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Partitioning measure of quasi-symmetry for square contingency tables

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Abstract. For the analysis of square contingency tables, we propose the Kullback–Leibler information type measure to represent the degree of departure from the quasi-symmetry (QS) model. We introduce the global quasi-symmetry (GQS) model, and show that the QS model holds if and only if both the GQS and extended quasi-symmetry (EQS) models hold. Furthermore, we propose a measure of departure from each of the GQS and the EQS models, and show that the value of measure of QS is equal to the sum of the value of measure of GQS and that of EQS.

1 Introduction

Consider an $R \times R$ square contingency table with same row and column classifications. Let p_{ij} denote the probability that an observation will fall in the ith row and the jth column of the table ($i = 1, \dots, R$; $j = 1, \dots, R$). Caussinus (1966) proposed the quasi-symmetry (QS) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \qquad \text{for } i = 1, \dots, R;\ j = 1, \dots, R,$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\{\alpha_i = \beta_i\}$ is the symmetry model (Bowker, 1948; Bishop, Fienberg and Holland, 1975, p. 282). We note that the symmetric association model proposed by Goodman (1979) is equivalent to the QS model.

Denote the odds ratio for rows i and j $(> i)$, and columns s and t $(> s)$ by $\theta_{ij;st} = (p_{is}p_{jt})/(p_{js}p_{it})$. Using odds ratios, the QS model is expressed as, for example,

$$\theta_{ij;jk} = \theta_{jk;ij} \qquad \text{for } i < j < k. \tag{1.1}$$

This indicates the symmetry of odds ratios with respect to the main diagonal of the table. From equation (1.1), the QS model is further expressed as

$$D_{ijk} = D_{kji} \qquad \text{for } i < j < k, \tag{1.2}$$

where $D_{ijk} = p_{ij}p_{jk}p_{ki}$ and $D_{kji} = p_{kj}p_{ji}p_{ik}$. See Caussinus (1966) for more details about the QS model.

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Tomizawa (1984) considered the extended quasi-symmetry (EQS) model defined by

$$D_{ijk} = \gamma D_{kji} \qquad \text{for } i < j < k.$$

A special case of this model obtained by putting $\gamma = 1$ is the QS model. If the QS model holds, then the EQS model holds; but the converse does not hold. Therefore, we are interested in seeing what structure is necessary for obtaining the QS model in addition to the structure of EQS. The decomposition of the QS model into two models may be useful for seeing the reason for the poor fit of the QS model when the QS model does not hold for the given data.

Some topics related to quasi-symmetry are described in many articles (e.g., Kateri and Papaioannou, 1997; Caussinus, 2002; Tomizawa and Tahata, 2007). For instance, some models that are extension of quasi-symmetry have been proposed in Tahata and Tomizawa (2006) and Tomizawa et al. (2007).

Consider the data in Table 1, taken from Tominaga (1979, p. 130). These data describe the cross-classification of Japanese father's and his son's academic background which were examined in 1955 and 1975. Note that category (1) is elementary school; (2) junior high school; (3) high school; and (4) university. We denote the move to the son's level j from his father's level i by "$i \to j$." For Table 1, the QS model indicates that for a given order $i < j < k$, the probability that $i \to j$, $j \to k$ and $k \to i$ (we shall call the probability for right circulation for convenience), is equal to the probability that $k \to j$, $j \to i$ and $i \to k$ (we shall call the probability for left circulation). Namely, the QS model states that for a given order $i < j < k$, the probability that for each of two father–son pairs the son's level is

Table 1 *Cross-classification of Japanese father's and his son's academic background; taken from Tominaga* (1979, *p.* 130)

Father's educational level	Son's educational level (1)	(2)	(3)	(4)	Total
	(a) Examined in 1955				
(1)	374	602	170	64	1210
(2)	18	255	139	71	483
(3)	4	23	42	55	124
(4)	2	6	17	53	78
Total	398	886	368	243	1895
	(b) Examined in 1975				
(1)	161	569	386	107	1223
(2)	11	262	318	112	703
(3)	2	43	168	144	357
(4)	0	8	55	128	191
Total	174	882	927	491	2474

