

OPTIMAL TRANSPORTATION UNDER CONTROLLED STOCHASTIC DYNAMICS¹

BY XIAOLU TAN AND NIZAR TOUZI

Ecole Polytechnique, Paris

We consider an extension of the Monge–Kantorovitch optimal transportation problem. The mass is transported along a continuous semimartingale, and the cost of transportation depends on the drift and the diffusion coefficients of the continuous semimartingale. The optimal transportation problem minimizes the cost among all continuous semimartingales with given initial and terminal distributions. Our first main result is an extension of the Kantorovitch duality to this context. We also suggest a finite-difference scheme combined with the gradient projection algorithm to approximate the dual value. We prove the convergence of the scheme, and we derive a rate of convergence.

We finally provide an application in the context of financial mathematics, which originally motivated our extension of the Monge–Kantorovitch problem. Namely, we implement our scheme to approximate no-arbitrage bounds on the prices of exotic options given the implied volatility curve of some maturity.

1. Introduction. In the classical mass transportation problem of Monge–Kantorovich, we fix at first an initial probability distribution μ_0 and a terminal distribution μ_1 on \mathbb{R}^d . An admissible transportation plan is defined as a random vector (X_0, X_1) (or, equivalently, a joint distribution on $\mathbb{R}^d \times \mathbb{R}^d$) such that the marginal distributions are, respectively, μ_0 and μ_1 . By transporting the mass from the position $X_0(\omega)$ to the position $X_1(\omega)$, an admissible plan transports a mass from the distribution μ_0 to the distribution μ_1 . The transportation cost is a function of the initial and final positions, given by $\mathbb{E}[c(X_0, X_1)]$ for some function $c: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^+$. The Monge–Kantorovich problem consists in minimizing the cost among all admissible transportation plans. Under mild conditions, a duality result is established by Kantorovich, converting the problem into an optimization problem under linear constraints. We refer to Villani [36] and Rachev and Ruschendorf [32] for this classical duality and the richest development on the classical mass transportation problem.

Received July 2011; revised June 2012.

¹Supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and CA-CIB.

MSC2010 subject classifications. Primary 60H30, 65K99; secondary 65P99.

Key words and phrases. Mass transportation, Kantorovich duality, viscosity solutions, gradient projection algorithm.

As an extension of the Monge–Kantorovitch problem, Mikami and Thieullen [30] introduced the following stochastic mass transportation mechanism. Let X be an \mathbb{R}^d -continuous semimartingale with decomposition

$$(1.1) \quad X_t = X_0 + \int_0^t \beta_s ds + W_t,$$

where W_t is a d -dimensional standard Brownian motion under the filtration \mathbb{F}^X generated by X . The optimal mass transportation problem consists in minimizing the cost of transportation defined by some cost functional ℓ along all transportation plans with initial distribution μ_0 and final distribution μ_1 :

$$V(\mu_0, \mu_1) := \inf \mathbb{E} \int_0^1 \ell(s, X_s, \beta_s) ds,$$

where the infimum is taken over all semimartingales given by (1.1) satisfying $\mathbb{P} \circ X_0^{-1} = \mu_0$ and $\mathbb{P} \circ X_1^{-1} = \mu_1$. Mikami and Thieullen [30] proved a strong duality result, thus extending the classical Kantorovitch duality to this context.

Motivated by a problem in financial mathematics, our main objective is to extend [30] to a larger class of transportation plans defined by continuous semimartingales with absolutely continuous characteristics:

$$X_t = X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s,$$

where the pair process $(\alpha := \sigma\sigma^T, \beta)$ takes values in some closed convex subset U of $\mathbb{R}^{d \times d} \times \mathbb{R}^d$, and the transportation cost involves the drift and diffusion coefficients as well as the trajectory of X .

The simplest motivating problem in financial mathematics is the following. Let X be the price process of some tradable security, and consider some path-dependent derivative security $\xi(X_t, t \leq 1)$. Then, by the no-arbitrage theory, any martingale measure \mathbb{P} (i.e., probability measure under which X is a martingale) induces an admissible no-arbitrage price $\mathbb{E}^{\mathbb{P}}[\xi]$ for the derivative security ξ . Suppose further that the prices of all 1-maturity European call options with all possible strikes are available. This is a standard assumption made by practitioners on liquid options markets. Then, the collection of admissible martingale measures is reduced to those which are consistent with this information, that is, $c_1(y) := \mathbb{E}^{\mathbb{P}}[(X_1 - y)^+]$ is given for all $y \in \mathbb{R}$ or, equivalently, the marginal distribution of X_1 under \mathbb{P} is given by $\mu_1[y, \infty) := -\partial^- c_1(y)$, where $\partial^- c_1$ denotes the left-hand side derivative of the convex function c_1 . Hence, a natural formulation of the no-arbitrage lower and upper bounds is $\inf \mathbb{E}^{\mathbb{P}}[\xi]$ and $\sup \mathbb{E}^{\mathbb{P}}[\xi]$ with optimization over the set of all probability measures \mathbb{P} satisfying $\mathbb{P} \circ (X_0)^{-1} = \delta_x$ and $\mathbb{P} \circ (X_1)^{-1} = \mu_1$, for some initial value of the underlying asset price $X_0 = x$. We refer to Galichon, Henry-Labordère and Touzi [21] for the connection to the so-called model-free superhedging problem. In Section 5.4 we shall provide some applications of our results

in the context of variance options $\xi = \langle \log X \rangle_1$ and the corresponding weighted variance options extension.

This problem is also intimately connected to the so-called Skorokhod Embedding Problem (SEP) that we now recall; see Obloj [31] for a review. Given a one-dimensional Brownian motion W and a centered $|x|$ -integrable probability distribution μ_1 on \mathbb{R} , the SEP consists in searching for a stopping time τ such that $W_\tau \sim \mu_1$ and $(W_{t \wedge \tau})_{t \geq 0}$ is uniformly integrable. This problem is well known to have infinitely many solutions. However, some solutions have been proved to satisfy some optimality with respect to some criterion (Azéma and Yor [1], Root [33] and Rost [34]). In order to recast the SEP in our context, we specify the set U , where the characteristics take values, to $U = \mathbb{R} \times \{0\}$, that is, transportation along a local martingale. Indeed, given a solution τ of the SEP, the process $X_t := W_{\tau \wedge t / (1-t)}$ defines a continuous local martingale satisfying $X_1 \sim \mu_1$. Conversely, every continuous local martingale can be represented as time-changed Brownian motion by the Dubins–Schwarz theorem (see, e.g., Theorem 4.6, Chapter 3 of Karatzas and Shreve [26]).

