The Annals of Probability
2014, Vol. 42, No. 3, 1257–1284
DOI: 10.1214/12-AOP829

STATIONARY DISTRIBUTIONS FOR A CLASS OF GENERALIZED FLEMING–VIOT PROCESSES

BY KENJI HANDA

Saga University

We identify stationary distributions of generalized Fleming–Viot processes with jump mechanisms specified by certain beta laws together with a parameter measure. Each of these distributions is obtained from normalized stable random measures after a suitable biased transformation followed by mixing by the law of a Dirichlet random measure with the same parameter measure. The calculations are based primarily on the well-known relationship to measure-valued branching processes with immigration.

1. Introduction. In the study of population genetics models, it is of great importance to identify their stationary distributions. Such identifications provide us with basic information of possible equilibria of the models and are needed prior to quantitative discussions on statistical inference. Since [5, 14] and [1], theory of generalized Fleming–Viot processes has served as a new area to be cultivated and has been developed considerably. (See [2] for an exposition.) In view of such progress, it seems that we are in a position to explore the aforementioned problems for some appropriate subclass of those models. In this respect, it would be natural to think of the one-dimensional Wright–Fisher diffusion with mutation as a prototype. This celebrated process is prescribed by its generator

$$A := \frac{1}{2}x(1-x)\frac{d^2}{dx^2} + \frac{1}{2}[c_1(1-x) - c_2 x]\frac{d}{dx}, \qquad x \in [0,1], \tag{1.1}$$

where c_1 and c_2 are positive constants interpreted as mutation rates. The stationary distribution is a beta distribution

$$B_{c_1,c_2}(dx) := \frac{\Gamma(c_1+c_2)}{\Gamma(c_1)\Gamma(c_2)}x^{c_1-1}(1-x)^{c_2-1}\,dx, \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function. In addition, the process associated with (1.1) admits an infinite-dimensional generalization known as the Fleming–Viot process with parent-independent mutation, whose stationary distribution is identified with the law of a Dirichlet random measure.

In the present paper, we consider a problem of finding a class of generalized Fleming–Viot processes whose stationary distributions can be identified. As far as

Received May 2012; revised November 2012.

MSC2010 subject classifications. Primary 60J75; secondary 60G57.

Key words and phrases. Generalized Fleming–Viot process, measure-valued branching process, stable random measure, Dirichlet random measure.

the first term on the right-hand side of (1.1) is concerned, its jump-type version has been discussed in population genetics as the generator of a model with "occasional extreme reproduction". (See Section 1.2 of [2] for a comprehensive account.) We additionally need to look for an appropriate modification of the second term, which should correspond to a generalization of the mutation mechanism. With these situations in mind, our problems can be described as follows.

(I) By modifying both mechanisms of reproduction and mutation, find a jump process on [0, 1] whose generator extends (1.1) and whose stationary distribution can be identified.

(II) Establish an analogous generalization for the Fleming–Viot process with parent-independent mutation.

Since these problems are rather vague, it may be worth showing now the generator we will believe to give an "answer" to (I). For each $\alpha \in (0,1)$, define an operator A_α by

$$\begin{aligned} A_\alpha G(x) = {} & \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{u^2}\big[xG\big((1-u)x+u\big) \\ & \qquad + (1-x)G\big((1-u)x\big) - G(x)\big] \\ & + \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{(\alpha+1)u}\big[c_1 G\big((1-u)x+u\big) \\ & \qquad + c_2 G\big((1-u)x\big) - (c_1+c_2)G(x)\big], \end{aligned} \tag{1.3}$$

where G are smooth functions on [0, 1]. Observe that $A_\alpha G(x) \to AG(x)$ as $\alpha \uparrow 1$. It should be noted that A_α is a one-dimensional version of the generator of the process studied in [3] if $c_1 = c_2 = 0$. See also [12] and [13]. The reader, however, is cautioned that our notation α is in conflict with that of these papers, in which α plays the same role as $\alpha + 1$ in our notation. (We adopt such notation in order for the formulae below to be simpler.) The constant c_1 (resp., c_2) in (1.3) can be interpreted as the rate of "simultaneous mutation" from one type to the other type and a proportion u of the individuals with that type, which are supposed to have the frequency $1-x$ (resp., x) in the population, are involved in this "mutation" event with intensity $B_{1-\alpha,\alpha}(du)/((\alpha+1)u)$. [Note that $(1-u)x+u = x+u(1-x)$.] As will be seen in Proposition 3.1 below for more general case, the closure of (1.3) with a suitable domain generates a Feller semigroup on $C([0,1])$, and our main concern is the equilibrium state of the associated Markov process. It will be shown in the forthcoming section that a unique stationary distribution of the process governed by (1.3) is identified with

$$\begin{aligned} & P_{\alpha,(c_1,c_2)}(dx) \\ & \quad := \Gamma(\alpha+1)\int_0^1 B_{c_1,c_2}(dy) E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right], \end{aligned} \tag{1.4}$$

where $E_{\alpha,y}$ denotes the expectation with respect to (Y_1, Y_2) with law determined by $\log E_{\alpha,y}[e^{-\lambda_1 Y_1 - \lambda_2 Y_2}] = -y\lambda_1^\alpha - (1-y)\lambda_2^\alpha$ $(\lambda_1, \lambda_2 \geq 0)$. Again we see that (1.4) with $\alpha = 1$ reduces to (1.2).

One might think that (1.3) is one of many possible generalizations of (1.1). In fact it arises naturally in the following manner. It is well-known [20] that the Fleming–Viot process with parent-independent mutation can be obtained by way of a normalization and a random time change from a measure-valued branching diffusion with immigration. (See also [6] and [18].) An extension of this significant result was shown in [3] for a class of generalized Fleming–Viot processes, which in the one-dimensional setting corresponds to (1.3) with $c_1 = c_2 = 0$. Moreover, [3] proved that such a jump mechanism is necessary for a generalized Fleming–Viot process to have the above mentioned link to a measure-valued branching process with immigration (henceforth MBI-process). Recently, [13] showed essentially that the second term of (1.3) is required when we additionally take a generalization of the mutation mechanism into account. Our argument will be crucially based on this kind of relationship between the generalized Fleming–Viot process associated with a natural generalization of (1.3) and a certain ergodic MBI-process. That relationship can be reformulated as a factorization result on the level of generators and hence is expected to yield also an explicit connection between stationary distributions. In principle, the problems (I) and (II) can be considered in a unified way. Nevertheless, we shall discuss (I) and (II) separately. This is mainly because the factorization identity will turn out to yield a correct answer only for certain restricted cases and in one dimension one can avoid its use by taking an analytic approach instead (although this does not reveal clearly the mathematical structure underlying).

The organization of this paper is as follows. Section 2 is devoted to derivation of (1.4) by purely analytic argument. Exploiting the relationship to MBI-processes, we show in Section 3 that the above mentioned answer to (I) has a natural generalization which settles (II). The irreversibility of the processes we consider is discussed in Section 4.

2. The one-dimensional model.

Let $0 < \alpha < 1$, $c_1 > 0$ and $c_2 > 0$ be given. The purpose of this section is to show that (1.4) is a unique stationary distribution of the process with generator (1.3). Analytically, we shall prove that a probability measure P on [0, 1] satisfying

$$\int_0^1 A_\alpha G(x) P(dx) = 0 \quad \text{for all } G(x) = \varphi_n(x) := x^n \text{ with } n = 1, 2, \ldots \tag{2.1}$$

is uniquely identified with (1.4). Actual starting point of the calculations below is

$$\int_0^1 A_\alpha G(x) P(dx) = 0 \quad \text{for all } G(x) = G_t(x) := (1 + tx)^{-1} \text{ with } t > 0. \tag{2.2}$$

The equivalence of (2.1) and (2.2) is a consequence of uniform estimates

$$\left|A_\alpha \varphi_n(x)\right| \le \left(1+\frac{c_1+c_2}{\alpha+1}\right)2^n, \qquad n=1,2,\dots,$$

which can be shown by observing that

$$\begin{aligned}
&c_1\big((1-u)x+u\big)^n + c_2\big((1-u)x\big)^n - (c_1+c_2)x^n \\
&\quad = c_1\big[\big((1-u)x+u\big)^n - \big((1-u)x+ux\big)^n\big] \\
&\qquad + c_2 x^n\big[(1-u)^n - \big((1-u)+u\big)^n\big] \\
&\quad = c_1 \sum_{k=1}^n \binom{n}{k} (1-u)^{n-k} x^{n-k} u^k \big(1-x^k\big) \\
&\qquad - c_2 x^n \sum_{k=1}^n \binom{n}{k} (1-u)^{n-k} u^k \\
&\quad = \sum_{k=1}^n \binom{n}{k} (1-u)^{n-k} u^k \big[c_1 x^{n-k} - (c_1+c_2)x^n\big] \\
&\quad = u \sum_{k=1}^n \binom{n}{k} (1-u)^{n-k} u^{k-1} \big[c_1 x^{n-k} - (c_1+c_2)x^n\big]
\end{aligned} \tag{2.3}$$

and in particular

$$\begin{aligned}
&x\big((1-u)x+u\big)^n + (1-x)\big((1-u)x\big)^n - x^n \\
&\quad = \sum_{k=2}^n \binom{n}{k} (1-u)^{n-k} u^k \big(x^{n-k+1} - x^n\big) \\
&\quad = u^2 \sum_{k=2}^n \binom{n}{k} (1-u)^{n-k} u^{k-2} \big(x^{n-k+1} - x^n\big).
\end{aligned}$$

Indeed, these bounds ensure that the function

$$t \mapsto \int_0^1 A_\alpha G_t(x) P(dx) = \sum_{n=1}^\infty (-t)^n \int_0^1 A_\alpha \varphi_n(x) P(dx)$$

is real analytic at least for $-1/2 < t < 1/2$. We prepare a simple lemma in order to calculate $A_\alpha G_t$.

LEMMA 2.1. *Assume that $b > 0$ and $a + b > 0$.*

(i) *It holds that for any $\theta_1 > 0$ and $\theta_2 > 0$*

$$\int_0^1 \frac{B_{\theta_1,\theta_2}(du)}{(au+b)^{\theta_1+\theta_2}} = (a+b)^{-\theta_1} b^{-\theta_2}. \tag{2.4}$$

(ii) *In addition, suppose that* $a' \neq a$ *and* $a' + b > 0$. *Then*

$$\int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{(au+b)(a'u+b)} = \frac{1}{\alpha(a-a')b^{1+\alpha}}\big[(a+b)^\alpha - (a'+b)^\alpha\big]. \tag{2.5}$$

Equation (2.4) is a one-dimensional version of the formula due to [4], which is sometimes referred to as the Markov–Krein identity. (See, e.g., [22] or (3.6) below.) We will give a self-contained proof based essentially on the well-known relationship between beta and gamma laws.

PROOF OF LEMMA 2.1. The proof of (2.4) is simply done by noting that

$$(a+b)^{-\theta_1} b^{-\theta_2} = \int_0^\infty \frac{dz_1}{\Gamma(\theta_1)} z_1^{\theta_1 - 1} e^{-(a+b)z_1} \int_0^\infty \frac{dz_2}{\Gamma(\theta_2)} z_2^{\theta_2 - 1} e^{-bz_2}$$

and then by change of variables to $u := z_1/(z_1 + z_2)$, $v := z_1 + z_2$. The proof of (2.5) can be deduced from (2.4) with $\theta_1 = 1 - \alpha$ and $\theta_2 = \alpha$ since $B_{1-\alpha,1+\alpha}(du) = B_{1-\alpha,\alpha}(du)(1-u)/\alpha$ and

$$\frac{1-u}{(au+b)(a'u+b)} = \frac{1}{(a-a')b}\left(\frac{a+b}{au+b} - \frac{a'+b}{a'u+b}\right). \qquad \square$$

We proceed to calculate $A_\alpha G_t$.