higher than his father's level (i.e., $i \to j$ and $j \to k$), and for one pair the son's level is lower than his father's level (i.e., $k \to i$), is equal to the probability that for each of two pairs the son's level is lower than his father's level (i.e., $k \to j$ and $j \to i$), and for one pair the son's level is higher than his father's level (i.e., $i \to k$). Also, the EQS model indicates that for a given order $i < j < k$, the probability that $i \to j$, $j \to k$ and $k \to i$ is γ times higher than the probability that $k \to j$, $j \to i$ and $i \to k$. Namely, the EQS model states that for any three father–son pairs, the probability that for each of two pairs the son's level is higher than his father's level, and for one pair the son's level is lower than his father's level, is γ times higher than the probability that for each of two pairs the son's level is lower than his father's level, and for one pair the son's level is higher than his father's level. Therefore, as the value of γ approaches the infinity (or zero), the *stochastic circular structure* tends to arise stronger among any three father–son pairs, where the *stochastic circular structure* means that the probability of left circulation is greater (or less) than that of right circulation. When the QS model holds, there is not the stochastic circular structure in the table in a sense that the probability of left circulation is equal to that of right circulation.

When the QS model does not hold, we may be interested in a measure to represent the degree of departure from QS. Similarly, when the EQS model does not hold, we are interested in measuring what degree the departure from EQS is. The QS model further may be expressed as $Q_{ijk} = Q_{kji}$ for $i < j < k$ where $Q_{ijk} = p^c_{ij} p^c_{jk} p^c_{ki}$ and $Q_{kji} = p^c_{kj} p^c_{ji} p^c_{ik}$ with $p^c_{st} = p_{st}/(p_{st} + p_{ts})$ for $s \neq t$. Tahata, Miyamoto and Tomizawa (2004) considered the measure to represent the degree of departure from QS (see Appendix A). This is the power-divergence type measure and is expressed as a function of $\{Q_{lmn}\}$. Since the QS model also is expressed as (1.2), we are interested in a measure to represent the degree of departure from QS as a function of $\{D_{lmn}\}$, namely, in a measure to represent the degree of departure from the equality of the probability for right circulation and the probability for left circulation for any three row-column pairs of observations.

The purpose of this paper is (i) to introduce the global quasi-symmetry (GQS) model, (ii) to consider the decomposition of QS model, and (iii) to propose the measures that represent the degree of departure from each of the QS, the GQS and the EQS models.

Section 2 describes the decomposition of the QS model. Section 3 proposes the measures to represent the degree of departure from each of the QS, the GQS and the EQS models. Moreover, Section 4 shows that the value of measure of QS is equal to the sum of the value of measure of GQS and that of EQS. The measures of departure from symmetry and decompositions of model have been proposed by, for example, Tahata et al. (2004) and Tomizawa (1984). However, we point out that using these methods we cannot decompose the QS model and measure the degree of departure from QS and EQS.

2 Decomposition of quasi-symmetry model

Consider the global quasi-symmetry (GQS) model defined by

$$\sum_{i<j<k} D_{ijk} = \sum_{i<j<k} D_{kji}.$$

This indicates that the sum of the probabilities for right circulation is equal to the sum of the probabilities for left circulation (although, for some $i < j < k$ the probability for right circulation would be greater than that for left circulation, and for the others the probability for right circulation would be less than that for left circulation). We note that (i) the QS model indicates that the probability for right circulation is equal to that for left circulation for all $i < j < k$, and (ii) the EQS model indicates that the probability for right circulation is γ times higher than that for left circulation for all $i < j < k$. Namely, (i) when the QS model holds, the GQS model holds, and (ii) when the EQS model with $\gamma \neq 1$ holds, the GQS model does not hold. We obtain the following theorem.

Theorem 1. *The QS model holds if and only if both the EQS and GQS models hold.*

Proof. If the QS model holds, then the EQS and GQS models hold. Assume that both the EQS and GQS models hold. Then we see

$$\sum_{i<j<k} D_{ijk} - \sum_{i<j<k} D_{kji} = (\gamma - 1) \sum_{i<j<k} D_{kji} = 0.$$

Thus, we obtain $\gamma = 1$ because $\sum_{i<j<k} D_{kji} \neq 0$. Therefore, the QS model holds. The proof is completed. □

3 Measures for three quasi-symmetric models

3.1 Measure for quasi-symmetry

We shall define a measure that represents the degree of departure from the QS model. Note that it is different from the measure proposed by Tahata et al. (2004).

