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BSDE with jumps and non-Lipschitz coefficients: Application to large deviations

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Abstract. In this work, we deal with a backward stochastic differential equation with respect to a Brownian motion and a Poisson random measure. We first establish existence and uniqueness of solution in the case of non-Lipschitz drift. In the second part, we prove a Large Deviation Principle for a family of solutions.

1 Introduction

It is well known that backward stochastic differential equation (BSDE in short) provided stochastic representation of solution some classes of partial differential equations (PDEs in short) of second order. With the help of BSDEs with respect to a Brownian motion and a Poisson random measure (BSDEP in short), some authors generalized this result to integro-partial differential equations. The pioneer result on BSDEs, established by Pardoux and Peng (1990) require Lipschitz condition on the drift of the equation. Several authors interested in weakening this assumption. See among others Kobylanski (2000), Mao (1995) and Wang and Wang (2003). Recently Wang and Huang (2009), improve Wang and Wang's result by weakening the continuity required. Our aim in the first part of the present work is to extend Wang and Huang's result to BSDEs with jumps with a generator satisfying weaker conditions.

Large deviation Principle (LDP) for stochastic processes was first discuss by Freidlin and Wentzell (1984). Later some authors investigate successfully a family of stochastic differential equation (see among others Baldi (1991), Baxendale and Stroock (1988), Makhno (1995)). Baldi (1991) considered a family of periodic diffusion processes with homogenization and a small parameter multiplying the diffusion coefficient. He established a large deviations principle and as an application, he derived an iterated logarithm law for periodic diffusions. Freidlin and Sowers (1999) studied the combined effects of homogenization and large deviations in a stochastic differential equation. The authors show some large deviations type estimates, and then apply these results to study wavefronts in both a single reaction–diffusion equation and in a system of reaction–diffusion equations.

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Many researchers interested in extending these results to solution weakly coupled forward and backward stochastic differential equations. Essaky (2008) proved that the solution of a BSDE, which involves a subdifferential operator and associated to a family of reflecting diffusion processes, converges to the solution of a deterministic backward equation and satisfies a large deviation principle. Rainero (2006) prove a LDP for a family of solutions of BSDEs with Lipschitzian drift. The key point is a suitable estimate on the solution of BSDE.

Inspired by the works of Barles et al. (1996), and applying our results in the first part, we interested in the second part in this paper in extending the result established in Rainero (2006) to BSDEs with jumps. The paper is organized as follows. We introduce some preliminaries and prove existence and uniqueness of solutions of BSDEP in Section 2. In Section 3, we prove a large deviations principle for a family of solutions of BSDEP under weaker conditions than those in Essaky (2008) and Rainero (2006).

2 BSDE with jumps

2.1 Definitions and notations

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{0\le t\le T}, \mathbf{P})$, $T > 0$, be a stochastic basis such that $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$ and $\mathcal{F}_{t+} = \bigcap_{\delta>0} \mathcal{F}_{t+\delta} = \mathcal{F}_t$ and suppose that the filtration is generated by two mutually independent processes: a d-dimensional Brownian motion $(B_t)_{0\le t\le T}$ and a Poisson random measure μ on $E \times \mathbf{R}_+$. The space $E = \mathbf{R}^\ell - \{0\}$ is equipped with its Borel field $\mathcal{E}$ with compensator $\nu(dt, de) = dt\lambda(de)$ such that $\{\widetilde{\mu}([0, t] \times A) = (\mu - \nu)[0, t] \times A\}$ is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a σ-finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1 \wedge |e|^2)\lambda(de) < \infty.$$

Given ξ a $\mathcal{F}_T$-measurable $\mathbf{R}^k$ valued random variable and $f : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k \to \mathbf{R}^k$, we are interested in the BSDEP

$$\begin{aligned} Y_t = \xi + \int_t^T f(r, Y_r, Z_r, U_r)\, dr - \int_t^T Z_r\, dB_r \\ - \int_t^T \int_E U_r(e)\widetilde{\mu}(dr, de), \qquad 0 \le t \le T. \end{aligned} \tag{2.1}$$

For $Q \in \mathbf{N}^*$, $|\cdot|$ and $\langle\cdot\rangle$ stand for the euclidian norm and the inner product in $\mathbf{R}^Q$.

We consider the following sets:

- $S^2_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ adapted càdlàg processes

$$\Psi : [0, T] \times \Omega \longrightarrow \mathbf{R}^Q, \qquad \|\Psi\|_2^2 = \mathbf{E}\Big(\sup_{0\le t\le T} |\Psi_t|^2\Big) < \infty.$$

- $H^2_{[0,T]}(\mathbf{R}^Q)$ the space of $\mathcal{F}_t$ progressively measurable processes

$$\Psi : [0,T] \times \Omega \longrightarrow \mathbf{R}^Q, \qquad \|\Psi\|^2 = \mathbf{E}\int_0^T |\Psi_t|^2\,dt < \infty.$$

- $L^2_{[0,T]}(\widetilde{\mu}, \mathbf{R}^Q)$ the space of mappings $U : \Omega \times [0,T] \times E \longrightarrow \mathbf{R}^Q$ which are $\mathcal{P} \otimes \mathcal{E}$-measurable s.t.

$$\|U\|^2 = \mathbf{E}\int_0^T \int_E U_t(e)^2 \lambda(de)\,dt < \infty,$$

  where $\mathcal{P} \otimes \mathcal{E}$ denotes the σ-algebra of predictable sets of $\Omega \times [0,T]$.

