

# ON THE STABILITY OF SOME CONTROLLED MARKOV CHAINS AND ITS APPLICATIONS TO STOCHASTIC APPROXIMATION WITH MARKOVIAN DYNAMIC

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We develop a practical approach to establish the stability, that is, the recurrence in a given set, of a large class of controlled Markov chains. These processes arise in various areas of applied science and encompass important numerical methods. We show in particular how individual Lyapunov functions and associated drift conditions for the parametrized family of Markov transition probabilities and the parameter update can be combined to form Lyapunov functions for the joint process, leading to the proof of the desired stability property. Of particular interest is the fact that the approach applies even in situations where the two components of the process present a time-scale separation, which is a crucial feature of practical situations. We then move on to show how such a recurrence property can be used in the context of stochastic approximation in order to prove the convergence of the parameter sequence, including in the situation where the so-called stepsize is adaptively tuned. We finally show that the results apply to various algorithms of interest in computational statistics and cognate areas.

**1. Introduction: Recurrence of controlled MC and compound drifts.** The class of controlled Markov chain processes underpins numerous models or algorithms encountered in various areas of engineering or science (e.g., control, EM algorithm, adaptive MCMC). Consider the space  $(X, \mathcal{B}(X))$  where  $X \subset \mathbb{R}^{n_x}$  for some  $n_x \geq 1$ , a parametrized family of Markov transition probabilities  $\{P_\theta, \theta \in \Theta\}$  (for some set  $\Theta \subset \mathbb{R}^{n_\theta}$ ) such that for any  $\theta, x \in \Theta \times X$ ,  $P_\theta(x, \cdot)$  is a probability distribution on  $(X, \mathcal{B}(X))$ . The class of controlled Markov chains we consider in this paper consists of the class of processes defined on  $((\Theta \times X)^\mathbb{N}, (\mathcal{B}(\Theta) \otimes \mathcal{B}(X))^{\otimes \mathbb{N}})$  initialized at some  $(\theta_0, X_0) = (\theta, x) \in \Theta \times X$ , with probability distribution denoted  $\mathbb{P}_{\theta,x}(\cdot)$  [and associated expectation  $\mathbb{E}_{\theta,x}(\cdot)$ ] and defined recursively for  $i \geq 0$  as

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follows:

$$(1.1) \quad \begin{aligned} X_{i+1} | (\theta_0, X_0, X_1, \dots, X_i) &\sim P_{\theta_i}(X_i, \cdot), \\ \theta_{i+1} &:= \phi_{i+1}(\theta_0, X_0, X_1, \dots, X_{i+1}), \end{aligned}$$

for a family of mappings  $\{\phi_i : \Theta \times \mathbf{X}^{i+1} \rightarrow \Theta\}$ . The present paper is concerned with the stability of the sequence  $\{\theta_i, X_i\}$ , or more precisely, the recurrence of such a process in a set  $\mathcal{C} \subset \mathbf{X} \times \Theta$ ; that is, we aim to develop practically relevant tools to establish that  $\{\theta_i, X_i\}$  visits  $\mathcal{C}$  infinitely often  $\mathbb{P}_{\theta, x}$ -a.s. Such a form of stability is central to establish important properties of the process which, depending on the context, range from the existence of an invariant distribution for the process or its marginals to the convergence of the parameter sequence  $\{\theta_i\}$  to a set of values of particular interest. This is largely an open problem despite its practical relevance as illustrated and discussed later in the paper. The following toy example illustrates the potential difficulties one may face. Let  $\mathbf{X} = \{0, 1\}$  and consider the transition matrix

$$P_\theta = \begin{bmatrix} 1 - \exp(-|\theta|) & \exp(-|\theta|) \\ \exp(-|\theta|) & 1 - \exp(-|\theta|) \end{bmatrix},$$

with  $\Theta = \mathbb{R}$ . This transition matrix has  $\pi = (1/2, 1/2)$  as invariant distribution, and its second eigenvalue is  $\lambda = 1 - 2 \exp(-|\theta|)$ . Set  $\theta_{i+1} = \theta_i + a/i[1/2 - X_{i+1}]$  for some  $a > 0$ . One could expect  $\{\theta_i\}$  to converge to a finite value, but following the argument in [16], Section 6.3, one can in fact show that for some values of  $a$ , with positive probability,  $\{X_i\}$  may get stuck in either states while  $\{\theta_i\}$  diverges. Ergodicity is lost here due to the fact that  $\mathcal{C}_0 = \Theta \times \{0\}$  or  $\mathcal{C}_1 = \Theta \times \{1\}$  is not visited infinitely often with probability one.

The remainder of the paper is organized as follows. In Section 2 we introduce our methodology, which relies on a classical Lyapunov function/drift argument to establish recurrence of the joint process  $\{\theta_i, X_i\}$  to a set  $\mathcal{C}$  (Lemma 1). Our main result in this section is Theorem 1 where it is shown how individual drift conditions of type (2.2) and (2.3) characterizing the evolution from  $X_i$  to  $X_{i+1}$  and  $\theta_i$  to  $\theta_{i+1}$  in (1.1), respectively, can be combined into a joint drift condition in order to characterize the joint dynamic and establish recurrence to a set  $\mathcal{C}$ . It is worth pointing out that the result applies even in situations where this dynamic exhibits a time-scale separation, which, as we shall see in the application section, is of practical interest. This result captures the main ideas behind our general strategy to establish recurrence, but for simplicity and clarity, remain unspecific about how the abstract conditions may be relevant in practice.

Section 3 contains the main practical results of the paper, Theorem 2 and its corollary, where we show how familiar (e.g., [13]), but  $\theta$ -dependent, drift conditions characterizing the evolution of homogeneous Markov chains with transitions  $\{P_\theta, \theta \in \Theta\}$  [see assumption (A2)] can be combined with a class of drift conditions

characterizing the evolution of the parameter  $\theta$  [see assumption (A3)] in order to apply our earlier abstract results for the stability of the joint process.

In Section 4 we focus on a practically important class of updates for the parameter  $\theta$ , known as stochastic approximation [7], which covers all our subsequent applications. The corresponding processes aim to find the zeroes of a function of the parameter  $\theta$  and can be seen as noisy gradient algorithms. The aims of the section are to introduce sufficient background for the application section and to establish Theorem 3. The result of this theorem highlights the central role played by recurrence in an appropriate set  $\mathcal{C}$  in this scenario, in order to ensure that such numerical methods are stable and that they achieve their goal.

Finally in Section 5 we show how the results established earlier apply in the context of adaptive Markov chain Monte Carlo (MCMC) algorithms, a particular type of MCMC algorithms which aim to optimize their performance “on the fly.” More specifically, we show that our general results apply to both the AM algorithm of [11] but also the coerced acceptance probability algorithm [3, 6] and a novel variation.

**2. Compound Lyapunov functions for some two timescale controlled Markov chains.** The approach we adopt throughout this paper relies on a classical Lyapunov function and drift argument commonly used in the (homogeneous) Markov chain setting [13]. Due to the potential time inhomogeneity of the process above, it is useful to consider a sequence of Lyapunov functions  $\{W_i\}$  satisfying a sequence of drift conditions and leading to the following classical result, provided here together with its proof (in Appendix A) for completeness only. Hereafter, for any  $i \geq 0$  we let  $\mathcal{F}_i := \sigma(\theta_0, X_0, X_1, \dots, X_i)$  and for any  $u, v \in \mathbb{R}^2$  we define  $u \vee v := \max\{u, v\}$  and  $u \wedge v := \min\{u, v\}$ .

**LEMMA 1.** *Let  $\{W_i\}$  be a sequence of functions  $W_i : \Theta \times \mathbf{X} \rightarrow [0, \infty)$  such that for the controlled Markov chain defined in (1.1) for all  $\theta, x \in \Theta \times \mathbf{X}$ :*

- (1) *for all  $i \geq 0$ ,  $\mathbb{E}_{\theta, x}[W_i(\theta_i, X_i)] < \infty$ ,*
- (2) *there exist  $\mathcal{C} \subset \Theta \times \mathbf{X}$ , a sequence  $\{\delta_i, i \geq 1\}$  of nonnegative scalars such that  $\sum_{i=1}^{\infty} \delta_i = \infty$  and an integer  $i_w < \infty$  such that for all  $i \geq i_w$ , and whenever  $(\theta_i, X_i) \notin \mathcal{C}$ ,  $\mathbb{P}_{\theta, x}$ -a.s.*

$$(2.1) \quad \mathbb{E}_{\theta, x}[W_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \leq W_i(\theta_i, X_i) - \delta_{i+1}.$$

*Then  $\sum_{i=1}^n \mathbb{I}\{(\theta_i, X_i) \in \mathcal{C}\} = \infty$ ,  $\mathbb{P}_{\theta, x}$ -a.s.*

The main result of this section consists of showing that it is possible to construct joint Lyapunov function sequences  $\{W_i\}$  which satisfy drifts to a set  $\mathcal{C}$ , such that the conditions of Lemma 1 hold, from two separate Lyapunov functions  $w(\theta)$  and  $V(x)$  each satisfying an individual drift condition characterizing the two respective updates involved in the definition of  $\{\theta_i, X_i\}$  in (1.1). The form of these individual

drifts is given below in (2.2) and (2.3): it is worth pointing out that we allow the drift on  $w(\theta)$  to vanish with time since  $\{\gamma_i\}$  may be allowed to vanish. This is practically very relevant since in many situations of interest the “size” of the increments  $|\theta_{i+1} - \theta_i|$  may vanish as  $i \rightarrow \infty$  while that of  $|X_{i+1} - X_i|$  may not. The role of the sequence  $\{\gamma_i\}$  is to accommodate the possibility of two distinct timescales for the two updates in (1.1)—examples are numerous and some will be presented later in Sections 4 and 5. We will consider two scenarios which share very similar assumptions, and will be labeled with  $s \in \{0, 1\}$ .

(A1) Suppose  $V : \mathbf{X} \rightarrow [1, \infty)$  and  $w : \Theta \rightarrow [1, \infty)$  are two functions such that there exist functions  $\Delta_w, \Delta_V : \Theta \times \mathbf{X} \rightarrow \mathbb{R}$ , a set  $\mathcal{C} \subset \Theta \times \mathbf{X}$ , a sequence of strictly positive integers  $\{\gamma_i, i \geq 1\}$  such that:

(1)  $\{\gamma_i\}$  is bounded,

(2) for some integer  $i_0 \geq 0$ ,  $\mathbb{P}_{\theta, x}$ -a.s. the following individual drifts hold for all  $i \geq i_0$ :

$$(2.2) \quad \mathbb{E}_{\theta, x}[w(\theta_{i+1}) \mid \mathcal{F}_i] \leq w(\theta_i) - \gamma_{i+1} \Delta_w(\theta_i, X_i),$$

$$(2.3) \quad \mathbb{E}_{\theta, x}[V(X_{i+1}) \mid \mathcal{F}_i] \leq V(X_i) - \Delta_V(\theta_i, X_i)$$

and  $\mathbb{E}_{\theta, x}[w(\theta_i)] < \infty$  and  $\mathbb{E}_{\theta, x}[V(X_i)] < \infty$ ,

(3) there exist constants  $\delta \in (0, \infty)$  and  $\nu_v, \nu_w \in (0, 1]$  such that

$$(2.4) \quad \nu_w \frac{\Delta_w(\theta, x)}{w^{1-\nu_w}(\theta)} + \nu_v \frac{\Delta_V(\theta, x)}{V^{1-\nu_v}(x)} \geq \delta w^{\nu_w} V^{\nu_v}(\theta, x) \quad \text{for } (\theta, x) \notin \mathcal{C}$$

and

$$\sup_{(\theta, x) \in \mathcal{C}} |\Delta_w(\theta, x)| \vee |\Delta_V(\theta, x)| < \infty.$$

The following theorem establishes two recurrence results for  $\{\theta_i, X_i\}$  to  $\mathcal{C}$ . The first result requires the strongest set of assumptions but also establishes a stronger result, namely that the first moment of the return times to  $\mathcal{C}$  are uniformly bounded in time. The second result requires weaker assumptions but does not guarantee the existence of a uniform in time upper bound on characteristics of the return times. A particular contribution here is the rescaling of either the Lyapunov function  $w(\theta)$  or  $V(x)$  in order to allow for their respective drift terms to be compared on the same time scale.

**THEOREM 1.** *Consider the controlled Markov chain defined in (1.1). Define the sequences of functions  $\{W_i : \Theta \times \mathbf{X} \rightarrow [1, \infty)\}$  and  $\{U_i : \Theta \times \mathbf{X} \rightarrow [1, \infty)\}$  for  $i \geq 1$  and  $\theta, x \in \Theta \times \mathbf{X}$  as follows:*

$$W_i(\theta, x) := V^{\nu_v}(x) + w^{\nu_w}(\theta)/\gamma_i \quad \text{and} \quad U_i(\theta, x) := \gamma_i W_i(\theta, x),$$

where  $\{\gamma_i\}$ ,  $w(\cdot)$ ,  $V(\cdot)$ ,  $\nu_v$  and  $\nu_w$  are as in (A1), which is assumed to hold. Then:

(1) if  $s = 1$  and  $\bar{\ell} := \limsup_{i \rightarrow \infty} (\gamma_{i+1}^{-1} - \gamma_i^{-1}) < \delta$ , then for any  $\delta_W \in (0, \delta - \bar{\ell})$  there exists  $i_W \geq i_0$  such that for any  $i \geq i_W$ , whenever  $(\theta_i, X_i) \notin \mathcal{C}$ ,  $\mathbb{P}_{\theta, x}$ -a.s.

$$(2.5) \quad \mathbb{E}_{\theta, x}[W_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \leq W_i(\theta_i, X_i) - \delta_W,$$

and  $\mathbb{E}_{\theta, x}[W_i(\theta_i, X_i)] < \infty$ , and  $\sum_{i=1}^{\infty} \mathbb{I}\{(\theta_i, X_i) \in \mathcal{C}\} = \infty$ ,  $\mathbb{P}_{\theta, x}$ -almost surely,

(2) if  $s = 0$ ,  $\{\gamma_i\}$  is nonincreasing then for any  $i \geq i_0$ , whenever  $(\theta_i, X_i) \notin \mathcal{C}$ ,  $\mathbb{P}_{\theta, x}$ -a.s.

$$(2.6) \quad \mathbb{E}_{\theta, x}[U_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \leq U_i(\theta_i, X_i) - \delta\gamma_{i+1}$$

and moreover  $\mathbb{E}_{\theta, x}[U_i(\theta_i, X_i)] < \infty$ . If in addition  $\sum_{i=1}^{\infty} \gamma_i = \infty$ , then  $\sum_{i=1}^{\infty} \mathbb{I}\{(\theta_i, X_i) \in \mathcal{C}\} = \infty$ ,  $\mathbb{P}_{\theta, x}$ -almost surely.

PROOF. By (A1), Jensen's inequality and the classical concavity identity  $(1+x)^v \leq 1 + vx$  for  $x \in [-1, \infty)$  and  $v \in (0, 1]$ , we have for any  $i \geq i_0$  and  $\mathbb{P}_{\theta, x}$ -a.s.

$$(2.7) \quad \begin{aligned} & \mathbb{E}_{\theta, x}[W_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \\ & \leq V^{\nu_v}(X_i) \left(1 - \frac{\Delta_V(\theta_i, X_i)}{V(X_i)}\right)^{\nu_v} + \gamma_{i+1}^{-1} w^{\nu_w}(\theta_i) \left(1 - \gamma_{i+1} \frac{\Delta_w(\theta_i, X_i)}{w(\theta_i)}\right)^{\nu_w} \\ & \leq V^{\nu_v}(X_i) \left(1 - \nu_v \frac{\Delta_V(\theta_i, X_i)}{V(X_i)}\right) + \gamma_{i+1}^{-1} w^{\nu_w}(\theta_i) \left(1 - \gamma_{i+1} \nu_w \frac{\Delta_w(\theta_i, X_i)}{w(\theta_i)}\right) \\ & = W_i(\theta_i, X_i) + (\gamma_{i+1}^{-1} - \gamma_i^{-1}) w^{\nu_w}(\theta_i) - \left(\nu_v \frac{\Delta_V(\theta_i, X_i)}{V^{1-\nu_v}(X_i)} + \nu_w \frac{\Delta_w(\theta_i, X_i)}{w^{1-\nu_w}(\theta_i)}\right). \end{aligned}$$

Now consider the scenario where  $s = 1$ . Let  $\delta_W \in (0, \delta - \bar{\ell})$  and  $i_W \geq i_0$  be such that  $\sup_{i \geq i_W} (\gamma_{i+1}^{-1} - \gamma_i^{-1}) < \delta - \delta_W$ . Then, for all  $i \geq i_W$  and  $(\theta_i, X_i) \notin \mathcal{C}$ ,  $\mathbb{P}_{\theta, x}$ -a.s.

$$\mathbb{E}_{\theta, x}[W_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \leq W_i(\theta_i, X_i) - \delta_W.$$

Let  $C := [\sup_{i \geq i_0} \gamma_i (\gamma_{i+1}^{-1} - \gamma_i^{-1})] \vee [\sup_{(\theta, x) \in \mathcal{C}} |\bar{\gamma}_{i_0} \Delta_w(\theta, x)| \vee |\Delta_V(\theta, x)|]$  with  $\bar{\gamma}_{i_0} = \sup_{i \geq i_0} \gamma_i$ . Now for any  $i \geq i_0$  and  $(\theta_i, X_i) \in \Theta \times \mathsf{X}$  we have, starting with the first inequality in (2.7),

$$\begin{aligned} & \mathbb{E}_{\theta, x}[W_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \\ & \leq (1+C)^{\nu_v} V^{\nu_v}(X_i) + (1+C)^{\nu_w} [\gamma_i (\gamma_{i+1}^{-1} - \gamma_i^{-1}) + 1] w^{\nu_w}(\theta_i) / \gamma_i \\ & \leq (1+C)^2 W_i(\theta_i, X_i) \end{aligned}$$

for  $s \in \{0, 1\}$ . From these inequalities we therefore deduce that for any  $i \geq i_0$ ,  $\mathbb{E}_{\theta, x}[W_i(\theta_i, X_i)] \leq (1+C)^{2(i-i_0)} \mathbb{E}_{\theta, x}[W_{i_0}(\theta_{i_0}, X_{i_0})] < \infty$  where the last inequality follows from our assumptions. For the scenario where  $s = 0$  with  $U_i(\theta, x) = \gamma_i W_i(\theta, x)$ , we obtain from (2.7)

$$\begin{aligned} & \mathbb{E}_{\theta, x}[U_{i+1}(\theta_{i+1}, X_{i+1}) \mid \mathcal{F}_i] \leq U_i(\theta_i, X_i) + (\gamma_{i+1} - \gamma_i) W_i(\theta_i, X_i) \\ & \quad + \gamma_{i+1} (\gamma_{i+1}^{-1} - \gamma_i^{-1}) w^{\nu_w}(\theta_i) \\ & \quad - \gamma_{i+1} \left(\nu_v \frac{\Delta_V(\theta_i, X_i)}{V^{1-\nu_v}(X_i)} + \nu_w \frac{\Delta_w(\theta_i, X_i)}{w^{1-\nu_w}(\theta_i)}\right) \end{aligned}$$

and since

$$(\gamma_{i+1} - \gamma_i)W_i(\theta, x) + \gamma_{i+1}(\gamma_{i+1}^{-1} - \gamma_i^{-1})w^{\nu_w}(\theta) = (\gamma_{i+1} - \gamma_i)V(x)$$

and  $\{\gamma_i\}$  is nonincreasing, we conclude (2.6) for  $(\theta_i, X_i) \in \mathcal{C}^c$ . Notice further that  $U_i(\theta, x) \leq \gamma_i W_i(\theta, x)$ , implying  $\mathbb{E}_{\theta, x}[U_i(\theta_i, X_i)] < \infty$  for any  $i \geq i_0$ . We now conclude in both scenarios with Lemma 1.  $\square$

Some comments are in order concerning the choice of the Lyapunov functions and the assumptions. First we clarify the role of  $\nu_v$  and  $\nu_w$ , which are additional degrees of freedom one may find helpful to establish (2.4) in regions of  $\Theta \times \mathbf{X}$  where  $\Delta_v(\theta, x)$  [resp.,  $\Delta_w(\theta, x)$ ] is negative and of large magnitude, but  $V$  (resp.,  $w$ ) is itself large. Notice also that more general concave transformations of  $V$  and  $w$  could be considered for the definition of  $W_i$  and  $U_i$ , but we do not pursue this here. We would also like to point out that other Lyapunov functions of the form  $U_i^\alpha(\theta, x) := \gamma_i^\alpha W_i(\theta, x)$  for  $\alpha \geq 0$  may be considered but we have found the scenarios  $\alpha = 0$  and  $\alpha = 1$  to be of interest only. Finally whereas it is clear that (2.4) is stronger for  $s = 1$  than  $s = 0$ , we also note that  $\limsup_{i \rightarrow \infty} (\gamma_{i+1}^{-1} - \gamma_i^{-1}) < \delta - \delta_W$  implies  $\sum_{i=1}^{\infty} \gamma_i = \infty$ .

