx_l\cdots dx_d,
\end{aligned}$$

for some bounded function ϕ with exponential decay at $\pm\infty$ and a function $\psi\in C_p^0(\mathbb{R}^{d-l+1})$. Hence

$$\int_{\mathbb{R}^{d-l+1}}\psi(x_l,\dots,x_d)\phi(x_l)\prod_{m=l+1}^d \sup_{M^{-1}\le z\le M}\varphi_z(x_m)\,dx_l\cdots dx_d<\infty,$$

and we conclude that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|Q^l(k)_t^n\big|\Big]=0.$$

This finishes step (v), and we are done with the proof of Proposition 4.3. □

Notice that an additional $\mathcal{F}$-conditional bias would appear in the limiting process L if we would drop the assumption that H is even in each coordinate. The corresponding asymptotic theory for the case $d=1$ has been studied in [22]; see also [17].

REMARK 3. Combining limit theorems for semimartingales with the empirical distribution function approach is probably the most efficient way of proving Proposition 4.3. Nevertheless, we shortly comment on alternative methods of proof.

Treating the multiple sum in the definition of $U'^n(H)$ directly is relatively complicated, since at a certain stage of the proof one will have to deal with partial sums of functions of α_j^n weighted by an anticipative process. This anticipation of the weight process makes it impossible to apply martingale methods directly.

Another approach to proving Proposition 4.3 is a *pseudo* Hoeffding decomposition. This method relies on the application of the classical Hoeffding decomposition to $U'^n(H)$ by pretending that the scaling components $\sigma_{(\mathbf{i}-1)/n}$ are nonrandom. However, since the random variables α_j^n are not independent when the process σ is stochastic, the treatment of the error term connected with the pseudo Hoeffding decomposition will not be easy, because the usual orthogonality arguments of the Hoeffding method do not apply in our setting.

REMARK 4. In the context of Proposition 4.3 we would like to mention a very recent work by Beutner and Zähle [3]. They study the empirical distribution function approach to U- and V-statistics for unbounded kernels H in the classical i.i.d. or weakly dependent setting. Their method relies on the application of the functional delta method for quasi-Hadamard differentiable functionals. In our setting it would require the functional convergence

$$\mathbb{G}_n(t,\cdot) \xrightarrow{\text{st}} \mathbb{G}(t,\cdot),$$

where the convergence takes place in the space of càdlàg functions equipped with the weighted sup-norm $\|f\|_\lambda := \sup_{x\in\mathbb{R}} |(1+|x|^\lambda)f(x)|$ for some $\lambda > 0$. Although we do not really require such a strong result in our framework (as can be seen from the proof of Proposition 4.3), it would be interesting to prove this type of convergence for functionals of high frequency data; cf. the comment before Remark 2.

To conclude this section, we finally present the main result: A functional stable central limit theorem for the original U-statistic $U(H)^n$.

THEOREM 4.4. *Assume that the symmetric function $H \in C_p^1(\mathbb{R}^d)$ is even in each* (*or, equivalently, in one*) *argument. If σ satisfies conditions* (16) *and* (17), *we obtain the functional stable central limit theorem*

$$\sqrt{n}\big(U(H)^n - U(H)\big) \xrightarrow{\text{st}} L, \tag{34}$$

where the convergence takes place in $\mathbb{D}([0,T])$ equipped with the uniform topology and the limiting process L is defined at (27).

PROOF. In Section 7.2 we will show the following statement: under condition (16) it holds that

$$\sqrt{n}\big|U(H)^n - \widetilde{U}(H)^n\big| \xrightarrow{\text{u.c.p.}} 0. \tag{35}$$

In view of Proposition 4.3, it remains to prove that $\sqrt{n}|\widetilde{U}(H)_t^n - U_t'^n(H)| \xrightarrow{\text{u.c.p.}} 0$. But due to the symmetry of H, we obtain as in the proof of Theorem 3.3

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\widetilde{U}(H)_t^n - U_t'^n(H)\big|\Big] \le \frac{C_T}{n}.$$

This completes the proof of Theorem 4.4. $\square$

We remark that the stable convergence at (34) is not *feasible* in its present form, since the distribution of the limiting process L is unknown. In the next section we will explain how to obtain a feasible central limit theorem that opens the door to statistical applications.

5. Estimation of the conditional variance. In this section we present a standard central limit theorem for the U-statistic $U(H)_t^n$. We will confine ourselves to the presentation of a result in finite distributional sense. According to Remark 2(iii) applied to

$$f_t(x) := d \int_{\mathbb{R}^{d-1}} H(x, x_2, \ldots, x_d) F(t, dx_2) \cdots F(t, dx_d),$$

the conditional variance of the limit L_t is given by

$$V_t := \mathbb{E}'[|L_t|^2 | \mathcal{F}] = \int_0^t \left(\int_{\mathbb{R}} f_t^2(x) \varphi_{\sigma_s}(x)\, dx - \left(\int_{\mathbb{R}} f_t(x) \varphi_{\sigma_s}(x)\, dx \right)^2 \right) ds.$$

Hence, the random variable L_t is nondegenerate when

$$\operatorname{var}(\mathbb{E}[H(x_1 U_1, \ldots, x_d U_d) | U_1]) > 0, \qquad (U_1, \ldots, U_d) \sim \mathcal{N}_d(0, \mathbf{I}_d),$$

for all $x_1, \ldots, x_d \in \{\sigma_s | s \in A \subseteq [0,t]\}$ and some set A with positive Lebesgue measure. This essentially coincides with the classical nondegeneracy condition for U-statistics of independent random variables.

We define the functions $G_1 : \mathbb{R}^{2d-1} \to \mathbb{R}$ and $G_2 : \mathbb{R}^2 \times \mathbb{R}^{2d-2} \to \mathbb{R}$ by

$$G_1(\mathbf{x}) = H(x_1, x_2, \ldots, x_d) H(x_1, x_{d+1}, \ldots, x_{2d-1}), \tag{36}$$

$$G_2(\mathbf{x}; \mathbf{y}) = H(x_1, y_1, \ldots, y_{d-1}) H(x_2, y_d, \ldots, y_{2d-2}), \tag{37}$$

respectively. Then V_t can be written as

$$\begin{aligned} V_t &= d^2 \int_{[0,t]^{2d-1}} \rho_{\sigma_{\mathbf{s}}}(G_1)\, d\mathbf{s} \\ &\quad - d^2 \int_{[0,t]^{2d-2}} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{\sigma_{\mathbf{s}}}(G_2(x_1, x_2; \cdot)) \varphi_{\sigma_q}(x_1) \varphi_{\sigma_q}(x_2)\, dx_1\, dx_2\, dq\, d\mathbf{s}. \end{aligned}$$

We denote the first and second summand on the right-hand side of the preceding equation by $V_{1,t}$ and $V_{2,t}$, respectively. Let $\widetilde{G}_1$ denote the symmetrization of the function G_1. By Theorem 3.3 it holds that

$$V_{1,t}^n = d^2 U(\widetilde{G}_1)_t^n \xrightarrow{\text{u.c.p.}} d^2 U(\widetilde{G}_1)_t = V_{1,t}.$$

The multiple integral $V_{2,t}$ is almost in the form of the limit in Theorem 3.3, and it is indeed possible to estimate it by a slightly modified U-statistic as the following proposition shows. The statistic presented in the following proposition is a generalization of the bipower concept discussed, for example, in [2] in the case $d = 1$.