We note that the seminal paper by Hobson [23] is crucially based on the connection between the SEP and the above problem of no-arbitrage bounds for a specific restricted class of derivatives prices (e.g., variance options, lookback option, etc.). We refer to Hobson [24] for an overview on some specific applications of the SEP in the context of finance.

Our first main result is to establish the Kantorovitch strong duality for our semimartingale optimal transportation problem. The dual value function consists in the minimization of $\mu_0(\lambda_0) - \mu_1(\lambda_1)$ over all continuous and bounded functions λ_1 , where λ_0 is the initial value of a standard stochastic control problem with final cost λ_1 . In the Markovian case, the function λ_0 can be characterized as the unique viscosity solution of the corresponding dynamics programming equation with terminal condition λ_1 .

Our second main contribution is to exploit the dual formulation for the purpose of numerical approximation of the optimal cost of transportation. To the best of our knowledge, the first attempt for the numerical approximation of an intimately related problem, in the context of financial mathematics, was initiated by Bonnans and Tan [10]. In this paper, we follow their approach in the context of a bounded set of admissible semimartingale characteristics. Our numerical scheme combines the finite difference scheme and the gradient projection algorithm. We prove convergence of the scheme, and we derive a rate of convergence. We also implement our numerical scheme and give some numerical experiments.

The paper is organized as follows. Section 2 introduces the optimal mass transportation problem under controlled stochastic dynamics. In Section 3 we extend the Kantorovitch duality to our context by using the classical convex duality approach. The convex conjugate of the primal problem turns out to be the value function of a classical stochastic control problem with final condition given by the

Lagrange multiplier lying in the space of bounded continuous functions. Then the dual formulation consists in maximizing this value over the class of all Lagrange multipliers. We also show, under some conditions, that the Lagrange multipliers can be restricted to the subclass of C^∞ -functions with bounded derivatives of any order. In the Markovian case, we characterize the convex dual as the viscosity solution of a dynamic programming equation in the Markovian case in Section 4. Further, when the characteristics are restricted to a bounded set, we use the probabilistic arguments to restrict the computation of the optimal control problem to a bounded domain of \mathbb{R}^d .

Section 5 introduces a numerical scheme to approximate the dual formulation in the Markovian case. We first use the finite difference scheme to solve the control problem. The maximization is then approximated by means of the gradient projection algorithm. We provide some general convergence results together with some control of the error. Finally, we implement our algorithm and provide some numerical examples in the context of its applications in financial mathematics. Namely, we consider the problem of robust hedging weighted variance swap derivatives given the prices of European options of all strikes. The solution of the last problem can be computed explicitly and allows to test the accuracy of our algorithm.

NOTATION. Given a Polish space E , we denote by $\mathbf{M}(E)$ the space of all Borel probability measures on E , equipped with the weak topology, which is also a Polish space. In particular, $\mathbf{M}(\mathbb{R}^d)$ is the space of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. S_d denotes the set of $d \times d$ positive symmetric matrices. Given $u = (a, b) \in S_d \times \mathbb{R}^d$, we define $|u|$ by its L^2 -norm as an element in \mathbb{R}^{d^2+d} . Finally, for every constant $C \in \mathbb{R}$, we make the convention $\infty + C = \infty$.

2. The semimartingale transportation problem. Let $\Omega := C([0, 1], \mathbb{R}^d)$ be the canonical space, X be the canonical process

$$X_t(\omega) := \omega_t \quad \text{for all } t \in [0, 1],$$

and $\mathbb{F} = (\mathcal{F}_t)_{1 \leq t \leq 1}$ be the canonical filtration generated by X . We recall that \mathcal{F}_t coincides with the Borel σ -field on Ω induced by the seminorm $|\omega|_{\infty, t} := \sup_{0 \leq s \leq t} |\omega_s|$, $\omega \in \Omega$ (see, e.g., the discussions in Section 1.3, Chapter 1 of Stroock and Varadhan [35]).

Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}_1) under which the canonical process X is a \mathbb{F} -continuous semimartingale. Then, we have the unique continuous decomposition w.r.t. \mathbb{F} :

$$(2.1) \quad X_t = X_0 + B_t^{\mathbb{P}} + M_t^{\mathbb{P}}, \quad t \in [0, 1], \mathbb{P}\text{-a.s.},$$

where $B^{\mathbb{P}} = (B_t^{\mathbb{P}})_{0 \leq t \leq 1}$ is the finite variation part and $M^{\mathbb{P}} = (M_t^{\mathbb{P}})_{0 \leq t \leq 1}$ is the local martingale part satisfying $B_0 = M_0 = 0$. Denote by $A_t^{\mathbb{P}} := \langle M^{\mathbb{P}} \rangle_t$ the quadratic variation of $M^{\mathbb{P}}$ between 0 and t and $A^{\mathbb{P}} = (A_t^{\mathbb{P}})_{0 \leq t \leq 1}$. Then, following Jacod and

Shiryaev [25], we say that the \mathbb{P} -continuous semimartingale X has characteristics $(A^\mathbb{P}, B^\mathbb{P})$.