In the next section we consider a practically relevant scenario encountered in practice, for which we identify  $\mathcal{C}$  and  $\{W_i\}$ , but also establish an even stronger drift than in (2.5). We will show in Section 5 that such results are satisfied in realistic scenarios.

**3. Simultaneous  $\theta$ -dependent drift conditions and stability.** The results presented in the previous section are rather abstract since  $\Delta_w$ ,  $\Delta_v$  and  $\mathcal{C}$  are not specified. Here we add some structure, and in particular, show how a simultaneous drift condition on the family of Markov transition probabilities  $P_\theta$  for  $\theta \in \Theta$ , where the dependence on  $\theta$  is explicit, can be used to prove the stability of the sequence  $\{\theta_i, X_i\}$  to a well-identified set  $\mathcal{C} \subset \Theta \times \mathbf{X}$ . For ease of exposition we focus throughout this section on the situation where  $\phi_i := \phi_{\gamma_i}$  for some family of updates  $\{\phi_\gamma : \Theta \times \mathbf{X} \rightarrow \Theta, \gamma \in (0, \gamma^+]\}$  and a positive sequence  $\{\gamma_i\} \subset (0, \gamma^+]^{\mathbb{N}}$ , allowing us to define the update  $\theta_{i+1} = \phi_{\gamma_{i+1}}(\theta_i, X_{i+1})$  for  $i \geq 0$ . This directly covers most relevant applications in computational statistics and can be easily generalized. As we shall see, the realistic assumptions we use lead, in fact, to stronger results than those of the previous section. For any  $f : \mathbf{X} \rightarrow \mathbb{R}$  we use the standard notation  $P_\theta f(x) := \int_{\mathbf{X}} P_\theta(x, dy) f(y)$ . The  $\theta$ -dependent simultaneous drift conditions we consider here are as follows:

(A2) The family of Markov transition probabilities  $\{P_\theta, \theta \in \Theta\}$  is such that there exist:

- (1)  $V : \mathbf{X} \rightarrow [1, +\infty)$  and  $\mathcal{C} \subset \mathbf{X}$  such that  $\sup_{x \in \mathcal{C}} V(x) < +\infty$ ,
- (2)  $a(\cdot), b(\cdot) : \Theta \rightarrow [0, +\infty)$  and  $\iota \in [0, 1]$

such that for any  $\theta, x \in \Theta \times \mathsf{X}$ ,

$$P_\theta V(x) \leq [V(x) - a^{-1}(\theta)V^\iota(x)]\mathbb{I}\{x \notin \mathsf{C}\} + b(\theta)\mathbb{I}\{x \in \mathsf{C}\}.$$

For functions  $v(\cdot): \Theta \rightarrow \mathbb{R}$  we define the level sets  $\mathcal{V}_M := \{\theta \in \Theta : v(\theta) \leq M\}$  for all  $M \geq 0$ , and for any set  $A$  we will denote  $A^c$  the complement of  $A$  in either  $\Theta$  or  $\mathsf{X}$ . Notice that assumption (A2) implies that  $\inf_{\theta \in \Theta} a(\theta) > 0$ . The situations we are interested in are those for which  $\mathcal{A}_M^c \neq \emptyset$  for any  $M > 0$ . Hereafter it will be convenient to denote for any  $\theta, x \in \Theta \times \mathsf{X}$ ,  $\gamma \in (0, \gamma^+]$  and  $f: \Theta \times \mathsf{X} \rightarrow \mathbb{R}^{n_f}$

$$P_{\theta, \gamma} f(\theta, x) := \int_{\mathsf{X}} P_\theta(x, dy) f(\phi_\gamma(\theta, y), y).$$

(A3) The family of mappings  $\{\phi_\gamma: \Theta \times \mathsf{X} \rightarrow \Theta, \gamma \in (0, \gamma^+]\}$  is such that there exists a Lyapunov function  $w(\cdot): \Theta \rightarrow [1, \infty)$  such that [with  $\{P_\theta, \theta \in \Theta\}$ ,  $V(\cdot), \mathsf{C}, a(\cdot), b(\cdot)$  and  $\iota$  as in (A2)]:

- (1) for any  $M > 0$ ,  $\sup_{\theta \in \mathcal{V}_M} a(\theta) < \infty$  and  $\lim_{M \rightarrow \infty} \sup_{\theta \in \mathcal{V}_M^c} b(\theta)/w(\theta) = 0$ ,
- (2) there exists  $\beta \in [0, 1]$ ,  $c(\cdot), d(\cdot): \Theta \rightarrow [0, \infty)$  and  $e(\cdot): \Theta \rightarrow [1, \infty)$  such that for all  $\gamma \in (0, \gamma^+]$  and  $\theta, x \in \Theta \times \mathsf{X}$ ,

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) - \gamma w(\theta) \Delta(V_\beta(\theta, x)\mathbb{I}\{x \notin \mathsf{C}\} + d(\theta)\mathbb{I}\{x \in \mathsf{C}\}),$$

where  $V_\beta(\theta, x) := c(\theta) + V^\beta(x)/e(\theta)$  and

- (3)  $c(\cdot), d(\cdot): \Theta \rightarrow [0, \infty)$  are bounded,  $\lim_{M \rightarrow \infty} \sup_{\theta \in \mathcal{V}_M^c} [c(\theta) \vee d(\theta)] = 0$  and for any  $M > 0$ ,  $\sup_{\theta \in \mathcal{V}_M} e(\theta) < \infty$ ,
- (4)  $\Delta(\cdot): [0, \infty) \rightarrow \mathbb{R}$  is such that:

- (a)  $\Delta(0) > 0$  and it is continuous in a neighborhood of 0,
- (b) there exists  $p_\Delta \in (0, \iota/\beta]$  such that for all  $M > 0$  there exists  $C_{\Delta, M} > 0$ , such that for all  $z \geq M$

$$|\Delta(z)| \leq C_{\Delta, M} \times z^{p_\Delta},$$

- (5) for any  $\epsilon > 0$ ,

$$\sup_{\theta, x \in \tilde{\mathcal{V}}_\epsilon} \frac{a(\theta)w(\theta)e^{-p_\Delta(\theta)}}{V^{\iota-p_\Delta\beta}(x)} < \infty,$$

where  $\tilde{\mathcal{V}}_\epsilon := \{\theta, x : V^\beta(x)/e(\theta) \geq \epsilon\}$ .

REMARK 1. The conditions above may appear abstract, but are motivated by the following concrete situations:

- (1) The simultaneous fixed- $\theta$  drift conditions (A2) can be established in numerous situations of practical interest. Examples are given in Section 5, where the transition probabilities share the same invariant distribution, but it should be pointed out that such drift conditions can also be established in situations of interest where each transition kernel  $P_\theta$  has its own invariant distribution  $\pi_\theta$ ; this is the

case for example in the context of the stochastic approximation implementation of the EM algorithm in [4]. Other examples can be found in [7] for algorithms used in the area of digital communications, although the dependence on  $\theta$  is never used.

(2) Typically the function  $\Delta(\cdot)$  in (A3) will take the form of a polynomial, as a byproduct of a tractable approximation of  $w(\phi_\gamma(\theta, y))$  in terms of  $w(\theta)$ . For example, in the situation where  $\vartheta = \phi_\gamma(\theta, y) = \theta + \gamma H(\theta, y)$ , which corresponds to the standard stochastic approximation framework (see Section 4), a Taylor expansion of  $w(\vartheta)$  around  $\theta$  will lead to

$$w(\vartheta) \leq w(\theta) + \gamma \langle H(\theta, y), \nabla w(\theta) \rangle + \frac{1}{2} \gamma^2 \bar{w}'' \times |H(\theta, y)|^2$$

whenever  $\bar{w}'' := \sup_{\theta \in \Theta} |\nabla^2 w(\theta)| < \infty$ . With appropriate assumptions on  $H(\theta, y)$ , one can apply  $P_\theta$  to both sides of this inequality and hence obtain a drift condition on  $w(\cdot)$  of the form given in (A3).

(3) The condition required on  $p_\Delta \in (0, \iota/\beta]$  can be understood as being a trade-off between the strength of the drift in (A2) and the strength of unfavorable updates  $\theta_+ = \phi_\gamma(\theta, x_+)$  such that  $w(\theta_+) \gg w(\theta)$ .

Hereafter for any  $\varepsilon \in (0, \Delta(0))$  we will denote

$$\Gamma_\varepsilon := \{\gamma, \bar{\gamma} \in (0, \gamma^+]: \gamma^{-1} - \bar{\gamma}^{-1} < \Delta(0) - \varepsilon\},$$

where we omit the dependence on  $\gamma^+$  for simplicity.

**THEOREM 2.** *Assume that  $\{P_\theta, \theta \in \Theta\}$  and  $\{\phi_\gamma, \gamma \in (0, \gamma^+)\}$  satisfy (A2) and (A3). Then for any  $\varepsilon \in (0, \Delta(0))$  there exist  $\lambda_* \in [1, \infty)$ ,  $\delta, M_* \in (0, +\infty)$  such that for any  $\gamma, \bar{\gamma} \in \Gamma_\varepsilon$  and  $\theta, x \notin \mathcal{W}_{M_*} \times \mathbf{C}$ ,*

$$(3.1) \quad P_{\theta, \gamma} \{\lambda_* V + w/\gamma\}(\theta, x) \leq \lambda_* V(x) + w(\theta)/\bar{\gamma} - \delta[V^\iota(x)/a(\theta) + w(\theta)].$$

**COROLLARY 1.** *Let  $\{\theta_i, X_i\}$  be the controlled Markov chain process as described in equation (1.1) with for any  $i \geq 1$   $\phi_i(\theta_0, x_0, x_1, \dots, x_i) := \phi_{\gamma_i}(\theta_{i-1}, x_i)$  for a family  $\{\phi_\gamma: \Theta \times \mathbf{X} \rightarrow \Theta, \gamma \in (0, \gamma^+)\}$  and some real positive sequence  $\{\gamma_i\}$ . Assume further that  $\{P_\theta, \theta \in \Theta\}$  and  $\{\phi_\gamma, \gamma \in (0, \gamma^+)\}$  satisfy (A2), (A3) and that  $\{\gamma_i\}$  is such that*

$$(3.2) \quad \bar{\varepsilon} := \Delta(0) - \limsup_{i \rightarrow \infty} (\gamma_{i+1}^{-1} - \gamma_i^{-1}) > 0.$$

*Then, for any  $\varepsilon \in (0, \bar{\varepsilon})$  there exists  $M_*$  as in Theorem 2 such that the set  $\mathcal{W}_{M_*} \times \mathbf{C}$  is visited infinitely often  $\mathbb{P}_{\theta, x}$ -a.s. by  $\{\theta_i, X_i\}$ .*

**PROOF.** Let  $\varepsilon \in (0, \bar{\varepsilon})$ ,  $\delta \in (0, 1]$ ,  $\lambda_* > 1$  and  $M_* > 0$  be as in Theorem 2, and define the family of (Lyapunov) functions  $\{W_i(\theta, x) := \lambda_* V(x) + w(\theta)/\gamma_i\}$ .

From the assumption on  $\{\gamma_i\}$  there exists  $i_0 \in \mathbb{N}$  such that for any  $i \geq i_0$  and  $\theta, x \notin \mathcal{W}_{M_*} \times \mathbf{C}$

$$P_{\theta, \gamma_i} W_i(\theta, x) \leq W_{i-1}(\theta, x) - \delta[V^l(x)/a(\theta) + w(\theta)].$$

The result follows from Lemma 1 since  $\inf_{\theta \in \Theta} w(\theta) > 0$ .  $\square$

REMARK 2. One can notice that:

(1) in the case where  $\gamma_i = c_0/(c_1 + i)^a$  (3.2) is satisfied for any  $c_0 > 0$  and  $a \in (0, 1)$ , and for  $c_0 < \Delta(0)$  when  $a = 1$ ;

(2) in the case where  $\{\gamma_i = \gamma \leq \gamma^+\}$  is constant,  $W_0(\theta, x) = \lambda_* V(x) + w(\theta)$  and for any  $\theta \in \Theta$ ,  $a(\theta) \leq Cw^\varkappa(\theta)$  for  $\varkappa > 0$ , then one may show that for any  $i \geq i_0$  and  $\theta, x \notin \mathcal{W}_{M_*} \times \mathbf{C}$

$$P_{\theta, \gamma} W_0(\theta, x) \leq W_0(\theta, x) - \delta' W_0^{l/(1+\varkappa)}(\theta, x).$$

Indeed, from a standard convexity inequality, for any  $l \in (0, 1]$ ,

$$(3.3) \quad l \frac{V^l(x)}{w(\theta)^\varkappa} + (1-l)w(\theta) \geq \left( \frac{V^l(x)}{w(\theta)^\varkappa} \right)^l w^{1-l}(\theta),$$

which, with the choice  $\bar{l} = 1/(1+\varkappa)$ , leads to

$$V^l(x)/w^\varkappa(\theta) + w(\theta) \geq \bar{l} V^l(x)/w^\varkappa(\theta) + (1-\bar{l})w(\theta) \geq V^{l/(1+\varkappa)}(x).$$

As a result we obtain [noting that, without loss of generality, one can always take  $C \geq 1$  in the upper bound of  $a(\cdot)$  above]

$$\begin{aligned} V^l(x)/a(\theta) + w(\theta) &\geq C^{-1} 2 \frac{1}{2} [V^l(x)/w^\varkappa(\theta) + w(\theta)] \\ &\geq \frac{1}{2C} [V^{l/(1+\varkappa)}(x) + w(\theta)] \\ &\geq \frac{1}{2C} [V^{l/(1+\varkappa)}(x) + w^{l/(1+\varkappa)}(\theta)] \\ &\geq C^{-1} 2^{-1-l/(1+\varkappa)} (V(x) + w(\theta))^{l/(1+\varkappa)}, \end{aligned}$$

and we conclude. This suggests the possibility of precisely characterizing the return times to  $\mathcal{W}_{M_*} \times \mathbf{C}$ , as this form of drift condition is known to lead to the existence of polynomial moments of return times.

PROOF OF THEOREM 2. Choose  $\varepsilon \in (0, \Delta(0))$  and  $\varepsilon_- > 0$  such that for any  $|z| \leq \varepsilon_-$ ,  $|\Delta(0) - \Delta(z)| \leq \varepsilon/2$ . This implies

$$(3.4) \quad \sup_{\gamma, \tilde{\gamma} \in \Gamma_\varepsilon} (\gamma^{-1} - \tilde{\gamma}^{-1}) - \inf_{\{z: |z| \leq \varepsilon_-\}} \Delta(z) \leq \Delta(0) - \varepsilon + \varepsilon/2 - \Delta(0) = -\varepsilon/2.$$

Now let  $M_0 \geq 0$  be such that  $\sup_{\theta \in \mathcal{W}_{M_0}^c} d(\theta) \leq \epsilon_-$  and  $\sup_{\theta \in \mathcal{W}_{M_0}^c} c(\theta) \leq \epsilon_-/2$ . From (A2) and (A3) we have for  $(\theta, x) \in \Theta \times \mathbf{X}$  and  $\lambda \in (0, \infty)$

$$\begin{aligned} & P_{\theta, \gamma} \{ \lambda V + w/\gamma \} (\theta, x) \\ & \leq \lambda [V(x) - a^{-1}(\theta) V^\iota(x)] \mathbb{I}\{x \notin \mathbf{C}\} + \lambda b(\theta) \mathbb{I}\{x \in \mathbf{C}\} \\ & \quad + w(\theta)/\gamma - w(\theta) \Delta(V_\beta(\theta, x) \mathbb{I}\{x \notin \mathbf{C}\} + d(\theta) \mathbb{I}\{x \in \mathbf{C}\}). \end{aligned}$$

Note that for  $(\theta, x) \in \mathcal{W}_{M_0} \times \mathbf{C}^c$ ,  $V_\beta(\theta, x) \geq \bar{M}_0^{-1} := 1/\sup_{\theta \in \mathcal{W}_{M_0}} e(\theta) > 0$  from (A3)(3), and therefore from (A3)(4)(b)

$$w(\theta) |\Delta(V_\beta(\theta, x))| \leq C_{\Delta, \bar{M}_0^{-1}} M_0 \sup_{\theta \in \Theta} (e^{-1}(\theta) + c(\theta))^{p_\Delta} \times V^{p_\Delta \beta}(x),$$

and  $\sup_{\theta \in \mathcal{W}_{M_0}} \Delta(d(\theta)) < \infty$  as  $\Delta(\cdot)$  is bounded on compact sets. In addition  $\sup_{\theta \in \Theta} d(\theta) < \infty$ . Let now

$$C'_{\Delta, M_0} := \left[ M_0 C_{\Delta, \bar{M}_0^{-1}} \sup_{\theta \in \Theta} (e^{-1}(\theta) + c(\theta))^{p_\Delta} \right] \vee \left[ M_0 \sup_{\theta \in \mathcal{W}_{M_0}} \Delta(d(\theta)) \right] < \infty,$$

notice that  $\iota \geq p_\Delta \beta$ , and recall that  $V \geq 1$ . Then we have for  $\gamma, \bar{\gamma} \in \Gamma_\varepsilon$

$$P_{\theta, \gamma} \{ \lambda V + w/\gamma \} (\theta, x) \leq \lambda V(x) + w(\theta)/\bar{\gamma} + \Lambda(\theta, x),$$

with

$$\begin{aligned} \Lambda(\theta, x) & := -\lambda V(x) + \lambda [V(x) - a^{-1}(\theta) V^\iota(x)] \mathbb{I}\{x \notin \mathbf{C}\} \\ & \quad + \lambda b(\theta) \mathbb{I}\{x \in \mathbf{C}\} + (\Delta(0) - \varepsilon) w(\theta) \\ (3.5) \quad & \quad - \Delta(V_\beta(\theta, x) \mathbb{I}\{x \notin \mathbf{C}\} + d(\theta) \mathbb{I}\{x \in \mathbf{C}\}) w(\theta) \mathbb{I}\{\theta \in \mathcal{W}_{M_0}^c\} \\ & \quad + C'_{\Delta, M_0} V^\iota(x) \mathbb{I}\{\theta \in \mathcal{W}_{M_0}\}. \end{aligned}$$

It will be convenient below to refer to the following inequality:

$$(3.6) \quad \Lambda(\theta, x) \leq -\tilde{\delta} [V^\iota(x)/a(\theta) + w(\theta)],$$

for  $(\theta, x) \notin \mathcal{W}_{\tilde{M}} \times \mathbf{S}$  and various instantiations of  $\tilde{\delta}, \tilde{M}, \lambda > 0$  and  $\mathbf{S} \subset \mathbf{X}$ . Our ultimate aim is to prove that under the stated assumptions there exist  $\delta, \lambda_* \in (0, +\infty)$  and  $M_* \geq M_0$  such that (3.6) holds for  $(\theta, x) \in (\mathcal{W}_{M_*} \times \mathbf{C})^c$ . For any  $M \geq M_0$ , we use the following partition:

$$(\mathcal{W}_M \times \mathbf{C})^c = (\mathcal{W}_{M_0} \times \mathbf{C}^c) \cup (\mathcal{W}_{M_0}^c \times \mathbf{C}^c) \cup (\mathcal{W}_M^c \times \mathbf{C}),$$

which leads us to consider three cases, (a), (b) and (c), from left to right.

(a) For  $(\theta, x) \in \mathcal{W}_{M_0} \times \mathbf{C}^c$  and any  $\lambda > 0$ , we have

$$\begin{aligned} \Lambda(\theta, x) & \leq [\Delta(0) - \varepsilon] w(\theta) - \lambda V^\iota(x)/a(\theta) + C'_{\Delta, M_0} V^\iota(x) \\ & \leq [\Delta(0) - \varepsilon] \sup_{\theta \in \mathcal{W}_{M_0}} w(\theta) + V^\iota(x) \left[ C'_{\Delta, M_0} - \lambda / \sup_{\vartheta \in \mathcal{W}_{M_0}} a(\vartheta) \right], \end{aligned}$$

where we note that  $\sup_{\vartheta \in \mathcal{W}_{M_0}} a(\vartheta) < \infty$  from (A3)(1). Now, from our choice of  $M_0$  and since  $V \geq 1$  and  $\inf_{\vartheta \in \Theta} a(\vartheta) > 0$ , we conclude about the existence of  $\lambda_a, \delta_a > 0$  such that for all  $\lambda \geq \lambda_a, (\theta, x) \in \mathcal{W}_{M_0} \times \mathbf{C}^c$

$$\begin{aligned} \Lambda(\theta, x) &\leq [\Delta(0) - \varepsilon]M_0 + V^l(x) \left[ C'_{\Delta, M_0} - \lambda / \sup_{\vartheta \in \mathcal{W}_{M_0}} a(\vartheta) \right] \\ &\leq -\delta_a [V^l(x)/a(\theta) + w(\theta)]. \end{aligned}$$

Therefore (3.6) is satisfied with  $\tilde{M} = M_0$ , any  $\lambda \geq \lambda_a$  and  $\tilde{\delta} = \delta_a$ .

