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Weighted approximations for Studentized U-statistics

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Abstract. In this article, we employ the jackknife method of estimation and the concept of Studentized U-statistics to derive a new weak convergence result for nondegenerate U-statistics on the space $D[0, 1]$. We drop the classical condition that the second moment of the kernel of the underlying U-statistic exists and derive a weighted weak convergence result for these Studentized statistics. This weak convergence is concluded from a weighted approximation in sup-norm $\|/q\|$, in probability, of Studentized U-statistics by partial sums of i.i.d. projections, where q is in an appropriate class of positive weight functions.

1 Introduction

Let $X_1, X_2, \dots,$ be a sequence of nondegenerate real-valued independent and identically distributed (i.i.d.) random variables with distribution F. Consider the parameter (parametric function) $\theta = \theta(F)$ for which there is an unbiased estimator in terms of a Borel-measurable real-valued function $h = h(x_1, \dots, x_m)$, which is symmetric in its arguments. That is to say, given a random sample $X_1, \dots, X_n$ of size $n \ge m$ on F, we have

$$\theta = \theta(F) := E\bigl(h(X_1, \dots, X_m)\bigr) = \int_{\mathbb{R}^m} h(x_1, \dots, x_m)\, dF(x_1) \cdots dF(x_m) < \infty.$$

The corresponding U-statistic (cf. Serfling (1980) or Hoeffding (1948)) is

$$\begin{aligned} U_n = U(X_1, \dots, X_n) &= \binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_m \le n} h(X_{i_1}, \dots, X_{i_m}) \\ &= [n]^{-m} \sum_{1 \le i_1 \ne \cdots \ne i_m \le n} h(X_{i_1}, \dots, X_{i_m}), \end{aligned} \tag{1.1}$$

where $[n]^{-m} := \frac{(n-m)!}{n!}$. For further use throughout, we define the projection of h as follows.

$$\tilde{h}_1(x) = \mathbb{E}\bigl(h(X_1, \dots, X_m) - \theta | X_1 = x\bigr).$$

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A large number of well-known statistics are U-statistics. Statistics such as the sample variance, deleted jackknife variance estimator, Fisher's k-statistic for estimation of cumulants, Kendal's τ, Gini's mean difference are few examples of U-statistics.

The first central limit theorem for nondegenerate U-statistics, that is, when $\mathrm{Var}(\tilde{h}_1(X)) > 0$, was proved by Hoeffding (1948) when the kernel h is square integrable. Under the same moment conditions, Miller and Sen (1972) established a weak convergence result for this class of U-statistics on the space $C[0, 1]$. Korolyuk and Borovskikh (1988) showed that the central limit theorem for nondegenerate U-statistics continues to hold true when $\mathbb{E}|h(X_1, \ldots, X_m)|^{4/3} < \infty$ and $0 < \mathrm{Var}(\tilde{h}_1(X)) < \infty$. The common feature of all of these contributions is that they rely on the unknown parameter $\mathrm{Var}(\tilde{h}_1(X))$. Bentkus et al. (2009) obtained a Berry–Esseen type bound for nondegenerate U-statistics of order 2. Also, by estimating $\mathrm{Var}(\tilde{h}_1(X))$ using the jackknife estimation, they proved a central limit theorem for the resulting Studentized U-statistics of order 2. Another example of the use of jackknife method of estimation for U-statistics is the paper Jing et al. (2009) in which the authors establish the validity of *jackknife empirical likelihood* for U-statistics when the kernel of the underlying U-statistic is assumed to be square integrable and $\mathrm{Var}(\tilde{h}_1(X)) > 0$. U-statistics of order 2 were also considered by Csörgő et al. (2008b) when they studied the weak convergence of the Studentized version of Csörgő-Horváth test statistic for detecting change in distribution (cf. Csörgő and Horváth (1997) and references therein) when the variance of the underlying kernel $h(\cdot, \cdot)$ is not necessarily finite. Inspired by this work of Csörgő et al. (2008b), in this exposition we establish a new weak convergence result for nondegenerate U-statistics on the space $D[0, 1]$, when the second moment of the kernel h is not necessarily finite (cf. Theorem 1). An application of the main result of this exposition, that is, Theorem 1, to constructing asymptotic confidence interval for the parameter $\theta = Eh(X_1, \ldots, X_m)$ is presented. Theorem 1 generalizes all of the above mentioned contributions to weak convergence and central limit theorems of U-statistics.

The material in this paper is organized as follows. In Section 2, we present our main results, Theorem 1 and Theorem 2 and outline their proofs via the auxiliary Theorem 3 and Theorem 4 whose proofs will be given in Section 4. Applications of Theorem 1, the main result, are given in Section 3. Finally, in Section 4, we state some required results on weighted weak convergence of partial sums in the domain of attraction of the normal law which are crucial in establishing our results and also present some technical tools, followed by the proofs of our auxiliary results Theorems 3 and 4 of Section 2.

2 Main results

For $i = 1, \ldots, n$, let U^i_{n-1} be the *jackknifed* version of U_n based on $X_1, \ldots, X_{i-1}$, $X_{i+1}, \ldots, X_n$, defined as follows

$$U^i_{n-1} := \frac{1}{\binom{n-1}{m}} \sum_{\substack{1 \le j_1 < \cdots < j_m \le n \\ j_1, \ldots, j_m \ne i}} h(X_{j_1}, \ldots, X_{j_m}).$$

Also define the *Studentized U-statistic* process, $U^{\text{stu}}_{[nt]}$, as follows.

$$U^{\text{stu}}_{[nt]} = \begin{cases} 0, & 0 \le t < \dfrac{m}{n}, \\ \dfrac{[nt](U_{[nt]} - \theta)}{\sqrt{n(n-1)\sum_{i=1}^n (U^i_{n-1} - U_n)^2}}, & \dfrac{m}{n} \le t \le 1. \end{cases} \tag{2.1}$$

In the following Theorem 1, which is the main result in this article, we drop the classical assumption of having finite variance of the kernel h of the underlying U-statistic U_n. Consequently, the variance of the projections $\tilde{h}_1(\cdot)$ is not necessarily finite. Hence, we may and will assume that $\tilde{h}_1(\cdot) \in DAN$, where DAN stands for the domain of attraction of the normal law (cf. Section 4). Moreover, we present our weak convergence results for Studentized U-statistics, in presence of weight functions. We now define the class of weights that will be used in our results.

Throughout this article, we let Q be the class of functions $q(t)$, which are positive on $(0, 1]$, that is, $\inf_{\delta \le t \le 1} q(t) > 0$ for $0 < \delta < 1$, and nondecreasing in a neighborhood of zero (cf. the below Lemmas 1 and 2 for characterization of Q). Moreover, we define the integral $I(q, c)$ as follows that is also used to characterize the class Q (cf. Section 4).

$$I(q, c) := \int_{0^+}^1 t^{-1} \exp\bigl(-cq^2(t)/t\bigr)\, dt, \qquad 0 < c < \infty.$$

The next result establishes a weak convergence result for Studentized U-statistics processes via weighted weak approximations as follows.

Theorem 1. *Let $q \in Q$. If*

(a) $E|h(X_1, \ldots, X_m)|^{5/3} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$,

then, as $n \to \infty$, we have

(b) $U^{\text{stu}}_{[nt]} \Rightarrow W(t)$ *on* $(\mathrm{D}, \mathfrak{D}, \|/q\|)$ *if and only if* $I(q, c) < \infty$ *for all* $c > 0$, *where* $\{W(t), 0 \le t \le 1\}$ *is a standard Wiener process and* $\|/q\|$ *is the weighted sup-norm metric for functions in* $D[0, 1]$;

(c) *On an appropriate probability space for* $X_1, X_2, \ldots$, *we can construct a standard Wiener process* $\{W(t), 0 \le t < \infty\}$ *such that*

$$\sup_{0 < t \le 1} \left| U^{\text{stu}}_{[nt]} - \frac{W(nt)}{n^{1/2}} \right| \Big/ q(t) = o_P(1),$$

if and only if $I(q,c)<\infty$.

Remark 1. The notation of $\Rightarrow$ in part (b) of the preceding theorem is as it was defined in Remark 7 below.

Remark 2. Note that taking $q(t)=1$ results in finiteness of $I(q,c)$ for all $c>0$, that is, Theorem 1 remains valid for nonweighted Studentized U-statistic processes. Moreover, in this case, $\|/q\|$-metric will coincide with the usual sup-norm metric and the notion $\Rightarrow$ of part (b) of Theorem 1, as it is defined above, will coincide with the convergence in distribution of functionals definition of weak convergence on $D[0,1]$ with respect to the sup-norm metric.

Theorem 2. *Let* $\{W(t), 0\le t<\infty\}$ *be a standard Wiener process*, $E|h(X_1,\dots,X_m)|^{5/3}<\infty$ *and* $\tilde{h}_1(X_1)\in DAN$. *If* $q\in Q$ *and* $q(t)$ *is nondecreasing on* $(0,1]$, *then as* $n\to\infty$,

$$\sup_{0<t\le 1}\left|U^{\text{stu}}_{[nt]}\right|/q(t)\longrightarrow_d \sup_{0<t\le 1}\left|W(t)\right|/q(t)$$

if and only if $I(q,c)<\infty$ *for some* $c>0$. *Consequently as* $n\to\infty$, *we have*

$$\sup_{0<t\le 1}\left|U^{\text{stu}}_{[nt]}\right|\Big/\left(t\log\log\left(\frac{1}{t}\right)\right)^{1/2}\longrightarrow_d \sup_{0<t\le 1}\left|W(t)\right|\Big/\left(t\log\log\left(\frac{1}{t}\right)\right)^{1/2}.$$

The preceding Theorem 2 is a U-statistic version of the below Lemma 4.

It can be readily seen that in Theorem 1, (c) implies (b). By virtue of Lemma 3 (cf. Section 4), with $\tilde{h}_1(X)$ instead of X, and in presence of the below conclusion (4.1), it becomes clear that to prove Theorem 1 it suffices to prove the following two results namely Theorems 3 and 4. Moreover, as it will be seen in the proof of Theorem 3, we only require that the weight function $q\in Q$ satisfy the condition that $\lim_{t\downarrow 0}\frac{t^{1/2}}{q(t)}=0$. In view of Lemma 1 (cf. Section 4), the latter relation holds whenever $I(q,c)<\infty$ for some $c>0$. Hence, in view of Lemma 4 in Section 4, Theorem 2 will also follow from the following Theorems 3 and 4.

Theorem 3. *Let* $q\in Q$ *and* $I(q,c)<\infty$ *for some* $c>0$. *Assume*

$$E\left(\left|h(X_1,\dots,X_m)\right|^{4/3}\log\left|h(X_1,\dots,X_m)\right|\right)<\infty$$

and that $\tilde{h}_1(X_1)\in DAN$. *Then, as* $n\to\infty$ *we have*

$$\sup_{0<t\le 1}\left|\frac{[nt]}{m\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde{h}_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)=o_P(1).$$

Theorem 4. *If* $E|h(X_1,\dots,X_m)|^{5/3}<\infty$ *and* $\tilde{h}_1(X_1)\in DAN$, *then, as* $n\to\infty$,

$$\left|\frac{(n-1)}{m^2\ell^2(n)}\sum_{i=1}^{n}(U^i_{n-1}-U_n)^2-\frac{1}{n\ell^2(n)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\right|=o_P(1).$$

Consequently, the preceding approximation combined with (4.1) of Section 4, the conclusion of Raikov's theorem, yields a Raikov type result for the distribution free jackknifed version of U-statistics which is of interest on its own (cf. Remark 3), and it reads as follows.

Corollary 1. *If* $E|h(X_1, \ldots, X_m)|^{5/3} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$, *then, as* $n \to \infty$,

$$\frac{(n-1)}{m^2 \ell^2(n)} \sum_{i=1}^{n} (U_{n-1}^i - U_n)^2 \longrightarrow_P 1.$$

Remark 3. When $Eh^2(X_1, \ldots, X_m) < \infty$, which in turn implies that $E\tilde{h}_1^2(X_1) < \infty$, then $\ell^2(n) = E\tilde{h}_1^2(X_1) > 0$ and, as $n \to \infty$, Corollary 1 implies that

$$\frac{(n-1)}{m^2} \sum_{i=1}^{n} (U_{n-1}^i - U_n)^2 \longrightarrow_P E\tilde{h}_1^2(X_1).$$

The latter version of Corollary 1 coincides with one of the results obtained by Arvesen (1969) who extended the idea of the so-called (by Tukey) pseudo-values to U-statistics and studied the asymptotic distribution of nondegenerate U-statistics via jackknifing.

