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PATH PROPERTIES OF THE DISORDERED PINNING MODEL IN THE DELOCALIZED REGIME

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We study the path properties of a random polymer attracted to a defect line by a potential with disorder, and we prove that in the delocalized regime, at any temperature, the number of contacts with the defect line remains in a certain sense "tight in probability" as the polymer length varies. On the other hand we show that at sufficiently low temperature, there exists a.s. a subsequence where the number of contacts grows like the log of the length of the polymer.

1. Introduction. The disordered pinning model has attracted significant attention in recent years. One reason is that it is one of the very few models where the effect of disorder on the critical properties can be identified with large precision. In particular, there exists a fairly satisfactory knowledge on whether and how much the critical point, which separates its localized and delocalized regime, changes under the presence of disorder [1, 10]. Furthermore, the mechanism that defines it is present in multiple physical models, and therefore it provides a step to understand the effect of disorder in more complicated systems—we refer to the recent monograph [8] for related references.

Before going into detail let us define the model. We first consider a sequence of i.i.d. variables $(\omega_n)_{n\in\mathbb{Z}}$, which play the role of disorder. The assumptions on this sequence are in general mild, for example, mean zero and exponential moments. We denote the joint distribution of this sequence by $\mathbb{P}$. The model involves also a renewal sequence $(\tau_n)_{n\in\mathbb{N}}$ on $\mathbb{N} = \{0, 1, 2, \ldots\}$, that is, a point process such that the gaps (or interarrival times) $\sigma_n := \tau_{n+1} - \tau_n$ are independent and identically distributed. This renewal process should be viewed physically as the set of contact points with $\{0\} \times \mathbb{N}$ of the space-time trajectory of a Markov process $(X_n)_{n\in\mathbb{N}}$ whose state space contains a designated site 0, with this trajectory representing the spatial configuration of the polymer. Since the interaction between the Markov process and the disorder comes only at contact times with $\{0\} \times \mathbb{N}$, the only relevant information is the renewal sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$, consisting of the contact points

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of the path $(X_n)_{n\in\mathbb{N}}$ with $\{0\} \times \mathbb{N}$. Therefore we only need to define the statistics of this renewal process, whose law we will denote by P. In particular, we define $\tau_0 = 0$ and assume that for some $\alpha \geq 0$ and slowly varying function $\phi(n)$,

$$K(n) := P(\tau_1 = n) = \frac{\phi(n)}{n^{1+\alpha}}, \qquad n \geq 1.$$

We will assume that $\sum_{n\geq 1} K(n) = 1$, that is, that the renewal is recurrent. We will also need the quantity $K^+(l) = \sum_{n>l} K(n)$.

The polymer measure can now be defined by

$$dP^{\beta,h}_{n,\omega} := \frac{1}{Z^{\beta,h}_{n,\omega}} e^{\mathcal{H}^{\beta,u}_{n,\omega}} \, dP,$$

where $\mathcal{H}^{\beta,u}_{n,\omega} := \sum_{i=0}^{n} (\beta\omega_i + h)\delta_i$ and $\delta_i = 1_{i\in\tau}$. The partition function $Z^{\beta,h}_{n,\omega}$ is defined by

$$Z^{\beta,h}_{n,\omega} = E\big[e^{\mathcal{H}^{\beta,h}_{N,\omega}}\big].$$

The polymer measure rewards paths for which the ω_i values are large at the times of renewals. It will also be useful to consider the constrained polymer measure

$$dP^{\beta,h,c}_{n,\omega} := \frac{1}{Z^{\beta,h,c}_{n,\omega}} e^{\mathcal{H}^{\beta,u}_{n,\omega}} \delta_n \, dP,$$

where we restrict the polymer to have a renewal at time n. Here the constrained partition function is

$$Z^{\beta,h,c}_{n,\omega} = E\big[e^{\mathcal{H}^{\beta,h}_{N,\omega}} \delta_n\big].$$

More generally for a collection A of trajectories we define

$$Z^{\beta,h}_{n,\omega}(A) = E\big[e^{\mathcal{H}^{\beta,h}_{N,\omega}}; A\big].$$

We will also need the notation

$$Z^{\beta,h}_{[m,n],\omega} = Z^{\beta,h}_{n-m,\theta_m\omega},$$

where $n \geq m$ and $\theta_m\omega(i) = \omega(i+m)$, for $i = 1, 2, \ldots.$

As already mentioned, the pinning polymer exhibits a nontrivial localization/delocalization transition, which is often quantified via the strict positivity of the free energy. To be more precise, let us define the quenched free energy of the pinning polymer to be the $\mathbb{P}$-a.s. limit

$$f_q(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\beta,h}_{n,\omega}.$$

We refer the reader to [8], Chapter 3, for the existence of this limit. The localized regime is defined as

$$\mathcal{L} = \big\{(\beta, h) : f_q(\beta, h) > 0\big\},$$

and the delocalized regime as

$$\mathcal{D} = \{(\beta, h) : f_q(\beta, h) = 0\}.$$

The free energy is monotone in h so the two regimes are separated by a critical line and we can define the quenched critical point $h_c(\beta)$ as

$$h_c(\beta) = \sup\{h : f_q(\beta, h) = 0\}.$$

Let $M(\beta) = \mathbb{E}[e^{\beta\omega_1}]$ be the moment generating function of ω_1. For the corresponding annealed model, with partition function $\mathbb{E}Z_{n,\omega}^{\beta,h}$ and free energy

$$f_a(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}Z_{n,\omega}^{\beta,h},$$

