CONCENTRATION OF MEASURE FOR BROWNIAN PARTICLE SYSTEMS INTERACTING THROUGH THEIR RANKS

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We consider a finite or countable collection of one-dimensional Brownian particles whose dynamics at any point in time is determined by their rank in the entire particle system. Using transportation cost inequalities for stochastic processes we provide uniform fluctuation bounds for the ordered particles, their local time of collisions and various associated statistics over intervals of time. For example, such processes, when exponentiated and rescaled, exhibit power law decay under stationarity; we derive concentration bounds for the empirical estimates of the index of the power law over large intervals of time. A key ingredient in our proofs is a novel upper bound on the Lipschitz constant of the Skorokhod map that transforms a multidimensional Brownian path to a path which is constrained not to leave the positive orthant.

1. Introduction. Define the set I as $\{1, \ldots, K\}$ for some $K \in \mathbb{N}$ or as the set of natural numbers $\mathbb{N}$ and let δ_i, $i \in I$, be a finite or countable collection of real constants. Consider the following system of stochastic differential equations:

$$(1.1) \qquad dX_i(t) = \sum_{j \in I} \delta_j \cdot 1_{\{X_i(t) = X_{(j)}(t)\}} \, dt + dW_i(t), \qquad i \in I.$$

Here, $X_{(1)}(t) \leq X_{(2)}(t) \leq \cdots$ are the coordinates of the process $X_i(t)$, $i \in I$, in the increasing order, and $W = (W_i : i \in I)$ is a system of jointly independent one-dimensional standard Brownian motions. The equations in (1.1) model the movement of the particles by interacting Brownian motions such that at every time point, if we order the positions of the particles, then the ith ranked particle from the bottom gets a drift δ_i for every $i \in I$. As time evolves, the Brownian motions switch ranks and drifts, and, hence, their motion is determined by these time-dependent interactions. When I is finite, the existence and uniqueness in law of such processes is a consequence of Girsanov's theorem; see, for example, Lemma 6 in [23]. The countable case is subtle and requires constraints on the initial positions of the particles. In particular, for (1.1) to make sense, the number of particles on every interval of the form $(-\infty, x]$ has to be finite at any point in time with probability one. We discuss this issue in more detail later in the text.

Different versions of the particle system in (1.1) have been considered in several recent articles. Among the more recent ones, see Banner, Fernholz and Karatzas [3], Banner and Ghomrasni [4], McKean and Shepp [21], Pal and Pitman [23], Jourdain and Malrieu [18], Chatterjee and Pal [5, 6], Ichiba and Karatzas [16], Ichiba et al. [17] and Shkolnikov [27, 28]. We refer the reader to the above articles for the full list of applications of such processes. Related discrete time processes are studied in the context of the Sherrington–Kirkpatrick model of spin glasses by Arguin and Aizenman [2], Ruzmaikina and Aizenman [25] and Shkolnikov [26].

The case of distinct drift parameters differs from the latter models in several remarkable ways. For example, if $I = \{1, \ldots, K\}$ for some $K \in \mathbb{N}$ and

$$\frac{1}{j} \sum_{k=1}^{j} \delta_k > \frac{1}{K - j} \sum_{k=j+1}^{K} \delta_k$$

for all $j = 1, \ldots, K - 1$, then there exists an invariant distribution for the process of gaps between consecutive particles $(X_{(2)}(t) - X_{(1)}(t), \ldots, X_{(K)}(t) - X_{(K-1)}(t))$, $t \geq 0$ (see Theorem 8 in [23]) which is not the case if $\delta_1 = \cdots = \delta_K$. When $I = \mathbb{N}$, one can consider the so-called *Atlas* model in which $\delta_1 = \delta > 0$ and $\delta_i = 0$ for every $i \neq 1$ in I. In this case it is shown in [23] that, if the initial positions of the particles are chosen according to a standard Poisson process of rate 2δ, then the joint distribution of the gaps between consecutive particles $X_{(2)}(t) - X_{(1)}(t), X_{(3)}(t) - X_{(2)}(t), \ldots$ is the same for all $t \geq 0$. However, the same statement is not true when $\delta_i = \delta$ for all $i \in I$; see Theorem 4.2 in [25]. Moreover, if we consider the motion of the left-most particle $X_{(1)}(t)$, $t \geq 0$, in the Atlas model, no estimates on its growth and fluctuations are known. Our article is a step in the latter direction. Using techniques from the theory of concentration of measures we give estimates on fluctuations of the paths of the distances between ordered particles and associated statistics.

One such statistic is given by the *market weights*. The latter can be defined for $I = \{1, \ldots, K\}$ with an arbitrary $K \in \mathbb{N}$ and any choice of $\delta_1, \ldots, \delta_K$ by setting

$$(1.2) \qquad \mu_i(t) = \frac{e^{X_{(K-i+1)}(t)}}{\sum_{j=1}^{K} e^{X_{(j)}(t)}}, \qquad i = 1, \ldots, K$$

for all $t \geq 0$. It is clear that for any fixed t the sequence $\mu_1(t), \ldots, \mu_K(t)$ is a nonincreasing sequence of positive numbers that add up to one. These numbers, in econometric models, go by the name of market weights and have an interesting history.

Fernholz in his 2002 book [13] introduces solutions of (1.1) to model the time dynamics of the logarithmic market capitalizations of different companies in an equity market. In other words, he considers a stock market with K companies whose total worths in stocks are given by exponentials of the one-dimensional components of the solution of equation (1.1). A major objective of his work is to
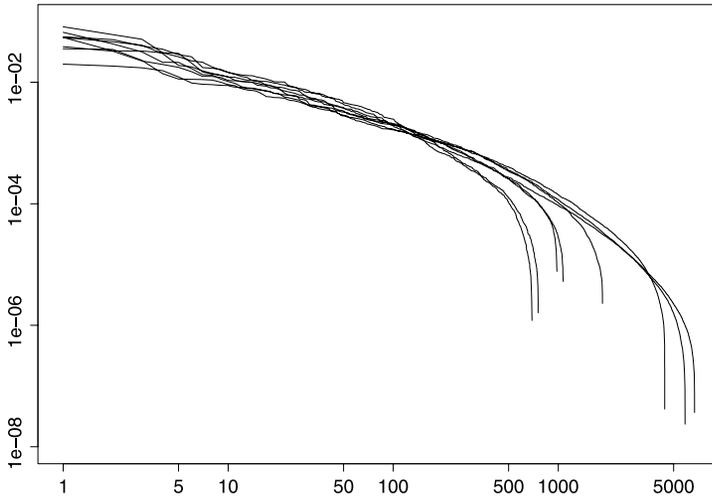
FIG. 1.   *Capital distribution curves*: 1929–1999.

explain the following curious empirical fact. Consider the shape one obtains by plotting $\log \mu_i$ versus $\log i$. This log–log plot is referred to as the *capital distribution curve*. See Figure 1 above (reproduced from [13]) which shows the capital distribution curves between 1929 and 1999 for all the major US stock markets (NYSE, AMEX and NASDAQ) combined. Empirically, the left part of the plot exhibits nearly linear decay. This corresponds to the market weights, in decreasing order, displaying power law (or Zipf's law) decay. More strikingly, the slope of the decay is nearly constant over eight decades, something truly remarkable in the volatile world of financial markets.

In [5] the authors explain the linearity by proving (under suitable assumptions) that the possible limiting stationary laws of the market weights, as K grows to infinity, are given by a subset of the Poisson–Dirichlet family of distributions. The masses of this family have a polynomial decay with size, which corresponds to the linear decay in the log–log plot. However, this does not quite address the temporal stability of the picture above. Our analysis below captures some of its subtleties.

To analyze the stability of the shape of the capital distribution curve, consider the process of market weights $(\mu_i(t), i = 1, \ldots, K)$, $t \geq 0$ and fix a $J \ll K$. At any point of time $t \geq 0$ we introduce the linear regression between the pairs of data $\{(\log i, \log \mu_i(t)), i = 1, \ldots, J\}$, passing through the first point $(0, \log \mu_1(t))$. In other words, we deal with regressions of the form

$$(1.3) \qquad \log \mu_1(t) - \log \mu_i(t) = \alpha(t) \log i + \varepsilon_i(t), \qquad i = 2, \ldots, J$$

for fixed values of $t \geq 0$. Clearly, the resulting ordinary least squares estimator for the slope parameter corresponds to the slope of the linear part of the curve at time t in Figure 1. With a minor abuse of notation we will denote this estimator by $\alpha(t)$.

Our objective is to estimate the fluctuations of the above parameter when the spacing process is running close to its stationary law. We choose the Atlas model to have a specific sequence of drifts $\delta_1, \ldots, \delta_K$, although other values of the drift parameters can be easily substituted. We consider initial configurations the spacings of which are close to their unique invariant distribution found in [23]; see Lemma 3.1 below.

THEOREM 1.1. *Let $I = \{1, \ldots, K\}$, and consider the Atlas model in which $\delta_1 = \delta$ and $\delta_i = 0$ for all $i = 2, \ldots, K$. Define the initial spacings between the particles by*

$$(1.4) \qquad X_{(K)}(0) - X_{(K-i+1)}(0) = \frac{K}{\delta} \log i, \qquad i = 2, \ldots, K.$$

Assume that K is sufficiently large. Fix a $J < K/15$ and consider the process $\alpha(t)$, $t \geq 0$, of ordinary least squares estimators resulting from regressions in (1.3) for different values of t. Then the process $\alpha(t)$, $0 \leq t \leq \delta^{-2} K$ satisfies the following concentration of measure property. Let $\bar{\alpha} = \sup_{0 \leq s \leq \delta^{-2} K} [\alpha(s)]$.