PROPOSITION 5.1. *Assume that $H \in C_p^0(\mathbb{R}^d)$. Let*

$$\begin{aligned} V_{2,t}^n &:= \frac{d^2}{n} \binom{n}{2d-2}^{-1} \\ &\quad \times \sum_{\mathbf{i} \in \mathcal{A}_t^n(2d-2)} \sum_{j=1}^{[nt]-1} \widetilde{G}_2(\sqrt{n}\Delta_j^n X, \sqrt{n}\Delta_{j+1}^n X; \sqrt{n}\Delta_{i_1}^n X, \ldots, \sqrt{n}\Delta_{i_{2d-2}}^n X), \end{aligned}$$

where $\widetilde{G}_2$ denotes the symmetrization of G_2 with respect to the **y**-*values*, *that is*,

$$\widetilde{G}_2(\mathbf{x};\mathbf{y}) = \frac{1}{(2d-2)!}\sum_{\pi} G_2(\mathbf{x};\pi\mathbf{y}),$$

for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{y} \in \mathbb{R}^{2d-2}$, and where the sum runs over all permutations of $\{1,\dots,2d-2\}$. Then

$$V_2^n \xrightarrow{\text{u.c.p.}} V_2.$$

PROOF. The result can be shown using essentially the same arguments as in the proofs of Proposition 3.2 and Theorem 3.3. We provide a sketch of the proof. Similarly to (7) we define

$$\widetilde{V}_{2,t}^n := \frac{d^2}{n}\binom{n}{2d-2}^{-1} \sum_{\mathbf{i}\in\mathcal{A}_t^n(2d-2)} \sum_{j=1}^{[nt]-1} \widetilde{G}_2\big(\alpha_j^n, \alpha_{j+1}'^n; \alpha_{i_1}^n, \dots, \alpha_{i_{2d-2}}^n\big),$$

where $\alpha_{j+1}'^n := \sqrt{n}\sigma_{(j-1)/n}\Delta_{i+1}^n W$. Analogously to (8) we introduce the random process

$$V_{2,t}'^n := d^2 \int_{\mathbb{R}^{2d-2}}\int_{\mathbb{R}^2} \widetilde{G}_2(\mathbf{x};\mathbf{y})\widetilde{F}_n(t, d\mathbf{x}) F_n^{\otimes(2d-2)}(t, d\mathbf{y}),$$

where

$$\widetilde{F}_n(t, x_1, x_2) = \frac{1}{n}\sum_{j=1}^{[nt]-1} \mathbb{1}_{\{\alpha_j^n \le x_1\}}\mathbb{1}_{\{\alpha_{j+1}'^n \le x_2\}}.$$

Writing out $V_{2,t}'^n$ as a multiple sum over nondecreasing multi-indices in the **y** arguments, one observes as before that $V_{2,t}'^n$ and $\widetilde{V}_{2,t}^n$ differ in at most $O(n^{2d-3})$ summands. Therefore, using the same argument as in the proof of Theorem 3.3

$$\widetilde{V}_{2,t}^n - V_{2,t}'^n \xrightarrow{\text{u.c.p.}} 0.$$

For any fixed $x, y \in \mathbb{R}$ it holds that

$$\widetilde{F}_n(t,x,y) \xrightarrow{\text{u.c.p.}} \widetilde{F}(t,x,y) := \int_0^t \Phi_{\sigma_s}(x)\Phi_{\sigma_s}(y)\,ds.$$

This can be shown similarly to the proof of Proposition 3.2 as follows. Let $\xi_j^n = n^{-1}\mathbb{1}_{\{\alpha_j^n \le x_1\}}\mathbb{1}_{\{\alpha_{j+1}'^n \le x_2\}}$. Then

$$\sum_{j=1}^{[nt]-1} \mathbb{E}\big[\xi_j^n | \mathcal{F}_{(j-1)/n}\big] = \frac{1}{n}\sum_{j=1}^{[nt]-1} \Phi_{\sigma_{(j-1)/n}}(x_1)\Phi_{\sigma_{(j-1)/n}}(x_2) \xrightarrow{\text{u.c.p.}} \widetilde{F}(t,x,y).$$

On the other hand, we trivially have that $\sum_{j=1}^{[nt]-1} \mathbb{E}[|\xi_j^n|^2|\mathcal{F}_{(j-1)/n}] \xrightarrow{\mathbb{P}} 0$, for any fixed $t > 0$. Hence, the Lenglart's domination property (see [21], page 35) implies the convergence

$$\sum_{j=1}^{[nt]-1} (\xi_j^n - \mathbb{E}[\xi_j^n|\mathcal{F}_{(j-1)/n}]) \xrightarrow{\text{u.c.p.}} 0,$$

which in turn means that $\widetilde{F}_n(t, x, y) \xrightarrow{\text{u.c.p.}} \widetilde{F}(t, x, y)$.

We know now that $V_{2,t}'^n$ converges to the claimed limit if G_2 is compactly supported. For a general G_2 with polynomial growth one can proceed exactly as in Proposition 3.2. To complete the proof, one has to show that $V_{2,t}^n - V_{2,t}'^n \xrightarrow{\text{u.c.p.}} 0$. This works exactly as in Section 7.1. □

The properties of stable convergence immediately imply the following theorem.

THEOREM 5.2. *Let the assumptions of Theorem 4.4 be satisfied. Let $t > 0$ be fixed. Then we obtain the standard central limit theorem*

$$\frac{\sqrt{n}(U(H)_t^n - U(H)_t)}{\sqrt{V_t^n}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{38}$$

where $V_t^n = V_{1,t}^n - V_{2,t}^n$ using the notation defined above.

The convergence in law in (38) is a feasible central limit theorem that can be used in statistical applications. It is possible to obtain similar multivariate central limit theorems for finite-dimensional vectors $\sqrt{n}(U(H)_{t_j}^n - U(H)_{t_j})_{1\le j\le k}$; we leave the details to the interested reader.

6. Statistical applications. In this section we present some statistical applications of the limit theory for U-statistics of continuous Itô semimartingales.

6.1. *Gini's mean difference.* Gini's mean difference is a classical measure of statistical dispersion, which serves as robust measure of variability of a probability distribution [7]. Recall that for a given distribution $\mathbb{Q}$, Gini's mean difference is defined as

$$\text{MD} := \mathbb{E}\big[|Y_1 - Y_2|\big],$$

where Y_1, Y_2 are independent random variables with distribution $\mathbb{Q}$. In the framework of i.i.d. observations $(Y_i)_{i\ge1}$, the measure MD is consistently estimated by the U-statistic $\frac{2}{n(n-1)} \sum_{1\le i<j\le n} |Y_i - Y_j|$. Gini's mean difference is connected to questions of stochasic dominance as shown by [27]. We refer to the recent paper [24] for the estimation theory for Gini's mean difference under long range dependence.

In the setting of continuous Itô semimartingales we conclude by Theorem 3.3 that

$$U(H)_t^n \xrightarrow{\text{u.c.p.}} \mathrm{MD}_t := m_1 \int_{[0,t]^2} |\sigma_{s_1}^2 + \sigma_{s_2}^2|^{1/2} \, ds_1 \, ds_2,$$

where the function H is given by $H(x,y) = |x-y|$, and m_p is the pth absolute moment of $\mathcal{N}(0,1)$. In mathematical finance the quantity MD_t may be viewed as an alternative measure of price variability, which is more robust to outliers than the standard quadratic variation $[X,X]_t$.