the corresponding critical point is

$$h_c^{\mathrm{ann}}(\beta) = -\log M(\beta). \tag{1.1}$$

The question of the path behavior of the quenched model for $h < h_c(\beta)$ is of particular interest when $h_c^{\mathrm{ann}}(\beta) < h_c(\beta)$, so we summarize what has been proved about such an inequality. It is known from [1, 16] (for Gaussian disorder) and from [13] (for general disorder) that for small β, $h_c(\beta) = h_c^{\mathrm{ann}}(\beta)$, for $\alpha < 1/2$ as well as for $\alpha = 1/2$ and $\sum_{n\geq 1}(n\phi(n)^2)^{-1} < \infty$. On the other hand, from [1–3, 7], for Gaussian disorder, for $1/2 < \alpha < 1$, there exists a constant c and a slowly varying function ψ related to ϕ and α such that for all small β,

$$c^{-1}\beta^{2\alpha/(2\alpha-1)}\psi\left(\frac{1}{\beta}\right) < h_c(\beta) - h_c^{\mathrm{ann}}(\beta) < c\beta^{2\alpha/(2\alpha-1)}\psi\left(\frac{1}{\beta}\right),$$

while for $\alpha = 1$,

$$c^{-1}\beta^2\psi\left(\frac{1}{\beta}\right) < h_c(\beta) - h_c^{\mathrm{ann}}(\beta).$$

A matching upper bound is also expected to hold but has not been proved. For $\alpha > 1$,

$$c^{-1}\beta^2 < h_c(\beta) - h_c^{\mathrm{ann}}(\beta) < c\beta^2.$$

The case $\alpha = 1/2$ is marginal and not fully understood. It is believed that $h_c(\beta) > h_c^{\mathrm{ann}}(\beta)$ for every β, as long as $\sum_n 1/(n\phi(n)^2) = \infty$. This inequality has been confirmed under some stronger hypotheses in [3, 9], for Gaussian disorder, and (most nearly optimally, for general disorder) in [10]. For all $\alpha > 0$, for large β the critical points are shown in [15] to be distinct provided the disorder is unbounded, but for $\alpha = 0$ they are equal for all $\beta > 0$ [4]. Theorem 1.5 of [6] shows that for $\alpha > 1/2$ the critical points are different for all values of $\beta > 0$.

The use of the terms localization/delocalization can be understood better by relating the quenched free energy to the portion of time the polymer spends on the defect line $\{0\} \times \mathbb{N}$. In particular, from [12], $f_q(\beta, \cdot)$ is differentiable for all $h \neq h_c(\beta)$ with

$$\frac{d}{dh} f_q(\beta, h) = \lim_{n\to\infty} E^{P_{n,\omega}^{\beta,h}}\left[\frac{1}{n}\sum_{i=1}^{n} \delta_i\right],$$

and therefore we can interpret the localized regime as the regime where the polymer spends a positive fraction of time on the defect line, while in the delocalized regime it spends a zero fraction of time on the defect line. While this is quite satisfactory in the localized regime, and further detailed studies on the path properties in the localized regime have been made in [12], it provides a rather incomplete picture in the delocalized one—it only allows one to conclude that the number of contacts is $o(n)$. It was proven in [11] that the number of contacts is at most of order $\log n$ in the delocalized regime. This was actually done for the related copolymer model, but its extension to the pinning model is straightforward [8]. More precisely, for every $h < h_c(\beta)$, there exists a constant $C_{\beta,h}$ such that

$$\limsup_{n\to\infty} \mathbb{E} P_{n,\omega}^{\beta,h}\big(|\tau \cap [1,n]| > C_{\beta,h} \log n\big) = 0.$$

This result was further extended to an a.s. statement in [14]: for $h < h_c(\beta)$ and for every $C > (1+\alpha)/(h_c(\beta) - h)$, we have

$$\limsup_{n\to\infty} P_{n,\omega}^{\beta,h}\big(|\tau \cap [1,n]| > C \log n\big) = 0, \qquad \mathbb{P}\text{-a.s.}$$

By analogy to the homogeneous pinning model (see [8], Chapter 8), one might expect that the number of contacts with the defect line should remain bounded in the whole delocalized regime. Nevertheless, the picture has been unclear in the disordered case, since stretches of unusual disorder values could typically attract the polymer back to the defect line a number of times growing to infinity with n. The open questions are discussed in [8], Section 8.5. In this work we clarify and complete the picture for behavior in probability. In fact, we will prove a stronger result, namely, that the last contact of the polymer happens at distance $O(1)$ from the origin. In particular, let

$$\tau_{\text{last}} = \max\{j \le n : \delta_j = 1\}.$$

We then have the following theorem.

THEOREM 1.1. *Suppose $\alpha > 0$, $\sum_n K(n) = 1$ and that ω_1 has exponential moments of all orders. For all $\beta, \varepsilon > 0$ and for all $h < h_c(\beta)$ we have that*

$$\limsup_{N\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(P_{n,\omega}^{\beta,h}(\tau_{\text{last}} > N) > \varepsilon\big) = 0.$$

One may ask whether this can be made an almost-sure result for $h < h_c(\beta)$, of the form

$$\limsup_{N\to\infty}\limsup_{n\to\infty} P_{n,\omega}^{\beta,h}(\tau_{\mathrm{last}} > N) = 0, \qquad \mathbb{P}\text{-a.s.},$$

or if the number of contacts is a.s. finite, that is,

$$\limsup_{N\to\infty}\limsup_{n\to\infty} P_{n,\omega}^{\beta,h}(|\tau \cap [0,n]| > N) = 0, \qquad \mathbb{P}\text{-a.s.}$$

The next theorem shows that the answer is no, at least for large β. Instead, for h between $h_c^{\mathrm{ann}}(\beta)$ and $h_c(\beta)$, infinitely often as $n \to \infty$, there will be an exceptionally rich segment of ω near n, which will (with high $P_{n,\omega}^{\beta,h}$-probability) induce the polymer to come to 0 and then make a number of returns of order $\log n$. For $t > 0$ let

$$h_t(\beta) := -(1+t\alpha)\log M\left(\frac{\beta}{1+t\alpha}\right). \tag{1.2}$$

Since $\log M$ is nondecreasing and convex on $[0,\infty)$ with $\log M(0) = 0$, it is easy to see that $h_t(\beta)$ is nondecreasing in t for fixed β. Recall (1.1); by [5], equation (3.7), for all $\beta > 0$ we have

$$-\log M(\beta) = h_c^{\mathrm{ann}}(\beta) = h_0(\beta) \le h_c(\beta) \le h_1(\beta). \tag{1.3}$$

By [15], Theorem 3.1, given $0 < \varepsilon < 1$, for large β we have

$$h_c(\beta) > h_{1-\varepsilon}(\beta). \tag{1.4}$$

We are now ready to state our second main result.