Notice that the space $S^2_{[0,T]}(\mathbf{R}^Q) \times H^2_{[0,T]}(\mathbf{R}^Q) \times L^2_{[0,T]}(\widetilde{\mu}, \mathbf{R}^Q)$ is a Banach space.

Definition 2.1. A triplet of processes $(Y_t, Z_t, U_t)_{0\le t\le T}$ is called a solution to (2.1), if $(Y_t, Z_t, U_t) \in S^2_{[0,T]}(\mathbf{R}^k) \times H^2_{[0,T]}(\mathbf{R}^{k\times d}) \times L^2_{[0,T]}(\widetilde{\mu}, \mathbf{R}^k)$ and satisfies (2.1).

We say that the coefficient $f : \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k \to \mathbf{R}^k$ of the BSDEP satisfies assumption (H1) if the following hold:

(H1.1) For all $(y,z,u) \in \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k$, $f(\cdot, y, z, u) \in H^2_{[0,T]}(\mathbf{R}^k)$.

(H1.2) There exists $K > 1$ s.t. for $0 \le t \le T$, $(y, y') \in (\mathbf{R}^k)^2$, $(z, z') \in (\mathbf{R}^{k\times d})^2$, $(u, u') \in (\mathbf{R}^k)^2$,

$$\left|f(t,y,z,u) - f(t,y',z',u')\right|^2 \le \rho\left(t, |y-y'|^2\right) + K\left(|z-z'|^2 + |u-u'|^2\right),$$

where $\rho(t,v) : [0,T] \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies

- For fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing s.t. $\rho(t,0) = 0$.
- The ordinary differential equation

$$v' = -\rho(t,v), \qquad v(T) = 0, \tag{2.2}$$

  has a unique solution $v(t) = 0, 0 \le t \le T$.
- There exists $a(t) \ge 0$, $b(t) \ge 0$ s.t. $\rho(t,v) \le a(t) + b(t)v$ and $\int_0^T [a(t) + b(t)]\,dt < \infty$.

Let us mention that assumptions (H1) are weaker than Lipschitz conditions required on the coefficients in Barles et al. (1996). Some examples satisfying these conditions (H1) are given in Wang and Huang (2009).

2.2 Existence and uniqueness of a solution

Our strategy in the proof of existence is to use the Picard approximate sequence. To this end, we recall the following result given in Tang and Li (1994, Lemma 2.4).

Proposition 2.2. *Assume that* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ *and* f *is* Γ*-Lipschitz for some constant* $\Gamma > 0$ *and* $f(t, \cdot, \cdot, \cdot) \in H^2_{[0,T]}(\mathbf{R}^k)$. *Then* (2.1) *has a unique solution* $(Y_t, Z_t, U_t)_{0 \le t \le T}$.

We consider now the sequence $(Y_t^n, Z_t^n, U_t^n)_{n\ge 0}$ given by

$$\begin{cases} Y_t^0 = 0; \\ Y_t^n = \xi + \int_t^T f(s, Y_s^{n-1}, Z_s^n, U_s^n)\, ds \\ \qquad - \int_t^T Z_s^n \, dB_s - \int_t^T \int_E U_s^n(e) \widetilde{\mu}(ds, de), \qquad n \ge 1. \end{cases} \tag{2.3}$$

Thanks to Proposition 2.2, this sequence is well defined since the function $(z, u) \mapsto f(s, Y_s^{n-1}, z, u)$ is K-Lipschitz. In what follows, we establish two results which will be useful in the sequel.

Lemma 2.3. *Assume that* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ *and* (H1) *holds. Then for* $n, m \ge 1$ *we have*

$$\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 \le \frac{1}{K} e^{K(T-t)} \int_t^T \rho\big(s, \mathbf{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2\big)\, ds, \qquad 0 \le t \le T.$$

