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SENSITIVITY ANALYSIS FOR DIFFUSION PROCESSES CONSTRAINED TO AN ORTHANT[1]

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This paper studies diffusion processes constrained to the positive orthant under infinitesimal changes in the drift. Our first main result states that any constrained function and its (left) drift-derivative is the unique solution to an augmented Skorohod problem. Our second main result uses this characterization to establish a basic adjoint relationship for the stationary distribution of the constrained diffusion process jointly with its left-derivative process.

1. Introduction. This paper is motivated by a desire to better understand the relation between performance metrics and control variables in a network with shared but limited resources. We are specifically interested in service networks, where customers seeking a certain service may suffer from delays as a result of temporary insufficient service capacity. The control variables are the service capacities at the individual stations. Many service processes can be modeled by stochastic (or queueing) networks, and an important question is how resources should be allocated, given random fluctuations in the arrivals and their interplay with potentially random service times. When planning horizons are long so that static allocation rules are required, questions of this type are readily answered if the network has a product-form structure Kleinrock (1964), Wein (1989). However, few results have been obtained when this assumption fails Dieker, Ghosh and Squillante (2014), Pollett (2009). It is the goal of this paper to introduce new tools in this context, which could be used in the context of both sensitivity analysis and system optimization.

We study diffusion processes and their "derivatives," defined as the change in the process under an infinitesimal change in the drift. Although some of our results are stated more generally, this paper focuses on diffusion processes for two reasons. First, this framework allows us to explain key concepts in a tractable yet relatively general setting. Second, diffusion processes are rooted in heavy-traffic approximations for stochastic networks, and the heavy-traffic assumption seems

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reasonable in the context of resource allocation problems with systems operating close to their capacity. This paper studies the stationary distribution of diffusions and their derivatives, as a proxy for the long-term (steady-state) behavior. Although it is certainly desirable to obtain time-dependent tools as well, given the vast body of work on stationary results, making this assumption is a natural first step. The techniques developed in this paper are likely to be also relevant in the time-dependent case.

We have two main results. The first is a statement on the behavior of deterministic functions under the well-known Skorohod reflection map with oblique reflection (regulation), and states that the map and its "derivative" are the unique solution to an augmented version of the Skorohod problem. Our proof of this result relies on recent insights into directional derivatives by Mandelbaum and Ramanan (2010), which have been developed in the context of time-inhomogeneous systems but are shown here to be useful for sensitivity analysis as well.

Our second main result specializes to diffusion processes and studies the stationary distribution of solutions to the augmented Skorohod problem. Given a constrained diffusion process Z representing the dynamics of the underlying stochastic network (i.e., the queue lengths at each of the stations), let the stochastic process A represent the change in Z under an infinitesimal change in the drift. The two results combined say that the stationary distribution of the joint processes (Z, A) satisfies a kind of basic adjoint relation, which is the analog of the equation $\pi' Q = 0$ for continuous-time Markov processes on a discrete state space. The proof relies on a delicate analysis of the jumps of A; the process A has jumps even if Z is continuous.

The intuition behind the program carried out in this paper can be summarized as follows. Suppose Z^ϵ is a constrained diffusion process with drift coefficient $\mu(\cdot) - \epsilon v$ in the interior of the orthant, where v is an arbitrary nonnegative vector. Suppose the processes $\{Z^\epsilon\}$ are driven by the same Brownian motion for every $\epsilon \geq 0$, so that they are coupled. The processes $Z \equiv Z^0$ and Z^ϵ are Markovian, and one can therefore expect to be able to give a basic adjoint relationship for their stationary distributions (should they exist). Moreover, (Z, Z^ϵ) and therefore $(Z, (Z - Z^\epsilon)/\epsilon)$ can be expected to be Markovian as a result of the coupling. Provided one can make sense of the pointwise limit (Z, A) of $(Z, (Z - Z^\epsilon)/\epsilon)$ as $\epsilon \to 0+$, one can expect that the distribution of (Z, A) satisfies a similar relationship. This results in an "augmented" basic adjoint relationship, which we state in Theorem 3. The constrained diffusion processes studied in this paper are pathwise solutions to stochastic differential equations with reflection; see Dupuis and Ishii (1991), Ramanan (2006). We only consider left derivatives in this paper, although one could develop similar tools and obtain similar results for right derivatives. This would affect our two main results as follows. On a sample-path level, the right derivative is the left-continuous modification of the (right-continuous) left derivative, see Section 4.1 for a detailed discussion. On a probabilistic level, studying the (left-continuous) right derivative requires a different set of technical

tools since one ordinarily works with right-continuous stochastic processes. We should expect that this change does not affect the stationary distribution or the basic adjoint relationship.

When carrying out the aforementioned approach, we were surprised to find that, even though Z is known not to spend any time on low-dimensional faces, it is critical to incorporate the jumps of A when Z reaches those faces in order to formulate the basic adjoint relationship.

This work has the potential to lead to new numerical methods in the context of optimization and sensitivity analysis for queueing networks, which relieve or remove the need for computationally intensive or numerically unstable operations such as gradient estimation. To explain, due to the division by ϵ, any performance metric of $(Z - Z^\epsilon)/\epsilon$ suffers from numerical instability issues for small $\epsilon > 0$. Researchers in stochastic optimization have developed several techniques to mitigate this effect; see, for example, Asmussen and Glynn (2007). The approach taken in this paper is to analytically describe and investigate the dynamics of the limit. Our experience with state-of-the-art stochastic optimization implementations in the context of resource capacity management, as documented in part in Dieker, Ghosh and Squillante (2014), is that it is computationally very costly to obtain reliable gradient estimates and that the use of "quick and dirty" estimates can have disastrous effects on the compute time of a stochastic optimization procedure due to bias and inherent random fluctuations. Therefore, reliable (numerical) tools that give merely a rough idea of the gradient can be desirable and useful. In particular, from an implementation perspective, heavy-traffic gradient information can be valuable even if a stochastic network is in moderate traffic. (A light-traffic setting is not of prime interest since one is typically interested in fine-tuning networks operating in a regime where servers are idling relatively rarely.)

The framework of this paper is related to a body of literature known as infinitesimal perturbation analysis Glasserman (1991, 1994, 1993), Heidergott (2006). Infinite perturbation analysis also aims to perform sensitivity analysis or gradient estimation for performance metrics in (say) a queueing network, and it does so by formulating conditions under which an expectation and a derivative operator can be interchanged. Here, however, it is not our objective to seek such an interchange involving a performance metric, but instead we study the (whole) stationary distribution of a stochastic process with its derivative process.

This paper is outlined as follows. Section 2 summarizes our approach in the one-dimensional case, which serves as a guide for our multidimensional results. Section 3 discusses two technical preliminaries: oblique reflection maps and their derivatives. In Section 4 we formulate our two main results. Section 5 is devoted to the proof of the first main result, while Section 6 gives the proof of the second main result. A key role is played by jump measures, for which we obtain a description in Section 7. The appendices contain several technical digressions.

Notation. For $J \in \mathbb{N}$, $\mathbb{R}^J$ denotes the J-dimensional Euclidean space. We denote the space of real $n \times m$ matrices by $\mathbb{M}^{n\times m}$, and the subset of nonnegative

matrices by $\mathbb{M}_+^{n\times m}$. All vectors are to be interpreted as column vectors, and we write M^j and M_i for the jth column and the ith row of a matrix M, respectively. In particular, v_i is the ith element of a vector v, and M_i^j is element (i, j) of a matrix M. Similarly, given a set $I \subseteq \{1, \ldots, J\}$, we write M_I and M^I for the matrices consisting of the rows and columns of M, respectively, with indices in I. Throughout, E stands for the identity matrix and we write δ_i^j for E_i^j. We use the symbol $'$ for transpose. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ stand for entrywise 1-norm and 2-norm, respectively, and are used for both vectors and matrices.

Given a measure space $(S, \mathcal{S})$, a measurable vector-valued function $h : S \to \mathbb{R}^J$ on $(S, \mathcal{S})$, and a vector of measures $\nu = (\nu_1, \ldots, \nu_J)$ on $(S, \mathcal{S})$, we set

$$\int h(x)\nu(dx) = \int h(x) \cdot \nu(dx),$$

provided the right-hand side exists. We shall also employ this notation when h and ν are matrix-valued. That is, we write for $h : S \to \mathbb{M}^{J\times J}$ and an $\mathbb{M}^{J\times J}$-valued measure ν on $(S, \mathcal{S})$,

$$\int h(x)\nu(dx) = \int \langle h(x), \nu(dx)\rangle_{\mathrm{HS}},$$

where $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ is the Hilbert–Schmidt inner product on $\mathbb{M}^{J\times J}$ given by

$$\langle M_1, M_2\rangle_{\mathrm{HS}} = \operatorname{tr}(M_1' M_2).$$

For a function $g : \mathbb{M}^{J\times J} \to \mathbb{R}$, we define $\nabla g : \mathbb{M}^{J\times J} \to \mathbb{M}^{J\times J}$ as the function for which element (i, j) is given by the directional derivative of g in the direction of the matrix with only zero entries except for element (i, j), where its entry is 1. We also write, for $i = 1, \ldots, J$, $F_i = \{(z, a) \in \mathbb{R}_+^J \times \mathbb{M}^{J\times J} : z_i = 0\}$, $F_i^a = \{(z, a) \in \mathbb{R}_+^J \times \mathbb{M}^{J\times J} : a_i = 0\}$. The space of functions $f : \mathbb{R}_+^J \times \mathbb{M}_+^{J\times J} \to \mathbb{R}$ which are twice continuously differentiable with bounded derivatives is denoted by $C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$.