The condition that $E|h(X_1, \ldots, X_m)|^{5/3} < \infty$ in Theorem 1 is assumed to guarantee the nearness of the $\frac{(n-1)}{m^2} \sum_{i=1}^{n} (U_{n-1}^i - U_n)^2$ and $\frac{1}{n} \sum_{i=1}^{n} \tilde{h}_1^2(X_i)$. The method of truncations is used to establish the latter nearness. Having tried several different truncations the author have found the sequence $a_n = n^{3m/5}$, cf. the proof of Theorem 4, to be an equilibrium point for the truncation $I(|h| \leq a_n)$ and its tail $I(|h| > a_n)$. Any other truncation results in requiring a moment higher than $5/3$. The truncation $I(|h| \leq n^{3m/5})$ is associated with the moment condition $E|h(X_1, \ldots, X_m)|^{5/3} < \infty$. Hence this condition seems to be minimal.

Remark 4. When $m = 1$, the projection $\tilde{h}_1(X_1)$ will coincide with $h(X_1) - \theta$, and Theorem 1, correspond to Corollary 5 of Csörgő et al. (2008a) on taking the weight function $q = 1$ for the therein studied Studentized process $T_{n,t}(X - \mu)$, that is, when $m = 1$, the studentized U-process $U_{[nt]}^{\text{stu}}$ coincides with $T_{n,t}(X - \mu)$. Hence, in this exposition, we shall state our proofs for nondegenerate U-statistics with order $m \geq 2$. Also, when $m = 2$, the two conditions in (a) of Theorem 1 as well as the idea of its proof by truncation, coincide with the corresponding ones of Theorem 2 of Csörgő et al. (2008b).

3 Application of Theorem 1 to confidence intervals

In view of Theorem 1, for a function $g : D[0, 1] \to \mathbb{R}$ as in Remark 7, one can use the pivot $g(U_{[n\cdot]}^{\text{stu}}/q(\cdot))$ to establish an asymptotic confidence interval for the parameter of interest θ. More precisely, in this section, we take $q(t) = 1$ and we derive an asymptotic confidence interval of level $1-\alpha$, $0<\alpha<1$ for θ using the pivot $\sup_{0\le t\le 1} U_{[nt]}^{\mathrm{stu}}$. This can be done since from Theorem 1, with $q(t)=1$, we have

$$\sup_{0\le t\le 1} U_{[nt]}^{\mathrm{stu}} \longrightarrow_d \sup_{0\le t\le 1} \big|W(t)\big|.$$

By virtue of the last relation and by defining w_α to be $P(\sup_{0\le t\le 1}|W(t)| > w_\alpha) = \alpha$, we establish the following $(1-\alpha)\%$ asymptotic confidence interval for θ as follows.

$$\bigcap_{k=1}^{n}\bigg[U_k - \frac{w_\alpha}{k\sqrt{n}}\hat{\sigma}, U_k + \frac{w_\alpha}{k\sqrt{n}}\hat{\sigma}\bigg], \tag{3.1}$$

where $\hat{\sigma} = \sqrt{(n-1)\sum_{i=1}^{n}(U_{n-1}^{i} - U_n)^2}$.

The distribution of $\sup_{0\le t\le 1}|W(t)|$ was tabulated by Csörgő and Horváth (1984). Confidence intervals of the form (3.1) which are shorter than those obtained from the central limit theorem, were also derived by Martsynyuk (2009), where she established functional asymptotic confidence intervals for a common mean of independent random variables.

One can use Theorem 1 to construct the confidence interval (3.1) for the parameter θ associated with a U-statistic that is built on a random sample whose variance is $+\infty$. Obviously, this can not be derived neither by classical weak convergence results for U-statistics such as Miller and Sen (1972), nor from the central limit theorem of Korolyuk and Borovskikh (1988) in which the variance of the U-statistic need not be finite, however, $\tilde{h}_1$ should have a finite positive variance. Moreover, this result can not be derived from Bentkus et al. (2009) either as the latter requires that the variance of $0 < \mathrm{Var}(\tilde{h}_1(X_1)) < \infty$.

To demonstrate this application of Theorem 1, we let $X_1, X_2, \ldots,$ be a sequence of i.i.d. random variables with the density function

$$f(x) = \begin{cases} 2a^2x^{-3}, & x \ge a, \\ 0, & \text{elsewhere}, \end{cases}$$

where $a>0$. Consider the parameter $\theta = E^m(X_1) = 2^m a^m$, where $m\ge 1$ is a positive integer, and the kernel $h(X_1,\ldots,X_m) = \prod_{i=1}^m X_i$. Then with m, n satisfying $n \ge m$, the corresponding U-statistic is

$$U_n = \binom{n}{m}^{-1} \sum_{1\le i_1<\cdots<i_m\le n} \prod_{j=1}^{m} X_{i_j}.$$

Simple calculations show that $\tilde{h}_1(X_1) = X_1 2^{m-1}a^{m-1} - 2^m a^m$.

It is easy to check that $E|h(X_1,\ldots,X_m)|^{5/3} < \infty$ and that $\tilde{h}_1(X_1) \in DAN$ (cf. Gut (2005)).

The asymptotic $(1-\alpha)\%$ confidence interval (3.1) for the parameter $\theta = 2^m a^m$ is of the form:

$$\bigcap_{k=1}^{n}\Bigg[\binom{k}{m}^{-1}\sum_{1\le i_1<\cdots<i_m\le k}\prod_{j=1}^{m}X_{i_j}-\frac{w_\alpha}{k\sqrt{n}}\hat{\sigma},$$

$$\binom{k}{m}^{-1}\sum_{1\le i_1<\cdots<i_m\le k}\prod_{j=1}^{m}X_{i_j}+\frac{w_\alpha}{k\sqrt{n}}\hat{\sigma}\Bigg],$$

where, by (4.18) of Section 4 we have

$$\begin{aligned}\hat{\sigma}^2 &= (n-1)\sum_{i=1}^{n}\big(U_{n-1}^{i}-U_n\big)^2\\ &= \frac{m^2(n-1)}{(n-m)^2}\Bigg\{\sum_{i=1}^{n}X_i^2\Bigg[\binom{n-1}{m-1}^{-1}\sum_{\substack{1\le i_2<\cdots<i_m\le n\\ i_2,\dots,i_m\ne i}}\prod_{j=2}^{m}X_{i_j}\Bigg]^2\\ &\qquad -n\Bigg[\binom{n}{m}^{-1}\sum_{C(n,m)}\prod_{j=1}^{m}X_{i_j}\Bigg]^2\Bigg\}.\end{aligned}\tag{3.2}$$

In addition to the above application to confidence intervals, for the Studentized U-process here, which is defined as

$$U_{[nt]}^{\text{stu}}=\begin{cases}0, & 0\le t<\dfrac{m}{n},\\ \dfrac{[nt]\big(\binom{[nt]}{m}^{-1}\sum_{1\le i_1<\cdots<i_m\le[nt]}\prod_{j=1}^{m}X_{i_j}-\theta\big)}{\hat{\sigma}\sqrt{n}}, & \dfrac{m}{n}\le t\le 1,\end{cases}$$

where $\hat{\sigma}$ is defined by (3.2), Theorem 1 with $q(t)=1$, for example, implies that

$$U_{[nt]}^{\text{stu}}\Rightarrow W(t)\qquad \text{on } \big(D[0,1],\rho\big),$$

where ρ is the sup-norm for functions in $D[0,1]$ and $\{W(t), 0\le t\le 1\}$ is a standard Wiener process. Also, applying Theorem 2, as $n\to\infty$, we conclude the following central limit theorem:

$$\sup_{0<t\le 1}U_{[nt]}^{\text{stu}}\Big/\bigg(t\log\log\Big(\frac{1}{t}\Big)\bigg)\longrightarrow_d \sup_{0<t\le 1}\big|W(t)\big|\Big/\bigg(t\log\log\Big(\frac{1}{t}\Big)\bigg).$$

Remark 5. It is noteworthy to note that on replacing the parameter θ by U_n, the Studentized U-statistic $U_{[n\cdot]}^{\text{stu}}$, which is defined in (2.1), is a desirable candidate in studying convergence in distribution of *bootstrapped* U-statistics.

4 Tools and proofs

The following is the formal definition of the concept of *DAN*.

Definition 1. A sequence $X, X_1, X_2, \ldots,$ of i.i.d. random variables is said to be in the domain of attraction of the normal law ($X \in DAN$) if there exist sequences of constants A_n and $B_n > 0$ such that, as $n \to \infty$,

$$\frac{\sum_{i=1}^{n} X_i - A_n}{B_n} \longrightarrow_d N(0, 1).$$

Remark 6. Further to this definition of *DAN*, it is known that A_n can be taken as $n\mathbb{E}(X)$ and $B_n = n^{1/2}\ell_X(n)$, where $\ell_X(n)$ is a slowly varying function at infinity (i.e., $\lim_{n\to\infty} \frac{\ell_X(nk)}{\ell_X(n)} = 1$ for any $k > 0$), defined by the distribution of X. Moreover, $\ell_X(n) = \sqrt{\mathrm{Var}(X)} > 0$, if $\mathrm{Var}(X) < \infty$, and $\ell_X(n) \to \infty$, as $n \to \infty$, if $\mathrm{Var}(X) = \infty$. Also X has all moments less than 2, and the variance of X is positive, but need not be finite.

Thus, in view of Remark 6, if $X \in DAN$, then as $n \to \infty$, with the numeric sequence $n^{1/2}\ell(n)$ we have

$$\frac{\sum_{i=1}^{n} (X_i - E(X_1))}{\sqrt{n}\ell(n)} \longrightarrow_d N(0, 1),$$

and

$$\frac{\sum_{i=1}^{n} (X_i - E(X_1))^2}{n\ell^2(n)} \longrightarrow_P 1. \tag{4.1}$$

The result (4.1) is known as Raikov's theorem (cf. Giné et al. (1997), who also give a nice proof in this formulation).

The following two lemmas, which characterize the class Q, are due to Csörgő et al. (1986) (cf. also Lemmas 2 and 3 in Csörgő et al. (2008a)).

Lemma 1. *Let $q \in Q$. If $I(q, c) < \infty$ for some $c > 0$, then*

$$\lim_{t\downarrow 0} \frac{t^{1/2}}{q(t)} = 0.$$

Lemma 2. *Let $\{W(t), 0 \le t < \infty\}$ be a standard Wiener process and $q \in Q$. Then,*

(a) *$I(q, c) < \infty$ for all $c > 0$ if and only if*

$$\limsup_{t\downarrow 0} \frac{|W(t)|}{q(t)} = 0 \qquad a.s.$$

(b) *$I(q,c)<\infty$ for some $c>0$ if and only if*

$$\limsup_{t\downarrow 0}\frac{|W(t)|}{q(t)}<\infty \qquad a.s.$$

The presence of the weight function q requires a definition of a proper metric on $D[0,1]$ by which weak convergence is defined. This definition is given as follows.

Definition 2. For the functions x, y on $[0,1]$ and $q\in Q$, define the weighted sup-norm metric

$$\|(x-y)/q\| = \sup_{0<t\le 1}\big|x(t)-y(t)\big|/q(t),$$

whenever it is well defined, that is, $\limsup_{t\downarrow 0}|x(t)-y(t)|/q(t)<\infty$. A short hand notation for this metric will be $\|/q\|$.

For more on the subject of weight functions we refer the interested reader to Csörgő et al. (1986) and Csörgő and Horváth (1993). Another good source for literature in this regard is Szyszkowicz (1992).

The next two results, namely Lemmas 3 and 4 are weak convergence results for partial sums of i.i.d. random variables in *DAN* in presence of weight functions $q\in Q$. We shall state and later on use these results in order to establish U-statistics versions of them in Theorems 1 and 2. The following Lemma 3 coincides with Corollary 1 of Csörgő et al. (2008a) that is a direct consequence of their Theorem 1 (cf. also Proposition 3.1 of Csörgő et al. (2004)).

Lemma 3. *Let $q\in Q$. As $n\to\infty$, the following statements are equivalent*:

(a) *$X_1\in DAN$ and $EX_1=\mu$;*

(b) $\frac{\sum_{i=1}^{[nt]}(X_i-\mu)}{\sqrt{n}\ell(n)}\Rightarrow W(t)$ *on* $(\mathrm{D},\mathfrak{D},\|/q\|)$ *if and only if $I(q,c)<\infty$ for all $c>0$, where $\{W(t),0\le t\le 1\}$ is a standard Wiener process*;

(c) *On an appropriate probability space for $X_1,X_2,\ldots,$ a standard Wiener process $\{W(t),0\le t\le\infty\}$ can be constructed in such a way that as $n\to\infty$,*

$$\sup_{0<t\le 1}\left|\frac{\sum_{i=1}^{[nt]}(X_i-\mu)}{\sqrt{n}\ell(n)}-\frac{W(nt)}{\sqrt{n}}\right|\Big/ q(t)=o_P(1)$$

if and only if $I(q,c)<\infty$ for all $c>0$.

