

## MOMENTS AND LYAPUNOV EXPONENTS FOR THE PARABOLIC ANDERSON MODEL

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We study the parabolic Anderson model in  $(1 + 1)$  dimensions with nearest neighbor jumps and space–time white noise (discrete space/continuous time). We prove a contour integral formula for the second moment and compute the second moment Lyapunov exponent. For the model with only jumps to the right, we prove a contour integral formula for all moments and compute moment Lyapunov exponents of all orders.

### 1. Introduction and main results.

1.1. *Nearest-neighbor parabolic Anderson model.* The nearest-neighbor parabolic Anderson model on  $\mathbb{Z}$  is the solution to a coupled system of diffusions on  $[0, \infty)$  given by

$$(1) \quad \frac{d}{dt} Z_\beta(t, n) = \frac{1}{2} \Delta^{p,q} Z_\beta(t, n) + \beta Z_\beta(t, n) dW_n(t).$$

We focus here on delta function initial data  $Z_\beta(0, n) = \mathbf{1}_{n=0}$ . Here  $t \in \mathbb{R}_+$ ,  $n \in \mathbb{Z}$ , and the operator  $\Delta^{p,q}$  (which is the generator for a nearest neighbor continuous time random walk) acts on functions  $f(n)$  as

$$(2) \quad \Delta^{p,q} f(n) = pf(n - 1) + qf(n + 1) - (p + q)f(n).$$

We assume that  $p, q \geq 0$  and  $p + q = 2$ . The collection  $\{W_n(\cdot)\}_{n \in \mathbb{Z}}$  are independent Brownian motions and  $\beta \in \mathbb{R}_+$ .

1.1.1. *Population growth in random environment.* The coupled diffusions can be considered as modeling population growth in a random, quickly changing environment at each spatial location, and with migration between locations. Consider a population of many small particles living on the sites of  $\mathbb{Z}$ . There are three forces acting upon this system:

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- (1) Each particle at time  $t$  and lattice site  $n$  independently duplicates itself at rate  $r_+(t, n)$ ;
- (2) Each particle at time  $t$  and lattice site  $n$  independently dies at rate  $r_-(t, n)$ ;
- (3) Each particle at time  $t$  and lattice site  $n$  independently jumps to a neighboring site  $n - 1$  with rate  $q/2$  and  $n + 1$  with rate  $p/2$ .

Letting  $m(t, n)$  be the expected population size at time  $t$  and location  $n$ , one finds that [8]

$$\frac{d}{dt}m(t, n) = \frac{1}{2}\Delta^{p,q}m(t, n) + (r_+(t, n) - r_-(t, n))m(t, n).$$

If the duplication and death rates are independent in space and quickly mixing in time, the factor  $(r_+(t, n) - r_-(t, n))$  is well modeled by  $\beta dW_n(t)$  where  $\beta$  modulates the relative rates of jumping and duplication/death. The delta function initial data translates into starting with all the particles clustered at the origin and then allowing them to spread over time.

As explained in [8, 14], it is of physical interest for these models to understand the structure of regions in space–time in which the population size is significantly larger than expected. This phenomenon is called *intermittency*. Generally, one seeks to measure the effect of changing various parameters with respect to this phenomenon. Of specific interest are the spatial dimension (replacing  $\mathbb{Z}$  by  $\mathbb{Z}^d$ ), the strength of  $\beta$ , and the type of environmental noise (replacing space–time noise by spatially varying but constant in time noise, or noise which is itself built out of interacting particle systems); see part I of [12] and [9] for reviews of these various directions. In the present paper we restrict ourselves to the one-dimensional, space–time independent case and offer a new approach to computing the moments of this model. Section 1.2 below explains the relevance of the moments to the intermittency phenomenon.

1.1.2. *Directed polymers.* Closely related to the above branching diffusion representation, the Feynman–Kac representation for this coupled system of diffusions writes  $Z_\beta(t, n)$  as point to point partition functions for a random polymer model

$$(3) \quad Z_\beta(t, n) = \mathcal{E}_{\pi(0)=0} \left[ \mathbf{1}_{\pi(t)=n} \exp \left( \int_0^t \beta dW_{\pi(s)}(s) ds - \frac{\beta^2 t}{2} \right) \right],$$

where  $\pi(s)$  is a Markov process with state space  $\mathbb{Z}$  and generator given by  $\frac{1}{2}\Delta^{q,p}$  (which is the adjoint of  $\frac{1}{2}\Delta^{p,q}$ ), and  $\mathcal{E}_{\pi(0)=0}$  is the expectation with respect to starting  $\pi(0) = 0$ . We write  $\mathbb{E}$  for the expectation over the disorder. The polymer measure on paths  $\pi(\cdot)$  is defined as the argument of the above expectation, normalized by  $Z_\beta(t, n)$ .

Directed polymer models are important from a number of perspectives; see [10, 11] and references therein. They were introduced to study the domain walls

of Ising models with impurities at high temperature and have been applied to other problems like vortices in superconductors, roughness of crack interfaces, Burgers turbulence, and interfaces in competing bacterial colonies. They also provide a unified mathematical framework for studying a variety of different abstract and physical problems including some in stochastic optimization, bio-statistics, queuing theory and operations research, interacting particle systems and random growth models.

The above defined class of directed polymers is a generalization of the model (at  $p = 2$  and  $q = 0$ ) introduced by O’Connell–Yor [20]. The primary interest in the study of directed polymers is to understand the free energy fluctuations [i.e.,  $\log Z_\beta(t, n)$ ] and the transversal path fluctuations of the polymer measure under the limit at  $t$  and  $n$  go to infinity. For the special  $p = 2$  and  $q = 0$  case, there has been significant progress in both of these directions coming from the work of [4, 6, 19, 21]. The model is now known to be in the Kardar–Parisi–Zhang universality class, which predicts these asymptotic fluctuation behaviors. It is expected that this asymptotic behavior should not depend on the values of  $p, q$  and  $\beta$ . The present analysis of the moments of  $Z_\beta(t, n)$  constitute a step toward an analysis of this class. On the other hand, one should note that by virtue of the intermittency which we prove herein, one knows that these moments will not determine the distribution of  $Z_\beta(t, n)$ .

From the above polymer representation for  $Z_\beta(t, n)$  and the Gaussian nature of the noise, one sees (by interchanging the path expectations with the expectation over the disorder) that

$$\begin{aligned} & \mathbb{E} \left[ \prod_{i=1}^2 Z_\beta(t, n_i) \right] \\ &= \mathcal{E}_{\pi_1(0)=\pi_2(0)=0} \left[ \mathbf{1}_{\pi_1(t)=n_1, \pi_2(t)=n_2} \exp \left( \frac{\beta^2}{2} \int_0^t \mathbf{1}_{\pi_1(s)=\pi_2(s)} ds \right) \right]. \end{aligned}$$

In other words, letting  $\pi = \pi_1 - \pi_2$  and  $\mathcal{E}$  be the associated expectation, we find that

$$\mathbb{E} \left[ \prod_{i=1}^2 Z_\beta(t, n_i) \right] = \mathcal{E} \left[ \mathbf{1}_{\pi(t)=n_1-n_2} \exp \left( \frac{\beta^2}{2} \int_0^t \mathbf{1}_{\pi(s)=0} ds \right) \right].$$

This is the first moment of the partition function for a random walk  $\pi$  which feels a pinning potential of strength  $\beta^2/2$  at the origin; see [2] and references therein for more discussion on this model.

1.2. *Lyapunov exponents and intermittency.* In order to introduce and explain the mathematical definition of intermittency, we introduce two types of the Lyapunov exponents for the parabolic Anderson model. Consider a velocity  $v \in \mathbb{R}$ .

Then the *almost sure Lyapunov exponent* with respect to velocity  $\nu$  is given by

$$(4) \quad \tilde{\gamma}_1(\beta; \nu) = \lim_{t \rightarrow \infty} \frac{1}{t} \log Z_\beta(t, \lfloor \nu t \rfloor).$$

The existence of this almost sure limit is due to a sub-additivity argument (see [8], Section IV.1). The *p*th *moment Lyapunov exponent* with respect to velocity  $\nu$  is given by

$$(5) \quad \gamma_k(\beta; \nu) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[(Z_\beta(t, \lfloor \nu t \rfloor))^k].$$

If the initial data  $Z_\beta(0, n)$  is stationary with respect to shifts in  $n$ , then the exponents are, in fact, independent of  $\nu$ . We, however, consider initial data in which  $Z_\beta(0, n) = \mathbf{1}_{n=0}$ , and hence the exponents will depend on the velocity  $\nu$  nontrivially.