We write $\mathbb{D}_+^J$ for the space of $\mathbb{R}_+^J$-valued functions on $\mathbb{R}_+$ which are right-continuous on $\mathbb{R}_+$ with left limits in $(0, \infty)$. The subset of continuous functions is written as C^J, and C_+^J denotes the set of nonnegative continuous functions. Similarly, we write $\mathbb{D}^{J\times J}$ for the space of $\mathbb{M}^{J\times J}$-valued right-continuous functions on $\mathbb{R}_+$ with left limits. The subset of $\mathbb{M}_+^{J\times J}$-valued functions is denoted by $\mathbb{D}_+^{J\times J}$.

2. A motivating one-dimensional result. Fix some $\theta < 0$. For any $\epsilon \geq 0$, we let Z^ϵ be a one-dimensional reflected Brownian motion with drift $\theta - \epsilon < 0$ and variance σ^2. That is,

$$Z^\epsilon(t) = X^\epsilon(t) + Y^\epsilon(t) \geq 0,$$

where X^ϵ is a Brownian motion with drift $\theta - \epsilon$ and variance σ^2, and the regulating term Y^ϵ is given by

$$Y^\epsilon(t) = \max\Big(\sup_{0\leq s\leq t}\big[-X^\epsilon(s)\big], 0\Big).$$

Suppose the family $\{Z^\epsilon : \epsilon \geq 0\}$ is coupled in the sense that $X^\epsilon(t) = W(t) + (\theta - \epsilon)t$ for some driftless Brownian motion W. Write $Z \equiv Z^0$.

It follows from Theorem 1.1 in Mandelbaum and Ramanan (2010) [see also Lemma 5.2 and equation (5.7) in Mandelbaum and Massey (1995)] that, for each $t \geq 0$, the limit

$$A(t) \equiv \lim_{\epsilon \to 0+} \frac{1}{\epsilon}\big(Z(t) - Z^\epsilon(t)\big) \tag{2.1}$$

exists. We also have the following explicit formula:

$$A(t) = t - B(t), \tag{2.2}$$

where

$$B(t) = \sup\big\{s \in [0, t] : Z(s) = 0\big\}$$

and $\sup \varnothing = 0$ by convention. In view of the definition of A in (2.1), we call it the derivative process of Z.

We now relate these notions to sensitivity analysis. Our investigations are motivated by the following sequence of equalities: for any "smooth" function (performance measure) ϕ, one could expect that

$$\frac{d}{d\epsilon}\mathbb{E}\big[\phi\big(Z^\epsilon(\infty)\big)\big] = \mathbb{E}\bigg[\frac{d}{d\epsilon}\phi\big(Z^\epsilon(\infty)\big)\bigg] = \mathbb{E}\big[A(\infty)\phi'\big(Z(\infty)\big)\big]. \tag{2.3}$$

Thus, to study (infinitesimal) changes in the steady-state performance measure under infinitesimal changes in the drift θ, one is led to investigating the stationary distribution of (Z, A) (assuming it exists). We are able to justify the interchange of expectation and derivative in the above equalities in the one-dimensional case (see below), but a justification in the setting of general multidimensional constrained diffusions requires a different set of techniques and falls outside the scope of this paper.

One readily checks that the sample paths of the process B are nondecreasing, that they are right-continuous with left-hand limits and that A has positive drift and negative jumps. In particular, the process A is of finite variation, and (Z, A) is a semimartingale with jumps. An illustration of the process (Z, A) is given in Figure 1. From Ito's formula in conjunction with sample path properties of A,

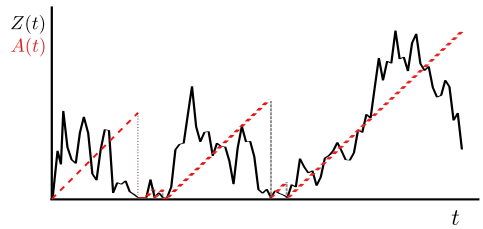

FIG. 1. *Sample paths of* (Z, A) *as a function of time. The solid black curve is* Z, *while the dashed red curve is* A. *The slope of* A *is* 1 *whenever it is continuous, and* A *jumps to* 0 *whenever* Z *hits* 0.

we obtain the following result. We suppress further details of the proof, since this program is carried out in greater generality in Section 6.

THEOREM 1. *Let Z be a one-dimensional reflected Brownian motion with drift θ and variance σ^2. Let A be defined in (2.2). Suppose that the process (Z, A) has a unique stationary distribution π. For any $f \in C_b^2(\mathbb{R}_+ \times \mathbb{R}_+)$, we have the following relationship*:

$$0 = \int_0^\infty \int_0^\infty \Big[\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial z^2} f(z,a) + \theta \frac{\partial}{\partial z} f(z,a) + \frac{\partial}{\partial a} f(z,a) - \frac{\partial}{\partial a} f(0,a) \Big] \pi(dz, da) - \frac{\partial}{\partial z} f(0,0)\theta. \tag{2.4}$$

One can go further and derive the Laplace transform of π using this theorem; see Appendix A. One then finds that, for any $\alpha, \eta > 0$,

$$\int_0^\infty \int_0^\infty e^{-\alpha z - \eta a} \pi(dz, da) = \frac{-2\theta}{\alpha\sigma^2 - \theta + \sqrt{2\eta\sigma^2 + \theta^2}}. \tag{2.5}$$

In particular, the theorem completely determines the stationary measure π. It is also possible to derive this result immediately from standard fluctuation identities for Brownian motion with drift, using results from Dębicki, Dieker and Rolski (2007). In fact, since the corresponding densities are known explicitly (or can be found by inverting the Laplace transform), it is possible to write down the density of $(Z(\infty), A(\infty))$ in closed form. Using the resulting expression, it can be verified directly that (2.3) indeed holds.

3. Oblique reflection maps and their directional derivatives. This section contains the technical preliminaries to formulate a multidimensional analog of Theorem 1. We need the following definition to introduce the analogs of the processes A and B.

DEFINITION 1 (Oblique reflection map). Suppose a given $J \times J$ real matrix R can be written as $R = E - P$, where P is a nonnegative matrix with spectral radius less than one and zeros on the diagonal. Then for every $x \in \mathbb{D}^J$, there exists a unique pair $(y, z) \in \mathbb{D}_+^J \times \mathbb{D}_+^J$ satisfying the following conditions:

(1) $z(t) = x(t) + Ry(t) \geq 0$ for $t \geq 0$;
(2) $y(0) = 0$, y is componentwise nondecreasing and

$$\int_0^\infty z(t)\, dy(t) = 0.$$

We write $y = \Phi(x)$ and $z = \Gamma(x)$ for the oblique reflection map.

The reflection map gives rise to left derivatives as formalized in the following definition. Existence of the derivatives is guaranteed by Theorem 1.1 in Mandelbaum and Ramanan (2010).

DEFINITION 2 (Derivatives of the reflection map). Let $\chi(t) = tE$ and define the $\mathbb{M}^{J\times J}$-valued functions a and b by defining $a = \lim_{\epsilon\to 0+} a_\epsilon$ and $b = \lim_{\epsilon\to 0+} b_\epsilon$, where the limits are to be understood as pointwise limits and, for $j = 1, \ldots, J$,

$$a_\epsilon^j \equiv \frac{1}{\varepsilon}\big[\Gamma(x) - \Gamma\big(x - \epsilon\chi^j\big)\big], \qquad b_\epsilon^j \equiv -\frac{1}{\varepsilon}\big[\Phi(x) - \Phi\big(x - \epsilon\chi^j\big)\big]. \tag{3.1}$$

Then we have for each $t \geq 0$,

$$a(t) = tE - Rb(t). \tag{3.2}$$

For notational convenience, we write $a = \Gamma'(x)$ and $b = -\Phi'(x)$.

4. Main results. This section states the main results of this paper. The first result makes the connection between derivatives and an augmented Skorohod problem, which we define momentarily. The second result is a basic adjoint relationship for the stationary distribution of solutions to the augmented Skorohod problem with diffusion input. The basic adjoint relationship is the analog of the equation $\pi' Q = 0$ for Markov chains on a countable state space as mentioned in the Introduction.

4.1. *Augmented Skorohod problems and derivatives.* In this section we introduce the augmented Skorohod problem and connect it with derivatives of the oblique reflection map.

DEFINITION 3 (Augmented Skorohod problem). Suppose we are given two $J \times J$ real matrices $R = E - P$ and $\widetilde{R} = E - \widetilde{P}$, where both P and $\widetilde{P}$ are nonnegative matrices with spectral radius less than one and zeros on the diagonal. Given $(x, \chi) \in C^J \times C^{J\times J}$ with χ componentwise nonnegative and nondecreasing, we say that $(z, y, a, b) \in C_+^J \times C_+^J \times \mathbb{D}_+^{J\times J} \times \mathbb{D}_+^{J\times J}$ satisfies the augmented Skorohod problem associated with $(R, \widetilde{R})$ for (x, χ) if the following conditions are satisfied:

(1) $z(t) = x(t) + Ry(t)$ for $t \geq 0$;
(2) $y(0) = 0$, y is componentwise nondecreasing and

$$\int_0^\infty z(t)\, dy(t) = 0;$$

(3) $a(t) = \chi(t) - \widetilde{R}b(t)$ for $t \geq 0$;

(4) $b(0) = 0$, $b(t) \geq 0$, b is componentwise nondecreasing and, for $j = 1, \ldots, J$,

$$\int_0^\infty z(t)\, db^j(t) = 0; \tag{4.1}$$

(5) For $i = 1, \ldots, J$ and $t \geq 0$, $z_i(t) = 0$ implies $a_i(t) = 0$.