Then there are constants $m_\alpha \in \mathbb{R}$ and $C > 0$ such that

$$\mathbb{P}(\bar{\alpha} < m_\alpha) \leq 1/2 + C e^{-K/C}$$

and for all positive r sufficiently large, one has

$$\mathbb{P}(\bar{\alpha} > m_\alpha + r\sqrt{K}) \leq 2 \exp\left(-\frac{r^2 \delta^2}{\mu C_\alpha}\right).$$

Hereby, μ is an absolute positive constant, and C_α is a positive constant depending on J and given by

$$C_\alpha(J) = J^3 \cdot \left(\frac{\sum_{i=1}^{J-1} \log(J!/i!)}{\sum_{i=2}^{J} (\log i)^2}\right)^2.$$

REMARK. The above result, although novel, does not capture fully the extent of concentration that is seen in the real world data (see Figure 1). We suspect that the reason for this is the empirically observed unequal diffusion coefficients of the ordered particles. See, for example, the discussion in [3] which mentions that the diffusion coefficient for the ranked particles decays linearly with increasing rank. Thus, particles at the top are more stable than the average which should lead to a higher concentration.

REMARK. Another obvious way to improve the bound is to use true stationarity. One can divide up a long time interval in small subintervals of appropriate size, use the above theorem on the small intervals and take a union bound. However, in this case it is not clear if the Gaussian concentration is preserved when the invariant distribution has exponential tails. However, we expect that variations of our method and the argument in this remark can lead to exponential concentration over much larger intervals.

Next, we set $I = \mathbb{N}$ in (1.1) and consider a sequence of drift coefficients $\delta_1, \delta_2, \ldots$ which satisfies

$$\text{(1.5)} \qquad\qquad \delta_M = \delta_{M+1} = \cdots$$

for some $M \in \mathbb{N}$. Under the assumption that the sequence of initial positions of the particles $X_1(0), X_2(0), \ldots$ is deterministic, nondecreasing and such that

$$\text{(1.6)} \qquad\qquad \liminf_{i \to \infty} \frac{X_i(0)}{i} > 0,$$

the system of stochastic differential equations (1.1) has a unique weak solution; see Proposition 3.1 in [27]. We let $L_{(i,i+1)}(t)$, $t \geq 0$, be the local time process at zero of the process $X_{(i+1)}(t) - X_{(i)}(t)$, $t \geq 0$, for each $i \in \mathbb{N}$ and are interested in the concentration properties of the vector-valued processes $(X_{(2)}(t) - X_{(1)}(t), \ldots, X_{(n)}(t) - X_{(n-1)}(t))$, $t \in [0, T]$, and $(L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t))$, $t \in [0, T]$, for arbitrary values of $n \in \mathbb{N}$ and $T \geq 0$, which we consider to be fixed from now on. For our main result on the latter we introduce the following assumption on the initial particle configuration $(X_1(0), X_2(0), \ldots)$.

ASSUMPTION 1.2. The sequence of initial positions of the particles is nondecreasing with probability one, and there exists a deterministic constant $c > 0$ such that

$$\text{(1.7)} \qquad\qquad X_k(0) - X_N(0) \geq c(k - N)$$

holds for all $k \geq N$ almost surely, where $N = \max(n, M)$.

We note that Assumption 1.2 ensures the existence of a unique weak solution for the system (1.1) by Proposition 3.1 in [27].

To state our second main theorem we define the norm

$$\text{(1.8)} \qquad\qquad \|f\|_{T,2} = \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \sup_{0 \leq t \leq T} f_i(t)^2 \right)^{1/2}$$

on the space $C([0, T], \mathbb{R}^{n-1})$ of continuous $\mathbb{R}^{n-1}$-valued functions on $[0, T]$ where f_i, $i = 1, \ldots, n-1$, are the component functions of f. Our result then reads as follows.

THEOREM 1.3. *Let Assumption 1.2 be satisfied with a constant $c > 0$. Moreover, let A be a measurable subset of $(C([0, T], \mathbb{R}^{n-1}), \|\cdot\|_{T,2})$ such that*

$$\text{(1.9)} \qquad \mathbb{P}\big(\big((L_{(i,i+1)}(t), i = 1, \ldots, n-1), t \in [0, T]\big) \in A\big) > \tfrac{1}{2}$$

and for any $r > 0$ set

$$\text{(1.10)} \qquad A_r = \left\{ h \in C([0, T], \mathbb{R}^{n-1}) \,\middle|\, \inf_{\tilde{h} \in A} \|h - \tilde{h}\|_{T,2} \leq r \right\}.$$

Then, there exists a constant $C > 0$ depending on c, $\Delta = \max_{j=1,\ldots,M-1} |\delta_j - \delta_M|$, M, n, T and the value of the left-hand side of (1.9) *such that for all $r > C$ it holds*

$$\mathbb{P}\big(\big((L_{(i,i+1)}(t), i = 1, \ldots, n-1), t \in [0, T]\big) \notin A_r\big) \leq C \exp\left(-r^{4/7} \cdot \frac{n^{2/7} c^{10/7}}{CT}\right).$$

Moreover, the same statement is true for the spacings $(X_{(i+1)} - X_{(i)}, i = 1, \ldots, n-1)$.

REMARK. The local time of collisions between two consecutive ordered particles, as considered in Theorem 1.3, is interesting both mathematically and in applications. Mathematically, say in the Atlas model, the local time (compared to the case $\delta_i = 0$, $i \in I$) measures the *push* felt by the various particles due to the drift at the bottom. Its significance in economic models is discussed in Section 13 of [14], which also mentions the somewhat surprising fact that these local times can indeed be measured from data.

The rest of the article is organized as follows. In the next section we recall some facts about the concentration of measure phenomenon and the Skorokhod problem in the orthant, and provide an upper bound on the Lipschitz constant for the Skorokhod map of interest in Lemma 2.4. The latter is the key to the proofs of the two main results. Its proof relies on the construction of the Skorokhod map by Harrison and Reiman [15] and applies to other Skorokhod problems in the orthant as well. In Section 3 we use a relation between transportation cost inequalities and the concentration of measure phenomenon to complete the proof of Theorem 1.1 and provide the remainder of the proof of Theorem 1.3 in Section 4.

2. Preliminaries.

2.1. *Some facts about measure concentration for stochastic processes.* Suppose $(\mathcal{X}, d)$ is a complete separable metric space equipped with the Borel σ-algebra. For a Borel subset A of $\mathcal{X}$ and a positive real number r, define the r-neighborhood of A by

$$A_r := \big\{x \in \mathcal{X} : d(x, A) \leq r\big\}.$$

We say that a probability measure μ on $(\mathcal{X}, d)$ has the *measure concentration* property if for any Borel subset A with $\mu(A) \geq 1/2$ the value of $\mu(A_r)$ is *very close* to one for large values of r. This closeness is usually expressed as a sub-Gaussian tail in terms of r.

The concentration of measure phenomenon has become one of the most important concepts in modern probability theory. For an excellent introduction to this area we refer the reader to the beautiful article by Talagrand [29]. A considerable effort has been spent by probabilists on identifying distributions that have the

measure concentration property. A (somewhat dated) survey can be found in the monograph by Ledoux [19].

One technique for proving the measure concentration property, originally proposed by Marton [20], involves the so-called transportation cost inequalities (TCI) that we describe below; see also Talagrand [30], Dembo [7], Dembo and Zeitouni [8]. Consider, as before, a complete separable metric space $(\mathcal{X}, d)$ endowed with its Borel σ-algebra. For a real number $p \geq 1$ and all probability measures P and Q on the latter space, define the pth Wasserstein distance

$$\mathcal{W}_p(P, Q) = \inf_{\pi}[Ed(X, X')^p]^{1/p},$$

where the infimum is taken over all couplings of a pair of random elements (X, X') such that the marginal law of X is P and that of X' is Q.

Next, fix a particular probability measure P. Suppose there is a constant $C > 0$ such that for all probability measures $Q \ll P$ we have

$$(2.1) \qquad\qquad \mathcal{W}_2(P, Q) \leq \sqrt{2CH(Q \mid P)},$$

where H refers to the relative entropy $H(Q \mid P) = E^Q \log(dQ/dP)$. In this case we say that P satisfies the *quadratic transportation cost inequality* (QTCI) with the constant C.

A function $f : \mathcal{X} \to \mathbb{R}$ will be called Lipschitz if there is a positive constant α for which

$$|f(x) - f(y)| \leq \alpha d(x, y), \qquad x, y \in \mathcal{X}.$$

The smallest such constant α is then referred to as the Lipschitz constant of f. We shall call a function 1-Lipschitz if α can be taken to be one. Let $\mathcal{L}$ denote the set of all 1-Lipschitz functions on $(\mathcal{X}, d)$. The (very short) proof of the following theorem can be found in Ledoux [19], page 118, and the original article by Marton [20].

THEOREM 2.1. *Suppose that P satisfies the QTCI with constant C. Then one has the following concentration estimates for all $r \geq 2\sqrt{2C \log 2}$:*

(i) *For any measurable set A such that $\mu(A) \geq 1/2$, it holds*

$$(2.2) \qquad\qquad \mu(A_r) \geq 1 - \exp(-r^2/8C).$$

(ii) *For any $f \in \mathcal{L}$, one has*

$$(2.3) \qquad\qquad P(x : |f(x) - m_f| \geq r) \leq 2e^{-r^2/8C},$$

where m_f is the median of f with respect to P.

In addition to Theorem 2.1, the following lemma will be useful in the later text. Its (short) proof can be found in Lemma 2.1 of [9].

LEMMA 2.2. *Suppose μ is a measure on a metric space (E, d_E) that satisfies the QTCI with a constant C. Let (F, d_F) be another metric space. If the map $\Psi : (E, d_E) \to (F, d_F)$ is Lipschitz, that is,*

$$d_F\big(\Psi(x), \Psi(y)\big) \leq \alpha d_E(x, y), \qquad x, y \in E$$

for some constant $\alpha > 0$, then $\tilde{\mu} = \mu \circ \Psi^{-1}$ satisfies the QTCI on (F, d_F) with the constant $C\alpha^2$.

In this article we are interested in the choice of $\mathcal{X} = C([0, T], \mathbb{R}^{K-1})$, the space of continuous maps from the interval $[0, T]$ to $\mathbb{R}^{K-1}$, where K is the number of interacting particles as before. The latter space is typically referred to as the *path space*. It will be endowed with different variants of the uniform metric which are described below. We shall use quadratic transportation cost inequalities satisfied by the laws of stochastic processes with continuous paths, especially multidimensional reflected Brownian motions.