Let

$$\Delta = \sum_{i<j<k} (D_{ijk} + D_{kji}),$$

and let for $i < j < k$,

$$D_{ijk}^{(1)} = \frac{D_{ijk}}{\Delta}, \qquad D_{ijk}^{(2)} = \frac{D_{kji}}{\Delta}.$$

Assuming that $\{D_{ijk} + D_{kji} \neq 0\}$, consider the measure defined by

$$\Phi_{\mathrm{QS}} = \frac{1}{\log 2} \operatorname*{minimum}_{\{E_{ijk}^{(1)}, E_{ijk}^{(2)}\}} I_1,$$

where

$$I_1 = \sum_{i<j<k} \left[D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{E_{ijk}^{(1)}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{E_{ijk}^{(2)}}\right) \right],$$

$$\sum_{i<j<k} \left(E_{ijk}^{(1)} + E_{ijk}^{(2)}\right) = 1, \qquad E_{ijk}^{(1)} > 0, \qquad E_{ijk}^{(2)} > 0, \qquad E_{ijk}^{(1)} = E_{ijk}^{(2)}.$$

This measure indicates, essentially, the *minimum* value of Kullback–Leibler information between $\{D_{ijk}^{(1)}, D_{ijk}^{(2)}\}$ and an arbitrary $\{E_{ijk}^{(1)}, E_{ijk}^{(2)}\}$ with the structure of QS.

Subject to $\sum_{i<j<k}(E_{ijk}^{(1)} + E_{ijk}^{(2)}) = 1$ and $E_{ijk}^{(1)} = E_{ijk}^{(2)}$, we consider minimizing I_1. Then we can obtain $\{E_{ijk}^{(1)} = E_{ijk}^{(2)} = (D_{ijk}^{(1)} + D_{ijk}^{(2)})/2\}$. Thus, the measure may be expressed as

$$\Phi_{\mathrm{QS}} = \frac{1}{\log 2} \sum_{i<j<k} \left[D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{C_{ijk}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{C_{ijk}}\right) \right],$$

where $C_{ijk} = (D_{ijk}^{(1)} + D_{ijk}^{(2)})/2$. We see that $\{D_{ijk}^{(1)} = D_{ijk}^{(2)} = C_{ijk}\}$ when the QS model holds. The quantities $\{C_{ijk}, C_{ijk}\}$ minimize the value of Kullback–Leibler information between $\{D_{ijk}^{(1)}, D_{ijk}^{(2)}\}$ and an arbitrary $\{E_{ijk}^{(1)}, E_{ijk}^{(2)}\}$ with the structure of QS.

We see that $0 \leq \Phi_{\mathrm{QS}} \leq 1$ because $0 \leq I_1 \leq \log 2$. If the QS model holds, then $I_1 = 0$ since $\{D_{ijk}^{(1)} = D_{ijk}^{(2)} = C_{ijk}\}$. Also, if $I_1 = 0$, then $\{D_{ijk}^{(1)} = D_{ijk}^{(2)} = C_{ijk}\}$ hold (i.e., the QS model holds). Therefore, we can obtain that $\Phi_{\mathrm{QS}} = 0$ if and only if the QS model holds. On the other hand, if $D_{ijk}^{(1)} = 0$ (then $D_{ijk}^{(2)} > 0$) or $D_{ijk}^{(2)} = 0$ (then $D_{ijk}^{(1)} > 0$), then $I_1 = \log 2$ holds, and the converse is also true. So, we can obtain that $\Phi_{\mathrm{QS}} = 1$ if and only if the degree of departure from QS is the largest in a sense that $D_{ijk}^{(1)} = 0$ (then $D_{ijk}^{(2)} > 0$) or $D_{ijk}^{(2)} = 0$ (then $D_{ijk}^{(1)} > 0$).

We shall call the structure, which for some (or all) $i < j < k$, the probability for right circulation is zero (and then the probability for left circulation is not zero), and for the others (or all) the probability for left circulation is zero (and then the probability for right circulation is not zero), the *strongest stochastic circular structure*. Namely, the *strongest stochastic circular structure* arises when $\Phi_{\mathrm{QS}} = 1$.