Remark 7. The statement (b) of Lemma 3 stands for the following functional central limit theorem on $(\mathrm{D},\mathfrak{D},\|/q\|)$, where $\mathfrak{D}$ is the σ-field of subsets of $\mathrm{D}=\mathrm{D}[0,1]$ generated by its finite-dimensional subsets, and $\|/q\|$ stands for the *weighted sup-norm metric* in $\mathrm{D} = \mathrm{D}[0, 1]$ with $q \in Q$. With $\longrightarrow_d$ standing for convergence in distribution as $n \to \infty$, we have

$$g\left(\frac{\sum_{i=1}^{[n\cdot]} X_i - \mu}{V_n q(\cdot)}\right) \to_d g\left(\frac{W(\cdot)}{q(\cdot)}\right)$$

for all $q \in Q$ and $g : \mathrm{D} = \mathrm{D}[0, 1] \to \mathbb{R}$ that are $(\mathrm{D}, \mathfrak{D})$ measurable and $\|/q\|$-continuous or $\|/q\|$-continuous except at points forming a set of Wiener measure zero on $(\mathrm{D}, \mathfrak{D})$, generated by a standard Wiener process $W(\cdot)$ on the unit interval $[0, 1]$.

For a larger class of weight functions that imply the finiteness of $I(q, c)$ and characterized by Lemma 1 and part (b) of Lemma 2, the next lemma coincides with (b) of Corollary 2 of Csörgő et al. (2008a) and it is a consequence of their Theorem 1 therein. We note in passing that the following Lemma 4 does not follow from Lemma 3. Moreover, it can not be obtained via classical method of weak convergence (cf. page 311 of Csörgő et al. (2008a)).

Lemma 4. *Let $E(X_1) = \mu$. If $q \in Q$ and $q(t)$ is nondecreasing on* $(0, 1]$, *then, as* $n \to \infty$

$$\frac{1}{\sqrt{n}\ell(n)} \sup_{0<t\le 1} \left|\sum_{i=1}^{[nt]}(X_i - \mu)\right| \Big/ q(t) \longrightarrow_d \sup_{0<t\le 1} |W(t)|/q(t)$$

if and only if $I(q, c) < \infty$ for some $c > 0$. Consequently, as $n \to \infty$, we have

$$\frac{1}{\sqrt{n}\ell(n)} \sup_{0<t\le 1} \left|\sum_{i=1}^{[nt]}(X_i - \mu)\right| \Big/ \left(t \log\log\left(\frac{1}{t}\right)\right)^{1/2}$$

$$\longrightarrow_d \sup_{0<t\le 1} |W(t)| \Big/ \left(t \log\log\left(\frac{1}{t}\right)\right)^{1/2},$$

where $\log(x) = \log(\max\{x, e\})$.

The concept of *complete degeneracy* is essential in our proofs. Below is our definition of this concept.

Definition 3. The Borel-measurable function $L(x_1, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}$, $m \ge 2$, with mean $\mu = \mathbb{E}L(X_1, \ldots, X_m)$, is said to be complete degenerate if for every proper subset $\{\alpha_1, \ldots, \alpha_j\}$ of $\{1, \ldots, m\}$, $j = 1, \ldots, m - 1$, we have

$$\mathbb{E}\big(L(X_1, \ldots, X_m) - \mu | X_{\alpha_1}, \ldots, X_{\alpha_j}\big) = 0 \qquad \text{a.s.}$$

Remark 8. When a summand is complete degenerate, we shall call the associated sum a complete degenerate one. In other words, complete degeneracy is inherited by the sum from the summand.

We note in passing that if L were symmetric in its arguments, then the associated U-statistic with such a kernel would be a complete degenerate one. Hence our terminology for L in this definition.

Remark 9. Concerning complete degenerate U-statistics, it is known that, with respect to the filtration $\sigma(X_1, \ldots, X_n)$, they are martingales (cf., e.g., Serfling (1980) or Borovskikh (1996)). This property will be used in our proofs. We note in passing that it can also be shown that the martingale property remains valid for multiple sums with not necessarily symmetric summands.

Along the lines of our proofs we shall use of the followings technical Propositions 1 and 2 which deal with approximation of the second moment of complete degenerate sums.

Proposition 1. *If* $L : \mathbb{R}^m \to \mathbb{R}$, $m \geq 2$, *is complete degenerate with mean* $\mu = EL(X_1, \ldots, X_m)$ *and* $EL^2(X_1, \ldots, X_m) < \infty$, *then,*

$$E\bigg([n]^{-m} \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq n} \big(L(X_{i_1}, \ldots, X_{i_m}) - \mu\big)\bigg)^2$$

$$\leq [n]^{-m} E\big(L(X_1, \ldots, X_m) - \mu\big)^2.$$

A companion of Proposition 1 when the summand L is symmetric and depends on the index i_m, where $1 \leq i_1 < \cdots < i_m \leq n$, reads as follows.

Proposition 2. *If* $L_{i_m} : \mathbb{R}^m \to \mathbb{R}$, $m \geq 2$, *is symmetric, centered and complete degenerate such that* $EL^2_{i_m}(X_{i_1}, \ldots, X_{i_m}) < \infty$, *then*

$$E\bigg(\sum_{1 \leq i_1 < \cdots < i_m \leq n} L_{i_m}(X_{i_1}, \ldots, X_{i_m})\bigg)^2 \leq \sum_{i_m = m}^{n} \binom{i_m}{m-1} E\big(L^2_{i_m}(X_1, \ldots, X_m)\big).$$

Proof of Proposition 1. Let $\hat{L}_{1,\ldots,m} := \frac{1}{m!} \sum_{C_m} L_{\sigma_1,\ldots,\sigma_m}$, where $L_{\sigma_1,\ldots,\sigma_m} := L(X_{\sigma_1}, \ldots, X_{\sigma_m})$ and C_m denotes the set of all permutations $\sigma_1, \ldots, \sigma_m$ of $1, \ldots, m$. It is clear that

$$\sum_{1 \leq i_1 \neq \cdots \neq i_m \leq n} (\hat{L}_{i_1,\ldots,i_m} - \mu) = \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq n} (L_{i_1,\ldots,i_m} - \mu).$$

Now write

$$E\bigg([n]^{-m} \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq n} (\hat{L}_{i_1,\ldots,i_m} - \mu)\bigg)^2$$

$$= ([n]^{-m})^2 \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq n} E(\hat{L}_{i_1,\ldots,i_m} - \mu)^2$$

$$
\begin{aligned}
&+ \big([n]^{-m}\big)^2 \sum_{j=1}^{m-1} \sum_{1\le i_1\ne\cdots\ne i_{2m-j}\le n} E\big[(\hat{L}_{i_1,\ldots,i_j,i_{j+1},\ldots,i_m} - \mu) \\
&\qquad\qquad \times (\hat{L}_{i_1,\ldots,i_j,i_{m+1},\ldots,i_{2m-j}} - \mu)\big] \\
&+ \big([n]^{-m}\big)^2 \sum_{1\le i_1\ne\cdots\ne i_{2m}\le n} E\big((\hat{L}_{i_1,\ldots,i_m} - \mu)(\hat{L}_{i_{m+1},\ldots,i_{2m}} - \mu)\big) \\
&= [n]^{-m} E(\hat{L}_{1,\ldots,m} - \mu)^2 \\
&+ \big([n]^{-m}\big)^2 \sum_{j=1}^{m-1} \sum_{1\le i_1\ne\cdots\ne i_{2m-j}\le n} E\big\{E[\hat{L}_{i_1,\ldots,i_j,i_{j+1},\ldots,i_m} - \mu | X_{i_1},\ldots,X_{i_j}] \\
&\qquad\qquad \times E[\hat{L}_{i_1,\ldots,i_j,i_{m+1},\ldots,i_{2m-j}} - \mu | \\
&\qquad\qquad\qquad X_{i_1},\ldots,X_{i_j}]\big\} \\
&+ \big([n]^{-m}\big)^2 \sum_{1\le i_1\ne\cdots\ne i_{2m}\le n} E(\hat{L}_{i_1,\ldots,i_m} - \mu) E(\hat{L}_{i_{m+1},\ldots,i_{2m}} - \mu) \\
&= [n]^{-m} E(\hat{L}_{1,\ldots,m} - \mu)^2 \\
&\le [n]^{-m} \frac{m!}{(m!)^2} \sum_{C_m} E(L_{\sigma(1),\ldots,\sigma(m)} - \mu)^2 \\
&= [n]^{-m} E(L_{1,\ldots,m} - \mu)^2.
\end{aligned}
$$

The last equality above is true provided that $EL^2_{\sigma_1,\ldots,\sigma_m} = EL^2_{1,\ldots,m}$. $\square$

It is easy to observe that when L is symmetric in its arguments, the inequality in Proposition 1 becomes equality.

Proof of Proposition 2. First, let

$$
\sum_{i_m=m}^{n} \sum_{i_{m-1}=m-1}^{i_m-1} \cdots \sum_{i_1=1}^{i_2-1} L_{i_m}(X_{i_1},\ldots,X_{i_m}) := \sum_{i_m=m}^{n} Y_{i_m},
$$

and for $i_m \ne i'_m$ write

$$
E\Bigg(\sum_{i_m=m}^{n} Y_{i_m}\Bigg)^2 = \sum_{i_m=m}^{n} E(Y_{i_m})^2 + \sum_{i_m=m}^{n} \sum_{i'_m=m}^{n} E(Y_{i_m} Y_{i'_m}). \tag{4.2}
$$

We now show that $E(Y_{i_m} Y_{i'_m}) = 0$. To do so, assume that $i_m < i'_m$ and write

$$
\begin{aligned}
E(Y_{i_m} Y_{i'_m}) &= E\big[E(Y_{i_m} Y_{i'_m}) | X_1,\ldots,X_{i_m}\big] \\
&= E\big[Y_{i_m} E(Y_{i'_m} | X_1,\ldots,X_{i_m})\big].
\end{aligned} \tag{4.3}
$$

To deal with the conditional expectation $E(Y_{i'_m}|X_1,\ldots,X_{i_m})$ in the latter relation, first recall that

$$Y_{i'_m}=\sum_{i'_{m-1}=m-1}^{i'_m}\cdots\sum_{i'_1=1}^{i'_2-1}L_{i'_m}(X_{i'_1},\ldots,X_{i'_{m-1}},X_{i'_m}).$$

Now consider the case when $i'_m-i_m<m-1$, then $\{i'_1,\ldots,i'_{m-1}\}\cap\{i_1,\ldots,i_m\}\neq\phi$. Hence, the complete degeneracy of $L_{i'_m}$ implies that

$$E\bigl(L_{i'_m}(X_{i'_1},\ldots,X_{i'_m})|X_1,\ldots,X_{i_m}\bigr)=0 \qquad \text{a.s.},$$

and this implies that

$$E(Y_{i'_m}|X_1,\ldots,X_{i_m})=0.$$

The other case is when $i'_m-i_m\geq m-1$. In this case $Y_{i'_m}$ can be written as

$$Y_{i'_m}=\sum_{I_1}L_{i'_m}(X_{i'_1},\ldots,X_{i'_m})+\sum_{I_2}L_{i'_m}(X_{i'_1},\ldots,X_{i'_m}),$$

where $I_1=\{i'_{m-1},\ldots,i'_1\}\subset\{i_1,\ldots,i_m\}$ and $I_2=\{i'_{m-1},\ldots,i'_1\}\subset\{i_{m+1},\ldots,i'_m\}$. The same argument as that of the previous case implies that on I_1 we have

$$E\bigl(L_{i'_m}(X_{i'_1},\ldots,X_{i'_m})|X_1,\ldots,X_{i_m}\bigr)=0 \qquad \text{a.s.}$$

Observe that on I_2, $\{i'_1,\ldots,i'_{m-1}\}\cap\{i_1,\ldots,i_m\}=\phi$. Therefore,

$$E\bigl(L_{i'_m}(X_{i'_1},\ldots,X_{i'_m})|X_1,\ldots,X_{i_m}\bigr)=E\bigl(L_{i'_m}(X_{i'_1},\ldots,X_{i'_m})\bigr)=0.$$

The last relation is true since $L_{i'_m}$ is centered. Thus,

$$E(Y_{i'_m}|X_1,\ldots,X_{i_m})=0.$$

The last relation together with (4.3) and (4.2) implies that

$$\begin{aligned}E&\Bigl(\sum_{1\leq i_1<\cdots<i_m\leq n}L_{i_m}(X_{i_1},\ldots,X_{i_m})\Bigr)^2\\&=\sum_{i_m=m}^{n}E\Bigl(\sum_{\substack{1\leq i_1<\cdots<i_{m-1}\\ i_{m-1}<i_m}}L_{i_m}(X_{i_1},\ldots,X_{i_m})\Bigr)^2.\end{aligned}$$

We now proceed via a similar argument as in the proof of Proposition 1, for estimating

$$E\Bigl(\sum_{\substack{1\leq i_1<\cdots<i_{m-1}\\ i_{m-1}<i_m}}L_{i_m}(X_{i_1},\ldots,X_{i_{m-1}},X_{i_m})\Bigr)^2$$

as follows.