2.2. *RBM and the Skorokhod map.* The reflected Brownian motions (RBMs) we are interested in have a constant drift vector, a constant diffusion matrix and are reflected whenever they hit the boundary of the positive orthant. On each face of the boundary of the latter the direction of reflection is constant. The theory of such processes is well developed. In particular, their existence and pathwise uniqueness follows from the existence of a deterministic transformation mapping Brownian paths to the corresponding reflected paths. This is the so-called Skorokhod map, whose one-dimensional version is due to Skorokhod. We lift the following description from the article by Harrison and Reiman [15]. For more details and generalizations see [10, 11] and [12].

Define $C([0, \infty), \mathbb{R}^{K-1})$ as the space of continuous functions $x : [0, \infty) \to \mathbb{R}^{K-1}$, endowed with the topology of the locally uniform convergence. For each such function x we denote its component functions by x_i for $i = 1, \ldots, K-1$. Let C_S denote the subset of functions in $C([0, \infty), \mathbb{R}^{K-1})$ for which $x_i(0) \geq 0$, $i = 1, \ldots, K-1$, and let $Q = (q_{ij})$ be a $(K-1) \times (K-1)$ matrix with nonnegative entries, zeros on the diagonal and spectral radius strictly less than one.

THEOREM 2.3 ([15]). *For each $x \in C_S$ there exists a unique pair of functions $y, z \in C([0, \infty), \mathbb{R}^{K-1})$ satisfying*

$$(2.4) \qquad z_j(t) = x_j(t) + y_j(t) - \sum_{i=1}^{K-1} q_{ij} y_i(t), \qquad t \geq 0,$$

$$(2.5) \qquad z_j(t) \geq 0, \qquad t \geq 0$$

for all $j = 1, \ldots, K-1$ and such that for every $i = 1, \ldots, K-1$, the function y_i is nondecreasing and increases only at those times t for which $z_i(t) = 0$.

Moreover, suppose that the matrix Q satisfies

$$(2.6) \qquad \|Q\|_{cs} := \max_{j=1,\ldots,K-1} \sum_{i=1}^{K-1} q_{ij} < 1.$$

Then y is the unique function in $C([0,\infty), \mathbb{R}^{K-1})$ that for all $t \geq 0$ satisfies the vector equation $y(t) = \sup_{0 \leq s \leq t}[y(s)Q - x(s)]_+$, and is given by the limit (in the locally uniform topology) of the following iterative scheme:

$$(2.7) \qquad y^{[0]}(t) \equiv 0,$$

$$(2.8) \qquad y^{[i+1]}(t) = \sup_{0 \leq s \leq t} \left(y^{[i]}(s)Q - x(s)\right)_+, \qquad t \geq 0.$$

Hereby, the supremum and the positive part are taken componentwise.

It will become apparent later that the matrix Q of our choice will be given by

$$Q^{(K-1)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Since $Q^{(K-1)}$ is a finite, irreducible, substochastic matrix, it is immediate (by adding an absorbing point) that the spectral norm of $Q^{(K-1)}$ is strictly less than one. As was shown in [15], this implies that the corresponding Skorokhod map, which transforms the input path x into the reflected path z (or, the path of the "local time" y), is Lipschitz with respect to a variant of the norm $\| \cdot \|_{T,2}$ defined in (1.8). We would like to obtain an explicit upper bound on its Lipschitz constant with respect to the norm $\| \cdot \|_{T,2}$.

Note that $Q^{(K-1)}$ does not satisfy assumption (2.6) and, hence, the iterative scheme of Theorem 2.3 cannot be applied directly to construct the corresponding Skorokhod map. The way to get around this is to define a $(K-1) \times (K-1)$ diagonal matrix

$$D^{(K-1)} = \begin{pmatrix} w^{(K-1)}\left(\frac{1}{K}\right) & 0 & \cdots & 0 \\ 0 & w^{(K-1)}\left(\frac{2}{K}\right) & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & w^{(K-1)}\left(\frac{K-1}{K}\right) \end{pmatrix}$$

with a strictly concave function $w^{(K-1)}: [0, 1] \to [0, \infty)$ such that $w^{(K-1)}(u) = 0$ if and only if $u \in \{0, 1\}$. Viewing the set of equations in (2.4) for different values of j as an equation between row vectors, and multiplying both sides of it by the matrix $[D^{(K-1)}]^{-1}$ from the right, we obtain the equations

$$(2.9) \qquad z_j'(t) = x_j'(t) + y_j'(t) - \sum_{i=1}^{K-1} r_{ij} y_i'(t), \qquad t \geq 0$$

for $j = 1, \ldots, K - 1$ where $x' = x[D^{(K-1)}]^{-1}$, $y' = y[D^{(K-1)}]^{-1}$, $z' = z[D^{(K-1)}]^{-1}$ and

$$(2.10) \qquad R^{(K-1)} = (r_{ij}) = D^{(K-1)} Q^{(K-1)} [D^{(K-1)}]^{-1}.$$

Note that the coordinates of x', y' and z' can be computed by a simple rescaling of the corresponding coordinates of x, y and z, respectively. Moreover, it holds

$$(2.11) \quad \|R^{(K-1)}\|_{cs} = \max_{l=1,\ldots,K-1} \frac{w^{(K-1)}((l-1)/K) + w^{(K-1)}((l+1)/K)}{2w^{(K-1)}(l/K)} < 1$$

due to the strict concavity of the function $w^{(K-1)}$. Thus the Skorokhod map corresponding to the matrix $R^{(K-1)}$ can be obtained using the iterative scheme of Theorem 2.3. It would be interesting to determine whether the iterative scheme of Theorem 2.3 converges in the case of $Q^{(K-1)}$ itself and, if so, to obtain a bound on the Lipschitz constant of the corresponding Skorokhod map directly in order to compare it with our bound below. However, at this point we are not able to resolve this question.

From now on we fix a terminal time $T > 0$ and equip the space $C([0, T], \mathbb{R}^{K-1})$ of continuous $\mathbb{R}^{K-1}$-valued functions on $[0, T]$ with the norm

$$(2.12) \qquad \|x\|_{T,2} := \left(\frac{1}{K-1} \sum_{i=1}^{K-1} \sup_{0 \leq t \leq T} x_i(t)^2 \right)^{1/2},$$

where x_i, $i = 1, \ldots, K - 1$, are the component functions of x. We write $\Phi_L^{Q^{(K-1)}}$ and $\Phi_R^{Q^{(K-1)}}$ for the maps that take a path $x \in C([0, T], \mathbb{R}^{K-1})$ to the local time path $y \in C([0, T], \mathbb{R}^{K-1})$ and the reflected path $z \in C([0, T], \mathbb{R}^{K-1})$, respectively, corresponding to the Skorokhod problem with reflection matrix $Q^{(K-1)}$ defined above. The following lemma is one of the crucial steps in the proofs of Theorems 1.1 and 1.3. It provides an upper bound on the Lipschitz constant of the Skorokhod map and is a significant improvement on an earlier attempt by Pal [22].

LEMMA 2.4. *For all natural numbers $K \geq 2$ the map $\Phi_L^{Q^{(K-1)}}$ is Lipschitz on $C([0, T], \mathbb{R}^{K-1})$ with respect to the norm $\|\cdot\|_{T,2}$ defined in (2.12). Moreover, its Lipschitz constant $\text{Lip}_L^{(K-1)}$ satisfies*

$$(2.13) \qquad \text{Lip}_L^{(K-1)} \leq 2 \cdot (K-1)^{5/2}$$

for all K large enough.

PROOF.   We fix a natural number K as in the statement of the lemma and will prove the inequality (2.13) for that value of K. From the considerations preceding the lemma we know that for each $g \in C([0, T], \mathbb{R}^{K-1})$ the map $\Phi_L^{R^{(K-1)}}$ corresponding to the Skorokhod problem with reflection matrix $R^{(K-1)}$, evaluated at g, is given by the limit of the iterative scheme of Theorem 2.3 with input g, restricted to the interval $[0, T]$. Now, let $\tilde{g}$ be another function in $C([0, T], \mathbb{R}^{K-1})$, for each $k \in \mathbb{N}$ define f^k and $\tilde{f}^k$ as the results of the kth step of the iterative scheme of Theorem 2.3 with inputs g and $\tilde{g}$, respectively, and set f and $\tilde{f}$ for $\Phi_L^{R^{(K-1)}}(g)$ and $\Phi_L^{R^{(K-1)}}(\tilde{g})$, respectively. Finally, define the norm $\|\cdot\|_{T,\max}$ on $C([0, T], \mathbb{R}^{K-1})$ by

$$(2.14) \qquad \|x\|_{T,\max} = \max_{i=1,\ldots,K-1} \sup_{0 \le t \le T} |x_i(t)|,$$

where x_i, $i = 1, \ldots, K - 1$, denote the component functions of a function $x \in C([0, T], \mathbb{R}^{K-1})$ as before. By applying the triangle inequality, the fact that the operation of taking the positive part is 1-Lipschitz and the definition of the norm $\|\cdot\|_{cs}$ in (2.6), one obtains the following chain of inequalities:

$$\|f^{k+1} - \tilde{f}^{k+1}\|_{T,\max}$$

$$= \max_{i=1,\ldots,K-1} \sup_{0 \le t \le T} \left| \sup_{0 \le s \le t} \left((f^k(s) R^{(K-1)})_i - g_i(s) \right)_+ \right.$$

$$\left. - \sup_{0 \le s \le t} \left((\tilde{f}^k(s) R^{(K-1)})_i - \tilde{g}_i(s) \right)_+ \right|$$

$$\le \max_{i=1,\ldots,K-1} \sup_{0 \le t \le T} \sup_{0 \le s \le t} \left| \left((f^k(s) R^{(K-1)})_i - g_i(s) \right)_+ \right.$$

$$\left. - \left((\tilde{f}^k(s) R^{(K-1)})_i - \tilde{g}_i(s) \right)_+ \right|$$

$$\le \max_{i=1,\ldots,K-1} \left(\sup_{0 \le t \le T} \left| (f^k(t) R^{(K-1)})_i - (\tilde{f}^k(t) R^{(K-1)})_i \right| \right.$$

$$\left. + \sup_{0 \le t \le T} \left| g_i(t) - \tilde{g}_i(t) \right| \right)$$

$$\le \|R^{(K-1)}\|_{cs} \cdot \max_{i=1,\ldots,K-1} \sup_{0 \le t \le T} |f_i^k(t) - \tilde{f}_i^k(t)| + \|g - \tilde{g}\|_{T,\max}.$$