Proof. Using Itô's formula for discontinuous processes, we have for $n, m \ge 1$ and $0 \le t \le T$,

$$\begin{aligned} &\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 + \mathbf{E}\int_t^T |Z_s^{n+m} - Z_s^n|^2\, ds \\ &\quad + \mathbf{E}\int_t^T \int_E \big|U_s^{n+m}(e) - U_s^n(e)\big|^2 \lambda(de)\, ds \\ &\quad = 2\mathbf{E}\int_t^T \langle Y_s^{n+m} - Y_s^n, \Delta f^{(n,m)}(s)\rangle\, ds, \end{aligned} \tag{2.4}$$

where $\Delta f^{(n,m)}(s) = f(s, Y_s^{n+m-1}, Z_s^{n+m}, U_s^{n+m}) - f(s, Y_s^{n-1}, Z_s^n, U_s^n)$. Using standard estimates and assumption (H1.2), we have

$$\begin{aligned} 2\langle Y_s^{n+m} - Y_s^n, \Delta f^{(n,m)}(s)\rangle &\le 2K|Y_s^{n+m} - Y_s^n|^2 + \frac{1}{2K}|\Delta f^{(n,m)}(s)|^2 \\ &\le 2K|Y_s^{n+m} - Y_s^n|^2 + \frac{1}{2K}\rho\big(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\big)\, ds \\ &\quad + \frac{1}{2}|Z_s^{n+m} - Z_s^n|^2 + \frac{1}{2}|U_s^{n+m} - U_s^n|^2. \end{aligned}$$

Plugging this last inequality in (2.4), we deduce from Gronwall's lemma

$$\begin{aligned} &\mathbf{E}|Y_t^{n+m} - Y_t^n|^2 \le \frac{1}{2K} e^{2K(T-t)} \int_t^T \rho\big(s, \mathbf{E}|Y_s^{n+m-1} - Y_s^{n-1}|^2\big)\, ds, \\ &0 \le t \le T. \end{aligned} \tag{2.5}$$

□

Lemma 2.4. *Assume that* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ *and* (H1) *holds. Then there exists* $0 \le T_1 < T$ *not depending on* ξ *and a constant* $M \ge 0$ *such that*

$$\mathbf{E}|Y_t^n|^2 \le M, \qquad T_1 \le t \le T.$$

Proof. Using again Itô's formula, we deduce that for $n \ge 1$,

$$\begin{aligned}
\mathbf{E}|Y_t^n|^2 + \mathbf{E}\int_t^T |Z_s^n|^2\,ds + \mathbf{E}\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)\,ds \\
\le \mathbf{E}|\xi|^2 + 2K\mathbf{E}\int_t^T |Y_s^n|^2\,ds \\
+ 2K\mathbf{E}\int_t^T |f(s, Y_s^{n-1}, Z_s^n, U_s^n)|^2\,ds, \qquad 0 \le t \le T.
\end{aligned}$$

Assumption (H1.2) and standard estimates imply

$$\begin{aligned}
\mathbf{E}|Y_t^n|^2 \le \mathbf{E}|\xi|^2 + 2K\mathbf{E}\int_t^T |Y_s^n|^2\,ds + \frac{1}{K}\mathbf{E}\int_t^T |f(s,0,0,0)|^2\,ds \\
+ \frac{1}{K}\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)\,ds.
\end{aligned}$$

Applying Gronwall's lemma, we deduce that

$$\begin{aligned}
\mathbf{E}|Y_t^n|^2 \le e^{2K(T-t)}\left[\mathbf{E}|\xi|^2 + \frac{1}{K}\mathbf{E}\int_t^T |f(s,0,0,0)|^2\,ds\right] \\
+ \frac{1}{K}e^{2K(T-t)}\int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)\,ds.
\end{aligned} \tag{2.6}$$

Putting $\overline{T}_1 = (T - \frac{\ln K}{2K}) \vee 0$, we deduce from (2.6)

$$\begin{aligned}
\mathbf{E}|Y_t^n|^2 \le K\mathbf{E}|\xi|^2 + \mathbf{E}\int_t^T |f(s,0,0,0)|^2\,ds \\
+ \int_t^T \rho(s, \mathbf{E}|Y_s^{n-1}|^2)\,ds, \qquad \overline{T}_1 \le t \le T.
\end{aligned}$$

Let $M = 2K\mathbf{E}|\xi|^2 + 2K\mathbf{E}\int_0^T |f(s,0,0,0)|^2\,ds + \int_0^T a(s)\,ds$, it remains to use the argument developed in Wang and Huang (2009), to complete the proof. □

Thanks to these two previous results, we have the following theorem.

Theorem 2.5. *Let* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$. *Under* (H1), (2.1) *has a unique solution* $(Y_t, Z_t, U_t)_{0 \le t \le T}$.