LEMMA 2.2. *For any* $t > 0$ *and* $x \in [0, 1]$,

$$\begin{aligned} A_\alpha G_t(x) &= t \cdot \frac{(1+t)^\alpha - 1}{\alpha} \cdot \frac{x(1-x)}{(1+tx)^{2+\alpha}} \\ &\quad - \frac{t}{\alpha+1} \cdot \frac{c_1(1-x)(1+t)^{\alpha-1} - c_2 x}{(1+tx)^{1+\alpha}}. \end{aligned} \tag{2.6}$$

PROOF. By straightforward calculations

$$\begin{aligned} &c_1 G_t\big((1-u)x+u\big) + c_2 G_t\big((1-u)x\big) - (c_1+c_2)G_t(x) \\ &\quad = -\frac{tu}{1+tx}\left[\frac{c_1(1-x)}{1+t(1-u)x+tu} - \frac{c_2 x}{1+t(1-u)x}\right]. \end{aligned}$$

Replacing c_1 and c_2 by x and $1-x$, respectively, we get

$$\begin{aligned} &x G_t\big((1-u)x+u\big) + (1-x) G_t\big((1-u)x\big) - G_t(x) \\ &\quad = \frac{t^2 u^2 x(1-x)}{1+tx} \cdot \frac{1}{(1+t(1-u)x+tu)(1+t(1-u)x)}. \end{aligned}$$

Plugging these equalities into (1.3) with $G = G_t$ and then applying Lemma 2.1 yield

$$
\begin{aligned}
A_\alpha G_t(x) &= \frac{t^2 x(1-x)}{1+tx} \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{(1+t(1-u)x+tu)(1+t(1-u)x)} \\
&\quad - \frac{t}{(\alpha+1)(1+tx)} \cdot c_1(1-x) \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{1+t(1-u)x+tu} \\
&\quad + \frac{t}{(\alpha+1)(1+tx)} \cdot c_2 x \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{1+t(1-u)x} \\
&= \frac{t^2 x(1-x)}{1+tx} \cdot \frac{1}{\alpha t(1+tx)^{1+\alpha}} \cdot [(1+t)^\alpha - 1] \\
&\quad - \frac{t}{(\alpha+1)(1+tx)} \left[\frac{c_1(1-x)}{(1+t)^{1-\alpha}(1+tx)^\alpha} - \frac{c_2 x}{(1+tx)^\alpha} \right],
\end{aligned}
$$

which equals the right-hand side of (2.6). □

Next, we are going to characterize stationary distributions P in terms of

$$
S_\alpha(t) := \int_0^1 \frac{P(dx)}{(1+tx)^\alpha}, \qquad t \ge 0, \tag{2.7}
$$

which is a variant of the generalized Stieltjes transform of order α.

PROPOSITION 2.3. *A probability measure P on* $[0, 1]$ *is a stationary distribution of the process associated with* (1.3) *if and only if S_α defined by* (2.7) *satisfies for all $t > 0$*

$$
\begin{aligned}
&\frac{(1+t)^\alpha - 1}{\alpha}(1+t) S_\alpha''(t) \\
&\quad + \left[\left(c_1 + 1 + \frac{1}{\alpha} \right) ((1+t)^\alpha - 1) + c_1 + c_2 \right] S_\alpha'(t) \\
&\quad + \alpha c_1 (1+t)^{\alpha-1} S_\alpha(t) = 0.
\end{aligned} \tag{2.8}
$$

PROOF. By virtue of Theorem 9.17 in Chapter 4 of [9], P is a stationary distribution of the process associated with A_α if and only if (2.1) [or (2.2)] holds. By Lemma 2.2, (2.2) now reads for all $t > 0$

$$
\begin{aligned}
&-\frac{(1+t)^\alpha - 1}{\alpha} \int_0^1 \frac{x(1-x)}{(1+tx)^{2+\alpha}} P(dx) \\
&\quad + \frac{c_1}{\alpha+1} (1+t)^{\alpha-1} \int_0^1 \frac{1-x}{(1+tx)^{1+\alpha}} P(dx) \\
&\quad - \frac{c_2}{\alpha+1} \int_0^1 \frac{x}{(1+tx)^{1+\alpha}} P(dx) = 0.
\end{aligned}
$$

This equation becomes (2.8) by substituting the equalities

$$-\int_0^1 \frac{x(1-x)}{(1+tx)^{2+\alpha}} P(dx) = \frac{1+t}{\alpha(\alpha+1)} S''_\alpha(t) + \frac{1}{\alpha} S'_\alpha(t),$$

$$\int_0^1 \frac{1-x}{(1+tx)^{1+\alpha}} P(dx) = \frac{1+t}{\alpha} S'_\alpha(t) + S_\alpha(t)$$

and

$$\int_0^1 \frac{x}{(1+tx)^{1+\alpha}} P(dx) = -\frac{1}{\alpha} S'_\alpha(t),$$

all of which are verified easily. $\square$

We now derive (1.4) as the unique stationary distribution we are looking for. Recall that for each $y \in (0,1)$ we denote by $E_{\alpha,y}$ the expectation with respect to the two-dimensional random variable (Y_1, Y_2) with joint law determined by

$$E_{\alpha,y}\big[e^{-\lambda_1 Y_1 - \lambda_2 Y_2}\big] = e^{-y\lambda_1^\alpha - (1-y)\lambda_2^\alpha}, \qquad \lambda_1, \lambda_2 \geq 0.$$

By using $t^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty dv v^{\alpha-1} e^{-vt}$ $(t > 0)$ and Fubini's theorem, observe that

$$\begin{aligned} & E_{\alpha,y}\big[(tY_1 + Y_2)^{-\alpha}\big] \\ & \quad = \Gamma(\alpha)^{-1} \int_0^\infty dv v^{\alpha-1} \exp\big[-y(vt)^\alpha - (1-y)v^\alpha\big] \\ & \quad = \frac{1}{\Gamma(\alpha+1)} \cdot \frac{1}{1+(t^\alpha - 1)y} \end{aligned} \tag{2.9}$$

for $t \geq 0$. In particular, $E_{\alpha,y}[(Y_1+Y_2)^{-\alpha}] = 1/\Gamma(\alpha+1)$ and hence

$$\begin{aligned} & P_{\alpha,(c_1,c_2)}(dx) \\ & \quad = \Gamma(\alpha+1) \int_0^1 B_{c_1,c_2}(dy) E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right] \end{aligned} \tag{2.10}$$

defines a probability measure on $[0,1]$. Although for each $y \in (0,1)$ an expression of the distribution function

$$[0,1] \ni x \mapsto \Gamma(\alpha+1) E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \leq x\right]$$

is given as the formula (3.2) in [23], that is,

$$\frac{\sin\alpha\pi}{\pi} \int_0^x \frac{(1-y)(x-u)^{\alpha-1} u^\alpha \, du}{(1-y)^2 u^{2\alpha} + y^2(1-u)^{2\alpha} + 2y(1-y)u^\alpha(1-u)^\alpha \cos\alpha\pi},$$

we do not have any explicit form concerning $P_{\alpha,(c_1,c_2)}$ except the case $c_1 + c_2 = 1$. [See Remark (ii) at the end of this section.]

The main result of this section is the following.

THEOREM 2.4. *The process associated with* (1.3) *has a unique stationary distribution*, *which coincides with* $P_{\alpha,(c_1,c_2)}$.

PROOF. Notice that the existence of a stationary distribution follows from compactness of the state space [0, 1]. (See, e.g., Remark 9.4 in Chapter 4 of [9].) Let P be an arbitrary stationary distribution of the process associated with (1.3) and S_α be defined by (2.7). Put

$$T_\alpha(u) = S_\alpha\big((1+u)^{1/\alpha} - 1\big)$$

for $u \geq 0$. Setting $t = (1+u)^{1/\alpha} - 1$ or $u = (1+t)^\alpha - 1$, observe that for $u > 0$

$$T'_\alpha(u) = \frac{1}{\alpha}(1+u)^{(1/\alpha)-1} S'_\alpha(t)$$

and

$$\begin{aligned} T''_\alpha(u) &= \frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)(1+u)^{(1/\alpha)-2} S'_\alpha(t) + \left[\frac{1}{\alpha}(1+u)^{(1/\alpha)-1}\right]^2 S''_\alpha(t) \\ &= \left(\frac{1}{\alpha} - 1\right)(1+u)^{-1} T'_\alpha(u) + \frac{1}{\alpha^2}(1+u)^{(2/\alpha)-2} S''_\alpha(t). \end{aligned}$$

Hence, $S'_\alpha(t) = \alpha(1+u)^{1-(1/\alpha)} T'_\alpha(u)$ and

$$S''_\alpha(t) = \alpha^2(1+u)^{2-(2/\alpha)}\left[T''_\alpha(u) - \left(\frac{1}{\alpha} - 1\right)(1+u)^{-1} T'_\alpha(u)\right].$$

Also, (2.8) can be rewritten as

$$\begin{aligned} &\frac{u}{\alpha}(1+u)^{1/\alpha} S''_\alpha(t) + \left[\left(c_1 + 1 + \frac{1}{\alpha}\right)u + c_1 + c_2\right] S'_\alpha(t) \\ &\qquad + \alpha c_1 (1+u)^{1-(1/\alpha)} S_\alpha(t) = 0. \end{aligned}$$

From these preliminary observations, it is direct to see that the equation (2.8) is transformed into a hypergeometric equation of the form

$$u(1+u)T''_\alpha(u) + \big[(c_1+c_2) + (c_1+2)u\big]T'_\alpha(u) + c_1 T_\alpha(u) = 0, \qquad u > 0. \tag{2.11}$$

Clearly, $T_\alpha(0) = S_\alpha(0) = 1$. In addition,

$$T'_\alpha(0) = S'_\alpha(0)/\alpha = -\int_0^1 P(dx)x = -c_1/(c_1+c_2),$$

where the last equality follows from (2.1) with $n = 1$. These facts together imply that

$$T_\alpha(u) = \int_0^1 \frac{B_{c_1,c_2}(dy)}{1+uy}, \qquad u \geq 0$$

or

$$S_\alpha(t) = \int_0^1 \frac{B_{c_1,c_2}(dy)}{1 + \{(1+t)^\alpha - 1\}y}, \qquad t \geq 0.$$

(See, e.g., Sections 7.2 and 9.1 in [16].) Combining this with

$$\frac{1}{1 + \{(1+t)^\alpha - 1\}y} = \Gamma(\alpha+1) \int_0^1 \frac{1}{(1+tx)^\alpha} E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right],$$

which is immediate from (2.9), we arrive at

$$S_\alpha(t) = \int_0^1 \frac{P_{\alpha,(c_1,c_2)}(dx)}{(1+tx)^\alpha}, \qquad t \geq 0 \tag{2.12}$$

in view of (2.10). Therefore, we conclude that $P = P_{\alpha,(c_1,c_2)}$ and the proof of Theorem 2.4 is complete. □

REMARKS. (i) In the case where $c_1 + c_2 > 1$, an alternative expression for $P_{\alpha,(c_1,c_2)}$ exists:

$$\begin{aligned} &P_{\alpha,(c_1,c_2)}(dx) \\ &\quad = \Gamma(\alpha+1)(c_1+c_2-1) E\left[(Z_1+Z_2)^{-\alpha}; \frac{Z_1}{Z_1+Z_2} \in dx\right] \\ &\quad =: \widetilde{P}_{\alpha,(c_1,c_2)}(dx), \end{aligned} \tag{2.13}$$

where Z_1 and Z_2 are independent random variables with Laplace transforms

$$E[e^{-\lambda Z_i}] = \exp[-c_i \log(1+\lambda^\alpha)], \qquad \lambda \geq 0. \tag{2.14}$$

This reflects the fact that the solution to (2.11) with the same initial conditions $T_\alpha(0) = 1$ and $T'_\alpha(0) = -c_1/(c_1+c_2)$ admits another integral expression of the form

$$T_\alpha(u) = \int_0^1 \frac{B_{1,c_1+c_2-1}(dy)}{(1+uy)^{c_1}}, \qquad u \geq 0$$

and accordingly by (2.12)

$$\int_0^1 \frac{P_{\alpha,(c_1,c_2)}(dx)}{(1+tx)^\alpha} = \int_0^1 \frac{B_{1,c_1+c_2-1}(dy)}{[1 + \{(1+t)^\alpha - 1\}y]^{c_1}}, \qquad t \geq 0. \tag{2.15}$$

On the other hand, it is not difficult to show that (2.15) with $\widetilde{P}_{\alpha,(c_1,c_2)}$ in place of $P_{\alpha,(c_1,c_2)}$ holds, too. In fact, we prove in Lemma 3.5 below a generalization of the coincidence (2.13) in the setting of random measures. Also, the role of Z_1 and Z_2 will be made clear in connection with branching processes with immigration related closely to the process generated by (1.3). [Compare (2.14) with (3.9) below.]