DEFINITION 1.1. A parabolic Anderson model shows intermittency if the Lyapunov exponents are strictly ordered as

$$\tilde{\gamma}_1(\beta; \nu) < \gamma_1(\beta; \nu) < \frac{\gamma_2(\beta; \nu)}{2} < \frac{\gamma_3(\beta; \nu)}{3} < \dots$$

The weak ordering of exponents is a consequence of Jensen's inequality (for the first inequality) and Hölder's inequality (for all subsequent inequalities). A useful fact is recorded in the following (cf. [8], Theorem III.1.2):

LEMMA 1.2. *If for any  $k \geq 1$ ,*

$$(6) \quad \frac{\gamma_k(\beta; \nu)}{k} < \frac{\gamma_{k+1}(\beta; \nu)}{k+1}$$

*then for all  $p \geq k$*

$$(7) \quad \frac{\gamma_p(\beta; \nu)}{p} < \frac{\gamma_{p+1}(\beta; \nu)}{p+1}.$$

As explained in [8], intermittent random fields are distinguished by the formation of strong pronounced spatial structures such as sharp peaks which give the main contribution to the physical processes in such media. A popular example cited therein is the observation of Zeldovich that the Solar magnetic field is intermittent since more than 99% of the magnetic energy concentrates on less than 1% of the surface area.

The above mathematical definition of intermittency is related to the presence of high peaks of  $Z_\beta(t, n)$  that dominate large time moment asymptotics. In particular,

fix  $\alpha$  such that  $\frac{\gamma_k(\beta;v)}{k} < \alpha < \frac{\gamma_{k+1}(\beta;v)}{k+1}$ ; then we know that  $\mathbb{P}(Z_\beta(t, \lfloor vt \rfloor) > e^{\alpha t}) > 0$  as  $t \rightarrow \infty$ . Writing

$$\begin{aligned} &\mathbb{E}[Z_\beta(t, \lfloor vt \rfloor)^{k+1}] \\ &= \mathbb{E}[Z_\beta(t, \lfloor vt \rfloor)^{k+1} \mathbf{1}_{Z_\beta(t, \lfloor vt \rfloor) < e^{\alpha t}}] + \mathbb{E}[Z_\beta(t, \lfloor vt \rfloor)^{k+1} \mathbf{1}_{Z_\beta(t, \lfloor vt \rfloor) > e^{\alpha t}}], \end{aligned}$$

we observe that the first term is  $\leq e^{\alpha(k+1)t}$ , but the sum of the two is asymptotically  $e^{\gamma_{k+1}(\beta;v)t}$ , which is exponentially (as  $t$  grows) larger than  $e^{\alpha(k+1)t}$ . This means that the event  $\{Z_\beta(t, \lfloor vt \rfloor) > e^{\alpha t}\}$  gives overwhelming contribution to the  $(k + 1)$ st moment. On the other hand,

$$\mathbb{E}[Z_\beta(t, \lfloor vt \rfloor)^k] \geq e^{\alpha kt} \mathbb{P}(Z_\beta(t, \lfloor vt \rfloor) > e^{\alpha t})$$

and hence, for large  $t$

$$\mathbb{P}(Z_\beta(t, \lfloor vt \rfloor) > e^{\alpha t}) \leq \frac{e^{\gamma_k(\beta;v)t}}{e^{\alpha kt}} = \exp\left\{-\left(\alpha - \frac{\gamma_k(\beta;v)}{k}\right)t\right\},$$

which is exponentially small.

In the case of spatially translation invariant ergodic solutions  $Z_\beta(t, n)$ , the consequences of intermittency may be interpreted via spatial averages over large balls at a fixed (large) time. Thus one can talk about islands where the solution is at least  $e^{((\gamma_k(\beta;v))/k)t}$  (as opposed to the typical value of  $e^{\gamma_1(\beta;v)t}$ ) whose spatial density is not more than  $e^{-(((\gamma_{k+1}(\beta;v))/(k+1)) - ((\gamma_k(\beta;v))/k))t}$ . Our results are for delta initial data  $Z_\beta(0, n) = \mathbf{1}_{n=0}$  and not stationary initial data.

In terms of the population model interpretation of  $Z_\beta(t, n)$ , the above discussion implies that knowledge of the Lyapunov exponents translates into detailed information about the spatial frequency of large clusters of population growth in space.

In this direction, the primary contribution of this paper is the precise calculation of the first two moment Lyapunov exponents for the general  $p, q$  model and the calculation of all moment Lyapunov exponents for the special  $p = 2$  and  $q = 0$  case. From the above considerations, this provides detailed information about the intermittent structure of the corresponding population growth models.

1.3. *Main results.* All of our results pertain to the nearest-neighbor parabolic Anderson model with delta initial data:  $Z_\beta(0, n) = \mathbf{1}_{n=0}$ . Our first result is a formula for the two-point moment of the model.

**THEOREM 1.3.** For  $n_1 \geq n_2$ ,

$$\begin{aligned} (8) \quad \mathbb{E}\left[\prod_{i=1}^2 Z_\beta(t, n_i)\right] &= \frac{1}{(2\pi t)^2} \oint \oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} \\ &\quad \times F_{t, n_1}^{p, q}(z_1) F_{t, n_2}^{p, q}(z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2}, \end{aligned}$$

where

$$F_{l,n}^{p,q}(z) = z^{-n} e^{(t/2)(pz+qz^{-1}-2)}$$

and where the contour of  $z_1$  is the unit circle, and the contour for  $z_2$  is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from  $z_2 = 0$ .

This theorem is proved in Section 2. This also provides an exact formula for the first moment of the pinned polymer partition function discussed above in Section 1.1.2. The following corollary follows immediately from the exact result above along with Lemma 1.2 applied for  $k = 1$ . In the symmetric case  $p = q$  this was established as a special case of the results in [8], Chapter III.

**COROLLARY 1.4.** *The  $p, q$  nearest-neighbor parabolic Anderson model displays intermittency at the velocity  $p - q$ .*

Via asymptotic analysis, Theorem 1.3 enables us to calculate the first and second moment Lyapunov exponents for the parabolic Anderson model (as well as the first moment Lyapunov for the pinned polymer partition function).

**THEOREM 1.5.** *The first moment Lyapunov exponent at velocity  $p - q$ , for the nearest-neighbor parabolic Anderson model is given by  $\gamma_1(\beta; p - q) = 0$ . The second moment Lyapunov exponent at velocity  $p - q$ , for the nearest-neighbor parabolic Anderson model is given by*

$$\gamma_2(\beta; p - q) = H_2(z_2^0),$$

where

$$H_2(z) = \frac{1}{2}(ps(z) + q(s(z))^{-1} - 2 - (p - q) \log(s(z)) + pz + qz^{-1} - 2 - (p - q) \log z)$$

with

$$s(z) = \frac{(pz - qz^{-1} + 2\beta^2) + \sqrt{(pz - qz^{-1} + 2\beta^2)^2 + 4pq}}{2p}$$

and where  $z_2^0$  is the unique solution to  $H_2'(z) = 0$  over  $z \in (0, \infty)$ .

When  $p = q = 1$ ,

$$z_2^0 = \frac{1}{2}(-\beta^2 + \sqrt{4 + \beta^4}), \quad s(z_0) = \frac{1}{2}(\beta^2 + \sqrt{4 + \beta^4}),$$

$$H_2(z_k^0) = 2(\sqrt{4 + \beta^4} - 2),$$

which implies that

$$\gamma_2(\beta; 0) = 2(\sqrt{4 + \beta^4} - 2)$$

for the standard ( $p = q$ ) parabolic Anderson model.

This theorem is proved in Section 2 via asymptotic analysis of Theorem 1.3. We include the full details only for the case  $p = q = 1$ .

REMARK 1.6. The above theorem is stated only for a velocity given by  $v = p - q$ . For general  $p - q \neq 0$  the same approach as given in the proof provides the exact values of the first and second moment Lyapunov exponents, but we forgo including this herein.

We now turn our attention to the one-sided case of the nearest-neighbor parabolic Anderson model, where  $p = 2$  and  $q = 0$ . In this case we may extend the result of Theorem 1.3 to arbitrary joint moments. For  $k \geq 1$ , define

$$(9) \quad W_{\geq 0}^k = \{\vec{n} = (n_1, n_2, \dots, n_k) \in (\mathbb{Z}_{>0})^k : n_1 \geq n_2 \geq \dots \geq n_k \geq 0\}.$$

THEOREM 1.7. For all  $k \geq 1$  and  $\vec{n} \in W_{\geq 0}^k$ ,

$$(10) \quad \mathbb{E} \left[ \prod_{i=1}^k Z_{\beta}(t, n_i) \right] = \frac{1}{(2\pi i)^k} \oint \dots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^k \frac{e^{t(z_i-1)} dz_i}{z_i^{n_i} z_i},$$

where the integration contour for  $z_a$  is a closed curve containing 0, and the image under addition by  $\beta^2$  of the integration contours for  $z_b$  for all  $b > a$  (for an illustration of possible contours see Figure 1).

This theorem is proved in Section 3. Asymptotics of this formula yield all the moment Lyapunov exponents. By Brownian scaling it suffices to consider just  $\beta = 1$ .