Building on results from Mandelbaum and Ramanan (2010), we show in Appendix B that the augmented Skorohod problem has a unique solution. To interpret solutions to the augmented Skorohod problem, we found it easiest to think of the dynamics of (z, a^j) for each $j = 1, \ldots, J$ separately. When z hits the face $z_I = 0$, then a^j jumps to the face $a_I^j = 0$ in the direction of the unique vector in the column space of $\widetilde{R}^I$ which brings it to that face. We refer to Figure 2 for an illustrative example in the two-dimensional case.

Unlike requirements 2 and 4 in Definition 3, requirement 5 is not a "complementarity" condition. In view of the sample path dynamics in Figure 2, it may seem reasonable to replace requirement 5 by $\int_0^\infty a^j(t)\, dy(t) = 0$ or another complementarity condition between (y, z) and (a, b). In that case, however, the augmented Skorohod will fail to have a unique solution. This can be seen by verifying that both the left derivative and the right-derivative of the reflection map satisfy $\int_0^\infty a^j(t)\, dy(t) = 0$ but only the left derivative (as defined in Definition 3) satisfies requirement 5.

We now make a connection between derivatives (sensitivity analysis) and solutions to the augmented Skorohod problem. Note that, unlike in Figure 2, one always has $a(0) = \chi(0) = 0$ in this case.

THEOREM 2. *Fix some $x \in C^J$, and let $z = \Gamma(x)$ and $y = \Phi(x)$ be given by the oblique reflection map. Define the derivatives $a = \Gamma'(x)$ and $b = -\Phi'(x)$ as in Definition 2. Set $\chi(t) = tE$ for $t \geq 0$. Then (z, y, a, b) satisfies the augmented Skorohod problem associated with (R, R) for (x, χ).*

4.2. *Stationary distribution of constrained diffusions and their derivatives.* Our second main result specializes to diffusion processes and studies the stationary distribution of solutions to the augmented Skorohod problem. We show that it satisfies a generalized version of the basic adjoint relationship (BAR) for reflected Brownian motion. The proof relies on Ito's formula in conjunction with properties developed in the previous section. All results are formulated in terms of solutions to the augmented Skorohod problem, and the special case $\widetilde{R} = R$ is of primary interest for the derivative process.

We first discuss the construction of constrained diffusion processes. We work with a d-dimensional standard Brownian motion $W = \{W(t) : t \geq 0\}$ adapted to some filtration $\{\mathcal{F}_t\}$, on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are given

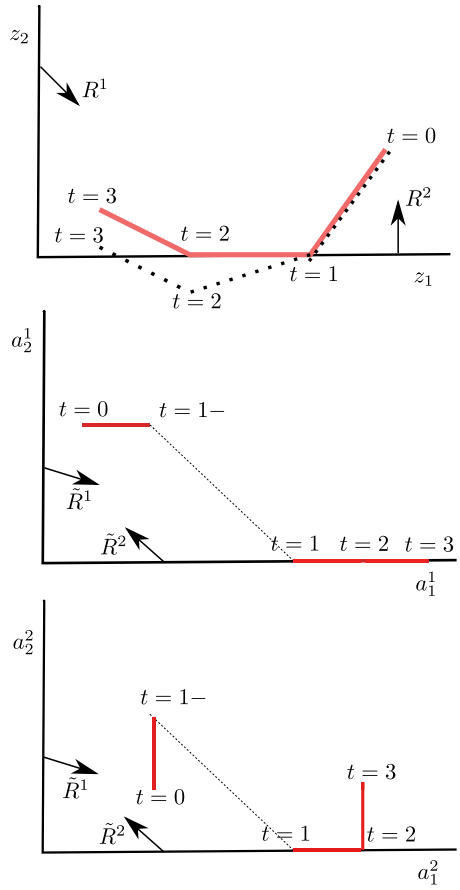

FIG. 2. *The first diagram depicts a trajectory of z, with corresponding "free" path x (dotted). In the second and third diagrams, the trajectories of a^1 and a^2 travel at unit rate right and up, respectively, until z hits $\partial\mathbb{R}^2_+$. The face $z_2 = 0$ is hit at time $t = 1$, causing a^1 and a^2 to jump to the faces $a^1_2 = 0$ and $a^2_2 = 0$, respectively, in direction $\tilde{R}^2$. Note that both $z(0)$ and $a(0) = \chi(0)$ are nonzero in these diagrams.*

functions θ and σ on $\mathbb{R}^J_+$ taking values in $\mathbb{R}^J$ and $\mathbb{M}^{J\times d}$, respectively, which satisfy the following standard Lipschitz and growth conditions: (1) For some $L < \infty$, we have $\|\sigma(x) - \sigma(y)\|_2 + \|\theta(x) - \theta(y)\|_2 \le L\|x - y\|_2$ for all $x, y \in \mathbb{R}^J_+$. (2) For some $K < \infty$, we have $\|\theta(x)\|_2^2 + \|\sigma(x)\|_2^2 \le K(1 + \|x\|_2^2)$ for $x \in \mathbb{R}^J_+$. Given any initial condition $Z(0)$ with $\mathbb{E}\|Z(0)\|_2^2 < \infty$, there exists a pathwise unique, strong solution $\{Z(t) : t \ge 0\}$ to the stochastic differential equation with reflection (SDER)

$$dZ(t) = \theta\big(Z(t)\big)\,dt + \sigma\big(Z(t)\big)\,dW(t) + R\,dY(t). \tag{4.2}$$

This equation is shorthand for the statement that, almost surely, $Z = \Gamma(X)$ and $X(t) = Z(0) + \int_0^t \theta(Z(s))\,ds + \int_0^t \sigma(Z(s))\,dW(s)$ for $t \ge 0$. Moreover, $\mathbb{E}\|Z(t)\|_2^2$

is locally bounded as a function of t. For these and related results, see Anderson and Orey (1976), Dupuis and Ishii (1991), Karatzas and Shreve (1991), Ramanan (2006). In particular, we have $Z(t) \in \mathbb{R}^J_+$ for all $t \geq 0$. We define the diffusion matrix Σ through $\Sigma(z) = \sigma(z)\sigma(z)'$ for $z \in \mathbb{R}^J_+$. The special case of reflected Brownian motion follows upon taking constant functions σ and θ. Throughout this paper, we only work with constrained diffusion processes that can be obtained through the oblique reflection map of Definition 1, and for which the time Z spends $\partial\mathbb{R}^J_+$ has Lebesgue measure zero almost surely (this is only used in Section 7). Although the notions of SDER and their solutions can be defined more generally, our results cannot be extended to other settings using the present framework.

We next introduce an $\mathbb{M}^{J\times J}_+$-valued process $A = \{A(t) : t \geq 0\}$ through an augmented Skorohod problem. Although the special choice $\widetilde{R} = R$ is most relevant for us given the connection with the derivative process, our treatment is not restricted to that case. Given some $A(0)$, suppose that (Z, Y, A, B) satisfies the augmented Skorohod problem associated with $(R, \widetilde{R})$ for (X, χ) with $\chi(t) = A(0) + Et$ and X as before. Also suppose $(Z(0), A(0))$ has some distribution u satisfying $\int \|z\|_2^2 u(dz, da) < \infty$. This assumption guarantees existence of Z on a sample-path level, and therefore we do not need moment assumptions on $A(0)$ in order to guarantee existence of the process A. The derivative process always starts at the origin (i.e., the zero matrix), but here we have defined A with an arbitrary initial distribution since we are interested in stationary distributions for (Z, A). Recall that π is said to be a stationary distribution for (Z, A) if all marginal distributions of (Z, A) are π when $(Z(0), A(0))$ has distribution π, that is, for every bounded measurable function $f : \mathbb{R}^J_+ \times \mathbb{M}^{J\times J} \to \mathbb{R}$ and for every $t \geq 0$,

$$\mathbb{E}\big[f\big(Z(t), A(t)\big)\big] = \int f(z, a)\pi(dz, da). \tag{4.3}$$

In view of Theorem 2, although a justification is outside the scope of this paper, we think of the stationary distribution of (Z, A) with $\widetilde{R} = R$ as the limiting distribution of Z jointly with its derivative process.

We define the following operators: Q_I is a projection operator with the following property. The matrix $Q_I(a)$ is obtained from a by subtracting columns of $\widetilde{R}^I$, in such a way that the rows of $Q_I(a)$ with indices in I become zero. That is, we have

$$Q_I(a) = a - \widetilde{R}^I\big(\widetilde{R}^I_I\big)^{-1} a_I, \tag{4.4}$$

where $\widetilde{R}^I_I$ is the principal submatrix of $\widetilde{R}$ obtained by removing rows and columns from $\widetilde{R}$ which do not lie in I. When $I = \varnothing$, we set $Q_I(a) = a$ for $a \in \mathbb{M}^{J\times J}$.

We also define operators L and T on $C^2_b(\mathbb{R}^J_+ \times \mathbb{M}^{J\times J}_+)$ through

$$\begin{aligned} Lf(\cdot) &= \tfrac{1}{2}\big\langle \Sigma(\cdot), H_z f(\cdot)\big\rangle_{\mathrm{HS}} + \big\langle \theta(\cdot), \nabla_z f(\cdot)\big\rangle, \\ Tf(\cdot) &= Lf(\cdot) + \mathrm{tr}\big(\nabla_a f(\cdot)\big), \end{aligned} \tag{4.5}$$

where $\nabla_z f$ and $H_z f$ denote the gradient and Hessian, respectively, with respect to the first argument of f, and we use $\nabla_a f$ as discussed in Section 1. Thus $\mathrm{tr}(\nabla_a)$ is shorthand for $\sum_{i=1}^{J} d/da_{ii}$.

We can now formulate the following theorem, which is our second main result. We write I^c for the complement of a set I. We write z_I for the subvector of z consisting of the components with indices in I as before, and we also let $z|_I$ denote the projection of z to $\{z : z_{I^c} = 0\}$.