(ii) It will be shown in the Remark after Lemma 3.5 below that $P_{\alpha,(c_1,c_2)} = B_{\alpha c_1,\alpha c_2}$ holds whenever $c_1 + c_2 = 1$. At least at a formal level, this would be seen by letting $c_1 + c_2 \downarrow 1$ in (2.15) and then by making use of (2.4).

(iii) In contrast with the case of the Wright–Fisher diffusion mentioned in the Introduction, $P_{\alpha,(c_1,c_2)}$ with $0 < \alpha < 1$ is not a reversible distribution for the generator (1.3) at least in case $c_1 \neq c_2$. This will be seen in Section 4.

3. The measure-valued process case. The main subject of this section is an extension of Theorem 2.4 to a class of generalized Fleming–Viot processes. But the strategy will be different from that in the previous section, and so an alternative proof of Theorem 2.4 will be given as a by-product. To discuss in the setting of measure-valued processes, we need new notation. Let E be a compact metric space having at least two distinct points and $C(E)$ [resp., $B_+(E)$] the set of continuous (resp., nonnegative, bounded Borel) functions on E. Define $\mathcal{M}(E)$ to be the totality of finite Borel measures on E, and we equip $\mathcal{M}(E)$ with the weak topology. Denote by $\mathcal{M}(E)^\circ$ the set of nonnull elements of $\mathcal{M}(E)$. The set $\mathcal{M}_1(E)$ of Borel probability measures on E is regarded as a subspace of $\mathcal{M}(E)$. We also use the notation $\langle \eta, f\rangle$ to stand for the integral of a function f with respect a measure η. For each $r \in E$, let δ_r denote the delta distribution at r. Given a probability measure Q, we write also $E^Q[\cdot]$ for the expectation with respect to Q.

Let $0 < \alpha < 1$ and $m \in \mathcal{M}(E)$ be given. We shall discuss in this section an $\mathcal{M}_1(E)$-valued Markov process associated with

$$\begin{aligned}
&\mathcal{A}_{\alpha,m}\Phi(\mu)\\
&\quad := \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{u^2}\int_E \mu(dr)\bigl[\Phi\bigl((1-u)\mu + u\delta_r\bigr) - \Phi(\mu)\bigr]\\
&\qquad + \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{(\alpha+1)u}\int_E m(dr)\bigl[\Phi\bigl((1-u)\mu + u\delta_r\bigr) - \Phi(\mu)\bigr],\\
&\qquad\qquad \mu \in \mathcal{M}_1(E),
\end{aligned} \tag{3.1}$$

where Φ belongs to the class $\mathcal{F}_1$ of functions of the form $\Phi_f(\mu) := \langle \mu^{\otimes n}, f\rangle$ for some positive integer n and $f \in C(E^n)$. Equation (3.1) shows clearly that $\mathcal{A}_{\alpha,m}$ satisfies the positive maximum principle and hence is dissipative. (See Lemma 2.1 in Chapter 4 of [9].) We begin by seeing that $\mathcal{A}_{\alpha,m}$ defines a Markov process on $\mathcal{M}_1(E)$ in an appropriate sense. For this purpose, we need an expression for $\mathcal{A}_{\alpha,m}\Phi_f$ with $f \in C(E^n)$. Set $(a)_b = \Gamma(a+b)/\Gamma(a)$ for $a > 0$ and $b \geq 0$, and let $|\cdot|$ stand for the cardinality. It holds that for any $\theta \geq 0$ and $\nu \in \mathcal{M}_1(E)$

$$\begin{aligned}
&\mathcal{A}_{\alpha,\theta\nu}\Phi_f(\mu)\\
&\quad = \sum_{k=2}^n \frac{(1-\alpha)_{k-2}(\alpha+1)_{n-k}}{\Gamma(n)} \sum_{I:|I|=k} \bigl(\bigl\langle \mu^{\otimes n}, \Theta_I^{(n)} f\bigr\rangle - \Phi_f(\mu)\bigr)\\
&\qquad + \theta \sum_{k=1}^n \frac{(1-\alpha)_{k-1}(\alpha)_{n-k}}{(\alpha+1)\Gamma(n)} \sum_{I:|I|=k} \bigl(\bigl\langle \mu^{\otimes n}, \Xi_{I,\nu}^{(n)} f\bigr\rangle - \Phi_f(\mu)\bigr),
\end{aligned} \tag{3.2}$$

where I are nonempty subsets of $\{1,\dots,n\}$, $\Theta_I^{(n)}: C(E^n) \to C(E^n)$ is defined by letting $\Theta_I^{(n)} f$ be the function obtained from f by replacing all the variables r_i with $i \in I$ by $r_{\min I}$ and $\Xi_{I,\nu}^{(n)}: C(E^n) \to C(E^n)$ is defined by letting $\Xi_{I,\nu}^{(n)} f$ be the function obtained from f by replacing all the variables r_i with $i \in I$ by r and then by integrating with respect to $\nu(dr)$. (For a degenerate ν, (3.2) is a special case of the corresponding expression found in the proof of Lemma 11 in [12].) Equation (3.2) can be deduced from the following identities [cf. (2.3)] among signed measures on E^n:

$$\begin{aligned}
&\bigotimes_{i=1}^{n}\big((1-u)\mu(dr_i)+u\delta_r(dr_i)\big)-\bigotimes_{i=1}^{n}\mu(dr_i)\\
&\quad=\bigotimes_{i=1}^{n}\big((1-u)\mu(dr_i)+u\delta_r(dr_i)\big)-\bigotimes_{i=1}^{n}\big((1-u)\mu(dr_i)+u\mu(dr_i)\big)\\
&\quad=\sum_{I\neq\varnothing}\bigotimes_{j\notin I}\big((1-u)\mu(dr_j)\big)\bigg[\bigotimes_{i\in I}\big(u\delta_r(dr_i)\big)-\bigotimes_{i\in I}\big(u\mu(dr_i)\big)\bigg]\\
&\quad=\sum_{I\neq\varnothing}u^{|I|}(1-u)^{n-|I|}\bigotimes_{j\notin I}\mu(dr_j)\bigg[\bigotimes_{i\in I}\delta_r(dr_i)-\bigotimes_{i\in I}\mu(dr_i)\bigg].
\end{aligned}$$

As for the Fleming–Viot process with parent-independent mutation, the result corresponding to the next proposition is a special case of Theorem 3.4 in [10].

PROPOSITION 3.1. *For each $m \in \mathcal{M}(E)$ the closure of $\mathcal{A}_{\alpha,m}$ defined on $\mathcal{F}_1$ generates a Feller semigroup on $C(\mathcal{M}_1(E))$.*

PROOF. Let $\theta \ge 0$ and $\nu \in \mathcal{M}_1(E)$ be such that $m = \theta\nu$. We simply mimic the proof of Theorem 3.4 in [10]. In particular, the Hille–Yosida theorem (Theorem 2.2 in Chapter 4 of [9]) will be applied. Let n be an arbitrary positive integer. Rewrite (3.2) as

$$\mathcal{A}_{\alpha,\theta\nu}\Phi_f(\mu)=\langle\mu^{\otimes n},\Theta^{(n)}f\rangle+\theta\langle\mu^{\otimes n},\Xi_\nu^{(n)}f\rangle-c_n(\alpha,\theta)\Phi_f(\mu),$$

where $\Theta^{(n)}$, $\Xi_\nu^{(n)}: C(E^n) \to C(E^n)$ and $c_n(\alpha,\theta)$ are, respectively, the nonnegative operators and the positive constant-defined implicitly by the above equation combined with (3.2). Let $\lambda > 0$ be arbitrary. Given $g \in C(E^n)$, define

$$h=\big(\lambda+c_n(\alpha,\theta)\big)^{-1}\sum_{k=0}^{\infty}\big[\big(\lambda+c_n(\alpha,\theta)\big)^{-1}\big(\Theta^{(n)}+\theta\,\Xi_\nu^{(n)}\big)\big]^k g.$$

Then $h \in C(E^n)$ since the operator norm of $\Theta^{(n)}+\theta\,\Xi_\nu^{(n)}$ equals $c_n(\alpha,\theta)$. Moreover,

$$\big(\lambda+c_n(\alpha,\theta)\big)h-\big(\Theta^{(n)}+\theta\,\Xi_\nu^{(n)}\big)h=g,$$

so $(\lambda - \mathcal{A}_{\alpha,\theta\nu})\Phi_h = \Phi_g$. This implies that the range of $\lambda - \mathcal{A}_{\alpha,\theta\nu}$ contains $\mathcal{F}_1$, which is dense in $C(\mathcal{M}_1(E))$. The rest of the proof is the same as that of Theorem 3.4 in [10]. □

For simplicity, we call the $\mathcal{A}_{\alpha,m}$-process the Markov process governed by $\mathcal{A}_{\alpha,m}$ in the sense of Proposition 3.1. This process is a natural generalization of the process generated by (1.3) in the following sense. Suppose that E consists of two points, say r_1 and r_2, set $m = c_1\delta_{r_1} + c_2\delta_{r_2}$, and let $\{X(t) : t \geq 0\}$ be the process generated by (1.3). Then, verifying the identity $\mathcal{A}_{\alpha,m}\Phi(\mu) = A_\alpha G(x)$ for $\mu = x\delta_{r_1} + (1-x)\delta_{r_2}$ and $\Phi(\mu) = G(x)$, we see that the process $\{X(t)\delta_{r_1} + (1 - X(t))\delta_{r_2} : t \geq 0\}$ defines an $\mathcal{A}_{\alpha,m}$-process. We note that [13] discusses the case where $E = [0, 1]$ and $m = c\delta_0$ for some $c > 0$.