THEOREM 1.8. For any  $k \geq 1$  and  $v > 0$ , the  $k$ th moment Lyapunov exponent at velocity  $v$  for the one-sided ( $p = 2$  and  $q = 0$ ) nearest-neighbor parabolic

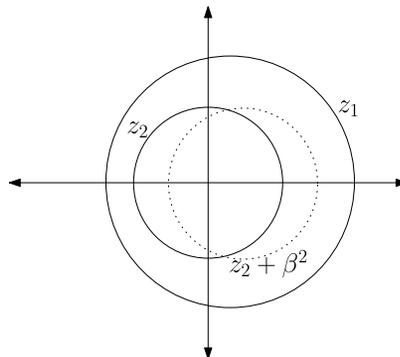


FIG. 1. Valid contours for equation (10) with  $k = 2$ . The inner contour is  $z_2$  and the  $z_1$  contour contains the image of  $z_2$  plus  $\beta^2$ .

Anderson model with  $\beta = 1$  is given by

$$\gamma_k(1; \nu) = H_k(z_k^0),$$

where

$$H_k(z) = \frac{k(k-3)}{2} + kz - \nu \log\left(\prod_{i=0}^{k-1} (z+i)\right)$$

and where  $z_k^0$  is the unique solution to  $H'_k(z) = 0$  with  $z \in (0, \infty)$ .

This theorem is proved in Section 3 via asymptotics of Theorem 1.7. Figure 2 records the plot of the various Lyapunov exponents.

Note that in the one-sided case, the almost sure Lyapunov exponent defined in (4) was conjectured in [20] and proved in [18], and it is given by (for  $\beta = 1$ )

$$\tilde{\gamma}_1(1; \nu) = -\frac{3}{2} + \inf_{t>0} (t - \nu\Psi(t)),$$

where  $\Psi(t) := [\log \Gamma]'(t)$  is the digamma function.

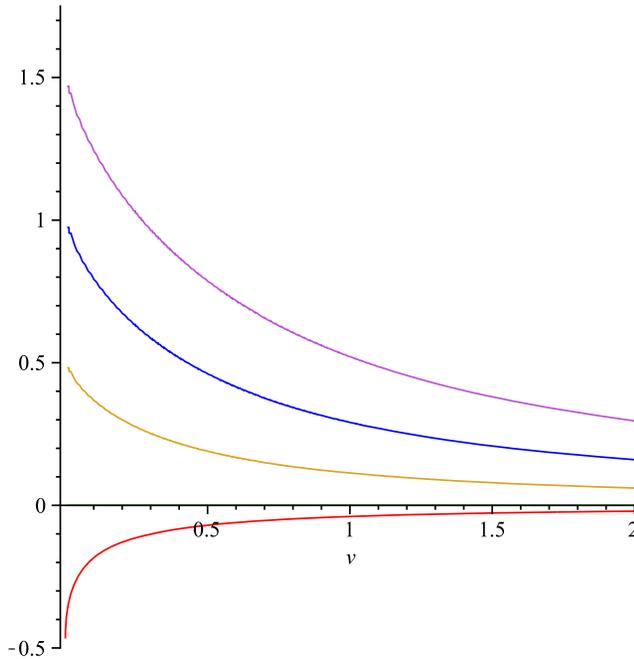


FIG. 2. Plot of one-sided parabolic Anderson model Lyapunov exponents versus the velocity  $\nu$  (which plays the role of the diffusion constant). Here we have normalized  $Z_1(t, \lfloor \nu t \rfloor)$  by the zero noise solution, so that  $\gamma_1(1; \nu) = 0$ . The lowest curve in the plot is the normalized  $\tilde{\gamma}_1(1; \nu)$ , and the higher curves are the normalized  $\gamma_k(1; \nu)/k$  (increasing in height with  $k$ ). This demonstrates the intermittency of this parabolic Anderson model, and the shape of this plot is very similar to those on page 105 of [8].

There are two ideas which are behind the results of this paper. The first idea is the content of Propositions 2.2 and 3.1 which show that one can compute the moments of the parabolic Anderson model via solving a system of coupled ODEs with spatial variables  $\vec{n} \in W_{\geq 0}^k$  and specific boundary conditions. This reduction to solving ODEs on  $W_{\geq 0}^k$  only works for  $k = 1, 2$  with the general  $p, q$  nearest-neighbor model. However, for  $p = 2$  and  $q = 0$ , the reduction holds for all  $k$ . The second idea is that the system of ODEs can be explicitly solved via a certain nested-contour integral ansatz that originated from [4]. This is the content of Propositions 2.3 and 3.3.

1.4. *Outline.* The rest of the paper is as follows: in Section 2 we show how the moments of the parabolic Anderson model can be computed via a coupled system of ODEs. We then solve this system and use this solution to prove Theorems 1.3 and 1.5. In Section 3 we show how in the one-sided model, all moments can be computed via ODEs, and we provide integral formulas which solve these ODEs. From this we are able to prove Theorems 1.7 and 1.8. In the Appendix we include a nonrigorous replica trick calculation (used extensively in the physics literature) and show how from this calculation one recovers the almost sure Lyapunov exponent for the one-sided model; we also briefly discuss the continuous space parabolic Anderson model (i.e., the stochastic heat equation with multiplicative noise) and record its moments and Lyapunov exponents.

**2. Nearest-neighbor parabolic Anderson model.** The first step in our computation of the moment Lyapunov exponents of the parabolic Anderson model is the following reduction to a coupled system of ODEs with two-body delta interaction. Recall the definition of the nearest-neighbor parabolic Anderson model  $Z_\beta(t, n)$  and the operator  $\Delta^{p,q}$  given in the Introduction. Write  $[\Delta^{p,q}]_i$  for the operator which acts as  $\Delta^{p,q}$  on the  $i$ th spatial coordinate.

PROPOSITION 2.1. Assume  $v : \mathbb{R}_+ \times \mathbb{Z}^k \rightarrow \mathbb{R}$  solves:

(1) for all  $\vec{n} \in \mathbb{Z}^k$  and  $t \in \mathbb{R}_+$ ,

$$\frac{d}{dt} v(t; \vec{n}) = \mathbf{H} v(t; \vec{n}), \quad \mathbf{H} = \frac{1}{2} \sum_{i=1}^k [\Delta^{p,q}]_i + \frac{1}{2} \beta^2 \sum_{\substack{a,b=1 \\ a \neq b}}^k \mathbf{1}_{n_a=n_b};$$

(2) for all permutations of indices  $\sigma \in S_k$ ,  $v(t; \sigma \vec{n}) = v(t; \vec{n})$ ;

(3) for all  $\vec{n} \in \mathbb{Z}^k$ ,  $\lim_{t \rightarrow 0} v(t; \vec{n}) = \prod_{i=1}^k \mathbf{1}_{n_i=0}$ ;

(4) for all  $T > 0$ , there exists  $c, C > 0$  such that for all  $\vec{n} \in \mathbb{Z}^k$  and all  $t \in [0, T]$ ,

$$|v(t; \vec{n})| \leq c e^{C \|n\|_1}.$$

Then for  $\vec{n} \in \mathbb{Z}^k$ ,  $v(t; \vec{n}) = \mathbb{E}[\prod_{i=1}^k Z_\beta(t, n_i)]$ .

PROOF. This result is well known and can be found, for instance, in Proposition 6.1.3 of [4]. The purpose of the fourth hypothesis on  $v$  is to ensure uniqueness of solutions to the system of ODEs given by the first three hypotheses. The fact that this exponential growth hypothesis is sufficient for uniqueness can be proved in the same manner as given in the proof of Proposition 4.9 in [7].

One way to see why this should be true is to consider the Feynman–Kac representation for  $Z_\beta(t, n)$  which is given in equation (3). The  $k$  factors of  $Z_\beta$  lead to  $k$  paths. The expectation  $\mathbb{E}$  over the Gaussian disorder (white-noise) can be taken inside the path expectations  $\mathcal{E}$  and calculated exactly yielding the exponential of the pair-wise local time for the  $k$  paths. This accounts for the delta interaction seen above.  $\square$

It is a priori not clear how one would start to solve the system of ODEs in the above proposition, one reason being that it is inhomogeneous in space. An idea from integrable systems (related to the coordinate Bethe Ansatz) is to instead try to solve a homogeneous system of ODEs and put the inhomogeneity into a boundary condition. If the number of boundary conditions is  $k - 1$ , then there is generally hope in solving the system by combining fundamental solutions of the homogeneous system in such a way that the initial data and boundary conditions are met.

For the general  $p, q$  case, it appears that this reduction to  $k - 1$  boundary conditions only works when  $k = 2$  (in which case there is just one boundary condition). When  $p = 2$  and  $q = 0$  the reduction works for all  $k$ ; see Section 3.