THEOREM 3 (Basic adjoint relationship). *Let the processes Z and A be defined as above, and suppose that (Z, A) has a unique stationary distribution π with $\int(\|z\|_2^2 + \|a\|_1)\pi(dz, da) < \infty$. Then there exists a finite Borel measure ν on $\bigcup_i (F_i \cap F_i^a)$ and, for $I \subseteq \{1, \dots, J\}$, finite Borel measures u_I on $(0,\infty)^{|I^c|} \times \mathbb{M}_+^{J\times J}$ such that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$, the following relationship holds*:

$$\begin{aligned}\int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} Tf(z,a)\,d\pi(z,a) + \int_{\bigcup_i (F_i \cap F_i^a)} \bigl[R' \nabla_z f(z,a)\bigr]\,d\nu(z,a)\\ + \sum_{I \subseteq \{1,\dots,J\}:\, I \neq \varnothing} \int_{(0,\infty)^{|I^c|} \times \mathbb{M}_+^{J\times J}} \bigl[f\bigl(z|_{I^c}, Q_I(a)\bigr)\\ - f(z|_{I^c}, a)\bigr]\,du_I(z_{I^c}, a) = 0,\end{aligned} \tag{4.6}$$

where the operators Q_I and T are given in (4.4) *and* (4.5), *respectively.*

Section 6.2 shows that the measures ν and u_I, $I \subseteq \{1, \dots, J\}$ are completely determined by π, and expresses these measures in terms of π. We believe that (4.6) fully determines π, ν and the u_I measures, but it is outside the scope of this paper to prove this. For recent developments along these lines, see Dai and Dieker (2011), Kang and Ramanan (2012).

Theorem 3 does not have the same form as Theorem 1, and our next result brings these two forms closer. It is obtained by substituting a special class of functions in (4.6) so that the last term in (4.6) vanishes. To formulate the result, we need the following family of operators: for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$ and each set $I \subseteq \{1, 2, \dots, J\}$, let

$$(O_I f)(z,a) = \sum_{S \subseteq \{1,\dots,J\} \setminus I} (-1)^{|S|} f\bigl(\Pi_{S\cup I} z, Q_I(a)\bigr), \tag{4.7}$$

$$O = \sum_{I \subseteq \{1,\dots,J\}} O_I, \tag{4.8}$$

where $\Pi_{S\cup I}$ is the projection operator which sets the coordinates in $S \cup I$ equal to 0.

COROLLARY 1. *Let the processes Z and A be defined as above, and suppose that (Z, A) has a unique stationary distribution π with $\int(\|z\|_2^2 + \|a\|_1)\pi(dz, da) < \infty$. Then there exists a finite Borel measure ν such that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$, the following relationship holds*:

$$\begin{aligned}&\int_{\mathbb{R}_+^J\times\mathbb{M}_+^{J\times J}}[T\circ Of](z,a)\,d\pi(z,a)\\&\quad+\int_{\bigcup_i(F_i\cap F_i^a)}\big[R'\nabla_z(Of)(z,a)\big]\,d\nu(z,a)=0,\end{aligned}\tag{4.9}$$

where the operators T and O are given in (4.5) *and* (4.8).

We remark that the proof of this corollary shows that (4.9) is equivalent to several equations. That is, for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ and each set $I \subseteq \{1, 2, \ldots, J\}$, π and ν must satisfy

$$\begin{aligned}&\int_{\mathbb{R}_+^J\times\mathbb{M}_+^{J\times J}}[T\circ O_I f](z,a)\,d\pi(z,a)\\&\quad+\int_{\bigcup_i(F_i\cap F_i^a)}\big[R'\nabla_z(O_I f)(z,a)\big]\,d\nu(z,a)=0,\end{aligned}\tag{4.10}$$

where the operators O_I are defined in (4.7). Note that (4.10) produces 2^J equations, one of which is trivial. We refer to (4.10) as BAR_I.

We first check that (4.9) yields the classical BAR for the stationary distribution of the reflected Brownian motion Z when choosing $f(z, a) \equiv g(z)$ for some smooth g. One readily checks that in this case,

$$(Of)(z,a)=\sum_{I\subseteq\{1,2,\ldots,J\}}\sum_{S\subseteq\{1,\ldots,J\}\setminus I}(-1)^{|S|}g(\Pi_{S\cup I}z)=g(z).$$

Substituting the above equation in (4.9), we immediately obtain the well-known basic adjoint relationship as introduced in Harrison and Williams (1987a) for reflected Brownian motion,

$$\int_{\mathbb{R}_+^J}Lg(z)\,d\pi(z)+\int_{\bigcup_i F_i}\big[R'\nabla_z g(z)\big]\,d\nu(z)=0,\tag{4.11}$$

where $d\pi(z) = \int_{a\in\mathbb{M}^{J\times J}} d\pi(z, a)$ is the stationary distribution for Z and the Borel measure $d\nu(z)$ is given by $d\nu(z) = \int_{a\in\mathbb{M}^{J\times J}} d\nu(z, a)$.

We next specialize (4.9) to the one-dimensional case, and we verify that we recover Theorem 1. This shows in particular that (4.9) fully determines π if $J = 1$. Indeed, it is readily seen that

$$(Of)(z,a)=(O_\varnothing f)(z,a)+(O_{\{1\}}f)(z,a)=f(z,a)-f(0,a)+f(0,0).$$

Combining this with (4.9) gives (2.4), but with $-\partial/\partial z f(0, 0)\theta$ replaced with $c\partial/\partial z f(0, 0)$ for some constant $c = \nu(\{0, 0\}) > 0$. One can further show that $c = -\theta$, but we suppress the argument.

We next argue that none of the $2^J - 1$ nontrivial equations in (4.10) can be dropped, but we leave open the question whether they characterize π. We do so by illustrating the interplay between the different BAR_I in a simple example. Let $J = 3$ and consider $Z = (Z_1, Z_2, Z_3)$, where Z_1, Z_2, and Z_3 are three independent one-dimensional standard reflected Brownian motions. We do not need the second argument A, and therefore we make no distinction between (4.10) and a "classical" analog of BAR_I in (4.10). This classical analog is obtained by considering (4.10) for f that do not depend on the second argument a; cf. how (4.11) was obtained from (4.9). The process Z has a unique stationary distribution π, which is a product form; see, for example, Harrison and Williams (1987b) for details. $\mathrm{BAR}_{\{1,2\}}$ is equivalent with the third marginal distribution of π being exponential, with similar conclusions for $\mathrm{BAR}_{\{1,3\}}$ and $\mathrm{BAR}_{\{2,3\}}$. On the other hand, $\mathrm{BAR}_{\varnothing}$ and $\mathrm{BAR}_{\{j\}}$ for any $j \in \{1, 2, 3\}$ contain no information on the marginal distributions, in the sense that $O_{\varnothing} g = 0$ and $O_{\{j\}} g = 0$ for functions of the form $g(z) = f_1(z_1) + f_2(z_2) + f_3(z_3)$ (assuming appropriate smoothness). Still, $\mathrm{BAR}_{\{1\}}$ with $\mathrm{BAR}_{\{1,2\}}$ and $\mathrm{BAR}_{\{1,3\}}$ together imply that the push-forward of π under the projection map onto the last two coordinates has a product form solution since the two-dimensional reflected Brownian motion (Z_2, Z_3) satisfies the so-called skew-symmetry condition; see Harrison and Williams (1987b), Theorem 6.1, and Williams (1987), Theorem 1.2. Consequently, one can think of $\mathrm{BAR}_{\{1\}}$ as describing the dependencies between the second and third components of π, with marginal distributions determined by $\mathrm{BAR}_{\{1,2\}}$ and $\mathrm{BAR}_{\{1,3\}}$, respectively. Similarly, $\mathrm{BAR}_{\varnothing}$ describes the dependencies of the three two-dimensional push-forward measures of π.

5. Characteristics of derivatives and proof of Theorem 2. In this section, we prove Theorem 2. We also collect additional sample path properties of derivatives, with an emphasis on their jump behavior. These properties will be used in the proof of Theorem 3.

Throughout this section, we work under the conditions of Theorem 2. That is, we assume that $x \in C^J$ is given, and we write $z = \Gamma(x)$, $y = \Phi(x)$, $a = \Gamma'(x)$ and $b = -\Phi'(x)$. We also set $\chi(t) = tE$ for $t \geq 0$.

5.1. *Complementarity.* This section connects the augmented Skorohod problem associated with (R, R) for (x, χ) with (z, a). Note that, in view of Definitions 1 and 2, the first two requirements of the augmented Skorohod problem in Definition 3 are immediately satisfied for (x, y, z). It is immediate that $a = \chi - Rb$ by definition of a, so we must indeed choose $\widetilde{R} = R$. We proceed with showing that a and b lie in $\mathbb{D}_+^{J \times J}$ as required for the augmented Skorohod problem, but it is convenient to first establish part of the fourth requirement.

LEMMA 1. *The $\mathbb{M}^{J \times J}$-valued function b is componentwise nonnegative and nondecreasing.*

PROOF. Since $\chi(t) = tE$ for $t \geq 0$, χ is evidently nonnegative and nondecreasing. The monotonicity result in Theorem 6 of Kella and Whitt (1996) shows that for any fixed $\epsilon > 0$, each component of b_ϵ is nonnegative and nondecreasing. The lemma follows from the fact that b is the pointwise limit of the sequences $\{b_\epsilon\}$ as $\epsilon \to 0+$. $\square$

LEMMA 2. *The $\mathbb{M}^{J\times J}$-valued functions a and b lie in $\mathbb{D}_+^{J\times J}$.*

PROOF. Since b is nonnegative in view of Lemma 1, we will have shown the claim for b if we verify that $b \in \mathbb{D}^{J\times J}$. We deduce from Theorem 1.1 in Mandelbaum and Ramanan (2010) that each component of b is upper semicontinuous and that it has left and right limits everywhere. Since b is nondecreasing by Lemma 1, these properties imply that $b \in \mathbb{D}^{J\times J}$.