Formally, we cannot directly apply Theorem 4.4 to obtain a weak limit theory for the statistic $U(H)_t^n$, since the function $H(x,y) = |x-y|$ is not differentiable, and H is not even in *each* component. Since $Y_1 - Y_2$ and $Y_1 + Y_2$ have the same distribution for centered independent normally distributed random variables Y_1, Y_2, the modification

$$\overline{H}(x,y) := \tfrac{1}{2}\big(|x-y| + |x+y|\big),$$

which is even in each component, has the same limit, that is, $U(\overline{H})_t^n \xrightarrow{\text{u.c.p.}} \mathrm{MD}_t$. Moreover, using sub-differential calculus and defining

$$\operatorname{grad} \overline{H}(x,y) := \tfrac{1}{2}\big(\operatorname{sign}(x-y) + \operatorname{sign}(x+y), \operatorname{sign}(x-y) + \operatorname{sign}(x+y)\big),$$

all the proof steps remain valid (we also refer to [2], who prove the central limit theorem for nondifferentiable functions). Thus, by the assertion of Theorem 4.4, we deduce the stable convergence

$$\sqrt{n}\big(U(\overline{H})_t^n - \mathrm{MD}_t\big) \xrightarrow{\text{st}} L_t = \int_{\mathbb{R}^2} \big(|x_1 - x_2| + |x_1 + x_2|\big) \mathbb{G}(t, dx_1) F(t, dx_2),$$

where the stochastic fields $\mathbb{G}(t,x)$ and $F(t,x)$ are defined in Proposition 4.2 and (9), respectively. Now, we follow the route proposed in Section 5 to obtain a standard central limit theorem. We compute the symmetrization $\widetilde{G}_1, \widetilde{G}_2$ of the functions G_1, G_2 defined at (36) and (37), respectively:

$$\begin{aligned}
\widetilde{G}_1(x_1,x_2,x_3) = \tfrac{1}{6}\Big(&\big(|x_1-x_2| + |x_1+x_2|\big)\big(|x_1-x_3| + |x_1+x_3|\big) \\
&+ \big(|x_2-x_1| + |x_2+x_1|\big)\big(|x_2-x_3| + |x_2+x_3|\big) \\
&+ \big(|x_3-x_1| + |x_3+x_1|\big)\big(|x_3-x_2| + |x_3+x_2|\big)\Big), \\
\widetilde{G}_1(x_1,x_2;y_1,y_2) = \tfrac{1}{4}\Big(&\big(|x_1-y_1| + |x_1+y_1|\big)\big(|x_2-y_2| + |x_2+y_2|\big) \\
&+ \big(|x_1-y_2| + |x_1+y_2|\big)\big(|x_2-y_1| + |x_2+y_1|\big)\Big).
\end{aligned}$$

Using these functions we construct the statistics $V_{1,t}^n$ and $V_{2,t}^n$ (see Section 5). Finally, for any fixed $t > 0$ we obtain a feasible central limit theorem

$$\frac{\sqrt{n}(U(\overline{H})_t^n - \mathrm{MD}_t)}{\sqrt{V_{1,t}^n - V_{2,t}^n}} \xrightarrow{d} \mathcal{N}(0,1).$$

The latter enables us to construct confidence regions for mean difference statistic MD_t.

6.2. $\mathbb{L}^p$*-type tests for constant volatility.* In this subsection we propose a new homoscedasticity test for the volatility process σ^2. Our main idea relies on a certain distance measure, which is related to $\mathbb{L}^p$-norms; we refer to [12, 13] for similar testing procedures in the $\mathbb{L}^2$ case. Let us define

$$h(s_1, \dots, s_d) := \sum_{i=1}^{d} \sigma_{s_i}^2, \qquad s_1, \dots, s_d \in [0, 1],$$

and consider a real number $p > 1$. Our test relies on the $\mathbb{L}^p$-norms

$$\|h\|_{\mathbb{L}^p} := \left(\int_{[0,1]^d} |h(\mathbf{s})|^p \, d\mathbf{s} \right)^{1/p}.$$

Observe the inequality $\|h\|_{\mathbb{L}^p} \geq \|h\|_{\mathbb{L}^1}$ and, when the process h is continuous, equality holds if and only if h is constant. Applying this intuition, we introduce a distance measure $\mathcal{M}^2$ via

$$\mathcal{M}^2 := \frac{\|h\|_{\mathbb{L}^p}^p - \|h\|_{\mathbb{L}^1}^p}{\|h\|_{\mathbb{L}^p}^p} \in [0, 1].$$

Notice that a continuous process σ^2 is constant if and only if $\mathcal{M}^2 = 0$. Furthermore, the measure $\mathcal{M}^2$ provides a quantitative account of the deviation from the homoscedasticity hypothesis, as it takes values in [0, 1].

For simplicity of exposition we introduce an empirical analogue of $\mathcal{M}^2$ in the case $d = 2$. We define the functions

$$H_1(x) := \tfrac{1}{2}\big(|x_1 - x_2|^{2p} + |x_1 + x_2|^{2p}\big), \qquad H_2(x) := x_1^2 + x_2^2$$

with $x \in \mathbb{R}^2$. Notice that both functions are continuously differentiable and even in each component; hence they satisfy the assumptions of Theorems 3.3 and 4.4. In particular, Theorem 3.3 implies the convergence in probability

$$U(H_1)_1^n \xrightarrow{\mathbb{P}} U(H_1)_1 = m_{2p}\|h\|_{\mathbb{L}^p}^p, \qquad U(H_2)_1^n \xrightarrow{\mathbb{P}} U(H_2)_1 = \|h\|_{\mathbb{L}^1},$$

where the constant m_{2p} has been defined in the previous subsection. The main ingredient for a formal testing procedure is the following result.

PROPOSITION 6.1. *Assume that conditions of Theorem 4.4 hold. Then we obtain the stable convergence*

$$\begin{aligned} &\sqrt{n}\big(U(H_1)_1^n - m_{2p}\|h\|_{\mathbb{L}^p}^p, U(H_2)_1^n - \|h\|_{\mathbb{L}^1}\big) \\ &\quad \xrightarrow{\text{st}} 2\bigg(\int_{\mathbb{R}^2} H_1(x_1, x_2)\mathbb{G}(1, dx_1)F(1, dx_2), \\ &\qquad \int_{\mathbb{R}^2} H_2(x_1, x_2)\mathbb{G}(1, dx_1)F(1, dx_2)\bigg). \end{aligned} \tag{39}$$

Furthermore, the $\mathcal{F}$-conditional covariance matrix $V = (V_{ij})_{1\le i,j\le 2}$ of the limiting random variable is given as

$$V_{ij} = \int_0^1 \biggl(\int_{\mathbb{R}} f_i(x) f_j(x) \varphi_{\sigma_s}(x)\,dx - \biggl(\int_{\mathbb{R}} f_i(x)\varphi_{\sigma_s}(x)\,dx \biggr) \biggl(\int_{\mathbb{R}} f_j(x)\varphi_{\sigma_s}(x)\,dx \biggr) \biggr)\,ds \tag{40}$$

with

$$f_i(x) := 2\int_{\mathbb{R}} H_i(x,y) F(1,dy), \qquad i = 1,2.$$

PROOF. As in the proof of Theorem 4.4 we deduce that

$$\sqrt{n}\bigl(U(H_i)_1^n - U(H_i)_1\bigr) = L_1'^n(i) + o_{\mathbb{P}}(1), \qquad i = 1,2,$$

where $L_1'^n(i)$ is defined via

$$L_1'^n(i) = 2\int_{\mathbb{R}^2} H_i(x_1,x_2)\mathbb{G}_n(1,dx_1)F(1,dx_2).$$

Now, exactly as in steps (ii)–(v) of the proof of Proposition 4.3, we conclude the joint stable convergence in (39). The $\mathcal{F}$-conditional covariance matrix V is obtained from Remark 2(iii) as in the beginning of Section 5. □