Taking the limit $k \to \infty$ and rearranging terms we conclude

$$(2.15) \qquad \|f - \tilde{f}\|_{T,\max} \le \frac{1}{1 - \|R^{(K-1)}\|_{cs}} \cdot \|g - \tilde{g}\|_{T,\max}.$$

Recalling that the map $\Phi_L^{Q^{(K-1)}}$ can be obtained from the map $\Phi_L^{R^{(K-1)}}$ by a rescaling of the coordinates of $\mathbb{R}^{K-1}$ according to the matrix $D^{(K-1)}$ [see equation (2.9)], we deduce

$$(2.16) \qquad \|\Phi_L^{Q^{(K-1)}}(g) - \Phi_L^{Q^{(K-1)}}(\tilde{g})\|_{T,\max} \le \frac{\varrho}{1 - \|R^{(K-1)}\|_{cs}} \cdot \|g - \tilde{g}\|_{T,\max},$$

where

$$\varrho = \frac{\max_{l=1,\ldots,K-1} w^{(K-1)}(l/K)}{\min_{l=1,\ldots,K-1} w^{(K-1)}(l/K)}. \tag{2.17}$$

Thus, the obvious equivalence of norm inequalities between the norms $\|\cdot\|_{T,\max}$ and $\|\cdot\|_{T,2}$ on $C([0,T],\mathbb{R}^{K-1})$ show

$$\|\Phi_L^{Q^{(K-1)}}(g) - \Phi_L^{Q^{(K-1)}}(\tilde{g})\|_{T,2} \le \|\Phi_L^{Q^{(K-1)}}(g) - \Phi_L^{Q^{(K-1)}}(\tilde{g})\|_{T,\max}$$

$$\le \frac{\varrho}{1 - \|R^{(K-1)}\|_{cs}} \cdot \|g - \tilde{g}\|_{T,\max}$$

$$\le \frac{(K-1)^{1/2}\varrho}{1 - \|R^{(K-1)}\|_{cs}} \cdot \|g - \tilde{g}\|_{T,2}.$$

Since the functions g and $\tilde{g}$ were chosen arbitrarily in $C([0,T],\mathbb{R}^{K-1})$, we conclude

$$\mathrm{Lip}_L^{(K-1)} \le \frac{(K-1)^{1/2}\varrho}{1 - \|R^{(K-1)}\|_{cs}}. \tag{2.18}$$

To complete the proof, we consider a $K \ge 4$ and need to bound the right-hand side in (2.18) from above for a suitable function $w^{(K-1)}$ in the definition of the matrix $D^{(K-1)}$. We let $w^{(K-1)}(x) = 1 + x(1 - x)$ for $x \in [K^{-1}, 1 - K^{-1}]$, set $w^{(K-1)}(0) = w^{(K-1)}(1) = 0$ and interpolate linearly on $[0, K^{-1}]$ and $[1 - K^{-1}, 1]$. It is not hard to check that for this choice $\varrho = \frac{5}{4+4K^{-1}(1-K^{-1})}$ and $\|R^{(K-1)}\|_{cs} \le 1 - \frac{4}{5K^2}$ due to (2.11). Combining these with (2.18) and choosing K large enough, we end up with (2.13). $\quad\square$

3. Concentration of the shape of the market weights. Consider the particle system in (1.1) with $I = \{1, \ldots, K\}$ for some $K \in \mathbb{N}$ and a choice of constants $\delta_1, \ldots, \delta_K$ that satisfy the following condition. Setting $\bar{\delta} = \frac{1}{K} \sum_{i=1}^K \delta_i$, one has

$$\alpha_j := \sum_{i=1}^j (\bar{\delta} - \delta_{K-i+1}) > 0, \qquad j = 1, \ldots, K - 1. \tag{3.1}$$

In our analysis of the market weights we will use the following result from [23].

LEMMA 3.1. *Under condition* (3.1) *the process of spacings*

$$\begin{aligned}
\big(\xi_i(t) : i &= 1, \ldots, K - 1\big) \\
&:= \big(X_{(K-i+1)}(t) - X_{(K-i)}(t) : i = 1, \ldots, K - 1\big), \qquad t \ge 0,
\end{aligned} \tag{3.2}$$

has a unique stationary distribution which is that of independent exponential random variables with rates $2\alpha_i$, for $i = 1, \ldots, K - 1$. Moreover, the system of spacings is reversible at equilibrium.

A situation in which the above lemma applies is given by the *Atlas model* with $\delta_1 = \delta > 0$ and $\delta_i = 0$ for all $i = 2, \ldots, K$. In that case one easily computes

$$
(3.3) \qquad \alpha_j = \sum_{i=1}^{j} \left(\frac{\delta}{K} \right) = \frac{\delta j}{K}, \qquad j = 1, \ldots, K-1.
$$

Thus, under the stationary distribution we have

$$
\mathbb{E}[\xi_j(t)] = \frac{K}{2\delta j}, \qquad j = 1, \ldots, K-1,
$$

$$
\mathbb{E}[X_{(K-i+1)}(t) - X_{(K-j+1)}(t)] = \frac{K}{2\delta} \sum_{l=i}^{j-1} \frac{1}{l} \approx \frac{K}{2\delta} \log(j/i), \qquad 1 \le i < j \le K
$$

for all $t \ge 0$.

Next, consider the linear regression in (1.3). The corresponding ordinary least squares estimator for the slope parameter is given by the formula

$$
\alpha(t) = \frac{\sum_{i=2}^{J} (\log i)(\log \mu_1(t) - \log \mu_i(t))}{\sum_{i=2}^{J} \log^2 i}
$$

$$
(3.4) \qquad = \frac{\sum_{i=2}^{J} (\log i)(X_{(K)}(t) - X_{(K-i+1)}(t))}{\sum_{i=2}^{J} \log^2 i} = \frac{\sum_{i=2}^{J} (\log i) \sum_{j=1}^{i-1} \xi_j(t)}{\sum_{i=2}^{J} \log^2 i}
$$

$$
= \frac{\sum_{j=1}^{J-1} \xi_j(t) \sum_{i=j+1}^{J} \log i}{\sum_{i=2}^{J} \log^2 i} = \frac{\sum_{i=1}^{J-1} (\log J!/i!) \xi_i(t)}{\sum_{i=1}^{J-1} \log^2(i+1)}.
$$

We use this formula in the proof below.

PROOF OF THEOREM 1.1.   The proof is broken down into several steps.

Step (1) A quadratic transportation cost inequality for the process of spacings is derived in the Atlas model with K particles.

Step (2) Next, we assume K to be very large compared to J. We prove a localization lemma that shows that the process $\alpha(t)$, $0 \le t \le \delta^{-2} K$, is determined only by the particles corresponding to the $5J$ topmost indices with very high probability.

Step (3) Finally, we show that the law of the $5J$ particles in step 2 is approximately that of another rank-based process, so that we can use the estimates obtained in step 1, with K replaced by $5J$, to bound concentration of measure probabilities under the event constructed in step 2.

Step 1. Consider the process $(X_{(1)}(t), \ldots, X_{(K)}(t))$, $t \in [0, T]$, of ordered particles in the system (1.1) with $I = \{1, \ldots, K\}$. From Lemma 4 in [23] we know

that there exist i.i.d. standard Brownian motions $\beta_1, \ldots, \beta_K$ such that for all $i = 1, \ldots, K$ it holds

$$(3.5) \qquad dX_{(i)}(t) = \delta_i \, dt + d\beta_i(t) + \tfrac{1}{2} dL_{(i-1,i)}(t) - \tfrac{1}{2} dL_{(i,i+1)}(t),$$
$$t \in [0, T]$$

with $L_{i,i+1}(t)$, $t \in [0, T]$, being the local time process at zero of the process $X_{(i+1)}(t) - X_{(i)}(t)$, $t \in [0, T]$ for $i = 1, \ldots, K-1$ and the convention $L_{0,1}(t) = L_{K,K+1}(t) = 0$ for all $t \in [0, T]$.

Hence, for all $i = 1, \ldots, K-1$ one has the dynamics

$$d\big(X_{(i+1)}(t) - X_{(i)}(t)\big) = (\delta_{i+1} - \delta_i) \, dt + d\beta_{i+1}(t) - d\beta_i(t)$$
$$+ dL_{(i,i+1)}(t) - \tfrac{1}{2} dL_{(i+1,i+2)}(t) - \tfrac{1}{2} dL_{(i-1,i)}(t)$$

on $[0, T]$. In particular, we can conclude from this representation as in Section 2 of [23] that the process

$$\big(X_{(2)}(t) - X_{(1)}(t), \ldots, X_{(K)}(t) - X_{(K-1)}(t)\big), \qquad t \in [0, T],$$

is a reflected Brownian motion in the $(K-1)$-dimensional positive orthant with reflection matrix $Q^{(K-1)}$ in the sense of Section 1 in [15]. By reversing the labeling we see that the process $\xi(t) := (\xi_1(t), \ldots, \xi_{K-1}(t))$, $t \in [0, T]$, is also an RBM in the positive orthant with reflection matrix $Q^{(K-1)}$. Thus, the process ξ can be obtained as the image of the process $\gamma^*(t) := (\beta_{i+1}(t) + \delta_{i+1}t - \beta_i(t) - \delta_i t, i = 1, \ldots, K-1)$, $t \in [0, T]$, under the map $\Phi_R^{Q^{(K-1)}}$.