In this paper, we further restrict to the case where the processes $A^\mathbb{P}$ and $B^\mathbb{P}$ are absolutely continuous in t w.r.t. the Lebesgue measure, \mathbb{P} -a.s. Then there are \mathbb{F} -progressive processes $\nu^\mathbb{P} = (\alpha^\mathbb{P}, \beta^\mathbb{P})$ (see, e.g., Proposition I.3.13 of [25]) such that

$$(2.2) \quad A_t^\mathbb{P} = \int_0^t \alpha_s^\mathbb{P} ds, \quad B_t^\mathbb{P} = \int_0^t \beta_s^\mathbb{P} ds, \quad \mathbb{P}\text{-a.s. for all } t \in [0, 1].$$

REMARK 2.1. By Doob’s martingale representation theorem (see, e.g., Theorem 4.2 in Chapter 3 of Karatzas and Shreve [26]), we can find a Brownian motion $W^\mathbb{P}$ (possibly in an enlarged space) such that X has an Itô representation:

$$X_t = X_0 + \int_0^t \beta_s^\mathbb{P} ds + \int_0^t \sigma_s^\mathbb{P} dW_s^\mathbb{P},$$

where $\sigma_t^\mathbb{P} = (\alpha_t^\mathbb{P})^{1/2}$ [i.e., $\alpha_t^\mathbb{P} = \sigma_t^\mathbb{P}(\sigma_t^\mathbb{P})^T$].

REMARK 2.2. With the unique processes $(A^\mathbb{P}, B^\mathbb{P})$, the progressively measurable processes $\nu^\mathbb{P} = (\alpha^\mathbb{P}, \beta^\mathbb{P})$ may not be unique. However, they are unique in sense $d\mathbb{P} \times dt$ -a.e. Since the transportation cost defined below is a $d\mathbb{P} \times dt$ integral, then the choice of $\nu^\mathbb{P} = (\alpha^\mathbb{P}, \beta^\mathbb{P})$ will not change the cost value and then is not essential.

We next introduce the set U defining some restrictions on the admissible characteristics:

$$(2.3) \quad U \text{ closed and convex subset of } S_d \times \mathbb{R}^d,$$

and we denote by \mathcal{P} the set of probability measures \mathbb{P} on Ω under which X has the decomposition (2.1) and satisfies (2.2) with characteristics $\nu_t^\mathbb{P} := (\alpha_t^\mathbb{P}, \beta_t^\mathbb{P}) \in U$, $d\mathbb{P} \times dt$ -a.e.

Given two arbitrary probability measures μ_0 and μ_1 in $\mathbf{M}(\mathbb{R}^d)$, we also denote

$$(2.4) \quad \mathcal{P}(\mu_0) := \{\mathbb{P} \in \mathcal{P} : \mathbb{P} \circ X_0^{-1} = \mu_0\},$$

$$(2.5) \quad \mathcal{P}(\mu_0, \mu_1) := \{\mathbb{P} \in \mathcal{P}(\mu_0) : \mathbb{P} \circ X_1^{-1} = \mu_1\}.$$

REMARK 2.3. (i) In general, $\mathcal{P}(\mu_0, \mu_1)$ may be empty. However, in the one-dimensional case $d = 1$ and $U = \mathbb{R}^+ \times \mathbb{R}$, the initial distribution $\mu_0 = \delta_{x_0}$ for some constant $x_0 \in \mathbb{R}$, and the final distribution satisfies $\int_{\mathbb{R}} |x| \mu_1(dx) < \infty$, we now verify that $\mathcal{P}(\mu_0, \mu_1)$ is not empty. First, we can choose any constant in \mathbb{R} for the drift part, hence, we can suppose, without loss of generality, that $x_0 = 0$ and μ_1 is centered distributed, that is, $\int_{\mathbb{R}} x \mu_1(dx) = 0$. Then, given a Brownian motion W , by Skorokhod embedding (see, e.g., Section 3 of Obloj [31]), there is a stopping

time τ such that $W_\tau \sim \mu_1$ and $(W_{t \wedge \tau})_{t \geq 0}$ is uniformly integrable. Therefore, $M = (M_t)_{0 \leq t \leq 1}$ defined by $M_t := W_{\tau \wedge t / (1-t)}$ is a continuous martingale with marginal distribution $\mathbb{P} \circ M_1^{-1} = \mu_1$. Moreover, its quadratic variation $\langle M \rangle_t = \tau \wedge \frac{t}{1-t}$ is absolutely continuous in t w.r.t. Lebesgue for every fixed ω , which can induce a probability on Ω belonging to $\mathcal{P}(\mu_0, \mu_1)$.

(ii) Let $d = 1$, $U = \mathbb{R}^+ \times \{0\}$, $\mu_0 = \delta_{x_0}$ for some constant $x_0 \in \mathbb{R}$, and μ_1 as in (i) with $\int x \mu_1(dx) = x_0$. Then, by the above discussion, we also have $\mathcal{P}(\mu_0, \mu_1) \neq \emptyset$.

The semimartingale X under \mathbb{P} can be viewed as a vehicle of mass transportation, from the \mathbb{P} -distribution of X_0 to the \mathbb{P} -distribution of X_1 . We then associate \mathbb{P} with a transportation cost

$$(2.6) \quad J(\mathbb{P}) := \mathbb{E}^{\mathbb{P}} \int_0^1 L(t, X, v_t^{\mathbb{P}}) dt,$$

where $\mathbb{E}^{\mathbb{P}}$ denotes the expectation under the probability measure \mathbb{P} , and $L : [0, 1] \times \Omega, \times U \rightarrow \mathbb{R}$. The above expectation is well defined on $\mathbb{R}^+ \cup \{+\infty\}$ in view of the subsequent Assumption 3.1 which states, in particular, that L is nonnegative.

Our main interest is on the following optimal mass transportation problem, given two probability measures $\mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d)$:

$$(2.7) \quad V(\mu_0, \mu_1) := \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} J(\mathbb{P}),$$

with the convention $\inf \emptyset = \infty$.

3. The duality theorem. The main objective of this section is to prove a duality result for problem (2.7) which extends the classical Kantorovitch duality in optimal transportation theory.

This will be achieved by classical convex duality techniques which require to verify that the function V is convex and lower semicontinuous. For the general theory on duality analysis in Banach spaces, we refer to Bonnans and Shapiro [9] and Ekeland and Temam [18]. In our context, the value function of the optimal transportation problem is defined on the Polish space of measures on \mathbb{R}^d , and our main reference is Deuschel and Stroock [17].