PROPOSITION 2.2. Assume  $u : \mathbb{R}_+ \times \mathbb{Z}^2 \rightarrow \mathbb{R}$  solves:

(1) For all  $\vec{n} \in \mathbb{Z}^2$  and  $t \in \mathbb{R}_+$ ,

$$\frac{d}{dt}u(t; \vec{n}) = \frac{1}{2} \sum_{i=1}^2 [\Delta^{p,q}]_i u(t; \vec{n});$$

(2) For  $\vec{n}$  such that  $n_1 = n_2 = n$

$$T_\beta u(t; \vec{n}) := \beta^2 u(t; n, n) + \frac{p}{2} u(t; n, n - 1) + \frac{q}{2} u(t; n + 1, n) - \frac{p}{2} u(t; n - 1, n) - \frac{q}{2} u(t; n, n + 1) = 0;$$

(3) For all  $\vec{n} \in \mathbb{Z}^k$  such that  $n_1 \geq n_2$ ,  $\lim_{t \rightarrow 0} u(t; \vec{n}) = \prod_{i=1}^2 \mathbf{1}_{n_i=0}$ ;

(4) For all  $T > 0$ , there exists  $c, C > 0$  such that for all  $\vec{n} \in \mathbb{Z}^k$  such that  $n_1 \geq n_2$  and all  $t \in [0, T]$ ,

$$|u(t; \vec{n})| \leq ce^{C\|\vec{n}\|_1}.$$

Then for  $\vec{n} \in \mathbb{Z}^2$  such that  $n_1 \geq n_2$ ,  $u(t; \vec{n}) = v(t; \vec{n}) = \mathbb{E}[\prod_{i=1}^2 Z_\beta(t, n_i)]$ .

PROOF. We show that restricted to  $n_1 \geq n_2$ ,  $u(t; \vec{n})$  symmetrically extended to  $\mathbb{Z}^2$  solves the system of ODEs in Proposition 2.1 and hence  $u(t; \vec{n}) = v(t; \vec{n})$ . For  $n_1 > n_2$  it is clear that  $u$  and  $v$  solve the same equation. For  $n_1 = n_2 = n$ ,

$$\begin{aligned} \frac{d}{dt}u(t; n, n) &= \frac{1}{2}(pu(t; n - 1, n) + qu(t; n + 1, n) \\ &\quad + pu(t; n, n - 1) + qu(t; n, n + 1) - 4u(t; n, n)) \\ &= (\beta^2 - 2)u(t; n, n) + pu(t; n, n - 1) + qu(t; n + 1, n), \end{aligned}$$

where the second line followed from the relation imposed by the assumption (2). Now compare this to the equation  $v(t; n, n)$  that satisfies:

$$\begin{aligned} \frac{d}{dt}v(t; n, n) &= \frac{1}{2}(pv(t; n - 1, n) + qv(t; n + 1, n) - 2v(t; n, n)) \\ &\quad + \frac{1}{2}(pv(t; n, n - 1) + qv(t; n, n + 1) - 2v(t; n, n)) + \beta^2v(t; n, n) \\ &= (\beta^2 - 2)v(t; n, n) + pv(t; n, n - 1) + qv(t; n + 1, n), \end{aligned}$$

where the second line followed from the symmetry hypothesis on  $v$ . Observe that on the diagonal  $n_1 = n_2$ , both  $u$  and  $v$  solve the same equation. Therefore they solve the same equation for all  $n_1 \geq n_2$  and hence (since the other hypotheses of Proposition 2.1 are satisfied)  $u(t; \vec{n}) = v(t; \vec{n})$ .  $\square$

We may now explicitly solve the system of ODEs defined in Proposition 2.2.

PROPOSITION 2.3. For  $k = 2$  and  $n_1 \geq n_2$ , the system of ODEs in Proposition 2.2 is uniquely solved by

$$\begin{aligned} (11) \quad u(t; n_1, n_2) &= \frac{1}{(2\pi i)^2} \oint \oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} \\ &\quad \times F_{t, n_1}^{p, q}(z_1) F_{t, n_2}^{p, q}(z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2}, \end{aligned}$$

where

$$F_{t, n}^{p, q}(z) = z^{-n} e^{(t/2)(pz + qz^{-1} - 2)}$$

and where the contour of  $z_1$  is the unit circle and the contour for  $z_2$  is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from  $z_2 = 0$ ; see Figure 1.

REMARK 2.4. The right-hand side of (11) is easy to generalize to all  $k$ ; see, for example, Section 6.1.2 of [4] or Proposition 3.3 below in the totally asymmetric case where  $p = 2$  and  $q = 0$ . However, it is not at all clear if such a generalization would have anything to do with  $\mathbb{E}[\prod_{i=1}^k Z_\beta(t, n_i)]$ .

Before proving this proposition, we note that Theorem 1.3 follows as an immediate corollary of the above result and Proposition 2.2.

PROOF OF PROPOSITION 2.3. We prove this proposition by checking the hypotheses of Proposition 2.2. Hypothesis 1 follows from the Leibniz rule and the observation that

$$\frac{d}{dt} F_{t,n}^{p,q}(z) = \frac{1}{2} \Delta^{p,q} F_{t,n}^{p,q}(z).$$

To check hypothesis 2, we apply  $T_\beta$  to  $u(t; \vec{n})$  when  $n_1 = n_2 = n$ . The operator  $T_\beta$  can be taken inside the integration. It acts on  $F_{t,n}^{p,q}(z_1) F_{t,n}^{p,q}(z_2)$  as

$$\begin{aligned} T_\beta(F_{t,n}^{p,q}(z_1) F_{t,n}^{p,q}(z_2)) \\ = -\frac{1}{2} F_{t,n}^{p,q}(z_1) F_{t,n}^{p,q}(z_2) ((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2). \end{aligned}$$

The factor  $((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2)$  cancels with the same term in the denominator of the integrand, yielding

$$\begin{aligned} T_\beta u(t; \vec{n}) = \frac{1}{(2\pi i)^2} \oint \oint ((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})) \\ \times F_{t,n}^{p,q}(z_1) F_{t,n}^{p,q}(z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2}. \end{aligned}$$

Since the integration contours are the same for both  $z_1$  and  $z_2$  and since  $((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}))$  is antisymmetric in  $z_1$  and  $z_2$ , while the rest of the integrand is symmetric, one immediately sees that the integral is zero as desired to check hypothesis 2.

Hypothesis 3 is checked via residue calculus. The  $t \rightarrow 0$  limit can be taken inside the integrand, and we are left to show that for  $n_1 \geq n_2$ ,

$$(12) \quad \frac{1}{(2\pi i)^2} \oint \oint C(z_1, z_2) z_1^{-n_1} z_2^{-n_2} \frac{dz_1}{z_1} \frac{dz_2}{z_2} = \mathbf{1}_{n_1=0} \mathbf{1}_{n_2=0},$$

where

$$C(z_1, z_2) = \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2}$$

and where the contour of  $z_1$  is the unit circle and the contour for  $z_2$  is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from  $z_2 = 0$ .

(1) If  $n_2 < 0$ , then in (12) we may shrink the  $z_2$  contour to 0. Observe that for  $z_1$  fixed on the specified contour,  $C(z_1, z_2)$  is analytic in  $z_2$  in a small neighborhood of  $z_2 = 0$ , with a value of  $C(z_1, 0) = 1$ . The term  $z_2^{-n_2} \frac{dz_2}{z_2}$  does not have a pole at 0 (because  $n_2 < 0$ ) and hence in this case  $u(0; \vec{n}) = 0$ .

(2) If  $n_2 = 0$ , then in (12) let us shrink the  $z_2$  contour to 0. The term  $z_2^{-n_2} \frac{dz_2}{z_2}$  has a simple pole at 0 and hence the integral evaluates as

$$\frac{1}{2\pi i} \oint z_1^{-n_1} \frac{dz_1}{z_1} = \mathbf{1}_{n_1=0}.$$

(3) If  $n_2 > 0$  this implies that  $n_1 > 0$  as well. Then we can expand  $z_1$  to infinity. As we do this, we encounter a pole at  $z_1$  such that

$$(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2 = 0.$$

For each  $z_2$  there is only one such pole  $r(z_2)$  which comes when  $z_1 \approx \frac{q}{p}z_2^{-1}$  for small  $|z_2|$ . The reason why only one pole is crossed is because the other pole coming from this term is approximately  $z_2$ , which is already contained inside the  $z_1$  contour. Before analyzing the residue, observe that because  $n_1 > 0$ , there is at least quadratic decay in  $z_1$  at infinity, so there is no pole at infinity. Thus, the integral in  $z_1$  is given by its negative residue at  $r(z_2)$ .

The negative residue at  $z_1 = r(z_2)$  is evaluated as

$$(13) \quad -\frac{1}{2\pi i} \oint \frac{2\beta^2 r(z_2)}{pr(z_2) + q(r(z_2))^{-1}} (r(z_2))^{-n_1-1} z_2^{-n_2-1} dz_2,$$

where the integral in  $z_2$  is over a small circle around the origin. It is easy to see that

$$\frac{2\beta^2 r(z_2)}{pr(z_2) + q(r(z_2))^{-1}}$$

is analytic in a neighborhood of  $z_2 = 0$  and its value at  $z_2 = 0$  is  $2\beta^2/p$ . Thus the entire integral (13) can be evaluated as the residue at  $z_2 = 0$ . Since  $n_1 \geq n_2$ , there is, in fact, no pole at  $z_2 = 0$ , thus the integral equals 0.

Combining the above cases we see that the only case in which  $u(0; \vec{n})$  is nonzero is when  $n_1 = n_2 = 0$ , in which case it is 1. This confirms the initial data of hypothesis 3.

