

A note on the infinite divisibility of a class of transformations of normal variables

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Abstract. This note examines the infinite divisibility of density-based transformations of normal random variables. We characterize a class of such transformations of normal variables which produces non-infinitely divisible distributions. We relate our result with some known skewing mechanisms.

1 Introduction

In recent years, the use of skewed distributions has attracted the attention of statisticians both from the applied and theoretical points of view, see, for example, <http://azzalini.stat.unipd.it/SN/list-publ.pdf> for the case of skew-normal distributions. Several skewing mechanisms have been proposed to obtain skewed distributions by transforming symmetric ones (Marshall and Olkin, 1997; Jones, 2004; Wang et al., 2004; Arellano-Valle et al., 2005). Thus, it is of interest per se to analyze theoretical properties of the transformed distributions. Here, we present a characterization (see Theorem 1 below) for a class of density-based transformations that produces non-infinitely divisible distributions when applied to the normal distribution. This characterization can be easily related to some skewing mechanisms. It is important to note that, although the method of proof used in this note is similar to the one used in Domínguez-Molina and Rocha-Arteaga (2007), our result is general enough to cover several known skewing mechanisms. It is worth mentioning that our result has implications in statistical modeling because it rules out the use of several skew-normal distributions in models defined in terms of infinitely divisible distributions (cf. Steutel et al., 1979). In particular, having an infinitely divisible distribution opens the door to the use of Lévy processes, which is a well-studied class of stochastic processes. For instance, applications of Lévy processes in finance have received great attention (see, e.g., Schoutens, 2003). However, Theorem 1 below points out that some attention has to be paid when dealing with the kind of skewing mechanisms studied in this note.

In Section 2, a general representation of density-based transformations proposed in Ferreira and Steel (2006) and its relationship with four skewing mechanisms is presented. Using this representation, in Section 3 we offer a sufficient

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condition on such density-based transformations which destroys the property of infinite divisibility of normal distributions. Our results partially extends the work of Domínguez-Molina and Rocha-Arteaga (2007), Kozubowski and Nolan (2008) and Abtahi et al. (2012). We specialize our characterization to some skewing mechanisms which are of interest in the statistical literature.

2 Skewing mechanisms

Let S and F be any two distribution functions on the real line and let P be a distribution function on $(0, 1)$, densities (which are assumed to exist) will be denoted by the corresponding lowercase letters. Ferreira and Steel (2006) show that for any pair of absolutely continuous distributions S and F with support on \mathbb{R} , there exists a distribution P such that $S = P \circ F$. This implies that the transformation from a random variable X , with distribution function F , to a random variable Y , with distribution function S , can be represented as a density-based transformation as follows

$$s(y|F, P) = f(y)p[F(y)], \quad y \in \mathbb{R}. \quad (2.1)$$

Ferreira and Steel (2006) prove that S is equal to F only when P is the uniform distribution on $(0, 1)$ and that, for a fixed F , it is possible to obtain any S by using an appropriate P . If the distribution F is symmetric and S is asymmetric, then S is said to be an asymmetric version of F generated by the skewing mechanism P (Ferreira and Steel, 2006). Several related skewing mechanisms have been proposed in the statistical literature. Next, we provide the relationship of four known skewing mechanisms with representation (2.1):

- (i) Skew-symmetric construction (Wang et al., 2004). Such transformation is defined as follows

$$\begin{aligned} s(y|F, P) &= 2f(y)\pi(y), \\ p(x) &= 2\pi(F^{-1}(x)), \end{aligned}$$

where π is a function that satisfies $0 \leq \pi(y) \leq 1$, $\pi(-y) = 1 - \pi(y)$ and $x \in [0, 1]$.

Particular cases of this transformation are the Hidden Truncation skewing mechanism (Arnold and Beaver, 2000) and Azzalini's skew-normal (Azzalini, 1985).

- (ii) Order Statistics (Jones, 2004). This transformation introduces two new parameters $\psi_1 > 0$ and $\psi_2 > 0$ as follows

$$\begin{aligned} s(y|F, P) &= [\beta(\psi_1, \psi_2)]^{-1} f(y)F(y)^{\psi_1-1}(1-F(y))^{\psi_2-1}, \\ p(x|\psi_1, \psi_2) &= [\beta(\psi_1, \psi_2)]^{-1} x^{\psi_1-1}(1-x)^{\psi_2-1}, \end{aligned}$$

where β denotes the beta function and $x \in [0, 1]$.

- (iii) Marshall–Olkin transformation (Marshall and Olkin, 1997). Such transformation is given through a positive parameter γ as follows

$$s(y|F, P) = \frac{\gamma f(y)}{[F(y) + \gamma(1 - F(y))]^2},$$

$$p(x|\gamma) = \frac{\gamma}{[x + \gamma(1 - x)]^2},$$

where $x \in [0, 1]$.