3.1. *The main duality result.* We first formulate the assumptions needed for our duality result.

ASSUMPTION 3.1. The function $L : (t, \mathbf{x}, u) \in [0, 1] \times \Omega \times U \mapsto L(t, \mathbf{x}, u) \in \mathbb{R}^+$ is nonnegative, continuous in (t, \mathbf{x}, u) , and convex in u .

Notice that we do not impose any progressive measurability for the dependence of L on the trajectory \mathbf{x} . However, by immediate conditioning, we may reduce the problem so that such a progressive measurability is satisfied.

The next condition controls the dependence of the cost functional on the time variable.

ASSUMPTION 3.2. The function L is uniformly continuous in t in the sense that

$$\Delta_t L(\varepsilon) := \sup \frac{|L(s, \mathbf{x}, u) - L(t, \mathbf{x}, u)|}{1 + L(t, \mathbf{x}, u)} \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where the supremum is taken over all $0 \leq s, t \leq 1$ such that $|t - s| \leq \varepsilon$ and all $\mathbf{x} \in \Omega, u \in U$.

We finally need the following coercivity condition on the cost functional.

ASSUMPTION 3.3. There are constants $p > 1$ and $C_0 > 0$ such that

$$|u|^p \leq C_0(1 + L(t, \mathbf{x}, u)) < \infty \quad \text{for every } (t, \mathbf{x}, u) \in [0, 1] \times \Omega \times U.$$

REMARK 3.4. In the particular case $U = \{I_d\} \times \mathbb{R}^d$, the last condition coincides with Assumption A.1 of Mikami and Thieullen [30]. Moreover, whenever U is bounded, Assumption 3.3 is a direct consequence of Assumption 3.1.

Let $C_b(\mathbb{R}^d)$ denote the set of all bounded continuous functions on \mathbb{R}^d and

$$\mu(\phi) := \int_{\mathbb{R}^d} \phi(x) \mu(dx) \quad \text{for all } \mu \in \mathbf{M}(\mathbb{R}^d) \text{ and } \phi \in \mathcal{L}^1(\mu).$$

We define the dual formulation of (2.7) by

$$(3.1) \quad \mathcal{V}(\mu_0, \mu_1) := \sup_{\lambda_1 \in C_b(\mathbb{R}^d)} \{\mu_0(\lambda_0) - \mu_1(\lambda_1)\},$$

where

$$(3.2) \quad \lambda_0(x) := \inf_{\mathbb{P} \in \mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 L(s, X, v_s^{\mathbb{P}}) ds + \lambda_1(X_1) \right],$$

with $\mathcal{P}(\delta_x)$ defined in (2.4). We notice that $\mu_0(\lambda_0)$ is well defined since λ_0 takes value in $\mathbb{R} \cup \{\infty\}$, is bounded from below and is measurable by the following lemma.

LEMMA 3.5. Let Assumptions 3.1 and 3.2 hold true. Then, λ_0 is measurable w.r.t. the Borel σ -field on \mathbb{R}^d completed by μ_0 , and

$$\mu_0(\lambda_0) = \inf_{\mathbb{P} \in \mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 L(s, X, v_s^{\mathbb{P}}) ds + \lambda_1(X_1) \right].$$

The proof of Lemma 3.5 is based on a measurable selection argument and is reported at the end of Section 4.1.1. We now state the main duality result.

THEOREM 3.6. *Let Assumptions 3.1, 3.2 and 3.3 hold. Then*

$$V(\mu_0, \mu_1) = \mathcal{V}(\mu_0, \mu_1) \quad \text{for all } \mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d),$$

and the infimum is achieved by some $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ for the problem $V(\mu_0, \mu_1)$ of (2.7).

The proof of this result is reported in the subsequent subsections.

We finally state a duality result in the space $C_b^\infty(\mathbb{R}^d)$ of all functions with bounded derivatives of any order:

$$(3.3) \quad \bar{\mathcal{V}}(\mu_0, \mu_1) := \sup_{\lambda_1 \in C_b^\infty(\mathbb{R}^d)} \{\mu_0(\lambda_0) - \mu_1(\lambda_1)\}.$$

ASSUMPTION 3.7. The function L is uniformly continuous in \mathbf{x} in the sense that

$$\Delta_x L(\varepsilon) := \sup \frac{|L(t, \mathbf{x}^1, u) - L(t, \mathbf{x}^2, u)|}{1 + L(t, \mathbf{x}^2, u)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

where the supremum is taken over all $0 \leq t \leq 1, u \in U$ and all $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ such that $|\mathbf{x}^1 - \mathbf{x}^2|_\infty \leq \varepsilon$.

THEOREM 3.8. *Under the conditions of Theorem 3.6 together with Assumption 3.7, we have $\mathcal{V} = \bar{\mathcal{V}}$ on $\mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d)$.*

The proof of the last result follows exactly the same arguments as those of Mikami and Thieullen [30] in the proof of their Theorem 2.1. We report it in Section 3.6 for completeness.

3.2. An enlarged space. In preparation of the proof of Theorem 3.6, we introduce the enlarged canonical space

$$(3.4) \quad \bar{\Omega} := C([0, 1], \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^d),$$

following the technique used by Haussmann [22] in the proof of his Proposition 3.1.

On $\bar{\Omega}$, we denote the canonical filtration by $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \leq t \leq 1}$ and the canonical process by (X, A, B) , where X, B are d -dimensional processes and A is a d^2 -dimensional process.