(b) For  $(\theta, x) \in \mathcal{W}_{M_0}^c \times \mathbf{C}^c$  and any  $\lambda > 0$ , we have

$$\Lambda(\theta, x) \leq -\lambda V^l(x)/a(\theta) + [\Delta(0) - \varepsilon - \Delta(V_\beta(\theta, x))]w(\theta),$$

and we seek to show that there exists  $\lambda_b = \lambda_i \vee \lambda_{ii}$  and  $\delta_b = \delta_i \wedge \delta_{ii} > 0$  (where  $\lambda_i, \lambda_{ii} > 0$  and  $\delta_i, \delta_{ii} > 0$  are given below in the proof) such that for all  $\lambda \geq \lambda_b$  and  $(\theta, x) \in \mathcal{W}_{M_0}^c \times \mathbf{C}^c$ , (3.6) is satisfied with  $\tilde{\delta} = \delta_b$ . In what follows we will use the following intermediate results. From (A3)(5) we have that for our earlier choice of  $\epsilon_-$  and any  $(\theta, x) \in \mathcal{W}_{M_0}^c \times \mathbf{C}^c$ , the condition

$$V_\beta(\theta, x) = \frac{V^\beta(x)}{e(\theta)} + c(\theta) \geq \epsilon_- \quad \text{implies} \quad \frac{V^\beta(x)}{e(\theta)} \geq \epsilon_- - \sup_{\vartheta \in \mathcal{W}_{M_0}^c} c(\vartheta) \geq \epsilon_-/2,$$

and therefore that for  $q \in \{0, p_\Delta\}$

$$(3.7) \quad \sup_{(\mathcal{W}_{M_0}^c \times \mathbf{C}^c) \cap \{(\theta, x) : V_\beta(\theta, x) \geq \epsilon_-\}} \frac{a(\theta)w(\theta)e^{-q(\theta)}}{V^{l-q\beta}(x)} \leq C_{\epsilon_-} < \infty.$$

Indeed the case  $q = p_\Delta$  is true by assumption and for  $V^\beta(x)/e(\theta) \geq \epsilon_-/2$

$$\frac{a(\theta)w(\theta)e^{-p_\Delta(\theta)}}{V^{l-p_\Delta\beta}(x)} = \frac{a(\theta)w(\theta)}{V^l(x)} \left( \frac{V^\beta(x)}{e(\theta)} \right)^{p_\Delta} \geq \frac{a(\theta)w(\theta)}{V^l(x)} (\epsilon_-/2)^{p_\Delta}$$

from which we conclude. We now partition  $\mathcal{W}_{M_0}^c \times \mathbf{C}^c$  by considering the two following subsets:

(i) From our choice of  $M_0$  and  $\epsilon_-$  and (3.4), we deduce that on the subset  $(\mathcal{W}_{M_0}^c \times \mathbf{C}^c) \cap \{(\theta, x) : V_\beta(\theta, x) < \epsilon_-\}$

$$\Delta(0) - \varepsilon - \Delta(V_\beta(\theta, x)) \leq -\varepsilon/2,$$

and consequently

$$\Lambda(\theta, x) \leq -\lambda V^l(x)/a(\theta) - w(\theta)\varepsilon/2,$$

and we conclude about the existence of  $\lambda_i, \delta_i > 0$  such that (3.6) holds for any  $\lambda \geq \lambda_i$  and  $\tilde{\delta} = \delta_i$ .

(ii) By our assumption on  $\Delta(\cdot)$  there exists  $C'_{\Delta, \epsilon_-} > 0$  such that for any  $z \geq \epsilon_-$

$$\Delta(0) - \varepsilon - \Delta(z) \leq C'_{\Delta, \epsilon_-} z^{p_\Delta}.$$

Consequently we deduce that on  $(\mathcal{W}_{M_0}^c \times \mathbf{C}^c) \cap \{\theta, x : V_\beta(\theta, x) \geq \epsilon_-\}$

$$\begin{aligned} \Lambda(\theta, x) &\leq \frac{V^\iota(x)}{a(\theta)} \left[ C'_{\Delta, \epsilon_-} \frac{a(\theta)w(\theta)}{V^\iota(x)} \left( \frac{V^\beta(x)}{e(\theta)} + c(\theta) \right)^{p_\Delta} - \lambda \right] \\ &\leq \frac{V^\iota(x)}{a(\theta)} \left[ 2^{p_\Delta} C'_{\Delta, \epsilon_-} \frac{a(\theta)w(\theta)e(\theta)^{-p_\Delta}}{V^{\iota-p_\Delta\beta}(x)} - \lambda \right]. \end{aligned}$$

We now choose  $\lambda > 2^{p_\Delta} C'_{\Delta, \epsilon_-} C_{\epsilon_-}$ , and from (3.7) with  $q = 0$  we have  $V^\iota(x)/a(\theta) \geq (\epsilon_-/2)^{p_\Delta} C_{\epsilon_-}^{-1} w(\theta)$  and therefore

$$\Lambda(\theta, x) \leq \frac{1}{2} \left[ \frac{V^\iota(x)}{a(\theta)} + w(\theta)(\epsilon_-/2)^{p_\Delta} C_{\epsilon_-}^{-1} \right] (2^{p_\Delta} C'_{\Delta, \epsilon_-} C_{\epsilon_-} - \lambda).$$

We conclude about the existence of  $\lambda_{ii}, \delta_{ii} > 0$  such that (3.6) holds for any  $\lambda \geq \lambda_{ii}$  and  $\tilde{\delta} = \delta_{ii}$ .

(c) First, we note from our choice of  $M_0$ , (3.4) and (3.5) that for any  $\theta, x \in \mathcal{W}_{M_0}^c \times \mathbf{C}$  and  $\lambda > 0$  the function  $\Lambda(\theta, x)$  is upper bounded by

$$\begin{aligned} \Lambda(\theta, x) &\leq -\lambda V(x) + \lambda b(\theta) + [\Delta(0) - \varepsilon - \Delta(d(\theta))]w(\theta) \\ &\leq -\lambda V(x) + [\lambda b(\theta)/w(\theta) - \varepsilon/2]w(\theta). \end{aligned}$$

We now show that for any  $\lambda \in (0, +\infty)$  there exist  $M_\lambda \in [M_0, +\infty)$  and  $\delta_\lambda \in (0, +\infty)$  such that for all  $\theta, x \in \mathcal{W}_{M_\lambda}^c \times \mathbf{C}$ , (3.6) is satisfied with  $\tilde{\delta} = \delta_\lambda$ , a function of  $\lambda$ . From our last inequality and since  $\iota \in [0, 1]$  and  $V \geq 1$ , for any  $M \geq M_0$  and  $\theta, x \in \mathcal{W}_M^c \times \mathbf{C}$

$$\Lambda(\theta, x) \leq -\lambda \left[ \inf_{\vartheta \in \Theta} a(\vartheta) \right] V^\iota(x)/a(\theta) + \left[ \lambda \sup_{\vartheta \in \mathcal{W}_M^c} b(\vartheta)/w(\vartheta) - \varepsilon/2 \right] w(\theta).$$

We conclude about the existence of  $M_\lambda$  and  $\delta_\lambda$  as above from our assumption on  $b(\cdot)$ .

We now conclude by letting  $\lambda_* \geq \lambda_{a,b} = \lambda_a \vee \lambda_b$ ,  $M_* \geq M_{\lambda_a \vee \lambda_b}$  and  $\delta = \delta_{\lambda_a \vee \lambda_b} \wedge \delta_a \wedge \delta_b$ .  $\square$

**4. The central role of stability in the context of stochastic approximation with Markovian dynamic.** In this section we illustrate the central role played by the form of stability considered in this paper to establish that some controlled Markov chains of practical relevance possess some desired properties. We focus on a particular class of controlled Markov chains driven by a so-called stochastic approximation recursion (also known as the Robbins–Monro algorithm). The motivation for such algorithms, described below, is to find the roots of the function  $h(\cdot) : \Theta \rightarrow \mathbb{R}^{n_\theta}$

$$h(\theta) := \int_{\mathbf{X}} H(\theta, x) \pi_\theta(dx),$$

for a family of functions  $\{H(\theta, x) : \Theta \times \mathbf{X} \rightarrow \Theta\}$  and a family of probability distributions  $\{\pi_\theta, \theta \in \Theta\}$  defined on some space  $\mathbf{X} \times \mathcal{B}(\mathbf{X})$ . This is a ubiquitous problem in statistics, engineering and computer science. These roots are rarely available analytically and a way of finding them numerically consists of considering the following controlled Markov chain on  $((\Theta \times \mathbf{X})^{\mathbb{N}}, (\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbf{X}))^{\otimes \mathbb{N}})$ , initialized at some  $(\theta_0, X_0) = (\theta, x) \in \Theta \times \mathbf{X}$  and defined recursively for a sequence of stepsizes  $\{\gamma_i\}$  for  $i \geq 0$ ,

$$(4.1) \quad \begin{aligned} X_{i+1} | \mathcal{F}_i &\sim P_{\theta_i}(X_i, \cdot), \\ \theta_{i+1} &= \theta_i + \gamma_{i+1} H(\theta_i, X_{i+1}), \end{aligned}$$

where  $\{P_\theta, \theta \in \Theta\}$  (for some set  $\Theta \subset \mathbb{R}^{n_\theta}$ ) is a family of Markov transition probabilities such that for each  $\theta \in \Theta$ ,  $P_\theta$  leaves  $\pi_\theta$  invariant, that is, is such that  $\pi_\theta P_\theta = \pi_\theta$ . The rationale for this recursion is as follows. Let us first rewrite the Robbins–Monro recursion as

$$\theta_{i+1} = \theta_i + \gamma_{i+1} [h(\theta_i) + \xi_{i+1}],$$

where  $\{\xi_{i+1} = H(\theta_i, X_{i+1}) - h(\theta_i)\}$ , which is traditionally referred to as the “noise.” Then  $\{\theta_i\}$  can be thought as being a noisy version of the sequence  $\{\bar{\theta}_i\}$  defined as  $\bar{\theta}_{i+1} = \bar{\theta}_i + \gamma_{i+1} h(\bar{\theta}_i)$ , and it is believable that the properties of  $\{\theta_i\}$  are closely related to those of the noiseless sequence  $\{\bar{\theta}_i\}$  provided the average effect of the noise on this sequence is negligible. This requires some form of averaging, or ergodicity, property on  $\{\xi_i\}$ .

The convergence of such sequences has been well studied by various authors, starting with the seminal work of [7], under various assumptions on all the quantities involved. A crucial step of such convergence analyses, however, consists of assuming that the sequence  $\{\theta_i\}$  remains bounded in a compact set of  $\Theta$  with probability one. This problem has traditionally been either ignored or circumvented by means of modifications of the recursion (4.1). Indeed, one of the major difficulties specific to the Markovian dynamic scenario is that  $\{\theta_i\}$  governs the ergodicity of  $\{X_i\}$  (and hence  $\{\xi_i\}$ ) and that stability properties of  $\{\theta_i\}$  relying on those of  $\{\bar{\theta}_i\}$  require good ergodicity properties which might vanish whenever  $\{\theta_i\}$  approaches a set  $\partial\Theta$  away from the zeroes of  $h(\theta)$ , resulting in instability. Most existing results rely on modifications of the updates  $\{\phi_\gamma\}$  designed to ensure a form of ergodicity of  $\{\xi_i\}$  which in turn ensures that  $\{\theta_i\}$  inherits the stability properties of  $\{\bar{\theta}_i\}$ . The only known results we are aware of where stability is established for (4.1) without any modification are [7], Part II, Section 1.9, where assumption (1.9.3) may not be satisfied in numerous cases of interest or directly verifiable, and [14] in a particular scenario.

The approach we follow here is significantly different from that developed in the aforementioned works and consists of dividing the difficult problem of proving boundedness away from  $\partial\Theta$  into two simpler tasks. First using the results of Sections 2 or 3, one may establish that the sequence  $\{\theta_i, X_i\}$  visits some set

$\mathcal{W} \times C \subset \Theta \times \mathsf{X}$  infinitely often  $\mathbb{P}_{\theta,x}$ -a.s., a set which has the particularity that transition probabilities  $\{P_\theta, \theta \in \mathcal{W}\}$  have uniformly good ergodicity properties. Then, using these facts, one can show that  $\{\theta_i\}$  follows the trajectories of the deterministic recursion  $\{\bar{\theta}_i\}$  more and more accurately at each visit of  $\mathcal{W} \times C$ , and eventually remains in a set only slightly larger than  $\mathcal{W}$  provided the deterministic sequence is itself stable. The advantage of our approach is that instead of aiming to establish ergodicity properties of  $\{\xi_i\}$  in worse case scenarios for the sequence  $\{\theta_i\}$ , we decouple the analysis of the behavior of  $\{\theta_i\}$  when it approaches  $\partial\Theta$  from the study of the ergodicity properties of  $\{\xi_i\}$ , which need to be studied for “reasonable” values of  $\theta$  only. Before stating the main result of this section we state our assumptions.

(A4) Let  $\{H(\theta, x)\}$ ,  $\{\gamma_i\}$ ,  $\{\pi_\theta\}$  and  $\{P_\theta\}$  be as above. We assume that:

(1) there exists  $\gamma_+ > 0$  such that:

- (a)  $\gamma := \{\gamma_i\} \subset [0, \gamma^+]^{\mathbb{N}}$ ,
- (b) for any  $\theta, x \in \Theta \times \mathsf{X}$  and  $\gamma \in [0, \gamma^+]$

$$\theta + \gamma H(\theta, x) \in \Theta,$$

(2)  $H : \Theta \times \mathsf{X} \rightarrow \mathbb{R}^{n_\theta}$  is such that for any  $\theta \in \Theta$ ,  $\int_{\mathsf{X}} |H(\theta, x)| \pi_\theta(dx) < +\infty$ ,

(3) and for any  $\theta \in \Theta$ ,  $\pi_\theta P_\theta = \pi_\theta$ .

A practical technique to prove the boundedness of the noiseless sequence consists, whenever possible, of determining a Lyapunov function  $w : \Theta \rightarrow [0, \infty)$  such that  $\langle \nabla w(\theta), h(\theta) \rangle \leq 0$  away from the roots of  $h(\theta)$ , where  $\nabla w$  denotes the gradient of  $w$  with respect to  $\theta$ , and for  $u, v \in \mathbb{R}^n$ ,  $\langle \cdot, \cdot \rangle$  is their Euclidean inner product (we will later on also use the notation  $|v| = \sqrt{\langle v, v \rangle}$  to denote the Euclidean norm of  $v$ ). Note that although we use here the same symbol  $w$  as in Sections 1 and 3, the Lyapunov function below might be different.

(A5)  $\Theta$  is an open subset of  $\mathbb{R}^{n_\theta}$ ,  $h : \Theta \rightarrow \mathbb{R}^{n_\theta}$  is continuous and there exists a continuously differentiable function  $w : \Theta \rightarrow [0, \infty)$  such that:

(1) there exists  $M_0 > 0$  such that

$$\mathcal{L} := \{\theta \in \Theta, \langle \nabla w(\theta), h(\theta) \rangle = 0\} \subset \{\theta \in \Theta, w(\theta) < M_0\},$$

(2) there exists  $M_1 \in (M_0, \infty]$  such that  $\mathcal{W}_{M_1}$  is a compact set,

(3) for any  $\theta \in \Theta \setminus \mathcal{L}$ ,  $\langle \nabla w(\theta), h(\theta) \rangle < 0$ .

We now introduce some additional notation needed to describe the ergodicity properties of  $\{\xi_i\}$  every time the sequence  $\{\theta_i, X_i\}$  visits some set  $\mathcal{W} \times C$ . More precisely, consider the stochastic processes  $\{\vartheta_i, \mathfrak{X}_i\}$  defined on  $((\Theta \times \mathsf{X})^{\mathbb{N}}, (\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathsf{X}))^{\otimes \mathbb{N}})$  which use the stepsize sequence  $\boldsymbol{\gamma}^{\leftarrow l} := \{\gamma_{i+l}, i \geq 0\}$  for some  $l \geq 0$ , initialized with  $\vartheta_0, \mathfrak{X}_0 \in \Theta \times \mathsf{X}$  and such that for  $i \geq 0$ ,

$$(4.2) \quad \mathfrak{X}_{i+1} | (\vartheta_0, \mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_i) \sim P_{\vartheta_i}(\mathfrak{X}_i, \cdot),$$

$$\vartheta_{i+1} = \vartheta_i + \gamma_{i+1+l} H(\vartheta_i, \mathfrak{X}_{i+1}).$$

In order to take the shift in the stepsize sequence into account, we denote by  $\bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y}^{\leftarrow l}}$  and  $\bar{\mathbb{E}}_{\theta,x}^{\mathcal{Y}^{\leftarrow l}}$  the associated probability distribution and expectation operator for  $\vartheta_0 = \theta \in \Theta$  and  $\mathfrak{X}_0 = x \in \mathbf{X}$ , and point out that in contrast to  $\mathbb{P}_{\theta,x}$  defined earlier for  $\{\theta_i, X_i\}$ , the notational dependence on  $\mathcal{Y}^{\leftarrow l}$  for  $l \geq 0$  is here crucial. For any  $M > 0$  we define the exit time from the level set  $\mathcal{W}_M$ ,  $\sigma(\mathcal{W}_M) := \inf\{k \geq 1 : \vartheta_k \notin \mathcal{W}_M\}$  (with the standard convention that  $\inf\{\emptyset\} = +\infty$ ), and for any  $j \geq 1$  we define  $\varsigma_j := H(\vartheta_{j-1}, \mathfrak{X}_j) - h(\vartheta_{j-1})$ . We extend the result of [2], Proposition 5.2 (see also [1]) in order to establish the following result.

**THEOREM 3.** *Assume (A4) and (A5), that  $\{\gamma_i\}$  is such that  $\limsup_{i \rightarrow \infty} \gamma_i = 0$ , and let  $M \in (M_0, M_1]$ . Assume that there exists  $\mathbf{C} \subset \mathbf{X}$  such that:*

(1) *for any  $\epsilon > 0$ ,*

$$(4.3) \quad \limsup_{l \rightarrow \infty} \sup_{\theta, x \in \mathcal{W}_{M_0} \times \mathbf{C}} \bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y}^{\leftarrow l}} \left( \sup_{k \geq 1} \mathbb{I}\{\sigma(\mathcal{W}_M) \geq k\} \left| \sum_{j=1}^k \gamma_{j+l} \varsigma_j \right| > \epsilon \right) < 1;$$

(2) *for any  $\theta, x \in \Theta \times \mathbf{X}$ ,*

$$\mathbb{P}_{\theta,x} \left( \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \{(\theta_i, X_i) \in \mathcal{W}_{M_0} \times \mathbf{C}\} \right) = 1,$$

*that is,  $\{\theta_i, X_i\}$  defined by equation (4.1) visits  $\mathcal{W}_{M_0} \times \mathbf{C}$  infinitely often  $\mathbb{P}_{\theta,x}$ -a.s.*

*Then the sequence  $\{\theta_i\}$ , as defined by equation (4.1), is such that*

$$\mathbb{P}_{\theta,x} \left( \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} \{\theta_i \in \mathcal{W}_M\} \right) = 1,$$

*that is,  $\{\theta_i\}$  eventually remains in  $\mathcal{W}_M$ ,  $\mathbb{P}_{\theta,x}$ -a.s.*

**REMARK 3.** Proving equation (4.3) is now rather well understood in general scenarios as soon as some form of local (in  $\theta$ ) uniform ergodicity of  $\{P_\theta\}$  is satisfied and can be checked in practice; see [1] and [2], for example, and the recent results in [5]. In the present paper, we rather focus on finding verifiable conditions on  $\{\gamma_i\}$ ,  $\{H(\theta, x)\}$  and  $\{P_\theta\}$  which ensure that  $\{\theta_i, X_i\}$ , as defined by equation (4.1), visits  $\mathcal{W}_{M_0} \times \mathbf{C}$  infinitely often  $\mathbb{P}_{\theta,x}$ -a.s., which in combination with the aforementioned existing results will allow us to conclude about the stability of a large class of controlled MCMC algorithms.