We next show that $a \in \mathbb{D}_+^{J\times J}$. Clearly, since $b \in \mathbb{D}_+^{J\times J}$, we only need to show that a is nonnegative. Again by the monotonicity result in Theorem 6 of Kella and Whitt (1996), for any fixed $\epsilon > 0$, each component of a_ϵ is nonnegative. This completes the proof of the lemma after letting $\epsilon \to 0+$. $\square$

We next investigate the fourth and fifth requirement of Definition 3. To this end, we need a characterization of b which relies heavily on Mandelbaum and Ramanan (2010).

LEMMA 3. *b is the unique solution to the following system of equations: for $i, j = 1, \ldots, J$ and $t \geq 0$,*

$$b_i^j(t) = \sup_{s\in\Phi_{(i)}(t)} \bigl[\delta_i^j s + \bigl[P' b^j\bigr]_i(s)\bigr],$$

where the supremum over an empty set should be interpreted as zero and

$$\Phi_{(i)}(t) = \bigl\{s \in [0,t] : z_i(s) = 0\bigr\}. \tag{5.1}$$

PROOF. We use Theorem 1.1 of Mandelbaum and Ramanan (2010), which can be simplified in view of Lemma 1 and the nonnegativity of the matrix P. This theorem states that

$$b_i^j(t) = \begin{cases} 0, & \text{if } t \in (0, t_{(i)}), \\ \sup\limits_{s\in\Psi_{(i)}(t)} \bigl[\delta_i^j s + \bigl[P' b^j\bigr]_i(s)\bigr], & \text{if } t \in [t_{(i)}, \infty), \end{cases} \tag{5.2}$$

where $t_{(i)} = \inf\{t \geq 0 : z_i(t) = 0\}$ and $\Psi_{(i)}(t) = \{s \in [0,t] : z_i(s) = 0, y_i(s) = y_i(t)\}$. Observe that, again using Lemma 1, the supremum must be attained at the rightmost end of the closed interval $\Psi_{(i)}(t)$. Since y is nondecreasing and $\int_0^t z_i(s)\, dy_i(s) = 0$, this is also the rightmost point of the closed set $\Phi_{(i)}(t)$. This establishes the lemma in view of the convention used for the supremum of an empty set. $\square$

LEMMA 4. *Fix any* $j = 1, \dots, J$, *and we have*

$$\int_0^\infty z(t)\, db^j(t) = 0. \tag{5.3}$$

PROOF. *Fix some* $i = 1, \dots, J$. *Note that if* $z_i(t) > 0$ *at time* t, *we deduce from the path continuity of* z *that there exists some* $\epsilon > 0$ *such that* $z_i(s) > 0$ *for* $s \in (t-\epsilon, t+\epsilon)$. *This implies that* $\Phi_{(i)}(s)$ *is constant as a set-valued function for* $s \in (t-\epsilon, t+\epsilon)$. *Thus* $b_i(s)$ *is constant for* $s \in (t-\epsilon, t+\epsilon)$ *by* (5.2). *Since* i *is arbitrary, this yields* (5.3). □

LEMMA 5. *If* $z_i(t) = 0$ *for some* i, *then we have* $a_i(t) = 0$.

PROOF. Suppose $z_i(t) = 0$. In view of Lemma 1, we deduce from (5.2) that, for any $j = 1, \dots, J$,

$$b_i^j(t) = \delta_i^j t + \left[P' b^j\right]_i(t).$$

Now it follows from (3.2) and $R = E - P'$ that

$$a_i^j(t) = \delta_i^j t - \left[R b^j\right]_i(t) = \delta_i^j t - b_i^j(t) + \left[P' b^j\right]_i(t) = 0,$$

which completes the proof of the lemma. □

The above two lemmas together with Lemma 2 yield two further complementarity conditions.

COROLLARY 2. *For any* $j = 1, \dots, J$, *we have*

$$\int^\infty a^j(t)\, dy(t) = 0, \qquad \int_0^\infty a^j(t)\, db^j(t) = 0. \tag{5.4}$$

PROOF OF THEOREM 2. The claim is now immediate from (3.1) in conjunction with Lemmas 1, 2, 4 and 5. □

5.2. *Jumps of* a. In this section, we collect sample path properties of a related to its jump behavior. This plays a critical role in the derivation of Theorem 3 and Corollary 1.

The next lemma states that a is linear whenever z is in the interior of $\mathbb{R}_+^J$.

LEMMA 6. *If* $z(t) \in \mathbb{R}_+^J \setminus \partial\mathbb{R}_+^J$ *for* $t \in [\alpha, \beta]$, *then we have for* $t \in [\alpha, \beta]$

$$a(t) = a(\alpha) + (t - \alpha)E.$$

In particular, a *is continuous on* (α, β) *and can only have jumps when* $z \in \partial\mathbb{R}_+^J$.

PROOF. In view of (3.2), it suffices to show that b is constant for $t \in [\alpha, \beta]$. Since $z(t) \in \mathbb{R}_+^J \setminus \partial\mathbb{R}_+^J$ for $t \in [\alpha, \beta]$, we obtain from (5.1) that for each $i = 1, \ldots, J$, $\Phi_{(i)}(t)$ is constant as a set-valued function. Therefore, we deduce from (5.2) that $b(t)$ is a constant in $\mathbb{M}^{J\times J}$ for $t \in [\alpha, \beta]$. The proof of the lemma is complete. □

For any function g on $\mathbb{R}_+$, we write $\Delta g(t) = g(t) - g(t-)$. In view of the above lemma, we can characterize the continuous part of the function a. Formally, we write

$$a(t) = a^c(t) + a^d(t),$$

where

$$a^d(t) = \sum_{s\le t} \Delta a(s).$$

We have the following corollary.

COROLLARY 3. *$a^c(t) = a(0) + tE$ for any $t \ge 0$.*

We next characterize the jump direction of a when a jump occurs.

LEMMA 7. *Fix a nonempty set $I \subseteq \{1, 2, \ldots, J\}$ and some $t > 0$. Suppose that $z_k(t) = 0$ for $k \in I$ and $z_i(t) > 0$ for $i \notin I$. If $\Delta a(t) \neq 0$, then we must have*

$$\Delta a(t) = -\sum_{k\in I} R^k [\Delta b]_k(t).$$

PROOF. Since $z_i(t) > 0$ for $i \notin I$, we deduce from the sample path continuity of z that there exists some $\epsilon > 0$ such that for $i \notin I$, $z_i(s) > 0$ for $s \in (t-\epsilon, t]$. This yields that for $i \notin I$, $\Phi_{(i)}(s)$ is a constant as a set-valued function for $s \in (t-\epsilon, t]$. From (5.2) we infer that for $i \notin I$, $b_i(s)$ is constant for $s \in (t-\epsilon, t]$. This implies that $[\Delta b]_i(t) = 0$ for $i \notin I$, and therefore that

$$\Delta a(t) = -R\Delta b(t) = -\sum_{k=1}^{J} R^k [\Delta b]_k(t) = -\sum_{k\in I} R^k [\Delta b]_k(t).$$

This completes the proof of the lemma. □

6. A basic adjoint relationship and proof of Theorem 3. This section is devoted to the proof of Theorem 3 and Corollary 1. The key idea is to apply Ito's formula to the semimartingale (Z, A) and use sample path properties of (Z, A) to analyze the stationary measure. This is a standard approach in the context of reflected Brownian motion, but the analysis here exposes new features due to the presence of jumps in the process A. Throughout, we work with the augmented filtration generated by W and $(Z(0), A(0))$.

6.1. *Ito's formula for the semimartingale* (Z, A). In this section, we apply Ito's formula to the semimartingale (Z, A). We first show that (Z, A) is a semimartingale, that is, each of its components is a semimartingale. Recall that a semimartingale is an adapted process which is the sum of a local martingale and a finite variation process, with sample paths in $\mathbb{D}$. For more detail, we refer readers to Protter (2005), Chapter 3, or Jacod and Shiryaev (2003), Chapter 1.

LEMMA 8. (Z, A) *is a semimartingale.*

PROOF. The process (Z, A) is adapted. This is a well-known property of Z, and $A(t)$ is a deterministic functional of $\{Z(s) : 0 \le s \le t\}$ and $A(0)$ since it arises from an augmented Skorohod problem. We know from Lemma 2 that each component of the process (Z, A) lies in $\mathbb{D}$. Since Z is a semimartingale, to show (Z, A) is a semimartingale, it suffices to show that A is a semimartingale. In fact, from Lemma 1 and (3.2) we immediately deduce that A is a finite variation process, that is, the paths of A are almost surely of finite variation on $[0, T]$ for any $T > 0$. In particular, A is a semimartingale. □

By Ito's formula, for example, Jacod and Shiryaev (2003), Section I.4, we deduce from (4.2) that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$, we have

$$\begin{aligned} f\big(Z(t), A(t)\big) &= f\big(Z(0), A(0)\big) \\ &\quad + \int_0^t \big[\sigma\big(Z(s)\big)' \nabla_z f\big(Z(s), A(s-)\big)\big]\, dW(s) \\ &\quad + \int_0^t \big[R' \nabla_z f\big(Z(s), A(s-)\big)\big]\, dY(s) \\ &\quad + \int_0^t Lf\big(Z(s), A(s-)\big)\, ds \\ &\quad + \int_0^t \nabla_a f\big(Z(s), A(s-)\big)\, dA^c(s) \\ &\quad + \sum_{s\le t} \big[f\big(Z(s), A(s)\big) - f\big(Z(s), A(s-)\big)\big]. \end{aligned} \tag{6.1}$$

Compared to the formulation in Theorem I.4.57 of Jacod and Shiryaev (2003), we have absorbed the last sum of the jump part into the integral

$$\int_0^t \nabla_a f\big(Z(s), A(s-)\big)\, dA^c(s).$$

This is justified by noting that, since $\Delta A(s) = -\widetilde{R}\Delta B(s)$ for some nonnegative and (componentwise) nondecreasing process B according to Definition 3,

$$\sum_{s\le t} \big\|\Delta A(s)\big\|_1 \le C \sum_{s\le t} \big\|\Delta B(s)\big\|_1 = C\big\|B(t)\big\|_1 < \infty, \tag{6.2}$$

where C denotes some constant depending on $\widetilde{R}$. Note that this also implies that the last term on the right-hand side of (6.1) is absolutely convergent. Indeed, combining the above bound with $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$ yields $\sum_{s\le t} |f(Z(s), A(s)) - f(Z(s), A(s-))| < \infty$.