Hypothesis 4 follows via easy bounds of the integrand of  $u(t; \vec{n})$ .  $\square$

Having proved the two-point moment formula for the nearest-neighbor parabolic Anderson model, we can now extract the second moment Lyapunov exponent via asymptotic analysis.

**PROOF OF THEOREM 1.5.** We will present a complete proof only in the case  $p = q = 1$  since this simplifies the (rather technical) analysis. For the moment we keep the  $p$  and  $q$  and only set them equal when necessary.

Let us start by proving  $\gamma_1 = 0$  from the formula

$$\mathbb{E}[Z_\beta(t, n)] = \frac{1}{2\pi i} \oint F_{t,n}^{p,q}(z) \frac{dz}{z},$$

which one easily checks via either Proposition 2.1 or 2.2. Let  $n = \lfloor (p - q)t \rfloor$  and observe that [up to an insignificant correction coming from the fractional difference between  $n$  and  $(p - q)t$ ]

$$\mathbb{E}[Z_\beta(t, n)] = \frac{1}{2\pi i} \oint e^{tG(z)} \frac{dz}{z}, \quad G(z) = pz + qz^{-1} - 2 - (p - q) \log z.$$

We want to study this as  $t \rightarrow \infty$ ; thus we can use the standard Laplace method (see Lemma A.1 for  $\ell = 1$ ) to perform the asymptotics. The critical point equation for  $G(z)$  is

$$G'(z) = p - qz^{-2} - (p - q)z^{-1} = 0,$$

which is solved by  $z = 1$  or  $z = -q/p$ . The critical point  $z = 1$  corresponds to the larger value of  $G(z)$ , namely  $G(1) = 0$ . Observe that we can deform the  $z$  contour to lie on the unit circle  $e^{i\theta}$ . As a function of  $\theta$  along this contour  $\text{Re}[G(z(\theta))] = 2 \cos(\theta) - 2$ . This shows that the real part of  $G(z)$  decays monotonically with respect to the angle  $\theta$  away from  $z = 1$ . In the vicinity of  $z = 1$ ,  $\text{Re}[G(z)]$  decays quadratically in the imaginary directions. Invoking Lemma A.1 for  $\ell = 1$  shows that

$$\gamma_1 := \lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}[Z_\beta(t, (p - q)t)]) = 0.$$

To calculate  $\gamma_2$  we use the formula for  $\mathbb{E}[Z_\beta(t, n_1)Z_\beta(t, n_2)]$  proved in Theorem 1.3. In order to perform the asymptotics we would like to deform the  $z_2$  contour to coincide with the  $z_1$  contour. While doing this we encounter a pole and must take into account the associated residue, in addition to the evaluation of the remaining integral on the new contours. The result of this manipulation is

$$\mathbb{E}[Z_\beta(t, (p - q)t)^2] = (A) + (B),$$

where

$$(A) = \frac{1}{(2\pi i)^2} \oint \oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} \times F_{t,(p-q)t}^{p,q}(z_1) F_{t,(p-q)t}^{p,q}(z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

with the  $z_2$  contour coinciding with the  $z_1$  contour, and

$$(14) \quad (B) = \frac{1}{2\pi i} \oint \frac{2\beta^2}{p + q[s(z_2)]^{-2}} e^{tH(z_2)} \frac{dz_2}{z_2}.$$

Note that this residue term comes from  $z_1 = s(z_2)$  where  $s(z_2)$  is given in the statement of the theorem. Since the definition of  $s(z)$  involves a square-root, for  $z$  complex we specify that for  $z = re^{i\theta}$  with  $\theta \in (-\infty, \infty)$ ,  $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ .

As it will turn out, the residue term (B) has a larger exponential growth rate than the integral term (A) and hence accounts entirely for the value of  $\gamma_2$ . To see this, we compute the exponent for both terms. We claim that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log[(A)] = 0.$$

It is easy to see why this is true. The contours for  $z_1$  and  $z_2$  can be chosen so that there exists a positive constant  $C$  such that along the contours

$$|g(z_1, z_2)| < C,$$

where

$$g(z_1, z_2) = \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2}.$$

Likewise, in a neighborhood of  $(z_1, z_2) = (1, 1)$ , one checks that

$$c|z_1^2 z_2 - z_2 - z_1 z_2^2 + z_1| \leq |g(z_1, z_2)|$$

for a small, yet positive  $c$ . One easily checks the remaining assumptions necessary to apply Lemma A.1 (with  $\ell = 2$ ) and therefore finds that as  $t \rightarrow \infty$ , the growth of the integral defining (A) is governed by the value of  $G(z_1) + G(z_2)$  at the critical point  $(1, 1)$ . By comparison to the calculation performed above for  $\gamma_1$ , we find that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log[(A)] = 2\gamma_1.$$

Since  $\gamma_1 = 0$ , the claimed result for (A) follows.

Turn now to the residue term (B) and call  $z_2$  simply  $z$ . From this point on we will assume that  $p = q = 1$  to simplify the analysis. A similar, albeit lengthy, analysis can be performed for all  $p$  and  $q$ . Notice that

$$z_2^0 = \frac{1}{2}(-\beta^2 + \sqrt{4 + \beta^4})$$

is a critical point of  $H_2(z)$  and that the contour for  $z$  can be deformed without crossing any singularities of (14) to the contour  $\Gamma$  parameterized by  $z = z_2^0 e^{i\theta}$  for  $\theta \in [0, 2\pi]$ . We wish to use Laplace’s method by applying Lemma A.1 with  $\ell = 1$ . It is straightforward to check hypotheses (1), (3) and (4) of the lemma. Hypothesis (2) requires more work.

To check hypothesis (2a) of Lemma A.1 observe that  $H_2'(z_2^0) = 0$ ,  $H_2''(z_2^0) \neq 0$  and  $H_2(z)$  is analytic in a neighborhood of  $z_2^0$ . Thus it immediately follows that it behaves locally like  $H(z_2^0) + c(z - z_2^0)^2 + o((z - z_2^0)^2)$ . Hypothesis (2b) requires that

$$\rho(\theta) := 2 \operatorname{Re}[H_2(z) - H_2(z_2^0)]$$

(the factor of 2 is irrelevant) is strictly negative for all  $z \in \Gamma \setminus \{z_2^0\}$ . In fact, by symmetry of  $H$  through the real axis,  $\rho(\theta) = \rho(2\pi - \theta)$  and hence this strict negativity needs only be checked for  $\theta \in (0, \pi]$ . By utilizing the fact that

$$(z_2^0)^{-1} = \frac{1}{2}(\beta^2 + \sqrt{4 + \beta^4}),$$

$$s(z)^{-1} = \frac{-(z - z^{-1} + 2\beta^2) + \sqrt{(z - z^{-1} + 2\beta^2)^2 + 4}}{2}$$

we find that

$$\begin{aligned} \rho(\theta) = & \sqrt{4 + \beta^4}(\cos(\theta) - 2) \\ & + \operatorname{Re}[(\beta^4(2 - \cos(\theta))^2 - (4 + \beta^4) \sin(\theta)^2 \\ & + 4 + i2\beta^2(2 - \cos(\theta))\sqrt{4 + \beta^4} \sin(\theta))^{1/2}]. \end{aligned}$$

Since for  $a, b$  real,

$$\operatorname{Re} \sqrt{a + ib} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}},$$

checking hypothesis (2b) reduces to checking that for  $\theta \in (0, \pi]$

$$\rho(\theta) = \sqrt{4 + \beta^4}(\cos(\theta) - 2) + \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} < 0,$$

where

$$\begin{aligned} a = & \beta^4(2 - \cos(\theta))^2 - (4 + \beta^4) \sin(\theta)^2 + 4, \\ b = & 2\beta^2(2 - \cos(\theta))\sqrt{4 + \beta^4} \sin(\theta). \end{aligned}$$

It is straight-forward to check that  $\rho(\pi) < 0$  hence by continuity of  $\rho(\theta)$  it suffices to prove that  $\rho(\theta) \neq 0$  for  $\theta \in (0, \pi]$ . A simple calculation shows that this is equivalent to showing that

$$128(4 + \beta^4)(\cos(\theta) - 2)^2 \sin(\theta/2)^2 \neq 0$$

on  $\theta \in (0, \pi]$  which is immediately verified. Thus we have proved hypothesis (2b) of Lemma A.1.

Applying Lemma A.1 to (14) we find that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(A) = H_2(z_2^0).$$

Since  $H_2(z_2^0)$  is positive [as compared to the contribution from the (A) term asymptotics] we conclude that  $\gamma_2(\beta; 0) = H_2(z_2^0)$ .  $\square$

**3. The one-sided parabolic Anderson model.** We now focus on the one-sided case where  $p = 2$  and  $q = 0$ . Recall the definition of  $W_{\geq 0}^k$  given in (9) and the definition of  $\Delta^{p,q}$  given in (2). In particular,  $\Delta^{2,0} f(n) = f(n - 1) - f(n)$ .

PROPOSITION 3.1. Fix  $k \in \mathbb{Z}_{>0}$ .