**PROOF OF THEOREM 3.** For  $M \in (M_0, M_1]$  we let  $\delta_0 > 0$  and  $\lambda_0 > 0$  be as in Theorem 7 from [2], Proposition 5.2, given in Appendix B for convenience. We consider the sequence  $\{T_i, i \geq 1\}$  of successive return times to  $\mathcal{W}_{M_0} \times \mathbf{C}$  “separated by at least an exit from  $\mathcal{W}_M$ ,” formally defined for  $i \geq 0$  as

$$T_{i+1} = \inf\{j \geq T_i + 1 : \exists l \in \{T_i + 1, \dots, j\} / \theta_l \notin \mathcal{W}_M \text{ and } (\theta_j, X_j) \in \mathcal{W}_{M_0} \times \mathbf{C}\},$$

with the conventions  $T_0 = 0$  and  $\inf\{\emptyset\} = +\infty$ . It will be useful below to note that for any  $i \geq 1$ ,  $T_i \geq i$ . Let  $n_0 \in \mathbb{N}$  be such that  $\sup_{k \geq n_0} \gamma_k \leq \lambda_0$ . We first show that for any  $\theta, x \in \Theta \times \mathbf{X}$ ,

$$(4.4) \quad \mathbb{P}_{\theta,x} \left( \bigcup_{k \geq 1} \{T_k = +\infty\} \right) = 1,$$

and to achieve this, we establish a bound on  $\sup_{\theta,x \in \Theta \times \mathbf{X}} \mathbb{P}_{\theta,x}(T_n < +\infty)$  for  $n \geq n_0$ . Notice that from the strong Markov property, for any  $\theta, x \in \Theta \times \mathbf{X}$  and  $l \geq n_0$

$$\mathbb{P}_{\theta,x}(T_{l+1} < +\infty) = \mathbb{E}_{\theta,x}(\mathbb{I}\{T_l < +\infty\} \mathbb{P}_{\theta_{T_l}, X_{T_l}}(T_1 < +\infty)).$$

In addition, for any  $\theta, x \in \Theta \times \mathbf{X}$ , we have

$$\begin{aligned} & \mathbb{I}\{T_l < +\infty\} \mathbb{P}_{\theta_{T_l}, X_{T_l}}(T_1 < +\infty) \\ & \leq \mathbb{I}\{T_l < +\infty\} \bar{\mathbb{P}}_{\theta_{T_l}, X_{T_l}}^{\mathcal{Y} \leftarrow T_l}(\sigma(\mathcal{W}_M) < +\infty), \quad \mathbb{P}_{\theta,x}\text{-a.s.} \end{aligned}$$

and for any  $q \geq 0$ ,

$$\bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y} \leftarrow q}(\sigma(\mathcal{W}_M) < +\infty) = \bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y} \leftarrow q} \left( \bigcup_{k \geq 1} \{\sigma(\mathcal{W}_M) = k\} \right).$$

From Theorem 7 we deduce that for any  $q \geq n_0$ ,

$$\bigcup_{k \geq 1} \{\sigma(\mathcal{W}_M) = k\} \subset \left\{ \sup_{k \geq 1} \mathbb{I}\{\sigma(\mathcal{W}_M) \geq k\} \left| \sum_{j=1}^k \gamma_{j+q} \varsigma_j \right| > \delta_0 \right\},$$

which implies that for any  $T_l \geq l \geq n_0$ ,

$$\begin{aligned} & \mathbb{I}\{T_l < +\infty\} \mathbb{P}_{\theta_{T_l}, X_{T_l}}(T_1 < +\infty) \\ & \leq \sup_{q \geq l} \sup_{\theta, x \in \mathcal{W}_{M_0} \times C} \bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y} \leftarrow q} \left( \sup_{k \geq 1} \mathbb{I}\{\sigma(\mathcal{W}_M) \geq k\} \left| \sum_{j=1}^k \gamma_{j+q} \varsigma_j \right| > \delta_0 \right). \end{aligned}$$

Consequently by induction one obtains that for any  $n > n_0$ ,

$$\begin{aligned} & \mathbb{P}_{\theta,x}(T_n < +\infty) \\ & \leq \prod_{l=n_0}^{n-1} \sup_{q \geq l} \sup_{\theta, x \in \mathcal{W}_{M_0} \times C} \bar{\mathbb{P}}_{\theta,x}^{\mathcal{Y} \leftarrow q} \left( \sup_{k \geq 1} \mathbb{I}\{\sigma(\mathcal{W}_M) \geq k\} \left| \sum_{j=1}^k \gamma_{j+q} \varsigma_j \right| > \delta_0 \right). \end{aligned}$$

Result (4.4) then follows by a standard Borel–Cantelli argument under the condition of the theorem. We now prove that  $\{\theta_k\}$  eventually remains in  $\mathcal{W}_M$ ,  $\mathbb{P}_{\theta,x}$ -a.s.

First notice that by construction of  $\{T_i\}$ ,

$$\begin{aligned}
& \bigcup_{k \geq 1} \{T_k = +\infty\} \\
&= \bigcup_{k \geq 1} \{T_{k-1} < +\infty, T_k = +\infty\} \\
&= \bigcup_{m \geq 0} \bigcup_{k \geq 1} \{T_{k-1} = m, T_k = +\infty\} \\
&= \bigcup_{m \geq 0} \{(\theta_m, X_m) \in \mathcal{W}_{M_0} \times C\} \\
&\quad \cap (\{\theta_l \in \mathcal{W}_M, \forall l \geq m+1\} \cup \{\theta_l \notin \mathcal{W}_{M_0}, \forall l \geq m+1\}).
\end{aligned}$$

Now, since by assumption  $\mathbb{P}_{\theta, x}(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \{(\theta_i, X_i) \in \mathcal{W}_{M_0} \times C\}) = 1$ , we deduce that for any  $\theta, x \in \Theta \times \mathcal{X}$

$$\mathbb{P}_{\theta, x} \left( \bigcup_{m \geq 0} \{(\theta_m, X_m) \in \mathcal{W}_{M_0} \times C, \theta_l \in \mathcal{W}_M, \forall l \geq m+1\} \right) = 1,$$

and we conclude.  $\square$

We briefly discuss here other applications of our stability results, particularly in the situation where the step-size sequence is held constant. Such fixed stepsize algorithms have been popular in engineering since they provide the algorithms with both some form of robustness and a “tracking” ability. The analysis of these algorithms naturally requires one to establish stability first [7]. We would like, however, to point out another important application in the context of adaptive step-size algorithms. Indeed, the choice of  $\{\gamma_i\}$  is known to have an important impact on the convergence properties of  $\{\theta_i\}$ . In particular it is well known that if  $\{\gamma_i\}$  vanishes too quickly in the early iterations of (4.1), convergence may be very slow. A natural way to address this problem consists of adaptively selecting the sequence of stepsizes  $\{\gamma_i\}$ . A strategy due to Kesten and further extended by Delyon and Juditsky in [8] is as follows. Given a nonincreasing function  $\gamma(\cdot) : \rightarrow (0, \infty)$  and  $s_0 = 0$ , consider Algorithm 1 which is a modification of (4.1).

Assume for brevity that the root of  $h(\theta) = 0$ ,  $\theta_*$ , is unique. The rationale behind this recursion is that for sufficiently regular scenarios, one may expect the event  $\{\langle H(\theta_{i-1}, X_i), H(\theta_i, X_{i+1}) \rangle < 0\}$  to occur with higher probability when  $\theta_i$  is in

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**Algorithm 1** Adaptive step-size algorithm

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- $X_{i+1} \sim P_{\theta_i}(X_i, \cdot)$
  - $\theta_{i+1} = \theta_i + \gamma(s_i)H(\theta_i, X_{i+1})$
  - $s_{i+1} = s_i + \mathbb{I}\{\langle H(\theta_{i-1}, X_i), H(\theta_i, X_{i+1}) \rangle < 0\}$
-

a neighborhood  $B(\theta_*, \epsilon)$  of  $\theta_*$  than when outside this neighborhood. As a result  $\gamma_i := \gamma(s_i)$  decreases slowly as long as  $\{\theta_i\}$  is outside this neighborhood of  $\theta_*$ , and decreases much faster whenever  $\{\theta_i\}$  approaches  $\theta^*$ . Convergence to  $\theta_*$  requires that  $\gamma_i \rightarrow 0$  or equivalently that  $s_i \rightarrow \infty$  with probability one. This means that one should show that for any  $\gamma \in \{\gamma(0), \gamma(1), \gamma(2), \dots\}$ , the fixed stepsize sequence  $\vartheta_i^\gamma = \vartheta_{i-1}^\gamma + \gamma H(\vartheta_{i-1}^\gamma, X_i^\gamma)$  for  $i \geq 1$  is recurrent in the aforementioned neighborhood, which is the essence of the proof of [8]. Our results allow one to establish that the homogeneous Markov chain  $\{\vartheta_i^\gamma, X_i^\gamma\}$  is recurrent in some set  $\mathcal{W}_M \times \mathcal{C}$ , the first crucial step of the proof of [8]. A detailed analysis of such a result is, however, beyond the scope of the present paper.

**5. Examples: Some adaptive MCMC algorithms.** In this section we illustrate how the results established in Sections 3–4 can be straightforwardly applied to a variety of adaptive Markov chain Monte Carlo (MCMC) algorithms, where the aim is to automatically optimally choose the parameter  $\theta$  of a family of Markov chain transition probabilities  $\{P_\theta, \theta \in \Theta\}$ , defined on some  $X \subset \mathbb{R}$  sharing a common invariant distribution with density  $\pi(\cdot)$  with respect to the Lebesgue measure. More specifically, we focus here on the symmetric random walk Metropolis (SRWM) algorithm with transition probability defined for  $(\theta, x, A) \in \Theta \times X \times \mathcal{B}(X)$  as

$$(5.1) \quad P_\theta(x, A) = \int_{A-x} \alpha(x, x+z) q_\theta(z) dz + \mathbb{I}\{x \in A\} \int_{X-x} (1 - \alpha(x, x+z)) q_\theta(z) dz,$$

where for any  $x, y \in X^2$ ,  $\alpha(x, y) := 1 \wedge \pi(y)/\pi(x)$  and  $\{q_\theta(\cdot), \theta \in \Theta\}$  is a family of symmetric increment probability densities with respect to the Lebesgue measure defined on  $Z \times \mathcal{B}(Z)$  for some  $Z \subset X$ . Various choices for  $q_\theta(\cdot)$  are possible.

The AM (adaptive Metropolis) algorithm [10, 11] is concerned with the situation where  $X = \mathbb{R}^{n_x}$  for some  $n_x \geq 1$  and  $\theta = [\mu | \Gamma] \in \Theta = \mathbb{R}^{n_x} \times \mathcal{C}$  where  $\mathcal{C} \subset \mathbb{R}^{n_x \times n_x}$  is the cone of symmetric positive definite matrices and  $q_\theta(z) := \det^{-1/2}((2.38^2/n_x)(\Gamma + \epsilon_{AM} I_{n_x \times n_x})) \times q(((2.38^2/n_x)(\Gamma + \epsilon_{AM} I_{n_x \times n_x}))^{-1/2} z)$  for  $q(z) = \mathcal{N}(z; 0, I_{n_x \times n_x})$  and some  $\epsilon_{AM} \in (0, 1)$ . In fact other choices for  $q(\cdot)$  are possible as long as it is symmetric, that is,  $q(z) = q(-z)$  for all  $z \in Z$ . In [9] it is shown that in some circumstances the “optimal” covariance matrix for the Normal-SRWM is  $\Gamma_\pi$ , where  $\Gamma_\pi$  is the true covariance matrix of the target distribution  $\pi(\cdot)$ , assumed here to exist. The AM algorithm of [11] essentially implements the following algorithm to estimate  $\Gamma$  on the fly. Let  $\epsilon_{AM} > 0$  and let  $X_0 = x \in X$ , then for  $i \geq 0$  and with  $\theta_i := [\mu_i | \Gamma_i]$  here one can consider Algorithm 2.

It was realized in [3] that this algorithm is a particular instance of (4.1) where  $H : \Theta \times X \rightarrow \Theta$  is

$$(5.3) \quad H(\theta, x) := [x - \mu | (x - \mu)(x - \mu)^\top - \Gamma]^\top,$$

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**Algorithm 2** AM algorithm, iteration  $i + 1$ 

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- Sample  $X_{i+1} \sim P_{\theta_i}(X_i, \cdot)$
- Update of the tuning parameter

$$(5.2) \quad \begin{aligned} \mu_{i+1} &= \mu_i + \gamma_{i+1}(X_{i+1} - \mu_i), \\ \Gamma_{i+1} &= \Gamma_i + \gamma_{i+1}((X_{i+1} - \mu_i)(X_{i+1} - \mu_i)^\top - \Gamma_i). \end{aligned}$$


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and the corresponding mean field is

$$h(\theta) = [\mu_\pi - \mu | (\mu_\pi - \mu)(\mu_\pi - \mu)^\top + \Gamma_\pi - \Gamma]^\top.$$

We show in Section 5.2 (Theorem 5) that the stability of these recursions is a direct consequence of the result of Section 3 and a result from [14], which establishes (A2) for a class of target distribution densities  $\pi(\cdot)$ . The result of Section 4 then directly applies to the AM algorithm, leading to the conclusion that  $\{\theta_i\}$  eventually remains in a compact set with probability one. While the boundedness of  $\{\theta_i\}$  has already been established in [14] using different arguments, our results are more general in several ways. For example, Theorem 5 shows that the AM algorithm is stable when the sequence of stepsizes  $\{\gamma_i\}$  is constant, which opens up the way for the analysis of more sophisticated and robust versions of the AM algorithm. Theorem 5 also shows that the AM algorithm is also stable for heavier tailed distributions than in [14], in the situation where  $\mathsf{X} = \mathbb{R}$ , for both decreasing or constant stepsize sequences. As should be clear from our current analysis, a full study of the multivariate scenario is a different (and significant) research project.

We now turn to another type of popular adaptive scheme for the SRWM. Let  $\mathsf{X} = \mathbb{R}^{n_x}$  and  $\Theta = \mathbb{R}$ . Suppose  $q(\cdot)$  is a symmetric probability density on  $\mathsf{X}$ , and define the family of proposal distributions  $\{q_\theta(\cdot), \theta \in \Theta\}$  as  $q_\theta(z) := \exp(-\theta)q(\exp(-\theta)z)$ . A possible increment probability density is again  $q_\theta(z) = \mathcal{N}(z, \sigma := \exp(\theta))$ . Let  $\alpha_* \in (0, 1)$  be a desired mean acceptance probability for the SRWM. The following algorithm aims to optimize  $\theta^*$  in order to achieve an expected acceptance rate of  $\alpha_*$  [3] and is often used as one of the components of more sophisticated schemes. The resulting procedure is displayed in Algorithm 3.

In Section 5.1 we prove the stability of  $\{\theta_i, X_i\}$  for a broad class of probability densities  $\pi(\cdot)$ , including a heavy-tailed scenario and situations where the stepsize sequence is constant (Theorem 6). It should be pointed out that in this case, in contrast with the AM algorithm scenario, we do not require a lower bound on the scaling factor  $\exp(\theta)$ , which requires establishing (A2) for both arbitrarily large and small values of  $\exp(\theta)$  and leads us to proving the new result Theorem 4 (a stability result has been proved in [15], but in a less general scenario). In fact the theory we have developed suggests improvements on this standard algorithm whose stability can be easily established thanks to the theory developed earlier in the paper. An example is given below: the rationale behind the algorithm is that for

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**Algorithm 3** Coerced acceptance probability RWM, iteration  $i + 1$ 


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- Update the state  $X_i, Y_i$ , with  $Z_{i+1} \sim q_{\theta_i}(\cdot)$

$$Y_{i+1} = X_i + Z_{i+1},$$

$$X_{i+1} = \begin{cases} Y_{i+1} & \text{with probability } \alpha(X_i, Y_{i+1}), \\ X_i & \text{otherwise} \end{cases}$$

- Update the scaling parameter

$$\theta_{i+1} = \theta_i + \gamma_{i+1}(\alpha(X_i, Y_{i+1}) - \alpha_*).$$


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very poor initializations, the increments on the parameter are initially large, while still leading to a stable dynamic. See Algorithm 4.

The proofs of stability of the three algorithms above rely on common key intermediate results. In Section 5.1 we establish (A2) for the SRWM under two different sets of assumptions on  $\pi(\cdot)$  and  $q(\cdot)$ . In Section 5.2 we establish (A3) for the AM algorithm and conclude with Theorem 5, while in Section 5.3 we establish (A3) for the coerced acceptance algorithms, and conclude with Theorem 6.

### 5.1. Establishing (A2) for SRWM algorithms.

5.1.1. *The superexponential “ $\Gamma + \epsilon_{\text{AM}} I_{n_x \times n_x}$ ” scenario.* In this section we state a result which establishes that (A2) is satisfied for the SRWM transition probability on  $\mathbf{X} = \mathbb{R}^{n_x}$  under suitable conditions on  $\pi(\cdot)$  and  $q(\cdot)$  [12].

(A6) The probability distribution  $\pi(\cdot)$  has the following properties:

- (1) it is positive on every compact set and continuously differentiable;
- (2) there exists  $\rho > 1$  such that

$$(5.4) \quad \lim_{R \rightarrow +\infty} \sup_{\{x: |x| \geq R\}} \frac{x}{|x|^\rho} \cdot \nabla \log \pi(x) = -\infty;$$

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**Algorithm 4** Fast coerced acceptance probability RWM, iteration  $i + 1$ 


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- Update the state  $X_i, Y_i$ , with  $Z_{i+1} \sim q_{\theta_i}(\cdot)$

$$Y_{i+1} = X_i + Z_{i+1},$$

$$X_{i+1} = \begin{cases} Y_{i+1} & \text{with probability } \alpha(X_i, Y_{i+1}), \\ X_i & \text{otherwise} \end{cases}$$

- Update the scaling parameter

$$\theta_{i+1} = \theta_i + \gamma_{i+1}(|\theta_i| + 1)(\alpha(X_i, Y_{i+1}) - \alpha_*).$$


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(3) the contours  $\partial A(x) = \{y : \pi(y) = \pi(x)\}$  are asymptotically regular, that is, for some  $R > 0$ ,

$$(5.5) \quad \sup_{\{x:|x|\geq R\}} \frac{x}{|x|} \cdot \frac{\nabla\pi(x)}{|\nabla\pi(x)|} < 0;$$

(4) the proposal distribution density  $q$  is that of a standardized Gaussian or Student's  $t$ -distribution.

The following theorem quantifies the way in which ergodicity of the SRWM vanishes under (A6) as some of its eigenvalues become large. The norm used for matrices below is  $|A| = \sqrt{\text{Tr}(AA^T)}$  and recall that here  $\theta = [\mu|\Gamma] \in \Theta = \mathbb{R}^{n_x} \times \mathcal{C}$ .

PROPOSITION 1. *Let  $\eta \in (0, 1)$  and  $V(x) \propto \pi^{-\eta}(x)$ . Under (A6) one can choose  $V \geq 1$  and there exist  $a, b \in (0, \infty)$  and  $\mathcal{C} = B(0, R)$  for some  $R > 0$  such that for any  $\theta, x \in \Theta \times \mathcal{X}$ ,*

$$P_\theta V(x) \leq (1 - a/\sqrt{\det(\Gamma + \epsilon_{\text{AM}} I_{n_x \times n_x})})V(x) + b\mathbb{I}\{x \in \mathcal{C}\},$$

and we note that for any  $\Gamma \in \mathcal{C}$ ,

$$\sqrt{\det(\Gamma)} \leq n_x^{-n_x/4} |\Gamma|^{n_x/2}.$$

PROOF. The first statement is proved in [14], Proposition 15, and the second statement follows from the standard arithmetic/geometric mean inequality applied to the eigenvalues of  $\Gamma^2$ ,

$$(5.6) \quad \det(\Gamma^2) \leq \left(\frac{\text{Tr}(\Gamma^2)}{n_x}\right)^{n_x} = \left(\frac{|\Gamma|^2}{n_x}\right)^{n_x}. \quad \square$$

5.1.2. *Establishing (A2) for the AM algorithms with weak tail assumptions.* In this section we prove (A2) for the SRWM algorithm on  $\mathcal{X} = \mathbb{R}$  in the situation where no lower bound on the scaling parameter of the proposal distribution is assumed and under a weaker assumption on the vanishing rate of the tails of the target density  $\pi(\cdot)$  than in the previous subsection. More precisely, let  $P_\theta$  denote here the random-walk Metropolis kernel with symmetric proposal distribution  $q_\theta(z) = \exp(-\theta)q(z/\exp(\theta))$  for  $\theta \in \Theta := (-\infty, \infty)$ . For notational simplicity, in this subsection, we introduce the piece of notation  $\sigma = \exp(\theta)$  and use  $P_\sigma$  instead of  $P_{\log \sigma}$ ,  $q_\sigma$  instead of  $q_{\log \sigma}$  throughout and say that  $\sigma \in \exp(\Theta)$ . We will also use the piece of notation  $\ell(x) := \log \pi(x)$ . We require the following assumptions on  $\pi(\cdot)$  and the increment proposal density  $q_\sigma$ :

(A7)

(1) The target distribution  $\pi(\cdot)$  on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$  has the following properties:

(a) It has a density  $\pi(x)$  with respect to the Lebesgue measure,

- (b)  $\pi(x)$  is bounded away from 0 on any compact set of  $\mathbb{R}$ ,  
(c)  $\ell(x)$  is twice differentiable. We denote  $\ell'(x) := \nabla \ell(x)$  and  $\ell''(x) := \nabla^2 \ell(x)$ ,  
(d) for any  $M > 0$ , defining  $\epsilon_x := M/|\ell'(x)|$ ,

$$\lim_{R \rightarrow \infty} \sup_{x \in B^c(0, R)} \sup_{|t| \leq \epsilon_x} \frac{|\ell''(x+t)|}{|\ell'(x+t)|^2} = 0,$$

$$\lim_{R \rightarrow \infty} \sup_{x \in B^c(0, R)} \sup_{|t| \leq \epsilon_x} \frac{|\ell''(x+t)|}{|\ell'(x)|^2} = 0,$$

$$\lim_{R \rightarrow \infty} \sup_{x \in B^c(0, R)} \sup_{0 \leq t \leq \epsilon_x} \left| \frac{|\ell'(x-t)|}{|\ell'(x+t)|} - 1 \right| = 0.$$

- (2) The tails of  $\pi(x)$  decay at a minimum rate characterized as follows: there exist  $p \in (0, 1)$  such that

$$\lim_{R \rightarrow \infty} \sup_{x \in B^c(0, R)} \frac{\ell'(x)}{|x|^{p-1}} < 0 \quad \text{and}$$

$$\forall \gamma \in (0, 1) \quad \lim_{R \rightarrow \infty} \inf_{x \in B^c(0, R)} \frac{\pi^{-\gamma}(x)}{|\ell'(x)|} > 0.$$

- (3) The increment proposal density  $q_\sigma(z)$  is of the form  $q_\sigma(z) = \frac{1}{\sigma} q(z/\sigma)$  for some symmetric probability density  $q(z)$ , such that  $\text{supp}(q) = [-1, 1] =: \mathbf{Z}$  and  $q(\cdot) : \mathbf{Z} \rightarrow [q, \bar{q}]$  for  $q, \bar{q} \in (0, \infty)$ .