Suppose that (Z, A) is positive recurrent and has a unique stationary distribution π. Henceforth we assume that $(Z(0), A(0))$ has distribution π, and we write $\mathbb{E}_\pi$ instead of $\mathbb{E}$. After taking an expectation with respect to π on both sides of (6.1), the term involving dW vanishes since it is a martingale term. We next analyze the second to last term on the right-hand side. From Corollary 3 and the fact that A has countably many jumps (Lemma 1), we deduce that

$$\begin{aligned}\mathbb{E}_\pi \int_0^t \nabla_a f\big(Z(s), A(s-)\big)\, dA^c(s) &= \mathbb{E}_\pi \int_0^t \nabla_a f\big(Z(s), A(s-)\big)\, d(sE)\\ &= \mathbb{E}_\pi \int_0^t \nabla_a f\big(Z(s), A(s)\big)\, d(sE)\\ &= \mathbb{E}_\pi \int_0^t \operatorname{tr}\big(\nabla_a f\big(Z(s), A(s)\big)\big)\, ds.\end{aligned}$$

Since $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$, we have from Fubini's theorem and the definition of stationarity in (4.3) that

$$\begin{aligned}\mathbb{E}_\pi \int_0^t \operatorname{tr}\big(\nabla_a f\big(Z(s), A(s)\big)\big)\, ds &= \int_0^t \mathbb{E}_\pi \operatorname{tr}\big(\nabla_a f\big(Z(s), A(s)\big)\big)\, ds\\ &= t \int \operatorname{tr}\big(\nabla_a f(z,a)\big)\, d\pi(z,a).\end{aligned}$$

Thus we obtain

$$\mathbb{E}_\pi \int_0^t \nabla_a f\big(Z(s), A(s-)\big)\, dA^c(s) = t \int \operatorname{tr}\big(\nabla_a f(z,a)\big)\, d\pi(z,a).$$

A similar argument applies to the fourth term on the right-hand side of (6.1). We conclude that, for each $t \ge 0$ and each $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$,

$$\begin{aligned}0 = t \int \big[Tf(z,a)\big]\, d\pi(z,a) &+ \mathbb{E}_\pi \int_0^t \big[R' \nabla_z f\big(Z(s), A(s-)\big)\big]\, dY(s)\\ &+ \mathbb{E}_\pi \sum_{s\le t} \big[f\big(Z(s), A(s)\big) - f\big(Z(s), A(s-)\big)\big],\end{aligned} \tag{6.3}$$

where T is given in (4.5). This equation serves as the starting point for proving Theorem 3.

6.2. *The boundary term.* In this section we rewrite the boundary term in (6.3), that is, the term involving dY. Let $\nu = (\nu_1, \dots, \nu_J)$ be the unique vector of measures on $\partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J}$ for which

$$\int h(z,a)\nu(dz, da) = \mathbb{E}_\pi \int_0^1 h\big(Z(s), A(s-)\big)\, dY(s)$$

for all continuous $h : \partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}^J$ with compact support. This is a well-defined measure by the following lemma. For a different proof in the reflected Brownian motion case, see Harrison and Williams (1987a), Section 8.

LEMMA 9. *We have* $\mathbb{E}_\pi Y(1) < \infty$ *componentwise*.

PROOF. Since $Y(1) \geq 0$, it is enough to show that $\mathbb{E}_\pi \|RY(1)\|_1 < \infty$. We prove the stronger statement that $\mathbb{E}_\pi \|RY(1)\|_2^2 < \infty$. From the fact that Z satisfies the SDER (4.2), we obtain

$$\mathbb{E}_\pi \|RY(1)\|_2^2 = \mathbb{E}_\pi \left\| Z(1) - Z(0) - \int_0^1 \theta\big(Z(s)\big)\,ds - \int_0^1 \sigma\big(Z(s)\big)\,dW(s) \right\|_2^2.$$

It follows from the fact that $t \mapsto \mathbb{E}_\pi \|Z(t)\|_2^2$ is locally bounded and the growth condition on θ that $\mathbb{E}_\pi \| \int_0^1 \theta(Z(s))\,ds\|_2^2 < \infty$. Similarly, we have

$$\mathbb{E}_\pi \left\| \int_0^1 \sigma\big(Z(s)\big)\,dW(s) \right\|_2^2 = \mathbb{E}_\pi \int_0^1 \operatorname{tr} \Sigma\big(Z(s)\big)\,ds < \infty,$$

where the finiteness follows from the growth condition on σ. □

Our next goal is to give a characterization of measure ν in terms of π, which we carry out through Laplace transforms. We start with determining the support of ν.

LEMMA 10. *The support of* ν *is* $\bigcup_i (F_i \cap F_i^a)$.

PROOF. In view of Lemma 2, it is clear that A can have at most countably many jumps. For any continuous $h : \partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}^J$ with compact support, we have

$$\int_0^1 h\big(Z(s), A(s-)\big)\,dY(s) = \int_0^1 h\big(Z(s), A(s)\big)\,dY(s),$$

since the measure dY is continuous and the integrand has countably many jumps by Lemma 1. It follows from the definition of ν that

$$\int h(z,a)\nu(dz,da) = \mathbb{E}_\pi \int_0^1 h\big(Z(s), A(s)\big)\,dY(s). \tag{6.4}$$

The complementarity conditions $\int_0^\infty Z(t)\,dY(t) = 0$ and (5.4) imply the lemma. □

On combining equations (6.3) and (6.4) we obtain that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$,

$$\begin{aligned} 0 = {} & \int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} \big[Tf(z,a)\big]\,d\pi(z,a) + \int_{\bigcup_i (F_i \cap F_i^a)} \big[R'\nabla_z f(z,a)\big]\,d\nu(z,a) \\ & + \frac{1}{t}\mathbb{E}_\pi \sum_{s\leq t} \big[f\big(Z(s), A(s)\big) - f\big(Z(s), A(s-)\big)\big]. \end{aligned} \tag{6.5}$$

We now express the Laplace transform of ν in terms of the Laplace transform of π. Set $f(z,a) = \exp(-\eta' z - \langle \alpha, a \rangle_{\mathrm{HS}}) \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ where $(\eta, \alpha) \in \mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}$. After substituting f in (6.5), we obtain

$$\pi^*(\eta,\alpha) - \sum_{j=1}^{J} \big(R'\eta\big)_j \nu_j^*(\eta,\alpha) + H(\eta,\alpha) = 0, \tag{6.6}$$

where

$$\pi^*(\eta,\alpha) = \int_{\mathbb{R}_+^J \times \mathbb{M}^{J\times J}} \left[\frac{1}{2}\eta'\Sigma(z)\eta + \eta'\theta(z) - \sum_{i=1}^{J}\alpha_i\right] e^{-\eta\cdot z - \alpha\cdot a}\, d\pi(z,a),$$

$$\nu_j^*(\eta,\alpha) = \int_{F_j \cap F_j^a} e^{-\eta\cdot z - \alpha\cdot a}\, d\nu_j(z,a),$$

$$H(\eta,\alpha) = \mathbb{E}_\pi \sum_{s\le 1} \big[e^{-\eta\cdot Z(s)} \cdot \big(e^{-\alpha\cdot A(s)} - e^{-\alpha\cdot A(s-)}\big)\big].$$

Dividing (6.6) by $\eta_j > 0$ and letting $\eta_j \to \infty$, we deduce that

$$\nu_j^*(\eta,\alpha) = \frac{1}{2}\lim_{\eta_j\to\infty} \eta_j \int_{\mathbb{R}_+^J \times \mathbb{M}^{J\times J}} \Sigma_{jj}(z) e^{-\eta\cdot z - \alpha\cdot a}\, d\pi(z,a), \tag{6.7}$$

where we have used the fact that $\nu_j(F_j \cap F_i) = 0$ for $i \neq j$ so that $\lim_{\eta_j\to\infty} \nu_i^*(\eta,\alpha) = 0$ by the dominated convergence theorem. Since all terms in (6.6) vanish in the limit by dominated convergence except for the term with ν_j^* and the term with π^*, existence of the limit in (6.7) follows immediately from the fact that $\nu_j(\eta,\alpha)$ does not depend on η_j. Under further regularity conditions on π, one can use the initial value theorem for Laplace transforms to show that $d\nu_j = \frac{1}{2}\Sigma_{jj}\, d\pi_j$ for an appropriate restriction π_j of π. Carrying out this procedure provides little additional insight, and we therefore suppress further details.