By Theorem 6 in [22] the process of independent Brownian motions $(\beta_1(t) + \delta_1 t, \ldots, \beta_K(t) + \delta_K t)$, $t \in [0, T]$, satisfies a QTCI with respect to the norm $\|\cdot\|_{T,2}$ with the constant $4K^{-1}T$. Moreover, the map that takes the vector of these Brownian motions to the process $\gamma^*(t)$, $t \in [0, T]$, is Lipschitz with respect to the norm $\|\cdot\|_{T,2}$ with Lipschitz constant $2\sqrt{K/(K-1)}$. Hence, by Lemma 2.2 the process $\gamma^*(t)$, $t \in [0, T]$, satisfies a QTCI with respect to the norm $\|\cdot\|_{T,2}$ with the constant

$$(3.6) \qquad C_K^*(0) := 16K^{-1}T \frac{K}{K-1} = 16(K-1)^{-1}T.$$

Now, we will use Lemma 2.4. Consider equation (2.4) with the matrix $Q^{(K-1)}$ for a fixed value of t and two different unconstrained processes x and $\tilde{x}$. Writing $y, \tilde{y}, z, \tilde{z}$ for $\Phi_L^{Q^{(K-1)}}(x)$, $\Phi_L^{Q^{(K-1)}}(\tilde{x})$, $\Phi_R^{Q^{(K-1)}}(x)$, $\Phi_R^{Q^{(K-1)}}(\tilde{x})$, respectively, we easily deduce

$$\big|z_j(t) - \tilde{z}_j(t)\big| \le \big|x_j(t) - \tilde{x}_j(t)\big|$$
$$+ \Big(\max_{i=1,\ldots,K-1} |y_i - \tilde{y}_i| \Big) \cdot \underbrace{\sum_{i=1}^{K-1} \big|(I^{(K-1)} - Q^{(K-1)})_{ij}\big|}_{\le 2},$$
$$t \in [0, T],$$

where $I^{(K-1)}$ is a $(K-1) \times (K-1)$ identity matrix. Hence,

$$(3.7) \qquad \|z - \tilde{z}\|_{T,\max} \leq \|x - \tilde{x}\|_{T,\max} + 2\|y - \tilde{y}\|_{T,\max}.$$

On the other hand, due to (2.16) and the bounds following it in the proof of Lemma 2.4 we get

$$(3.8) \qquad \|y - \tilde{y}\|_{T,\max} \leq 2(K-1)^2 \|x - \tilde{x}\|_{T,\max}$$

for all $K \geq 4$. Putting inequalities (3.7) and (3.8) together we obtain

$$(3.9) \qquad \begin{aligned} \|z - \tilde{z}\|_{T,\max} &\leq \left(1 + 4(K-1)^2\right) \cdot \|x - \tilde{x}\|_{T,\max} \\ &\leq 5(K-1)^2 \|x - \tilde{x}\|_{T,\max} \end{aligned}$$

for all $K \geq 4$. Recall that by applying the map $\Phi_R^{Q^{(K-1)}}$ to the paths of the process γ^* and by reversing the order of the coordinates thereafter, one gets the process ξ. Combining this observation with Lemma 2.2, (3.6) and (3.9), we see that the process $\xi(t)$, $t \in [0, T]$, satisfies a QTCI with respect to the norm $\|\cdot\|_{T,\max}$ with the constant

$$(3.10) \qquad C_K^* := 400(K-1)^3 T.$$

Now, we restrict ourselves to the first $(J-1)$ coordinates of the process ξ, since only those appear in (3.4). By Lemma 2.2 the vector-valued process $(\xi_1(t), \ldots, \xi_{J-1}(t))$, $t \in [0, T]$, also satisfies a QTCI with respect to the norm $\|\cdot\|_{T,\max}$ with the constant C_K^*.

Step 2. Consider the formula for $\alpha(t)$ in (3.4). The value of $\alpha(t)$ depends only on the top J spacings, whereby we have assumed that J is very much smaller than K. In this case, for a large enough m and with high probability, the top J processes during the time interval $[0, T]$ are identical to the top J processes among the processes which start off at the top $J + m$ positions at time zero. The following lemma makes this idea precise. $\qquad \square$

LEMMA 3.2. *Consider the particle system of Theorem* 1.1, *and for all* $m \in \mathbb{N}$ *define* σ_m *as the first time* t *at which, for some* $i \geq J + m$ *and some* $1 \leq j \leq J$, *it holds* $X_{K-i+1}(t) = X_{(K-j+1)}(t)$. *Then, there exists a uniform constant* $C > 0$ *such that for all* $K \geq C$, *we have*

$$(3.11) \qquad \mathbb{P}\left(\sigma_{4J+1} \leq \delta^{-2} K\right) \leq C J K^{1/2} e^{-K/C}.$$

PROOF. For any $T > 0$, the event $\{\sigma_m \leq T\}$ implies that for some $i \geq J + m$ and some $1 \leq j \leq J$, the processes X_{K-i+1} and X_{K-j+1} cross paths during the time interval $[0, T]$. Using the union bound and bounding the drift of the lower

particle by the constant δ, we have the following estimate:

$$\mathbb{P}(\sigma_m \leq T) \leq \sum_{i=J+m}^{K} \sum_{j=1}^{J} \mathbb{P}\Big(\sup_{0 \leq t \leq T} \big(W_{K-i+1}(t) - W_{K-j+1}(t)\big) $$

$$\geq -\delta T + X_{K-j+1}(0) - X_{K-i+1}(0)\Big)$$

(3.12)

$$\leq J \sum_{i=J+m}^{K} \mathbb{P}\Big(\sup_{0 \leq t \leq T} \big(W_{K-i+1}(t) - W_{K-J+1}(t)\big)$$

$$\geq -\delta T + X_{K-J+1}(0) - X_{K-i+1}(0)\Big).$$

Next, we use the fact that the supremum of a standard Brownian motion up to time T has the same law as the absolute value of a normal random variable with mean zero and variance T. Thus with a standard normal random variable Z, one has

$$\mathbb{P}\Big(\sup_{0 \leq t \leq T} \big(W_{K-i+1}(t) - W_{K-J+1}(t)\big) \geq -\delta T + X_{K-J+1}(0) - X_{K-i+1}(0)\Big)$$

$$= \mathbb{P}\Big(\sqrt{2T}|Z| \geq \frac{K}{\delta}\log(i/J) - \delta T \Big) = 2\overline{\Phi}\Big(\Big(\frac{K}{\delta\sqrt{2T}}\log(i/J) - \delta\sqrt{\frac{T}{2}}\Big)_+\Big).$$

Here, $\overline{\Phi}$ is the one minus the cumulative distribution function of a standard normal random variable.

Plugging this into (3.12), we get

$$(3.13) \qquad \mathbb{P}(\sigma_m \leq T) \leq 2J \sum_{i=J+m}^{K} \overline{\Phi}\Big(\Big(\frac{K}{\delta\sqrt{2T}}\log(i/J) - \delta\sqrt{\frac{T}{2}}\Big)_+\Big).$$

We note that for $m = 4J + 1$ and $T = \delta^{-2}K$ we may omit taking the positive part in the latter formula. Moreover, for this choice there exists a uniform constant $C > 0$ such that for all $K \geq C$ the latter upper bound can be estimated from above further by

$$2J \sum_{i=J+m}^{K} \overline{\Phi}\Big(\frac{K}{C\delta\sqrt{T}}\log\Big(\frac{i}{J}\Big)\Big)$$

$$\leq 2J \sum_{i=J+m}^{K} \frac{C\delta\sqrt{T}}{K}\log\Big(\frac{i}{J}\Big)^{-1}\exp\Big(-\frac{K^2}{C\delta^2 T}\log\Big(\frac{i}{J}\Big)^2\Big)$$

$$\leq 2J \sum_{i=J+m}^{K} \frac{C\delta\sqrt{T}}{K}\log\Big(\frac{i}{J}\Big)^{-1}\Big(\frac{i}{J}\Big)^{-K^2/(C\delta^2 T)} \leq CJ\delta\sqrt{T}\exp\Big(-\frac{K^2}{C\delta^2 T}\Big)$$

$$\leq CJK^{1/2}\exp\Big(-\frac{K}{C}\Big),$$

where we have been increasing the value of C suitably in every step. □

Step 3. Consider the particle system of Theorem 1.1. We proceed with another *localization* lemma.

LEMMA 3.3. *Define* $\tilde{\sigma}_m$ *to be the first time* t *at which, for some* $i \geq m$, *we have* $X_i(t) = X_{(1)}(t)$. *Then, there exists a uniform constant* $C > 0$ *such that*

$$\mathbb{P}(\tilde{\sigma}_{2K/3+1} \leq \delta^{-2}K) \leq CK^{1/2}e^{-K/C}.$$

PROOF. With $m = 2K/3 + 1$, and letting Z be a standard normal random variable and $\overline{\Phi}$ be one minus its cumulative distribution function as before, we have

$$\mathbb{P}(\tilde{\sigma}_m \leq \delta^{-2}K) \leq \sum_{i=m}^{K} \mathbb{P}\left(\sup_{0 \leq t \leq \delta^{-2}K} (W_1(t) - W_i(t)) \geq -\delta^{-1}K + X_i(0) - X_1(0)\right)$$

$$= \sum_{i=m}^{K} \mathbb{P}\left(\delta^{-1}\sqrt{2K}|Z| \geq \delta^{-1}K \log\left(\frac{K}{K-i+1}\right) - \delta^{-1}K\right)$$

$$= \sum_{i=m}^{K} 2\overline{\Phi}\left(\left(\sqrt{\frac{K}{2}}\log\left(\frac{K}{K-i+1}\right) - \sqrt{\frac{K}{2}}\right)_+\right)$$

$$= 2\sum_{i=1}^{K-m+1} \overline{\Phi}\left(\sqrt{\frac{K}{2}}\log\left(\frac{K}{i}\right) - \sqrt{\frac{K}{2}}\right).$$