- (iv) Two-piece distributions (Arellano-Valle et al., 2005). This transformation consists of scaling by different factors, $a(\gamma)$ and $b(\gamma)$, either side of the symmetry point of a unimodal density f . If f is symmetric about 0, this transformation is given by

$$s(y|F, P) = \frac{2}{a(\gamma) + b(\gamma)} \left[f\left(\frac{y}{b(\gamma)}\right) I(y < 0) + f\left(\frac{y}{a(\gamma)}\right) I(y \geq 0) \right],$$

$$p(x|\gamma) = \frac{2}{a(\gamma) + b(\gamma)} \times \frac{f(F^{-1}(x)/b(\gamma)) I(x < 0.5) + f(F^{-1}(x)/a(\gamma)) I(x \geq 0.5)}{f(F^{-1}(x))},$$

with $x \in [0, 1]$, $a(\gamma)$ and $b(\gamma)$ are positive functions of the parameter $\gamma \in \Gamma$; where Γ depends on the choice of the functions $\{a(\gamma), b(\gamma)\}$.

This class of transformations includes the Inverse Scale Factors presented in Fernández and Steel (1998) and the ε -skew normal given in Mudholkar and Hutson (2000).

The skew-normals obtained with these skewing mechanisms have been used for modeling data presenting departures from symmetry in medicine, psychology, genetics, engineering, finance, among others (for a compendium of literature about applications of these models see, for example, <http://azzalini.stat.unipd.it/SN/list-publ.pdf>).

3 On infinite divisibility

The goal of this section is twofold. Firstly, we consider $f = \phi$ in (2.1), where ϕ denotes the normal density, then we derive a sufficient condition on the density p such that the skewed distribution S is not infinitely divisible. Secondly, we relate this result with the skewing mechanisms described in Section 2.

Theorem 1. *Let Φ and ϕ be the distribution and density functions of a standard normal variable, respectively. Consider the transformation given in (2.1), with $f = \phi$, $F = \Phi$ and p bounded almost everywhere. Under these conditions, we have that S is not infinitely divisible when p is not the density of a uniform variable on $(0, 1)$.*

Before we proceed to the proof of this theorem, we cite the following result which plays a key role in what follows.

Lemma 1 (Steutel and Van Harn, 2004, Corollary 9.9). *A non-degenerate infinitely divisible random variable X has a normal distribution if and only if it satisfies*

$$\limsup_{x \rightarrow \infty} \frac{-\log \mathbb{P}(|X| > x)}{x \log x} = \infty.$$

Proof of Theorem 1. Let us assume that S is infinitely divisible. By assumption, there exists a positive constant M such that $p \leq M$, almost everywhere. Note that for $y > 0$

$$\begin{aligned} S(-y|\Phi, P) &= \int_{-\infty}^{-y} \phi(t)p[\Phi(t)] dt \leq M \int_{-\infty}^{-y} \phi(t) dt = M\Phi(-y), \\ 1 - S(y|\Phi, P) &= \int_y^{\infty} \phi(t)p[\Phi(t)] dt \leq M \int_y^{\infty} \phi(t) dt = M[1 - \Phi(y)]. \end{aligned}$$

Then

$$S(-y|\Phi, P) + 1 - S(y|\Phi, P) \leq M[\Phi(-y) + 1 - \Phi(y)].$$

Therefore, for $y > 1$

$$-\frac{\log[S(-y|\Phi, P) + 1 - S(y|\Phi, P)]}{y \log(y)} \geq -\frac{\log[M[\Phi(-y) + 1 - \Phi(y)]]}{y \log(y)}.$$

Hence, taking limits on both sides we get

$$\begin{aligned} \limsup_{y \rightarrow \infty} -\frac{\log[S(-y|\Phi, P) + 1 - S(y|\Phi, P)]}{y \log(y)} \\ \geq \limsup_{y \rightarrow \infty} -\frac{\log[M[\Phi(-y) + 1 - \Phi(y)]]}{y \log(y)} = \infty, \end{aligned}$$

which together with the characterization of the normal distribution given in Lemma 1 implies that S is normal. This contradicts the fact that S is not normal as p is not the uniform density in $(0, 1)$, see Theorem 1 in Ferreira and Steel (2006). Therefore, S is not infinitely divisible. \square

Theorem 1 has the following immediate.

Corollary 1. *Skew normals obtained by the skew-symmetric construction, the Marshall–Olkin transformation and the Order Statistics transformation for $\psi_1 > 1$ and $\psi_2 > 1$, are non-infinitely divisible.*

Proof. It is enough to note that, in each case, the corresponding density p is bounded. \square

Domínguez-Molina and Rocha-Arteaga (2007) and Kozubowski and Nolan (2008) prove that, in particular, Azzalini’s skew normal is not infinitely divisible. Theorem 1 together with Corollary 1 extend this result to the family of skew-normal distributions obtained by the skew-symmetric construction (Wang et al., 2004), from which Azzalini’s skew-normal is a particular case. Another example of this is the skew-normal analyzed in Abtahi et al. (2012). Moreover, the skewed normal distribution obtained by any skewing mechanism which satisfies the condition given in Theorem 1 will lose the infinite divisibility property.