We consider a probability measure $\bar{\mathbb{P}}$ on $\bar{\Omega}$ such that X is an $\bar{\mathbb{F}}$ -semimartingale characterized by (A, B) and, moreover, (A, B) is $\bar{\mathbb{P}}$ -a.s. absolutely continuous w.r.t. t and $\nu_t \in U, d\bar{\mathbb{P}} \times dt$ -a.e., where $\nu = (\alpha, \beta)$ is defined by

$$(3.5) \quad \alpha_t := \limsup_{n \rightarrow \infty} n(A_t - A_{t-1/n}) \quad \text{and} \quad \beta_t := \limsup_{n \rightarrow \infty} n(B_t - B_{t-1/n}).$$

We also denote by $\overline{\mathcal{P}}$ the set of all probability measures $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ satisfying the above conditions, and

$$\begin{aligned} \overline{\mathcal{P}}(\mu_0) &:= \{\overline{\mathbb{P}} \in \overline{\mathcal{P}} : \overline{\mathbb{P}} \circ X_0^{-1} = \mu_0\}, \\ \overline{\mathcal{P}}(\mu_0, \mu_1) &:= \{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0) : \overline{\mathbb{P}} \circ X_1^{-1} = \mu_1\}. \end{aligned}$$

Finally, we denote

$$\overline{J}(\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}} \int_0^1 L(t, X, v_t) dt.$$

LEMMA 3.9. *The function \overline{J} is lower semicontinuous on $\overline{\mathcal{P}}$.*

PROOF. We follow the lines in Mikami [29]. By exactly the same arguments for proving inequality (3.17) in [29], under Assumptions 3.1 and 3.2, we get

$$\begin{aligned} (3.6) \quad & \int_0^1 L(s, \mathbf{x}, \eta_s) ds \\ & \geq \frac{1}{1 + \Delta_t L(\varepsilon)} \int_0^{1-\varepsilon} L\left(s, \mathbf{x}, \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \eta_t dt\right) ds - \Delta_t L(\varepsilon) \end{aligned}$$

for every $\varepsilon < 1$, $\mathbf{x} \in \Omega$ and \mathbb{R}^{d^2+d} -valued process η .

Suppose now $(\overline{\mathbb{P}}^n)_{n \geq 1}$ is a sequence of probability measures in $\overline{\mathcal{P}}$ which converges weakly to some $\overline{\mathbb{P}}^0 \in \overline{\mathcal{P}}$. Replacing (\mathbf{x}, η) in (3.6) by (X, v) , taking expectation under $\overline{\mathbb{P}}^n$, by the definition of v_t in (3.5) as well as the absolute continuity of (A, B) in t , it follows that

$$\begin{aligned} \overline{J}(\overline{\mathbb{P}}^n) &= \mathbb{E}^{\overline{\mathbb{P}}^n} \int_0^1 L(s, X, v_s) ds \\ &= \frac{1}{1 + \Delta_t L(\varepsilon)} \mathbb{E}^{\overline{\mathbb{P}}^n} \left[\int_0^{1-\varepsilon} L\left(s, X, \frac{1}{\varepsilon}(A_{s+\varepsilon} - A_s), \frac{1}{\varepsilon}(B_{s+\varepsilon} - B_s)\right) ds \right] \\ &\quad - \Delta_t L(\varepsilon). \end{aligned}$$

Next, by Fatou’s lemma, we find that

$$(X, A, B) \mapsto \int_0^{1-\varepsilon} L\left(s, X, \frac{1}{\varepsilon}(A_{s+\varepsilon} - A_s), \frac{1}{\varepsilon}(B_{s+\varepsilon} - B_s)\right) ds$$

is lower-semicontinuous. It follows by $\overline{\mathbb{P}}^n \rightarrow \overline{\mathbb{P}}^0$ that

$$\liminf_{n \rightarrow \infty} \overline{J}(\overline{\mathbb{P}}^n) \geq \frac{1}{1 + \Delta_t L(\varepsilon)} \mathbb{E}^{\overline{\mathbb{P}}^0} \left[\int_0^{1-\varepsilon} L\left(s, X, \frac{1}{\varepsilon} \int_s^{s+\varepsilon} v_t dt\right) ds \right] - \Delta_t L(\varepsilon).$$

Note that by the absolute continuity assumption of (A, B) in t under $\overline{\mathbb{P}}^0$,

$$\frac{1}{\varepsilon} \int_s^{s+\varepsilon} v_t(\omega) dt \longrightarrow v_s(\omega) \quad \text{as } \varepsilon \rightarrow 0, \text{ for } d\overline{\mathbb{P}}^0 \times dt\text{-a.e. } (\omega, s) \in \Omega \times [0, 1),$$

and $\Delta_t L(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ from Assumption 3.2; we then finish the proof by sending ε to zero and using Fatou's lemma. \square

REMARK 3.10. In the Markovian case $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$, for some deterministic function ℓ , we observe that Assumption 3.2 is stronger than Assumption A2 in Mikami [29]. However, we can easily adapt this proof by introducing the trajectory set $\{\mathbf{x} : \sup_{0 \leq t, s \leq 1, |t-s| \leq \varepsilon} |\mathbf{x}(t) - \mathbf{x}(s)| \leq \delta\}$ and then letting $\varepsilon, \delta \rightarrow 0$ as in the proof of inequality (3.17) in [29].

Our next objective is to establish a one-to-one connection between the cost functional J defined on the set $\mathcal{P}(\mu_0, \mu_1)$ of probability measures on Ω and the cost functional \bar{J} defined on the corresponding set $\bar{\mathcal{P}}(\mu_0, \mu_1)$ on the enlarged space $\bar{\Omega}$.

PROPOSITION 3.11. (i) For any probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, there exists a probability $\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) = \bar{J}(\bar{\mathbb{P}})$.

(ii) Conversely, let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0, \mu_1)$ be such that $\mathbb{E}^{\bar{\mathbb{P}}} \int_0^1 |\beta_s| ds < \infty$. Then, under Assumption 3.1, there exists a probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) \leq \bar{J}(\bar{\mathbb{P}})$.

PROOF. (i) Given $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, define the processes $A^{\mathbb{P}}, B^{\mathbb{P}}$ from decomposition (2.1) and observe that the mapping $\omega \in \Omega \mapsto (X_t(\omega), A_t^{\mathbb{P}}(\omega), B_t^{\mathbb{P}}(\omega)) \in \mathbb{R}^{2d+d^2}$ is measurable for every $t \in [0, 1]$. Then the mapping $\omega \in \Omega \mapsto (X(\omega), A^{\mathbb{P}}(\omega), B^{\mathbb{P}}(\omega)) \in \bar{\Omega}$ is also measurable; see, for example, discussions in Chapter 2 of Billingsley [7] at page 57.