Note that we could drop the positive part in the last identity due to our assumption $m = 2K/3 + 1$. Now, we estimate the latter upper bound further by $K\overline{\Phi}(K^{1/2}/C) \leq CK^{1/2}\exp(-K/C)$. This finishes the proof of the lemma. □

To complete the proof of Theorem 1.1, we recall from [23] that a weak solution for the Atlas model as in Theorem 1.1 can be obtained by the following application of Girsanov's Theorem. Let $Z_1, \ldots, Z_K$ be independent Brownian motions such that $Z_i(0) = X_i(0)$, $i = 1, \ldots, K$. Set $T = \delta^{-2}K$, and let Q^0 denote their joint law during the time interval $[0, T]$ on the canonical sample space of continuous $\mathbb{R}^K$-valued functions on $[0, T]$ with the usual Brownian filtration. Consider the martingale

$$M(t) = \sum_{i=1}^{K} \int_0^t 1_{\{Z_i(s)=Z_{(1)}(s)\}} dZ_i(s), \qquad t \geq 0.$$

Note that its quadratic variation at any fixed time $t \geq 0$ is given by

$$\langle M \rangle(t) = \left(\sum_{i=1}^{K} \int_0^t 1_{\{Z_i(s)=Z_{(1)}(s)\}}\right) ds = t,$$

since two independent Brownian particles can simultaneously be the leftmost ones only on a set of Lebesgue measure zero. Hence, by Lévy's characterization of Brownian motion, the process M is a standard Brownian motion under Q^0. If we now change the measure to Q^δ according to the formula

$$(3.14) \qquad \frac{dQ^\delta}{dQ^0} = \exp\left(\delta M(T) - \frac{\delta^2 T}{2}\right),$$

then, under Q^δ, the law of the process $(Z_1(t), \dots, Z_K(t))$, $t \in [0, T]$, is that of the Atlas model during the time interval $[0, T]$.

Let $\mathcal{F}_T$ denote the σ-algebra generated by the entire process $(Z_1(t), \dots, Z_K(t))$, $t \in [0, T]$. Clearly, one has the decomposition

$$\mathcal{F}_T = \mathcal{G}_T \vee \mathcal{H}_T,$$

where $\mathcal{G}_T$ is the σ-algebra generated by the top $5J$ indexed coordinate processes

$$(Z_{K-5J+1}(t), \dots, Z_K(t)), \qquad t \in [0, T],$$

$\mathcal{H}_T$ is the σ-algebra generated by the rest of the coordinate processes and $\vee$ refers to the smallest σ-algebra containing the two.

By our assumption, $5J < K/3$. Hence, the process $(Z_1(t), \dots, Z_{2K/3+1}(t))$, $t \in [0, T]$, is measurable with respect to $\mathcal{H}_T$. For any fixed $t \in [0, T]$ define

$$\widetilde{Z}_{(1)}(t) = \min_{i=1,\dots,2K/3+1} Z_i(t).$$

Then, the process $\widetilde{Z}_{(1)}(t)$, $t \in [0, T]$, is also measurable with respect to $\mathcal{H}_T$.

Now, consider an arbitrary $\mathcal{G}_T$-measurable function F such that $0 \le F \le 1$. To simplify the notation, we will denote expectations with respect to the measures Q and Q^δ by $Q(\cdot)$ and $Q^\delta(\cdot)$, respectively. By Lemma 3.3 and the change of measure formula (3.14) we have

$$(3.15) \qquad
\begin{aligned}
Q^\delta(F) &\le Q^\delta(F1_{\{\tilde\sigma_{2K/3+1} > T\}}) + Q^\delta(1_{\{\tilde\sigma_{2K/3+1} \le T\}}) \\
&\le Q^0(Fe^{\delta M(T) - \delta^2 T/2} 1_{\{\tilde\sigma_{2K/3+1} > T\}}) + CK^{1/2}e^{-K/C}.
\end{aligned}$$

Now, on the set $\{\tilde\sigma_{2K/3+1} > T\}$, the process $M(t)$, $t \in [0, T]$, is identical to the process $\widetilde{M}(t)$, $t \in [0, T]$, where the latter is defined by

$$\widetilde{M}(t) = \sum_{i=1}^{2K/3+1} \int_0^t 1_{\{Z_i(s) = \widetilde{Z}_{(1)}(s)\}} dZ_i(s), \qquad t \in [0, T].$$

Hence, it holds

$$
\begin{aligned}
Q^0(Fe^{\delta M(T) - \delta^2 T/2} 1_{\{\tilde\sigma_{2K/3+1} > T\}}) &= Q^0(Fe^{\delta \widetilde{M}(T) - \delta^2 T/2} 1_{\{\tilde\sigma_{2K/3+1} > T\}}) \\
&\le Q^0(Fe^{\delta \widetilde{M}(T) - \delta^2 T/2}).
\end{aligned}
$$

Note that $\widetilde{M}(T)$ is measurable with respect to $\mathcal{H}_T$ while F is measurable with respect to $\mathcal{G}_T$. Moreover, under Q^0, the σ-algebras $\mathcal{G}_T$ and $\mathcal{H}_T$ are independent of each other. Using this observation and $\langle \widetilde{M} \rangle(T) = T$, we obtain

$$Q^0\big(F e^{\delta \widetilde{M}(T) - \delta^2 T/2}\big) = Q^0(F) Q^0\big(e^{\delta \widetilde{M}(T) - \delta^2 T/2}\big) = Q^0(F).$$

Combining this with the previous inequality and (3.15), we get

$$(3.16) \qquad\qquad Q^\delta(F) \le Q^0(F) + C K^{1/2} e^{-K/C}.$$

For the rest of the argument we will assume that the particles indexed by the top $5J$ indices are independent Brownian motions starting from their respective initial conditions, the idea being that all probabilities under the actual measure Q^δ can be bounded from above as in (3.16).

Back to the K particles Atlas model, consider the event $\{\sigma_{4J+1} > \delta^{-2} K\}$ as in Lemma 3.2. On this event, during the time interval $[0, \delta^{-2} K]$ the top J processes are identical to the top J processes among those that started at the top $5J$ positions at time zero. Let $Y_1', \ldots, Y_{5J}'$ be the ranked processes $X_{K-5J+1}, \ldots, X_K$ in the increasing order. Also, set

$$\alpha'(t) = \frac{\sum_{i=2}^{J} (\log i)(Y_{5J}'(t) - Y_{5J+1-i}'(t))}{\sum_{i=2}^{J} \log^2 i}, \qquad t \ge 0.$$

Then, by Lemma 3.2,

$$(3.17) \qquad \mathbb{P}\big(\alpha(t) = \alpha'(t), \text{ for all } 0 \le t \le \delta^{-2} K\big) \ge 1 - C J K^{1/2} e^{-K/C}.$$

We now prove a concentration of measure property for $\alpha'(t)$, $t \in [0, \delta^{-2} K]$. Relying on (3.16), we can assume first that $X_{K-5J+1}, \ldots, X_K$ evolve according to independent standard Brownian motions. If we let

$$(\xi_i'(t) : i = 1, \ldots, J-1) = (Y_{5J-i+1}'(t) - Y_{5J-i}'(t) : i = 1, \ldots, J-1),$$

$$t \in [0, \delta^{-2} K],$$

then $(\xi_1'(t), \ldots, \xi_{J-1}'(t))$, $t \in [0, \delta^{-2} K]$ can be viewed as a vector of $(J-1)$ component processes of a reflected Brownian motion in the $(5J)$-dimensional positive orthant with zero drift vector and a constant diffusion matrix.

By (3.10) and the paragraph following it, we know that the process $\xi'(t)$, $t \in [0, \delta^{-2} K]$, satisfies a QTCI with respect to the norm $\| \cdot \|_{\delta^{-2} K, \max}$ with the constant $\mu' J^3 \delta^{-2} K$. Here, we have abbreviated $5^3 \times 400$ by μ'.

Note that for any fixed $t \in [0, \delta^{-2} K]$, $\alpha'(t)$ can be written in terms of $\xi'(t)$ as

$$\alpha'(t) = \frac{\sum_{i=1}^{J-1} \xi_i'(t) \sum_{j=i+1}^{J} \log j}{\sum_{i=2}^{J} \log^2 i} = \frac{\sum_{i=1}^{J-1} \xi_i'(t) \log(J!/i!)}{\sum_{i=2}^{J} \log^2 i}.$$

Hence, the function that takes the paths of $\xi'(t)$, $t \in [0, \delta^{-2}K]$, to the paths of $\alpha'(t)$, $t \in [0, \delta^{-2}K]$, is Lipschitz with respect to $\|\cdot\|_{\delta^{-2}K, \max}$ norms with the Lipschitz constant

$$\widetilde{C}_\alpha(J) = \frac{\sum_{i=1}^{J-1} \log(J!/i!)}{\sum_{i=2}^{J} \log^2 i}.$$

It follows that the random variable $\bar{\alpha}' := \sup_{0 \le t \le \delta^{-2}K} \alpha'(t)$ can be viewed as the image of $\xi'(t)$, $t \in [0, \delta^{-2}K]$, under a Lipschitz function with the Lipschitz constant $\widetilde{C}_\alpha(J)$. Thus, by Lemma 2.2 its law satisfies a QTCI with the constant

$$\widetilde{C}_\alpha(J)^2 \mu' J^3 \delta^{-2} K.$$

Let m_α be the median of $\bar{\alpha}'$ under Q^0. Setting $\mu = 8\mu'$, we deduce from Theorem 2.1 and (3.16),

$$Q^\delta(\bar{\alpha}' > m_\alpha + r\sqrt{K}) \le \exp\left(-\frac{r^2\delta^2}{\mu \widetilde{C}_\alpha(J)^2 J^3}\right) + CK^{1/2}e^{-K/C}$$

for all r greater than a constant depending only on J and δ. Combining this estimate with (3.17), bounding J by K and suitably increasing the value of C we get

$$Q^\delta(\bar{\alpha} > m_\alpha + r\sqrt{K}) \le \exp\left(-\frac{r^2\delta^2}{\mu \widetilde{C}_\alpha(J)^2 J^3}\right) + Ce^{-K/C}.$$

The observation that the sum of the last two summands is smaller than twice the first summand for all sufficiently large K yields

$$Q^\delta(\bar{\alpha} > m_\alpha + r\sqrt{K}) \le 2\exp\left(-\frac{r^2\delta^2}{\mu \widetilde{C}_\alpha(J)^2 J^3}\right).$$

We note that m_α is not the median of $\bar{\alpha}$ under Q^δ. However, by (3.16) and (3.17) one has

$$Q^\delta(\bar{\alpha} < m_\alpha) \le Q^0(\bar{\alpha} < m_\alpha) + Ce^{-K/C} \le Q^0(\bar{\alpha}' < m_\alpha) + Ce^{-K/C}$$
$$\le 1/2 + Ce^{-K/C},$$

where we have been increasing the value of C suitably in every step. This completes the proof of Theorem 1.1.