Let now $\mathcal{M}_n^2$ be the empirical analogue of $\mathcal{M}^2$, that is,

$$\mathcal{M}_n^2 := \frac{m_{2p}^{-1} U(H_1)_1^n - (U(H_2)_1^n)^p}{m_{2p}^{-1} U(H_1)_1^n} \xrightarrow{\mathbb{P}} \mathcal{M}^2.$$

Observe the identities

$$\mathcal{M}_n^2 = r\bigl(U(H_1)_1^n, U(H_2)_1^n\bigr), \qquad \mathcal{M}^2 = r\bigl(m_{2p}\|h\|_{\mathbb{L}^p}^p, \|h\|_{\mathbb{L}^1}\bigr),$$

where $r(x,y) = 1 - m_{2p}\frac{y^p}{x}$. Applying Proposition 6.1 and delta method for stable convergence, we conclude that $\sqrt{n}(\mathcal{M}_n^2 - \mathcal{M}^2)$ converges stably in law toward a mixed normal distribution with mean 0 and $\mathcal{F}$-conditional variance given by

$$v^2 := \nabla r\bigl(m_{2p}\|h\|_{\mathbb{L}^p}^p, \|h\|_{\mathbb{L}^1}\bigr) V \nabla r\bigl(m_{2p}\|h\|_{\mathbb{L}^p}^p, \|h\|_{\mathbb{L}^1}\bigr)^\star,$$

where the random variable $V \in \mathbb{R}^{2\times 2}$ is defined at (40).

For an estimation of V we can proceed as in Section 5. Define the functions $G_1^{ij}:\mathbb{R}^3 \to \mathbb{R}$ and $G_2^{ij}:\mathbb{R}^4 \to \mathbb{R}$ by

$$G_1^{ij}(x_1,x_2,x_3) = H_i(x_1,x_2)H_j(x_1,x_3),$$
$$G_2^{ij}(x_1,x_2,y_1,y_2) = H_i(x_1,y_1)H_j(x_2,y_2), \qquad i,j = 1,2.$$

Let further $\widetilde{G}_1^{ij}$ be the symmetrization of G_1^{ij} and $\widetilde{G}_2^{ij}$ the symmetrization of G_2^{ij} with respect to the $\mathbf{y}$-values. With

$$W_{ij} := \frac{4}{n}\binom{n}{2}^{-1} \sum_{i_1=1}^{n-1} \sum_{1\le i_2<i_3\le n} \widetilde{G}_2^{ij}\big(\sqrt{n}\Delta_{i_1}^n X, \sqrt{n}\Delta_{i_1+1}^n X, \sqrt{n}\Delta_{i_2}^n X, \sqrt{n}\Delta_{i_3}^n X\big),$$

we can, exactly as in Section 5, deduce that

$$V^n := \big(4U\big(\widetilde{G}_1^{ij}\big)_1^n - W_{ij}\big)_{i,j=1,2} \xrightarrow{\mathbb{P}} V.$$

Using the previous results, we directly get

$$v_n^2 := \nabla r\big(U(H_1)_1^n, U(H_2)_1^n\big) V^n \nabla r\big(U(H_1)_1^n, U(H_2)_1^n\big)^\star \xrightarrow{\mathbb{P}} v^2.$$

Now the properties of stable convergence yield the following feasible central limit theorem:

$$\frac{\sqrt{n}(\mathcal{M}_n^2 - \mathcal{M}^2)}{\sqrt{v_n^2}} \xrightarrow{d} \mathcal{N}(0,1). \tag{41}$$

With these formulas at hand, we can derive a formal test procedure for the hypothesis

$$H_0 : \sigma_s^2 \text{ is constant on } [0,1] \quad \text{vs.} \quad H_1 : \sigma_s^2 \text{ is not constant on } [0,1].$$

These hypotheses are obviously equivalent to

$$H_0 : \mathcal{M}^2 = 0 \quad \text{vs.} \quad H_1 : \mathcal{M}^2 > 0.$$

Defining the test statistic S_n via

$$S_n := \frac{\sqrt{n}\mathcal{M}_n^2}{\sqrt{v_n^2}},$$

we reject the null hypothesis at level $\gamma \in (0,1)$ whenever $S_t^n > c_{1-\gamma}$, where $c_{1-\gamma}$ denotes the $(1-\gamma)$-quantile of $\mathcal{N}(0,1)$. Now, (41) implies that

$$\lim_{n\to\infty} \mathbb{P}_{H_0}(S_n > c_{1-\gamma}) = \gamma, \qquad \lim_{n\to\infty} \mathbb{P}_{H_1}\big(S_n^n > c_{1-\gamma}\big) = 1.$$

In other words, our test statistic is consistent and keeps the level γ asymptotically.

6.3. *Wilcoxon test statistic for structural breaks.* Change-point analysis has been an active area of research for many decades; we refer to [6] for a comprehensive overview. The Wilcoxon statistic is a standard statistical procedure for testing structural breaks in location models. Let $(Y_i)_{1\le i\le n}$, $(Z_i)_{1\le i\le m}$ be mutually independent observations with $Y_i \sim \mathbb{Q}_{\theta_1}$, $Z_i \sim \mathbb{Q}_{\theta_2}$, where $\mathbb{Q}_\theta(A) = \mathbb{Q}_0(A-\theta)$ for all

$A \in \mathcal{B}(\mathbb{R})$ and $\mathbb{Q}_0$ be a nonatomic probability measure. In this classical framework the Wilcoxon statistic is defined by

$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{1}_{\{Y_i \le Z_j\}}.$$

Under the null hypothesis $\theta_1 = \theta_2$, the test statistic is close to $1/2$, while deviations from this value indicate that $\theta_1 \neq \theta_2$. We refer to the recent work [8] for change-point tests for long-range dependent data.

Applying the same intuition we may provide a test statistic for structural breaks in the volatility process σ^2. Assume that the semimartingale X is observed at high frequency on the interval $[0, 1]$ and the volatility is constant on the intervals $[0, t)$ and $(t, 1]$ for some $t \in (0, 1)$, that is, $\sigma_s^2 = \sigma_0^2$ on $[0, t)$ and $\sigma_s^2 = \sigma_1^2$ on $(t, 1]$. Our aim is to test the null hypothesis $\sigma_0^2 = \sigma_1^2$ or to infer the change-point t when $\sigma_0^2 \neq \sigma_1^2$. In this framework the Wilcoxon type statistic is defined via

$$\mathrm{WL}_t^n := \frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}\mathbb{1}_{\{|\Delta_i^n X| \le |\Delta_j^n X|\}}.$$

Notice that the kernel is neither symmetric nor continuous. Nevertheless, we deduce the following result.