REMARK 4. Consider  $\ell(x) = C - |x|^\alpha$ , for  $|x| \geq R_\ell$  and  $\alpha > 0$ . Then for  $x \geq R_\ell$ ,  $\ell'(x) = -\alpha x^{\alpha-1}$  and  $\ell''(x) = -\alpha(\alpha-1)x^{\alpha-2}$  and all the conditions in (A7) are satisfied.

REMARK 5. The support condition on  $q$  can be removed, but this requires one to control additional “tail integral” terms in the proofs of this section, which would add further to already long arguments. We have opted for this presentation for brevity and clarity since it is the terms that we handle which are both crucial and difficult to control.

The following theorem establishes (A2) for the scalar SRWM with  $V(x) \propto \pi^{-\beta}(x)$  for some  $\beta \in (0, 1)$  under (A7).

THEOREM 4. Consider the SRWM targeting  $\pi(\cdot)$  and with increment proposal density  $q_\sigma$ . Assume they satisfy (A7), and let  $V(x) := c\pi^{-\eta}(x)$  for some  $\eta \in (0, 1)$  and  $c \in (0, \infty)$  such that  $V \geq 1$ . Then for any  $\iota \in (0, 1)$  there exists  $R \geq 0$  and  $a_0 \in (0, \infty)$  such that:

(1) for any  $x \in B^c(0, R)$ , with  $\tilde{a}^{-1}(\sigma) := a_0/(\sigma \vee \sigma^{-2})$ ,

$$P_\sigma V(x) \leq V(x) - \tilde{a}^{-1}(\sigma) V'(x),$$

(2) there exists a constant  $b \in (0, \infty)$  such that for all  $\sigma, x \in \exp(\Theta) \times B(0, R)$  we have  $P_\sigma V(x) \leq b$ .

PROOF. Let  $\iota \in (0, 1)$  and  $R \geq R_0$  such that  $\inf_{x \in B^c(0, R)} V^{1-\iota}(x) |\ell'(x)|^2 > 0$  and  $\inf_{x \in B^c(0, R)} V^{1-\iota}(x) / |\ell'(x)| > 0$ , where  $R_0$  is given in Proposition 2. The existence of  $R$  is ensured by (A7)(2) and the choice of  $V$ . Indeed, from the assumption, for  $x \in B^c(0, R)$ , we have from Lemma 2 that  $V(x) \geq C_{\Upsilon, 2} \exp(\eta C_{\Upsilon, 1}^{-1} |x|^p)$  for some constant  $C > 0$  and  $|\ell'(x)| \geq C_\ell |x|^{p-1}$  for  $x \geq R_\ell$  for some  $C_\ell, R_\ell > 0$ , and we can conclude. From Proposition 2 below, for  $x \in B^c(0, R)$  and  $\sigma \leq c_0 / |\ell'(x)|$ , we have

$$\begin{aligned} P_\sigma V(x) &\leq V(x) - a'_0 \sigma^2 |\ell'(x)|^2 V(x) \\ &\leq V(x) - a'_0 \inf_{x_0 \in B^c(0, R)} |\ell'(x_0)|^2 V^{1-\iota}(x_0) \times \sigma^2 V'(x). \end{aligned}$$

For  $|x| \geq \sigma \geq c_0 / |\ell'(x)|$ , we have

$$\begin{aligned} P_\sigma V(x) &\leq V(x) - a'_0 V(x) / |\sigma \ell'(x)| \\ &\leq V(x) - a'_0 \inf_{x_0 \in B^c(0, R)} V^{1-\iota}(x_0) / |\ell'(x_0)| \sigma^{-1} V'(x). \end{aligned}$$

For  $\sigma \geq |x| \geq R$ , we have

$$\begin{aligned} P_\sigma V(x) &\leq V(x) - a'_0 |x| \times V(x) / \sigma \\ &\leq V(x) - a'_0 R \times V(x) / \sigma. \end{aligned}$$

Now we can use the trivial inequalities  $\sigma^2 \leq \sigma \leq 1 \leq \sigma^{-1}$  (case  $\sigma \in (0, 1]$ ) and  $\sigma^{-1} \leq 1 \leq \sigma \leq \sigma^2$  [case  $\sigma \in (1, \infty)$ ] which lead to the following upper bound:

$$P_\sigma V(x) \leq V(x) - a_0 (\sigma^{-1} \wedge \sigma^2) V'(x).$$

The second claim follows immediately from the bound  $PV(x) \leq 2V(x)$  easily obtained from (C.4) and the fact that  $\sup_{x \in B(0, R)} V(x) < \infty$ .  $\square$

PROPOSITION 2. Consider the SRWM targetting  $\pi(\cdot)$  satisfying (A7). Let  $V(x) := c\pi^{-\eta}(x)$  for some  $\eta \in (0, 1)$  and  $c$  such that for all  $x \in \mathbf{X}$ ,  $V(x) \geq 1$ . Then there exist  $a'_0, R_0 > 0$  such that for any  $x \in B^c(0, R_0)$  and any  $\sigma \in \exp(\Theta) = (0, \infty)$ ,

$$\frac{P_\sigma V(x)}{V(x)} - 1 \leq -a'_0 \times \begin{cases} \sigma^2 |\ell'(x)|^2, & \text{if } \sigma |\ell'(x)| < c_0, \\ 1 / |\sigma \ell'(x)|, & \text{if } |x| \geq \sigma \geq c_0 / |\ell'(x)|, \\ |x| / \sigma, & \text{if } 1 > |x| / \sigma. \end{cases}$$

PROOF. Without loss of generality we detail the situation where  $x > 0$  since the case  $x < 0$  can be straightforwardly addressed by considering the density  $\pi_-(x) := \pi(-x)$  which also satisfies (A7), and hence Lemmata 3, 4 and 5 (given below). In what follows the terms  $T_i(\sigma, x)$  for  $i = 1, 2, 3, 4$  are defined in Lemma 3. Choose  $R \geq R_{PV} \vee R_\psi \vee R_T$  such that for  $x \in B^c(0, R)$ ,  $x \geq c_0/|\ell'(x)|$ , where  $R_{PV}$ ,  $R_\psi$ ,  $R_T$  and  $c_0$  are as in Lemmata 3, 4 and 5. First from Lemma 4, we have for  $x \geq c_0/|\ell'(x)|$  and any  $\sigma \in \exp(\Theta)$ ,

$$\begin{aligned} T_1(\sigma, x) &= \int_0^{\sigma \wedge c_0/|\ell'(x)|} \psi_x(z) q_\sigma(z) \, dz \\ &\quad + \mathbb{I}\{\sigma \geq c_0/|\ell'(x)|\} \int_{c_0/|\ell'(x)|}^{\sigma \wedge x} \psi_x(z) q_\sigma(z) \, dz \\ &\leq -\epsilon_\psi |\ell'(x)|^2 \int_0^{\sigma \wedge c_0/|\ell'(x)|} z^2 q_\sigma(z) \, dz \\ &\quad - \epsilon_\psi \mathbb{I}\{\sigma \geq c_0/|\ell'(x)|\} \int_{c_0/|\ell'(x)|}^{\sigma \wedge x} q_\sigma(z) \, dz \\ &\leq -\epsilon_\psi \underline{q}/3 |\ell'(x)|^2 [\sigma \wedge c_0/|\ell'(x)|]^3 / \sigma \\ &\quad - \epsilon_\psi \underline{q} \mathbb{I}\{\sigma \geq c_0/|\ell'(x)|\} [\sigma \wedge x - c_0/|\ell'(x)|] / \sigma, \end{aligned}$$

and therefore for  $\sigma \leq x$

$$\begin{aligned} T_1(\sigma, x) &\leq -\epsilon_\psi \underline{q} \left[ \mathbb{I}\left\{\sigma < \frac{c_0}{|\ell'(x)|}\right\} \frac{\sigma^2 |\ell'(x)|^2}{3} \right. \\ &\quad \left. + \mathbb{I}\left\{\sigma \geq \frac{c_0}{|\ell'(x)|}\right\} \left( \frac{c_0^3}{3\sigma |\ell'(x)|} + 1 - \frac{c_0}{\sigma |\ell'(x)|} \right) \right] \\ &\leq \frac{-\epsilon_\psi \underline{q}}{3} \left[ \sigma^2 |\ell'(x)|^2 \times \mathbb{I}\left\{\sigma < \frac{c_0}{|\ell'(x)|}\right\} \right. \\ &\quad \left. + \frac{c_0^3}{\sigma |\ell'(x)|} \times \mathbb{I}\left\{x \geq \sigma \geq \frac{c_0}{|\ell'(x)|}\right\} \right]. \end{aligned}$$

Now from Lemma 5 for  $\sigma \geq x \geq R$  we have

$$\begin{aligned} T_1(\sigma, x) + T_2(\sigma, x) &\leq -\epsilon_T \times x / \sigma, \\ T_3(\sigma, x) &\leq 0, \end{aligned}$$

and for  $\sigma \geq x - \Upsilon(x)$  we have

$$T_3(\sigma, x) + T_4(\sigma, x) \leq -\epsilon_T \times (-\Upsilon(x)) / \sigma$$

and we conclude with Lemma 3 and by treating the case where  $x < 0$  in a similar fashion.  $\square$

Note that, as pointed out in the proof of Proposition 2, it is sufficient to specialize most of the results of Lemmata 2, 3, 4 and 5 (stated below and proved in Appendix C) to the case  $x > 0$ . The following lemma establishes some key properties implied specifically by (A7)(2), which will also be used in Section 5.3.

LEMMA 2. *Assume that  $\pi(\cdot) > 0$ , is differentiable and satisfies (A7)(2), define for any  $\gamma > 0$ ,*

$$I_\gamma(x) := \int_0^\infty \left( \frac{\pi(x + \operatorname{sgn}(x)z)}{\pi(x)} \right)^\gamma dz \quad \text{and}$$

$$J_\gamma(x) := \int_0^{|x|} \left( \frac{\pi(x)}{\pi(x - \operatorname{sgn}(x)z)} \right)^\gamma dz,$$

with  $\operatorname{sgn}(x) := x/|x|$  for  $x \neq 0$  and for any  $x > 0$ ,  $\Upsilon(x) := \inf\{y \in \mathbf{X} : \pi(y) = \pi(x)\}$ . Then:

(1) *the function  $\Upsilon(\cdot)$  has the following properties:*

- (a)  $\lim_{x \rightarrow \infty} \Upsilon(x) = -\infty$ ,
- (b) there exist constants  $C_{\Upsilon,1}, C_{\Upsilon,2} \in (0, \infty)$  such that for all  $|x| \geq R_\Upsilon$

$$|\Upsilon(x)| \vee |x| \leq C_{\Upsilon,1} (-\log(\pi(x)/C_{\Upsilon,2}))^{1/p}$$

[or equivalently  $\pi^{-1}(x) \geq C_{\Upsilon,2} \exp(C_{\Upsilon,1}^{-1}[|\Upsilon(x)|^p \vee |x|^p])$ ],

(2) *and there exists a constant  $C_\gamma \in (0, \infty)$  such that for any  $x \in \mathbf{X}$ ,  $I_\gamma(x) \vee J_\gamma(x) \leq C_\gamma |x|^{1-p}$ .*

We now find a convenient expression for  $P_\sigma V(x)/V(x)$  valid for sufficiently large  $x$  and all  $\sigma$ 's.

LEMMA 3. *Assume (A7), and for  $x, \eta, s, z \in \mathbf{X} \times (0, 1) \times \{-1, 1\} \times \mathbf{Z}$  define  $\phi_{x,\eta,s}(z) := [\pi(x + sz)/\pi(x)]^\eta$  and*

$$\psi_x(z) := (\phi_{x,-\eta,-1}(z) - 1) + (\phi_{x,1-\eta,1}(z) - 1) - (\phi_{x,1,1}(z) - 1).$$

For any  $x \geq 0$ , define  $\Upsilon(x) := \inf\{y \in \mathbf{X} : \pi(x) = \pi(y)\}$ , and let  $V(x) \propto \pi^{-\eta}(x)$ . Then there exists  $R_{PV} > 0$  such that for all  $x \geq R_{PV}$  and any  $\sigma \in \exp(\Theta)$

$$\frac{P_\sigma V(x)}{V(x)} - 1 = \sum_{i=1}^4 T_i(\sigma, x)$$

with

$$T_1(\sigma, x) = \int_0^{\sigma \wedge x} \psi_x(z) q_\sigma(z) dz,$$

$$T_2(\sigma, x) = \mathbb{I}\{\sigma \geq x\} \int_x^\sigma [\phi_{x,1-\eta,1}(z) - \phi_{x,1,1}(z)] q_\sigma(z) dz,$$

$$\begin{aligned}
T_3(\sigma, x) &= \mathbb{I}\{\sigma \geq x\} \int_{\Upsilon(x)}^{(\sigma-x+\Upsilon(x)) \wedge 0} [\phi_{\Upsilon(x), -\eta, -1}(z) - 1] q_\sigma(z + x - \Upsilon(x)) dz, \\
T_4(\sigma, x) &= \mathbb{I}\{\sigma \geq x - \Upsilon(x)\} \\
&\quad \times \int_0^{\sigma-(x-\Upsilon(x))} [\phi_{\Upsilon(x), 1-\eta, -1}(z) - 1 + 1 - \phi_{\Upsilon(x), 1, -1}(z)] \\
&\quad \times q_\sigma(z + x - \Upsilon(x)) dz.
\end{aligned}$$

Here we prove some properties of  $\psi_x(z)$  which will allow us to upper bound the term  $T_1(\sigma, x)$  in the case where  $\sigma \leq x$ .

LEMMA 4. *Assume (A7)(1) and for  $\eta \in (0, 1)$  let  $\psi_x(z)$  be as in Lemma 3. Then there exist constants  $c_0, \epsilon_\psi, R_\psi > 0$  such that for all  $x \geq R_\psi$ ,  $\psi_x(z) \leq 0$  for  $z \in [0, x]$  and  $\psi_x(z)$  satisfies the following upper bounds:*

$$(5.7) \quad \psi_x(z) \leq -\epsilon_\psi \times \begin{cases} |\ell'(x)|^2 z^2, & \text{for } 0 \leq z \leq c_0/|\ell'(x)|, \\ 1, & \text{for } c_0/|\ell'(x)| \leq z \leq x. \end{cases}$$

Now in the following lemma we address the situation where  $\sigma \geq x$  and require an additional assumption on the vanishing speed of  $\ell'(x)$ .

LEMMA 5. *Assume (A7), and let  $T_i(\sigma, x)$  for  $i = 1, \dots, 4$  be as defined in Lemma 3. Then there exist  $C_T, R_T, \epsilon_T > 0$  such that for  $x \geq R_T$  and:*

(1) for  $\sigma \geq x$

$$T_1(\sigma, x) + T_2(\sigma, x) \leq -\epsilon_T \times x/\sigma,$$

(2) for  $\sigma \geq x$

$$T_3(\sigma, x) \leq 0,$$

(3) and for  $\sigma \geq x - \Upsilon(x)$

$$T_3(\sigma, x) + T_4(\sigma, x) \leq -\epsilon_T(-\Upsilon(x))/\sigma.$$

5.2. *Stability of the AM algorithms.* Thanks to Theorem 2 and its corollary we know that recurrence is ensured as soon as (A2) and (A3) are satisfied. In the previous section we established conditions on  $\pi(\cdot)$  and  $q_\theta(\cdot)$  under which (A2) is satisfied for the transition probabilities underpinning Algorithm 2. We therefore focus on checking that (A3) is satisfied. First we start with a result which, together with Theorem 3, leads to the same conclusions as [14] when  $\{\gamma_i\}$  is not constant, but also to the additional stability of the time-homogeneous Markov chain  $\{\theta_i, X_i\}$  when  $\gamma_i = \gamma_0$  for any  $i \geq 0$ .

**THEOREM 5.** Consider the controlled MC defined by Algorithm 2 for  $\mathbf{X} = \mathbb{R}^{n_x}$  with  $n_x \geq 1$  (resp., Algorithm 2 for  $\mathbf{X} = \mathbb{R}$ ), assume that  $\pi(\cdot)$  and  $q(\cdot)$  satisfy (A6) [resp., (A7)] and that  $\{\gamma_i\}$  is such that  $\limsup_{i \rightarrow \infty} \gamma_{i+1}^{-1} - \gamma_i^{-1} < 1$ . Then for any  $\epsilon > 0$  there exist  $M, R > 0$  such that with  $\mathcal{W}_M := \{\theta \in \Theta : w(\theta) \leq M\}$  for  $w(\theta) = 1 + |\mu|^{2+\epsilon} + |\Gamma|$

$$\mathbb{P}_{\theta, x} \left( \bigcap_{k=0}^{\infty} \bigcup_{i \geq k} \{(\theta_i, X_i) \in \mathcal{W}_M \times B(0, R)\} \right) = 1.$$

The proofs of the theorem for the two sets of assumptions rely on the following proposition, which establishes (A3) for a suitable Lyapunov function  $w(\cdot)$ . Note that despite its dependence on (A2) the result does not depend on the expression for  $a(\cdot)$ .