THEOREM 1.2. *Suppose ω is unbounded with all exponential moments finite. Given $\varepsilon > 0$, there exists $\beta_0(\varepsilon)$ and $\nu(\beta,h) > 0$ such that for*

$$\beta > \beta_0 \quad \text{and} \quad h > h_\varepsilon(\beta),$$

we have

$$\limsup_{n\to\infty} P_{n,\omega}^{\beta,h}(|\tau \cap [0,n]| > \nu \log n) = 1, \qquad \mathbb{P}\text{-a.s.}$$

By (1.4), Theorem 1.2 with $\varepsilon < 1/2$ includes at least the interval of values $h \in [h_\varepsilon(\beta), h_c(\beta)]$ below $h_c(\beta)$, which in turn (for large β) includes $h \in [h_\varepsilon(\beta), h_{1-\varepsilon}(\beta)]$. The path behavior in the regime of Theorem 1.2 is therefore in contrast with that for $h < h_c^{\mathrm{ann}}(\beta)$, where, in fact, the number of contacts remains tight for the measures averaged over the disorder; see [11], Remark 1.5.

The next two sections are devoted to the proofs of each theorem, respectively.

2. Proof of Theorem 1.1. It will be convenient to introduce generic constants. Specifically, C will denote a generic constant whose value might be different in different appearances. If we want to distinguish between constants we will enumerate them, for example, C_1, C_2, etc. When we want to emphasize the dependence of a generic constant on some parameters, we will include the symbols of these parameters as a subscript. In particular, we use the notation C_α for a generic constant which will depend on the parameter α and the slowly varying function ϕ of the renewal process. To simplify the notation we will also defer from using the integer part $[x]$ and simply write x, which should not lead to any confusion in the contexts where we use it. Let us define the events

$$E_{n,N} = \{|\tau \cap [0,n]| > N\}, \qquad E_{[m,n],N} = \{|\tau \cap [m,n]| > N\}.$$

In proving Theorem 1.1 we will make use of the following theorem, which was proved in [14].

THEOREM 2.1 ([14]). *Let $\beta \geq 0$ and $h < h_c(\beta)$. Then*:

(i) *For $\mathbb{P}$-a.e. environment ω, we have*

$$\sum_{n=0}^{\infty} Z_{n,\omega}^{\beta,h,c} < +\infty.$$

(ii) *For every $\varepsilon > 0$ and for $\mathbb{P}$-a.e. environment ω, there exists $N_\varepsilon(\omega) > 0$ such that for all $N \geq N_\varepsilon$, we have that*

$$\sum_{n=0}^{\infty} Z_{n,\omega}^{\beta,h,c}(E_{n,N}) \leq \sum_{k=N}^{\infty} e^{-k(h_c(\beta)-h-\varepsilon)}.$$

(iii) *For every constant $C > \frac{1+\alpha}{h_c(\beta)-h}$ and for $\mathbb{P}$-a.e. environment ω, we have*

$$P_{n,\omega}^{\beta,h,c}(E_{n,C\log n}) \longrightarrow 0, \qquad \text{as } n \to \infty.$$

The quantity $\mathcal{Z}(\omega) = \sum_{n=0}^{\infty} Z_{n,\omega}^{\beta,h,c}$, which is a.s. finite, will play an important role, as will the reversed process $\mathcal{Z}_n(\omega) = \sum_{m=-\infty}^{n} Z_{[m,n],\omega}^{\beta,h,c}$, which for any fixed n has the same distribution as $\mathcal{Z}(\omega)$. Note that we think here of the polymer path starting at point n and going backwards in time, which is why we have defined the disorder on the whole of $\mathbb{Z}$.

Here is a sketch of the proof. The event $\{\tau_{\text{last}} > N\}$ is contained in the union of the following events, where $C_1, b > 0$ are constants, with b small:

(a) there are more than $C_1 \log n$ returns by time n;

(b) there are fewer than $C_1 \log n$ returns, and no gap between returns exceeds bn;

(c) $\tau_{\text{last}} > N$, there are fewer than $C_1 \log n$ returns, and some gap between returns inside $[0,n]$ exceeds bn;

(d) $\tau_{\text{last}} > N$, there are fewer than $C_1 \log n$ returns, and the incomplete gap $[\tau_{\text{last}}, n]$ exceeds bn.

The Gibbs probability of (a) can be controlled by a variant of Theorem 2.1(iii), (b) can be controlled using the small probability of the event under the free measure and (d) is relatively straightforward, so the main work is (c). The segment to the left of the size-bn gap corresponds to a term in the sum $\mathcal{Z}(\omega)$, and (after we "tie down" the right end of the polymer by adding a visit at time n) the segment to the right corresponds to a term in $\mathcal{Z}_n(\omega)$, so we make use of Theorem 2.1(i) and a bound for the probability of a big gap under the free measure.

Let us make note here of the trivial lower bound

$$Z_{n,\omega}^{\beta,h} \geq K^+(n) e^{\beta\omega_0 + h}, \tag{2.1}$$

which comes from the trajectory having no renewals after time 0.

We will need the following analog of Theorem 2.1(iii), for the free polymer measure.