Proof. Let us consider the sequence $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_0(t) = \int_t^T \rho(s, M)\, ds,$$

$$\varphi_{n+1}(t) = \int_t^T \rho\big(s, \varphi_n(s)\big)\, ds, \qquad n \geq 0, T_1 \leq t \leq T.$$

Using the argument developed in Wang and Huang (2009, Theorem 1), we shall prove that the sequence $(Y_t^n)_{n\geq 1}$ is a Cauchy sequence in $S^2_{[T_1,T]}(\mathbf{R}^k)$, $(Z_t^n)_{n\geq 1}$ is a Cauchy sequence in $H^2_{[T_1,T]}(\mathbf{R}^k)$ and $(U_t^n)_{n\geq 1}$ is a Cauchy sequence in $L^2_{[T_1,T]}(\widetilde{\mu}, \mathbf{R}^k)$, where T_1 is given by Lemma 2.4. Letting $n \to \infty$ in (2.3), we obtain

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, Y_s, Z_s, U_s)\, ds - \int_t^T Z_s\, dB_s \\ &- \int_t^T \int_E U_s(e) \widetilde{\mu}(ds, de), \qquad T_1 \leq t \leq T. \end{aligned} \tag{2.7}$$

Hence, the limiting process (Y_t, Z_t, U_t) satisfies (2.1) on $[T_1, T]$. Moreover by virtue of (H1), we have $(Z_t, U_t) \in H^2_{[T_1,T]}(\mathbf{R}^{k\times d}) \times L^2_{[T_1,T]}(\widetilde{\mu}, \mathbf{R}^k)$. As a consequence, by assumption (H1) and Doob's inequality, we deduce that

$$\begin{aligned} \mathbf{E}\int_t^T \big|f(s, Y_s, Z_s, U_s)\big|^2\, ds &+ \mathbf{E}\bigg(\sup_{T_1\leq t\leq T}\bigg|\int_{T_1}^t Z_s\, dB_s\bigg|^2\bigg) \\ &+ \mathbf{E}\bigg(\sup_{T_1\leq t\leq T}\bigg|\int_{T_1}^t \int_E U_s(e)\widetilde{\mu}(de, ds)\bigg|^2\bigg) < \infty. \end{aligned}$$

This implies essentially that $\mathbf{E}(\sup_{T_1\leq t\leq T} |Y_t|^2) < \infty$. Thus, the triplet (Y_t, Z_t, U_t) solves (2.1) on $[T_1, T]$. Using once again Lemma 2.4, one can deduce existence of solution on $[T_2, T_1]$, with $0 \leq T_2 < T_1$. Hence by iteration, we prove existence of solution on $[0, T]$.

Let us prove uniqueness. Let (Y_t, Z_t, U_t) and $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{U}_t)$ two solutions of (2.1) and define for $\mathcal{W} \in \{Y, Z, U\}$, $\overline{\mathcal{W}} = \mathcal{W}_t - \widetilde{\mathcal{W}}_t$ and $\Delta f(s) = f(s, Y_s, Z_s, U_s) - f(s, \widetilde{Y}_s, \widetilde{Z}_s, \widetilde{U}_s)$.

Itô's formula yields

$$\mathbf{E}|\overline{Y}_t|^2 + \mathbf{E}\int_t^T |\overline{Z}_s|^2\, ds + \mathbf{E}\int_t^T \int_E \big|\overline{U}_s(e)\big|^2 \lambda(de)\, ds = 2\mathbf{E}\int_t^T \langle \overline{Y}_s, \Delta f(s)\rangle\, ds.$$

Using assumption (H1.2) and standard estimates, the right-hand side is less than

$$\begin{aligned} 2K\mathbf{E}\int_t^T |\overline{Y}_s|^2\, ds &+ \frac{1}{2K}\int_t^T \rho\big(s, \mathbf{E}|\overline{Y}_s|^2\big)\, ds \\ &+ \frac{1}{2}\mathbf{E}\int_t^T |\overline{Z}_s|^2\, ds + \frac{1}{2}\mathbf{E}\int_t^T \int_E \big|\overline{U}_s(e)\big|^2 \lambda(de)\, ds. \end{aligned}$$

Putting pieces together, we deduce that

$$\begin{aligned}\mathbf{E}|\overline{Y}_t|^2 + \frac{1}{2}\mathbf{E}\int_t^T |\overline{Z}_s|^2\,ds + \frac{1}{2}\mathbf{E}\int_t^T \int_E |\overline{U}_s(e)|^2\lambda(de)\,ds \\ \le 2K\mathbf{E}\int_t^T |\overline{Y}_s|^2\,ds + \frac{1}{2K}\int_t^T \rho\big(s, \mathbf{E}|\overline{Y}_s|^2\big)\,ds.\end{aligned} \tag{2.8}$$

Applying Gronwall's lemma, we obtain

$$\mathbf{E}|\overline{Y}_t|^2 \le \frac{1}{2K}e^{2K(T-t)}\int_t^T \rho\big(s, \mathbf{E}|\overline{Y}_s|^2\big)\,ds, \qquad 0 \le t \le T. \tag{2.9}$$

Let $\delta = (2K)^{-1}\ln 2K > 0$ and $m = [T/\delta] + 1$. If $(t_j)_{0\le j\le m}$ denotes the uniform subdivision of $[0, T]$ given by $T_0 = 0$, $T_j = T - (m - j)\delta$, $j \ge 1$, we have

$$\mathbf{E}|\overline{Y}_t|^2 \le \int_t^T \rho\big(s, \mathbf{E}|\overline{Y}_s|^2\big)\,ds, \qquad T_{m-1} \le t \le T.$$