4. The infinite rank-based system. This section is devoted to the proof of Theorem 1.3. The first step in the proof is to understand the dynamics of the process $(X_{(1)}(t), \ldots, X_{(n)}(t))$, $t \in [0, T]$, of the n leftmost particles in the particle system of Theorem 1.3.

LEMMA 4.1. *There exist stopping times $0 = \tau_N \le \tau_{N+1} \le \cdots$ such that the following are true*:

(a) $\lim_{m\to\infty}\tau_{N+m}=\infty$ with probability one.

(b) For each $m\in\mathbb{N}$ there exists a system of i.i.d. standard Brownian motions $\beta_1^{(m)},\dots,\beta_n^{(m)}$ such that for all $i=1,\dots,n$ one has the dynamics

$$dX_{(i)}(t\wedge\tau_{N+m})=1_{\{\tau_{N+m}\geq t\}}\delta_i\,dt+d\beta_i^{(m)}(t\wedge\tau_{N+m})$$
$$+\tfrac{1}{2}\,dL_{(i-1,i)}(t\wedge\tau_{N+m})-\tfrac{1}{2}\,dL_{(i,i+1)}(t\wedge\tau_{N+m})$$

on $[0,T]$. Hereby, $a\wedge b$ denotes $\min(a,b)$ for any two real numbers a,b.

PROOF. We define inductively the sets $\Lambda_N\subset\Lambda_{N+1}\subset\cdots$ and the stopping times $0=\tau_N\leq\tau_{N+1}\leq\cdots$ by

(4.1) $\qquad \Lambda_{N+m}=\big\{k\geq 1|\exists 1\leq i\leq N,0\leq s\leq\tau_{N+m}:X_k(s)=X_{(i)}(s)\big\},$

(4.2) $\quad \tau_{N+m+1}=\inf\big\{s\geq\tau_{N+m}|\exists 1\leq i\leq N,k\notin\Lambda_{N+m}:X_k(s)=X_{(i)}(s)\big\}$

for all $m=0,1,\dots$. The proof of Proposition 3.1 in [27] shows that, with probability one, it holds $\lim_{m\to\infty}\tau_{N+m}=\infty$ and the sets Λ_{N+m} are finite for all $m=0,1,\dots$. Moreover, the same proof implies that for each such number m the paths of the process $X_{(1)}(t\wedge\tau_{N+m}),\dots,X_{(n)}(t\wedge\tau_{N+m})$, $t\in[0,T]$, are given by the paths of the n leftmost particles in a particle system as in (1.1) with $I=\{1,\dots,|\Lambda_{N+m}|\}$, which is stopped at time τ_{N+m}. Hence, by Lemma 4 in [23] we conclude that assertion (b) of the lemma is true for our choice of the stopping times $0=\tau_N\leq\tau_{N+1}\leq\cdots$. $\square$

Next, fix a $K\in\mathbb{N}$ and let $X'_{(1)},\dots,X'_{(K)}$ be the ranked particles in the system (1.1) with $I=\{1,\dots,K\}$. Also, let $L'_{(1,2)},\dots,L'_{(K-1,K)}$ be the local time processes at zero of the spacings processes in that system. Recall the definition of the norm $\|\cdot\|_{T,2}$ in (1.8). From Lemma 2.4 we can deduce the following concentration of measure property of the finite particle system.

COROLLARY 4.2. Let A be a measurable subset of $(C([0,T],\mathbb{R}^{n-1}),\|\cdot\|_{T,2})$ such that

(4.3) $\qquad\qquad \mathbb{P}\big(((L'_{(1,2)}(t),\dots,L'_{(n-1,n)}(t)),t\in[0,T])\in A\big)\geq\tfrac{1}{2}$

and for any $r>0$ set

(4.4) $\qquad\qquad A_r=\Big\{h\in C([0,T],\mathbb{R}^{n-1})|\inf_{\tilde h\in A}\|\tilde h-h\|_{T,2}\leq r\Big\}.$

Then, there exists a constant $C>0$, so that for all $r\geq C\frac{K^{5/2}T^{1/2}}{n^{1/2}}$ it holds

$$\mathbb{P}\big(((L'_{(1,2)}(t),\dots,L'_{(n-1,n)}(t)),t\in[0,T])\notin A_r\big)\leq\exp\Big(-\frac{r^2 n}{CK^5 T}\Big).$$

Moreover, the same statement is true for the spacings $(X'_{(i+1)}-X'_{(i)})$, $i=1,\dots,n-1$.

PROOF. From the considerations in step 1 of the proof of Theorem 1.1 we see that the process

$$(4.5) \qquad \left(L'_{(1,2)}(t), \ldots, L'_{(n-1,n)}(t)\right), \qquad t \in [0, T],$$

can be obtained by applying $\Phi_L^{Q^{(K-1)}}$ and then the canonical projection of $C([0, T], \mathbb{R}^{K-1})$ onto $C([0, T], \mathbb{R}^{n-1})$ to the process

$$(4.6) \qquad \begin{aligned} &\left((\delta_2 - \delta_1)t + \beta'_2(t) - \beta'_1(t), \ldots, (\delta_K - \delta_{K-1})t + \beta'_K(t) - \beta'_{K-1}(t)\right), \\ &\hspace{8cm} t \in [0, T]. \end{aligned}$$

Hereby, the Brownian motions $\beta'_1, \ldots, \beta'_K$ are defined analogously to the Brownian motions $\beta_1, \ldots, \beta_K$ in the proof of Theorem 1.1. There, we have seen that the process in (4.6) satisfies a QTCI with respect to the norm $\|\cdot\|_{T,2}$ with a constant of the form CT/K. Combining Lemmas 2.2 and 2.4, we conclude that a QTCI with a constant of the form

$$(4.7) \qquad C\left(\sqrt{\frac{K}{n}}\right)^2 \cdot (K^{5/2})^2 \cdot \frac{T}{K} = C\frac{K^5 T}{n}$$

applies to the process in (4.5) with respect to the norm $\|\cdot\|_{T,2}$. Hence, from Theorem 2.1 we obtain the concentration of measure result for the local times.

To show the corresponding result for the spacings, we recall from step (1) of the proof of Theorem 1.1 that the process

$$(4.8) \qquad \left(X'_{(2)}(t) - X'_{(1)}(t), \ldots, X'_{(n)}(t) - X'_{(n-1)}(t)\right), \qquad t \in [0, T],$$

is the image of the process in (4.6) under the successive application of the map $\Phi_R^{Q^{(K-1)}}$ and the canonical projection of $C([0, T], \mathbb{R}^{K-1})$ onto $C([0, T], \mathbb{R}^{n-1})$.

Moreover, we can rewrite (2.4) as

$$(4.9) \qquad \begin{aligned} &\left(\Phi_R^{Q^{(K-1)}}(h)\right)(t) = h(t) + \left(I^{(K-1)} - Q^{(K-1)}\right)\left(\left(\Phi_L^{Q^{(K-1)}}(h)\right)(t)\right), \\ &\hspace{9cm} t \in [0, T] \end{aligned}$$

for all $h \in C([0, T], \mathbb{R}^{K-1})$, where $I^{(K-1)}$ is the $(K-1) \times (K-1)$ identity matrix. It follows that for all $h_1, h_2 \in C([0, T], \mathbb{R}^{K-1})$, one has the estimates

$$\begin{aligned} &\left\|\left(\Phi_R^{Q^{(K-1)}}(h_2)\right) - \left(\Phi_R^{Q^{(K-1)}}(h_1)\right)\right\|_{T,2} \\ &\qquad \leq \|h_2 - h_1\|_{T,2} \\ &\qquad\quad + \left\|\left(I^{(K-1)} - Q^{(K-1)}\right)\left(\left(\Phi_L^{Q^{(K-1)}}(h_2)\right)(\cdot) - \left(\Phi_L^{Q^{(K-1)}}(h_1)\right)(\cdot)\right)\right\|_{T,2} \\ &\qquad \leq \|h_2 - h_1\|_{T,2} + \frac{3\sqrt{2}}{2} \cdot \left\|\left(\Phi_L^{Q^{(K-1)}}(h_2)\right) - \left(\Phi_L^{Q^{(K-1)}}(h_1)\right)\right\|_{T,2} \\ &\qquad \leq \left(1 + 3\sqrt{2}(K-1)^{5/2}\right)\|h_2 - h_1\|_{T,2}. \end{aligned}$$

In the second inequality we have combined the fact that the matrix $I^{(K-1)} - Q^{(K-1)}$ is tridiagonal with the elementary inequality $(a_1 + a_2 + a_3)^2 \leq 3(a_1^2 + a_2^2 + a_3^2)$, $a_1, a_2, a_3 \in \mathbb{R}$. The third inequality is a consequence of Lemma 2.4.

Now, it follows from Lemma 2.2 that the process in (4.8) satisfies a QTCI with respect to the norm $\| \cdot \|_{T,2}$ with the constant

$$(4.10) \qquad \left(\sqrt{\frac{K-1}{n-1}} \right)^2 \cdot (1 + 3\sqrt{2}(K-1)^{5/2})^2 \cdot \frac{16T}{K-1} \leq C \frac{K^5 T}{n}.$$

The result of the corollary for the spacings is a consequence of this and Theorem 2.1. $\square$

The last ingredient in the proof of Theorem 1.3 is an estimate on how fast the stopping times $0 = \tau_N \leq \tau_{N+1} \leq \cdots$ in the proof of Lemma 4.1 grow to infinity in terms of the initial positions of the particles.