We could also establish the well-posedness of the martingale problem for $\mathcal{A}_{\alpha,m}$ by modifying some existing arguments. More precisely, the existence could be shown through a limit theorem for suitably generalized Moran particle systems by modifying those considered in the proof of Theorem 2.1 [especially (2.2)] of [14], which took account of the jump mechanism describing simultaneous reproduction (sampling) only, so that simultaneous movement (mutation) of particles to a random location (type) distributed according to $m(dr)/m(E)$ is allowed. The uniqueness would follow by the duality argument employing a function-valued process as in the proof of Theorem 2.1 of [14]. Its possible transitions and the associated transition rates are found in (3.2). The duality would be useful in discussing (weak) ergodicity of the $\mathcal{A}_{\alpha,m}$-process. (See, e.g., Theorem 5.2 in [10] for such a result in the Fleming–Viot process case.)

The following argument is based primarily on the relationship between the $\mathcal{A}_{\alpha,m}$-process and a suitable MBI-process, which takes values in $\mathcal{M}(E)$. More precisely, the generator, say $\mathcal{L}_{\alpha,m}$, of the latter will be chosen so that for some constant $C > 0$

$$\mathcal{L}_{\alpha,m}\Psi(\eta) = C\eta(E)^{-\alpha}\mathcal{A}_{\alpha,m}\Phi\big(\eta(E)^{-1}\eta\big), \qquad \eta \in \mathcal{M}(E)^\circ, \tag{3.3}$$

where $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ and Φ is in the linear span $\mathcal{F}_0$ of functions of the form $\mu \mapsto \langle\mu, f_1\rangle \cdots \langle\mu, f_n\rangle$ with $f_i \in C(E)$, $i = 1, \ldots, n$ and n being a positive integer. In the case of the Fleming–Viot process (which corresponds to $\alpha = 1$ formally), such a relation is well known. For instance, it played a key role in [20]. As for the generalized Fleming–Viot process, factorizations of the form (3.3) have been shown in [3] for $m = 0$ (the null measure) and in [13] for degenerate measures m. From now on, suppose that $m \in \mathcal{M}(E)^\circ$. To exploit (3.3) in the study of stationary distributions, we further require the MBI-process associated with $\mathcal{L}_{\alpha,m}$ to be ergodic, that is, to have a unique stationary distribution, say $\widetilde{Q}_{\alpha,m}$, supported on $\mathcal{M}(E)^\circ$. Once these requirements are fulfilled, (3.3) suggests that

$$\widetilde{P}_{\alpha,m}(\cdot) := E^{\widetilde{Q}_{\alpha,m}}\big[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot\big]/E^{\widetilde{Q}_{\alpha,m}}\big[\eta(E)^{-\alpha}\big] \tag{3.4}$$

would give a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process provided that $\eta(E)^{-\alpha}$ is integrable with respect to $\widetilde{Q}_{\alpha,m}$. This conditional answer may be modified to be a general one, which must be consistent with the one-dimensional result (1.4).

To describe the answer, we need both the α-stable random measure with parameter measure m and the Dirichlet random measure with parameter measure m, whose laws on $\mathcal{M}(E)^\circ$ and $\mathcal{M}_1(E)$ are denoted by $Q_{\alpha,m}$ and $\mathcal{D}_m$, respectively. These infinite-dimensional laws are determined uniquely by the identities

$$\int_{\mathcal{M}(E)^\circ} Q_{\alpha,m}(d\eta)e^{-\langle \eta, f\rangle} = e^{-\langle m, f^\alpha\rangle} \tag{3.5}$$

and

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu)\langle \mu, 1+f\rangle^{-m(E)} = e^{-\langle m, \log(1+f)\rangle}, \tag{3.6}$$

where $f \in B_+(E)$ is arbitrary. A random measure with law $Q_{\alpha,m}$ is constructed from a Poisson random measure on $(0,\infty)\times E$. (See also Definition 6 in [22].) Observe from (3.5) that $E^{Q_{\alpha,m}}[\eta(E)^{-\alpha}] = 1/(m(E)\Gamma(\alpha+1))$. As in [11], $\mathcal{D}_m$ is defined originally to be the law of a random measure whose arbitrary finite-dimensional distributions are Dirichlet distributions with parameters specified by m. The useful identity (3.6) is due to [4] and reduces to (2.4) in one-dimension. We now state the main result of this paper.

THEOREM 3.2. *For any $m \in \mathcal{M}(E)^\circ$, the $\mathcal{A}_{\alpha,m}$-process has a unique stationary distribution, which is identified with*

$$P_{\alpha,m}(\cdot) := \Gamma(\alpha+1)\int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu)E^{Q_{\alpha,\mu}}[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot]. \tag{3.7}$$

To illustrate, consider the trivial case where $m = \theta\delta_r$ for some $\theta > 0$ and $r \in E$. Then it is verified easily that $P_{\alpha,m}$ concentrates at $\delta_r \in \mathcal{M}_1(E)$, and this is consistent with the equality $\mathcal{A}_{\alpha,m}\Phi(\delta_r) = 0$ in that case. Also, for every $m \in \mathcal{M}(E)^\circ$, we note that $P_{\alpha,m} \to \mathcal{D}_m$ as $\alpha \uparrow 1$ since by (3.5) $Q_{\alpha,\mu}$ converges weakly to the delta distribution at μ for each $\mu \in \mathcal{M}_1(E)$.

The proof of Theorem 3.2 will be divided into three steps. As mentioned earlier, we first find an ergodic MBI-process whose generator satisfies (3.3) and show, under necessary integrability condition, that $\widetilde{P}_{\alpha,m}$ in (3.4) gives a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process. [In fact, the condition will turn out to be that $m(E) > 1$. This motivates us to make a reparametrization $m =: \theta\nu$ with $\theta > 0$ and $\nu \in \mathcal{M}_1(E)$.] Second, for each $\nu \in \mathcal{M}_1(E)$, we prove that $\widetilde{P}_{\alpha,\theta\nu} = P_{\alpha,\theta\nu}$ for any $\theta > 1$. As the last step, we extend stationarity of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ to all $\theta > 0$ by interpreting the condition of stationarity as certain recursion equations among moment measures which are seen to be real analytic in $\theta > 0$. Also, the recursion equations will be shown to yield uniqueness of the stationary distribution.

For the first step, we prove in the next proposition that the MBI-process with the following generator is the desired one:

$$
\begin{aligned}
&\mathcal{L}_{\alpha,m}\Psi(\eta) \\
&\quad := \frac{\alpha+1}{\Gamma(1-\alpha)}\int_0^\infty \frac{dz}{z^{2+\alpha}}\int_E \eta(dr)\left[\Psi(\eta+z\delta_r)-\Psi(\eta)-z\frac{\delta\Psi}{\delta\eta}(r)\right] \\
&\qquad - \frac{1}{\alpha}\left\langle \eta, \frac{\delta\Psi}{\delta\eta}\right\rangle \\
&\qquad + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{dz}{z^{1+\alpha}}\int_E m(dr)\big[\Psi(\eta+z\delta_r)-\Psi(\eta)\big],
\end{aligned}
\tag{3.8}
$$

where Ψ is in the class $\mathcal{F}$ of functions of the form $\eta \mapsto F(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)$ for some $F \in C_b^2(\mathbf{R}^n)$, $f_i \in C(E)$ and a positive integer n, and $\frac{\delta\Psi}{\delta\eta}(r) = \frac{d}{d\varepsilon}\Psi(\eta + \varepsilon\delta_r)|_{\varepsilon=0}$. Up to this first order differential term, the operator (3.8) for $E = [0, 1]$ and $m = c\delta_0$ with $c > 0$ is the same as the one discussed in Lemma 5.5 of [13], in which the factorization (3.3) has been proved. Thus, our main observation in the next proposition is that, keeping the validity of (3.3), such an extra term yields the ergodicity. Note that the generator (3.8) is a special case of the one discussed in Chapter 9 of [17]. [See (9.25) combined with (7.12) there for an expression of the generator.] In particular, a unique solution to the martingale problem for $\mathcal{L}_{\alpha,m}$ defines an $\mathcal{M}(E)$-valued Markov process, which henceforth we call the $\mathcal{L}_{\alpha,m}$-process. Intuitively, because of absence of the "motion process", the law of this process is considered as continuum convolution of the continuous-state branching process with immigration (CBI-process) studied in [15]. [See (3.11) below.] In addition, Example 1.1 and Theorem 2.3 in [15] concern the one-dimensional version of the $\mathcal{L}_{\alpha,m}$-process without the drift. The latter proved that the offspring distribution and the distribution associated with immigration of the approximating branching processes may have probability generating functions of the form $s + c(1-s)^{\alpha+1}$ and $1 - d(1-s)^\alpha$, respectively.

PROPOSITION 3.3. *Let $m \in \mathcal{M}(E)^\circ$. Then $\mathcal{L}_{\alpha,m}$ in (3.8) and $\mathcal{A}_{\alpha,m}$ in (3.1) together satisfy (3.3) with $C = \Gamma(\alpha+2)$ and $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ for any $\Phi \in \mathcal{F}_0$. Moreover, the $\mathcal{L}_{\alpha,m}$-process has a unique stationary distribution $\widetilde{Q}_{\alpha,m}$ with Laplace functional*

$$
\int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)e^{-\langle \eta, f\rangle} = e^{-\langle m, \log(1+f^\alpha)\rangle}, \qquad f \in B_+(E). \tag{3.9}
$$

A random measure with law $\widetilde{Q}_{\alpha,m}$ may be called a Linnik random measure since it is an infinite-dimensional analogue of the random variable with law sometimes referred to as a (nonsymmetric) Linnik distribution, whose Laplace transform appeared already in (2.14). It is obtained by subordinating to an α-stable subordinator by a gamma process. (See, e.g., Example 30.8 in [19].) Namely, letting $\{Y_\alpha(t) : t \geq 0\}$ and $\{\gamma(t) : t \geq 0\}$ be independent Lévy processes such that

$$E\big[e^{-\lambda Y_\alpha(t)}\big] = e^{-t\lambda^\alpha} \quad \text{and} \quad E\big[e^{-\lambda\gamma(t)}\big] = e^{-t\log(1+\lambda)}, \qquad t, \lambda \geq 0,$$

we have for each $c > 0$

$$E\big[e^{-\lambda Y_\alpha(\gamma(c))}\big] = E\big[e^{-\gamma(c)\lambda^\alpha}\big] = e^{-c\log(1+\lambda^\alpha)}, \qquad \lambda \geq 0.$$

The first equality implies that

$$P\big(Y_\alpha(\gamma(c)) \in \cdot\big) = \int_0^\infty P\big(\gamma(c) \in dt\big)P\big(Y_\alpha(t) \in \cdot\big).$$

Equation (3.9) clearly shows an analogous structure underlying, that is,

$$\widetilde{Q}_{\alpha,m}(\cdot) = \int_{\mathcal{M}(E)^\circ} \mathcal{G}_m(d\eta)Q_{\alpha,\eta}(\cdot),$$

where $\mathcal{G}_m$ is the law of the standard gamma process on (E, m). (See Definition 5 in [22]). It is also obvious from (3.9) that, as $\alpha \uparrow 1$, $\widetilde{Q}_{\alpha,m}$ converges to $\mathcal{G}_m$. In addition, one can see that

$$\lim_{\alpha\uparrow 1} \mathcal{L}_{\alpha,m}\Psi(\eta) = \left\langle \eta, \frac{\delta^2\Psi}{\delta\eta^2}\right\rangle - \left\langle \eta, \frac{\delta\Psi}{\delta\eta}\right\rangle + \left\langle m, \frac{\delta\Psi}{\delta\eta}\right\rangle =: \mathcal{L}_m\Psi(\eta)$$

for "nice" functions Ψ, where $\frac{\delta^2\Psi}{\delta\eta^2}(r) = \frac{d^2}{d\varepsilon^2}\Psi(\eta + \varepsilon\delta_r)|_{\varepsilon=0}$. This is a special case of the generator of MBI-processes discussed in Section 3 of [21]. It has been proved there that $\mathcal{G}_m$ is a reversible stationary distribution of the process associated with $\mathcal{L}_m$.