LEMMA 2.2. *Let $\beta \geq 0$ and $h < h_c(\beta)$. Then for all $C_1 > \frac{\alpha}{h_c(\beta) - h}$ and for $\mathbb{P}$-a.e. environment ω, we have*

$$P_{n,\omega}^{\beta,h}(E_{n,C_1 \log n}) \longrightarrow 0, \qquad \text{as } n \to \infty.$$

PROOF. Let $\varepsilon > 0$ satisfy $C_1 > \frac{\alpha+\varepsilon}{h_c(\beta)-h-\varepsilon}$. Using Theorem 2.1(ii) and (2.1), for some $C_2 = C_2(\beta, h, \varepsilon, \alpha)$, we have for large n

$$\begin{aligned} Z_{n,\omega}^{\beta,h}(E_{n,C_1 \log n}) &= \sum_{j=1}^{n} Z_{j,\omega}^{\beta,h,c}(E_{j,C_1 \log n}) K^+(n-j) \\ &\leq \sum_{k=C_1 \log n}^{\infty} e^{-k(h_c(\beta)-h-\varepsilon)} \\ &\leq C_2 n^{-(\alpha+\varepsilon)} \\ &\leq C_2 K^+(n) e^{\beta\omega_0 + h} n^{-\varepsilon/2} \\ &\leq C_2 n^{-\varepsilon/2} Z_{n,\omega}^{\beta,h}, \end{aligned} \tag{2.2}$$

and the lemma follows. $\square$

Proposition 2.4 below will show that the probability is small for having fewer than $C_1 \log n$ renewals without some gap σ exceeding bn, when b is chosen sufficiently small. Let us denote this gap event by $A_{b,n}$; more precisely, let

$$\begin{aligned} A'_{b,n} = \{\tau &: \text{there exist } i, j \in [0,n],\ j - i \geq bn, \\ &\text{such that } \tau \cap [i,j] = \{i,j\}\} \end{aligned} \tag{2.3}$$

$$A''_{b,n} = \{\tau : \tau_{\text{last}} \leq n - bn\}$$
$$A_{b,n} = A'_{b,n} \cup A''_{b,n}.$$

We first prove an analogous statement for the free renewal process.

LEMMA 2.3. *Given C_1 as in Lemma 2.2 and $b \in (0, 1/2)$, for sufficiently large n, we have*

$$P(E^c_{n,C_1 \log n} \cap A^c_{b,n}) \leq n^{-\alpha/9b}.$$

PROOF. When the event $E^c_{n,C_1 \log n} \cap A^c_{b,n}$ occurs, there exists $l \leq C_1 \log n$ such that

$$\sigma_1 + \cdots + \sigma_l = \tau_{\text{last}} \in (n - bn, n] \quad \text{and} \quad \max_{i \leq l} \sigma_i < bn.$$

Among these first l jumps, the total length of all jumps having individual length $\sigma_i \leq n/4C_1 \log n$ is at most $n/4$, so the total length of all jumps with individual length $\sigma_i \in [n/4C_1 \log n, bn)$ is at least $n/4$. This means there must be at least $1/4b$ values $\sigma_i \geq n/4C_1 \log n$ among $\sigma_1, \dots, \sigma_{C_1 \log n}$. Presuming n is large, we have

$$p_n := P\left(\sigma_1 \geq \frac{n}{4C_1 \log n}\right) \leq n^{-\alpha/2}.$$

Let k_n be the integer part of $C_1 \log n$, and let r be the least integer greater than or equal to $1/4b$. Then for large n,

$$\begin{aligned} P(E^c_{n,C_1 \log n} \cap A^c_{b,n}) &\leq P\left(\left|\left\{i \leq k_n : \sigma_i \geq \frac{n}{4C_1 \log n}\right\}\right| \geq r\right) \\ &\leq \binom{k_n}{r} p_n^r \leq (k_n p_n)^r \leq n^{-\alpha/9b}. \end{aligned}$$

□

PROPOSITION 2.4. *Given $C_1 > 0$ as in Lemma 2.2 and given β, h, for $b > 0$ sufficiently small,*

$$P^{\beta,h}_{n,\omega}(E^c_{n,C_1 \log n} \cap A^c_{b,n}) \to 0 \qquad \text{a.s. as } n \to \infty.$$

PROOF. We have from Lemma 2.3 that if b is sufficiently small, then for large n,

$$\begin{aligned} &\frac{1}{K^+(n)} \mathbb{E}[Z^{\beta,h}(E^c_{n,C_1 \log n} \cap A^c_{b,n})] \\ &\quad \leq \frac{C_\alpha n^\alpha}{\phi(n)} e^{(\log M(\beta)+h)C_1 \log n} P(E^c_{n,C_1 \log n} \cap A^c_{b,n}) \qquad (2.4) \\ &\quad \leq \frac{1}{n^3}. \end{aligned}$$

Therefore for all $\eta > 0$,

$$
\begin{aligned}
&\mathbb{P}\big(P_{n,\omega}^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c) > \eta \text{ i.o.}\big)\\
&\quad\le \mathbb{P}\big(Z_{n,\omega}^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c) > \eta K^+(n) e^{\beta\omega_0+h} \text{ i.o.}\big)\\
&\quad\le \mathbb{P}\big(Z_{n,\omega}^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c) > \eta K^+(n) n^{-1} \text{ i.o.}\big)\\
&\qquad + \mathbb{P}\left(e^{\beta\omega_0+h} < \frac{1}{n} \text{ i.o.}\right).
\end{aligned}
\tag{2.5}
$$

Now the second probability on the right-hand side of (2.5) is 0, and by (2.4), for the first probability on the right-hand side, we have

$$
\begin{aligned}
&\mathbb{P}\big(Z_{n,\omega}^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c) > \eta K^+(n) n^{-1}\big)\\
&\quad\le \frac{n}{\eta K^+(n)} \mathbb{E}\big[Z^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c)\big]\\
&\quad\le \frac{1}{\eta n^2}.
\end{aligned}
\tag{2.6}
$$

Summing over n and applying the Borel–Cantelli lemma completes the proof. $\square$

The next proposition, together with Lemma 2.2 and Proposition 2.4, shows that with probability tending to one, the first big gap, of length at least bn, brings the polymer out of $[0, n]$.