PROPOSITION 6.2. *Assume that condition* (17) *holds. Then we obtain the convergence*:

$$\mathrm{WL}_t^n \xrightarrow{\text{u.c.p.}} \mathrm{WL}_t := \int_0^t \int_t^1 \left(\int_{\mathbb{R}^2} \mathbb{1}_{\{|\sigma_{s_1} u_1| \le |\sigma_{s_2} u_2|\}} \varphi_d(\mathbf{u})\, d\mathbf{u}\right) ds_1\, ds_2 \tag{42}$$

$$= \int_0^t \int_t^1 \left(1 - \frac{2}{\pi}\arctan\left|\frac{\sigma_{s_1}}{\sigma_{s_2}}\right|\right) ds_1\, ds_2. \tag{43}$$

PROOF. As in the proof of Theorem 3.3, we first show the convergence (42) for the approximations α_i^n of the scaled increments $\sqrt{n}\Delta_i^n X$. We define

$$U_t^{\prime n} := \int_{\mathbb{R}^2} \mathbb{1}_{\{|x| \le |y|\}} F_n(t, dx)\big(F_n(1, dy) - F_n(t, dy)\big).$$

Since condition (17) holds, the measure $F_n(t, dx)$ is nonatomic. Hence, we conclude that

$$U_t^{\prime n} \xrightarrow{\text{u.c.p.}} \mathrm{WL}_t$$

exactly as in the proof of Proposition 3.2. It remains to prove the convergence

$$\mathrm{WL}_t^n - U_t^{\prime n} \xrightarrow{\text{u.c.p.}} 0.$$

Observe the identity

$$\begin{aligned}\mathrm{WL}_t^n - U_t'^n &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}(\mathbb{1}_{\{|\Delta_i^n X|\le|\Delta_j^n X|\}} - \mathbb{1}_{\{|\alpha_i^n|\le|\alpha_j^n|\}})\\ &= \frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}(\mathbb{1}_{\{|\Delta_i^n X|\le|\Delta_j^n X|\}} - \mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|\le|\alpha_j^n|\}}\\ &\qquad + \mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|\le|\alpha_j^n|\}} - \mathbb{1}_{\{|\alpha_i^n|\le|\alpha_j^n|\}}).\end{aligned}$$

In the following we concentrate on proving that

$$\frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}(\mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|\le|\alpha_j^n|\}} - \mathbb{1}_{\{|\alpha_i^n|\le|\alpha_j^n|\}}) \xrightarrow{\text{u.c.p.}} 0,$$

as the other part is negligible by the same arguments. Using the identity

$$\begin{aligned}&\mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|\le|\alpha_j^n|\}} - \mathbb{1}_{\{|\alpha_i^n|\le|\alpha_j^n|\}}\\ &\quad = \mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|\le|\alpha_j^n|,|\alpha_i^n|>|\alpha_j^n|\}} - \mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|>|\alpha_j^n|,|\alpha_i^n|\le|\alpha_j^n|\}}\end{aligned}$$

we restrict our attention on proving

$$\frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}\mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|>|\alpha_j^n|,|\alpha_i^n|\le|\alpha_j^n|\}} \xrightarrow{\text{u.c.p.}} 0.$$

For an arbitrary $q \in (0, 1/2)$, we deduce the inequality

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|>|\alpha_j^n|,|\alpha_i^n|\le|\alpha_j^n|\}}] &\le \mathbb{E}\left[\frac{|\sqrt{n}\Delta_i^n X - \alpha_i^n|^q}{||\alpha_j^n| - |\alpha_i^n||^q}\right]\\ &\le \mathbb{E}\big[\big|\sqrt{n}\Delta_i^n X - \alpha_i^n\big|^{2q}\big]^{1/2}\mathbb{E}\big[\big||\alpha_j^n| - |\alpha_i^n|\big|^{-2q}\big]^{1/2}.\end{aligned}$$

For a standard normal random variable U, and for any $x > 0$, $y \ge 0$, define

$$g_q(x, y) := \mathbb{E}\big[\big|x|U| - y\big|^{-2q}\big].$$

Since $2q < 1$, we have

$$\begin{aligned}g_q(x, y) &= \mathbb{E}\big[\big|x|U| - y\big|^{-2q}\mathbb{1}_{\{|x|U|-y|\le 1\}}\big] + \mathbb{E}\big[\big|x|U| - y\big|^{-2q}\mathbb{1}_{\{|x|U|-y|>1\}}\big]\\ &\le \int_{\mathbb{R}}\big|x|u| - y\big|^{-2q}\mathbb{1}_{\{|x|u|-y|\le 1\}}\,du + 1 \qquad (44)\\ &\le \frac{C_q}{x} + 1 < \infty.\end{aligned}$$

Due to assumption (17) and by a localization argument, we can assume that σ_t is uniformly bounded away from zero. Therefore, and by (44) we obtain

$$\begin{aligned}\mathbb{E}[||\alpha_j^n| - |\alpha_i^n||^{-2q}] &= \mathbb{E}[\mathbb{E}[||\alpha_j^n| - |\alpha_i^n||^{-2q}|\mathcal{F}_{(j-1)/n}]] \\ &= \mathbb{E}[g_q(\sigma_{(j-1)/n}, \alpha_i^n)] \le C_q < \infty.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}\mathbb{E}[\mathbb{1}_{\{|\sqrt{n}\Delta_i^n X|>|\alpha_j^n|, |\alpha_i^n|\le|\alpha_j^n|\}}] \\ &\quad \le \frac{C}{n^2}\sum_{i=1}^{[nt]}\sum_{j=[nt]+1}^{n}\mathbb{E}[|\sqrt{n}\Delta_i^n X - \alpha_i^n|^{2q}]^{1/2} \xrightarrow{\text{u.c.p.}} 0,\end{aligned}$$

where the last convergence follows as in (45). This completes the proof of Proposition 6.2. □

Now, observe that when the process σ^2 has no change-point at time $t \in (0, 1)$ (i.e., $\sigma_0^2 = \sigma_1^2$) the limit at (42) is given by $\mathrm{WL}_t = \frac{1}{2}t(1-t)$. Thus, under the null hypothesis $\sigma_0^2 = \sigma_1^2$, we conclude that $\mathrm{WL}_t^n \xrightarrow{\text{u.c.p.}} \frac{1}{2}t(1-t)$. Since the time point $t \in (0, 1)$ is unknown in general, we may use the test statistic

$$\sup_{t\in(0,1)}\left|\mathrm{WL}_t^n - \frac{1}{2}t(1-t)\right|$$

to test for a possible change point. Large values of this quantity speak against the null hypothesis. On the other hand, under the alternative $\sigma_0^2 \neq \sigma_1^2$, the statistic $\hat{t}_n := \operatorname{argsup}_{t\in(0,1)} |\mathrm{WL}_t^n - \frac{1}{2}t(1-t)|$ provides a consistent estimator of the change-point $t \in (0, 1)$. A formal testing procedure would rely on a stable central limit theorem for WL_t^n, which is expected to be highly complex, since the applied kernel is not differentiable.

7. Proofs of some technical results. Before we start with the proofs of (13) and (35) we state the following lemma, which can be shown exactly as [2], Lemma 5.4.

LEMMA 7.1. *Let $f:\mathbb{R}^d \to \mathbb{R}^q$ be a continuous function of polynomial growth. Let further $\gamma_i^n, \gamma_i'^n$ be real-valued random variables satisfying $\mathbb{E}[(|\gamma_i^n| + |\gamma_i'^n|)^p] \le C_p$ for all $p \ge 2$ and*

$$\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_t^n(d)}\mathbb{E}[\|\gamma_{\mathbf{i}}^n - \gamma_{\mathbf{i}}'^n\|^2] \to 0.$$

Then we have for all $t > 0$,

$$\binom{n}{d}^{-1} \sum_{\mathbf{i} \in \mathcal{A}_t^n(d)} \mathbb{E}[\| f(\gamma_{\mathbf{i}}^n) - f(\gamma_{\mathbf{i}}'^n)\|^2] \to 0.$$

Recall that we assume (4) without loss of generality; in Sections 7.2 and 7.3 we further assume (18), that is, all the involved processes are bounded.