Part 1. Assume  $v : \mathbb{R}_+ \times (\mathbb{Z}_{\geq -1})^k$  solves:

(1) for all  $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$  and  $t \in \mathbb{R}_+$ ,

$$\frac{d}{dt} v(t; \vec{n}) = \mathbf{H} v(t; \vec{n}), \quad \mathbf{H} = \frac{1}{2} \sum_{i=1}^k [\Delta^{2,0}]_i + \frac{1}{2} \beta^2 \sum_{\substack{a,b=1 \\ a \neq b}}^k \mathbf{1}_{n_a = n_b};$$

(2) for all permutations of indices  $\sigma \in S_k$ ,  $v(t; \sigma \vec{n}) = v(t; \vec{n})$ ;

(3) for all  $\vec{n} \in W_{\geq 0}^k$ ,  $v(0; \vec{n}) = \prod_{i=1}^k \mathbf{1}_{n_i=0}$ ;

(4) for all  $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$  such that  $n_k = -1$ ,  $v(t; \vec{n}) \equiv 0$  for all  $t \in \mathbb{R}_+$ .

Then for all  $\vec{n} \in W_{\geq 0}^k$ ,  $\mathbb{E}[\prod_{i=1}^k Z_\beta(t, n_i)] = v(t; \vec{n})$ .

Part 2. Assume  $u : \mathbb{R}_+ \times (\mathbb{Z}_{\geq -1})^k \rightarrow \mathbb{R}$  solves:

(1) for all  $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$  and  $t \in \mathbb{R}_+$ ,

$$\frac{d}{dt} u(t; \vec{n}) = \frac{1}{2} \sum_{i=1}^k [\Delta^{2,0}]_i u(t; \vec{n});$$

(2) for all  $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$  such that for some  $i \in \{1, \dots, k - 1\}$ ,  $n_i = n_{i+1}$ ,

$$([\Delta^{2,0}]_i - [\Delta^{2,0}]_{i+1} - 2\beta^2) u(t; \vec{n}) = 0;$$

(3) for all  $\vec{n} \in (\mathbb{Z}_{> -1})^k$  such that  $n_k = -1$ ,  $u(t; \vec{n}) \equiv 0$  for all  $t \in \mathbb{R}_+$ ;

(4) for all  $\vec{n} \in W_{\geq 0}^k$ ,  $u(0; \vec{n}) = \prod_{i=1}^k \mathbf{1}_{n_i=0}$ .

Then for all  $\vec{n} \in W_{\geq 0}^k$ ,  $\mathbb{E}[\prod_{i=1}^k Z_\beta(t, n_i)] = u(t; \vec{n})$ .

PROOF. This is contained in Proposition 6.3 of [7].  $\square$

REMARK 3.2. Part 1 of the proposition is essentially a specialization of Proposition 2.1 to the case  $p = 2$ ,  $q = 0$ . However, due to the delta initial data and the one-sided nature of the operator  $\Delta^{2,0}$ ,  $v(t; \vec{n})$  with  $\vec{n} \in \{-1, 0, \dots, m\}^k$  evolves autonomously as a closed system of ODEs. This ensures uniqueness of solutions and explains why we no longer require the at-most exponential growth hypothesis which was present in Proposition 2.1. Part 2 of the proposition is an extension of Proposition 2.2 to general  $k$ , but only for  $p = 2$ ,  $q = 0$ . The fact that this holds for all  $k$  is what enables us to solve for higher than second moments in this one-sided case.

The system of ODEs in part 2 of Proposition 3.1 can be solved via a nested-contour integral ansatz introduced in [4] and further developed in [7]. This yields the following generalization of Proposition 2.3 to all  $k$  but only for the one-side ( $p = 2, q = 0$ ) case.

PROPOSITION 3.3. For all  $k \geq 1$  and  $\vec{n} \in W_{\geq 0}^k$ , the system of ODEs in part 2 of Proposition 3.1 is uniquely solved by

$$(15) \quad u(t; \vec{n}) = \frac{1}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^k \frac{e^{t(z_i-1)} dz_i}{z_i^{n_i}},$$

where the integration contour for  $z_a$  is a closed curve containing 0 and the image under addition by  $\beta^2$  of the integration contours for  $z_b$  for all  $b > a$ .

Before proving this proposition, we note that Theorem 1.7 follows as an immediate corollary of the above result and Proposition 3.1, part 2.

PROOF OF PROPOSITION 3.3. We check the four conditions for the system of ODEs in part 2 of Proposition 3.1. Condition (1) follows by Leibnitz rule and the fact that

$$\frac{d}{dt} \frac{e^{t(z-1)}}{z^n} = \frac{1}{2} \Delta^{2,0} \frac{e^{t(z-1)}}{z^n}$$

for all  $z \in \mathbb{C} \setminus \{0\}$ .

Condition (2) follows by applying  $([\Delta^{2,0}]_i - [\Delta^{2,0}]_{i+1} - 2\beta^2)$  to the integrand of the right-hand side of (15). The effect of this operator is to bring out an extra factor of  $2(z_i - z_{i+1} - \beta^2)$ . This factor cancels the corresponding term in the denominator of the product over  $a < b$ . Without the pole associated with this term, it is possible to deform the  $z_i$  and  $z_{i+1}$  contours to coincide, and since  $n_i = n_{i+1}$ , we find that

$$([\Delta^{2,0}]_i - [\Delta^{2,0}]_{i+1} - 2\beta^2)u(t; \vec{n}) = \oint dz_i \oint dz_{i+1} (z_i - z_{i+1}) f(z_i) f(z_{i+1}),$$

where  $f(z)$  includes all of the other integrations aside from those in  $z_i$  and  $z_{i+1}$ . The above integral is clearly 0 by skew-symmetry, thus confirming condition (2) as desired.

Condition (3) follows by observing that if  $n_k = -1$ , then on the right-hand side of (15) there is no pole at 0 in the  $z_k$  variable. By Cauchy’s theorem, this means that since the  $z_k$  contour only contains 0 and no other poles of the integrand, the entire integral is 0, as desired.

Condition (4) follows from three easy residue calculations. From above, if  $n_k < 0$ , the integral in (15) is zero. Similarly, if  $n_1 > 0$  the integrand in (15) has no pole at infinity and the  $z_1$  contour can be freely deformed to infinity. By Cauchy’s theorem this means that the entire integral is zero. The only possible nonzero value of  $u(0; \vec{n})$  is (due to the ordering of the elements in  $\vec{n} \in W_{\geq 0}^k$ ) when

$n_1 = \dots = n_k = 0$ . The value of  $u$  for this choice of  $\vec{n}$  is readily calculated via residues to equal 1, just as desired.  $\square$

We now show how asymptotics of the result of Theorem 1.7 yield a proof of the moment Lyapunov exponents claimed in Theorem 1.8.

**PROOF OF THEOREM 1.8.** The starting point of this proof is the moment formula of Theorem 1.7. Setting all  $n_i \equiv \lfloor vt \rfloor$  and  $\beta = 1$  we find that

$$(16) \quad \mathbb{E} \left[ \prod_{i=1}^k Z_{\beta}(t, \lfloor vt \rfloor) \right] = \frac{1}{(2\pi i)^k} \oint \dots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{i=1}^k F_{t, \lfloor vt \rfloor}^{2,0}(z_i) \frac{dz_i}{z_i},$$

where the integration contour for  $z_a$  is a closed curve containing 0 and the image under addition by 1 of the integration contours for  $z_b$  for all  $b > a$ . From now on we will study the right-hand side of the above equality, with  $F_{t, \lfloor vt \rfloor}^{2,0}(z_i)$  replaced by  $F_{t, vt}^{2,0}(z_i)$ , as the asymptotic effect of this modification is easily seen to be inconsequential.

In order to perform the asymptotic analysis necessary to compute the moment Lyapunov exponents, we would like to deform our contours to all coincide so as to apply Lemma A.1. This requires deforming all contours to pass through a specific critical point of  $\log F_{t, vt}^{2,0}(z)$ . However, due to the nesting of the contours, such a deformation requires passing a number of poles. The following lemma records the effect of such a deformation.

**LEMMA 3.4.** Consider a function  $f(z)$  which is analytic in  $\mathbb{C} \setminus \{0\}$ . For  $k \geq 1$ , set

$$\mu_k = \frac{1}{(2\pi i)^k} \oint \dots \oint \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - z_B - 1} \frac{f(z_1) \dots f(z_k)}{z_1 \dots z_k} dz_1 \dots dz_k,$$

where the integration contour for  $z_A$  contains 0 and the image under addition by 1 of the integration contours for  $z_B$  for all  $B > A$ ; see Figure 3. Then

$$\mu_k = k! \sum_{\lambda=1^{m_1} 2^{m_2} \dots \vdash k} I_{\lambda},$$

where

$$(17) \quad I_{\lambda} = \frac{1}{m_1! m_2! \dots (2\pi i)^{\ell(\lambda)}} \times \oint \dots \oint \det \left[ \frac{1}{\lambda_i + w_i - w_j} \right]_{i,j=1}^{\ell(\lambda)} \times \prod_{j=1}^{\ell(\lambda)} f(w_j) f(w_j + 1) \dots f(w_j + \lambda_j - 1) dw_j.$$

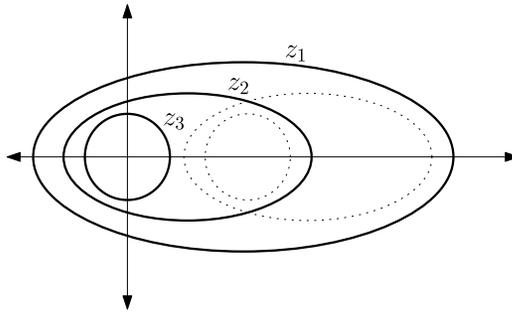


FIG. 3. Valid contours for Lemma 3.4 with  $k = 3$ . The inner contour is  $z_3$  and contains 0; the next contour is  $z_2$  and contains 0 and the image of the  $z_3$  contour plus one; the outer contour is  $z_1$  and contains 0 and the image of the  $z_2$  and  $z_3$  contours plus one. The images of the  $z_3$  and  $z_2$  contours after adding one are indicated by the dotted lines.