**PROPOSITION 3.** Let  $\epsilon > 0$ , and define  $w : \Theta \rightarrow [0, \infty)$  as

$$w(\theta) := 1 + |\mu|^{2+\epsilon} + |\Gamma|,$$

and assume that (A2) holds for some  $V : \mathbf{X} \rightarrow [1, \infty)$  such that for some  $\beta \in (0, 1)$ , we have  $V^\beta(x) \geq 1 + |x|^{2+\epsilon}$  for all  $x \in \mathbf{X}$ . Let  $\gamma^+ \in (0, 1)$ . Then there exists  $C > 0$  such that for any  $\gamma \in (0, \gamma^+]$ , and any  $\theta, x \in \Theta \times \mathbf{X}$

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta)$$

$$- \gamma w(\theta) \Delta \left( w(\theta)^{-\epsilon/(2+\epsilon)} + \frac{V^\beta(x) \mathbb{I}\{x \notin \mathbf{C}\} + b^\beta(\theta) \mathbb{I}\{x \in \mathbf{C}\}}{w(\theta)} \right),$$

where  $b : \Theta \rightarrow [0, \infty)$  is as in (A2) and  $\Delta : [0, \infty) \rightarrow \mathbb{R}$

$$\Delta(z) := 1 - C[z + z^{1/(2+\epsilon)}].$$

**PROOF.** For  $(x_+, \mu, \Gamma) \in \mathbf{X} \times \mathbb{R}^{n_x} \times \mathcal{C}$  and  $\gamma \in [0, 1]$ , let  $\mu_+ := \mu + \gamma[x_+ - \mu]$  and  $\Gamma_+ := \Gamma + \gamma[(\mu - x_+)(\mu - x_+)^T - \Gamma]$ . We have the two trivial inequalities

$$|\mu_+| = |\mu + \gamma[x_+ - \mu]| \leq (1 - \gamma)|\mu| + \gamma|x_+|,$$

$$\begin{aligned} |\Gamma_+| &= |\Gamma + \gamma[(\mu - x_+)(\mu - x_+)^T - \Gamma]| \\ &\leq (1 - \gamma)|\Gamma| + \gamma|(\mu - x_+)(\mu - x_+)^T| \end{aligned}$$

which imply that with  $w(\theta) := 1 + |\mu|^{2+\epsilon} + |\Gamma|$ , denoting  $\bar{\gamma} := \gamma/(1 - \gamma) < 1/(1 - \gamma^+)$ ,

$$\begin{aligned} w(\theta_+) - w(\theta) &\leq -|\mu|^{2+\epsilon} + \gamma[-|\Gamma| + |\mu - x_+|^2] \\ &\quad + (1 - \gamma)^{2+\epsilon} |\mu|^{2+\epsilon} \left[ 1 + \frac{\gamma}{1 - \gamma} \frac{|x_+|}{|\mu|} \right]^{2+\epsilon} \end{aligned}$$

$$\begin{aligned}
&\leq \gamma[-|\Gamma| + |\mu - x_+|^2] + |\mu|^{2+\epsilon}[(1 - \gamma)(1 + \bar{\gamma}|x_+|/|\mu|)^{2+\epsilon} - 1] \\
&\leq \gamma[-w(\theta) + 1 + |\mu|^{2+\epsilon} + 2(|\mu|^2 + |x_+|^2)] \\
&\quad + |\mu|^{2+\epsilon}[-\gamma(1 + \bar{\gamma}|x_+|/|\mu|)^{2+\epsilon} + (1 + \bar{\gamma}|x_+|/|\mu|)^{2+\epsilon} - 1].
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
|\mu|^{2+\epsilon}[(1 + \bar{\gamma}|x_+|/|\mu|)^{2+\epsilon} - 1] &\leq |\mu|^{2+\epsilon}(2 + \epsilon)\bar{\gamma}|x_+|/|\mu| \times (1 + \bar{\gamma}|x_+|/|\mu|)^{1+\epsilon} \\
&= \gamma \frac{2 + \epsilon}{1 - \gamma} \times |\mu|^{1+\epsilon}|x_+|(1 + \bar{\gamma}|x_+|/|\mu|)^{1+\epsilon},
\end{aligned}$$

and since  $|\mu|^{2+\epsilon}[1 - (1 + \bar{\gamma}|x_+|/|\mu|)^{2+\epsilon}] \leq 0$  we obtain the following bound:

$$\begin{aligned}
w(\theta_+) - w(\theta) &\leq \gamma w(\theta) \left[ -1 + \frac{1}{w(\theta)} + \frac{2|\mu|^2}{w(\theta)}(1 + |x_+|^2/|\mu|^2) \right. \\
&\quad \left. + \frac{(2 + \epsilon)|\mu|^{1+\epsilon}|x_+|}{1 - \gamma} \frac{1}{w(\theta)}(1 + \bar{\gamma}|x_+|/|\mu|)^{1+\epsilon} \right] \\
&\leq \gamma w(\theta)(-1 + C\Psi(\theta, x_+)),
\end{aligned}$$

for some  $C \in (0, \infty)$  and where

$$\Psi(\theta, x_+) := \left( \frac{|\mu|^2}{w(\theta)} + \frac{1 + |x_+|^2}{w(\theta)} \right) + \left( \frac{|\mu| \times |x_+|^{1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} + \frac{|x_+|^{1+1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} \right)^{1+\epsilon}.$$

Now from Jensen's inequality we have the identity  $(a + b)^{1+\epsilon} \leq 2^\epsilon(a^{1+\epsilon} + b^{1+\epsilon})$  for  $a, b > 0$  and the following equalities:

$$\begin{aligned}
\frac{|\mu| \times |x_+|^{1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} &= \frac{|\mu|}{w^{1/(2+\epsilon)}(\theta)} \left( \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right)^{1/[(1+\epsilon)(2+\epsilon)]}, \\
\frac{|x_+|^{1+1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} &= \left( \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right)^{1/(1+\epsilon)}
\end{aligned}$$

yield

$$\begin{aligned}
&\left( \frac{|\mu| \times |x_+|^{1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} + \frac{|x_+|^{1+1/(1+\epsilon)}}{w^{1/(1+\epsilon)}(\theta)} \right)^{1+\epsilon} \\
&\leq 2^\epsilon \left( \left( \frac{|\mu|^{2+\epsilon}}{w(\theta)} \right)^{(1+\epsilon)/(2+\epsilon)} \left( \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right)^{1/(2+\epsilon)} + \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right).
\end{aligned}$$

Note also that since  $|\mu|^{2+\epsilon}/w(\theta) \leq 1$ , we have

$$\frac{|\mu|^2}{w(\theta)} = \frac{(|\mu|^{2+\epsilon})^{2/(2+\epsilon)}}{w(\theta)} \leq w(\theta)^{-\epsilon/(2+\epsilon)},$$

and we therefore deduce that if for any  $x \in \mathbf{X}$ ,  $V^\beta(x) \geq 1 + |x|^{2+\epsilon}$ , then

$$\begin{aligned} \Psi(\theta, x_+) &\leq w(\theta)^{-\epsilon/(2+\epsilon)} + \frac{1 + |x_+|^2}{w(\theta)} + 2^\epsilon \left( \left( \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right)^{1/(2+\epsilon)} + \frac{|x_+|^{2+\epsilon}}{w(\theta)} \right) \\ &\leq w(\theta)^{-\epsilon/(2+\epsilon)} + 2 \frac{V^\beta(x_+)}{w(\theta)} + 2^\epsilon \left( \left( \frac{V^\beta(x_+)}{w(\theta)} \right)^{1/(2+\epsilon)} + \frac{V^\beta(x_+)}{w(\theta)} \right). \end{aligned}$$

Now by (A2) and Jensen's inequality we deduce that for some constant  $C > 1$

$$\begin{aligned} P_{\gamma, \theta} \Psi(\theta, x) &\leq w(\theta)^{-\epsilon/(2+\epsilon)} + C \left[ \left( \frac{V^\beta(x) \mathbb{I}\{x \notin \mathbf{C}\} + b^\beta(\theta) \mathbb{I}\{x \in \mathbf{C}\}}{w(\theta)} \right)^{1/(2+\epsilon)} \right. \\ &\quad \left. + \frac{V^\beta(x) \mathbb{I}\{x \notin \mathbf{C}\} + b^\beta(\theta) \mathbb{I}\{x \in \mathbf{C}\}}{w(\theta)} \right], \end{aligned}$$

and we conclude.  $\square$

### 5.2.1. Multivariate case and superexponential tails: (A6).

**PROOF OF THEOREM 5 UNDER (A6).** Let  $\epsilon > 0$ ,  $\beta \in (0, 1/(1 + n_x/2)]$  and  $V(x) \propto \pi^{-\eta}(x)$  for some  $\eta \in (0, 1)$  where the constant of proportionality is such that  $V^\beta(x) \geq 1 + |x|^{2+\epsilon}$  [which is possible as  $\pi^{-\eta}(x) \geq C_1 \exp(C_2|x|)$  for some  $C_1, C_2 > 0$ ]. From Proposition 1 there exists  $a', b' > 0$  and  $\mathbf{C} := B(0, R)$  for some  $R > 0$  such that for any  $x \in \mathbf{X}$  with  $w(\theta) := 1 + |\mu|^{2+\epsilon} + |\Gamma|$  and appropriate  $a', a'' > 0$ ,

$$\begin{aligned} P_\theta V(x) &\leq (1 - a'/|\Gamma| + \epsilon_{\text{AM}} I_{n_x \times n_x} |^{n_x/2}) V(x) + b' \mathbb{I}\{x \in \mathbf{C}\} \\ &\leq [1 - a''/(|\epsilon_{\text{AM}} I_{n_x \times n_x} |^{n_x/2} + w^{n_x/2}(\theta))] V(x) \mathbb{I}\{x \notin \mathbf{C}\} + b \mathbb{I}\{x \in \mathbf{C}\}, \end{aligned}$$

where  $b = b' + \sup_{x \in \mathbf{C}} V(x)$ . Naturally here  $\iota = 1$ . Now from Proposition 3 we have

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) - \gamma w(\theta) \Delta \left( w(\theta)^{-\epsilon/(2+\epsilon)} + \frac{V^\beta(x) \mathbb{I}\{x \notin \mathbf{C}\} + b^\beta \mathbb{I}\{x \in \mathbf{C}\}}{w(\theta)} \right)$$

with  $\Delta(z) := 1 - C[z + z^{1/(2+\epsilon)}]$ . Therefore here  $c(\theta) = w(\theta)^{-\epsilon/(2+\epsilon)}$ ,  $d(\theta) = w(\theta)^{-\epsilon/(2+\epsilon)} + b^\beta/w(\theta)$  and  $e(\theta) = w(\theta)$ . We note that  $p_\Delta = 1 \leq n_x/2 + 1 \leq \iota/\beta$ . The condition  $\beta \leq 1/(1 + n_x/2)$  implies (A3)(5) as for any  $\epsilon > 0$ , on  $V^\beta(x)/e(\theta) \geq \epsilon$

$$\frac{a(\theta)w(\theta)e^{-1}(\theta)}{V^{\iota-\beta}(x)} \leq C \frac{w^{n_x/2}(\theta)}{V^{\iota-\beta}(x)} \leq \frac{C\epsilon^{-n_x/2}}{V^{\iota-\beta(1+n_x/2)}(x)},$$

for some  $C > 0$ , and we conclude with Theorem 2.  $\square$

5.2.2. *Relaxed tail conditions, univariate scenario: (A7).* Now we draw the same conclusions when  $\mathbf{X} = \mathbb{R}$ , and  $\pi(\cdot)$  now satisfies less stringent tail conditions.

PROOF OF THEOREM 5 UNDER (A7). Let  $\iota, \eta \in (0, 1)$  and  $\beta \in (0, \iota/2]$ . Let  $V(x) \propto \pi^{-\eta}(x)$ , such that  $V^\beta(x) \geq 1 + |x|^{2+\epsilon}$  [which is possible as  $\pi^{-1}(x) \geq C_1 \exp(C_2|x|^p)$  for some  $C_1, C_2 > 0$  from Lemma 2]. From Theorem 4, there exist  $b, R > 0$  such that with  $\mathbf{C} = B(0, R)$  for any  $\theta, x \in \Theta \times \mathbf{X}$ ,

$$P_\theta V(x) \leq [V(x) - a^{-1}(\theta)V^\iota(x)]\mathbb{I}\{x \in \mathbf{C}\} + b\mathbb{I}\{x \in \mathbf{C}\},$$

with  $a^{-1}(\theta) = a_0/[(\Gamma + \epsilon_{\text{AM}})^{-1} \vee (\Gamma + \epsilon_{\text{AM}})^{1/2}] \geq a_0/[\epsilon_{\text{AM}}^{-1} \vee (\epsilon_{\text{AM}} + w(\theta))]$  and  $w(\theta) := 1 + |\mu|^{2+\epsilon} + |\Gamma|$ . From Proposition 3 we therefore have

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) - \gamma w(\theta) \Delta \left( w(\theta)^{-\epsilon/(2+\epsilon)} + \frac{V^\beta(x)\mathbb{I}\{x \notin \mathbf{C}\} + b^\beta \mathbb{I}\{x \in \mathbf{C}\}}{w(\theta)} \right)$$

with  $\Delta(z) := 1 - C[z + z^{1/(2+\epsilon)}]$ . Therefore here  $c(\theta) = w(\theta)^{-\epsilon/(2+\epsilon)}$ ,  $d(\theta) = w(\theta)^{-\epsilon/(2+\epsilon)} + b^\beta/w(\theta)$  and  $e(\theta) = w(\theta)$ . Note that we have  $p_\Delta = 1 \leq 2 \leq \iota/\beta$ . The condition  $\beta \leq \iota/2$  implies (A3)(5) as for any  $\epsilon > 0$ , on  $V^\beta(x)/e(\theta) \geq \epsilon$

$$\frac{a(\theta)w(\theta)e^{-1}(\theta)}{V^{\iota-\beta}(x)} \leq C \frac{w(\theta)}{V^{\iota-\beta}(x)} \leq \frac{C\epsilon^{-1}}{V^{\iota-2\beta}(x)},$$

for some  $C > 0$  and we conclude with Theorem 2.  $\square$

5.3. *Stability of the coerced acceptance probability algorithms.* In this subsection we establish the stability of Algorithm 3 and Algorithm 4 in a univariate setting. We proceed as in Section 5.2 and aim to apply Theorem 2 and its corollary which require (A2) and (A3) to be satisfied. A related result has been established in [15] under a more stringent condition on the decay of the tails of the target density, and not covering constant stepsize sequences  $\{\gamma_i\}$ .

THEOREM 6. *Consider the controlled MC as defined by either Algorithm 3 or Algorithm 4 for some  $\alpha_* \in (0, 1/2)$ . Assume that  $\pi(\cdot)$  and  $q(\cdot)$  satisfy (A7) and that the stepsize sequence  $\{\gamma_i\}$  is such that*

$$\left( \limsup_{i \rightarrow \infty} \gamma_{i+1}^{-1} - \gamma_i^{-1} \right) + \limsup_{i \rightarrow \infty} \gamma_i < \alpha_* \wedge \left( \frac{1}{2} - \alpha_* \right).$$

Let  $w(\theta) = \exp(|\theta|)$  for Algorithm 3 and  $w(\theta) := 1 + |\theta|^2$  for Algorithm 4. Then there exist  $M, R > 0$  such that for any  $\theta, x \in \Theta \times \mathbf{X}$ ,

$$\mathbb{P}_{\theta, x} \left( \bigcap_{k=0}^{\infty} \bigcup_{i \geq k} \{(\theta_i, X_i) \in \mathcal{W}_M : \times B(0, R)\} \right) = 1,$$

that is,  $\mathbb{P}_{\theta, x}$ -a.s.  $\{\theta_i, X_i\}$  visits  $\mathcal{W}_M \times B(0, R)$  infinitely often.

The proofs are given for the two scenarios in the following two subsections. The following lemma, whose proof can be found in Appendix C, will be useful in both scenarios.

LEMMA 6. *Assume that  $\pi(\cdot)$  is a strictly positive, differentiable probability density satisfying (A7)(2). Moreover, suppose that  $q_\sigma(z) := \sigma^{-1}q(z/\sigma)$  where  $q: \mathcal{Z} \rightarrow [0, \bar{q}]$  for  $\bar{q} > 0$  is symmetric and such that it has a finite absolute first order moment. Then, for any  $x \in \mathcal{X}$*

$$\alpha_\sigma(x) := \int_{\mathcal{Z}} \min\left\{1, \frac{\pi(x+z)}{\pi(x)}\right\} q_\sigma(z) dz,$$

there exist constants  $C_-, C_+ > 0$  such that

$$\alpha_\sigma(x) \geq 1/2 - C_- \sigma \quad \text{for } \sigma \leq 1 \text{ and } x \in \mathcal{X},$$

$$\alpha_\sigma(x) \leq C_+ \frac{(-\log \pi(x))^{1/p} \vee 1}{\sigma} \quad \text{for } \sigma \geq 1 \text{ and } x \in \mathcal{X}.$$

REMARK 6. Notice from the proof that the moment condition is assumed here in order to simplify our statement and that more general conditions are possible. Note that the restriction  $\alpha_* \in (0, 1/2)$  is practically harmless since this covers relevant values according to the scaling theory of the RWM [9].

5.3.1. *Proof in the standard scenario: Algorithm 3.* Before starting the proofs it is worth stressing on the fact that throughout

$$P_\theta(x, y; dx' \times dy') = q(x, dy')[\alpha(x, y')\delta_{y'}(dx') + (1 - \alpha(x, y'))\delta_x(dx')]$$

and hence that for any  $x, y \in \mathcal{X}$ ,  $P_\theta(x, y; \cdot) = P_\theta(x, \cdot)$  and that for notational simplicity the Lyapunov function  $V(x)$  should be understood as being the function  $V(x) \times 1$  defined on  $\mathcal{X}^2$ . The following proposition establishes part of (A3) under a condition implied by Lemma 6.

PROPOSITION 4. *Consider the controlled MC as defined in Algorithm 3 with  $\alpha_* \in (0, 1/2)$ ,  $V(x) := c\pi^{-\eta}(x)$  and assume that there exist  $C > 0$  and  $\beta \in [0, 1)$  such that for all  $(\theta, x) \in \Theta \times \mathcal{X}$*

$$\text{sgn}(\theta)(\alpha_{\exp(\theta)}(x) - \alpha_*) \leq -[\alpha_* \wedge (1/2 - \alpha_*)] + CV^\beta(x)/\exp(|\theta|).$$

Let  $\gamma_{\max} \in (0, \alpha_* \wedge (1/2 - \alpha_*))$ . Then for any  $\gamma \in (0, \gamma_{\max}]$  and  $\theta, x \in \Theta \times \mathcal{X}$  and the Lyapunov function  $w: \Theta \rightarrow [1, \infty)$  defined by  $w(\theta) := \exp(|\theta|)$

$$P_{\gamma, \theta} w(\theta, x) \leq w(\theta)$$

$$- \gamma w(\theta) \Delta \left( \mathbb{I}\{|\theta| \leq \gamma_{\max}\} C^{-1} \left( 2 + \alpha_* \wedge \left( \frac{1}{2} - \alpha_* \right) \right) + \frac{V^\beta(x)}{w(\theta)} \right),$$

with

$$\Delta(z) = (\alpha_* \wedge (\frac{1}{2} - \alpha_*)) - \gamma_{\max} - Cz.$$

PROOF. For  $|\theta| > \gamma$  and since for all  $x, y_+ \in \mathcal{X}^2$   $|\alpha(x, y_+) - \alpha_*| \leq 1$ , one can write (with  $\theta_+ = \theta + \gamma[\alpha(x, y_+) - \alpha_*]$ )

$$\begin{aligned} w(\theta_+) &= w(\theta + \gamma[\alpha(x, y_+) - \alpha_*]) \\ &= \exp(|\theta| + \operatorname{sgn}(\theta)\gamma[\alpha(x, y_+) - \alpha_*]). \end{aligned}$$

Now since  $\gamma \leq 1$ , from the inequality  $\exp(u) \leq 1 + u + u^2$  valid for  $|u| \leq 1$ , one obtains

$$w(\theta_+) \leq w(\theta)(1 + \gamma \operatorname{sgn}(\theta)[\alpha(x, y_+) - \alpha_*] + \gamma^2).$$

Taking the conditional expectations yields for  $|\theta| > \gamma$  and by assumption

$$\begin{aligned} P_{\gamma, \theta} w(\theta, x) &\leq w(\theta)(1 + \gamma \operatorname{sgn}(\theta)[\alpha_{\exp(\theta)}(x) - \alpha_*] + \gamma^2) \\ &\leq w(\theta) - \gamma w(\theta)([\alpha_* \wedge (1/2 - \alpha_*)] - CV^\beta(x)/\exp(|\theta|) - \gamma_{\max}). \end{aligned}$$

Also notice that for any  $\theta \in \Theta$  we have  $|\theta_+| \leq |\theta| + \gamma$ , whence  $w(\theta_+) \leq w(\theta) \exp(\gamma) \leq w(\theta)(1 + \gamma + \gamma^2) \leq w(\theta)(1 + 2\gamma)$  for all  $\gamma \leq \gamma_{\max} \leq 1$ . From this inequality and the display above, we deduce for all  $\theta, x \in \Theta \times \mathcal{X}$  and  $\gamma \in (0, \gamma_{\max}]$ ,

$$\begin{aligned} P_{\gamma, \theta} w(\theta, x) &\leq w(\theta) - \gamma w(\theta)([\alpha_* \wedge (\tfrac{1}{2} - \alpha_*)] - \gamma_{\max} \\ &\quad - \mathbb{I}\{|\theta| \leq \gamma_{\max}\}(2 + \alpha_* \wedge (\tfrac{1}{2} - \alpha_*)) \\ &\quad - CV^\beta(x)/w(\theta)]. \quad \square \end{aligned}$$

PROOF OF THEOREM 6 IN THE CASE OF ALGORITHM 3. First notice that there exists  $i_0 \in \mathbb{N}$  such that  $\sup_{i \geq i_0} \gamma_{i+1}^{-1} - \gamma_i^{-1} < \alpha_* \wedge (\frac{1}{2} - \alpha_*) - \gamma_{\max}$  with  $\gamma_{\max} := \sup_{i \geq i_0} \gamma_i$ . We show that (A2) and (A3) are satisfied and conclude with Theorem 2 and Corollary 1, for  $i \geq i_0$ . Let  $\eta, \iota \in (0, 1)$ ,  $\beta \in (0, \iota/3]$ , and define  $V(x) \propto \pi^{-\eta}(x)$ , such that  $V^\beta(x) \geq 1 \vee (-\log \pi(x))^{1/p}$  (which is possible since for any  $a_1, a_2, M > 0$ ,  $\sup_{0 \leq u \leq M} u^{a_1} |\log u|^{a_2} < \infty$ ). From Theorem 4, there exist  $b, R > 0$  such that for any  $\theta, x \in \Theta \times \mathcal{X}$ ,

$$P_\theta V(x) \leq [V(x) - a^{-1}(\theta)V^\iota(x)]\mathbb{I}\{x \notin \mathbf{C}\} + b\mathbb{I}\{x \in \mathbf{C}\},$$

with  $a(\theta) = [\exp(\theta) \vee \exp(-2\theta)]/a_0$  (for some  $a_0 > 0$ ) and  $\mathbf{C} = B(0, R)$ . Now with  $w(\theta) = \exp(|\theta|)$  from Lemma 6 there exists  $C > 0$  such that

$$\begin{aligned} \alpha_{\exp(\theta)}(x) &\geq 1/2 - C \frac{V^\beta(x)}{\exp(-\theta)} \quad \text{for } \theta \leq 0 \text{ and } x \in \mathcal{X}, \\ \alpha_{\exp(\theta)}(x) &\leq C + \frac{(-\log \pi(x))^{1/p} \vee 1}{\exp(\theta)} \leq C \frac{V^\beta(x)}{\exp(\theta)} \quad \text{for } \theta \geq 0 \text{ and } x \in \mathcal{X}. \end{aligned}$$

One can apply Proposition 4, leading to the existence of  $C > 0$  such that for any  $\theta, x \in \Theta \times \mathcal{X}$ ,

$$P_{\gamma, \theta} w(\theta, x) \leq w(\theta) - \gamma w(\theta) \Delta \left( \left[ c(\theta) + \frac{V^\beta(x)}{w(\theta)} \right] \mathbb{I}\{x \notin \mathbf{C}\} + d(\theta) \mathbb{I}\{x \in \mathbf{C}\} \right),$$

with

$$c(\theta) = C^{-1} \mathbb{I}\{|\theta| \leq \gamma_{\max}\} (2 + \alpha_* \wedge (\frac{1}{2} - \alpha_*)),$$

$$d(\theta) = \frac{\sup_{x \in \mathbb{C}} V^\beta(x)}{w(\theta)} + C^{-1} \mathbb{I}\{|\theta| \leq \gamma_{\max}\} \left( 2 + \alpha_* \wedge \left( \frac{1}{2} - \alpha_* \right) \right)$$

and

$$\Delta(z) = \alpha_* \wedge (\frac{1}{2} - \alpha_*) - \gamma_{\max} - Cz.$$

Notice that from our choice of  $i_0 \sup_{i \geq i_0} \gamma_{i+1}^{-1} - \gamma_i^{-1} < \Delta(0)$  and that we have  $\iota/\beta \geq 3 > p_\Delta = 1$  in (A3). Clearly here  $e(\theta) = w(\theta)$ . Now the condition  $\beta \leq \iota/3$  implies (A3)(5) as for any  $\epsilon > 0$ , on  $V^\beta(x)/e(\theta) \geq \epsilon$

$$\frac{a(\theta)w(\theta)e^{-1}(\theta)}{V^{\iota-\beta}(x)} = \frac{\exp(2|\theta|)}{a_0 V^{\iota-\beta}(x)} \leq \frac{\epsilon^{-2} a_0^{-1}}{V^{\iota-3\beta}(x)} < \infty,$$

and we conclude.  $\square$

*5.3.2. Proof for the accelerated version: Algorithm 4.* The arguments are similar to those of Section 5.3.1, but here  $w(\theta)$  is here of a different form. The following proposition is similar to Proposition 4 but takes this change of Lyapunov function into account.