7.1. *Proof of* (13). The Burkhölder inequality yields that $\mathbb{E}[(|\sqrt{n}\Delta_i^n X| + |\alpha_i^n|)^p] \le C_p$ for all $p \ge 2$. In view of the previous lemma $U(H)^n - \widetilde{U}(H)^n \xrightarrow{\text{u.c.p.}} 0$ is a direct consequence of

$$\binom{n}{d}^{-1} \sum_{\mathbf{i} \in \mathcal{A}_t^n(d)} \mathbb{E}[\|\sqrt{n}\Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n\|^2] \le \frac{C}{n} \sum_{j=1}^{[nt]} \mathbb{E}[|\sqrt{n}\Delta_j^n X - \alpha_j^n|^2] \to 0 \tag{45}$$

as it is shown in [2], Lemma 5.3.

7.2. *Proof of* (35). We divide the proof into several steps.

(i) We claim that

$$\sqrt{n}\big(U(H)^n - \widetilde{U}(H)^n\big) - P^n(H) \xrightarrow{\text{u.c.p.}} 0,$$

where

$$P_t^n(H) := \sqrt{n}\binom{n}{d}^{-1} \sum_{\mathbf{i} \in \mathcal{A}_t^n(d)} \nabla H(\alpha_{\mathbf{i}}^n)(\sqrt{n}\Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n).$$

Here, ∇H denotes the gradient of H. This can be seen as follows. Since the process σ is itself a continuous Itô semimartingale, we have

$$\mathbb{E}\big[|\sqrt{n}\Delta_i^n X - \alpha_i^n|^p\big] \le C_p n^{-p/2} \tag{46}$$

for all $p \ge 2$. By the mean value theorem, for any $\mathbf{i} \in \mathcal{A}_t^n(d)$, there exists a random variable $\chi_{\mathbf{i}}^n \in \mathbb{R}^d$ such that

$$H(\sqrt{n}\Delta_{\mathbf{i}}^n X) - H(\alpha_{\mathbf{i}}^n) = \nabla H(\chi_{\mathbf{i}}^n)(\sqrt{n}\Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n)$$

with $\|\chi_{\mathbf{i}}^n - \alpha_{\mathbf{i}}^n\| \le \|\sqrt{n}\Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n\|$. Therefore, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t \le T}\big|\sqrt{n}\big(U(H)_t^n - \widetilde{U}_t(H)^n\big) - P_t^n(H)\big|\Big]\\ &\quad \le C\sqrt{n}\binom{n}{d}^{-1} \sum_{\mathbf{i} \in \mathcal{A}_T^n(d)} \mathbb{E}[\|(\nabla H(\chi_{\mathbf{i}}^n) - \nabla H(\alpha_{\mathbf{i}}^n)\|\|(\sqrt{n}\Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n)\|]\end{aligned}$$

$$
\begin{aligned}
&\le C\sqrt{n}\binom{n}{d}^{-1}\Big(\sum_{\mathbf{i}\in\mathcal{A}_T^n(d)}\mathbb{E}\big[\big\|\big(\nabla H(\chi_{\mathbf{i}}^n)-\nabla H(\alpha_{\mathbf{i}}^n)\big)\big\|^2\big]\Big)^{1/2}\\
&\qquad\times\Big(\sum_{\mathbf{i}\in\mathcal{A}_T^n(d)}\mathbb{E}\big[\big\|\big(\sqrt{n}\Delta_{\mathbf{i}}^n X-\alpha_{\mathbf{i}}^n\big)\big\|^2\big]\Big)^{1/2}\\
&\le C\Big\{\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_T^n(d)}\mathbb{E}\big[\big\|\big(\nabla H(\chi_{\mathbf{i}}^n)-\nabla H(\alpha_{\mathbf{i}}^n)\big)\big\|^2\big]\Big\}^{1/2}\\
&\to 0
\end{aligned}
$$

by (45) and Lemma 7.1.

(ii) In this and the next step we assume that H has compact support. Now we split P_t^n up into two parts:

$$
\begin{aligned}
P_t^n &= \sqrt{n}\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_t^n(d)}\nabla H(\alpha_{\mathbf{i}}^n)v_{\mathbf{i}}^n(1)\\
&\quad+\sqrt{n}\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_t^n(d)}\nabla H(\alpha_{\mathbf{i}}^n)v_{\mathbf{i}}^n(2),
\end{aligned}
\tag{47}
$$

where $\sqrt{n}\Delta_{\mathbf{i}}^n X-\alpha_{\mathbf{i}}^n = v_{\mathbf{i}}^n(1)+v_{\mathbf{i}}^n(2)$ and $\mathbf{i}=(i_1,\dots,i_d)$, with

$$
\begin{aligned}
v_{i_k}^n(1) &= \sqrt{n}\Big(n^{-1}a_{(i_k-1)/n}+\int_{(i_k-1)/n}^{(i_k)/n}\big\{\tilde{\sigma}_{(i_k-1)/n}(W_s-W_{(i_k-1)/n})\\
&\qquad\qquad+\tilde{v}_{(i_k-1)/n}(V_s-V_{(i_k-1)/n})\big\}\,dW_s\Big),\\
v_{i_k}^n(2) &= \sqrt{n}\Big(\int_{(i_k-1)/n}^{(i_k)/n}(a_s-a_{(i_k-1)/n})\,ds\\
&\qquad+\int_{(i_k-1)/n}^{(i_k)/n}\Big\{\int_{(i_k-1)/n}^{s}\tilde{a}_u\,du\\
&\qquad\qquad+\int_{(i_k-1)/n}^{s}(\tilde{\sigma}_{u-}-\tilde{\sigma}_{(i_k-1)/n})\,dW_u\\
&\qquad\qquad+\int_{(i_k-1)/n}^{s}(\tilde{v}_{u-}-\tilde{v}_{(i_k-1)/n})\,dV_u\Big\}\,dW_s\Big).
\end{aligned}
$$

We denote the first and the second summand on the right-hand side of (47) by S_t^n and $\widetilde{S}_t^n$, respectively. First, we show the convergence $\widetilde{S}^n \xrightarrow{\text{u.c.p.}} 0$. Since the first derivative of H is of polynomial growth we have $\mathbb{E}[\|\nabla H(\alpha_{\mathbf{i}}^n)\|^2]\le C$ for all $\mathbf{i}\in\mathcal{A}_t^n(d)$. Furthermore, we obtain by using the Hölder, Jensen and Burkhölder inequalities

$$\begin{aligned}
&\mathbb{E}\big[\big|v^n_{i_k}(2)\big|^2\big] \\
&\quad \le \frac{C}{n^2} + \int_{(i_k-1)/n}^{(i_k)/n} (a_s - a_{[ns]/n})^2 + (\tilde{\sigma}_{s-} - \tilde{\sigma}_{[ns]/n})^2 + (\tilde{v}_{s-} - \tilde{v}_{[ns]/n})^2\, ds.
\end{aligned}$$