Here  $\lambda = 1^{m_1} 2^{m_2} \dots \vdash k$  means  $\lambda$  is a partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$  such that  $\sum \lambda_i = k$ ;  $m_i$  records the number of entries of  $\lambda$  equal to  $i$ ;  $\ell(\lambda)$  is the number of nonzero entries in  $\lambda$ ; and for  $1 \leq j \leq \ell(\lambda)$  the  $w_j$  contours are all chosen to be the same contour as  $z_k$ .

PROOF. This is given in [4], Proposition 6.2.7, and is proved by taking a scaling limit (with  $q \mapsto e^{-\varepsilon}$ ,  $z \mapsto e^{-\varepsilon z}$ ,  $w \mapsto e^{-\varepsilon w}$  and  $\varepsilon \rightarrow 0$ ) of [4], Proposition 3.2.1. Alternatively another proof is given in [5] as Proposition 5.1.  $\square$

The following lemma will also be helpful in completing our asymptotics.

LEMMA 3.5. Consider  $I_\lambda$  in (17) with  $f(z) = F_{t, \nu}^{2,0}(z)$  and write  $I_\lambda(t)$  to emphasize the  $t$  dependence. Then

$$\gamma_\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \log I_\lambda(t)$$

exists and is given by

$$(18) \quad \gamma_\lambda = \sum_{j=1}^{\ell(\lambda)} \gamma_{\lambda_j},$$

where

$$\gamma_r = H_r(z_r^0).$$

Here, as in the statement of Theorem 1.8,

$$H_r(z) = \frac{r(r-3)}{2} + rz - \nu \log \left( \prod_{i=0}^{r-1} (z+i) \right)$$

and  $z_r^0$  is the unique solution to  $H_r'(z) = 0$  with  $z \in (0, \infty)$ .

PROOF. First observe that we can take the contours of integration for  $w_j$  in  $I_\lambda$  to be a large circle containing  $\{0, -1, \dots, -\lambda_j + 1\}$ . This is because before having applied the identity in Lemma 3.4, we could take the  $z_j$  contours to be large enough nested circles so that  $z_k$  contains  $0, -1, \dots, -\lambda_j + 1$ .

Next we observe that the integrals defining  $I_\lambda$  match the form of (23) in Lemma A.1 with  $\ell = \ell(\lambda)$  and (using  $w$ 's instead of  $z$ 's)

$$g(w_1, \dots, w_\ell) = \frac{1}{m_1!m_2! \dots} \det \left[ \frac{1}{\lambda_i + w_i - w_j} \right]_{i,j=1}^{\ell(\lambda)}$$

and  $G_j(w_j) = H_{\lambda_j}(w_j)$ . If we can show that the four hypotheses of Lemma A.1 apply, then the result claimed in the present lemma follows immediately.

By convexity,  $G_j(w_j)$  has exactly one critical point along  $w_j \in (0, \infty)$ . Call this point  $w_j^0$ . Without changing the value of the integrals, we can freely deform the contour of integration for  $w_j$  to a contour  $\Gamma_j$  which is defined (see Figure 4 for an illustration) as the union of a long vertical line segment going through  $w_j^0$  and a semi-circle enclosing  $\{0, -1, \dots, -\lambda_j + 1\}$ , with radius large enough so that for all  $r \in \{0, -1, \dots, -\lambda_j + 1\}$  and all  $w_j \in \Gamma_j \setminus \{w_j^0\}$ ,  $|w_j - r| > |w_j^0 - r|$ . For this choice of contour it is clear that all of the hypotheses of Lemma A.1 hold. In particular hypothesis (2a) holds since  $\text{Re}(w_j)$  is constant along the vertical portion of the contour and decreasing on the circular part; and  $\text{Re}[\log(w_j + i)] > \text{Re}[\log(w_j^0 + i)]$  for all  $w_j \neq w_j^0$  along  $\Gamma_j$ .  $\square$

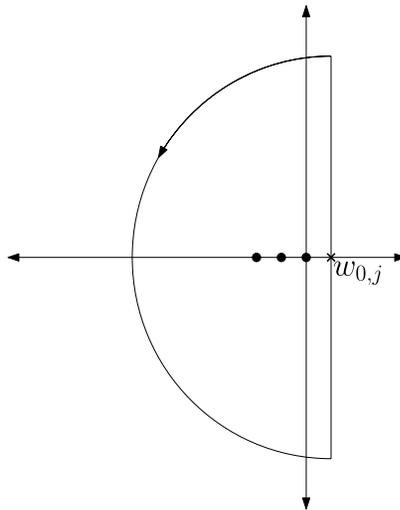


FIG. 4. The contour  $\Gamma_j$  is a vertical line segment with real part  $w_{0,j}$  joined with a semi-circle which encloses  $\{0, -1, \dots, -\lambda_j + 1\}$  as well has the property that  $r \in \{0, -1, \dots, -\lambda_j + 1\}$  and all  $w_j \in \Gamma_j \setminus \{w_{0,j}\}$ ,  $|w_j - r| > |w_{0,j} - r|$ . In words, this means that the distance from the points on  $\Gamma_j$  to the various elements of the set  $\{0, -1, \dots, -\lambda_j + 1\}$  is minimized for  $w_{0,j}$  and strictly larger otherwise.

We can now complete the proof of Theorem 1.8. From straightforward comparison of growth of exponentials it follows that

$$\gamma_k := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left( \sum_{\lambda \vdash k} I_\lambda(t) \right) = \max_{\lambda \vdash k} \gamma_\lambda.$$

Observe that by combining equation (16) with Lemmas 3.4 and 3.5 we find that

$$(19) \quad \gamma_k(1; \nu) = \max_{\lambda \vdash k} \gamma_\lambda,$$

where  $\gamma_k(1; \nu)$  is defined in (5), and  $\gamma_\lambda$  is defined above in (18). We claim now that this maximum is attained for all  $k$  at  $\gamma_k$  and hence  $\gamma_k(1; \nu) = \gamma_k$ . If we can show this, then the theorem is proved.

For  $k = 1$ , this result is immediate since there is only one partition of 1. For  $k = 2$  there are two partitions to consider  $\lambda = (1, 1)$  and  $\lambda = (2)$ . We claim that for all  $\nu > 0$ ,  $2\gamma_1 - \gamma_2 > 0$ . This can be proved by explicit evaluation. Observe that

$$\begin{aligned} \gamma_1 &= -1 + \nu + \nu \log \nu, \\ \gamma_2 &= -2 + \nu + \sqrt{1 + \nu^2} - \nu \log \left( \frac{1}{2} \nu (\nu + \sqrt{1 + \nu^2}) \right). \end{aligned}$$

The difference  $f(\nu) := \gamma_2 - 2\gamma_1$  goes to zero as  $\nu$  goes to infinity and has derivative  $\log(2\nu) - \log(\nu + \sqrt{1 + \nu^2})$  which is negative for all  $\nu > 0$ . This shows that  $f(\nu) > 0$  for all  $\nu > 0$  and hence

$$\gamma_2(1; \nu) = \max_{\lambda \vdash 2} \gamma_\lambda = \gamma_2.$$

Note that we have now shown that  $\gamma_1(1; \nu) < \gamma_2(1; \nu)/2$  and hence we can apply Lemma 1.2 to show the intermittency of all of the moment Lyapunov exponents. We now proceed by induction on  $k$ . Assume that for all  $j \leq k$ , we have proved that  $\gamma_k(1; \nu) = \gamma_k$ . As a base case we have  $k = 1$  and 2. By intermittency we know that for any partition of  $k + 1$  aside from  $\lambda = (k + 1)$ ,  $\gamma_{k+1}(1; \nu)$  must strictly exceed  $\sum_i \gamma_{\lambda_i}(1; \nu)$  which, by induction, equals  $\sum_i \gamma_{\lambda_i}$  as well. By (19) this implies that the maximum over  $\lambda \vdash k + 1$  must be attained for the partition  $\lambda = (k + 1)$  and hence  $\gamma_{k+1}(1; \nu) = \gamma_{k+1}$  as desired to prove the inductive step and complete the proof of Theorem 1.8.  $\square$

### APPENDIX

In the first subsection of this Appendix we show how (using our moment Lyapunov exponents for the one-side parabolic Anderson model) the physics replica trick leads nonrigorously to the correct formula for the almost sure Lyapunov exponent  $\tilde{\gamma}_1$ . This almost sure exponent is already-known rigorously [18], so the below calculation should be thought of as a nontrivial check of the efficacy of the replica trick.