PROOF OF PROPOSITION 3.3. As already remarked, if the term $-\alpha^{-1}\langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle$ in (3.8) would vanish, (3.3) can be shown by essentially the same calculations as in the proof of Lemma 17 in [13]. [In fact, the change of variable $z =: \eta(E)u/(1-u)$ in the integrals with respect to dz in (3.8) almost suffices for our purpose.] So, for the proof of (3.3), we only need to observe that $\langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle = 0$ for Ψ of the form $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ with $\Phi \in \mathcal{F}_0$. But this is readily done by giving a specific form of Φ. Indeed, for $\Phi(\mu) = \langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$ the function Ψ takes the form $\Psi(\eta) = \langle \eta, f_1\rangle \cdots \langle \eta, f_n\rangle\langle \eta, 1\rangle^{-n}$, from which it follows that

$$\frac{\delta\Psi}{\delta\eta}(r) = \sum_{i=1}^n \frac{f_i(r)\langle \eta, 1\rangle - \langle \eta, f_i\rangle}{\langle \eta, 1\rangle^{n+1}} \prod_{j\neq i}\langle \eta, f_j\rangle.$$

After integrating with respect to $\eta(dr)$, the numerator on the right-hand side vanishes.

The argument regarding ergodicity is based on a well-known formula for Laplace functionals of transition functions. (See (9.18) in [17] for a much more general case than ours.) To write it down, we need only auxiliary functions called

Ψ-semigroup [15] because there is no “motion process”. These functions form a one-parameter family $\{\psi(t,\cdot)\}_{t\geq 0}$ of nonnegative functions on $[0,\infty)$ and are determined by the equation

$$\frac{\partial\psi}{\partial t}(t,\lambda) = -\frac{1}{\alpha}\psi(t,\lambda)^{1+\alpha} - \frac{1}{\alpha}\psi(t,\lambda), \qquad \psi(0,\lambda)=\lambda \tag{3.10}$$

with $\lambda \geq 0$ being arbitrary. An explicit expression is found in Example 3.1 of [17]:

$$\psi(t,\lambda) = \frac{e^{-t/\alpha}\lambda}{[1+(1-e^{-t})\lambda^\alpha]^{1/\alpha}}.$$

Let $\{\eta_t : t \geq 0\}$ be an $\mathcal{L}_{\alpha,m}$-process, and for each $\eta \in \mathcal{M}(E)$ denote by E_η the expectation with respect to $\{\eta_t : t \geq 0\}$ starting at η. Then for any $f \in B_+(E)$ and $t \geq 0$

$$E_\eta\bigl[e^{-\langle \eta_t, f\rangle}\bigr] = \exp\biggl[-\langle \eta, V_t f\rangle - \int_0^t \langle m, (V_s f)^\alpha\rangle\, ds\biggr], \tag{3.11}$$

where $V_t f(r) = \psi(t, f(r))$. As $t \to \infty$ the right-hand side converges to

$$\exp\biggl[-\int_0^\infty \langle m, (V_t f)^\alpha\rangle\, dt\biggr] = \exp\bigl[-\langle m, \log(1+f^\alpha)\rangle\bigr]$$

since by (3.10)

$$\frac{d}{dt}\log\bigl(1+\bigl(V_t f(r)\bigr)^\alpha\bigr) = -\bigl(V_t f(r)\bigr)^\alpha.$$

This shows the ergodicity required and completes the proof. $\square$

PROPOSITION 3.4. *Suppose that $m(E) > 1$ and let $\widetilde{Q}_{\alpha,m}$ be as in Proposition 3.3. Then*

$$E^{\widetilde{Q}_{\alpha,m}}\bigl[\eta(E)^{-\alpha}\bigr] = \bigl(\Gamma(\alpha+1)\bigl(m(E)-1\bigr)\bigr)^{-1}.$$

Moreover,

$$\widetilde{P}_{\alpha,m}(\cdot) = \Gamma(\alpha+1)\bigl(m(E)-1\bigr)E^{\widetilde{Q}_{\alpha,m}}\bigl[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot\bigr] \tag{3.12}$$

is a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process.

PROOF. The first assertion is shown by using $t^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}e^{-vt}$ $(t > 0)$ and (3.9) with $f \equiv v$. Indeed, these equalities together with Fubini's theorem yield

$$\begin{aligned} E^{\widetilde{Q}_{\alpha,m}}\bigl[\eta(E)^{-\alpha}\bigr] &= \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}\exp\bigl[-m(E)\log\bigl(1+v^\alpha\bigr)\bigr] \\ &= \Gamma(\alpha+1)^{-1}\int_0^\infty dz \exp\bigl[-m(E)\log(1+z)\bigr] \\ &= \Gamma(\alpha+1)^{-1}\bigl(m(E)-1\bigr)^{-1}. \end{aligned}$$

As in the one-dimensional case, Theorem 9.17 in Chapter 4 of [9] reduces the proof of stationarity of (3.12) with respect to $\mathcal{A}_{\alpha,m}$ to showing that

$$\int_{\mathcal{M}_1(E)} \widetilde{P}_{\alpha,m}(d\mu)\mathcal{A}_{\alpha,m}\Phi(\mu) = 0 \tag{3.13}$$

for any Φ of the form $\Phi(\mu) = \langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$ with $f_i \in C(E)$ and n being a positive integer. Without any loss of generality, we can assume that $0 \le f_i(x) \le 1$ for any $x \in E$ and $i = 1, \dots, n$. Furthermore, we only have to consider the case where $f_1 = \cdots = f_n =: f$ because the coefficients of the monomial $t_1 \cdots t_n$ in $\langle \mu, t_1 f_1 + \cdots + t_n f_n\rangle^n$ equals $n!\langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$. Thus, we let $\Phi(\mu) = \langle \mu, f\rangle^n$ with $0 \le f(x) \le 1$ for any $x \in E$. Because of the basic relation (3.3) and (3.12) together, (3.13) can be rewritten as

$$\int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}_{\alpha,m}\Psi(\eta) = 0, \tag{3.14}$$

where $\Psi(\eta) = \langle \eta, f\rangle^n \langle \eta, 1\rangle^{-n}$. The main difficulty comes from the fact that Ψ does not belong to $\mathcal{F}$. For each $\varepsilon > 0$, introduce $\Psi_\varepsilon(\eta) := \langle \eta, f\rangle^n (\langle \eta, 1\rangle + \varepsilon)^{-n}$ and observe that $\Psi_\varepsilon \in \mathcal{F}$. Thanks to Proposition 3.3, we then have (3.14) with Ψ_ε in place of Ψ provided that $\mathcal{L}_{\alpha,m}\Psi_\varepsilon$ is bounded. Thus, the proof of (3.14) reduces to showing the following two assertions:

(i) For every $\varepsilon > 0$, $\mathcal{L}^{(1)}_{\alpha,m}\Psi_\varepsilon$, $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\varepsilon$ and $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon$ are bounded functions on $\mathcal{M}(E)$.

(ii) It holds that for each $k \in \{1, 2, 3\}$

$$\lim_{\varepsilon \downarrow 0} \int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}^{(k)}_{\alpha,m}\Psi_\varepsilon(\eta) = \int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}^{(k)}_{\alpha,m}\Psi(\eta). \tag{3.15}$$

Here, $\mathcal{L}_{\alpha,m} = \mathcal{L}^{(1)}_{\alpha,m} + \mathcal{L}^{(2)}_{\alpha,m} + \mathcal{L}^{(3)}_{\alpha,m}$, and the operators $\mathcal{L}^{(1)}_{\alpha,m}$, $\mathcal{L}^{(2)}_{\alpha,m}$ and $\mathcal{L}^{(3)}_{\alpha,m}$ correspond, respectively, to the first, second and last term on the right-hand side of (3.8).

First, we consider $\mathcal{L}^{(2)}_{\alpha,m}$. Observe that

$$\begin{aligned} \frac{\delta \Psi_\varepsilon}{\delta \eta}(r) &= \frac{n f(r)\langle \eta, f\rangle^{n-1}}{(\langle \eta, 1\rangle + \varepsilon)^n} - \frac{n\langle \eta, f\rangle^n}{(\langle \eta, 1\rangle + \varepsilon)^{n+1}} \\ &= \frac{n(f(r)\langle \eta, 1\rangle - \langle \eta, f\rangle + \varepsilon f(r))\langle \eta, f\rangle^{n-1}}{(\langle \eta, 1\rangle + \varepsilon)^{n+1}}, \end{aligned} \tag{3.16}$$

from which it follows that

$$\begin{aligned} \alpha\mathcal{L}^{(2)}_{\alpha,m}\Psi_\varepsilon(\eta) &= -\left\langle \eta, \frac{\delta \Psi_\varepsilon}{\delta \eta}\right\rangle \\ &= -\frac{n(\langle \eta, f\rangle\langle \eta, 1\rangle - \langle \eta, f\rangle\langle \eta, 1\rangle + \varepsilon\langle \eta, f\rangle)\langle \eta, f\rangle^{n-1}}{(\langle \eta, 1\rangle + \varepsilon)^{n+1}} \\ &= -n\varepsilon \frac{\Psi_\varepsilon(\eta)}{\langle \eta, 1\rangle + \varepsilon}. \end{aligned}$$

Hence, $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\varepsilon$ is a bounded function on $\mathcal{M}(E)$ and $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\varepsilon(\eta) \to 0 = \mathcal{L}^{(2)}_{\alpha,m}\Psi(\eta)$ boundedly as $\varepsilon \downarrow 0$. This proves that (i) and (ii) hold true for $\mathcal{L}^{(2)}_{\alpha,m}$.

In calculating $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon$, (3.16) is useful since $\frac{d}{dz}\Psi_\varepsilon(\eta + z\delta_r) = \frac{\delta\Psi_\varepsilon}{\delta(\eta+z\delta_r)}(r)$. Indeed, by Fubini's theorem

$$\begin{aligned}
&\int_0^\infty \frac{dz}{z^{1+\alpha}}\big[\Psi_\varepsilon(\eta+z\delta_r) - \Psi_\varepsilon(\eta)\big] \\
&\quad = \int_0^\infty \frac{dz}{z^{1+\alpha}} \int_0^z dw \frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r) \\
&\quad = \frac{1}{\alpha}\int_0^\infty w^{-\alpha}\,dw \frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r)
\end{aligned} \tag{3.17}$$

and combining with (3.16) yields

$$\begin{aligned}
&\left|\int_0^\infty \frac{dz}{z^{1+\alpha}}\big[\Psi_\varepsilon(\eta+z\delta_r) - \Psi_\varepsilon(\eta)\big]\right| \\
&\quad \le \frac{1}{\alpha}\int_0^\infty w^{-\alpha}\,dw \frac{n|f(r)\langle \eta+w\delta_r, 1\rangle - \langle \eta+w\delta_r, f\rangle + \varepsilon f(r)|\langle \eta+w\delta_r, f\rangle^{n-1}}{(\langle \eta+w\delta_r, 1\rangle + \varepsilon)^{n+1}} \\
&\quad \le \frac{n}{\alpha}\int_0^\infty w^{-\alpha}\,dw \frac{1}{\langle \eta, 1\rangle + w + \varepsilon} \\
&\quad = \frac{n}{\alpha}\int_0^\infty w^{-\alpha}\,dw \int_0^\infty dv e^{-v(\langle\eta,1\rangle + w + \varepsilon)} \\
&\quad = n\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\alpha}\big(\langle \eta, 1\rangle + \varepsilon\big)^{-\alpha}.
\end{aligned} \tag{3.18}$$

This shows not only that $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon$ is bounded but also

$$\big|\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon(\eta)\big| \le n\Gamma(\alpha)\cdot\frac{\langle m, 1\rangle}{\langle \eta, 1\rangle^\alpha},$$

which is integrable with respect to $\widetilde{Q}_{\alpha,m}$ as proved already. It can be seen also from (3.16) and (3.17) that $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon$ converges pointwise to $\mathcal{L}^{(3)}_{\alpha,m}\Psi$ as $\varepsilon \downarrow 0$. By Lebesgue's dominated convergence theorem we have proved (3.15) for $\mathcal{L}^{(3)}_{\alpha,m}$.