3.2 Measure for global quasi-symmetry

We shall consider a measure to represent the degree of departure from the GQS model. Let

$$D^{(1)} = \sum_{i<j<k} D^{(1)}_{ijk}, \qquad D^{(2)} = \sum_{i<j<k} D^{(2)}_{ijk}.$$

The GQS model can be expressed as

$$D^{(1)} = D^{(2)} (= \tfrac{1}{2}).$$

Assuming that $\{D_{ijk} + D_{kji} \neq 0\}$, consider the measure defined by

$$\Phi_{\mathrm{GQS}} = \frac{1}{\log 2}\left\{ D^{(1)} \log\left(\frac{D^{(1)}}{1/2}\right) + D^{(2)} \log\left(\frac{D^{(2)}}{1/2}\right)\right\}.$$

This is obtained by the similar manner to Section 3.1. We see that (i) $0 \leq \Phi_{\mathrm{GQS}} \leq 1$, (ii) $\Phi_{\mathrm{GQS}} = 0$ if and only if the GQS model holds, and (iii) $\Phi_{\mathrm{GQS}} = 1$ if and only if the degree of departure from GQS is the largest in a sense that $D^{(1)} = 0$ (then $D^{(2)} = 1$) or $D^{(2)} = 0$ (then $D^{(1)} = 1$). Namely, when $\Phi_{\mathrm{GQS}} = 1$, the *strongest stochastic circular structure* arises because the probability for right circulation is zero (and then the probability for left circulation is not zero) for all $i < j < k$ or the probability for left circulation is zero (and then the probability for right circulation is not zero) for all $i < j < k$.

3.3 Measure for extended quasi-symmetry

We shall define a measure that represents the degree of departure from the EQS model.

Assuming that $D^{(1)} \neq 0$, $D^{(2)} \neq 0$ and $\{D^{(1)}_{ijk} + D^{(2)}_{ijk} \neq 0\}$, consider the measure defined by

$$\Phi_{\mathrm{EQS}} = \frac{1}{\log 2} \underset{\{E^{(1)}_{ijk}, E^{(2)}_{ijk}\}}{\text{minimum}} I_2,$$

where

$$I_2 = \sum_{i<j<k} \left[D^{(1)}_{ijk} \log\left(\frac{D^{(1)}_{ijk}}{E^{(1)}_{ijk}}\right) + D^{(2)}_{ijk} \log\left(\frac{D^{(2)}_{ijk}}{E^{(2)}_{ijk}}\right)\right],$$

$$\sum_{i<j<k} (E^{(1)}_{ijk} + E^{(2)}_{ijk}) = 1, \qquad E^{(1)}_{ijk} > 0, \qquad E^{(2)}_{ijk} > 0, \qquad \frac{E^{(1)}_{ijk}}{E^{(2)}_{ijk}} = \gamma.$$

This measure indicates, essentially, the *minimum* value of Kullback–Leibler information between $\{D^{(1)}_{ijk}, D^{(2)}_{ijk}\}$ and an arbitrary $\{E^{(1)}_{ijk}, E^{(2)}_{ijk}\}$ with the structure of EQS.

Subject to $\sum_{i<j<k}(E_{ijk}^{(1)} + E_{ijk}^{(2)}) = 1$ and $E_{ijk}^{(1)} = \gamma E_{ijk}^{(2)}$, we consider minimizing I_2. Then we can obtain $\{E_{ijk}^{(1)} = D^{(1)}(D_{ijk}^{(1)} + D_{ijk}^{(2)})\}$ and $\{E_{ijk}^{(2)} = D^{(2)}(D_{ijk}^{(1)} + D_{ijk}^{(2)})\}$. Thus, the measure Φ_{EQS} may be expressed as

$$\Phi_{\mathrm{EQS}} = \frac{1}{\log 2} \sum_{i<j<k} \left[D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{G_{ijk}^{(1)}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{G_{ijk}^{(2)}}\right) \right],$$

where

$$G_{ijk}^{(1)} = D^{(1)}(D_{ijk}^{(1)} + D_{ijk}^{(2)}), \qquad G_{ijk}^{(2)} = D^{(2)}(D_{ijk}^{(1)} + D_{ijk}^{(2)}).$$

The quantities $\{G_{ijk}^{(1)}, G_{ijk}^{(2)}\}$ minimize the value of Kullback–Leibler information between $\{D_{ijk}^{(1)}, D_{ijk}^{(2)}\}$ and an arbitrary $\{E_{ijk}^{(1)}, E_{ijk}^{(2)}\}$ with the structure of EQS. We see that (i) $0 \le \Phi_{\mathrm{EQS}} \le 1$, (ii) $\Phi_{\mathrm{EQS}} = 0$ if and only if the EQS model holds, and (iii) $\Phi_{\mathrm{EQS}} = 1$ if and only if the degree of departure from EQS is the largest in a sense that $D_{ijk}^{(1)} = 0$ (then $D_{ijk}^{(2)} > 0$) or $D_{ijk}^{(2)} = 0$ (then $D_{ijk}^{(1)} > 0$) and $D^{(1)} = D^{(2)} = 1/2$; namely, although the sum of the probabilities for right circulation is equal to the sum of the probabilities for left circulation, the *strongest stochastic circular structure* arises for each $i < j < k$.