This implies from the comparison theorem of ordinary differential equation, $\mathbf{E}|\overline{Y}_t|^2 \le r(t)$ where $r(t)$ is the maximum of solution of (2.2). As a consequence, we have $Y_t = \widetilde{Y}_t, T_{m-1} \le t \le T$. From (2.8), we deduce $Z_t = \widetilde{Z}_t$ and $U_t = \widetilde{U}_t, T_{m-1} \le t \le T$. Using the same scheme, we prove uniqueness on $[T_j, T_{j+1}]$, $j = 0, \dots, m-2$. This completes the proof. □

3 BSDE with jumps and large deviations

Assume given $\beta : \mathbf{R}^d \to \mathbf{R}^d$, $\sigma : \mathbf{R}^d \to \mathbf{R}^d \times \mathbf{R}^d$ and $f : \Omega \times [0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k \to \mathbf{R}^k$. For $x \in \mathbf{R}^d$ and $0 \le s \le T$, we are interested in the equation

$$\begin{aligned}Y_t^{\varepsilon,s,x} &= g\big(X_T^{\varepsilon,s,x}\big) + \int_t^T f\big(r, \theta_r^{\varepsilon,s,x}\big)\,dr \\ &\quad - \int_t^T Z_r^{\varepsilon,s,x}\,dB_r - \int_t^T \int_E U_r^{\varepsilon,s,x}(e)\widetilde{\mu}(dr, de),\end{aligned} \tag{3.1}$$

where $(X_t^{\varepsilon,s,x})_{t\le s\le T}$, is the solution of the stochastic differential equation

$$\begin{aligned}X_t^{\varepsilon,s,x} &= x + \int_s^t \beta\big(X_r^{\varepsilon,s,x}\big)\,dr + \varepsilon\int_s^t \sigma\big(X_r^{\varepsilon,s,x}\big)\,dB_r, \\ &0 \le s \le t \le T, \varepsilon \ge 0\end{aligned} \tag{3.2}$$

and $\theta_r^{\varepsilon,s,x} = (X_r^{\varepsilon,s,x}, Y_r^{\varepsilon,s,x}, Z_r^{\varepsilon,s,x}, U_r^{\varepsilon,s,x})$, $t \le r \le T$.

We assume that β, σ, g and f satisfy assumption (H2) below

(H2.1) There exists $K > 1$ s.t. for $0 \le t \le T$, $(x, x') \in (\mathbf{R}^d)^2$, $(y, y') \in (\mathbf{R}^k)^2$, $(z, z') \in (\mathbf{R}^{k\times d})^2$ and $(u, u') \in (\mathbf{R}^k)^2$,

$$\begin{aligned}&\left|\beta(x) - \beta(x')\right| + \left|\sigma(x) - \sigma(x')\right| + \left|g(x) - g(x')\right| \\ &\quad \le K|x - x'| \\ &\left|f(t, x, y, z, u) - f(t, x', y', z', u')\right|^2 \\ &\quad \le \rho(t, |y - y'|^2) + K(|x - x'|^2 + |z - z'|^2 + |u - u'|^2)\end{aligned}$$

(H2.2) $a \equiv 0$ and $b : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous.

Let us recall the following result due to Freidlin and Wentzell (1984).

Theorem 3.1. *The solution* $(X_t^{\varepsilon,x})_{0\le t\le T}$ *of* (3.2) *satisfies whenever* $\varepsilon \to 0$ *a Large Deviation Principle in* $\mathcal{C}([0, T], \mathbf{R}^k)$ *associated to rate function* I_x *defined by* $\varphi \in \mathcal{C}([0, T], \mathbf{R}^k)$:

$$\begin{aligned}I_x(\varphi) := \inf\bigg\{&\frac{1}{2}\int_0^T |\dot{v}_t|^2\, dt, v \in H^1([0, T], \mathbf{R}^d) : \\ &\varphi_t = x + \int_0^t \beta(\varphi_s)\, ds + \int_0^t \sigma(\varphi_s)\dot{v}_s\, ds, 0 \le t \le T\bigg\}.\end{aligned} \tag{3.3}$$

In view of Theorem 3.1, $X_t^{\varepsilon,s,x} \overset{\varepsilon\to 0}{\longrightarrow} \varphi_t$ in probability where $\varphi^{s,x}$ is the solution of the deterministic equation

$$\varphi_t' = \beta(\varphi_t), \qquad 0 \le s \le t \le T, \varphi_s = x, \tag{3.4}$$

and satisfies a large deviation principle.