Let $\bar{\mathbb{P}}$ be the probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$ induced by $(\mathbb{P}, (X, A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)))$. In the enlarged space $(\bar{\Omega}, \bar{\mathcal{F}}_1, \bar{\mathbb{P}})$, the canonical process X is clearly a continuous semimartingale characterized by $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X))$. Moreover, $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)) = (A, B)$, $\bar{\mathbb{P}}$ -a.s., where (X, A, B) are canonical processes in $\bar{\Omega}$. It follows that, on the enlarged space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, X is a continuous semimartingale characterized by (A, B) . Also, (A, B) is clearly $\bar{\mathbb{P}}$ -a.s. absolutely continuous in t , with $\nu^{\bar{\mathbb{P}}}(X)_t = \nu_t$, $d\bar{\mathbb{P}} \times dt$ -a.e., where ν is defined in (3.5). Then $\bar{\mathbb{P}}$ is the required probability in $\bar{\mathcal{P}}(\mu_0, \mu_1)$ and satisfies $\bar{J}(\bar{\mathbb{P}}) = J(\mathbb{P})$.

(ii) Let us first consider the enlarged space $\bar{\Omega}$, and denote by $\bar{\mathbb{F}}^X = (\bar{\mathcal{F}}_t^X)_{0 \leq t \leq 1}$ the filtration generated by process X . Then for every $\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0, \mu_1)$, $(\bar{\Omega}, \bar{\mathbb{F}}^X, \bar{\mathbb{P}}, X)$ is still a continuous semimartingale, by the stability property of semimartingales. It follows from Theorem A.3 in the Appendix that the decomposition of X under filtration $\bar{\mathbb{F}}^X = (\bar{\mathcal{F}}_t^X)_{0 \leq t \leq 1}$ can be written as

$$X_t = X_0 + \bar{B}(X)_t + \bar{M}(X)_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}(X)_t,$$

with $\bar{A}(X)_t := \langle \bar{M}(X) \rangle_t = \int_0^t \bar{\alpha}_s ds$, $\bar{\beta}_s = \mathbb{E}^{\bar{\mathbb{P}}}[\beta_s | \bar{\mathcal{F}}_s^X]$ and $\bar{\alpha}_s = \alpha_s$, $d\bar{\mathbb{P}} \times dt$ -a.e. Moreover, by the convexity property (2.3) of the set U , it follows that $(\bar{\alpha}, \bar{\beta}) \in U$,

$d\bar{\mathbb{P}} \times dt$ -a.e. Finally, since $\bar{\mathcal{F}}_t^X = \mathcal{F}_t \otimes \{\emptyset, C([0, 1], \mathbb{R}^{d^2} \times \mathbb{R}^d)\}$, $\bar{\mathbb{P}}$ then induces a probability measure \mathbb{P} on (Ω, \mathcal{F}_1) by

$$\mathbb{P}[E] := \bar{\mathbb{P}}[E \times C([0, 1], \mathbb{R}^{d^2} \times \mathbb{R}^d)] \quad \forall E \in \mathcal{F}_1.$$

Clearly, $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ and $J(\mathbb{P}) \leq \bar{J}(\bar{\mathbb{P}})$ by the convexity of L in b of Assumption 3.1 and Jensen’s inequality. \square

REMARK 3.12. Let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}$ be such that $\bar{J}(\bar{\mathbb{P}}) < \infty$, then from the coercivity property of L in u in Assumption 3.3, it follows immediately that $\mathbb{E}^{\bar{\mathbb{P}}} \int_0^1 |\beta_s| ds < \infty$.

3.3. *Lower semicontinuity and existence.* By the correspondence between J and \bar{J} (Proposition 3.11) and the lower semicontinuity of \bar{J} (Lemma 3.9), we now obtain the corresponding property for V under the crucial Assumption 3.3, which guarantees the tightness of any minimizing sequence of our problem $V(\mu_0, \mu_1)$.

LEMMA 3.13. *Under Assumptions 3.1, 3.2 and 3.3, the map*

$$(\mu_0, \mu_1) \in \mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d) \longmapsto V(\mu_0, \mu_1) \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

is lower semicontinuous.

PROOF. We follow the arguments in Lemma 3.1 of Mikami and Thieullen [30]. Let (μ_0^n) and (μ_1^n) be two sequences in $\mathbf{M}(\mathbb{R}^d)$ converging weakly to $\mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d)$, respectively, and let us prove that

$$\liminf_{n \rightarrow \infty} V(\mu_0^n, \mu_1^n) \geq V(\mu_0, \mu_1).$$

We focus on the case $\liminf_{n \rightarrow \infty} V(\mu_0^n, \mu_1^n) < \infty$, as the result is trivial in the alternative case. Then, after possibly extracting a subsequence, we can assume that $(V(\mu_0^n, \mu_1^n))_{n \geq 1}$ is bounded, and there is a sequence $(\mathbb{P}_n)_{n \geq 1}$ such that $\mathbb{P}_n \in \mathcal{P}(\mu_0^n, \mu_1^n)$ for all $n \geq 1$ and

$$(3.7) \quad \sup_{n \geq 1} J(\mathbb{P}_n) < \infty, \quad 0 \leq J(\mathbb{P}_n) - V(\mu_0^n, \mu_1^n) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Assumption 3.3 it follows that $\sup_{n \geq 1} \mathbb{E}^{\mathbb{P}_n} \int_0^1 |v_s^{\mathbb{P}_n}|^p ds < \infty$. Then, it follows from Theorem 3 of Zheng [38] that the sequence $(\bar{\mathbb{P}}_n)_{n \geq 1}$, of probability measures induced by $(\mathbb{P}_n, X, A^{\mathbb{P}_n}, B^{\mathbb{P}_n})$ on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$, is tight. Moreover, under any one of their limit laws $\bar{\mathbb{P}}$, the canonical process X is a semimartingale characterized by (A, B) such that (A, B) are still absolutely continuous in t . Moreover, $v \in U$, $d\bar{\mathbb{P}} \times dt$ -a.e. since $\frac{1}{t-s}(A_t - A_s, B_t - B_s) \in U$, $d\bar{\mathbb{P}}$ -a.s. for every $t, s \in [0, 1]$, hence, $\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0, \mu_1)$. We then deduce from (3.7), Proposition 3.11 and Lemma 3.9 that

$$\liminf_{n \rightarrow \infty} V(\mu_0^n, \mu_1^n) = \liminf_{n \rightarrow \infty} J(\mathbb{P}_n) = \liminf_{n \rightarrow \infty} \bar{J}(\bar{\mathbb{P}}_n) \geq \bar{J}(\bar{\mathbb{P}}).$$