The final task is to deal with $\mathcal{L}^{(1)}_{\alpha,m}\Psi_\varepsilon$. Similar to (3.17)

$$\begin{aligned}
I_\varepsilon(\eta, r) &:= \int_0^\infty \frac{dz}{z^{2+\alpha}}\left[\Psi_\varepsilon(\eta+z\delta_r) - \Psi_\varepsilon(\eta) - z\frac{\delta\Psi_\varepsilon}{\delta\eta}(r)\right] \\
&= \int_0^\infty \frac{dz}{z^{2+\alpha}}\int_0^z dw\left[\frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r) - \frac{\delta\Psi_\varepsilon}{\delta\eta}(r)\right] \\
&= \frac{1}{1+\alpha}\int_0^\infty \frac{dw}{w^{1+\alpha}}\left[\frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r) - \frac{\delta\Psi_\varepsilon}{\delta\eta}(r)\right].
\end{aligned}$$

By (3.16) $\frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\varepsilon}{\delta\eta}(r)$ equals

$$\frac{(\langle\eta,1\rangle+\varepsilon)^{n+1}n(f(r)\langle\eta,1\rangle-\langle\eta,f\rangle+\varepsilon f(r))[\langle\eta+w\delta_r,f\rangle^{n-1}-\langle\eta,f\rangle^{n-1}]}{(\langle\eta,1\rangle+w+\varepsilon)^{n+1}(\langle\eta,1\rangle+\varepsilon)^{n+1}}$$
$$+\frac{[(\langle\eta,1\rangle+\varepsilon)^{n+1}-(\langle\eta,1\rangle+w+\varepsilon)^{n+1}]n(f(r)\langle\eta,1\rangle-\langle\eta,f\rangle+\varepsilon f(r))\langle\eta,f\rangle^{n-1}}{(\langle\eta,1\rangle+w+\varepsilon)^{n+1}(\langle\eta,1\rangle+\varepsilon)^{n+1}}.$$

Moreover, we have bounds

$$\begin{aligned}\left|\langle\eta+w\delta_r,f\rangle^{n-1}-\langle\eta,f\rangle^{n-1}\right|&=\left|\int_0^w dv(n-1)f(r)\langle\eta+v\delta_r,f\rangle^{n-2}\right|\\&\le w(n-1)\big(\langle\eta,1\rangle+w\big)^{n-2}\end{aligned}$$

and

$$\begin{aligned}\left|\big(\langle\eta,1\rangle+\varepsilon\big)^{n+1}-\big(\langle\eta,1\rangle+w+\varepsilon\big)^{n+1}\right|&=(n+1)\int_0^w dv\big(\langle\eta,1\rangle+v+\varepsilon\big)^n\\&\le w(n+1)\big(\langle\eta,1\rangle+w+\varepsilon\big)^n.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\frac{\delta\Psi_\varepsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\varepsilon}{\delta\eta}(r)\right|\\&\quad\le w\frac{n(\langle\eta,1\rangle+\varepsilon)^{n+2}(n-1)(\langle\eta,1\rangle+w)^{n-2}}{(\langle\eta,1\rangle+w+\varepsilon)^{n+1}(\langle\eta,1\rangle+\varepsilon)^{n+1}}\\&\qquad+w\frac{(n+1)(\langle\eta,1\rangle+w+\varepsilon)^n n(\langle\eta,1\rangle+\varepsilon)\langle\eta,1\rangle^{n-1}}{(\langle\eta,1\rangle+w+\varepsilon)^{n+1}(\langle\eta,1\rangle+\varepsilon)^{n+1}}\\&\quad\le w\frac{2n^2}{(\langle\eta,1\rangle+w+\varepsilon)(\langle\eta,1\rangle+\varepsilon)}.\end{aligned}$$

Therefore, analogous calculations to those in (3.18) lead to

$$\begin{aligned}\left|\mathcal{L}^{(1)}_{\alpha,m}\Psi_\varepsilon(\eta)\right|&=\left|\frac{\alpha+1}{\Gamma(1-\alpha)}\int_E I_\varepsilon(\eta,r)\eta(dr)\right|\\&\le 2n^2\Gamma(\alpha)\big(\langle\eta,1\rangle+\varepsilon\big)^{-\alpha}\cdot\frac{\langle\eta,1\rangle}{\langle\eta,1\rangle+\varepsilon}.\end{aligned}$$

This makes it possible to argue as in the case of $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\varepsilon$ to verify (i) and (ii) for $\mathcal{L}^{(1)}_{\alpha,m}$. We complete the proof of Proposition 3.4. $\square$

Next, we show the coincidence of two distributions (3.4) [or (3.12)] and (3.7). Before going to the proof, it is worth noting that

$$P_{\alpha,m}(\cdot)=\int_{\mathcal{M}_1(E)}\mathcal{D}_m(d\mu)\mathcal{D}^{(\alpha,\alpha)}_\mu(\cdot), \tag{3.19}$$

where in general, for $\theta > -\alpha$ and $m \in \mathcal{M}(E)$, $\mathcal{D}_m^{(\alpha,\theta)}$ is the law of the two-parameter generalization of the Dirichlet random measure with parameter (α, θ) and parameter measure m defined by

$$\mathcal{D}_m^{(\alpha,\theta)}(\cdot) = \frac{\Gamma(\theta+1)}{\Gamma((\theta/\alpha)+1)} E^{Q_{\alpha,m}}\bigl[\eta(E)^{-\theta}; \eta(E)^{-1}\eta \in \cdot\bigr].$$

(See, e.g., Section 5 of [22].) We will make use of the identity

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_m^{(\alpha,\alpha)}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} = \bigl\langle m, (1+f)^\alpha \bigr\rangle^{-1}, \qquad f \in B_+(E). \tag{3.20}$$

This is a special case of Theorem 4 in [22] and can be shown as follows:

$$\begin{aligned}
&\int_{\mathcal{M}_1(E)} \mathcal{D}_m^{(\alpha,\alpha)}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} \\
&\quad = \Gamma(\alpha+1) E^{Q_{\alpha,m}}\bigl[\langle \eta, 1\rangle^{-\alpha}\bigl(1+\langle \eta, 1\rangle^{-1}\langle \eta, f\rangle\bigr)^{-\alpha}\bigr] \\
&\quad = \Gamma(\alpha+1) E^{Q_{\alpha,m}}\bigl[\langle \eta, 1+f\rangle^{-\alpha}\bigr] \\
&\quad = \alpha \int_0^\infty dv v^{\alpha-1} \exp\bigl[-v^\alpha \bigl\langle m, (1+f)^\alpha\bigr\rangle\bigr] \\
&\quad = \bigl\langle m, (1+f)^\alpha \bigr\rangle^{-1}.
\end{aligned}$$

LEMMA 3.5. *If $m(E) > 1$, then $\widetilde{P}_{\alpha,m}$ in (3.12) coincides with $P_{\alpha,m}$ in (3.7).*

PROOF. It suffices to show that for any $f \in B_+(E)$

$$\begin{aligned}
\widetilde{I}(f) &:= \int_{\mathcal{M}_1(E)} \widetilde{P}_{\alpha,m}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} \\
&= \int_{\mathcal{M}_1(E)} P_{\alpha,m}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} =: I(f).
\end{aligned}$$

In view of (3.12), calculations similar to the proof of (3.20) show that

$$\begin{aligned}
&\bigl(\Gamma(\alpha+1)\bigl(m(E)-1\bigr)\bigr)^{-1}\widetilde{I}(f) \\
&\quad = E^{\widetilde{Q}_{\alpha,m}}\bigl[\langle \eta, 1+f\rangle^{-\alpha}\bigr] \\
&\quad = \Gamma(\alpha)^{-1} \int_0^\infty dv v^{\alpha-1} \exp\bigl[-\bigl\langle m, \log\bigl(1+v^\alpha(1+f)^\alpha\bigr)\bigr\rangle\bigr] \\
&\quad = \Gamma(\alpha+1)^{-1} \int_0^\infty dz \exp\bigl[-\bigl\langle m, \log\bigl(1+z(1+f)^\alpha\bigr)\bigr\rangle\bigr] \\
&\quad = \frac{1}{\Gamma(\alpha+1)} \int_0^1 du(1-u)^{-2} \exp\biggl[-\biggl\langle m, \log\biggl(1+\frac{u}{1-u}(1+f)^\alpha\biggr)\biggr\rangle\biggr] \\
&\quad = \frac{1}{\Gamma(\alpha+1)} \int_0^1 du(1-u)^{m(E)-2} \exp\bigl[-\bigl\langle m, \log\bigl(1+u\bigl((1+f)^\alpha-1\bigr)\bigr)\bigr\rangle\bigr]
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^1 du(1-u)^{m(E)-2}$$
$$\times \int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) \langle \mu, 1 + u\big((1+f)^\alpha - 1\big)\rangle^{-m(E)},$$

where the last equality follows from (3.6). Hence, by applying Fubini's theorem and (2.4)

$$\widetilde{I}(f) = \int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) \int_0^1 \frac{B_{1,m(E)-1}(du)}{\langle \mu, 1 + u((1+f)^\alpha - 1)\rangle^{m(E)}}$$
$$= \int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) \langle \mu, (1+f)^\alpha \rangle^{-1}.$$

On the other hand, combining (3.19) with (3.20), we get

$$I(f) = \int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) \langle \mu, (1+f)^\alpha \rangle^{-1} \tag{3.21}$$

and therefore $I(f) = \widetilde{I}(f)$ as desired. □

REMARK. The "semi-explicit" form (3.19) can be explicit if m is a probability measure. More precisely, we have $P_{\alpha,\nu} = \mathcal{D}_{\alpha\nu}$ for any $\nu \in \mathcal{M}_1(E)$. Indeed, observe that by (3.21) with $m = \nu$

$$\begin{aligned}\int_{\mathcal{M}_1(E)} P_{\alpha,\nu}(d\mu) \langle \mu, 1+f\rangle^{-\alpha} &= \int_{\mathcal{M}_1(E)} \mathcal{D}_\nu(d\mu) \langle \mu, (1+f)^\alpha \rangle^{-1} \\ &= \exp\big[-\langle \nu, \log\{(1+f)^\alpha\}\rangle\big] \\ &= \exp\big[-\langle \alpha\nu, \log(1+f)\rangle\big] \\ &= \int_{\mathcal{M}_1(E)} \mathcal{D}_{\alpha\nu}(d\mu) \langle \mu, 1+f\rangle^{-\alpha},\end{aligned}$$

where (3.6) has been applied twice. [A one-dimensional version of the identity $P_{\alpha,\nu} = \mathcal{D}_{\alpha\nu}$ is mentioned in Remark (ii) at the end of Section 2.] By (3.19) what we have just seen is rewritten as

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_\nu(d\mu) \mathcal{D}_\mu^{(\alpha,\alpha)}(\cdot) = \mathcal{D}_{\alpha\nu}(\cdot),$$

which is a special case of

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_\nu^{(\beta,\theta/\alpha)}(d\mu) \mathcal{D}_\mu^{(\alpha,\theta)}(\cdot) = \mathcal{D}_\nu^{(\alpha\beta,\theta)}(\cdot), \qquad \beta \in [0,1), \theta > -\alpha\beta.$$

Here notice that, in case $\beta = 0$, $\mathcal{D}_\nu^{(0,\theta)} = \mathcal{D}_{\theta\nu}$ by definition. This generalization can be proved analogously by virtue of the two-parameter generalization of (3.6) and (3.20). (See, e.g., Theorem 4 in [22].)