Our aim is to study the asymptotic behavior of $(Y_t^{\varepsilon,s,x}, Z_t^{\varepsilon,s,x}, U_t^{\varepsilon,s,x})_{\varepsilon\ge 0}$ solution of (3.1) whenever $\varepsilon \to 0$. To this end, we consider the triplet $(\psi^{s,x}, 0, 0)$ where $\psi^{s,x}$ solves the deterministic equation

$$\psi_t' = -f(t, \varphi_t^{s,x}, \psi_t, 0, 0), \qquad 0 \le s \le t \le T, \psi_T = g(\varphi_T^{s,x}). \tag{3.5}$$

By uniqueness, the triplet $(\psi^{s,x}, 0, 0)$ is the unique solution of the BSDEP

$$\begin{aligned}Y_t = g(\varphi_T^{s,x}) &+ \int_t^T f(r, \varphi_r^{s,x}, Y_r, Z_r)\, dr - \int_t^T Z_r\, dB_r \\ &- \int_t^T \int_E U_r(e)\widetilde{\mu}(dr, de), \qquad 0 \le s \le t \le T,\end{aligned} \tag{3.6}$$

where $\psi^{s,x}$ given by (3.5). We have the following proposition.

Proposition 3.2. *Let* $0 \le \varepsilon \le 1$, $(s,x) \in [0,T] \times \mathcal{K}$ *where* $\mathcal{K}$ *is a compact set of* $\mathbf{R}^k$. *There exists a constant* $C_{3.2} > 0$ *depending only on* T *and* K *such that*

$$\mathbf{E}\Big[\sup_{s\le t\le T}|Y_t^{\varepsilon,s,x} - \psi_t^{s,x}|^2 + \int_s^T |Z_r^{\varepsilon,s,x}|^2\,dr + \int_s^T\int_E |U_r^{\varepsilon,s,x}(e)|^2\lambda(de)\,dr\Big] \le C_{3.2}\varepsilon^2. \tag{3.7}$$

Proof. Applying Itô's formula, we have for $s \le t \le T$,

$$\begin{aligned} &|Y_t^{\varepsilon,s,x} - \psi_t^{s,x}|^2 + \int_t^T |Z_r^{\varepsilon,s,x}|^2\,dr + \int_t^T\int_E |U_r^{\varepsilon,s,x}(e)|^2\lambda(de)\,dr \\ &\quad\le |g(X_T^{\varepsilon,s,x}) - g(\varphi_T^{s,x})|^2 \\ &\qquad - 2\int_t^T (Y_r^{\varepsilon,s,x} - \psi_r^{s,x})\big[f(r,\theta_r^{\varepsilon,s,x}) - f(r,\varphi_r^{s,x},\psi_r^{s,x},0,0)\big]\,dr \\ &\qquad - 2\int_t^T \langle (Y_r^{\varepsilon,s,x} - \psi_r^{s,x}), Z_r^{\varepsilon,s,x}\,dB_r\rangle \\ &\qquad + 2\int_t^T\int_E \langle (Y_{r-}^{\varepsilon,s,x} - \psi_{r-}^{s,x}), U_r^{\varepsilon,s,x}(e)\widetilde{\mu}(de,dr)\rangle. \end{aligned} \tag{3.8}$$

Using assumption (H2.1) and standard estimates, there exist $\gamma(K) > 0$ and $\gamma'(K) > 0$ (which may change from line to line) s.t.

$$\begin{aligned} &\mathbf{E}|Y_t^{\varepsilon,s,x} - \psi_t^{s,x}|^2 + \mathbf{E}\int_t^T |Z_r^{\varepsilon,s,x}|^2\,dr + \mathbf{E}\int_t^T\int_E |U_r^{\varepsilon,s,x}(e)|^2\lambda(de)\,dr \\ &\quad\le \gamma'(K)\mathbf{E}\sup_{s\le t\le T}|X_t^{\varepsilon,s,x} - \varphi_t^{s,x}|^2 \\ &\qquad + \gamma(K)\mathbf{E}\int_t^T \big[|Y_r^{\varepsilon,s,x} - \psi_r^{s,x}|^2 + \rho(r,|Y_r^{\varepsilon,s,x} - \psi_r^{s,x}|^2)\big]\,dr \\ &\qquad + \frac{1}{2}\mathbf{E}\int_t^T \big(|Z_r^{\varepsilon,s,x}|^2 + |U_r^{\varepsilon,s,x}|^2\big)\,dr. \end{aligned}$$

Hence since b is bounded on $[t,T]$, by assumption (H2.2) we deduce that

$$\begin{aligned} &\mathbf{E}|Y_t^{\varepsilon,s,x} - \psi_t^{s,x}|^2 + \frac{1}{2}\mathbf{E}\int_t^T |Z_r^{\varepsilon,s,x}|^2\,dr + \frac{1}{2}\mathbf{E}\int_t^T\int_E |U_r^{\varepsilon,s,x}(e)|^2\lambda(de)\,dr \\ &\quad\le \gamma'(K)\mathbf{E}\sup_{s\le t\le T}|X_t^{\varepsilon,s,x} - \varphi_t^{s,x}|^2 + \gamma(K)\mathbf{E}\int_t^T |Y_r^{\varepsilon,s,x} - \psi_r^{s,x}|^2\,dr. \end{aligned} \tag{3.9}$$