PROPOSITION 2.5. *For every $b, \varepsilon > 0$ we have*

$$
\lim_{n\to\infty} \mathbb{P}\big(P_{n,\omega}^{\beta,h}(A'_{b,n}) > \varepsilon\big) = 0.
\tag{2.7}
$$

PROOF. Let $0 < \theta < 1$. Then, summing over possible locations $[n_1, n_2]$ for the interval of the first long jump, we have

$$
\begin{aligned}
Z_{n,\omega}^{\beta,h}(A'_{b,n}) &\le \sum_{n_1} \sum_{n_1+bn<n_2\le n} Z_{n_1,\omega}^{\beta,h,c} K(n_2-n_1) Z_{[n_2,n],\omega}^{\beta,h}\\
&= \sum_{n_1=0}^{n} \sum_{\max(n_1+bn,n-n^\theta)<n_2\le n} Z_{n_1,\omega}^{\beta,h,c} K(n_2-n_1) Z_{[n_2,n],\omega}^{\beta,h}\\
&\quad + \sum_{n_1=0}^{n} \sum_{n_1+bn<n_2\le n-n^\theta} Z_{n_1,\omega}^{\beta,h,c} K(n_2-n_1) Z_{[n_2,n],\omega}^{\beta,h}.
\end{aligned}
\tag{2.8}
$$

Using (2.1), we can bound the first term on the right-hand side of (2.8) by

$$
\begin{aligned}
&\sum_{n_1} \sum_{\max(n_1+bn, n-n^\theta)<n_2\le n} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} K(n_2-n_1) Z^{\beta,h,c}_{[n_2,n-l],\omega} K^+(l) \\
&\quad \le C K(bn) \sum_{n_1} \sum_{\max(n_1+bn, n-n^\theta)<n_2\le n} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n-l],\omega} K^+(l) \\
&\quad \le \frac{C}{b^{1+\alpha}} K(n) n^\theta \sum_{n_1} \sum_{\max(n_1+bn, n-n^\theta)<n_2\le n} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n-l],\omega} K(l) \qquad (2.9)\\
&\quad \le \frac{C}{b^{1+\alpha}} K^+(n) n^{\theta-1} e^{-(\beta\omega_n+h)} \sum_{n_1} \sum_{\max(n_1+bn, n-n^\theta)<n_2\le n} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n],\omega} \\
&\quad \le \frac{C}{b^{1+\alpha}} Z^{\beta,h}_{n,\omega} n^{\theta-1} e^{-(\beta\omega_0+h)} e^{-(\beta\omega_n+h)} \mathcal{Z}(\omega)\mathcal{Z}_n(\omega).
\end{aligned}
$$

The second term on the right-hand side of (2.8) is bounded by

$$
\begin{aligned}
&\sum_{n_1} \sum_{n_1+bn<n_2<n-n^\theta} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} K(n_2-n_1) Z^{\beta,h,c}_{[n_2,n-l],\omega} K^+(l) \\
&\quad \le C K(bn) \sum_{n_1} \sum_{n_1+bn<n_2\le n-n^\theta} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n-l],\omega} K^+(l) \\
&\quad \le \frac{C}{b^{1+\alpha}} K(n) n \sum_{n_1} \sum_{n_1+bn<n_2\le n-n^\theta} \sum_{l\le n-n_2} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n-l],\omega} K(l) \qquad (2.10)\\
&\quad \le \frac{C}{b^{1+\alpha}} K^+(n) e^{-(\beta\omega_n+h)} \sum_{n_1} \sum_{n_1+bn<n_2\le n-n^\theta} Z^{\beta,h,c}_{n_1,\omega} Z^{\beta,h,c}_{[n_2,n],\omega} \\
&\quad \le \frac{C}{b^{1+\alpha}} Z^{\beta,h}_{n,\omega} e^{-(\beta\omega_0+h)} e^{-(\beta\omega_n+h)} \sum_{n_1=0}^{\infty} Z^{\beta,h,c}_{n_1,\omega} \sum_{n_2=-\infty}^{n-n^\theta} Z^{\beta,h,c}_{[n_2,n],\omega} \\
&\quad \le \frac{C}{b^{1+\alpha}} Z^{\beta,h}_{n,\omega} e^{-(\beta\omega_0+h)} e^{-(\beta\omega_n+h)} \mathcal{Z}(\omega) \sum_{n_2=-\infty}^{n-n^\theta} Z^{\beta,h,c}_{[n_2,n],\omega}.
\end{aligned}
$$

From (2.8), (2.9) and (2.10) we have that

$$
\begin{aligned}
P^{\beta,h}_{n,\omega}(A'_{b,n}) &\le \frac{C}{b^{1+\alpha}} n^{\theta-1} e^{-(\beta\omega_0+h)} e^{-(\beta\omega_n+h)} \mathcal{Z}(\omega)\mathcal{Z}_n(\omega) \qquad (2.11)\\
&\quad + \frac{C}{b^{1+\alpha}} e^{-(\beta\omega_0+h)} e^{-(\beta\omega_n+h)} \mathcal{Z}(\omega) \sum_{n_2=-\infty}^{n-n^\theta} Z^{\beta,h,c}_{[n_2,n],\omega}.
\end{aligned}
$$

Now $\mathcal{Z}(\omega)$ and $\mathcal{Z}_n(\omega)$ are finite almost surely and equidistributed, so the first term on the right in (2.11) converges to 0 in $\mathbb{P}$-probability. The sum on the right-hand side of (2.11) has the same distribution as

$$\sum_{m=n^\theta}^{\infty} Z_{m,\omega}^{\beta,h,c},$$

so by Theorem 2.1(i), it converges to 0 in probability. Hence the second term on the right-hand side of (2.11) also converges to 0 in probability, and the proof is complete. □

We can now complete the proof of our first theorem.