6.3. *The jump term.* We now proceed investigating the jump term, that is, the term in (6.3) involving the countable sum. Lemma 6 implies that jumps in A can only occur when Z lies hits the boundary $\partial\mathbb{R}_+^J$ of the nonnegative orthant, which motivates the following definition. For $I \subseteq \{1,\dots,J\}$, $I \neq \varnothing$, we define measures u_I on $\mathbb{R}_+^{|I^c|} \times \mathbb{M}_+^{J\times J}$ with support in $(0,\infty)^{|I^c|} \times \mathbb{M}_+^{J\times J}$. We set, for Borel sets $G \subseteq (0,\infty)^{|I^c|}$, $C \subseteq \mathbb{M}_+^{J\times J}$,

$$u_I(G,C) = \mathbb{E}_\pi \sum_{s\le 1 \,:\, Z_I(s)=0, Z_{I^c}(s)\in G, A(s)\neq A(s-)} 1_C\big\{A(s-)\big\}.$$

This is a well-defined σ-finite measure because of (6.2) and $\mathbb{E}_\pi \|B(1)\|_1 = \mathbb{E}_\pi \|A(1) - A(0) - E\|_1 \le 2\mathbb{E}_\pi \|A(0)\|_1 + J < \infty$, so that

$$\mathbb{E}_\pi \left| \sum_{s\le 1} \big[f\big(Z(s), A(s)\big) - f\big(Z(s), A(s-)\big)\big] \right| < \infty$$

for $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$. It is possible to express these measures in terms of π using the theory of distributions; this is done in Section 7.

The primary objective of this subsection is to show that the jump term in (6.3) vanishes for a special class of functions, which is key in our proof of Corollary 1. Throughout, we fix a set $I \subseteq \{1,\dots,J\}$. Recall the definition of O_I in (4.7). It is our aim to show that the jump term vanishes for functions of the form $O_I f$, where $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$ as before. We first introduce a lemma.

LEMMA 11. *For any $f : \mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}$, if $z_j = 0$ for some $j \notin I$, then for any $a \in \mathbb{M}^{J\times J}$ we have*

$$\sum_{S\subseteq\{1,\dots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) = 0.$$

In particular, if $z_j = 0$ for some $j \notin I$, then we have $O_I f(z,a) = 0$.

PROOF. Suppose $z_j = 0$ for some $j \notin I$. Then for any set $S \subseteq \{1,\dots,J\}\setminus I$ with $j \notin S$, we have $\Pi_{S\cup I} z = \Pi_{S\cup I\cup\{j\}} z$. Using this observation, we deduce that

$$\begin{aligned}
&\sum_{S\subseteq\{1,\dots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) \\
&\quad= \sum_{S\subseteq\{1,\dots,J\}\setminus I\,:\, j\in S} (-1)^{|S|} f(\Pi_{S\cup I} z, a) \\
&\qquad+ \sum_{S\subseteq\{1,\dots,J\}\setminus I\,:\, j\notin S} (-1)^{|S|} f(\Pi_{S\cup I} z, a) \\
&\quad= \sum_{S\subseteq\{1,\dots,J\}\setminus I\,:\, j\in S} (-1)^{|S|} f(\Pi_{S\cup I} z, a) \\
&\qquad+ \sum_{S\subseteq\{1,\dots,J\}\setminus I\,:\, j\notin S} (-1)^{|S|} f(\Pi_{S\cup I\cup\{j\}} z, a) \\
&\quad= \sum_{S\subseteq\{1,\dots,J\}\setminus I\,:\, j\in S} (-1)^{|S|} f(\Pi_{S\cup I} z, a) \\
&\qquad+ \sum_{\widetilde{S}\subseteq\{1,\dots,J\}\setminus I\,:\, j\in \widetilde{S}} (-1)^{|\widetilde{S}|-1} f(\Pi_{\widetilde{S}\cup I} z, a) \\
&\quad= 0.
\end{aligned}$$

The proof of the lemma is complete. □

Now we are ready to show that the jump term vanishes for functions of the form $O_I f$. For any $K \subseteq \{1,\dots,J\}$, Z_K denotes the process whose components are those of Z with indices in K.

LEMMA 12. *For each $t \geq 0$ and any measurable $f : \mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}$, we have*

$$\mathbb{E}_\pi \sum_{s\leq t} \big[O_I f\big(Z(s), A(s)\big) - O_I f\big(Z(s), A(s-)\big)\big] = 0. \tag{6.8}$$

PROOF. By Lemmas 6 and 11, we have

$$\begin{aligned}
&\mathbb{E}_\pi \sum_{s\leq t} \big[O_I f\big(Z(s), A(s)\big) - O_I f\big(Z(s), A(s-)\big)\big] \\
&\quad = \sum_{\varnothing\neq K\subseteq\{1,\dots,J\}} \mathbb{E}_\pi \sum_{s\leq t\,:\, Z_K(s)=0, Z_{\{1,\dots,J\}\setminus K}(s)>0} \big[O_I f\big(Z(s), A(s)\big) \\
&\qquad\qquad - O_I f\big(Z(s), A(s-)\big)\big] \\
&\quad = \sum_{\varnothing\neq K\subseteq I} \mathbb{E}_\pi \sum_{s\leq t\,:\, Z_K(s)=0, Z_{\{1,\dots,J\}\setminus K}(s)>0} \big[O_I f\big(Z(s), A(s)\big) \\
&\qquad\qquad - O_I f\big(Z(s), A(s-)\big)\big].
\end{aligned}$$

Therefore, to show (6.8) it suffices to show for each nonempty set $K \subseteq I$, we have

$$\mathbb{E}_\pi \sum_{s\leq t\,:\, Z_K(s)=0, Z_{\{1,\dots,J\}\setminus K}(s)>0} \big[O_I f\big(Z(s), A(s)\big) - O_I f\big(Z(s), A(s-)\big)\big] = 0. \tag{6.9}$$

To prove (6.9) we first deduce from Definition 3 that when $Z_K(s) = 0$ and $Z_{\{1,\dots,J\}\setminus K}(s) > 0$,

$$Q_K\big(A(s-)\big) = A(s).$$

Next, since $K \subseteq I$, we use the projection property of the operator Q_I to obtain

$$Q_I\big(A(s)\big) = Q_I\big(Q_K\big(A(s-)\big)\big) = Q_I\big(A(s-)\big).$$

Now (6.9) readily follows from the definition of O_I as in (4.7). Thus we have completed the proof of the lemma. □

6.4. *Proofs of Theorem 3 and Corollary 1.* We now prove Theorem 3 and Corollary 1.

PROOF OF THEOREM 3. We rewrite the jump term in (6.5) using the jump measures. In view of Lemmas 5 and 7,

$$\begin{aligned}
&\mathbb{E}_\pi \sum_{s\leq 1} \big[f\big(Z(s), A(s)\big) - f\big(Z(s), A(s-)\big)\big] \\
&\quad = \sum_{\varnothing\neq K\subseteq\{1,\dots,J\}} \mathbb{E}_\pi \sum_{s\leq 1\,:\, Z_K(s)=0, Z_{K^c}(s)>0} \big[f\big(Z|_{K^c}(s), A(s)\big) \\
&\qquad\qquad - f\big(Z|_{K^c}(s), A(s-)\big)\big]
\end{aligned}$$

$$= \sum_{\varnothing \neq K \subseteq \{1,\dots,J\}} \mathbb{E}_\pi \sum_{s \le 1 : Z_K(s)=0, Z_{K^c}(s)>0, A(s)\neq A(s-)} \big[f\big(Z|_{K^c}(s), Q_K(A(s-))\big) - f\big(Z|_{K^c}(s), A(s-)\big)\big]$$

$$= \sum_{\varnothing \neq K \subseteq \{1,\dots,J\}} \int_{z_{K^c},a} \big[f\big(z|_{K^c}, Q_K(a)\big) - f\big(z|_{K^c}, a\big)\big] du_K(z_{K^c}, a).$$

Thus Theorem 3 follows from (6.5). □

PROOF OF COROLLARY 1. Equation (4.10) immediately follows from (6.5) and Lemma 12. Summing all the equations in (4.10) over the sets $I \subseteq \{1,\dots,J\}$, we obtain (4.9). □

7. Jump measures. In this section, we further investigate the jump term in (6.3), resulting in a characterization of jump measures u_I in terms of the stationary distribution π. We start with an auxiliary result on the measures u_I.

LEMMA 13. *For each $I \subseteq \{1,\dots,J\}$, $I \neq \varnothing$ and $k = 1,\dots,J$, we have $u_I(\{(z_{I^c}, a) : a_k = 0\}) = 0$.*

PROOF. We exploit the dynamics of the augmented Skorohod problem. Since $A_k(s-) = 0$ implies $Z_k(s) = 0$, we have $u_I(\{(z_{I^c}, a) : a_k = 0\}) = 0$ for $k \in I^c$. We next consider $k \in I$. Since the continuous part of A_k^k is strictly increasing when $Z_k > 0$, the only possibility for $Z_I(s) = 0$, $A_k(s-) = 0$, and $A(s) \neq A(s-)$ to occur simultaneously is for Z to hit the face $z_I = 0$ without having left the face $z_k = 0$ for some positive amount of time. Since the time Z spends on the boundary has Lebesgue measure zero, this cannot happen almost surely. □

To proceed with our description of the measures u_I, we need tools from theory of distributions (or generalized functions). For background on this theory, see Duistermaat and Kolk (2010), Rudin (1991). For $I \subseteq \{1,\dots,J\}$, we define the operator T_I^* on distributions through

$$T_I^* f = \frac{1}{2} \sum_{i,j \in I} \frac{\partial^2}{\partial z_i \partial z_j} \big[\Sigma_{ij}(\cdot) f\big] - \sum_{j \in I} \theta_j \frac{\partial}{\partial z_j} f - \operatorname{tr}(\nabla_a f)$$

for any distribution f. With the understanding that we identify any probability measure with the distribution it generates, we can differentiate (probability) measures and T_I^* can act on measures. We also define

$$d\pi_I(z_{I^c}, a) = \int_{z_I} d\pi(z, a).$$

The main result of this section is that u_I can be expressed in terms of π. Indeed, together with Lemma 13, it completely determines u_I.