Then by Gronwall's lemma, there exists $\gamma_{(K,T)} > 0$ which may change from line to line s.t.

$$\mathbf{E}|Y_t^{\varepsilon,s,x} - \psi_t^{s,x}|^2 \le \gamma_{(K,T)}\mathbf{E}\sup_{s\le t\le T}|X_t^{\varepsilon,s,x} - \varphi_t^{s,x}|^2.$$

Keeping in mind that $\mathbf{E}\sup_{s\le t\le T}|X_t^{\varepsilon,s,x}-\varphi_t^{s,x}|^2\le c\varepsilon^2$, $c>0$, we deduce from (3.9),

$$\mathbf{E}\big|Y_t^{\varepsilon,s,x}-\psi_t^{s,x}\big|^2+\mathbf{E}\int_t^T\big|Z_r^{\varepsilon,s,x}\big|^2\,dr+\mathbf{E}\int_t^T\int_E\big|U_r^{\varepsilon,s,x}(e)\big|^2\lambda(de)\,dr \\ \le\gamma_{(K,T)}\varepsilon^2. \tag{3.10}$$

Futhermore using (3.8), we obtain,

$$\begin{aligned}\mathbf{E}\Big(&\sup_{s\le t\le T}|Y_t^{\varepsilon,s,x}-\psi_t^{s,x}|^2\Big)\\ &\le\mathbf{E}\big|g\big(X_T^{\varepsilon,s,x}\big)-g\big(\varphi_T^{s,x}\big)\big|^2\\ &\quad+2\mathbf{E}\sup_{s\le t\le T}\bigg(\int_t^T\big(Y_r^{\varepsilon,s,x}-\psi_r^{s,x}\big)\\ &\qquad\qquad\times\big[f\big(r,\theta_r^{\varepsilon,s,x}\big)-f\big(r,\varphi_r^{s,x},\psi_r^{s,x},0,0\big)\big]\bigg)\\ &\quad+2\mathbf{E}\sup_{s\le t\le T}\bigg|\int_t^T\big\langle\big(Y_r^{\varepsilon,s,x}-\psi_r^{s,x}\big),Z_r^{\varepsilon,s,x}\,dB_r\big\rangle\bigg|\\ &\quad+2\mathbf{E}\sup_{s\le t\le T}\bigg|\int_t^T\int_E\big\langle\big(Y_{r^-}^{\varepsilon,s,x}-\psi_{r^-}^{s,x}\big),U_r^{\varepsilon,s,x}(e)\widetilde{\mu}(de,dr)\big\rangle\bigg|.\end{aligned} \tag{3.11}$$

Applying Burkhölder–Davis–Gundy inequality, there exists $\gamma>0$ such that

$$\begin{aligned}2\mathbf{E}&\sup_{s\le t\le T}\bigg|\int_t^T\big\langle\big(Y_r^{\varepsilon,s,x}-\psi_r^{s,x}\big),Z_r^{\varepsilon,s,x}\,dB_r\big\rangle\bigg|\\ &\le\frac14\mathbf{E}\sup_{s\le t\le T}|Y_t^{\varepsilon,s,x}-\psi_t^{s,x}|^2+\gamma\mathbf{E}\int_t^T|Z_r^{\varepsilon,s,x}|^2\,dr,\\ 2\mathbf{E}&\sup_{s\le t\le T}\bigg|\int_t^T\int_E\big\langle\big(Y_{r^-}^{\varepsilon,s,x}-\psi_{r^-}^{s,x}\big),U_r^{\varepsilon,s,x}(e)\widetilde{\mu}(de,dr)\big\rangle\bigg|\\ &\le\frac14\mathbf{E}\sup_{s\le t\le T}|Y_t^{\varepsilon,s,x}-\psi_t^{s,x}|^2+\gamma\mathbf{E}\int_t^T\int_E\big|U_r^{\varepsilon,s,x}(e)\big|^2\lambda(de)\,dr.\end{aligned}$$

Furthermore using (H2), the second term of the right-hand side of (3.11) is less than

$$c(K)\mathbf{E}\int_t^T|Y_r^{\varepsilon,s,x}-\psi_r^{s,x}|^2\,dr+\mathbf{E}\sup_{s\le t\le T}|X_t^{\varepsilon,s,x}-\varphi_t^{s,x}|^2 \\ +\mathbf{E}\int_t^T|Z_r^{\varepsilon,s,x}|^2\,dr+\mathbf{E}\int_t^T\int_E\big|U_r^{\varepsilon,s,x}(e)\big|^2\lambda(de)\,dr,$$

where $c(K) > 0$. Hence combining these inequalities with (3.10), we deduce from (3.11)

$$\mathbf{E}\Big(\sup_{s\le t\le T}|Y_t^{\varepsilon,s,x}-\psi_t^{s,x}|^2\Big)\le\gamma_{(K,T)}\varepsilon^2.$$