PROOF OF THEOREM 1.1. For $b > 0$ we have

$$\begin{aligned} P_{n,\omega}^{\beta,h}(\tau_{\text{last}} > N) \le{}& P_{n,\omega}^{\beta,h}(E_{n,C_1\log n}^c \cap A_{b,n}^c) + P_{n,\omega}^{\beta,h}(E_{n,C_1\log n}) \\ &+ P_{n,\omega}^{\beta,h}(A'_{b,n}) + P_{n,\omega}^{\beta,h}(\{\tau_{\text{last}} > N \cap A''_{b,n}). \end{aligned}$$

By Proposition 2.4, with the choice of sufficiently small $b > 0$, and Lemma 2.2, respectively, we have that the first and second terms in the above expression converge to zero $\mathbb{P}$-a.s., while by Proposition 2.5, the third term converges to 0 in $\mathbb{P}$-probability. Therefore, it only remains to check that

$$\limsup_{N\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(P_{n,\omega}^{\beta,h}(\{\tau_{\text{last}} > N\}\cap A''_{b,n}) > \varepsilon\big) = 0.$$

To this end we have

$$\begin{aligned} &\mathbb{P}\big(P_{n,\omega}^{\beta,h}(\{\tau_{\text{last}} \ge N\}\cap A''_{b,n}) > \varepsilon\big) \\ &\quad\le \mathbb{P}\big(Z_{n,\omega}^{\beta,h}(\{\tau_{\text{last}} \ge N\}\cap A''_{b,n}) > \varepsilon K^+(n)e^{\beta\omega_0+h}\big) \\ &\quad= \mathbb{P}\Bigg(\sum_{N\le n_1<n-bn} Z_{n_1,\omega}^{\beta,h,c} K^+(n-n_1) > \varepsilon K^+(n)e^{\beta\omega_0+h}\Bigg) \\ &\quad\le \mathbb{P}\Bigg(\sum_{N\le n_1<n-bn} Z_{n_1,\omega}^{\beta,h,c} > \varepsilon C_{\alpha,b}\, e^{\beta\omega_0+h}\Bigg) \\ &\quad\le \mathbb{P}\Bigg(\sum_{n_1\ge N} Z_{n_1,\omega}^{\beta,h,c} > \varepsilon C_{\alpha,b}\, e^{\beta\omega_0+h}\Bigg), \end{aligned}$$

and by Theorem 2.1(i), the latter tends to 0 as $N \to \infty$. □

The analog of Theorem 1.1 also holds for the constrained case, that is, for $P_{n,\omega}^{\beta,h,c}$, in the sense that the rightmost contact point in $[0, \frac{n}{2}]$ and the leftmost contact point in $[\frac{n}{2}, n]$ occur at distances $O(1)$ from 0 and n, respectively. To quantify things, let us denote

$$\hat{\tau}_{\text{last}} = \max\left\{ j \in \left[0, \frac{n}{2}\right] : \delta_j = 1 \right\}$$

and

$$\check{\tau}_{\text{last}} = \min\left\{ j \in \left[\frac{n}{2}, n\right] : \delta_j = 1 \right\}.$$

Then we have the following.

THEOREM 2.6. *Suppose $\alpha > 0$, $\sum_n K(n) = 1$ and that ω_1 has exponential moments of all orders. For all $\beta, \varepsilon, \delta > 0$ and for all $h < h_c(\beta)$ there exist $n_0(\varepsilon, \delta)$, $N_0(\varepsilon, \delta)$ and $M_0(\varepsilon, \delta)$, such that for all $n > n_0(\varepsilon, \delta)$, $N > N_0(\varepsilon, \delta)$, $M > M_0(\varepsilon, \delta)$*

$$\mathbb{P}\big(P_{n,\omega}^{\beta,h,c}(\{\hat{\tau}_{\text{last}} > N\} \cup \{\check{\tau}_{\text{last}} < n - M\}) > \varepsilon\big) < \delta.$$

PROOF. Notice that in the constrained case $A_{b,n} = A'_{b,n}$, and we have

$$\begin{aligned}
&P_{n,\omega}^{\beta,h,c}(\{\hat{\tau}_{\text{last}} > N\} \cup \{\check{\tau}_{\text{last}} < n - M\}) \\
&\quad \le P_{n,\omega}^{\beta,h,c}(E_{n,C_1 \log n}^c \cap A_{b,n}^c) + P_{n,\omega}^{\beta,h,c}(E_{n,C_1 \log n}) \\
&\qquad + P_{n,\omega}^{\beta,h,c}\big((\{\hat{\tau}_{\text{last}} > N\} \cup \{\check{\tau}_{\text{last}} < n - M\}) \cap A'_{b,n}\big).
\end{aligned}$$

By a straightforward modification of Proposition 2.4, Theorem 2.1(iii) and Lemma 2.2, the first two terms converge to zero as n tends to infinity, once b is chosen small enough. Regarding the third term, notice that by symmetry it is sufficient to control $Z_{n,\omega}^{\beta,h,c}(\{\hat{\tau}_{\text{last}} > N\} \cap A'_{b,n})$. We make two sums according to whether the first big gap $[n_1, n_2]$ ends before or after the midpoint $n/2$. Specifically, we have

$$\begin{aligned}
&Z_{n,\omega}^{\beta,h,c}(\{\hat{\tau}_{\text{last}} > N\} \cap A'_{b,n}) \\
&\quad \le \sum_{n_1 > N} \sum_{\max(n_1 + bn, n/2) < n_2 \le n} Z_{n_1,\omega}^{\beta,h,c} K(n_2 - n_1) Z_{[n_2,n],\omega}^{\beta,h,c} \\
&\qquad + \sum_{n_1} \sum_{n_1 + bn < n_2 \le n/2} Z_{n_1,\omega}^{\beta,h,c} K(n_2 - n_1) Z_{[n_2,n],\omega}^{\beta,h,c}.
\end{aligned}$$