We can now prove our main result, Theorem 3.2. In the proof, we write $\theta\nu$ $[\theta > 0,\ \nu \in \mathcal{M}_1(E)]$ for the parameter measure m.

PROOF OF THEOREM 3.2. Let $\nu \in \mathcal{M}_1(E)$ be given. We first show that, for arbitrary $\theta > 0$, $P_{\alpha,\theta\nu}$ is a stationary distribution of the $\mathcal{A}_{\alpha,\theta\nu}$-process. For the same reason as in the proof of Proposition 3.4 [cf. (3.13)], it is sufficient to prove that

$$\int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\mathcal{A}_{\alpha,\theta\nu}\Phi(\mu) = 0 \tag{3.22}$$

for Φ of the form $\Phi(\mu) = \langle \mu, f\rangle^n$ with $f \in C(E)$ and n being a positive integer. Since Proposition 3.4 and Lemma 3.5 together imply that (3.22) holds true for any $\theta > 1$, it is enough to show that the left-hand side of (3.22) defines a real analytic function of $\theta > 0$. We claim that

$$\begin{aligned}
&\mathcal{A}_{\alpha,\theta\nu}\Phi(\mu)\\
&\quad= \frac{1}{\Gamma(n)}\sum_{k=2}^{n}\binom{n}{k}(1-\alpha)_{k-2}(\alpha+1)_{n-k}\bigl(\langle\mu, f^k\rangle\langle\mu, f\rangle^{n-k} - \langle\mu, f\rangle^n\bigr)\\
&\qquad+ \frac{\theta}{(\alpha+1)\Gamma(n)}\\
&\qquad\quad\times\sum_{k=1}^{n}\binom{n}{k}(1-\alpha)_{k-1}(\alpha)_{n-k}\bigl(\langle\nu, f^k\rangle\langle\mu, f\rangle^{n-k} - \langle\mu, f\rangle^n\bigr)\\
&\quad= \frac{1}{\Gamma(n)}\sum_{k=2}^{n}\binom{n}{k}(1-\alpha)_{k-2}(\alpha+1)_{n-k}\langle\mu, f^k\rangle\langle\mu, f\rangle^{n-k}\\
&\qquad+ \frac{\theta}{(\alpha+1)\Gamma(n)}\sum_{k=1}^{n}\binom{n}{k}(1-\alpha)_{k-1}(\alpha)_{n-k}\langle\nu, f^k\rangle\langle\mu, f\rangle^{n-k}\\
&\qquad- \frac{(\alpha+1)_{n-1}}{(\alpha+1)\Gamma(n)}(\theta+n-1)\langle\mu, f\rangle^n.
\end{aligned}\tag{3.23}$$

The first equality is a special case of (3.2), and the second one can be shown with the help of Leibniz's formula

$$(\phi_1\phi_2)^{(n)}(0) = \sum_{k=0}^{n}\binom{n}{k}\phi_1^{(n-k)}(0)\phi_2^{(k)}(0)$$

for $\phi_1(t) = (1-t)^{-a}$ and $\phi_2(t) = (1-t)^{-b}$ with $(a,b) = (\alpha+1, -\alpha-1)$ or $(a,b) = (\alpha, -\alpha)$. In view of (3.23), it is clear that the proof reduces to verifying real analyticity of $\int P_{\alpha,\theta\nu}(d\mu)\langle\mu, f_1\rangle\cdots\langle\mu, f_n\rangle$ in θ for arbitrary $f_1,\dots,f_n \in C(E)$.

To this end, we shall exploit the following identity which is equivalent to (3.21):

$$\int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\langle\mu, 1+f\rangle^{-\alpha} = \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\langle\mu, (1+f)^{\alpha}\rangle^{-1}, \tag{3.24}$$

where $f \in B_+(E)$ is arbitrary. Clearly, this remains true for all bounded Borel functions f on E such that $\inf_{r\in E} f(r) > -1$. Therefore, for any $t_1, \dots, t_n \in \mathbf{R}$ with $|t_1| + \cdots + |t_n|$ being sufficiently small, (3.24) for $f = -\sum_{i=1}^n t_i f_i$ is valid, that is, $I(t_1, \dots, t_n) = J(t_1, \dots, t_n)$, where

$$I(t_1, \dots, t_n) = \int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\left(1 - \left\langle\mu, \sum_{i=1}^n t_i f_i\right\rangle\right)^{-\alpha} \tag{3.25}$$

and

$$J(t_1, \dots, t_n) = \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\left\langle\mu, \left(1 - \sum_{i=1}^n t_i f_i\right)^{\alpha}\right\rangle^{-1}. \tag{3.26}$$

Noting that $(1-t)^{-\alpha} = 1 + \sum_{k=1}^{\infty} (\alpha)_k t^k/k!$ as long as $|t|$ is small enough, we see from (3.25) that the coefficient of the monomial $t_1 \cdots t_n$ in the expansion of $I(t_1, \dots, t_n)$ is given by

$$(\alpha)_n \int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\langle\mu, f_1\rangle \cdots \langle\mu, f_n\rangle. \tag{3.27}$$

To find the corresponding coefficient for $J(t_1, \dots, t_n)$, define

$$h_\alpha(t) = 1 - (1-t)^{\alpha} = \alpha \sum_{l=1}^{\infty} (1-\alpha)_{l-1} t^l / l!$$

and observe from (3.26) that $J(t_1, \dots, t_n)$ equals

$$\begin{aligned}
&\int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\left\langle\mu, 1 - h_\alpha\left(\sum_{i=1}^n t_i f_i\right)\right\rangle^{-1} \\
&\quad = 1 + \sum_{k=1}^{\infty} \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\left\langle\mu, h_\alpha\left(\sum_{i=1}^n t_i f_i\right)\right\rangle^{k} \\
&\quad = 1 + \sum_{k=1}^{\infty} \alpha^k \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu) \sum_{l_1,\dots,l_k=1}^{\infty} \prod_{j=1}^{k} \left\{\frac{(1-\alpha)_{l_j-1}}{l_j!}\left\langle\mu, \left(\sum_{i=1}^n t_i f_i\right)^{l_j}\right\rangle\right\}.
\end{aligned}$$

One can see that the coefficient of the monomial $t_1 \cdots t_n$ in the expansion of $J(t_1, \dots, t_n)$ can be expressed as

$$\sum_{k=1}^{n} \alpha^k k! \sum_{\gamma\in\pi(n,k)} \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu) \prod_{j=1}^{k} \left\{\frac{(1-\alpha)_{|\gamma_j|-1}}{|\gamma_j|!}\left\langle\mu, \prod_{i\in\gamma_j} f_i\right\rangle\right\}, \tag{3.28}$$

where $\pi(n,k)$ is the set of partitions γ of $\{1,\dots,n\}$ into k unordered nonempty subsets $\gamma_1,\dots,\gamma_k$. By Lemma 2.2 of [7] (or equivalently by Lemma 2.4 of [8]), each integral in the above sum is a real analytic function of $\theta > 0$. Hence, so is the integral in (3.27) and the stationarity of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ follows.

It remains to prove the uniqueness of stationary distribution P of the $\mathcal{A}_{\alpha,\theta\nu}$-process for each $\theta > 0$. But this is an immediate consequence of (3.22) with P in place of $P_{\alpha,\theta\nu}$ and (3.23), which together determine uniquely $\int P(d\mu)\langle\mu, f\rangle^n$ and hence the nth moment measure

$$M_n(dr_1\cdots dr_n) := \int_{\mathcal{M}_1(E)} P(d\mu)\mu(dr_1)\cdots\mu(dr_n)$$

for any $n = 1, 2, \dots$. This completes the proof of Theorem 3.2. $\square$

It is not clear whether we can derive from (3.28) an extension of the Ewens sampling formula in some explicit and informative form. (See Remarks after the proof of Lemma 2.2 in [7].) In view of (3.19), one might think that Pitman's sampling formula would be applicable. But it is not the case since $\mathcal{D}_m(\mu \text{ is discrete}) = 1$. The expression (3.12) might be rather useful for such a purpose.

4. Irreversibility. In this section, we discuss reversibility of our processes. In contrast with the Fleming–Viot diffusion case, we guess that for any $0 < \alpha < 1$ and nondegenerate m the $\mathcal{A}_{\alpha,m}$-process would be irreversible. Unfortunately, the following result does not give an affirmative answer in all cases. However, this does not suggest any possibility of the reversibility in the exceptional case, which is believed to be dealt with a different choice of test functions.

THEOREM 4.1. *Let $m \in \mathcal{M}(E)^\circ$ be given. Assume that either of the following two conditions holds.*

(i) *The support of m has at least three distinct points.*

(ii) *The support of m has exactly two points, say r_1 and r_2 and $m(\{r_1\}) \neq m(\{r_2\})$.*

Then the stationary distribution $P_{\alpha,m}$ of the $\mathcal{A}_{\alpha,m}$-process is not a reversible distribution of it.