By Remark 3.12 and Proposition 3.11, we may find $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $\bar{J}(\bar{\mathbb{P}}) \geq J(\mathbb{P})$. Hence, $\liminf_{n \rightarrow \infty} V(\mu_0^n, \mu_1^n) \geq J(\mathbb{P}) \geq V(\mu_0, \mu_1)$, completing the proof. \square

PROPOSITION 3.14. *Let Assumptions 3.1, 3.2 and 3.3 hold true. Then for every $\mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d)$ such that $V(\mu_0, \mu_1) < \infty$, existence holds for the minimization problem $V(\mu_0, \mu_1)$. Moreover, the set of minimizers $\{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1) : J(\mathbb{P}) = V(\mu_0, \mu_1)\}$ is a compact set of probability measures on Ω .*

PROOF. We just let $(\mu_0^n, \mu_1^n) = (\mu_0, \mu_1)$ in the proof of Lemma 3.13, then the required existence result is proved by following the same arguments. \square

3.4. Convexity.

LEMMA 3.15. *Let Assumptions 3.1 and 3.3 hold, then the map $(\mu_0, \mu_1) \mapsto V(\mu_0, \mu_1)$ is convex.*

PROOF. Given $\mu_0^1, \mu_0^2, \mu_1^1, \mu_1^2 \in \mathbf{M}(\mathbb{R}^d)$ and $\mu_0 = \theta\mu_0^1 + (1 - \theta)\mu_0^2, \mu_1 = \theta\mu_1^1 + (1 - \theta)\mu_1^2$ with $\theta \in (0, 1)$, we shall prove that

$$V(\mu_0, \mu_1) \leq \theta V(\mu_0^1, \mu_1^1) + (1 - \theta)V(\mu_0^2, \mu_1^2).$$

It is enough to show that for both $\mathbb{P}_i \in \mathcal{P}(\mu_0^i, \mu_1^i)$ such that $J(\mathbb{P}_i) < \infty, i = 1, 2$, we can find $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ satisfying

$$(3.8) \quad J(\mathbb{P}) \leq \theta J(\mathbb{P}_1) + (1 - \theta)J(\mathbb{P}_2).$$

As in Lemma 3.13, let us consider the enlarged space $\bar{\Omega}$, on which the probability measures $\bar{\mathbb{P}}_i$ are induced by $(\mathbb{P}_i, X, A^{\mathbb{P}_i}, B^{\mathbb{P}_i}), i = 1, 2$. By Proposition 3.11, $(\bar{\mathbb{P}}_i)_{i=1,2}$ are probability measures under which X is a $\bar{\mathbb{F}}$ -semimartingale characterized by the same process (A, B) , which is absolutely continuous in t , such that $J(\mathbb{P}_i) = \bar{J}(\bar{\mathbb{P}}_i), i = 1, 2$.

By Corollary III.2.8 of Jacod and Shiryaev [25], $\bar{\mathbb{P}} := \theta\bar{\mathbb{P}}_1 + (1 - \theta)\bar{\mathbb{P}}_2$ is also a probability measure under which X is an $\bar{\mathbb{F}}$ -semimartingale characterized by (A, B) . Clearly, $\nu \in U, d\bar{\mathbb{P}} \times dt$ -a.e. since it is true $d\bar{\mathbb{P}}_i \times dt$ -a.e. for $i = 1, 2$. Thus, $\bar{\mathbb{P}} \in \mathcal{P}(\mu_0, \mu_1)$ and it satisfies that

$$\bar{J}(\bar{\mathbb{P}}) = \theta\bar{J}(\bar{\mathbb{P}}_1) + (1 - \theta)\bar{J}(\bar{\mathbb{P}}_2) = \theta J(\mathbb{P}_1) + (1 - \theta)J(\mathbb{P}_2) < \infty.$$

Finally, by Remark 3.12 and Proposition 3.11, we can construct $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) \leq \bar{J}(\bar{\mathbb{P}})$, and it follows that inequality (3.8) holds true. \square

3.5. *Proof of the duality result.* We follow the first part of the proof of Theorem 2.1 in Mikami and Thieullen [30]. If $V(\mu_0, \mu_1)$ is infinite for every $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$, then $J(\mathbb{P}) = \infty$ for all $\mathbb{P} \in \mathcal{P}(\mu_0)$. It follows from (3.1) and Lemma 3.5 that

$$V(\mu_0, \mu_1) = \mathcal{V}(\mu_0, \mu_1) = \infty.$$

Now, suppose that $V(\mu_0, \cdot)$ is not always infinite. Let $\overline{\mathbf{M}}(\mathbb{R}^d)$ be the space of all finite signed measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, equipped with weak topology, that is, the coarsest topology making $\mu \mapsto \mu(\phi)$ continuous for every $\phi \in C_b(\mathbb{R}^d)$. As indicated in Section 3.2 of [17], the topology inherited by $\mathbf{M}(\mathbb{R}^d)$ as a subset of $\overline{\mathbf{M}}(\mathbb{R}^d)$ is its weak topology. We then extend $V(\mu_0, \cdot)$ to $\overline{\mathbf{M}}(\mathbb{R}^d) \supset \mathbf{M}(\mathbb{R}^d)$ by setting $V(\mu_0, \mu_1) = \infty$ when $\mu_1 \in \overline{\mathbf{M}}(\mathbb{R}^d) \setminus \mathbf{M}(\mathbb{R}^d)$, thus, $\mu_1 \mapsto V(\mu_0, \mu_1)$ is a convex and lower semicontinuous function defined on $\overline{\mathbf{M}}(\mathbb{R}^d)$. Then, the duality result $V = \mathcal{V}$ follows from Theorem 2.2.15 and Lemma 3.2.3 in [17], together with the fact that for $\lambda_1 \in C_b(\mathbb{R}^d)$,