$$
\begin{aligned}
E\Bigg(&\sum_{\substack{1\le i_1<\cdots<i_{m-1}\\ i_{m-1}<i_m}} L_{i_m}(X_{i_1},\dots,X_{i_{m-1}},X_{i_m})\Bigg)^2\\
&=\sum_{\substack{1\le i_1<\cdots<i_{m-1}\\ i_{m-1}<i_m}} E\big(L_{i_m}^2(X_{i_1},\dots,X_{i_{m-1}},X_{i_m})\big)\\
&\quad+\sum_{j=1}^{m-2}\sum_{1\le i_1<\cdots<i_{2m-j}\le n} E\big[L_{i_m}(X_{i_1},\dots,X_{i_j},X_{i_{j+1}},\dots,X_{i_{m-1}},X_{i_m})|\\
&\qquad\qquad X_{i_1},\dots,X_{i_j},X_{i_m}\big]\\
&\qquad\qquad\times E\big[L_{i_m}(X_{i_1},\dots,X_{i_j},X_{i_{m+1}},\dots,X_{i_{2m-j}},X_{i_m})|\\
&\qquad\qquad X_{i_1},\dots,X_{i_j},X_{i_m}\big]\\
&\quad+\sum_{1\le i_1<\cdots<i_{2m-2}\le n} E\big[L_{i_m}(X_{i_1},\dots,X_{i_{m-1}},X_{i_m})|X_{i_m}\big]\\
&\qquad\qquad\times E\big[L_{i_m}(X_{i_{m+1}},\dots,X_{i_{2m-2}},X_{i_m})|X_{i_m}\big]\\
&=\sum_{\substack{1\le i_1<\cdots<i_{m-1}\\ i_{m-1}<i_m}} E\big(L_{i_m}^2(X_{i_1},\dots,X_{i_m})\big)\le\binom{i_m}{m-1}E\big(L_{i_m}^2(X_1,\dots,X_m)\big).
\end{aligned}
$$

Now the proof of Proposition 2 is complete. □

Proof of Theorem 3. To establish Theorem 3, we first observe that

$$
\begin{aligned}
&\sup_{0<t\le 1}\left|\frac{[nt]}{m\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde h_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)\\
&\quad\le\sup_{0<t\le m/n}\left|\frac{[nt]}{m\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde h_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)\\
&\qquad+\sup_{m/n<t\le 1}\left|\frac{[nt]}{m\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde h_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t).
\end{aligned}\tag{4.4}
$$

But as $n\to\infty$, definition of $U_{[nt]}^{\mathrm{stu}}$ together with Lemma 4 imply that

$$
\begin{aligned}
&\sup_{0<t\le m/n}\left|\frac{[nt]}{\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde h_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)\\
&\quad\le\sup_{0<t<m/n}\left|\frac{\sum_{i=1}^{[nt]}\tilde h_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{m}{m\ell(n)\sqrt{n}}\big(h(X_1,\ldots,X_m)-\theta\big)-\frac{\sum_{i=1}^{m}\tilde{h}_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q\Big(\frac{m}{n}\Big)\\
&=o_P(1).
\end{aligned}
$$

Therefore, according to the latter relation and (4.4), in order to prove Theorem 3, we need to show that

$$
\sup_{m/n<t\le 1}\left|\frac{[nt]}{m\ell(n)\sqrt{n}}(U_{[nt]}-\theta)-\frac{\sum_{i=1}^{[nt]}\tilde{h}_1(X_i)}{\ell(n)\sqrt{n}}\right|\Big/q(t)=o_P(1). \tag{4.5}
$$

Due to the fact that

$$
\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(\tilde{h}_1(X_{i_1})+\cdots+\tilde{h}_1(X_{i_m})\big)=\frac{m}{[nt]}\binom{[nt]}{m}\sum_{i=1}^{[nt]}\tilde{h}_1(X_i),
$$

it becomes clear that in order to establish (4.5), it will be enough to prove the following Proposition 3.

Proposition 3. *Let $q\in Q$. If*

$$
E\big(\big|h(X_1,\ldots,X_m)\big|^{4/3}\log\big|h(X_1,\ldots,X_m)\big|\big)<\infty.
$$

Then, as $n\to\infty$

$$
\begin{aligned}
\frac{n^{-1/2}}{\ell(n)}\sup_{m/n<t\le 1}&\left|\frac{[nt]}{\binom{[nt]}{m}}\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})-\theta\right.\\
&\left.-\tilde{h}_1(X_{i_1})-\cdots-\tilde{h}_1(X_{i_m})\big)\right|\Big/q(t)\\
&=o_P(1).
\end{aligned}
$$

Proof. Without loss of generality, assume that $\theta=0$. On taking n large enough, let $\delta\in(\frac{m}{n},1]$ be small enough so that $q(t)$ is nondecreasing on $(0,\delta)$. Observe that

$$
\begin{aligned}
\frac{n^{-1/2}}{\ell(n)}\sup_{m/n<t\le 1}&\left|\frac{[nt]}{\binom{[nt]}{m}}\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})\right.\\
&\left.-\tilde{h}_1(X_{i_1})-\cdots-\tilde{h}_1(X_{i_m})\big)\right|\Big/q(t)\\
\le\frac{n^{-1/2}}{\ell(n)}\sup_{m/n<t<\delta}&\left|\frac{[nt]}{\binom{[nt]}{m}}\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})-\tilde{h}_1(X_{i_1})\right.\\
&\left.-\cdots-\tilde{h}_1(X_{i_m})\big)\right|\Big/q(t)
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&+\frac{n^{-1/2}}{\ell(n)}\sup_{\delta\le t\le 1}\left|\frac{[nt]}{\binom{[nt]}{m}}\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})\right.\\
&\qquad\qquad\left.-\tilde h_1(X_{i_1})-\cdots-\tilde h_1(X_{i_m})\big)\right|\Big/q(t)\\
&:=I_1(n,t)+I_2(n,t).
\end{aligned}
$$

To prove Proposition 3, it will be enough to show asymptotic negligibility of both $I_1(n,t)$ and $I_2(n,t)$ in probability.

To deal with $I_1(n,t)$ write

$$
\begin{aligned}
I_1(n,t)\le &\sup_{m/n\le t\le 1}\frac{[nt]^{1/2}}{\binom{[nt]}{m}\ell(n)}\left|\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})\right.\\
&\qquad\left.-\tilde h_1(X_{i_1})-\cdots-\tilde h_1(X_{i_m})\big)\right|\\
&\times\sup_{m/n<t<\delta}\frac{t^{1/2}}{q(t)}.
\end{aligned}
$$

The last relation suggests that by virtue of Lemma 1 and from the fact that $\ell(n)$ is a slowly varying function at infinity, for large n, we have that $\ell(n)\sqrt{n}\ge\sqrt{n}$. To prove $I_1(n,t)=o_P(1)$ it suffices to show that

$$
\begin{aligned}
&\sup_{m/n\le t\le 1}\frac{[nt]^{1/2}}{\binom{[nt]}{m}}\left|\sum_{1\le i_1<\cdots<i_m\le[nt]}\big(h(X_{i_1},\ldots,X_{i_m})\right.\\
&\qquad\left.-\tilde h_1(X_{i_1})-\cdots-\tilde h_1(X_{i_m})\big)\right|\\
&\quad=O_P(1).
\end{aligned}
\tag{4.7}
$$

To establish (4.7), for $i_1<\cdots<i_m$, consider the following truncation setup:

$$
\begin{aligned}
H^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})&:=h(X_{i_1},\ldots,X_{i_m})\mathbf{1}_{(|h|\le i_m^{3/2})}-E\big(h(X_{i_1},\ldots,X_{i_m})\mathbf{1}_{(|h|\le i_m^{3/2})}\big),\\
H^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})&:=h(X_{i_1},\ldots,X_{i_m})\mathbf{1}_{(|h|> i_m^{3/2})}-E\big(h(X_{i_1},\ldots,X_{i_m})\mathbf{1}_{(|h|> i_m^{3/2})}\big),\\
g^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})&:=H^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})-E\big(H^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})|X_{i_1}\big)\\
&\qquad-\cdots-E\big(H^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})|X_{i_m}\big),\\
g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})&:=H^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})-E\big(H^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})|X_{i_1}\big)\\
&\qquad-\cdots-E\big(H^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})|X_{i_m}\big).
\end{aligned}
\tag{4.8}
$$

Having the above setup, to prove (4.7), we proceed by stating and proving the following Proposition 4.

Proposition 4. *If* $E|h(X_1,\ldots,X_m)|^{4/3}<\infty$, *then, as* $n\to\infty$, *we have*

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Big|\sum_{1\le i_1<\cdots<i_m\le K} g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})\Big|=O_P(1)\tag{4.9}$$

and

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Big|\sum_{1\le i_1<\cdots<i_m\le K} g^{tr}_{i_m}(X_{i_1},\ldots,X_{i_m})\Big|=O_P(1).\tag{4.10}$$

Proof. For throughout, use let A be a positive constant which may be different as each stage. To prove (4.9), we first represent the statistic $\sum_{1\le i_1<\cdots<i_m\le K} g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})$ in terms of 2^m-m-1 sums which their summand posses the property of complete degeneracy as follows.

$$\begin{aligned}
&\sum_{1\le i_1<\cdots<i_m\le K} g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})\\
&\quad=\sum_{1\le i_1<\cdots<i_m\le K}\Bigg\{\sum_{d=2}^{m}(-1)^{m-d}\sum_{1\le j_1<\cdots<j_d\le m}E\big(g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})|\\
&\qquad\qquad X_{i_{j_1}},\ldots,X_{i_{j_d}}\big)\\
&\qquad+\sum_{c=2}^{m-1}\sum_{1\le k_1<\cdots<k_c\le m}\sum_{d=2}^{c}(-1)^{c-d}\sum_{1\le j_1<\cdots<j_d\le c}E\big(g^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})|\\
&\qquad\qquad X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}\big)\Bigg\}\\
&\quad:=\sum_{1\le i_1<\cdots<i_m\le K} V^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})\\
&\qquad+\sum_{c=2}^{m-1}\sum_{1\le k_1<\cdots<k_c\le m}\sum_{1\le i_1<\cdots<i_m\le K} V^{ta}_{i_m}(X_{i_{k_1}},\ldots,X_{i_{k_c}}).
\end{aligned}$$

In view of the latter setup to prove (4.9), we need to show that

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Big|\sum_{1\le i_1<\cdots<i_m\le K} V^{ta}_{i_m}(X_{i_1},\ldots,X_{i_m})\Big|=O_P(1)\tag{4.11}$$

and for, $m\ge 3$, $c=2,\ldots,m-1$ and $1\le k_1<\cdots<k_c\le m$,

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Big|\sum_{1\le i_1<\cdots<i_m\le K} V^{ta}_{i_m}(X_{i_{k_1}},\ldots,X_{i_{k_c}})\Big|=O_P(1).\tag{4.12}$$

Due to similarity, we shall only give the proof of (4.11).

Noting that $\frac{K^{1/2}}{\binom{K}{m}}$ is decreasing in K, for $M>0$, we write,

$$
\begin{aligned}
&P\Bigg(\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Bigg|\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{ta}(X_{i_1},\ldots,X_{i_m})\Bigg|>M\Bigg)\\
&\quad\le P\Bigg(\max_{m\le K\le n}\sum_{i_m=m}^{K}\frac{i_m^{1/2}}{\binom{i_m}{m}}\Bigg|\sum_{1\le i_1<\cdots<i_{m-1}<i_m}V_{i_m}^{ta}(X_{i_1},\ldots,X_{i_m})\Bigg|>M\Bigg)\\
&\quad\le M^{-1}E\Bigg(\sum_{i_m=m}^{\infty}\frac{i_m^{1/2}}{\binom{i_m}{m}}\Bigg|\sum_{1\le i_1<\cdots<i_{m-1}<i_m}V_{i_m}^{ta}(X_{i_1},\ldots,X_{i_m})\Bigg|\Bigg)\\
&\quad\le AM^{-1}\sum_{i_m=m}^{\infty}i_m^{-1/2}E\big(|h(X_{i_1},\ldots,X_{i_m})|\mathbf{1}_{(|h|>i_m^{3/2})}\big)\\
&\quad\le AM^{-1}\sum_{j=m}^{\infty}E\big(|h(X_1,\ldots,X_m)|\mathbf{1}_{(j^{3/2}<|h|\le(j+1)^{3/2})}\big)\sum_{i_m=m}^{j}i_m^{-1/2}\\
&\quad\le AM^{-1}E\big|h(X_1,\ldots,X_m)\big|^{4/3}<\infty.
\end{aligned}
$$

This completes the proof of (4.11) and that of (4.9).