**PROPOSITION 5.** *Consider Algorithm 4 with  $\alpha_* \in (0, 1/2)$ , let  $w(\theta) := 1 + |\theta|^2$  and assume that there exists  $C > 0$  and  $\beta \in [0, 1)$  such that for any  $\theta, x \in \Theta \times \mathbb{X}$*

$$\text{sgn}(\theta)(\alpha_{\exp(\theta)}(x) - \alpha_*) \leq -[\alpha_* \wedge (1/2 - \alpha_*)] + CV^\beta(x)/\exp(|\theta|).$$

*Let  $\gamma_{\max} \in (0, \alpha_* \wedge (1/2 - \alpha_*))$ . Then there exists  $C' > 0$  such that for any  $\gamma \in (0, \gamma_{\max}]$  and  $\theta, x \in \Theta \times \mathbb{X}$ ,*

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta)$$

$$- \gamma w(\theta) \Delta(\mathbb{I}\{|\theta| \leq 1\}) C'^{-1} (\alpha_* \wedge (1/2 - \alpha_*)) + V^\beta(x)/\exp(|\theta|)$$

with

$$\Delta(z) = 2[\alpha_* \wedge (1/2 - \alpha_*) - \gamma_{\max} - C'z].$$

**PROOF.** With  $\theta_+ = \theta + \gamma(1 + |\theta|)[\alpha(x, y) - \alpha_*]$  we have

$$\begin{aligned} w(\theta_+) &= w(\theta) + 2\gamma|\theta|(|\theta| + 1) \text{sgn}(\theta)(\alpha(x, y) - \alpha_*) \\ &\quad + \gamma^2(1 + |\theta|)^2 [\alpha(x, y) - \alpha_*]^2 \\ &\leq w(\theta) + 2\gamma|\theta|(|\theta| + 1) \text{sgn}(\theta)(\alpha(x, y) - \alpha_*) + 2\gamma^2 w(\theta) \end{aligned}$$

so

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) + 2\gamma|\theta|(|\theta| + 1)[-[\alpha_* \wedge (1/2 - \alpha_*)] + CV^\beta(x)/\exp(|\theta|)] \\ + 2\gamma^2 w(\theta).$$

Notice that for  $|\theta| \geq 1$  we have  $|\theta|(1 + |\theta|) \geq 1 + |\theta|^2$ . Consequently for any  $|\theta| \geq 1$  and  $x \in \mathbf{X}$  such that  $-[\alpha_* \wedge (1/2 - \alpha_*)] + CV^\beta(x)/\exp(|\theta|) \leq 0$

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) + 2\gamma w(\theta)[-[\alpha_* \wedge (1/2 - \alpha_*)] + \gamma + CV^\beta(x)/\exp(|\theta|)] \\ \leq w(\theta) - 2\gamma w(\theta)[\alpha_* \wedge (1/2 - \alpha_*) - \gamma - 2CV^\beta(x)/\exp(|\theta|)].$$

Notice that for any  $\theta \in \Theta$ ,  $|\theta|(1 + |\theta|) \leq (1 + |\theta|)^2 \leq 2(1 + |\theta|^2)$ . For the specific case  $-[\alpha_* \wedge (1/2 - \alpha_*)] + CV^\beta(x)/\exp(|\theta|) \geq 0$ , we therefore have

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) \\ + 2\gamma 2w(\theta)[-[\alpha_* \wedge (1/2 - \alpha_*)] + \gamma/2 + CV^\beta(x)/\exp(|\theta|)] \\ \leq w(\theta) + 2\gamma w(\theta)[-[\alpha_* \wedge (1/2 - \alpha_*)] + \gamma + 2CV^\beta(x)/\exp(|\theta|)],$$

and for any  $\theta, x \in \Theta \times \mathbf{X}$  one has

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) - 2\gamma w(\theta)[-2CV^\beta(x)/\exp(|\theta|) - \gamma].$$

We can now combine these intermediate results, yielding for any  $\theta, x \in \Theta \times \mathbf{X}$

$$P_{\theta, \gamma} w(\theta, x) \leq w(\theta) - \gamma 2w(\theta)[\alpha_* \wedge (1/2 - \alpha_*) - \gamma_{\max} \\ - \mathbb{I}\{|\theta| \leq 1\}(\alpha_* \wedge (1/2 - \alpha_*)) \\ - 2CV^\beta(x)/\exp(|\theta|)],$$

and we conclude.  $\square$

**PROOF OF THEOREM 6 IN THE CASE OF ALGORITHM 4.** The beginning of the proof is similar to that of Algorithm 3 by using Proposition 5 and Lemma 6, but here we set  $\beta \in (0, \iota/2)$ . This leads to the existence of  $C > 0$  such that for any  $\theta, x \in \Theta \times \mathbf{X}$ ,

$$P_{\gamma, \theta} w(\theta, x) \leq w(\theta) - \gamma w(\theta) \Delta \left( \left[ c(\theta) + \frac{V^\beta(x)}{\exp(|\theta|)} \right] \mathbb{I}\{x \notin \mathbf{C}\} + d(\theta) \mathbb{I}\{x \in \mathbf{C}\} \right),$$

with

$$c(\theta) = C'^{-1} \mathbb{I}\{|\theta| \leq 1\}(\alpha_* \wedge (\frac{1}{2} - \alpha_*)), \\ d(\theta) = 2 \frac{\sup_{x \in \mathbf{C}} V^\beta(x)}{\exp(|\theta|)} + C'^{-1} \mathbb{I}\{|\theta| \leq 1\} \left( \alpha_* \wedge \left( \frac{1}{2} - \alpha_* \right) \right)$$

and

$$\Delta(z) = 2[(\alpha_* \wedge (\frac{1}{2} - \alpha_*)) - \gamma_{\max} - C'z].$$

Notice that from our choice of  $i_0$ ,  $\sup_{i \geq i_0} \gamma_{i+1}^{-1} - \gamma_i^{-1} < \Delta(0)$  and that we have  $\iota/\beta \geq 2 > p_\Delta = 1$  in (A3). Now the condition  $\beta < \iota/2$  implies (A3)(5) as for any  $\epsilon > 0$  there exist  $C'' \in (0, \infty)$  such that on  $V^\beta(x)/e(\theta) \geq \epsilon$  [since here  $e(\theta) = \exp(|\theta|)$ ]

$$\frac{a(\theta)w(\theta)e^{-1}(\theta)}{V^{\iota-\beta}(x)} = \frac{\exp(|\theta|)[1 + |\theta|^2]}{a_0 V^{\iota-\beta}(x)} \leq C'' \frac{1 + (\log V(x))^2}{V^{\iota-2\beta}(x)} < \infty,$$

and we conclude.  $\square$

## APPENDIX A: APPENDIX FOR SECTION 1

**PROOF OF LEMMA 1.** For any  $k \geq 1$ , we introduce the stopping times  $\tau(k) := \inf\{i > k : (\theta_i, X_i) \in \mathcal{C}\}$ . We proceed by contradiction and observe first that if the claim did not hold, then there would be an integer  $i_w \leq n < \infty$  such that with positive probability the stopping time  $\tau(n)$  would be infinite, that is,  $\mathbb{P}_{\theta,x}(\tau(n) = \infty) > 0$ . We establish a result similar to [13], Proposition 11.3.3, page 266, but take care of the inhomogeneity and do not require the same precision. We introduce the following notation for simplicity:  $W_i := W_i(\theta_i, X_i)$  and for any  $m \in \mathbb{N}$ ,  $\tau^m := \tau(n) \wedge m$  (we omit the dependence on  $n$  in order to alleviate notation). Assumption (2.1) implies that for  $i \geq n + 1$ ,

$$\begin{aligned} \mathbb{E}_{\theta,x}[W_{i+1}\mathbb{I}\{\tau^m \geq i+1\}] \\ &= \mathbb{E}_{\theta,x}[W_i\mathbb{I}\{\tau^m \geq i+1\}] + \mathbb{E}_{\theta,x}[W_{i+1} - W_i \mid \mathcal{F}_i]\mathbb{I}\{\tau^m \geq i+1\}] \\ &\leq \mathbb{E}_{\theta,x}[W_i\mathbb{I}\{\tau^m \geq i\}] - \mathbb{E}_{\theta,x}[\delta_{i+1}\mathbb{I}\{\tau^m \geq i+1\}], \end{aligned}$$

and consequently, we can establish

$$\mathbb{E}_{\theta,x} \left[ \sum_{i=n+1}^{\infty} \delta_{i+1}\mathbb{I}\{\tau^m - 1 \geq i\} \right] \leq \mathbb{E}_{\theta,x}[W_{n+1}] - \mathbb{E}_{\theta,x}[W_{\tau^m}] \leq \mathbb{E}_{\theta,x}[W_{n+1}].$$

Now, by using the trivial inequality  $\mathbb{E}_{\theta,x}[\mathbb{I}\{\tau(n) = \infty\} \sum_{i=n+1}^{\infty} \delta_{i+1}\mathbb{I}\{\tau^m - 1 \geq i\}] \leq \mathbb{E}_{\theta,x}[\sum_{i=n+1}^{\infty} \delta_{i+1}\mathbb{I}\{\tau^m - 1 \geq i\}]$  and the monotone convergence theorem (thanks to our assumptions on  $\{\delta_i\}$ ) we obtain the contradictory statement

$$\mathbb{P}_{\theta,x}(\tau(n) = \infty) \sum_{i=n+1}^{\infty} \delta_{i+1} \leq \mathbb{E}_{\theta,x}[W_{n+1}] < \infty.$$

We therefore conclude that for any  $i \geq i_w$ ,  $\mathbb{P}_{\theta,x}(\tau(i) = \infty) = 0$ , and the result follows.  $\square$

## APPENDIX B: APPENDIX FOR SECTION 3

We state the following result for the reader's convenience.

**THEOREM 7** (see [2] for a proof). *Assume (A5). For any  $M \in (M_0, M_1]$  there exist  $\delta_0 > 0$  and  $\lambda_0 > 0$  such that, for all  $n \geq 1$ , all  $\vartheta_0 \in \mathcal{W}_{M_0}$ , all sequences  $\rho = \{\rho_k\}$  of nonnegative real numbers and all sequences  $\{\zeta_k\} \subset \Theta^{\mathbb{N}}$  of  $n\theta$ -dimensional vectors satisfying*

$$\sup_{1 \leq k \leq n} \rho_k \leq \lambda_0 \quad \text{and} \quad \sup_{1 \leq k \leq n} \left| \sum_{j=1}^k \rho_j \zeta_j \right| \leq \delta_0,$$

we have for  $k = 1, \dots, n$ ,  $w(\vartheta_k) \leq M$ , where  $\vartheta_k = \vartheta_{k-1} + \rho_k h(\vartheta_{k-1}) + \rho_k \zeta_k$ .

## APPENDIX C: PROOFS FROM SECTION 5

Before proving Lemmas 2–5 we state and prove an intermediate result.

**LEMMA 7.** *Let  $c, p > 0$  be constants. Then there exist constants  $M = M(c, p) \in (0, \infty)$  and  $x_0 = x_0(c, p) \in (0, \infty)$  such that*

$$\int_0^\infty \exp(-c[(x+z)^p - x^p]) dz \leq Mx^{1-p} \quad \text{for all } x \geq x_0.$$

**PROOF.** By a change of variable  $u = c(x+z)^p$ , we obtain

$$\int_0^\infty \exp(-c[(x+z)^p - x^p]) dz = \frac{e^{cx^p}}{cp} \int_{cx^p}^\infty e^{-u} u^{1/p-1} du.$$

Integration by parts yields

$$(C.1) \quad \int_{cx^p}^\infty e^{-u} u^{1/p-1} du = e^{-cx^p} (cx^p)^{1/p-1} + \left(\frac{1}{p} - 1\right) \int_{cx^p}^\infty e^{-u} u^{1/p-2} du.$$

Now, if  $p \geq 1$ , this is enough to yield the claim. Suppose then  $p \in (0, 1)$ , and fix a constant  $\lambda \in (0, 1)$ . By (C.1),

$$(1 - \lambda) \int_{cx^p}^\infty e^{-u} u^{1/p-1} du = e^{-cx^p} (cx^p)^{1/p-1} + \int_{cx^p}^\infty e^{-u} u^{1/p-1} \left[ \left(\frac{1}{p} - 1\right) \frac{1}{u} - \lambda \right] du.$$

Now, if  $cx^p \geq \left(\frac{1}{p} - 1\right) \frac{1}{\lambda}$ , the latter integrand is negative. Setting

$$x_0 := \left[ \left(\frac{1}{p} - 1\right) \frac{1}{\lambda c} \right]^{1/p},$$

we therefore have for  $x \geq x_0$  the desired bound

$$\int_0^\infty \exp(-c[(x+z)^p - x^p]) dz \leq \frac{(cx^p)^{1/p-1}}{cp(1-\lambda)} = \frac{c^{1/p-2}}{p(1-\lambda)} x^{1-p}.$$

We remark that the constant  $\lambda \in (0, 1)$  can be used to optimize the value constants  $M$  and  $x_0$ .  $\square$

**PROOF OF LEMMA 2.** First from assumption (A7)(2) there exist  $R_\ell, C_\ell > 0$  such that for all  $x \in B^c(0, R_\ell)$ , we have

$$\ell'(x) \leq -C_\ell |x|^{p-1},$$

and consequently for all  $x \in B^c(0, R_\ell)$  and  $z \geq 0$ , we have

$$\begin{aligned} \frac{\pi(x + \operatorname{sgn}(x)z)}{\pi(x)} &= \exp\left(\operatorname{sgn}(x) \int_0^z \ell'(x + \operatorname{sgn}(x)t) dt\right) \\ &\leq \exp\left(-\frac{C_\ell}{p}[|x| + z]^p - |x|^p\right). \end{aligned}$$

Consequently for any  $x \in B^c(0, R_\ell)$  we deduce that

$$(C.2) \quad \pi(x) \leq [\pi(-R_\ell) \vee \pi(R_\ell)] \exp(C_\ell/p|R_\ell|^p) \exp(-C_\ell/p|x|^p).$$

We deduce that there exists  $R_1 \geq R_\ell$  such that

$$[\pi(-R_\ell) \vee \pi(R_\ell)] \exp(C_\ell/p|R_\ell|^p) \exp(-C_\ell/pR_1^p) \leq \inf_{x \in B(0, R_\ell)} \pi(x),$$

and from  $\pi(\cdot) > 0$ , its continuity and the fact that it is monotone on both  $(-\infty, -R_\ell]$  and  $[R_1, \infty)$  we deduce the first statement. Now from (C.2) we deduce that there exists  $C_1 > 0$  such that for  $x \in [R_1, \infty)$

$$\pi(x) = \pi(\Upsilon(x)) \leq C_1 \exp(-C_\ell |\Upsilon(x)|^p)$$

which implies the existence of  $C_{\Upsilon,1}, C_{\Upsilon,2}, R_\Upsilon > 0$  such that for any  $x \in X$  such that  $|x| \geq R_\Upsilon$

$$(C.3) \quad |\Upsilon(x)| \vee |x| \leq C_{\Upsilon,1} (-\log(\pi(x)/C_{\Upsilon,2}))^{1/p}.$$

From above we have the upper bound

$$I_\gamma(x) \leq \int_0^\infty \exp(-C_\ell \gamma/p[|x + \operatorname{sgn}(x)z|^p - |x|^p]) dz.$$

We can conclude with the result of Lemma 7. We proceed similarly with  $J_\gamma(x)$  by noticing that  $\ell(x) - \ell(x - \operatorname{sgn}(x)z) = \operatorname{sgn}(x) \int_{-z}^0 \ell'(x + \operatorname{sgn}(x)t) dt \leq -C_\ell/p[|x|^p - |x - \operatorname{sgn}(x)z|^p]$  and again conclude with Lemma 7 above.  $\square$

PROOF OF LEMMA 3. Let  $\eta \in (0, 1)$ , and consider

$$\begin{aligned} P_\sigma V(x) &= \int_{\mathcal{X}} V(y) \min\left\{1, \frac{\pi(y)}{\pi(x)}\right\} q_\sigma(x, y) dy \\ &\quad + V(x) \int_{\mathcal{X}} \left(1 - \min\left\{1, \frac{\pi(y)}{\pi(x)}\right\}\right) q_\sigma(x, y) dy \\ &= \int_{A_x} V(y) q_\sigma(x, y) dy + \int_{R_x} V(y) \frac{\pi(y)}{\pi(x)} q_\sigma(x, y) dy \\ &\quad + V(x) \int_{R_x} \left(1 - \frac{\pi(y)}{\pi(x)}\right) q_\sigma(x, y) dy, \end{aligned}$$

where  $A_x := \{y \in \mathbb{R} : \pi(y) \geq \pi(x)\}$  and  $R_x := \{y \in \mathbb{R} : \pi(y) < \pi(x)\}$  are the regions of (almost) sure acceptance and possible rejection, respectively. From this expression, we obtain

$$\begin{aligned} \frac{P_\sigma V(x)}{V(x)} - 1 &= \int_{A_x} \left(\frac{V(y)}{V(x)} - 1\right) q_\sigma(x, y) dy \\ &\quad + \int_{R_x} \left[\left(\frac{V(y)}{V(x)} \frac{\pi(y)}{\pi(x)} - 1\right) + \left(1 - \frac{\pi(y)}{\pi(x)}\right)\right] q_\sigma(x, y) dy \\ \text{(C.4)} \quad &= \int_{A_x} \left[\left(\frac{\pi(y)}{\pi(x)}\right)^{-\eta} - 1\right] q_\sigma(x, y) dy \\ &\quad + \int_{R_x} \left\{\left[\left(\frac{\pi(y)}{\pi(x)}\right)^{1-\eta} - 1\right] + \left[1 - \frac{\pi(y)}{\pi(x)}\right]\right\} q_\sigma(x, y) dy. \end{aligned}$$

Notice that thanks to (A7) and Lemma 2  $\lim_{x \rightarrow \infty} \Upsilon(x) = -\infty$  and that for  $R$  sufficiently large, for any  $x \geq R$  we have that  $A_x = [\Upsilon(x), x]$  and  $R_x = (-\infty, \Upsilon(x)) \cup (x, \infty)$ . Then with  $y = x \pm z$  and by taking into account that the support of  $q_\sigma(z)$  is included in  $[-\sigma, \sigma]$ , we have

$$\begin{aligned} \frac{P_\sigma V(x)}{V(x)} - 1 &= \int_0^{(x-\Upsilon(x)) \wedge \sigma} (\phi_{x, -\eta, -1}(z) - 1) q_\sigma(z) dz \\ &\quad + \int_0^\sigma [(\phi_{x, 1-\eta, 1}(z) - 1) - (\phi_{x, 1, 1}(z) - 1)] q_\sigma(z) dz \\ &\quad + \mathbb{I}\{\sigma \geq x - \Upsilon(x)\} \\ &\quad \times \int_{x-\Upsilon(x)}^\sigma [(\phi_{x, 1-\eta, -1}(z) - 1) - (\phi_{x, 1, -1}(z) - 1)] q_\sigma(z) dz \end{aligned}$$

and therefore, because  $x - \Upsilon(x) > x$ , we may write

$$\begin{aligned} \frac{P_\sigma V(x)}{V(x)} - 1 &= \int_0^{\sigma \wedge x} \psi_x(z) q_\sigma(z) dz \\ &\quad + \mathbb{I}\{\sigma \geq x\} \int_x^\sigma [(\phi_{x, 1-\eta, 1}(z) - 1) - (\phi_{x, 1, 1}(z) - 1)] q_\sigma(z) dz \end{aligned}$$

$$\begin{aligned}
& + \mathbb{I}\{\sigma \geq x\} \int_{x \wedge \sigma}^{(x - \Upsilon(x)) \wedge \sigma} (\phi_{x, -\eta, -1}(z) - 1) q_\sigma(z) dz \\
& + \mathbb{I}\{\sigma \geq x - \Upsilon(x)\} \\
& \quad \times \int_{x - \Upsilon(x)}^\sigma [(\phi_{x, 1-\eta, -1}(z) - 1) - (\phi_{x, 1, -1}(z) - 1)] q_\sigma(z) dz,
\end{aligned}$$

and we conclude by using that  $\pi(\Upsilon(x)) = \pi(x)$  and the intermediate change of variable  $z' = \Upsilon(x) - x + z$ .  $\square$

**PROOF OF LEMMA 4.** Note first that for  $s \in \{-1, 1\}$ , because  $\phi_{x, \eta, s}(z) := [\pi(x + sz)/\pi(x)]^\eta = \exp[\eta(\ell(x + sz) - \ell(x))]$ ,

$$\phi'_{x, \eta, s}(z) = \eta s \ell'(x + sz) \phi_{x, \eta, s}(z) \quad \text{and}$$

$$\phi''_{x, \eta, s}(z) = [\eta^2 |\ell'(x + sz)|^2 + \eta \ell''(x + sz)] \phi_{x, \eta, s}(z).$$

We now prove the desired upper bounds on  $\psi_x(z)$  by considering the following three cases: (a)  $0 \leq z \leq c_0/|\ell'(x)|$ , (b)  $c_0/|\ell'(x)| \leq z \leq C_0/|\ell'(x)|$  and (c)  $C_0/|\ell'(x)| \leq z \leq x$  for an appropriate choice of the constants  $c_0, C_0 > 0$  to be determined.