4 Relationships between the measures

Assuming that $D^{(1)} \neq 0$, $D^{(2)} \neq 0$ and $\{D_{ijk}^{(1)} + D_{ijk}^{(2)} \neq 0\}$, we obtain the following theorem.

Theorem 2. *The value of* Φ_{QS} *is equal to the sum of the value of* Φ_{GQS} *and the value of* Φ_{EQS}.

Proof. We see

$$\begin{aligned}
&\Phi_{\mathrm{GQS}} + \Phi_{\mathrm{EQS}} \\
&\quad = \frac{1}{\log 2} \sum_{i<j<k} \left[D_{ijk}^{(1)} \log\left(\frac{D^{(1)}}{1/2}\right) + D_{ijk}^{(2)} \log\left(\frac{D^{(2)}}{1/2}\right) \right. \\
&\qquad\qquad \left. + D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{G_{ijk}^{(1)}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{G_{ijk}^{(2)}}\right) \right] \\
&\quad = \frac{1}{\log 2} \sum_{i<j<k} \left[D_{ijk}^{(1)} \log 2 + D_{ijk}^{(2)} \log 2 \right.
\end{aligned}$$

$$+ D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{D_{ijk}^{(1)} + D_{ijk}^{(2)}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{D_{ijk}^{(1)} + D_{ijk}^{(2)}}\right)\Bigg]$$

$$= \frac{1}{\log 2} \sum_{i<j<k} \left[D_{ijk}^{(1)} \log\left(\frac{D_{ijk}^{(1)}}{C_{ijk}}\right) + D_{ijk}^{(2)} \log\left(\frac{D_{ijk}^{(2)}}{C_{ijk}}\right)\right].$$

This is equal to Φ_{QS}. The proof is completed. □

From Theorem 2, we obtain $\Phi_{\text{EQS}} = \Phi_{\text{QS}} - \Phi_{\text{GQS}}$. Therefore, the measure Φ_{EQS} also would indicate the degree of departure from the QS model excluding the influence of degree of departure from the GQS model.

Since the value of Φ_{EQS} is greater than or equal to zero, we also obtain the following theorem.

Theorem 3. *The value of Φ_{QS} is greater than or equal to the value of Φ_{GQS}. The equality holds if and only if there is a structure of EQS in the $R \times R$ table.*

From $0 \le \Phi_{\text{QS}} \le 1$ and $0 \le \Phi_{\text{GQS}} < 1$ (note that $\Phi_{\text{GQS}} \ne 1$ because $D^{(1)} > 0$ and $D^{(2)} > 0$ being the assumption), we see that $0 \le \Phi_{\text{EQS}} \le 1$. Moreover, from Theorems 2 and 3, (i) $\Phi_{\text{EQS}} = 0$ if and only if $\Phi_{\text{QS}} = \Phi_{\text{GQS}}$; namely, this indicates that the degree of departure from QS is equal to the degree of departure from GQS, (ii) $\Phi_{\text{EQS}} = 1$ if and only if $\Phi_{\text{QS}} = 1$ and $\Phi_{\text{GQS}} = 0$; namely, the degree of departure from QS is the largest and there is a structure of GQS. Note that $\Phi_{\text{EQS}} = 1$ indicates that $D_{ijk}/D_{kji} = \infty$ for some $i < j < k$ and $D_{ijk}/D_{kji} = 0$ for the other $i < j < k$, and $\sum_{i<j<k} D_{ijk} = \sum_{i<j<k} D_{kji}$.

5 Approximate confidence interval and model fitting

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \ldots, R$; $j = 1, \ldots, R$) with $n = \sum\sum n_{ij}$. Assume that $\{n_{ij}\}$ have a multinomial distribution. We shall consider the approximate confidence intervals for the measures Φ_{QS}, Φ_{GQS} and Φ_{EQS} (say, Φ) using the delta method as described by, for example, Bishop et al. (1975, Section 14.6). The sample version of Φ, that is, $\hat{\Phi}$, is given by Φ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$. Using the delta method, $\sqrt{n}(\hat{\Phi} - \Phi)$ has asymptotically (as $n \to \infty$) a normal distribution $N(0, \sigma^2)$. The variances are given in Appendix B. Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}/\sqrt{n}$ is an estimated approximate standard error for $\hat{\Phi}$. Thus, we obtain the approximate confidence intervals for the measures. Note that the approximate confidence intervals for the measures should be referred when the sample size is large.