Now we give the proof of (4.10). Similarly to what we had in the proof of (4.9), we write

$$
\begin{aligned}
&\sum_{1\le i_1<\cdots<i_m\le K}g_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})\\
&\quad=\sum_{1\le i_1<\cdots<i_m\le K}\Bigg\{\sum_{d=2}^{m}(-1)^{m-d}\\
&\qquad\qquad\times\sum_{1\le j_1<\cdots<j_d\le m}E\big(g_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})|X_{i_{j_1}},\ldots,X_{i_{j_d}}\big)\\
&\qquad+\sum_{c=2}^{m-1}\sum_{1\le k_1<\cdots<k_c\le m}\sum_{d=2}^{c}(-1)^{c-d}\sum_{1\le j_1<\cdots<j_d\le c}E\big(g_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})|\\
&\qquad\qquad X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}\big)\Bigg\}\\
&\quad:=\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})\\
&\qquad+\sum_{c=2}^{m-1}\sum_{1\le k_1<\cdots<k_c\le m}\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_{k_1}},\ldots,X_{i_{k_c}}).
\end{aligned}
$$

Therefore, to establish (4.10) we need to show that

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})=O_P(1) \tag{4.13}$$

and for, $m\ge 3$, $c=2,\ldots,m-1$ and $1\le k_1<\cdots<k_c\le m$,

$$\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_{k_1}},\ldots,X_{i_{k_c}})=O_P(1). \tag{4.14}$$

To prove (4.13), having the martingale property (cf. Remark 9), we apply Chow's maximal inequality for martingales (cf. Chow (1960)) and Proposition 2, for $M>0$ we write

$$\begin{aligned}
&P\Big(\max_{m\le K\le n}\frac{K^{1/2}}{\binom{K}{m}}\Big|\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})\Big|>M\Big)\\
&\le M^{-2}\frac{n}{\binom{n}{m}^2}E\Big(\sum_{1\le i_1<\cdots<i_m\le n}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})\Big)^2\\
&\quad+M^{-2}\sum_{K=m}^{n-1}\Big(\frac{K}{\binom{K}{m}^2}-\frac{K+1}{\binom{K+1}{m}^2}\Big)E\Big(\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_m})\Big)^2\\
&\le AM^{-2}\frac{n}{\binom{n}{m}^2}\sum_{i_m=m}^{n}\binom{i_m}{m-1}E\big(h^2(X_1,\ldots,X_{m-1},X_{i_m})\mathbf{1}_{(|h|\le i_m^{3/2})}\big)\\
&\quad+AM^{-2}\sum_{K=m}^{n-1}\frac{2m}{\binom{K}{m}^2}\sum_{i_m=m}^{K}\binom{i_m}{m-1}E\big(h^2(X_1,\ldots,X_{m-1},X_{i_m})\mathbf{1}_{(|h|\le i_m^{3/2})}\big)\\
&\le AM^{-2}\sum_{K=m}^{\infty}K^{-2}E\big(h^2(X_1,\ldots,X_{m-1},X_{i_m})\mathbf{1}_{(|h|\le K^{3/2})}\big)\\
&\le AM^{-2}\sum_{i=1}^{\infty}i^{-1}E\big(h^2(X_1,\ldots,X_m)\mathbf{1}_{((i-1)^{3/2}<|h|\le i^{3/2})}\big)\\
&\le AM^{-2}E\big|h(X_1,\ldots,X_m)\big|^{4/3}<\infty.
\end{aligned}$$

Now the proof of (4.13) is complete.

Due to similarity, to establish (4.14), we shall only state the proof for $k_1=1,\ldots,k_c=c$, where $c=2,\ldots,m-1$ and $m\ge 3$. But first observe that

$$\begin{aligned}
&\sum_{1\le i_1<\cdots<i_m\le K}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c})\\
&\qquad=\binom{K-m+c}{m-c-1}\sum_{i_m=m}^{K}\sum_{1\le i_1<\cdots<i_c<i_m}V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c})
\end{aligned}$$

$$\leq \binom{K}{m-c-1} \sum_{i_m=m}^{K} \sum_{1\leq i_1<\cdots<i_c<i_m} V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c}).$$

Once again an application of Chow's maximal inequality followed by an application of Proposition 2, yield

$$P\biggl(\max_{m\leq K\leq n} \frac{K^{1/2}}{\binom{K}{m}}\biggl|\sum_{1\leq i_1<\cdots<i_m\leq K} V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c})\biggr| > M\biggr)$$

$$\leq M^{-2}\biggl\{\frac{n}{\binom{n}{m}^2} E\biggl(\binom{n}{m-c-1}\sum_{i_m=m}^{n}\sum_{1\leq i_1<\cdots<i_c<i_m} V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c})\biggr)^2$$

$$+ \sum_{K=m}^{n-1}\biggl(\frac{K}{\binom{K}{m}^2} - \frac{K+1}{\binom{K+1}{m}^2}\biggr)$$

$$\times E\biggl(\sum_{i_m=m}^{K}\sum_{1\leq i_1<\cdots<i_c<i_m}\binom{K}{m-c-1} V_{i_m}^{tr}(X_{i_1},\ldots,X_{i_c})\biggr)^2\biggr\}$$

$$\leq AM^{-2}E\bigl|h(X_1,\ldots,X_m)\bigr|^{4/3} < \infty.$$

The latter relation completes the proof of (4.14) and that of Proposition 4. Hence, $I_1(n,t) = o_P(1)$.

By virtue of our notion of $I_1(n,t)$ and $I_2(n,t)$, so far, we have shown that $I_1(n,t) = o_P(1)$. To complete the proof of Proposition 3, we need to show that $I_2(n,t) = o_P(1)$. But observe that for $I_2(n,t)$ we can write

$$I_2(n,t) \leq \frac{n^{-1/2}}{\ell(n)} \sup_{\delta\leq t\leq 1} \frac{[nt]}{\binom{[nt]}{m}}\biggl|\sum_{1\leq i_1<\cdots<i_m\leq[nt]} \bigl(h(X_{i_1},\ldots,X_{i_m}) - \theta$$

$$- \tilde{h}_1(X_{i_1}) - \cdots - \tilde{h}_1(X_{i_m})\bigr)\biggr|$$

$$\times \sup_{\delta\leq t\leq 1} \frac{1}{q(t)}.$$

And since $\sup_{\delta\leq t\leq 1} \frac{1}{q(t)} = O(1)$, in order to show that $I_2(n,t) = o_P(1)$, we only need to show that

$$\frac{n^{-1/2}}{\ell(n)} \sup_{m/n\leq t\leq 1}\biggl|\frac{[nt]}{\binom{[nt]}{m}}\sum_{1\leq i_1<\cdots<i_m\leq[nt]} \bigl(h(X_{i_1},\ldots,X_{i_m})$$

$$- \tilde{h}_1(X_{i_1}) - \cdots - \tilde{h}_1(X_{i_m})\bigr)\biggr| \tag{4.15}$$

$$= o_P(1).$$

The proof of the preceding is similar to that of (4) of Nasari (2009). Hence, the detailed proof is omitted. Now the proof of Theorem 3 is complete. □

Proof of Theorem 4. To prove Theorem 4, it suffices to show that as $n \to \infty$,

$$\left|(n-1)\sum_{i=1}^{n}(U_{n-1}^{i}-U_n)^2-\frac{m^2}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\right|=o_P(1). \tag{4.16}$$

Before giving the proof of (4.16), we do some simplifications as follows.

$$\begin{aligned}
&(n-1)\sum_{i=1}^{n}(U_{n-1}^{i}-U_n)^2\\
&\quad=(n-1)\sum_{i=1}^{n}\Bigg(\frac{\binom{n}{m}}{\binom{n-1}{m}}U_n\\
&\qquad\qquad-\frac{1}{\binom{n-1}{m}}\sum_{\substack{1\le j_1<\cdots<j_{m-1}\le n\\ j_1,\ldots,j_{m-1}\neq i}}h(X_i,X_{j_1},\ldots,X_{j_{m-1}})-U_n\Bigg)^2\\
&\quad=\frac{m^2(n-1)}{(n-m)^2}\sum_{i_1=1}^{n}\Bigg(\frac{1}{\binom{n-1}{m-1}}\sum_{\substack{1\le i_2<\cdots<i_m\le n\\ i_2,\ldots,i_m\neq i_1}}h(X_{i_1},X_{i_2},\ldots,X_{i_m})-U_n\Bigg)^2 \qquad (4.17)\\
&\quad=\frac{m^2(n-1)}{(n-m)^2}\sum_{i_1=1}^{n}\Bigg(\frac{1}{\binom{n-1}{m-1}}\sum_{\substack{1\le i_2<\cdots<i_m\le n\\ i_2,\ldots,i_m\neq i_1}}h(X_{i_1},X_{i_2},\ldots,X_{i_m})\Bigg)^2\\
&\qquad-\frac{m^2n(n-1)}{(n-m)^2}U_n^2. \qquad (4.18)
\end{aligned}$$

Remark 10. In view of (4.17) in what will follow without loss of generality, we may and shall assume that $Eh(X_1,\ldots,X_m)=\theta=0$.

By virtue of (4.18) to prove (4.16), it will be enough to prove the following two propositions.

Proposition 5. *If $E|h(X_1,\ldots,X_m)|^{5/3}<\infty$, then, as $n\to\infty$,*

$$U_n^2 \longrightarrow 0 \qquad a.s.$$

Proof. The proof of this proposition follows from the SLLN for U-statistics (cf., e.g., Serfling (1980)). □

Proposition 6. *If* $E|h(X_1, \dots, X_m)|^{5/3} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$, *then, as* $n \to \infty$,

$$\left| \frac{(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \Bigg(\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \cdots < i_m \le n \\ i_2, \dots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \dots, X_{i_m}) \Bigg)^2 - \frac{1}{n} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \right|$$
$$= o_P(1).$$

Let $a_n \sim b_n$ stand for the asymptotic equivalency of the numerical sequences $(a_n)_n$ and $(b_n)_n$, that is, as $n \to \infty$, $\frac{a_n}{b_n} \to 1$.

To prove Proposition 6, observe that

$$\begin{aligned}
&\frac{(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \Bigg(\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \cdots < i_m \le n \\ i_2, \dots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \dots, X_{i_m}) \Bigg)^2 \\
&\quad = \frac{(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \Bigg([n-1]^{-m+1} \sum_{\substack{1 \le i_2 \ne \cdots \ne i_m \le n \\ i_2, \dots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \dots, X_{i_m}) \Bigg)^2 \\
&\quad \sim [n]^{-2m+1} \sum_{i_1=1}^{n} \Bigg(\sum_{\substack{1 \le i_2 \ne \cdots \ne i_m \le n \\ i_2, \dots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \dots, X_{i_m}) \Bigg)^2 \\
&\quad = [n]^{-2m+1} \sum_{1 \le i_1 \ne \cdots \ne i_m \le n} h^2(X_{i_1}, \dots, X_{i_m}) \\
&\qquad + [n]^{-2m+1} \sum_{j=2}^{m-1} \sum_{1 \le i_1 \ne \cdots \ne i_{2m-j} \le n} h(X_{i_1}, \dots, X_{i_j}, X_{i_{j+1}}, \dots, X_{i_m}) \\
&\qquad\qquad \times h(X_{i_1}, \dots, X_{i_j}, X_{i_{m+1}}, \dots, X_{i_{2m-j}}) \\
&\qquad + [n]^{-2m+1} \sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h(X_{i_1}, X_{i_2}, \dots, X_{i_m}) \\
&\qquad\qquad \times h(X_{i_1}, X_{i_{m+1}}, \dots, X_{i_{2m-1}}).
\end{aligned}$$

The first term and the second one, which obviously does not appear when $m = 2$, in the latter equality will be seen to be negligible in probability (cf. Propositions 7 and 8), thus the third term becomes the main term that will play the main role in establishing Proposition 6.

To complete the proof of Proposition 6, we shall state and prove the next three results, namely Propositions 7, 8 and Theorem 5.

Proposition 7. *If* $E|h(X_1,\ldots,X_m)|^{5/3}<\infty$, *then, as* $n\to\infty$,

$$[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_m\le n} h^2(X_{i_1},\ldots,X_{i_m})\to 0 \qquad a.s.$$

From the fact that for $m\ge 2$, $\frac{2m}{2m-1}<\frac{5}{3}$, it follows that

$$\mathbb{E}\big|h^2(X_1,\ldots,X_m)\big|^{m/(2m-1)}=\mathbb{E}\big|h(X_1,\ldots,X_m)\big|^{2m/(2m-1)}<\infty.$$

By this the proof of Proposition 7 follows from Theorem 1 of Giné and Zinn (1992).

Proposition 8. *For* $m\ge 3$, *If* $E|h(X_1,\ldots,X_m)|^{5/3}<\infty$, *then, as* $n\to\infty$,

$$[n]^{-2m+1}\sum_{j=2}^{m-1}\sum_{1\le i_1\ne\cdots\ne i_{2m-j}\le n} h(X_{i_1},\ldots,X_{i_j},X_{i_{j+1}},\ldots,X_{i_m}) \times h(X_{i_1},\ldots,X_{i_j},X_{i_{m+1}},\ldots,X_{i_{2m-j}})=o_P(1).$$

Proof. In order to prove Proposition 8, it suffices to show that as $n\to\infty$, for $j=2,\ldots,m-1$, we have

$$[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-j}\le n} h(X_{i_1},\ldots,X_{i_j},X_{i_{j+1}},\ldots,X_{i_m}) \times h(X_{i_1},\ldots,X_{i_j},X_{i_{m+1}},\ldots,X_{i_{2m-j}})=o_P(1).$$

Since the proof of the latter relation can be done by modifying, mutatis mutandis, that of the next theorem, that is, Theorem 5, hence the detailed proof is omitted. □

Theorem 5. *If* $E|h(X_1,\ldots,X_m)|^{5/3}<\infty$ *and* $\tilde{h}_1(X_1)\in DAN$, *then, as* $n\to\infty$,

$$\Bigg|[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} h(X_{i_1},X_{i_2},\ldots,X_{i_m})h(X_{i_1},X_{i_{m+1}},\ldots,X_{i_{2m-1}}) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\Bigg|=o_P(1).$$

Proof. Here we make use of the property of complete degeneracy of functions (summands) even though, here they will not be symmetric.