PROPOSITION 1. *For each* $I \subseteq \{1,\dots,J\}, I \neq \varnothing$, *we have, with* $z_{I^c} \in (0,\infty)^{|I^c|}, a \in \mathbb{M}_+^{J\times J}$ *and* $a_k \neq 0$ *for* $k = 1,\dots,J$,

$$du_I(z_{I^c},a) = \sum_{K\subseteq I, K\neq\varnothing} (-1)^{|I\setminus K|} \int_{z_{I\setminus K}} \big[T_{K^c}^* \, d\pi_K\big](z_{K^c},a).$$

PROOF. Equation (6.5) forms the basis of the proof, together with the identity

$$\mathbb{E}_\pi \sum_{s\le 1} \big[f\big(Z(s),A(s)\big) - f\big(Z(s),A(s-)\big)\big]$$
$$= \sum_{\varnothing\neq K\subseteq\{1,\dots,J\}} \int_{z_{K^c},a} \big[f\big(z|_{K^c}, Q_K(a)\big) - f(z|_{K^c},a)\big]\, du_K(z_{K^c},a),$$

which was established in Section 6.4. Fix some nonempty $I \subseteq \{1,\dots,J\}$. For $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ with the property that f vanishes on $\bigcup_{i\in I^c} F_i \cup \bigcup_i F_i^a$, (6.5) reduces to

$$\int_{\mathbb{R}_+^J\times\mathbb{M}_+^{J\times J}} Tf(z,a)\, d\pi(z,a) = \sum_{L\subseteq I\,:\,L\neq I} \int_{z_{I^c\cup L},a} f(z|_{I^c\cup L},a)\, du_{I\setminus L}(z_{I^c\cup L},a).$$

If moreover $f(z,a)$ does not depend on z_I, this can be simplified further,

$$\begin{aligned}(7.1)\qquad &\int_{\mathbb{R}_+^J\times\mathbb{M}_+^{J\times J}} Tf(z,a)\, d\pi(z,a)\\ &\quad = \sum_{L\subseteq I\,:\,L\neq I} \int_{z_{I^c},a} f(z|_{I^c},a) \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a).\end{aligned}$$

The left-hand side can be rewritten using the theory of differentiation for distributions Duistermaat and Kolk (2010), Chapter 4, or Rudin (1991), Section II.6.12. This leads to

$$\int_{\mathbb{R}_+^J\times\mathbb{M}^{J\times J}} Tf(z,a)\, d\pi(z,a) = \int_{z_{I^c},a} f(z|_{I^c},a)\big[T_{I^c}^*\, d\pi_I\big](z_{I^c},a).$$

Combining this with (7.1) and rearranging terms, we get

$$\begin{aligned}&\int_{z_{I^c},a} f(z|_{I^c},a)\, du_I(z_{I^c},a)\\ &\quad = \int_{z_{I^c},a} f(z|_{I^c},a)\big[T_{I^c}^*\, d\pi_I\big](z_{I^c},a)\\ &\qquad - \sum_{L\subseteq I\,:\,L\neq\varnothing, L\neq I} \int_{z_{I^c},a} f(z|_{I^c},a) \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a).\end{aligned}$$

This shows that, for $z_{I^c} \in (0,\infty)^{|I^c|}, a \in \mathbb{M}_+^{J\times J}$ and $a_k \neq 0$ for $k = 1,\dots,J$,

$$du_I(z_{I^c},a) = T_{I^c}^*\, d\pi_I(z_{I^c},a) - \sum_{L\subseteq I, L\neq\varnothing, L\neq I} \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a).$$

Since $|I \setminus L| < |I|$, this representation allows us to finish the proof of the proposition by an elementary induction argument on $|I|$. Alternatively, one could use a version of the inclusion-exclusion principle Stanley (1997), Section 2.1. $\square$

APPENDIX A: PROOF OF (2.5)

This appendix uses Theorem 1 to find the Laplace transform of the stationary distribution π of (Z, A) in the one-dimensional case, thereby showing in particular that Theorem 1 completely determines π. Writing $\mathcal{L}(\alpha, \eta)$ for the Laplace transform of π, Theorem 1 implies that

$$\tfrac{1}{2}\sigma^2\alpha^2 - \alpha\theta - \eta)\mathcal{L}(\alpha, \eta) + \eta\mathcal{L}(0, \eta) + \alpha\theta = 0. \tag{A.1}$$

In particular, on setting $\eta = \frac{1}{2}\sigma^2\alpha^2 - \alpha\theta$ we get

$$\left[\tfrac{1}{2}\sigma^2\alpha^2 - \alpha\theta\right]\mathcal{L}\left(0, \tfrac{1}{2}\sigma^2\alpha^2 - \alpha\theta\right) + \alpha\theta = 0.$$

After substitution of $\alpha = (\theta + \sqrt{\theta^2 + 2\sigma^2\eta})/\sigma^2$, we find that

$$\eta\mathcal{L}(0, \eta) = -\theta\left[\frac{\theta + \sqrt{\theta^2 + 2\sigma^2\eta}}{\sigma^2}\right].$$

Substituting this back into (A.1) and simplifying the resulting expression, we obtain the Laplace transform given in (2.5).

APPENDIX B: THE AUGMENTED SKOROHOD PROBLEM AND UNIQUENESS

In this appendix, we prove that the augmented Skorohod problem admits a unique solution. To this end, we employ a similar contraction map as in Lemma 3.6 of Mandelbaum and Ramanan (2010). Define a map Λ from $\mathbb{D}^{J\times J}$ to $\mathbb{D}^{J\times J}$ by setting, for $t \geq 0$,

$$\Lambda(b)_i^j(t) = \sup_{s\in\Phi_{(i)}(t)} \left[\chi_i^j(s) + \left[\widetilde{P}b^j\right]_i(s)\right]. \tag{B.1}$$

Momentarily we show that Λ is a contraction map, and thus Λ has a unique fixed point b. This also implies that, defining $b^{(0)} = 0$ and $b^{(n)} = \Lambda(b^{(n-1)})$ for $n \geq 1$, we have $\|b^{(n)} - b\|_T \to 0$ as $n \to \infty$ for every $T > 0$. Here and throughout this proof, we write $\|x\|_T = \sup_{t\in[0,T]} |x(t)|$; this should not be confused with the 1-norm and 2-norm used elsewhere in this paper. Since χ is nonnegative and nondecreasing and $\widetilde{P}$ is nonnegative, we deduce that $b^{(n)}$ is componentwise nonnegative and nondecreasing for each n. Therefore, we obtain that the fixed point b is also nonnegative and nondecreasing. Now let $a = \chi - \widetilde{R}b$, $z = \Gamma(x)$, and $y = \Phi(x)$. We now verify directly that (z, y, a, b) is a solution to the augmented

Skorohod problem. Only the fourth and fifth requirement in Definition 3 are not immediate. The fourth requirement can be shown to hold using the same argument as in the proof of Lemma 4. For the fifth requirement, we note that if $z_i(t) = 0$, (B.1) implies that for each j,

$$b_i^j(t) = \chi_i^j(t) + \big(\widetilde{P} b^j\big)_i(t),$$

which yields

$$a_i(t) = \chi_i(t) - (\widetilde{R} b)_i(t) = \chi_i(t) + (\widetilde{P} b)_i(t) - b_i(t) = 0.$$

To establish the uniqueness of solutions to the augmented Skorohod problem, we use the contraction map Λ. Suppose (z, y, a, b) solves the augmented Skorohod problem. Let $\tilde{b} = \Lambda(b)$. If we can show that $\tilde{b} = b$, meaning b is a fixed point of Λ, then it follows from the uniqueness of the fixed point that there must be a unique solution to the augmented Skorohod problem. Suppose there exists some i, j and t_0 such that $\tilde{b}_i^j(t_0) \neq b_i^j(t_0)$. We discuss two cases. If $z_i(t_0) = 0$, using nonnegativity and monotonicity of b, one can check from (B.1) that $\tilde{b}_i^j(t_0) = \chi_i^j(t_0) + [\widetilde{P} b^j]_i(t_0)$. From the definition of the augmented Skorohod problem, we also know that $z_i(t_0) = 0$ implies $a_i^j(t_0) = \chi_i^j(t_0) + [\widetilde{R} b^j]_i(t_0) = 0$. Therefore, we have $\tilde{b}_i^j(t_0) = b_i^j(t_0)$, a contradiction. Now consider the second case where we have $z_i(t_0) > 0$. If the set $\Phi_{(i)}(t_0)$ is empty, we have $\tilde{b}_i^j(t_0) = b_i^j(t_0) = b_i^j(0) = 0$. If not, let s be the maximal element in $\Phi_{(i)}(t_0)$. We deduce from the previous case in conjunction with the complementarily condition (4.1) that $b_i^j(t_0) = b_i^j(s) = \tilde{b}_i^j(s) = \tilde{b}_i^j(t_0)$. This is again a contradiction. Therefore, we obtain $\tilde{b} = b$ and infer that the augmented Skorohod problem has a unique solution.

It remains to show that Λ is a contraction map on $\mathbb{D}^{J \times J}$, which is equipped with the uniform norm on compact sets. As in the proof of Lemma 3.6 in Mandelbaum and Ramanan (2010) we assume that, without loss of generality, the maximum row sum of $\widetilde{P}$ is $\eta < 1$. It is easy to verify that for any fixed $T > 0$,

$$\big\|\Lambda(b) - \Lambda\big(b'\big)\big\|_T \leq \eta \big\|b - b'\big\|_T$$

for all $b, b' \in \mathbb{D}^{J \times J}$. Thus we have proved the existence and uniqueness of a fixed point for Λ.

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