The maximum likelihood estimates of expected frequencies under the models described in this paper could be obtained using the iterative procedures, for example, using the Newton–Raphson method to the log-likelihood equations. Each model can be tested for goodness of fit by, for example, the likelihood ratio chi-squared statistic with the corresponding degrees of freedom. Also, Lawal (2003, Chap. 11) discussed on model fitting with SAS and SPSS.

6 Example

Consider the data in Table 1 again. We are interested in seeing how strongly there is the stochastic circular structure in Tables 1(a) and 1(b) by using the proposed measures. From Table 2, we can infer that the stochastic circular structure arises because the confidence interval do not contain zero for Φ_{QS} applied to the data in Table 1(a). On the other hand, since the confidence interval for Φ_{QS} applied to the data in Table 1(b) contains zero, these may indicate that the stochastic circular structure does not arise, or if it is not so, that the stochastic circular structure arises very little. Also, the degrees of departure from QS for Tables 1(a) and 1(b) are estimated to be 34.1 percent and 6.4 percent of the maximum degree of departure from QS, respectively. When the degrees of departure from QS in Tables 1(a) and 1(b) are compared using the estimated measure, it is greater for Table 1(a) than for Table 1(b). Since the data in Table 1(a) rather than in Table 1(b) is estimated to be close to the strongest stochastic circular structure, we can see that the educational mobility for any three father–son pairs in 1955 is greater than that in 1975.

Moreover, we consider the data in Table 1 for more detail. We can decompose the value of Φ_{QS} into the values of Φ_{GQS} and Φ_{EQS} from Theorem 2. From Table 2, we see that $\hat{\Phi}_{\text{GQS}} = 0.311$ and $\hat{\Phi}_{\text{EQS}} = 0.030$ for Table 1(a). Thus, we would say that the degree of departure from QS for Table 1(a) strongly depends on the degree of departure from GQS. Namely, we can conjecture that the data in Table 1

Table 2 *Estimates of* Φ_{QS}, Φ_{GQS} *and* Φ_{EQS}, *their approximate standard errors and* 95% *confidence intervals, applied to Tables* 1(a) *and* 1(b)

Measure	Estimated measure	Standard error	Confidence interval
	(a) For Table 1(a)		
Φ_{QS}	0.341	0.149	(0.049, 0.632)
Φ_{GQS}	0.311	0.146	(0.024, 0.597)
Φ_{EQS}	0.030	0.031	(−0.030, 0.089)
	(b) For Table 1(b)		
Φ_{QS}	0.064	0.092	(−0.117, 0.245)
Φ_{GQS}	0.034	0.066	(−0.096, 0.164)
Φ_{EQS}	0.030	0.033	(−0.035, 0.094)

is close to the structure that the probability for right circulation is zero for all three father–son pairs or the probability for left circulation is zero for all three father–son pairs. Consequently, it may be inferred that for any three father–son pairs the educational mobility that for each of two father–son pairs the son's level is higher than his father's level and for one pair the son's level is lower than his father's level rather than the reverse educational mobility arises.

We see that $\hat{\Phi}_{\mathrm{GQS}} = 0.034$ and $\hat{\Phi}_{\mathrm{EQS}} = 0.030$ for Table 1(b). Thus, we would say that the degree of departure from GQS is nearly equal to that from EQS.

7 Concluding remarks

We note that Φ_{QS} is invariant under arbitrary permutations of row and column categories, but Φ_{GQS} and Φ_{EQS} are not. Thus, it is suitable to use Φ_{GQS} and Φ_{EQS} for analyzing the data on an ordinal scale. Also, it may be possible to use Φ_{QS} for analyzing the data on an ordinal scale when we may not use the information about category ordering. Similarly, it is suitable to use the GQS and EQS models for analyzing the data on an ordinal scale, and it may be possible to use the QS model for analyzing the data on an ordinal scale by ignoring the ordering of categories.