For further use in this proof, we consider the following truncation setup:

$$h_{1,\ldots,m}:=h(X_1,\ldots,X_m),$$
$$h^{(m)}_{1,\ldots,m}:=h_{1,\ldots,m}\mathbf{1}_{(|h|\le n^{3m/5})},$$

$$\begin{aligned}
h^*_{12,\ldots,2m-1} &:= h^{(m)}_{12,\ldots,m} h^{(m)}_{1m+1,\ldots,2m-1},\\
\tilde{h}^{(m)}_1(x) &:= E\big(h^{(m)}_{1,\ldots,m} | X_1 = x\big),\\
h^{(j)}_{1,\ldots,m} &:= h^{(m)}_{1,\ldots,m} \mathbf{1}_{(|h^{(m)}| \le n^{3j/5})}, \qquad j = 1, \ldots, m-1,\\
h^{(0)}_{1,\ldots,m} &:= h^{(m)}_{1,\ldots,m} \mathbf{1}_{(|h^{(m)}| \le \log(n))},\\
h^{(\ell)}_{1,\ldots,m} &:= h^{(m)}_{1,\ldots,m} \mathbf{1}_{(|\tilde{h}^{(m)}_1(x)| \le n^{1/2}\ell(n))},
\end{aligned}$$

where, again, $\mathbf{1}_A$ denotes the indicator function of the set A and $\ell(\cdot)$ is a slowly varying function at infinity associated to $\tilde{h}_1(X_1)$.

In view of the above set up, observe that as $n \to \infty$

$$\begin{aligned}
&P\Bigg(\sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h_{i_1 i_2,\ldots,i_m} h_{i_1 i_{m+1},\ldots,i_{2m-1}} \ne \sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h^*_{i_1,\ldots,i_{2m-1}} \Bigg)\\
&\quad \le n^m P\big(|h_{1,\ldots,m}| > n^{3m/5}\big) \le E\big[|h_{1,\ldots,m}|^{5/3} 1_{(|h_{1,\ldots,m}| > n^{3m/5})}\big] \longrightarrow 0.
\end{aligned}$$

Hence, the asymptotic equivalency of $\sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h_{i_1 i_2,\ldots,i_m} h_{i_1 i_{m+1},\ldots,i_{2m-1}}$ and its truncated version, that is, $\sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h^*_{i_1,\ldots,i_{2m-1}}$ in probability.

Having the asymptotic equivalency of the original statistic and its truncated version, to prove Theorem 5, we shall proceed by working with the truncated version. Noting that due to lack of symmetry, the statistic of our interest, that is, $\sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h^*_{i_1,\ldots,i_{2m-1}}$, is not a U-statistic, once again here, we extend the idea of Hoeffding procedure to represent U-statistics in terms of complete degenerate ones in our context. This extension shall be done by creating complete degenerate sums by adding and subtracting required terms. Then by employing proper new truncations and applying Proposition 1 we conclude the asymptotic negligibility of all of these *complete degenerate* sums in probability (cf. Propositions 9, 10 and 11) except for the last group of them which are of the form of sums of i.i.d. random variables. Among those the latter mentioned just one (cf. part (b) of Proposition 12) will asymptotically in probability coincide with $\frac{1}{n}\sum_{i=1}^n \tilde{h}^2_1(X_i)$ and that will complete the proof of Theorem 5.

Now as it was already mentioned, by adding and subtracting required terms we write

$$\begin{aligned}
&\sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} h^*_{i_1,\ldots,i_{2m-1}}\\
&\quad = \sum_{1 \le i_1 \ne \cdots \ne i_{2m-1} \le n} \Bigg\{ \sum_{d=1}^{2m-1} (-1)^{2m-1-d}\\
&\qquad\qquad \times \sum_{1 \le j_1 < \cdots < j_d \le 2m-1} E\big(h^*_{i_1,\ldots,i_{2m-1}} - E\big(h^*_{i_1,\ldots,i_{2m-1}}\big)|
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad X_{i_{j_1}}, \ldots, X_{i_{j_d}}\big) \\
&\quad + \sum_{c=1}^{2m-2} \sum_{1\le k_1<\cdots<k_c\le 2m-1} \sum_{d=1}^{c} (-1)^{c-d} \\
&\quad\times \sum_{1\le j_1<\cdots<j_d\le c} E\big(h^*_{i_1,\ldots,i_{2m-1}} - E\big(h^*_{i_1,\ldots,i_{2m-1}}\big)\big| \\
&\qquad\qquad X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}\big) \\
&\quad + E\big(h^*_{i_1,\ldots,i_{2m-1}}\big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
:= &\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1}) \\
&+ \sum_{c=1}^{2m-2} \sum_{1\le k_1<\cdots<k_c\le 2m-1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_{k_1},\ldots,i_{k_c}) \\
&+ \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} E\big(h^*_{i_1,\ldots,i_{2m-1}}\big).
\end{aligned}
$$

Proposition 9. *If $E|h_{1,\ldots,m}|^{5/3}<\infty$, then, as $n\to\infty$,*

$$
[n]^{-2m+1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1}) = o_P(1).
$$

Proof. Since $V(i_1,\ldots,i_{2m-1})$ posses the property of complete degeneracy, we can apply Proposition 1 for the associated statistics and write, for $\varepsilon>0$,

$$
\begin{aligned}
P&\Bigg(\Bigg|[n]^{-2m+1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1})\Bigg| > \varepsilon\Bigg) \\
&\le \varepsilon^{-2} E\Bigg[[n]^{-2m+1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1})\Bigg]^2 \\
&\le \varepsilon^{-2}[n]^{-2m+1} E\big[V(1,\ldots,2m-1)\big]^2 \\
&\le A\varepsilon^{-2}[n]^{-2m+1} n^{2m-1} n^{-2m+1} E\big[h^{(m)}_{12,\ldots,m} h^{(m)}_{1m+1,\ldots,2m-1}\big]^2 \\
&\le A\varepsilon^{-2}[n]^{-2m+1} n^{7m/5} E|h_{12,\ldots,m}|^{5/3} \\
&\longrightarrow 0, \qquad \text{as } n\to\infty.
\end{aligned}
$$

The estimation for $m\ge 3$ that occurs in our next proposition does not appear, and hence not needed, when $m=2$. □

Proposition 10. *For $m \geq 3$, if $E|h_{12,\dots,m}|^{5/3} < \infty$, then, as $n \to \infty$*

$$[n]^{-2m+1} \sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V(i_{k_1}, \dots, i_{k_c}) = o_P(1),$$

where $c = 3, \dots, 2m-2$ and $1 \leq k_1 < \cdots < k_c \leq 2m-1$.

Proof. Based on the way $i_{k_1}, \dots, i_{k_c}$ are distributed between $h^{(m)}_{i_1 i_2,\dots,i_m}$ and $h^{(m)}_{i_1 i_{m+1},\dots,i_{2m-1}}$ in two different cases when $k_1 = 1$ and $k_1 \neq 1$, the proof is stated as follows.

Case $k_1 = 1$. Let s and t be respectively, the number of elements of the sets

$$\{i_{k_1}, \dots, i_{k_c}\} \cap \{i_1, i_2, \dots, i_m\} \quad \text{and} \quad \{i_{k_1}, \dots, i_{k_c}\} \cap \{i_1, i_{m+1}, \dots, i_{2m-1}\}.$$

It is clear that in this case, that is, $k_1 = 1$, we have that $s, t \geq 1$ and $s + t = c + 1$. Now define

$$V^T(i_{k_1}, \dots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \cdots < j_d \leq c} E\big(h^{*^T}_{i_1,\dots,i_{2m-1}} - E\big(h^{*^T}_{i_1,\dots,i_{2m-1}}\big) | X_{i_{k_{j_1}}}, \dots, X_{i_{k_{j_d}}}\big), \tag{4.19}$$

$$V^{T'}(i_{k_1}, \dots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \cdots < j_d \leq c} E\big(h^{*^{T'}}_{i_1,\dots,i_{2m-1}} - E\big(h^{*^{T'}}_{i_1,\dots,i_{2m-1}}\big) | X_{i_{k_{j_1}}}, \dots, X_{i_{k_{j_d}}}\big), \tag{4.20}$$

where $h^{*^T}_{i_1,\dots,i_{2m-1}} = h^{(s)}_{i_1 i_2,\dots,i_m} h^{(m)}_{i_1 i_{m+1},\dots,i_{2m-1}}$ and $h^{*^{T'}}_{i_1,\dots,i_{2m-1}} = h^{(s)}_{i_1 i_2,\dots,i_m} \times h^{(t)}_{i_1 i_{m+1},\dots,i_{2m-1}}$.

Now observe that as, $n \to \infty$,

$$\begin{aligned}
&P\Big(\sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V(i_{k_1}, \dots, i_{k_c}) \neq \sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \dots, i_{k_c})\Big) \\
&\quad \leq P\Big(\sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V(i_{k_1}, \dots, i_{k_c}) \neq \sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V^T(i_{k_1}, \dots, i_{k_c})\Big) \\
&\qquad + P\Big(\sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V^T(i_{k_1}, \dots, i_{k_c}) \\
&\qquad\qquad \neq \sum_{1 \leq i_1 \neq \cdots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \dots, i_{k_c})\Big) \\
&\quad \leq n^s P\big(|h^{(m)}_{12,\dots,m}| > n^{3s/5}\big) + n^t P\big(|h^{(m)}_{1m+1,\dots,2m-1}| > n^{3t/5}\big) \\
&\quad \leq E\big[|h_{12,\dots,m}|^{5/3} \mathbf{1}_{(|h| > n^{3s/5})}\big] + E\big[|h_{1m+1,\dots,2m-1}|^{5/3} \mathbf{1}_{(|h| > n^{3t/5})}\big] \longrightarrow 0.
\end{aligned}$$

The latter relation suggests that

$$\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_{k_1},\dots,i_{k_c}) \quad \text{and} \quad \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},\dots,i_{k_c})$$

are asymptotically equivalent in probability.

Since $V^{T'}(i_{k_1},\dots,i_{k_c})$ is complete degenerate, Markov's inequality followed by an application of Proposition 1 yields,

$$\begin{aligned}
&P\bigg(\bigg|[n]^{-2m+1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},\dots,i_{k_c})\bigg| > \varepsilon\bigg) \\
&\quad\le \varepsilon^{-2} E\bigg([n]^{-2m+1} \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},\dots,i_{k_c})\bigg)^2 \\
&\quad\le A\varepsilon^{-2}\big[n-(2m-1-c)\big]^{-c} E\big(h^{(s)}_{12,\dots,m} h^{(t)}_{1m+1,\dots,2m-1}\big)^2 \\
&\quad\le A\varepsilon^{-2}\big[n-(2m-1-c)\big]^{-c} n^c n^{-c} n^{7(t+s)/10} E|h_{12,\dots,m}|^{5/3} \\
&\quad\longrightarrow 0, \qquad \text{as } n\to\infty.
\end{aligned}$$

The latter relation is true since when $c\ge 3$, we have $-c+\frac{7(t+s)}{10}<0$.

Case $k_1\ne 1$. Similarly to the previous case let s and t be, respectively, the number of elements of the sets $\{i_{k_1},\dots,i_{k_c}\}\cap\{i_1,i_2,\dots,i_m\}$ and $\{i_{k_1},\dots,i_{k_c}\}\cap\{i_1,i_{m+1},\dots,i_{2m-1}\}$. Clearly here we have $s,t\ge 0$ and $s+t=c$. It is obvious that in this case s,t can be zero but not simultaneously. More specifically, either $(s=c,t=0)$ or $(s=0,t=c)$ can happen and due to their similarity we shall only treat $(s=c,t=0)$.