Following the same (and actually more direct) steps as in the proof of Proposition 2.5 we can bound the above by

$$C_b e^{-(\beta\omega_0 + h)} e^{-(\beta\omega_n + h)} Z_{n,\omega}^{\beta,h,c}\left(\sum_{n_1 > N} Z_{n_1,\omega}^{\beta,h} \mathcal{Z}_n(\omega) + \mathcal{Z}(\omega) \sum_{n_2 < n/2} Z_{[n_2,n],\omega}^{\beta,h,c}\right),$$

and the rest follows as in Proposition 2.5. □

3. Proof of Theorem 1.2. We begin again with a sketch. Assume for simplicity that $K(1) > 0$. Suppose there is a "rich segment" of $[0, n]$ of length at least $\gamma \log n$ in which the average of the disorder is at least u; here γ is small and u is large. (We show that such a rich segment exists for infinitely many n.) We consider the contribution to the partition function from two different sets of trajectories:

(a) the single trajectory which returns at every site of the rich segment, and nowhere else;

(b) those trajectories which make at most $\nu \log n$ returns, with ν small.

We show that (up to slowly varying correction factors) the contribution from (a) is at least a certain inverse power $n^{-\alpha+\kappa}$, while a.s., except for finitely many n, the contribution from (b) is bounded by the smaller inverse power $n^{-\alpha+\kappa/2}$. The Gibbs probability of (b) is bounded by the ratio of the two contributions, hence by $n^{-\kappa/2}$, so it approaches 0.

We will need the following lemma, which is an elementary fact about convex functions.

LEMMA 3.1. *Suppose Ψ is nondecreasing and convex on $[0,\infty)$ with $\Psi(0) = 0$ and $\Psi'(x) \to \infty$ as $x \to \infty$. Then for all $s > 1$,*

$$\Psi(sx) - s\Psi(x) \to \infty \qquad \textit{as } x \to \infty.$$

PROOF. Since Ψ' is nondecreasing, for $s > 1$ and $x > 1$, we have

$$\Psi(sx) - s\Psi(x) = (s-1)\int_0^x \big(\Psi'\big(x+(s-1)t\big) - \Psi'(t)\big)\,dt \tag{3.1}$$

$$\geq (s-1)\int_0^1 \big(\Psi'\big(x+(s-1)t\big) - \Psi'(t)\big)\,dt \tag{3.2}$$

$$\geq (s-1)\big(\Psi'(x) - \Psi'(1)\big) \tag{3.3}$$

$$\to \infty \qquad \text{as } x \to \infty. \tag{3.4}$$

Here the first inequality follows from the fact that the integrand in nonnegative. $\square$

PROOF OF THEOREM 1.2. Recall the definition of $h_t(\beta)$ from (1.2). Suppose $h = h_t(\beta)$ with $t > \varepsilon$. If $t \geq 1$, then $h \geq h_c(\beta)$ by (1.3), so we need only consider $t < 1$. Let $r = \min\{j : K(j) > 0\}$, let $\gamma, u > 0$ to be specified, define

$$J_n = \{n - ir : 0 \leq i \leq \gamma \log n - 1\}, \qquad \overline{\omega}_{J_n} = \frac{1}{|J_n|}\sum_{j\in J_n} \omega_j,$$

and define the event

$$D_n^{u,\gamma} = \{\omega : \overline{\omega}_{J_n} \geq u\}.$$

We can bound $Z_{n,\omega}^{\beta,h}$ below by the contribution from the path which makes returns precisely at the times in J_n, obtaining that for large n, for all $\omega \in D_n^{u,\gamma}$,

$$\begin{aligned} Z_{n,\omega}^{\beta,h} &\geq e^{(\beta u+h)|J_n|} K(n - \gamma r\log n) K(r)^{|J_n|-1} \\ &\geq \frac{1}{2} n^{-(1+\alpha)}\phi(n)\exp\biggl(\gamma\biggl(\beta u + h - \log\frac{1}{K(r)}\biggr)\log n\biggr). \end{aligned} \tag{3.5}$$

Let Φ be the large deviation rate function related to ω, and let $\delta > 0$ to be specified. For large n we have

$$\mathbb{P}\big(D_n^{u,\gamma}\big) \geq e^{-(1+\delta)\Phi(u)\gamma \log n}. \tag{3.6}$$

Since all exponential moments of ω are finite, we have $\Phi(u)/u \to \infty$ as $u \to \infty$. Recalling that $\log M(\beta) = \sup\{\beta u - \Phi(u) : u \in \mathbb{R}\}$, we can therefore choose $u = u_\beta$ to satisfy

$$\beta u - \Phi(u) = \log M(\beta).$$

For β sufficiently large (depending on ε), since $\Phi'(u) \to \infty$ as $u \to \infty$, we have by Lemma 3.1 that

$$\log M(\beta) - (1 + \varepsilon\alpha) \log M\left(\frac{\beta}{1 + \varepsilon\alpha}\right) > \log \frac{1}{K(r)} \tag{3.7}$$

or equivalently,

$$\beta u_\beta + h_\varepsilon(\beta) - \log \frac{1}{K(r)} > \Phi(u_\beta).$$

We now choose δ to satisfy

$$\beta u_\beta + h_\varepsilon(\beta) - \log \frac{1}{K(r)} > (1 + \delta)\Phi(u_\beta)$$

and then γ to satisfy

$$\beta u_\beta + h_\varepsilon(\beta) - \log \frac{1}{K(r)} > \frac{1}{\gamma} > (1 + \delta)\Phi(u_\beta). \tag{3.8}$$

Define $\kappa > 0$ by

$$\gamma\left(\beta u_\beta + h_\varepsilon(\beta) - \log \frac{1}{K(r)}\right) = 1 + \kappa, \tag{3.9}$$

so that by (3.5), for all $\omega \in D_n^{u,\gamma}$,

$$Z_{n,\omega}^{\beta,h} \geq \tfrac{1}{2} n^{-\alpha+\kappa} \phi(n). \tag{3.10}$$