Consider the artificial data in Tables 3(a) and 3(b). From Table 4(a), the value of $\hat{\Phi}_{\mathrm{QS}}$ for Table 3(a) is close to that for Table 3(b). We see that (i) for Table 3(a), the value of $\hat{\Phi}_{\mathrm{GQS}}$ is large but the value of $\hat{\Phi}_{\mathrm{EQS}}$ is small; on the other hand, (ii) for Table 3(b), the value of $\hat{\Phi}_{\mathrm{GQS}}$ is small but the value of $\hat{\Phi}_{\mathrm{EQS}}$ is large. Let $\hat{D}_{ijk}$ ($\hat{D}_{kji}$) be given by D_{ijk} (D_{kji}) with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Since for Table 3(a) the values of $\hat{D}_{ijk}/\hat{D}_{kji}$ for all $i < j < k$ are greater than 1 (Table 4(b)), the probability for right circulation is estimated to be greater than the probability for left circulation for all $i < j < k$. On the other hand, for Table 3(b), $\hat{D}_{123}/\hat{D}_{321} = 15.10$ and $\hat{D}_{234}/\hat{D}_{432} = 16.13$ are greater than 1 but $\hat{D}_{124}/\hat{D}_{421} = 0.13$ (thus $(\hat{D}_{124}/\hat{D}_{421})^{-1} = 7.69$) and $\hat{D}_{134}/\hat{D}_{431} = 0.14$ (thus $(\hat{D}_{134}/\hat{D}_{431})^{-1} = 7.14$) are less than 1. These (Table 4(b)) indicate that (1) for Ta-

Table 3 *Artificial data* (*n is sample size*)

	(a) $n = 883$		
10	211	64	32
43	20	106	18
12	8	30	186
23	5	75	40
	(b) $n = 1022$		
10	63	54	72
93	20	86	28
112	8	30	6
120	245	35	40

Table 4 *Values of* $\hat{\Phi}_{\mathrm{QS}}$, $\hat{\Phi}_{\mathrm{GQS}}$ *and* $\hat{\Phi}_{\mathrm{EQS}}$, *and values of* $\hat{D}_{ijk}/\hat{D}_{kji}$ *applied to Tables* 3(a) *and* 3(b)

(a) Estimated measures

Table	$\hat{\Phi}_{\mathrm{QS}}$	$\hat{\Phi}_{\mathrm{GQS}}$	$\hat{\Phi}_{\mathrm{EQS}}$
3(a)	0.578	0.577	0.001
3(b)	0.533	0.082	0.451

(b) $\{\hat{D}_{ijk}/\hat{D}_{kji}\}$

	(i, j, k)			
Table	(1, 2, 3)	(1, 2, 4)	(1, 3, 4)	(2, 3, 4)
3(a)	12.19	12.70	9.51	9.13
3(b)	15.10	0.13	0.14	16.13

ble 3(a), it is likely that there is the structure of EQS but it is unlikely that there is the structure of GQS, and (2) for Table 3(b), it is unlikely that there is the structure of EQS but it is somewhat likely that there is the structure of GQS.

Theorem 2 is useful for seeing, e.g., that (1) for Table 3(a) the large departure from QS would be the influence of large departure from GQS (rather than EQS) and (2) for Table 3(b) it would be the influence of large departure from EQS (rather than GQS).

We see from Theorems 2 and 3 that the value of Φ_{EQS} increases as the difference between the degree of departure from QS and that from GQS increases (especially as the degree of departure from QS increases and that from GQS decreases). Also, the value of Φ_{EQS} attains the maximum value (equals 1) when the degree of departure from QS is maximum and that from GQS is minimum. Therefore, Φ_{EQS} would be useful when we want to see what degree the departure from EQS is toward the complete asymmetry with a structure of GQS when there is not a structure of EQS in the square table.

Appendix A

Let $\delta = \sum_{i<j<k}(Q_{ijk} + Q_{kji})$, and for $i < j < k$,

$$Q^*_{ijk} = \frac{Q_{ijk}}{\delta}, \qquad Q^*_{kji} = \frac{Q_{kji}}{\delta}, \qquad C^*_{ijk} = C^*_{kji} = \frac{1}{2}(Q^*_{ijk} + Q^*_{kji}).$$

Tahata et al. (2004) proposed the measure as follows: Assuming that $Q_{ijk} + Q_{kji} \neq 0$ for $i < j < k$, the measure is defined by

$$\phi^{(\lambda)}_{\mathrm{QS}} = \frac{\lambda(\lambda+1)}{2^{\lambda}-1} I^{(\lambda)} \qquad \text{for } \lambda > -1,$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{i<j<k} \left[Q^*_{ijk} \left\{ \left(\frac{Q^*_{ijk}}{C^*_{ijk}} \right)^{\lambda} - 1 \right\} + Q^*_{kji} \left\{ \left(\frac{Q^*_{kji}}{C^*_{kji}} \right)^{\lambda} - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$.