Case (a)  $0 \leq z \leq c_0/|\ell'(x)|$ . We consider a first-order Taylor expansion of  $\psi_x(z)$  at  $z_0 = 0$  with integral error form and obtain

$$\begin{aligned}
\psi_x(z) & = z\eta\ell'(x) + z(1 - \eta)\ell'(x) - z\ell'(x) \\
& \quad + \int_0^z [\phi''_{x, -\eta, -1}(t) + \phi''_{x, 1-\eta, 1}(t) - \phi''_{x, 1, 1}(t)](z - t) dt \\
& = \int_0^z a_{x, \eta}(t)(z - t) dt,
\end{aligned}$$

where

$$\begin{aligned}
a_{x, \eta}(t) & := \eta^2 [|\ell'(x - t)|^2 - \eta \ell''(x - t)] \phi_{x, -\eta, -1}(t) \\
& \quad + [(1 - \eta)^2 |\ell'(x + t)|^2 + (1 - \eta) \ell''(x + t)] \phi_{x, 1-\eta, 1}(t) \\
& \quad - [|\ell'(x + t)|^2 + \ell''(x + t)] \phi_{x, 1, 1}(t).
\end{aligned}$$

We seek to upperbound  $a_{x, \eta}(t)$ . We choose  $\epsilon_0 \in (0, \eta(1 - \eta))$  and first show that for any  $c_0 \in (0, \epsilon_0/2)$ ,

$$(C.5) \quad \lim_{x \rightarrow \infty} \inf_{0 \leq z \leq c_0/|\ell'(x)|} \phi_{x, 1, 1}(z) > 1 - \epsilon_0/2.$$

Indeed, for  $0 \leq z \leq c_0/|\ell'(x)|$  and  $x$  large enough to ensure  $\ell'(x) < 0$ , we have for some  $\xi_{x, z} \in [x, x + z]$ , the following Taylor expansion:

$$\begin{aligned}
\ell(x + z) - \ell(x) & = \ell'(x)z + \frac{1}{2}z^2\ell''(x + \xi_{x, z}) \\
& \geq -c_0 - \frac{c_0^2}{2} \frac{|\ell''(x + \xi_{x, z})|}{|\ell'(x)|^2}
\end{aligned}$$

and with (A7)(1) the last term vanishes as  $x \rightarrow \infty$ , and we conclude by the assumption that  $-c_0 > -\epsilon_0/2$ .

Now choose  $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ . From (C.5) and (A7)(1) there exists  $R > 0$  such that for any  $x \geq R$ ,  $\inf_{|z| \leq c_0/|\ell'(x)|} \phi_{x,1,1}(z) \geq 1 - \epsilon_0$ ,  $\sup_{|t| \leq c_0/|\ell'(x)|} |\ell''(x+t)|/|\ell'(x+t)|^2 \leq \epsilon_1$ ,  $\sup_{|t| \leq c_0/|\ell'(x)|} |\ell'(x-t)|^2/|\ell'(x+t)|^2 \leq 1 + \epsilon_2$  and  $\sup_{|t| \leq c_0/|\ell'(x)|} |\ell''(x+t)|/|\ell'(x)|^2 \leq \epsilon_3/c_0$ . With these, and observing that for the values considered here we have  $0 \leq \phi_{x,-\eta,-1}(t), \phi_{x,1-\eta,1}(t), \phi_{x,1,1}(t) \leq 1$ , we obtain the following upper bound:

$$a_{x,\eta}(t) \leq |\ell'(x)|^2 \frac{|\ell'(x+t)|^2}{|\ell'(x)|^2} [\eta(\eta + \epsilon_1)(1 + \epsilon_2) + (1 - \eta)(1 - \eta + \epsilon_1) - (1 - \epsilon_1)(1 - \epsilon_0)].$$

We consider then the case where  $\epsilon_0, \epsilon_1$  and  $\epsilon_3$  are chosen small enough so that the term in brackets in the last display is negative. We note now that since for some  $\xi_{x,t} \in [x, x+t]$ ,

$$\ell'(x+t) = \ell'(x) + t\ell''(x + \xi_{x,t}).$$

Then with  $0 \leq t \leq c_0/|\ell'(x)|^2$  we have

$$\frac{\ell'(x+t)}{\ell'(x)} \geq 1 - c_0 \frac{|\ell''(x + \xi_{x,t})|}{|\ell'(x)|^2},$$

which leads to the following upper bound:

$$a_{x,\eta}(t) \leq |\ell'(x)|^2 (1 - \epsilon_3) [\eta(\eta + \epsilon_1)(1 + \epsilon_2) + (1 - \eta)(1 - \eta + \epsilon_1) - (1 - \epsilon_1)(1 - \epsilon_0)].$$

Notice that by our choice of  $\epsilon_0$  above, we have  $\eta^2 + (1 - \eta)^2 - (1 - \epsilon_0) \leq -\eta(1 - \eta)$ . Now since  $\epsilon_1, \epsilon_2, \epsilon_3 > 0$  can be chosen arbitrarily small above, we conclude about the existence of  $M > 0, c_0 > 0$  and  $R > 0$  such that for any  $x \geq R$

$$\sup_{|t| \leq c_0/|\ell'(x)|} a_{x,\eta}(t) \leq -M |\ell'(x)|^2,$$

and we therefore conclude that in such a case, for  $0 \leq z \leq c_0/|\ell'(x)|$

$$\psi_x(z) \leq -M \frac{1}{2} z^2 |\ell'(x)|^2.$$

Case (b)  $c_0/|\ell'(x)| \leq z \leq C_0/|\ell'(x)|$ . First notice that  $\psi_x(0) = 0$  and inspect the derivative of this function and aim to prove that it is negative. For any  $x \in X$  we have

$$\begin{aligned} \psi'_x(z) &= \eta \ell'(x-z) \phi_{x,-\eta,-1}(z) + (1-\eta) \ell'(x+z) \phi_{x,1-\eta,1}(z) \\ &\quad - \ell'(x+z) \phi_{x,1,1}(z) \\ &= \ell'(x+z) \left[ \eta \frac{\ell'(x-z)}{\ell'(x+z)} \phi_{x,-\eta,-1}(z) + (1-\eta) \phi_{x,1-\eta,1}(z) - \phi_{x,1,1}(z) \right]. \end{aligned}$$

Because  $\ell'(x+z) < 0$  and the two first terms in brackets form a convex combination, the second line of (5.7) will be established for  $c_0/|\ell'(x)| \leq z \leq C_0/|\ell'(x)|$  once we will have shown that for  $x \geq 0$  sufficiently large,

$$\phi_{x,1,1}(z) \leq \left( \frac{\ell'(x-z)}{\ell'(x+z)} \phi_{x,-\eta,-1}(z) \right) \wedge \phi_{x,1-\eta,1}(z).$$

Clearly  $1 \geq \phi_{x,1-\eta,1}(z) = \phi_{x,1,1}^{1-\eta}(z) \geq \phi_{x,1,1}(z)$ , so we are left with showing that  $\phi_{x,1,1}(z) \leq \frac{\ell'(x-z)}{\ell'(x+z)} \phi_{x,-\eta,-1}(z)$ , or equivalently,

$$\frac{\pi(x+z)}{\pi(x)} \left( \frac{\pi(x-z)}{\pi(x)} \right)^\eta \leq \frac{\ell'(x-z)}{\ell'(x+z)}.$$

We consider the following Taylor expansion:

$$\begin{aligned} \ell(x+z) - \ell(x) + \eta[\ell(x-z) - \ell(x)] \\ &= z\ell'(x) + \frac{1}{2}z^2\ell''(x + \xi_{x,z}) + \eta[-z\ell'(x) + \frac{1}{2}z^2\ell''(x + \xi_{x,-z})] \\ &= (1-\eta)z\ell'(x) + \frac{1}{2}z^2[\ell''(x + \xi_{x,z}) + \eta\ell''(x + \xi_{x,-z})] \end{aligned}$$

for some  $\xi_{x,z} \in [0, z]$  and  $\xi_{x,-z} \in [-z, 0]$ . For now choose any  $C_0 > c_0$  and notice that for  $c_0/|\ell'(x)| \leq z \leq C_0/|\ell'(x)|$ , we have that

$$\begin{aligned} (1-\eta)z\ell'(x) &\leq -c_0(1-\eta), \\ z^2[|\ell''(x + \xi_{x,z})| + \eta|\ell''(x + \xi_{x,-z})|] &\leq C_0^2 \frac{|\ell''(x + \xi_{x,z})| + \eta|\ell''(x + \xi_{x,-z})|}{|\ell'(x)|^2}. \end{aligned}$$

Let  $\epsilon_1 \in (0, c_0(1-\eta))$ , and choose  $\epsilon_2 > 0$  such that  $\exp(-c_0(1-\eta) + \epsilon_1) < 1 - \epsilon_2$ . By (A7)(1) we can conclude by letting  $x$  be sufficiently large to ensure that for  $c_0/|\ell'(x)| \leq z \leq C_0/|\ell'(x)|$ ,

$$\frac{\pi(x+z)}{\pi(x)} \left( \frac{\pi(x-z)}{\pi(x)} \right)^\eta \leq \exp(-c_0(1-\eta) + \epsilon_1) < 1 - \epsilon_2 \leq \frac{\ell'(x-z)}{\ell'(x+z)}.$$

Now using the result of case (a) we conclude that

$$\psi_x(z) \leq \psi_x\left(\frac{c_0}{|\ell'(x)|}\right) \leq -\frac{M}{2}c_0^2.$$

Case (c)  $C_0/|\ell'(x)| \leq z < x$ . We have the following simple bound:

$$(C.6) \quad \psi_x(z) \leq \left( \frac{\pi(x)}{\pi(x-z)} \right)^\eta - 1 + \left( \frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta}.$$

We inspect, for  $C_0/|\ell'(x)| \leq z \leq x$  and  $x$  large enough, the following difference:

$$\begin{aligned} \ell(x+z) - \ell(x) &= \int_0^z \ell'(x+t) dt \\ &\leq \int_0^{C_0/|\ell'(x)|} \ell'(x+t) dt \\ &\leq -C_0 \sup_{0 \leq t \leq C_0/|\ell'(x)|} \left| \frac{\ell'(x+t)}{\ell'(x)} \right|, \end{aligned}$$

and we can similarly obtain a bound on

$$\ell(x) - \ell(x-z) \leq -C_0 \sup_{0 \leq t \leq C_0/|\ell'(x)|} \left| \frac{\ell'(x-t)}{\ell'(x)} \right|.$$

From (A7)(1) and the Taylor expansion  $\ell'(x+t) = \ell'(x) + z\ell''(x + \xi_{x,t})$ , we conclude that for  $C_0$  and  $x$  sufficiently large enough, we can ensure that the upper bound in (C.6) is negative.

The proof is now concluded by choosing  $c_0$  as in (a), which leads to the first line of (5.7),  $C_0$  as in (c) and  $R$  large enough to cover cases (b) and (c), which imply the second line of (5.7).  $\square$

**PROOF OF LEMMA 5.** We start with  $T_1(\sigma, x) + T_2(\sigma, x)$ , and with the notation of Lemma 2, we obtain

$$\begin{aligned} T_1(\sigma, x) + T_2(\sigma, x) &\leq \int_0^x \left[ \left( \frac{\pi(x-z)}{\pi(x)} \right)^{-\eta} - 1 \right] q_\sigma(z) dz \\ &\quad + \int_0^\infty \left( \frac{\pi(x+z)}{\pi(x)} \right)^{1-\eta} q_\sigma(z) dz \\ &\leq \frac{q}{\sigma} [J_\eta(x) - x] + \frac{\bar{q}}{\sigma} I_{1-\eta}(x). \end{aligned}$$

For  $\sigma \geq x$ , because  $\phi_{\Upsilon(x), -\eta, -1}(z) \leq 1$  in the integration domain,

$$T_3(\sigma, x) \leq 0.$$

For  $\sigma \geq x - \Upsilon(x) \geq x$  we have on the one hand

$$\begin{aligned} T_3(\sigma, x) &= \int_{\Upsilon(x)}^0 [\phi_{\Upsilon(x), -\eta, -1}(z) - 1] q_\sigma(z+x-\Upsilon(x)) dz \\ &\leq \frac{q}{\sigma} \left( \Upsilon(x) + \int_{\Upsilon(x)}^0 \phi_{\Upsilon(x), -\eta, -1}(z) dz \right) \\ &\leq \frac{q}{\sigma} \left( \Upsilon(x) + \int_0^{-\Upsilon(x)} \phi_{\Upsilon(x), -\eta, 1}(z) dz \right) \\ &\leq \frac{q}{\sigma} (\Upsilon(x) + C|\Upsilon(x)|^{1-p}), \end{aligned}$$

where we have used Lemma 2. On the other hand we also have

$$\begin{aligned} T_4(\sigma, x) &= \int_0^{\sigma - (x - \Upsilon(x))} \left[ \left( \frac{\pi(\Upsilon(x) - z)}{\pi(\Upsilon(x))} \right)^{1-\eta} - \frac{\pi(\Upsilon(x) - z)}{\pi(\Upsilon(x))} \right] \\ &\quad \times q_\sigma(z + x - \Upsilon(x)) \, dz \\ &\leq \frac{\bar{q}}{\sigma} \int_0^\infty \left( \frac{\pi(\Upsilon(x) - z)}{\pi(\Upsilon(x))} \right)^{1-\eta} \, dz \\ &\leq \frac{C}{\sigma} (-\Upsilon(x))^{1-p}, \end{aligned}$$

where we have again used Lemma 2. We now conclude.  $\square$

**PROOF OF LEMMA 6.** For any  $x \in \mathsf{X}$  let  $A_Z(x) := \{z \in \mathsf{Z} : \pi(x+z)/\pi(x) \geq 1\}$  and  $R_Z(x) := A_Z^c(x)$  (where the complement is with respect to  $\mathsf{Z}$ ) and  $A(x) := x + A_Z(x)$ . Without loss of generality we focus on the case  $x > 0$ . From Lemma 2 there exists  $R_1 > 0$  such that for any  $x \geq R_1$ ,  $R_Z(x) = (-\infty, -x + \Upsilon(x)) \cup (0, \infty)$  and  $A_Z(x) = [-x + \Upsilon(x), 0]$ , where  $\Upsilon(x)$  is as in Lemma 2. For  $x \geq R_1$  and  $\sigma \leq 1$ , we have the inequalities

$$\begin{aligned} \alpha_\sigma(x) &= \int_{\mathsf{Z}} \min \left\{ 1, \frac{\pi(x+z)}{\pi(x)} \right\} q_\sigma(z) \, dz \\ &= 1 + \int_{R_Z(x)} \left[ \frac{\pi(x+z)}{\pi(x)} - 1 \right] q_\sigma(z) \, dz \\ &\geq 1 - \int_{R_Z(x)} q_\sigma(z) \, dz \\ &= \frac{1}{2} - \int_{-\infty}^{(-x+\Upsilon(x))/\sigma} q(z) \, dz \\ &\geq \frac{1}{2} - \int_{-\infty}^{-x/\sigma} q(z) \, dz. \end{aligned}$$

Now with  $\mu_1 < \infty$  the first-order moment of  $q$ , we notice that from Chebyshev's inequality and for  $x \geq R_1$ ,

$$\int_{x/\sigma}^\infty q(z) \, dz \leq \sigma R_1^{-1} \times \mu_1$$

from which we deduce the first statement for  $\sigma \leq 1$  and  $x \geq R_1$ . Now for  $x \geq R_1$  and  $\sigma \geq 1$ ,

$$\begin{aligned} \alpha_\sigma(x) &\leq \frac{\bar{q}}{\sigma} \left( \int_{A_Z(x) \cup R_Z(x)} \min \left\{ 1, \frac{\pi(x+z)}{\pi(x)} \right\} \, dz \right) \\ &\leq \frac{\bar{q}}{\sigma} \left( 2C_{\Upsilon,1} (-\log(\pi(x)/C_{\Upsilon,2}))^{1/p} \right) \end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \frac{\pi(\Upsilon(x) - z)}{\pi(\Upsilon(x))} dz + \int_0^\infty \frac{\pi(x + z)}{\pi(x)} dz \\
& \leq C \frac{(-\log(\pi(x)/C_{\Upsilon,2}))^{1/p}}{\sigma},
\end{aligned}$$

where we have used the results of Lemma 2 to upper bound the Lebesgue measure of  $A_Z(x)$  and the last two integrals. We now turn to the case  $0 \leq x \leq R_1$ . Let  $M > 0$  such that  $\int_M^\infty q(z) dz \leq 1/4$  and  $\sigma \leq 1$  and with  $\phi_x(z) = \pi(x + z)/\pi(x)$

$$\begin{aligned}
\alpha_\sigma(x) &= 1 + \frac{1}{\sigma} \int_Z \left(1 \wedge \frac{\pi(x + z)}{\pi(x)} - 1\right) q\left(\frac{z}{\sigma}\right) dz \\
&\geq 1 + \int_{-\infty}^{-M} \left(1 \wedge \frac{\pi(x + \sigma z)}{\pi(x)} - 1\right) q(z) dz \\
&\quad + \int_M^\infty \left(1 \wedge \frac{\pi(x + \sigma z)}{\pi(x)} - 1\right) q(z) dz \\
&\quad - \int_{-M}^M \left| \frac{\pi(x + \sigma z)}{\pi(x)} - 1 \right| q(z) dz \\
&\geq 1 - 2 \int_M^\infty q(z) dz - 2M\bar{q} \sup_{x \in B(0, R_1), z \in B(0, M)} |\phi'_x(z)| \sigma,
\end{aligned}$$

and we deduce the first statement of the lemma. We now consider the case  $0 \leq x \leq R_1$  and  $\sigma \geq 1$ . There exists (cf. the proof of Lemma 2)  $R_2 > 0$  such that for all  $x \leq R_1$

$$\begin{aligned}
\alpha_\sigma(x) &\leq \sigma^{-1} \int_{-R_2}^{R_2} q(z/\sigma) dz + \sigma^{-1} \int_{-\infty}^{-R_2} \frac{\pi(x + z)}{\pi(x)} q(z/\sigma) dz \\
&\quad + \sigma^{-1} \int_{R_2}^\infty \frac{\pi(x + z)}{\pi(x)} q(z/\sigma) dz.
\end{aligned}$$

From the proof of Lemma 2, we have the bound  $\pi(x + z)/\pi(x) \leq C_1 \exp(-C_2|z|^p)$  for some  $C_1, C_2 > 0$  and since  $q(z) \leq \bar{q}$ , we deduce the existence of  $C > 0$  such that for  $x \leq R_1$  and  $\sigma \geq 1$  we have  $\alpha_\sigma(x) \leq C/\sigma$ .  $\square$

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