We select a subsequence of the events $\{D_n^{u,\gamma}\}$ that are independent, as follows. Fix n_0 and given $n_0, \dots, n_j$ define $n_{j+1} = n_j + 2r\gamma \log n_j$. Then $n_j \sim 2r\gamma j \log j$ as $j \to \infty$, and it is easily checked that, provided n_0 is sufficiently large, the events $\{D_{n_j}^{u,\gamma}, j \geq 0\}$ are independent. With (3.6) and (3.8) this shows that

$$\sum_j \mathbb{P}\big(D_{n_j}^{u,\gamma}\big) = \infty \quad \text{so} \quad \mathbb{P}\big(D_n^{u,\gamma} \text{ i.o.}\big) = 1. \tag{3.11}$$

Let us now choose

$$m > \frac{4}{\kappa}, \qquad \lambda = 2\left(\frac{1}{m} \log M(m\beta) + h\right), \qquad \nu = \frac{\kappa}{2\lambda}, \tag{3.12}$$

with m an integer. We claim that

$$\mathbb{P}\big(Z_{n,\omega}^{\beta,h}\big(E_{n,\nu\log n}^{c}\big) > n^{-\alpha+\lambda\nu}\phi(n) \text{ i.o.}\big) = 0. \tag{3.13}$$

This is plausible because for appropriate λ, $\nu\log n$ visits should not likely yield more than $\lambda\nu\log n$ energy above the "immediate escape" value, which is approximately the log of $K^{+}(n)$, that is, approximately $-\alpha\log n$. Assuming this claim, we use (3.11) to conclude that

$$\mathbb{P}\big(D_n \cap \big\{Z_{n,\omega}^{\beta,h}\big(E_{n,\nu\log n}^{c}\big) < n^{-\alpha+\lambda\nu}\phi(n)\big\} \text{ i.o.}\big) = 1,$$

which with (3.10) shows that

$$\mathbb{P}\big(P_{n,\omega}^{\beta,h}\big(E_{n,\nu\log n}^{c}\big) < 2n^{-\kappa/2} \text{ i.o.}\big) = 1,$$

which proves the theorem.

It remains to prove (3.13). Observe that by Chebyshev's inequality we have

$$\begin{aligned} &\mathbb{P}\big(Z_{n,\omega}^{\beta,h}\big(E_{n,\nu\log n}^{c}\big) > n^{-\alpha+\lambda\nu}\phi(n)\big) \\ &\qquad \le \big(n^{-\alpha}\phi(n)\big)^{-m} n^{-m\lambda\nu}\mathbb{E}\big[\big(Z_{n,\omega}^{\beta,h}\big(E_{n,\nu\log n}^{c}\big)\big)^{m}\big]. \end{aligned} \tag{3.14}$$

Denoting by $E^{\otimes m}$ the expectation over m independent copies of the renewal τ, we see that the expectation on the right-hand side of (3.14) can be written as

$$E^{\otimes m}\big[e^{\sum_{i=1}^{n}(\log M(\beta(\delta_i^{(1)}+\cdots+\delta_i^{(m)}))+h(\delta_i^{(1)}+\cdots+\delta_i^{(m)}))}; \big(E_{n,\nu\log n}^{c}\big)^{\otimes m}\big],$$

where $(E_{n,\nu\log n}^{c})^{\otimes m}$ is the m-fold product of $E_{n,\nu\log n}^{c}$. Using the convexity of $\log M(\beta)$ we have

$$\log M(\beta k) \le \frac{k}{m}\log M(\beta m) \qquad \text{for all } k \le m,$$

so we can bound the above expectation by

$$\begin{aligned} &E^{\otimes m}\big[e^{\sum_{i=1}^{n}((1/m)\log M(m\beta)+h)(\delta_i^{(1)}+\cdots+\delta_i^{(m)})}; \big(E_{n,\nu\log n}^{c}\big)^{\otimes m}\big] \\ &\qquad < e^{((1/m)\log M(m\beta)+h)m\nu\log n}\, P\big(E_{n,\nu\log n}^{c}\big)^{m}. \end{aligned} \tag{3.15}$$

We use $A_{b,n}$ from (2.3). By Lemma 2.3 we have for b sufficiently small and then n sufficiently large,

$$\begin{aligned} P\big(E_{n,\nu\log n}^{c}\big) &\le P\big(E_{n,\nu\log n}^{c} \cap A_{b,n}^{c}\big) + \sum_{j=1}^{\nu\log n} P(\sigma_j > bn) \\ &\le n^{-2\alpha} + \nu K^{+}(bn)\log n \\ &\le C_b\nu\log n\frac{\phi(n)}{n^{\alpha}}. \end{aligned}$$

Inserting this into (3.15) and the result into (3.14), and considering our choice of λ, m, ν, we obtain that

$$\mathbb{P}\big(Z_{n,\omega}^{\beta,h}(E_{n,\nu\log n}^c) > n^{-\alpha+\lambda\nu}\phi(n)\big) \le (C_b \nu \log n)^m n^{-m\kappa/4},$$

which, by the choice of m in (3.12) and the Borel–Cantelli lemma, completes the proof. $\square$

If we do not assume β large in Theorem 1.2, then in the proof, the entropy cost $\log 1/K(r)$ per visit to J_n will not be exceeded by the energy gain; in more concrete terms, (3.7) will fail. The entropy cost can be reduced by visiting only a small fraction of the sites in an interval of form $[n - \gamma \log n, n]$, but then the interval length $\gamma \log n$ (where the disorder average exceeds u_β) must be much larger than in the large-β proof, reducing the probability of such an interval. It is not clear whether there is a strategy (in place of the present "visit all sites of J_n") of sufficiently low entropy cost so that the interval of large average disorder values can be exploited, and therefore it seems unclear whether a variant of Theorem 1.2 should be true for small β.

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