Thus, for all $t > 0$, we have

$$\begin{aligned}
&\sqrt{n}\binom{n}{d}^{-1} \mathbb{E} \sum_{\mathbf{i}\in\mathcal{A}^n_t(d)} \big|\nabla H(\alpha^n_{\mathbf{i}}) v^n_{\mathbf{i}}(2)\big| \\
&\quad \le C\sqrt{n}\binom{n}{d}^{-1} \bigg(\mathbb{E}\bigg[\sum_{\mathbf{i}\in\mathcal{A}^n_t(d)} \big\|\nabla H(\alpha^n_{\mathbf{i}})\big\|^2\bigg]\bigg)^{1/2} \bigg(\mathbb{E}\bigg[\sum_{\mathbf{i}\in\mathcal{A}^n_t(d)} \big\|v^n_{\mathbf{i}}(2)\big\|^2\bigg]\bigg)^{1/2} \\
&\quad \le C\bigg(n\binom{n}{d}^{-1} \mathbb{E}\bigg[\sum_{i_1,\dots,i_d=1}^{[nt]} \big(\big|v^n_{i_1}(2)\big|^2 + \cdots + \big|v^n_{i_d}(2)\big|^2\big)\bigg]\bigg)^{1/2} \\
&\quad \le C\bigg(\mathbb{E}\bigg[\sum_{j=1}^{[nt]} \big|v^n_j(2)\big|^2\bigg]\bigg)^{1/2} \\
&\quad \le C\bigg(n^{-1} + \int_0^t (a_s - a_{[ns]/n})^2 + (\tilde{\sigma}_{s-} - \tilde{\sigma}_{[ns]/n})^2 + (\tilde{v}_{s-} - \tilde{v}_{[ns]/n})^2\, ds\bigg)^{1/2} \\
&\quad \to 0
\end{aligned}$$

by the dominated convergence theorem, and $\widetilde{S}^n \xrightarrow{\text{u.c.p.}} 0$ readily follows.

(iii) To show $S^n \xrightarrow{\text{u.c.p.}} 0$ we use

$$S^n_t = \sum_{k=1}^d \sqrt{n}\binom{n}{d}^{-1} \sum_{\mathbf{i}\in\mathcal{A}^n_t(d)} \partial_k H(\alpha^n_{\mathbf{i}}) v^n_{i_k}(1) =: \sum_{k=1}^d S^n_t(k).$$

Before we proceed with proving $S^n(k) \xrightarrow{\text{u.c.p.}} 0$, for $k = 1, \dots, d$, we make two observations: first, by the Burkhölder inequality, we deduce

$$\mathbb{E}\big[\big|\sqrt{n} v^n_{i_k}(1)\big|^p\big] \le C_p \qquad \text{for all } p \ge 2, \tag{48}$$

and second, for fixed $x \in \mathbb{R}^{d-k}$, and for all $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{A}^n_t(k)$, we have

$$\mathbb{E}\big[\partial_k H(\alpha^n_{\mathbf{i}}, x) v^n_{i_k}(1) | \mathcal{F}_{(i_k-1)/n}\big] = 0, \tag{49}$$

since $\partial_k H$ is an odd function in its kth component. Now, we will prove that

$$\sqrt{n} n^{-k} \sum_{\mathbf{i}\in\mathcal{A}^n_t(k)} \partial_k H(\alpha^n_{\mathbf{i}}, x) v^n_{i_k}(1) \xrightarrow{\text{u.c.p.}} 0, \tag{50}$$

for any fixed $x \in \mathbb{R}^{d-k}$. From (49) we know that it suffices to show that

$$\sum_{i_k=1}^{[nt]} \mathbb{E}\Big[\Big(\sum_{1 \le i_1 < \cdots < i_{k-1} < i_k} \chi_{i_1,\dots,i_k}\Big)^2 \Big| \mathcal{F}_{(i_k-1)/n}\Big] \xrightarrow{\mathbb{P}} 0,$$

where $\chi_{i_1,\dots,i_k} := \sqrt{n} n^{-k} \partial_k H(\alpha_{\mathbf{i}}^n, x) v_{i_k}^n(1)$. (Note that the sum in the expectation only runs over the indices $i_1, \dots, i_{k-1}$.) But this follows from the L^1 convergence and (48) via

$$\begin{aligned}
&\sum_{i_k=1}^{[nt]} \mathbb{E}\Big[\Big(\sum_{1 \le i_1 < \cdots < i_{k-1} < i_k} \chi_{i_1,\dots,i_k}\Big)^2\Big] \\
&\qquad \le \frac{C}{n^k} \sum_{i_k=1}^{[nt]} \sum_{1 \le i_1 < \cdots < i_{k-1} < i_k} \mathbb{E}\big[\big(\partial_k H\big(\alpha_{\mathbf{i}}^n, x\big) v_{i_k}^n(1)\big)^2\big] \\
&\qquad \le \frac{C}{n} \to 0.
\end{aligned}$$

Recall that we still assume that H has compact support. Let the support of H be a subset of $[-K, K]^d$ and further $-K = z_0 < \cdots < z_m = K$ be an equidistant partition of $[-K, K]$. We denote the set $\{z_0, \dots, z_m\}$ by Z_m. Also, let $\eta(\varepsilon) := \sup\{\|\nabla H(\mathbf{x}) - \nabla H(\mathbf{y})\|; \|\mathbf{x} - \mathbf{y}\| \le \varepsilon\}$ be the modulus of continuity of ∇H. Then we have

$$\begin{aligned}
\sup_{t \le T} \big|S_t^n(k)\big| &\le C\sqrt{n} n^{-k} \sup_{t \le T} \sup_{x \in [-K,K]^{d-k}} \Big| \sum_{\mathbf{i} \in \mathcal{A}_t^n(k)} \partial_k H\big(\alpha_{\mathbf{i}}^n, x\big) v_{i_k}^n(1) \Big| \\
&\le C\sqrt{n} n^{-k} \sup_{t \le T} \max_{x \in Z_m^{d-k}} \Big| \sum_{\mathbf{i} \in \mathcal{A}_t^n(k)} \partial_k H\big(\alpha_{\mathbf{i}}^n, x\big) v_{i_k}^n(1) \Big| \\
&\quad + C\sqrt{n} n^{-k} \sum_{\mathbf{i} \in \mathcal{A}_T^n(k)} \eta\Big(\frac{2K}{m}\Big) \big|v_{i_k}^n(1)\big|.
\end{aligned}$$

Observe that, for fixed m, the first summand converges in probability to 0 as $n \to \infty$ by (50). The second summand is bounded in expectation by $C\eta(2K/m)$ which converges to 0 as $m \to \infty$. This implies $S_t^n(k) \xrightarrow{\text{u.c.p.}} 0$ which finishes the proof of (35) for all H with compact support.

(iv) Now, let $H \in C_p^1(\mathbb{R}^d)$ be arbitrary and H_k be a sequence of functions in $C_p^1(\mathbb{R}^d)$ with compact support that converges pointwise to H and fulfills $H = H_k$ on $[-k, k]^d$. In view of step (i) it is enough to show that

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{E}\Big[\sup_{t \le T} \Big| \sqrt{n} \binom{n}{d}^{-1} \sum_{\mathbf{i} \in \mathcal{A}_t^n(d)} \nabla(H - H_k)\big(\alpha_{\mathbf{i}}^n\big)\big(\sqrt{n} \Delta_{\mathbf{i}}^n X - \alpha_{\mathbf{i}}^n\big) \Big|\Big] = 0.$$

Since $H - H_k$ is of polynomial growth and by (46), we get

$$\mathbb{E}\Big[\sup_{t\le T}\Big|\sqrt{n}\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_t^n(d)}\nabla(H-H_k)(\alpha_{\mathbf{i}}^n)(\sqrt{n}\Delta_{\mathbf{i}}^n X-\alpha_{\mathbf{i}}^n)\Big|\Big]$$
$$\le C\sqrt{n}\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_T^n(d)}\mathbb{E}\big[\|\nabla(H-H_k)(\alpha_{\mathbf{i}}^n)\|\|\sqrt{n}\Delta_{\mathbf{i}}^n X-\alpha_{\mathbf{i}}^n\|\big]$$
$$\le C\binom{n}{d}^{-1}\sum_{\mathbf{i}\in\mathcal{A}_T^n(d)}\mathbb{E}\Big[\Big(\sum_{l=1}^d\mathbb{1}_{\{|\alpha_{i_l}^n|>k\}}\Big)^2\|\nabla(H-H_k)(\alpha_{\mathbf{i}}^n)\|^2\Big]^{1/2}\le\frac{C}{k},$$

which finishes the proof.