Appendix B

Denote the asymptotic variances for $\hat{\Phi}_{\mathrm{QS}}$, $\hat{\Phi}_{\mathrm{GQS}}$ and $\hat{\Phi}_{\mathrm{EQS}}$ by σ^2_{QS}, σ^2_{GQS} and σ^2_{EQS}, respectively, divided by n. These are given as follows:

$$\sigma^2_{\mathrm{QS}} = \sum_{a<b} \left(\frac{1}{p_{ab}} (A^{\mathrm{QS}}_{ab})^2 + \frac{1}{p_{ba}} (B^{\mathrm{QS}}_{ab})^2 \right), \tag{B.1}$$

where

$$A^{\mathrm{QS}}_{ab} = \frac{1}{\log 2} \Bigg[\sum_{i<j<k} \Bigg\{ D^{(1)}_{ijk} \log\left(\frac{D^{(1)}_{ijk}}{C_{ijk}} \right) (I_{(i=a,\, j=b)} + I_{(j=a,k=b)}) + D^{(2)}_{ijk} \log\left(\frac{D^{(2)}_{ijk}}{C_{ijk}} \right) I_{(i=a,k=b)} \Bigg\} - \Phi_{\mathrm{QS}} \Delta^{(1)}_{ab} (\log 2) \Bigg],$$

with

$$\Delta^{(1)}_{ab} = \sum_{s<t<u} \left(D^{(1)}_{stu} I_{(s=a,t=b)} + D^{(1)}_{stu} I_{(t=a,u=b)} + D^{(2)}_{stu} I_{(s=a,u=b)} \right),$$

and where B^{QS}_{ab} is defined as A^{QS}_{ab} obtained by interchanging $D^{(1)}_{ijk}$ and $D^{(2)}_{ijk}$, and $I_{(\cdot)} = 1$ if true, 0 if not;

$$\sigma^2_{\mathrm{GQS}} = \sum_{a<b} \left(\frac{1}{p_{ab}} (A^{\mathrm{GQS}}_{ab})^2 + \frac{1}{p_{ba}} (B^{\mathrm{GQS}}_{ab})^2 \right), \tag{B.2}$$

where

$$A^{\mathrm{GQS}}_{ab} = \frac{1}{\log 2} \Bigg[\sum_{i<j<k} \Bigg\{ D^{(1)}_{ijk} \log\left(\frac{D^{(1)}}{1/2} \right) (I_{(i=a,\, j=b)} + I_{(j=a,k=b)}) + D^{(2)}_{ijk} \log\left(\frac{D^{(2)}}{1/2} \right) I_{(i=a,k=b)} \Bigg\} - \Phi_{\mathrm{GQS}} \Delta^{(1)}_{ab} (\log 2) \Bigg],$$

and where B^{GQS}_{ab} is defined as A^{GQS}_{ab} obtained by interchanging $D^{(1)}_{ijk}$ and $D^{(2)}_{ijk}$ and interchanging $D^{(1)}$ and $D^{(2)}$;

$$\sigma^2_{\mathrm{EQS}} = \sum_{a<b} \left(\frac{1}{p_{ab}} (A^{\mathrm{EQS}}_{ab})^2 + \frac{1}{p_{ba}} (B^{\mathrm{EQS}}_{ab})^2 \right), \tag{B.3}$$

where

$$A_{ab}^{\mathrm{EQS}} = \frac{1}{\log 2}\Bigg[\sum_{i<j<k}\Bigg\{D_{ijk}^{(1)}\log\Bigg(\frac{D_{ijk}^{(1)}}{G_{ijk}^{(1)}}\Bigg)(I_{(i=a,j=b)} + I_{(j=a,k=b)})$$

$$+ D_{ijk}^{(2)}\log\Bigg(\frac{D_{ijk}^{(2)}}{G_{ijk}^{(2)}}\Bigg)I_{(i=a,k=b)}\Bigg\} - \Phi_{\mathrm{EQS}}\Delta_{ab}^{(1)}(\log 2)\Bigg],$$

and where B_{ab}^{EQS} is defined as A_{ab}^{EQS} obtained by interchanging $D_{ijk}^{(1)}$ and $D_{ijk}^{(2)}$ and interchanging $G_{ijk}^{(1)}$ and $G_{ijk}^{(2)}$.

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