Using once again (3.10), we get the desired result. □

Let us consider the infinitesimal generator of $(X_t^{\varepsilon,s,x})_{s\le t\le T}$ defined by (where $a=\sigma\sigma^*$)

$$\mathcal{L}^{\varepsilon,x}=\frac{\varepsilon^2}{2}\sum_{i,j=1}^{k}a_{ij}(x)\partial^2_{x_ix_j}+\sum_{i=1}^{k}\beta_i(x)\partial_{x_i}$$

and the system of parabolic PDE

$$\begin{cases}\partial_t u_i^\varepsilon(t,x)+\mathcal{L}^{\varepsilon,x}u_i^\varepsilon(t,x)+f_i\big(t,x,u^\varepsilon(t,x),\big(\partial_x u_i^\varepsilon\sigma\big)(t,x)\big)=0;\\ \qquad (t,x)\in[0,T]\times\mathbf{R}^k,\ 1\le i\le k,\\ u_i^\varepsilon(T,x)=g_i(x).\end{cases}\tag{3.12}$$

Thanks to Barles et al. (1996, Theorem 3.4),

$$u^\varepsilon(s,x)=Y_s^{\varepsilon,s,x},\qquad 0\le s\le T,\ x\in\mathbf{R}^k\tag{3.13}$$

is continuous and it is a viscosity solution of (3.12). Moreover, $Y_t^{\varepsilon,s,x}=u^\varepsilon(t,X_t^{\varepsilon,s,x})$, $s\le t\le T$.

Definition 3.3. For $s\in[0,T]$ and $\varepsilon\ge 0$, we define the map $\Phi^\varepsilon:\mathcal{C}([s,T],\mathbf{R}^k)\to\mathcal{C}([s,T],\mathbf{R}^k)$ by

$$\Phi^\varepsilon(\varphi):=\big[t\mapsto u^\varepsilon(t,\varphi_t)\big],\qquad t\in[s,T],\ \varphi\in\mathcal{C}\big([s,T],\mathbf{R}^k\big),$$

where u^ε given by (3.13).

Thus, $Y_t^{\varepsilon,s,x}=\Phi^\varepsilon(X^{\varepsilon,s,x})(t)$, $s\le t\le T$ and $\varepsilon\ge 0$. We denote $u=u^0$ and $\Phi=\Phi^0$.

We claim the following theorem.

Theorem 3.4. *The process $Y_\cdot^{\varepsilon,s,x}$ satisfies in $\mathcal{C}([0,T],\mathbf{R}^k)$ a Large Deviation Principle associated to the function rate I'_x defined by*

$$I'_x(\psi)=\inf\big\{I_x(\varphi):\varphi\in H^1\big([0,T],\mathbf{R}^k\big)|\psi=\Phi(\varphi)\big\},\qquad \psi\in\mathcal{C}\big([0,T],\mathbf{R}^k\big).$$

Proof. By the contraction principle (see Varadhan (1984)), it suffices to prove that Φ^ε, $\varepsilon\ge 0$ is continuous and Φ^ε converge uniformly to Φ on compacts of $\mathcal{C}([0,T],\mathbf{R}^k)$ whenever $\varepsilon\to 0$. Using the method developed in Rainero (2006, Theorem 2.3), we establish continuity of Φ^ε, $\varepsilon \ge 0$. Applying Proposition 3.2, we have for any compact $\mathcal{K}$ of $\mathcal{C}([0,T],\mathbf{R}^k)$,

$$\sup_{\varphi\in\mathcal{K}} \|\Phi^\varepsilon(\varphi) - \Phi(\varphi)\|^2 \le \gamma_{(K,T)}\varepsilon^2$$

which is enough to get the desired result. □

Remark 3.5. If a is uniformly elliptic, then (3.3) reads to

$$I_x(\varphi) = \frac{1}{2}\int_0^T \mathbf{Q}^*_{\varphi_t}(\dot\varphi_t - \beta(\varphi_t))\,dt \qquad \text{if} \qquad \varphi \in H^1([0,T],\mathbf{R}^k) \quad \text{and} \quad \varphi_0 = x,$$
$$I_x(\varphi) = +\infty, \qquad \text{otherwise}$$

with $\mathbf{Q}^*_u(v) = \langle v, a^{-1}(u)v\rangle$, $(u,v) \in \mathbf{R}^k \times \mathbf{R}^k$.

This implies that the function rate $I'_x(\psi)$ is given by

$$I'_x(\psi) = \inf\biggl\{\frac{1}{2}\int_0^T \mathbf{Q}^*_{\varphi_t}(\dot\varphi_t - \beta(\varphi_t))\,dt : \varphi \in H^1([0,T],\mathbf{R}^k), \varphi_0 = x, \psi_t = u^0(t,\varphi_t), 0 \le t \le T\biggr\}.$$

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