PROOF. As in the proof of Theorem 3.2, we write $\theta\nu$ instead of m. Thus, $\theta > 0$ and $\nu \in \mathcal{M}_1(E)$. Recall that an equivalent condition to the reversibility of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ is the symmetry

$$E[\Phi\mathcal{A}_{\alpha,\theta\nu}\Phi'] = E[\Phi'\mathcal{A}_{\alpha,\theta\nu}\Phi], \qquad \Phi, \Phi' \in \mathcal{F}_0,$$

in which $E[\cdot]$ stands for the expectation with respect to $P_{\alpha,\theta\nu}$. (See the proof of Theorem 2.3 in [7].) In the rest of the proof, we suppress the suffix "$\alpha, \theta\nu$" for simplicity. Let $f \in C(E)$ be given and define $\Phi_n(\mu) = \langle \mu, f \rangle^n$ for each positive integer n. We are going to calculate

$$\Delta := E[\Phi_2 \mathcal{A} \Phi_1] - E[\Phi_1 \mathcal{A} \Phi_2]. \tag{4.1}$$

For this purpose, observe from (3.23) that

$$\mathcal{A}\Phi_1(\mu) = \frac{\theta}{\alpha+1}\bigl(\langle \nu, f\rangle - \langle \mu, f\rangle\bigr), \tag{4.2}$$

$$\begin{aligned} \mathcal{A}\Phi_2(\mu) = {} & \langle \mu, f^2\rangle + \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle\langle \mu, f\rangle \\ & + \frac{(1-\alpha)\theta}{\alpha+1}\langle \nu, f^2\rangle - (\theta+1)\langle \mu, f\rangle^2 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & \Gamma(3)\mathcal{A}\Phi_3(\mu) \\ & \quad = 3(\alpha+1)\langle \mu, f^2\rangle\langle \mu, f\rangle + (1-\alpha)\langle \mu, f^3\rangle \\ & \qquad + \frac{\theta}{\alpha+1}\cdot 3\alpha(\alpha+1)\langle \nu, f\rangle\langle \mu, f\rangle^2 \\ & \qquad + \frac{\theta}{\alpha+1}\cdot 3(1-\alpha)\alpha\langle \nu, f^2\rangle\langle \mu, f\rangle \\ & \qquad + \frac{\theta}{\alpha+1}\cdot (1-\alpha)(2-\alpha)\langle \nu, f^3\rangle - (\alpha+2)(\theta+2)\langle \mu, f\rangle^3. \end{aligned} \tag{4.4}$$

Combining (4.2) with the stationarity $E[\mathcal{A}\Phi_1] = 0$, we get $E[\langle \mu, f\rangle] = \langle \nu, f\rangle$. Therefore, it is possible to deduce from (4.3) and $E[\mathcal{A}\Phi_2] = 0$

$$(\theta+1)E\bigl[\langle \mu, f\rangle^2\bigr] = \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle^2 + \left(1 + \frac{(1-\alpha)}{\alpha+1}\theta\right)\langle \nu, f^2\rangle.$$

Moreover, this equality between quadratic forms is enough to imply the one between symmetric bilinear forms:

$$\begin{aligned} & (\theta+1)E\bigl[\langle \mu, f\rangle\langle \mu, g\rangle\bigr] \\ & \qquad = \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle\langle \nu, g\rangle + \left(1 + \frac{(1-\alpha)}{\alpha+1}\theta\right)\langle \nu, fg\rangle, \end{aligned} \tag{4.5}$$

where $g \in C(E)$ is also arbitrary. In the rest of the proof, we assume that $\langle \nu, f\rangle = 0$. This makes the calculations below considerably simple. By (4.5)

$$M_{1,2} := E\bigl[\langle \mu, f\rangle\langle \mu, f^2\rangle\bigr] = \frac{(\alpha+1) + (1-\alpha)\theta}{(\alpha+1)(\theta+1)}\langle \nu, f^3\rangle. \tag{4.6}$$

The equality $E[\mathcal{A}\Phi_3] = 0$ together with (4.4) implies that

$$\begin{aligned}(4.7)\qquad &(\alpha+2)(\theta+2)E\big[\langle\mu, f\rangle^3\big]\\ &\quad = 3(\alpha+1)M_{1,2} + (1-\alpha)\left(1+\frac{2-\alpha}{\alpha+1}\theta\right)\langle\nu, f^3\rangle.\end{aligned}$$

These preliminaries help us calculate Δ in (4.1) as follows. By (4.3) and (4.4)

$$\begin{aligned}\Delta &= E\left[\langle\mu, f\rangle^2\left(-\frac{\theta}{\alpha+1}\langle\mu, f\rangle\right)\right] - E\big[\langle\mu, f\rangle\big(\langle\mu, f^2\rangle - (\theta+1)\langle\mu, f\rangle^2\big)\big]\\ &= \frac{(\alpha+1)+\alpha\theta}{\alpha+1}E\big[\langle\mu, f\rangle^3\big] - M_{1,2}\end{aligned}$$

and hence (4.7) yields

$$\begin{aligned}&(\alpha+1)(\alpha+2)(\theta+2)\Delta\\ &\quad = \big[(\alpha+1)+\alpha\theta\big]\left[3(\alpha+1)M_{1,2} + (1-\alpha)\left(1+\frac{2-\alpha}{\alpha+1}\theta\right)\langle\nu, f^3\rangle\right]\\ &\qquad - (\alpha+1)(\alpha+2)(\theta+2)M_{1,2}\\ &\quad = (\alpha+1)(\alpha-1)(2\theta+1)M_{1,2}\\ &\qquad + \big[(\alpha+1)+\alpha\theta\big](1-\alpha)\left(1+\frac{2-\alpha}{\alpha+1}\theta\right)\langle\nu, f^3\rangle.\end{aligned}$$

Plugging (4.6) into this expression, we obtain

$$(\alpha+1)(\alpha+2)(\theta+2)\Delta = \frac{1-\alpha}{(\alpha+1)(\theta+1)}U(\alpha,\theta)\langle\nu, f^3\rangle,$$

where

$$\begin{aligned}U(\alpha,\theta) &= -(\alpha+1)(2\theta+1)\big[(\alpha+1)+(1-\alpha)\theta\big]\\ &\qquad + \big[(\alpha+1)+\alpha\theta\big](\theta+1)\big[(\alpha+1)+(2-\alpha)\theta\big]\\ &= \alpha\theta^2\big[(\alpha+4)+(2-\alpha)\theta\big] =: V(\alpha,\theta).\end{aligned}$$

[The second equality between quadratic functions of α is verified by checking that $U(-1,\theta) = -3\theta^2(\theta+1) = V(-1,\theta)$, $U(0,\theta) = 0 = V(0,\theta)$ and $U(1,\theta) = \theta^2(\theta+5) = V(1,\theta)$.] Consequently, whenever $\langle\nu, f\rangle = 0$, we have

$$\Delta = \frac{\alpha(1-\alpha)\theta^2[(\alpha+4)+(2-\alpha)\theta]}{(\alpha+1)^2(\alpha+2)(\theta+1)(\theta+2)}\langle\nu, f^3\rangle.$$

Thus, all that remains is to construct an $f \in C(E)$ such that $\langle\nu, f\rangle = 0$ and $\langle\nu, f^3\rangle > 0$. Because of the assumption, we can choose a closed subset E_0 of E such that $0 < \nu(E_0) < 1/2$. Indeed, in the case (ii) this is trivial while in the case (i) there exist disjoint closed subsets E_1, E_2 and E_3 of E such that

$\nu(E_1)\nu(E_2)\nu(E_3) > 0$ and so $0 < \nu(E_i) < 1/2$ for some $i \in \{1, 2, 3\}$. Letting g denote the indicator function of E_0, we observe that

$$\begin{aligned}\langle \nu, (g - \langle \nu, g\rangle)^3\rangle &= \langle \nu, g^3\rangle - 3\langle \nu, g^2\rangle\langle \nu, g\rangle + 3\langle \nu, g\rangle\langle \nu, g\rangle^2 - \langle \nu, g\rangle^3 \\ &= \nu(E_0) - 3\nu(E_0)^2 + 2\nu(E_0)^3 \\ &= \nu(E_0)(1 - \nu(E_0))(1 - 2\nu(E_0)) > 0.\end{aligned}$$

Finally, the required f exists since g can be approximated boundedly and pointwise by a sequence of functions in $C(E)$. The proof of the theorem is complete. □

It is worth noting that the exceptional case of Theorem 4.1 corresponds to a subclass of the one-dimensional case discussed in Section 1, more specifically, the process generated by (1.3) with $c_1 = c_2$. There is no reason why this class should be so special with respect to the reversibility, and it seems that such a "spatial symmetry" makes it more subtle to see the asymmetry in time. The actual difficulty in showing the irreversibility for these processes along similar lines to the above proof is that expressions of $E[\Phi_{n_1}\mathcal{A}\Phi_{n_2}]$ with $n_1 + n_2 \geq 4$ as functions of α and θ are too complicated to handle.

Acknowledgment. The author would like to thank the referees for pointing out some mistakes and helpful comments on the earlier version of the manuscript. He is indebted to Professor S. Mano for the reference to [2]. Part of the work was carried out under the ISM Cooperative Research Program (2012·ISM·CRP-5008).

REFERENCES

[1] BERTOIN, J. and LE GALL, J.-F. (2003). Stochastic flows associated to coalescent processes. *Probab. Theory Related Fields* **126** 261–288. MR1990057

[2] BIRKNER, M. and BLATH, J. (2009). Measure-valued diffusions, general coalescents and population genetic inference. In *Trends in Stochastic Analysis* 329–363. Cambridge Univ. Press, Cambridge. MR2562160

[3] BIRKNER, M., BLATH, J., CAPALDO, M., ETHERIDGE, A., MÖHLE, M., SCHWEINSBERG, J. and WAKOLBINGER, A. (2005). Alpha-stable branching and beta-coalescents. *Electron. J. Probab.* **10** 303–325 (electronic). MR2120246

[4] CIFARELLI, D. M. and REGAZZINI, E. (1990). Distribution functions of means of a Dirichlet process. *Ann. Statist.* **18** 429–442. MR1041402

[5] DONNELLY, P. and KURTZ, T. G. (1999). Particle representations for measure-valued population models. *Ann. Probab.* **27** 166–205. MR1681126

[6] ETHERIDGE, A. and MARCH, P. (1991). A note on superprocesses. *Probab. Theory Related Fields* **89** 141–147. MR1110534

[7] ETHIER, S. N. (1990). The infinitely-many-neutral-alleles diffusion model with ages. *Adv. in Appl. Probab.* **22** 1–24. MR1039374

[8] ETHIER, S. N. and GRIFFITHS, R. C. (1993). The transition function of a Fleming–Viot process. *Ann. Probab.* **21** 1571–1590. MR1235429

[9] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

[10] ETHIER, S. N. and KURTZ, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim*. **31** 345–386. MR1205982

[11] FERGUSON, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Ann*. *Statist*. **1** 209–230. MR0350949

[12] FOUCART, C. (2011). Distinguished exchangeable coalescents and generalized Fleming–Viot processes with immigration. *Adv. in Appl*. *Probab.* **43** 348–374. MR2848380

[13] FOUCART, C. and HÉNARD, O. (2013). Stable continuous-state branching processes with immigration and Beta–Fleming–Viot processes with immigration. *Electron*. *J. Probab.* **18** 1–21. MR3035751

[14] HIRABA, S. (2000). Jump-type Fleming–Viot processes. *Adv. in Appl*. *Probab.* **32** 140–158. MR1765166

[15] KAWAZU, K. and WATANABE, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl*. **16** 36–54.

[16] LEBEDEV, N. N. (1972). *Special Functions and Their Applications*, Revised ed. Dover, New York. MR0350075

[17] LI, Z. (2011). *Measure-Valued Branching Markov Processes*. Springer, Heidelberg. MR2760602

[18] PERKINS, E. A. (1992). Conditional Dawson–Watanabe processes and Fleming–Viot processes. In *Seminar on Stochastic Processes*, 1991 (*Los Angeles*, *CA*, 1991). *Progress in Probability* **29** 143–156. Birkhäuser, Boston, MA. MR1172149

[19] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press, Cambridge. MR1739520

[20] SHIGA, T. (1990). A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes. *J. Math*. *Kyoto Univ.* **30** 245–279. MR1068791

[21] STANNAT, W. (2003). Spectral properties for a class of continuous state branching processes with immigration. *J. Funct*. *Anal*. **201** 185–227. MR1986159

[22] VERSHIK, A. M., YOR, M. and TSILEVICH, N. V. (2004). The Markov–Kreĭn identity and the quasi-invariance of the gamma process. *J. Math*. *Sci*. **121** 2303–2310.

[23] YANO, Y. (2006). On the occupation time on the half line of pinned diffusion processes. *Publ*. *Res*. *Inst*. *Math*. *Sci*. **42** 787–802. MR2266997

DEPARTMENT OF MATHEMATICS
SAGA UNIVERSITY
SAGA 840-8502
JAPAN
E-MAIL: handa@ms.saga-u.ac.jp