7.3. *Proof of* (28). We can write

$$U(H)_t=\int_{[0,t]^d}\int_{\mathbb{R}^d}H(\mathbf{x})\varphi_{\sigma_{s_1}}(x_1)\cdots\varphi_{\sigma_{s_d}}(x_d)\,d\mathbf{x}\,d\mathbf{s}.$$

We also have

$$\overline{F}_n'(t,x)=\int_0^{[nt]/n}\varphi_{\sigma_{[ns]/n}}(x)\,ds,$$

where $\overline{F}_n'(t,x)$ denotes the Lebesgue density in x of $\overline{F}_n(t,x)$ defined at (28). So we need to show that $P^n(H)\xrightarrow{\text{u.c.p.}}0$, where

$$P_t^n(H):=\sqrt{n}\int_{[0,t]^d}\int_{\mathbb{R}^d}H(\mathbf{x})\big(\varphi_{\sigma_{s_1}}(x_1)\cdots\varphi_{\sigma_{s_d}}(x_d)$$
$$-\varphi_{\sigma_{[ns_1]/n}}(x_1)\cdots\varphi_{\sigma_{[ns_d]/n}}(x_d)\big)\,d\mathbf{x}\,d\mathbf{s}.$$

As previously we show the result first for H with compact support.

(i) Let the support of H be contained in $[-k,k]^d$. From [2], Section 8, we know that, for fixed $x\in\mathbb{R}$, it holds that

$$\sqrt{n}\int_0^t\big(\varphi_{\sigma_s}(x)-\varphi_{\sigma_{[ns]/n}}(x)\big)\,ds\xrightarrow{\text{u.c.p.}}0. \tag{51}$$

Also, with $\rho(z,x):=\varphi_z(x)$ we obtain, for $x,y\in[-k,k]$,

$$\Big|\int_0^t\big(\varphi_{\sigma_s}(x)-\varphi_{\sigma_{[ns]/n}}(x)\big)-\big(\varphi_{\sigma_s}(y)-\varphi_{\sigma_{[ns]/n}}(y)\big)\,ds\Big|$$
$$\le\int_0^t\big|\partial_1\rho(\xi_s,x)(\sigma_s-\sigma_{[ns]/n})-\partial_1\rho(\xi_s',y)(\sigma_s-\sigma_{[ns]/n})\big|\,ds$$
$$\le\int_0^t\big|\partial_{11}\rho(\xi_s'',\eta_s)(\xi_s-\xi_s')+\partial_{21}\rho(\xi_s'',\eta_s)(x-y)\big||\sigma_s-\sigma_{[ns]/n}|\,ds$$
$$\le C\int_0^t|\sigma_s-\sigma_{[ns]/n}|^2+|\sigma_s-\sigma_{[ns]/n}||y-x|\,ds,$$

where ξ_s, ξ_s', ξ_s'' are between σ_s and $\sigma_{[ns]/n}$ and η_s is between x and y. Now, let $Z_m = \{jk/m | j = -m, \dots, m\}$. Then, we get

$$\begin{aligned}\sup_{t\le T}\big|P_t^n(H)\big| &\le C_T \sup_{t\le T}\sqrt{n}\int_{[-k,k]}\bigg|\int_0^t \varphi_{\sigma_s}(x) - \varphi_{\sigma_{[ns]/n}}(x)\,ds\bigg|\,dx \\ &\le C_T \sup_{t\le T}\sup_{x\in[-k,k]}\sqrt{n}\bigg|\int_0^t \varphi_{\sigma_s}(x) - \varphi_{\sigma_{[ns]/n}}(x)\,ds\bigg| \\ &\le C_T \sup_{t\le T}\max_{x\in Z_m}\sqrt{n}\bigg|\int_0^t \varphi_{\sigma_s}(x) - \varphi_{\sigma_{[ns]/n}}(x)\,ds\bigg| \\ &\quad + C_T\sqrt{n}\int_0^T\Big(|\sigma_s - \sigma_{[ns]/n}|^2 + \frac{k}{m}|\sigma_s - \sigma_{[ns]/n}|\Big)\,ds \\ &\le C_T \sum_{x\in Z_m}\sup_{t\le T}\sqrt{n}\bigg|\int_0^t \varphi_{\sigma_s}(x) - \varphi_{\sigma_{[ns]/n}}(x)\,ds\bigg| \\ &\quad + C_T\sqrt{n}\int_0^T\Big(|\sigma_s - \sigma_{[ns]/n}|^2 + \frac{k}{m}|\sigma_s - \sigma_{[ns]/n}|\Big)\,ds.\end{aligned}$$

Observe that, for fixed m, the first summand converges in probability to 0 by (51). By the Itô isometry and (18) we get for the expectation of the second summand,

$$\begin{aligned}&\mathbb{E}\bigg[\sqrt{n}\int_0^T\Big(|\sigma_s - \sigma_{[ns]/n}|^2 + \frac{k}{m}|\sigma_s - \sigma_{[ns]/n}|\Big)\,ds\bigg] \\ &\quad = \sqrt{n}\int_0^T \mathbb{E}\Big[|\sigma_s - \sigma_{[ns]/n}|^2 + \frac{k}{m}|\sigma_s - \sigma_{[ns]/n}|\Big]\,ds \le C_T\Big(\frac{1}{\sqrt{n}} + \frac{1}{m}\Big).\end{aligned}$$

Thus, by choosing m large enough and then letting n go to infinity, we get $P_t^n(H) \xrightarrow{\text{u.c.p.}} 0$.

(ii) Now let $H \in C_p^1(\mathbb{R}^d)$ and H_k be an approximating sequence of functions in $C_p^1(\mathbb{R}^d)$ with compact support and $H = H_k$ on $[-k,k]^d$. Observe that, for $\mathbf{x}, \mathbf{s} \in \mathbb{R}^d$, we obtain by the mean value theorem that

$$\begin{aligned}&\mathbb{E}\big[\big|\varphi_{\sigma_{s_1}}(x_1)\cdots\varphi_{\sigma_{s_d}}(x_d) - \varphi_{\sigma_{[ns_1]/n}}(x_1)\cdots\varphi_{\sigma_{[ns_d]/n}}(x_d)\big|\big] \\ &\quad \le \psi(\mathbf{x})\sum_{i=1}^d \mathbb{E}|\sigma_{s_i} - \sigma_{[ns_i]/n}| \le \frac{C}{\sqrt{n}}\psi(\mathbf{x}),\end{aligned}$$

where the function ψ is exponentially decaying at $\pm\infty$. Thus

$$\begin{aligned}&\lim_{k\to\infty}\limsup_{n\to\infty}\mathbb{E}\Big[\sup_{t\le T}\big|P_t^n(H) - P_t^n(H_k)\big|\Big] \\ &\quad \le C_T \lim_{k\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^d}\big|(H - H_k)(\mathbf{x})\big|\psi(\mathbf{x})\,d\mathbf{x} = 0,\end{aligned}$$

which finishes the proof of (28).

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