Let $V^T(i_{k_1},\dots,i_{k_c})$ and $V^{T'}(i_{k_1},\dots,i_{k_c})$ be of the forms respectively, (4.19) and (4.20), where

$$h^{*T}_{i_1,\dots,i_{2m-1}} = h^{(s)}_{i_1 i_2,\dots,i_m} h^{(m)}_{i_1 i_{m+1},\dots,i_{2m-1}}$$

and

$$h^{*T'}_{i_1,\dots,i_{2m-1}} = h^{(s)}_{i_1 i_2,\dots,i_m} h^{(t)}_{i_1 i_{m+1},\dots,i_{2m-1}}.$$

Observe that as $n\to\infty$,

$$\begin{aligned}
&P\bigg(\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_{k_1},\dots,i_{k_c}) \ne \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},\dots,i_{k_c})\bigg) \\
&\quad\le \begin{cases} n^s P\big(|h^{(m)}_{12,\dots,m}|>n^{3s/5}\big)+n^t P\big(|h^{(m)}_{1m+1,\dots,2m-1}|>n^{3t/5}\big), \\ \qquad s,t>0,\, s+t=c, \\ n^c P\big(|h^{(m)}_{12,\dots,m}|>n^{3c/5}\big)+P\big(|h^{(m)}_{1m+1,\dots,2m-1}|>\log(n)\big), \\ \qquad s=c,\, t=0 \end{cases}
\end{aligned}$$

$$\leq \begin{cases} E\big(|h_{12,\dots,m}|^{5/3}\mathbf{1}_{(|h|>n^{3s/5})}\big) + E\big(|h_{1m+1,\dots,2m-1}|^{5/3}\mathbf{1}_{(|h|>n^{3t/5})}\big), \\ \qquad s,t>0, s+t=c, \\ E\big[|h_{12,\dots,m}|^{5/3}\mathbf{1}_{(|h|>n^{3c/5})}\big] + P\big(|h^{(m)}_{1m+1,\dots,2m-1}| > \log(n)\big), \\ \qquad s=c, t=0 \end{cases}$$

$$\longrightarrow 0.$$

Applying Markov's inequality followed by an application of Proposition 1 once again yields,

$$P\Bigg(\Bigg|[n]^{-2m+1} \sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1},\dots,i_{k_c})\Bigg| > \varepsilon\Bigg)$$

$$\leq A\varepsilon^{-2}\big[n-(2m-1-c)\big]^{-c} n^c n^{-c} E\big(h^{(s)}_{12,\dots,m} h^{(t)}_{1m+1,\dots,2m-1}\big)^2$$

$$\leq \begin{cases} A\varepsilon^{-2}\big[n-(2m-1-c)\big]^{-c} n^{7c/10} E|h_{12,\dots,m}|^{5/3}, \\ \qquad s,t>0, s+t=c, \\ A\varepsilon^{-2}\big[n-(2m-1-c)\big]^{-c} n^{7c/10}\log^{7/6}(n) E|h_{12,\dots,m}|^{5/3}, \\ \qquad s=c, t=0 \end{cases}$$

$$\longrightarrow 0, \qquad \text{as } n\to\infty.$$

This completes the proof of Proposition 10. □

Proposition 11. *If $E|h_{12,\dots,m}|^{5/3}<\infty$ and $\tilde{h}_1(X_1)\in DAN$, then, as $n\to\infty$,*

$$[n]^{-2m+1} \sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V(i_{k_1},i_{k_2}) = o_P(1),$$

where, $1\leq k_1<k_2\leq 2m-1$.

Proof. As it was the case in the proof of the last proposition, we shall state the proof for two cases $k_1=1$ and $k_1\neq 1$ separately.

Case $k_1=1$. Again let s and t be respectively the number of elements of the sets $\{i_{k_1},i_{k_2}\}\cap\{i_1,i_2,\dots,i_m\}$ and $\{i_{k_1},i_{k_2}\}\cap\{i_1,i_{m+1},\dots,i_{2m-1}\}$. It is clear that in this case we either have $(s=2,t=1)$ or $(s=1,t=2)$ which due to their similarity only $(s=2,t=1)$ will be treated as follows.

Define

$$V^T(i_{k_1},i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d} \sum_{1\leq j_1<\cdots<j_d\leq 2} E\big(h^{*^T}_{i_1,\dots,i_{2m-1}} - E\big(h^{*^T}_{i_1,\dots,i_{2m-1}}\big)\big| X_{i_{k_{j_1}}},\dots,X_{i_{k_{j_d}}}\big),$$

$$V^{T'}(i_{k_1},i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d} \sum_{1\leq j_1<\cdots<j_d\leq 2} E\big(h^{*^{T'}}_{i_1,\dots,i_{2m-1}} - E\big(h^{*^{T'}}_{i_1,\dots,i_{2m-1}}\big)\big| X_{i_{k_{j_1}}},\dots,X_{i_{k_{j_d}}}\big),$$

where $h^{*^T}_{i_1,\dots,i_{2m-1}} = h^{(2)}_{i_1 i_2,\dots,i_m} h^{(m)}_{i_1 i_{m+1},\dots,i_{2m-1}}$ and $h^{*^{T'}}_{i_1,\dots,i_{2m-1}} = h^{(2)}_{i_1 i_2,\dots,i_m} \times h^{(\ell)}_{i_1 i_{m+1},\dots,i_{2m-1}}$.

Having the above setup, as $n \to \infty$, we have

$$\begin{aligned}
&P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V(i_{k_1}, i_{k_2}) \neq \sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T'}(i_{k_1}, i_{k_2})\Big)\\
&\quad \le P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V(i_{k_1}, i_{k_2}) \neq \sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T}(i_{k_1}, i_{k_2})\Big)\\
&\qquad + P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^T(i_{k_1}, i_{k_2}) \neq \sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T'}(i_{k_1}, i_{k_2})\Big)\\
&\quad \le n^2 P\big(|h^{(m)}_{12,\dots,m}| > n^{6/5}\big) + nP\big(|\tilde{h}^{(m)}_1(X_1)| > n^{1/2}\ell(n)\big)\\
&\quad \le E\big(|h_{12,\dots,m}|^{5/3}\mathbf{1}_{(|h|>n^{6/5})}\big) + nP\big(|\tilde{h}^{(m)}_1(X_1)| > n^{1/2}\ell(n)\big)\\
&\quad := I_1(n) + I_2(n).
\end{aligned}$$

It can be easily seen that as n tends to infinity $I_1(n) \to 0$.

To deal with $I_2(n)$, we write

$$\begin{aligned}
&nP\big(|\tilde{h}^{(m)}_1(X_1)| > n^{1/2}\ell(n)\big)\\
&\quad \le nP\Big(\big|\tilde{h}_1(X_1)\big| > \frac{n^{1/2}\ell(n)}{2}\Big)\\
&\qquad + nP\Big(\big|E(h_{1m+1,\dots,2m-1}\mathbf{1}_{(|h|>n^{3m/5})}|X_1)\big| > \frac{n^{1/2}\ell(n)}{2}\Big)\\
&\quad \le nP\Big(\big|\tilde{h}_1(X_1)\big| > \frac{n^{1/2}\ell(n)}{2}\Big)\\
&\qquad + 2n^{1/2}\ell^{-1}(n)E\big(|h_{1m+1,\dots,2m-1}|\mathbf{1}_{(|h|>n^{3m/5})}\big)\\
&\quad \le nP\Big(\big|\tilde{h}_1(X_1)\big| > \frac{n^{1/2}\ell(n)}{2}\Big)\\
&\qquad + 2n^{1/2}n^{-2m/5}\ell^{-1}(n)E|h_{1m+1,\dots,2m-1}|^{5/3}\\
&\quad \longrightarrow 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

The latter relation is true since $\tilde{h}_1(X_1) \in DAN$ and $m \ge 2$, and it means that $I_2(n) = o(1)$. Hence the asymptotic equivalency of $\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V(i_{k_1}, i_{k_2})$ and $\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T'}(i_{k_1}, i_{k_2})$ in probability.

Before applying Proposition 1 for $[n]^{-2m+1}\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T'}(i_{k_1}, i_{k_2})$, since we know that $k_1 = 1$ and $s = 2$, due to symmetry of $h_{i_1 i_2,\dots,i_m}$, without loss of generality we assume that $k_2 = 2$.

Now for $\varepsilon > 0$, Markov's inequality and Proposition 1 lead to

$$
\begin{aligned}
&P\Big(\Big|[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_1,i_2)\Big| > \varepsilon\Big)\\
&\quad\le A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}\\
&\qquad\times E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1} - E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}\big)|X_1,X_2\big)\big)^2\\
&\qquad+ A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}\\
&\qquad\quad\times E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1} - E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}\big)|X_1\big)\big)^2\\
&\qquad+ A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}\\
&\qquad\quad\times E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1} - E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}\big)|X_2\big)\big)^2\\
&\quad:= A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2J_1(n)\\
&\qquad+ A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2J_2(n)\\
&\qquad+ A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2J_3(n).
\end{aligned}
$$

Considering that as $n\to\infty$, $[n-(2m-3)]^{-2}n^2\to 1$, we will show that

$$J_1(n),\ J_2(n),\ J_3(n) = o(1).$$

To deal with $J_1(n)$ write

$$
\begin{aligned}
J_1(n) &\le n^{-2}E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}|X_1,X_2\big)\big)^2\\
&= n^{-2}E\big(E^2\big(h^{(2)}_{12,\ldots,m}|X_1,X_2\big)E^2\big(h^{(\ell)}_{1m+1,\ldots,2m-1}|X_1\big)\big)\\
&= n^{-2}E\big(E^2\big(h^{(2)}_{12,\ldots,m}|X_1,X_2\big)E^2\big(h^{(m)}_{1m+1,\ldots,2m-1}|X_1\big)\mathbf{1}_{(|\tilde h^{(m)}_1(X_1)|\le n^{1/2}\ell(n))}\big)\\
&\le n^{-1}\ell^2(n)E\big(h^{(2)}_{12,\ldots,m}\big)^2\\
&\le n^{-3/5}\ell^2(n)E|h_{12,\ldots,m}|^{5/3}\\
&\longrightarrow 0, \qquad \text{as } n\to\infty,
\end{aligned}
$$

that is, $J_1(n) = o(1)$. A similar argument yields, $J_2(n) = o(1)$, hence the details are omitted.

As for $J_3(n)$ we write

$$
\begin{aligned}
J_3(n) &\le n^{-2}E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}|X_2\big)\big)^2\\
&= n^{-2}E\big\{E\big(E\big(h^{(2)}_{12,\ldots,m}h^{(\ell)}_{1m+1,\ldots,2m-1}|X_1,\ldots,X_m\big)|X_2\big)\big\}^2\\
&= n^{-2}E\big\{E\big(h^{(2)}_{12,\ldots,m}|X_2\big)E\big(h^{(\ell)}_{1m+1,\ldots,2m-1}|X_1\big)\big\}^2
\end{aligned}
$$

$$\leq \; n^{-3/5}\ell^2(n)E|h_{12,\ldots,m}|^{5/3}$$
$$\longrightarrow 0, \qquad \text{as } n \to \infty.$$

The latter relation means that $J_3(n) = o(1)$. By this the proof of Proposition 11, when $k_1 = 1$, is complete.

At this stage, we state the proof of Proposition 11, when $k_1 \neq 1$.

Case $k_1 \neq 1$. Once again let s and t be respectively, the number of elements of the sets $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_2, \ldots, i_m\}$ and $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_{m+1}, \ldots, i_{2m-1}\}$. It is obvious that in this case the possibilities are $s = t = 1$ and when $m \geq 3$, $(s = 2, t = 0)$ or $(s = 0, t = 2)$. We shall treat the cases $s = t = 1$ and when $m \geq 3$, $(s = 2, t = 0)$, separately as follows.

Case $k_1 \neq 1$: $s = t = 1$. We note that here we have $k_1 \in \{2, \ldots, m\}$ and $k_2 \in \{m+1, \ldots, 2m-1\}$.

Now define

$$V^T(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}$$
$$\times \sum_{1\leq j_1<\cdots<j_d\leq 2} E\big(h^{*^T}_{i_1,\ldots,i_{2m-1}} - E\big(h^{*^T}_{i_1,\ldots,i_{2m-1}}\big)|X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}\big),$$

$$V^{T'}(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}$$
$$\times \sum_{1\leq j_1<\cdots<j_d\leq 2} E\big(h^{*^{T'}}_{i_1,\ldots,i_{2m-1}} - E\big(h^{*^{T'}}_{i_1,\ldots,i_{2m-1}}\big)|X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}\big),$$

where $h^{*^T}_{i_1,\ldots,i_{2m-1}} = h^{(1)}_{i_1 i_2,\ldots,i_m} h^{(m)}_{i_1 i_{m+1},\ldots,i_{2m-1}}$ and $h^{*^{T'}}_{i_1,\ldots,i_{2m-1}} = h^{(1)}_{i_1 i_2,\ldots,i_m} \times h^{(1)}_{i_1 i_{m+1},\ldots,i_{2m-1}}$. Now observe that as $n \to \infty$, we have

$$P\Big(\sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V(i_{k_1}, i_{k_2}) \neq \sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1}, i_{k_2})\Big)$$
$$\leq P\Big(\sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V(i_{k_1}, i_{k_2}) \neq \sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V^{T}(i_{k_1}, i_{k_2})\Big)$$
$$+ P\Big(\sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V^T(i_{k_1}, i_{k_2}) \neq \sum_{1\leq i_1\neq\cdots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1}, i_{k_2})\Big)$$
$$\leq 2nP\big(|h^{(m)}_{12,\ldots,m}| > n^{3/5}\big)$$
$$\leq 2E\big[|h_{12,\ldots,m}|^{5/3}\mathbf{1}_{(|h|>n^{3/5})}\big]$$
$$\longrightarrow 0.$$

In view of the latter relation, we apply Proposition 1 to the sum

$$[n]^{-2m+1}\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})$$

and we get

$$\begin{aligned}
&P\Big([n]^{-2m+1}\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}\big|V^{T'}(i_{k_1},i_{k_2})\big|>\varepsilon\Big)\\
&\quad\le A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2n^{-2}E\big(h^{(1)}_{12,\ldots,m}h^{(1)}_{1m+1,\ldots,2m-1}\big)^2\\
&\quad\le A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2n^{-2}n^{7/5}E|h_{12,\ldots,m}|^{5/3}\\
&\quad\longrightarrow 0,\qquad \text{as } n\to\infty.
\end{aligned}$$

This completes the proof of Proposition 11 for the Case $k_1\neq 1$ when $s=t=1$.