In the second subsection of this Appendix we apply the nested contour integral methods to compute all of the moment Lyapunov exponents for the continuum parabolic Anderson model (i.e., stochastic heat equation with multiplicative noise). These exponents have been known for some time and were first computed in the physics literature by Kardar [16] and then in the math literature by Bertini and Cancrini’s; see [3] Theorem 2.6 and remark after it.

The final subsection of this Appendix contains a version of Laplace’s method for computing asymptotics of integrals (and was referenced earlier in the paper).

**A.1. The replica trick.** The replica trick is an idea which goes back to Kac [15] and which has received a great deal of attention within the statistical physics community. In its most basic form, one hopes to extract the almost sure Lyapunov exponent from the knowledge of all of the moment Lyapunov exponents. The reader should be warned that what follows is extremely nonrigorous. However, we include it since it is a validation of the replica trick in the context of the one-sided parabolic Anderson model; see [16] for this approach implemented in the continuous model discussed below in Section A.2.

We would like to compute the almost sure Lyapunov exponent

$$\tilde{\gamma}_1(1; \nu) := \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[\log Z_1(t, \nu t)].$$

Note that even though we have taken the expectation of  $\log Z_1(t, \nu t)$ , this should not affect the value of the almost sure exponent. Recall that for  $z \in \mathbb{C} \setminus \mathbb{R}_-$ ,

$$(20) \quad \log z = \lim_{k \rightarrow 0} \frac{z^k - 1}{k}.$$

We have shown in Theorem 1.8 that

$$\mathbb{E}[Z_1(t, \nu t)^k] \approx e^{t\gamma_k(1; \nu)} = e^{tH_k(z_k^0)}$$

for

$$H_k(z) = \frac{k(k-3)}{2} + kz - \nu \log \left( \prod_{i=0}^{k-1} (z+i) \right) = \frac{k(k-3)}{2} + kz - \nu \log \frac{\Gamma(z+k)}{\Gamma(z)}$$

and  $z_k^0$  is the unique minimum of  $H_k(z)$  for  $z \in (0, \infty)$ . This second expression has a clear analytic extension in  $k$ .

By using (20) and interchanging the two limits (without justification) we have

$$\tilde{\gamma}_1(1; \nu) = \lim_{t \rightarrow \infty} \frac{1}{t} \lim_{k \rightarrow 0} \frac{e^{tH_k(z_k^0)} - 1}{k}.$$

Notice that for  $k$  near 0,  $e^{tH_k(z_k^0)} \approx 1 + tH_k(z_k^0)$ , hence

$$\tilde{\gamma}_1(1; \nu) = \lim_{t \rightarrow \infty} \frac{1}{t} \lim_{k \rightarrow 0} \frac{tH_k(z_k^0)}{k}.$$

The limit in  $t$  can now be taken (since the factors of  $t$  cancel), and the limit in  $k$  is achieved via L'Hôpital's rule,

$$\lim_{k \rightarrow 0} \frac{H_k(z)}{k} = -\frac{3}{2} + z - \nu\Psi(z),$$

where  $\Psi(z) = [\log \Gamma]'(z)$  is the digamma function. This limit should be evaluated at the unique infimum over  $z \in (0, \infty)$  and hence

$$\tilde{\gamma}_1(1; \nu) = -\frac{3}{2} + \inf_{z>0}(z - \nu\Psi(z)).$$

This nonrigorous calculation does yield the proved value; cf. [20] and [18].

**A.2. The space–time continuum parabolic Anderson model.** Consider the solution  $\mathcal{Z}_\beta : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$  to the multiplicative stochastic heat equation with delta function initial data,

$$(21) \quad \frac{d}{dt} \mathcal{Z}_\beta = \frac{1}{2} \Delta \mathcal{Z}_\beta + \beta \dot{W} \mathcal{Z}_\beta, \quad \mathcal{Z}_\beta(0, x) = \delta_{x=0},$$

where  $\Delta$  is the Laplacian on  $\mathbb{R}$ , and  $\delta_{x=0}$  is the Dirac delta function. The solution  $\mathcal{Z}_\beta(t, x)$  can be thought of as the partition function for a space–time continuous directed polymer in a white-noise environment [1]. In fact, under a particular scaling, the parabolic Anderson model considered in the previous sections, converges to the SHE [17]. The following formula for joint moments can be found by applying this limit transition to Theorem 1.7. For all  $k \geq 1$  and all  $x_1 \leq x_2 \leq \dots \leq x_k$  in  $\mathbb{R}^k$ ,

$$(22) \quad \mathbb{E} \left[ \prod_{i=1}^k \mathcal{Z}_\beta(t, x_i) \right] = \frac{1}{(2\pi t)^k} \oint \dots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^k e^{(t/2)z_i^2 + x_i z_i} dz_i,$$

where  $z_j \in \beta^2 \alpha_j + t\mathbb{R}$  and  $\alpha_1 > \alpha_2 + 1 > \alpha_3 + 2 > \dots > \alpha_k + k - 1$ .

The formula of Theorem 1.7 solved the system of ODEs in Proposition 3.1. Likewise, (22) solves a PDE which is called the “quantum delta Bose gas”; see Section 6.2 of [4] for a discussion on this, as well as remarks on certain gaps in a rigorous statement to this effect.

From (22) it is possible to compute the moment Lyapunov exponents for the SHE (we restrict attention now to  $x_i \equiv 0$  and to  $\beta = 1$  since general  $x_i \equiv x$  and  $\beta$  can be achieved from the resulting formula via Brownian scaling). The result is that

$$\gamma_k := \lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}[\mathcal{Z}(t, 0)^k]) = \frac{k^3 - k}{24}.$$

This reproduces Kardar’s formula [16] and agrees with Bertini and Cancrini’s Theorem 2.6 and remark after it; see [3].

This calculation is done by deforming all contours to the imaginary axis and considering the growth in  $t$  of the various residue terms. As in Theorem 1.8, the Lyapunov exponent comes from the (ground-state) term when  $z_1 = z_2 + 1 = z_3 + 2 = \dots = z_k + k - 1$ . There remains only one free variable of integration in this residue term, and the main part of the integrand is the following exponential:

$$\begin{aligned} & \exp\left\{\frac{t}{2}(z^2 + (z + 1)^2 + \dots + (z + k - 1)^2)\right\} \\ &= \exp\left\{\frac{kt}{2}\left(z + \frac{k - 1}{2}\right)^2 + \frac{t}{2}\left(1 + 2^2 + \dots + (k - 1)^2 - k\frac{(k - 1)^2}{4}\right)\right\}. \end{aligned}$$

Deforming the  $z$ -contour to  $i\mathbb{R} - \frac{k-1}{2}$  shows that this term behaves like

$$\exp\left\{\frac{t}{2}\left(1 + 2^2 + \dots + (k - 1)^2 - k\frac{(k - 1)^2}{4}\right)\right\} = \exp\left\{t\left(\frac{k^3 - k}{24}\right)\right\}$$

from which the result readily follows.

**A.3. Laplace’s method.** The following lemma is a version of Laplace’s method for computing asymptotics of integrals. The proof is an easy modification of the usual proof of Laplace’s method [13].

LEMMA A.1. *Consider a contour integral*

$$(23) \quad I(t) = \frac{1}{(2\pi i)^\ell} \oint_{\Gamma_1} \dots \oint_{\Gamma_\ell} g(z_1, \dots, z_\ell) \exp\left(t \sum_{j=1}^\ell G_j(z_j)\right) dz_1 \dots dz_\ell.$$

Assume that:

- (1) For each  $j$ ,  $\Gamma_j$  is a closed piecewise smooth contour.
- (2) For each  $j$ , there exists  $z_j^0 \in \Gamma_j$  such that:
  - (a)  $\operatorname{Re}[G_j(z)] < \operatorname{Re}[G_j(z_j^0)]$  for all  $z \in \Gamma_j$  not equal to  $z_j^0$ ;
  - (b)  $G'_j(z_j^0) = 0$  and in a neighborhood of  $z_j^0$ ,  $G_j(z) = G_j(z_j^0) + c(z - z_j^0)^r + o((z - z_j^0)^r)$  for some  $r \geq 2$ .
- (3) There exists a (nonidentically zero) rational function  $R(z_1, \dots, z_\ell)$  such that in a neighborhood of  $(z_1^0, \dots, z_\ell^0)$ ,

$$|R(z_1, \dots, z_\ell)| \leq |g(z_1, \dots, z_\ell)|.$$

- (4) There exists a positive constant  $C$  such that for all  $z_j \in \Gamma_j$  ( $j = 1, \dots, \ell$ ),

$$|g(z_1, \dots, z_\ell)| \leq C.$$

Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log I(t) = \sum_{j=1}^{\ell} G_j(z_j^0).$$

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