LEMMA 4.3. *Let the Assumption 1.2 be satisfied with a constant $c > 0$. Then for all natural numbers $m \geq \frac{\max_{j=1,\ldots,M-1} |\delta_j - \delta_M| T}{c} + 1$ $(= \frac{\Delta T}{c} + 1)$ one has the inequality*

$$(4.11) \qquad \mathbb{P}(\tau_{N+m} \leq T) \leq \frac{NT}{c(cm - c - \Delta T)} \cdot \exp\left(-\frac{1}{2T}(cm - c - \Delta T)^2 \right).$$

In particular, there exists a constant $C(c, M, n, T, \Delta) > 0$ independent of m such that

$$(4.12) \qquad \mathbb{P}(\tau_{N+m} \leq T) \leq C(c, M, n, T, \Delta) e^{-(c^2/(3T))m^2}, \qquad m \in \mathbb{N}.$$

PROOF. We fix a natural number m as in the first statement of the lemma and note that on the event $\{\tau_{N+m} \leq T\}$ there exist numbers $1 \leq i \leq N$ and $j \geq N + m$ such that the particle, which was the ith from the left in the initial particle configuration, appears on the right or at the same position as the particle, which was the jth from the left in the initial particle configuration, at a time $t \in [0, T]$. Using this observation, the union bound and the definition of Δ (see the statement of Theorem 1.3), one has the chain of inequalities

$$\mathbb{P}(\tau_{N+m} \leq T) \leq \sum_{i=1}^{N} \sum_{j=N+m}^{\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq T} (W_i(t) - W_j(t)) \geq -\Delta T + X_j(0) - X_i(0) \Big)$$

$$\leq N \sum_{j=N+m}^{\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq T} (W_1(t) - W_j(t)) \geq -\Delta T + X_j(0) - X_N(0) \Big).$$

From Bernstein's inequality for Brownian motion (see page 145 in [24]), Assumption 1.2 and the assumption $m \geq \frac{\Delta c}{T} + 1$ it follows that the latter expression

can be bounded further by

$$N \sum_{j=N+m}^{\infty} \exp\left(-\frac{(X_j(0) - X_N(0) - \Delta T)^2}{2T}\right)$$

$$\leq N \sum_{k=m}^{\infty} \exp\left(-\frac{(ck - \Delta T)^2}{2T}\right) = N \sum_{k=m}^{\infty} \exp\left(-\frac{c^2}{2T}\left(k - \frac{\Delta T}{c}\right)^2\right)$$

$$\leq N \int_{m-1}^{\infty} \exp\left(-\frac{c^2}{2T}\left(y - \frac{\Delta T}{c}\right)^2\right) dy.$$

Next, we note that the latter integral is equal to the probability that a standard normal random variable exceeds $\frac{m-1-(\Delta T/c)}{\sqrt{T}/c}$ multiplied by $\sqrt{2\pi(T/c^2)}$. Using this and the standard estimate

(4.13) $$\int_y^{\infty} e^{-z^2/2} dz \leq \frac{1}{y} e^{-y^2/2}, \qquad y > 0,$$

one ends up with the first statement of the lemma.

Finally, to see (4.12), it suffices to observe that the argument of the exponential function on the right-hand side of inequality (4.11) is a quadratic polynomial in m, in which the coefficient of m^2 is given by $\frac{c^2}{2T}$. $\square$

We can now prove the following refined version of Theorem 1.3.

PROPOSITION 4.4. *Let the sets A and A_r, $r > 0$, be defined as in Theorem 1.3, and let the constant $C(c, M, n, T, \Delta)$ be as in Lemma 4.3. Moreover, let $m_1 \in \mathbb{N}$ be such that for all natural numbers $m \geq m_1$ the value of $C(c, M, n, T, \Delta)e^{-(c^2/(3T))m^2}$ is less or equal to the difference between the left-hand and the right-hand side of inequality (1.9). Then there is a uniform constant $C > 0$ such that the following is true. If one defines C_1 as the smallest positive real number such that*

(4.14) $$\frac{n^{1/7}C_1^{2/7}}{c^{2/7}} \geq Cm_1 \quad and \quad \forall r \geq C_1 : r \geq C\frac{(N + m(r) - 1)^{5/2}T^{1/2}}{n^{1/2}}$$

with $m(r) = \frac{n^{1/7}r^{2/7}}{c^{2/7}}$, then there exists a constant $C_2 > 0$ depending on c, Δ, M, n, T and the value on the left-hand side of (1.9) such that for all $r \geq C_1$ it holds

$$\mathbb{P}\big(\big((L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)), t \in [0, T]\big) \notin A_r\big) \leq C_2 \exp\left(-r^{4/7} \cdot \frac{n^{2/7}c^{10/7}}{CT}\right).$$

Moreover, the same statement is true for the spacings $(X_{(i+1)} - X_{(i)}, i = 1, \ldots, n-1)$.

PROOF.   We only provide the proof for the local times, since the proof for the spacings is the same. To this end, we fix an r as in the statement of the proposition, let $C > 0$ be a constant whose value will be chosen later, and set

$$(4.15) \qquad \tilde{m}_1 = \frac{1}{C} \frac{n^{1/7} r^{2/7}}{c^{2/7}}, \qquad K_1 = N + \tilde{m}_1.$$

We assume from now on that r is such that $\tilde{m}_1$ is an integer. If this is not the case, one merely needs to replace $\tilde{m}_1$ by the smallest integer which is larger than $\tilde{m}_1$. Moreover, we note that the inequality $r_1 \geq C_1$ and the first inequality in (4.14) imply $\tilde{m}_1 \geq m_1$.

Next, we recall the definition of the stopping time τ_{K_1} and observe

$$\mathbb{P}\big(\big((L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)\big), t \in [0, T]\big) \notin A_r\big)$$
$$\leq \mathbb{P}\big(\big((L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)\big), t \in [0, T]\big) \notin A_r, \tau_{K_1} \geq T\big)$$
$$+ \mathbb{P}(\tau_{N+\tilde{m}_1} \leq T)$$
$$\leq \mathbb{P}\big(\big((L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)\big), t \in [0, T]\big) \notin A_r, \tau_{K_1} \geq T\big)$$
$$+ C(c, M, n, T, \Delta) e^{-(c^2/(3T))\tilde{m}_1^2},$$

where the last inequality is a consequence of Lemma 4.3.

To obtain an upper bound on the first summand, which we call term (*), we remark first that for all triples of indices $1 \leq i < j < k$ it holds

$$(4.16) \qquad \mathbb{P}\big(\exists t \in [0, T] : X_{(i)}(t) = X_{(j)}(t) = X_{(k)}(t)\big) = 0.$$

Indeed, arguing as in the proof of Lemma 4.1, but replacing N by $\max(k, M)$, we deduce the existence of stopping times $0 = \tilde{\tau}_{\max(k,M)} \leq \tilde{\tau}_{\max(k,M)+1} \leq \cdots$ tending to infinity almost surely and such that for each $l \in \mathbb{N}$ the dynamics of the k leftmost particles, stopped at $\tilde{\tau}_{\max(k,M)+l}$, is given by the dynamics of the k leftmost particles in a finite particle system as in (1.1), stopped at $\tilde{\tau}_{\max(k,M)+l}$. Hence, (4.16) is a consequence of the considerations in Section 2.2 of [16], Proposition 1 in [16] and the fact that a countable union of $\mathbb{P}$-null sets is $\mathbb{P}$-null set.

Next, we recall from the proof of Lemma 4.1 that for each $m \in \mathbb{N}$ the paths of the process $X_{(1)}(t \wedge \tau_{N+m}), \ldots, X_{(n)}(t \wedge \tau_{N+m})$, $t \in [0, T]$, are the paths of the n leftmost particles in a particle system as in (1.1) with $|\Lambda_{N+m}|$ particles, stopped at τ_{N+m}. Moreover, (4.16) shows that $|\Lambda_{N+m}| = N + m$. Thus, on the event $\{\tau_{K_1} \geq T\}$ the paths of the process

$$\big(L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)\big), \qquad t \in [0, T],$$

can be written as the composition of the map $\Phi_L^{Q^{(K_1-1)}}$ with the canonical projection of $C([0, T], \mathbb{R}^{K_1-1})$ onto $C([0, T], \mathbb{R}^{n-1})$ applied to the paths of the process

$$\big((\delta_2 - \delta_1)t + \beta_2^{(\tilde{m}_1)}(t) - \beta_1^{(\tilde{m}_1)}(t), \ldots, (\delta_{K_1} - \delta_{K_1-1})t + \beta_{K_1}^{(\tilde{m}_1)}(t) - \beta_{K_1-1}^{(\tilde{m}_1)}(t)\big),$$
$$t \in [0, T],$$

where $\beta_1^{(\tilde{m}_1)}, \ldots, \beta_{K_1}^{(\tilde{m}_1)}$ are i.i.d. standard Brownian motions defined in Lemma 4.1. Hence, following the proof of the first statement of Corollary 4.2, and using the second inequality in (4.14), we conclude that term (*) is bounded from above by $\exp(-r^2 \frac{n}{C K_1^5 T})$ for a $C > 0$ large enough.

All in all, we have shown that $\mathbb{P}(((L_{(1,2)}(t), \ldots, L_{(n-1,n)}(t)), t \in [0, T]) \notin A_r)$ is bounded from above by

$$\exp\left(-r^2 \frac{n}{C K_1^5 T}\right) + C(c, M, n, T, \Delta) e^{-(c^2/(3T))\tilde{m}_1^2}.$$

Plugging in the values of $\tilde{m}_1$ and K_1 one observes that the leading order term in the variable r is the same for both exponents on the right-hand side of the latter inequality and is of the form $-r^{4/7} \cdot \frac{n^{2/7} c^{10/7}}{CT}$. This yields the proposition and completes the proof of Theorem 1.3. $\square$

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