Case $k_1\neq 1$: $(m\ge 3)$ $s=2, t=0$. In this case, we first note that $k_1,k_2\in\{2,\ldots,m\}$. Now define

$$\begin{aligned}
V^T(i_{k_1},i_{k_2})&=\sum_{d=1}^{2}(-1)^{2-d}\\
&\qquad\times\sum_{1\le j_1<\cdots<j_d\le 2}E\big(h^{*^T}_{i_1,\ldots,i_{2m-1}}-E\big(h^{*^T}_{i_1,\ldots,i_{2m-1}}\big)|X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}\big),\\
V^{T'}(i_{k_1},i_{k_2})&=\sum_{d=1}^{2}(-1)^{2-d}\\
&\qquad\times\sum_{1\le j_1<\cdots<j_d\le 2}E\big(h^{*^{T'}}_{i_1,\ldots,i_{2m-1}}-E\big(h^{*^{T'}}_{i_1,\ldots,i_{2m-1}}\big)|X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}\big),
\end{aligned}$$

where $h^{*^T}_{i_1,\ldots,i_{2m-1}} = h^{(2)}_{i_1i_2,\ldots,i_m}h^{(m)}_{i_1i_{m+1},\ldots,i_{2m-1}}$ and $h^{*^{T'}}_{i_1,\ldots,i_{2m-1}} = h^{(2)}_{i_1i_2,\ldots,i_m}\times h^{(0)}_{i_1i_{m+1},\ldots,i_{2m-1}}$. Now observe that as $n\to\infty$

$$\begin{aligned}
&P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V(i_{k_1},i_{k_2})\neq\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V^{T'}(i_{k_1},i_{k_2})\Big)\\
&\quad\le P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V(i_{k_1},i_{k_2})\neq\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V^{T}(i_{k_1},i_{k_2})\Big)\\
&\qquad+P\Big(\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V^T(i_{k_1},i_{k_2})\neq\sum_{1\le i_1\neq\cdots\neq i_{2m-1}\le n}V^{T'}(i_{k_1},i_{k_2})\Big)\\
&\quad\le n^2P\big(\big|h^{(m)}_{12,\ldots,m}\big|>n^{6/5}\big)+P\big(\big|h^{(m)}_{1m+1,\ldots,2m-1}\big|>\log(n)\big)
\end{aligned}$$

$$\le E\big(|h_{12,\dots,m}|^{5/3}\mathbf{1}_{(|h|>n^{6/5})}\big) + P\big(|h_{1m+1,\dots,2m-1}| > \log(n)\big)$$
$$\longrightarrow 0.$$

The latter relation together with degeneracy of $V^{T'}(i_{k_1}, i_{k_2})$ enable us to use Proposition 1 once again and arrive at

$$P\bigg([n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n}\big|V^{T'}(i_{k_1}, i_{k_2})\big| > \varepsilon\bigg)$$
$$\le A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2n^{-2}E\big(h^{(2)}_{12,\dots,m}h^{(0)}_{1m+1,\dots,2m-1}\big)^2$$
$$\le A\varepsilon^{-2}\big[n-(2m-3)\big]^{-2}n^2n^{-3/5}\log^{7/6}(n)E|h_{12,\dots,m}|^{5/3}$$
$$\longrightarrow 0, \qquad \text{as } n\to\infty.$$

Now the proof of Proposition 11 is complete. □

Remark 11. Before stating our next result, we note in passing that when $k_1 = 1$ then, $[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_{k_1})$ is of the form

$$\big[n-(2m-2)\big]^{-1}\sum_{i_1\in\{1,\dots,n\}/\{2,\dots,2m-1\}}^{n} E\big(h^*_{i_12,\dots,2m-1} - E\big(h^*_{i_12,\dots,2m-1}\big)|X_{i_1}\big),$$

otherwise, that is, when, for example, $k_1 = 2$ it has the following form

$$\big[n-(2m-2)\big]^{-1}\sum_{i_2\in\{1,\dots,n\}/\{1,3,\dots,2m-1\}}^{n} E\big(h^*_{1i_23,\dots,2m-1} - E\big(h^*_{1i_23,\dots,2m-1}\big)|X_{i_2}\big),$$

and so on for $k_1 \in \{2,\dots,2m-1\}$.

Proposition 12. *If $E|h_{1,\dots,m}|^{5/3} < \infty$ and $\tilde{h}_1(X_1)\in DAN$, then, as $n\to\infty$*

(a) $$[n]^{-2m+1}\sum_{1\le\ne i_1\ne\cdots\ne i_{2m-1}\le n} V(i_{k_1}) = o_P(1) \qquad \textit{for } k_1\in\{2,\dots,2m-1\},$$

(b) $$\Bigg|\big[n-(2m-2)\big]^{-1}\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}^{n} E\big(h^*_{i2,\dots,2m-1} - E\big(h^*_{i2,\dots,2m-1}\big)|X_i\big) + E\big(h^*_{12,\dots,2m-1}\big) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\Bigg| = o_P(1).$$

Proof. First we give the proof of part (a). Due to similarities, we state the proof only for $k_1 = 2$.

Define

$$V^T(i_2) = E\big(h^{*^T}_{i_1i_2,\dots,i_{2m-1}} - E\big(h^{*^T}_{i_1i_2,\dots,i_{2m-1}}\big)|X_{i_2}\big),$$
$$V^{T'}(i_2) = E\big(h^{*^{T'}}_{i_1i_2,\dots,i_{2m-1}} - E\big(h^{*^{T'}}_{i_1i_2,\dots,i_{2m-1}}\big)|X_{i_2}\big),$$

where $h^{*^T}_{i_1 i_2,\dots,i_{2m-1}} = h^{(1)}_{i_1 i_2,\dots,i_m} h^{(m)}_{i_1 i_{m+1},\dots,i_{2m-1}}$ and $h^{*^{T'}}_{i_1 i_2,\dots,i_{2m-1}} = h^{(1)}_{i_1 i_2,\dots,i_m} \times h^{(0)}_{i_1 i_{m+1},\dots,i_{2m-1}}$. Again observe that as $n \to \infty$,

$$
\begin{aligned}
P\Big(&\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_2) \ne \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_2)\Big) \\
&\le P\Big(\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V(i_2) \ne \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T}(i_2)\Big) \\
&\quad + P\Big(\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T}(i_2) \ne \sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_2)\Big) \\
&\le nP\big(\big|h^{(m)}_{12,\dots,m}\big| > n^{3/5}\big) + P\big(\big|h^{(m)}_{1m+1,\dots,2m-1}\big| > \log(n)\big) \\
&\le E\big(|h_{12,\dots,m}|^{5/3}\mathbf{1}_{(|h|>n^{3/5})}\big) + P\big(|h_{1m+1,\dots,2m-1}| > \log(n)\big) \\
&\longrightarrow 0.
\end{aligned}
$$

An application of Markov's inequality yields

$$
\begin{aligned}
P\Bigg(&\Bigg|[n]^{-2m+1}\sum_{1\le i_1\ne\cdots\ne i_{2m-1}\le n} V^{T'}(i_2)\Bigg| > \varepsilon\Bigg) \\
&\le A\varepsilon^{-2}\big[n-(2m-2)\big]^{-1} n n^{-1} E\big(h^{(1)}_{12,\dots,m} h^{(0)}_{1m+1,\dots,2m-1}\big)^2 \\
&\le A\varepsilon^{-2}\big[n-(2m-2)\big]^{-1} n n^{-3/10}\log^{7/6}(n) E|h_{12,\dots,m}|^{5/3} \\
&\longrightarrow 0, \qquad \text{as } n\to\infty.
\end{aligned}
$$

This complete the proof of part (a).

In the final stage of our proofs, to prove part (b) first define

$$\tilde{h}^*(x) = E(h_{12,\dots,m}\mathbf{1}_{(|h|>n^{3m/5})}|X_1 = x)$$

and write

$$
\begin{aligned}
&\Bigg|\frac{1}{n-2m+2}\sum^{n}_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} E\big(h^*_{i2,\dots,2m-1} - E\big(h^*_{i2,\dots,2m-1}\big)|X_i\big) \\
&\quad + E\big(h^*_{12,\dots,2m-1}\big) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\Bigg| \\
&\quad = \Bigg|\frac{1}{n-2m+2}\sum^{n}_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} E\big(h^*_{i2,\dots,2m-1}|X_i\big) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\Bigg| \\
&\quad \le \Bigg|\frac{1}{n-2m+2}\sum^{n}_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} E\big(h^*_{i2,\dots,2m-1}|X_i\big)
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
&- \frac{1}{n-2m+2}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\Bigg| + \frac{2m-2}{n(n-2m+2)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i) \\
&\le \Bigg|\frac{1}{n-2m+2}\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}^{n} E\big(h^*_{i2,\dots,2m-1}|X_i\big) \\
&\quad - \frac{1}{n-2m+2}\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\tilde{h}_1^2(X_i)\Bigg| \\
&\quad + \frac{1}{n-2m+2}\sum_{i=2}^{2m-1}\tilde{h}_1^2(X_i) + \frac{2m-2}{n(n-2m+2)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i) \\
&= \frac{1}{n-2m+2}\Bigg|\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\big(-\tilde{h}^*(X_i)\big)\big(2\tilde{h}_1^{(m)}(X_i)+\tilde{h}^*(X_i)\big)\Bigg| \\
&\quad + \frac{1}{n-2m+2}\sum_{i=2}^{2m-1}\tilde{h}_1^2(X_i) + \frac{2m-2}{n(n-2m+2)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i) \\
&\le \frac{1}{n-2m+2}\Bigg(\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\tilde{h}_1^2(X_i)\Bigg)^{1/2} \\
&\quad \times \Bigg(\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\tilde{h}^{*2}(X_i)\Bigg)^{1/2} \\
&\quad + \frac{1}{n-2m+2}\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\tilde{h}^{*2}(X_i) + \frac{1}{n-2m+2}\sum_{i=2}^{2m-1}\tilde{h}_1^2(X_i) \\
&\quad + \frac{2m-2}{n(n-2m+2)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i).
\end{aligned}
$$

It is easy to see that as $n\to\infty$, we have $\frac{1}{n-2m+2}\sum_{i=2}^{2m-1}\tilde{h}_1^2(X_i)=o_P(1)$. Also in view of (4.1), that is, Raikov's theorem in the present context, we have

$$
\frac{2m-2}{n(n-2m+2)}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)=o_P(1), \qquad \text{as } n\to\infty.
$$

Hence, in view of (4.21), in order to complete the proof of part (b), it suffices to show that as $n\to\infty$,

$$
\frac{1}{n-2m+2}\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}}\tilde{h}^{*2}(X_i)=o_P(1).
$$

To prove the latter relation, we first use Markov's inequality and conclude that

$$
\begin{aligned}
&P\Bigl(\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h}^{*2}(X_i) > \varepsilon(n-2m+2)\Bigr)\\
&\quad\le \varepsilon^{-1/2}(n-2m+2)^{-1/2} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} E\bigl|\tilde{h}^{*2}(X_i)\bigr|^{1/2}\\
&\quad\le \varepsilon^{-1/2}(n-2m+2)^{1/2} E\bigl|\tilde{h}^{*}(X_1)\bigr|\\
&\quad\le \varepsilon^{-1/2}(n-2m+2)^{1/2} n^{-1/2} n^{1/2} E\bigl(|h_{12,\dots,m}|\mathbf{1}_{(|h|>n^{3m/5})}\bigr)\\
&\quad\le \varepsilon^{-1/2}(n-2m+2)^{1/2} n^{-1/2} E\bigl(|h_{12,\dots,m}|^{5/(6m)+1}\mathbf{1}_{(|h|>n^{3m/5})}\bigr)\\
&\quad\longrightarrow 0, \qquad \text{as } n\to\infty.
\end{aligned}
$$

The last relation is true since for $m \ge 2$, we have that $\frac{5}{6m} + 1 < \frac{5}{3}$, and this completes the proof of part (b) and those of Proposition 12 and Theorem 5. □

□

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