Note that, for the skewing mechanism that produces two-piece distributions, the corresponding p is not necessarily bounded. Thus, an ad hoc proof of the non-infinite divisibility of the skew-normals obtained with this sort of transformation is presented in the following

Theorem 2. *The two-piece skew normal is non-infinitely divisible for $a(\gamma) \neq b(\gamma)$.*

Proof. Let us assume that $a(\gamma) \neq b(\gamma)$. Jones (2006) proves that the elements of the class of two-piece distributions are reparameterizations of each other. Therefore, it is enough to prove the result for the particular choice $\{a(\gamma), b(\gamma)\} = \{1 - \gamma, 1 + \gamma\}$, $\gamma \in (-1, 1)$, analyzed in Arellano-Valle et al. (2005). Note that $a(\gamma) = b(\gamma)$ if and only if $\gamma = 0$, which corresponds to the symmetric normal which is infinitely divisible. In addition, the density s obtained with a particular γ , corresponds to reflecting the density s with parameter $-\gamma$ around 0. Hence, it is enough to prove the result for $-1 < \gamma < 0$.

Note that

$$S(y|\Phi, P) = (1 + \gamma)\Phi\left(\frac{y}{1 + \gamma}\right)I(y < 0) + \left[-\gamma + (1 - \gamma)\Phi\left(\frac{y}{1 - \gamma}\right)\right]I(y \geq 0).$$

Then, given that $-1 < \gamma < 0$ and for any $y > 0$ we have that

$$\begin{aligned} S(-y|\Phi, P) + 1 - S(y|\Phi, P) &= (1 + \gamma)\Phi\left(-\frac{y}{1 + \gamma}\right) \\ &\quad + 1 + \gamma - (1 - \gamma)\Phi\left(\frac{y}{1 - \gamma}\right) \\ &< 2(1 - \gamma)\Phi\left(-\frac{y}{1 - \gamma}\right) \\ &< 4\Phi\left(-\frac{y}{2}\right). \end{aligned}$$

Then, for $y > 1$ we have

$$\begin{aligned} -\frac{\log[S(-y|\Phi, P) + 1 - S(y|\Phi, P)]}{y \log(y)} &> -\frac{\log[4\Phi(-y/2)]}{y \log(y)} \\ &= -\frac{\log[2(\Phi(-y/2) + 1 - \Phi(y/2))]}{y \log(y)}. \end{aligned}$$

The result follows by taking limits in both sides of this expression and following the same reasoning as in the proof of Theorem 1. \square

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References

- Abtahi, A., Behboodian, J. and Sharafi, M. (2012). A general class of univariate skew distributions considering Stein's lemma and infinite divisibility. *Metrika* **75**, 193–206. [MR2887562](#)
- Arellano-Valle, R. B., Gómez, H. W. and Quintana, F. A. (2005). Statistical inference for a general class of asymmetric distributions. *Journal of Statistical Planning and Inference* **128**, 427–443. [MR2102768](#)
- Arnold, B. C. and Beaver, R. J. (2000). Hidden truncation models. *Sankhya A* **62**, 23–35. [MR1769732](#)
- Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics* **12**, 171–178. [MR0808153](#)
- Domínguez-Molina, J. A. and Rocha-Ortega, A. (2007). On the infinite divisibility of some skewed symmetric distributions. *Statistics & Probability Letters* **77**, 644–648. [MR2345031](#)
- Fernández, C. and Steel, M. F. J. (1998). On Bayesian modeling of fat tails and skewness. *Journal of the American Statistical Association* **93**, 359–371. [MR1614601](#)
- Ferreira, J. T. A. S. and Steel, M. F. J. (2006). A constructive representation of univariate skewed distributions. *Journal of the American Statistical Association* **101**, 823–829. [MR2256188](#)
- Jones, M. C. (2004). Families of distributions arising from distributions of order statistics (with discussion). *Test* **13**, 1–43. [MR2065642](#)
- Jones, M. C. (2006). A note on rescalings, reparametrizations and classes of distributions. *Journal of Statistical Planning and Inference* **136**, 3730–3733. [MR2284672](#)
- Kozubowski, T. J. and Nolan, J. P. (2008). Infinite divisibility of skew Gaussian and Laplace laws. *Statistics & Probability Letters* **78**, 654–660. [MR2409529](#)
- Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika* **84**, 641–652. [MR1603936](#)
- Mudholkar, G. S. and Hutson, A. D. (2000). The epsilon-skew-normal distribution for analyzing near-normal data. *Journal of Statistical Planning and Inference* **83**, 291–309. [MR1748020](#)

- Schoutens, W. (2003). *Levy Processes in Finance: Pricing Financial Derivatives*. New York: Wiley.
- Steutel, F. W. (1979). Infinite divisibility in theory and practice. *Scandinavian Journal of Statistics* **6**, 57–64.
- Steutel, F. W. and Van Harn, K. (2004). *Infinite Divisibility of Probability Distributions on the Real Line*. New York: Dekker. [MR2011862](#)
- Wang, J., Boyer, J. and Genton, M. C. (2004). A skew symmetric representation of multivariate distributions. *Statistica Sinica* **14**, 1259–1270. [MR2126352](#)

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