$$\begin{aligned} & \sup_{\mu_1 \in \mathbf{M}(\mathbb{R}^d)} \{ \mu_1(-\lambda_1) - V(\mu_0, \mu_1) \} \\ &= - \inf_{\substack{\mu_1 \in \mathbf{M}(\mathbb{R}^d) \\ \mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)}} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 L(s, X, v_s^{\mathbb{P}}) ds + \lambda_1(X_1) \right] \\ &= - \inf_{\mathbb{P} \in \mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 L(s, X, v_s^{\mathbb{P}}) ds + \lambda_1(X_1) \right] \\ &= -\mu_0(\lambda_0), \end{aligned}$$

where the last equality follows by Lemma 3.5.

3.6. *Proof of Theorem 3.8.* The proof is almost the same as that of Theorem 2.1 of Mikami and Thieullen [30]; we report it here for completeness. Let $\psi \in C_c^\infty([-1, 1]^d, \mathbb{R}^+)$ be such that $\int_{\mathbb{R}^d} \psi(x) dx = 1$, and define $\psi_\varepsilon(x) := \varepsilon^{-d} \psi(x/\varepsilon)$. We claim that

$$(3.9) \quad \overline{\mathcal{V}}(\mu_0, \mu_1) \geq \frac{\mathcal{V}(\psi_\varepsilon * \mu_0, \psi_\varepsilon * \mu_1)}{1 + \Delta_x L(\varepsilon)} - \Delta_x L(\varepsilon).$$

Since the inequality $\mathcal{V} \geq \overline{\mathcal{V}}$ is obvious, the required result is then obtained by sending $\varepsilon \rightarrow 0$ and using Assumption 3.7 together with Lemma 3.13.

Hence, we only need to prove the claim (3.9). Let us denote $\delta := \Delta_x L(\varepsilon)$ in the rest of this proof. We first observe from Assumption 3.7 that

$$L(s, \mathbf{x}, u) \geq \frac{L(s, \mathbf{x} + z, u)}{1 + \delta} - \delta \quad \text{for all } z \in \mathbb{R} \text{ satisfying } |z| \leq \varepsilon,$$

where $\mathbf{x} + z := (\mathbf{x}(t) + z)_{0 \leq t \leq 1} \in \Omega$. For an arbitrary $\lambda_1 \in C_b(\mathbb{R}^d)$, we denote $\lambda_1^\varepsilon := (1 + \delta)^{-1} \lambda_1 * \psi_\varepsilon \in C_b^\infty$. Let $\mathbb{P} \in \mathcal{P}(\mu_0)$ and Z be a r.v. independent of X

with distribution defined by the density function ψ_ε under \mathbb{P} . Then the probability $\bar{\mathbb{P}}_\varepsilon$ on $\bar{\Omega}$ induced by $(\mathbb{P}, X + Z := (X_t + Z)_{0 \leq t \leq 1}, A^\mathbb{P}, B^\mathbb{P})$ is in $\bar{\mathcal{P}}(\psi_\varepsilon * \mu_0)$, and

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[\int_0^1 L(s, X, v_s^\mathbb{P}) ds + \lambda_1^\varepsilon(X_1) \right] \\ & \geq -\delta + \frac{1}{1 + \delta} \mathbb{E}^\mathbb{P} \left[\int_0^1 L(s, X + Z, v_s^\mathbb{P}) ds + \lambda_1(X_1 + Z) \right] \\ & = -\delta + \frac{1}{1 + \delta} \mathbb{E}^{\bar{\mathbb{P}}_\varepsilon} \left[\int_0^1 L(s, X, v_s) ds + \lambda_1(X_1) \right] \\ & \geq -\delta + \frac{1}{1 + \delta} \inf_{\tilde{\mathbb{P}} \in \mathcal{P}(\psi_\varepsilon * \mu_0)} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^1 L(s, X, v_s^{\tilde{\mathbb{P}}}) ds + \lambda_1(X_1) \right], \end{aligned}$$

where the last inequality follows from Proposition 3.11.

Notice that $\mu_1(\lambda_1^\varepsilon) = (1 + \delta)^{-1}(\psi_\varepsilon * \mu_1)(\lambda_1)$ by Fubini’s theorem. Then, by the arbitrariness of $\lambda_1 \in C_b(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}(\mu_0)$, the last inequality implies (3.9).

4. Characterization of the dual formulation. In the rest of the paper we assume that

$$L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u),$$

where the deterministic function $\ell : (t, x, u) \in [0, 1] \times \mathbb{R}^d \times U \mapsto \ell(t, x, u) \in \mathbb{R}^+$ is nonnegative and convex in u . Then, the function λ_0 in (3.2) is reduced to the value function of a standard Markovian stochastic control problem:

$$(4.1) \quad \lambda_0(x) = \inf_{\mathbb{P} \in \mathcal{P}(\delta_x)} \mathbb{E}^\mathbb{P} \left[\int_0^1 \ell(s, X_s, v_s^\mathbb{P}) ds + \lambda_1(X_1) \right].$$

Our main objective is to characterize λ_0 by means of the corresponding dynamic programming equations. Then in the case of bounded characteristics, we show more regularity as well as approximation properties of λ_0 , which serves as a preparation for the numerical approximation in Section 5.

4.1. PDE characterization of the dynamic value function. Let us consider the probability measures \mathbb{P} on the canonical space (Ω, \mathcal{F}_1) , under which the canonical process X is a semimartingale on $[t, 1]$, characterized by $\int_t^\cdot v_s^\mathbb{P} ds$ for some progressively measurable process $v^\mathbb{P}$. As discussed in Remark 2.2, $v^\mathbb{P}$ is unique on $\Omega \times [t, 1]$ in the sense of $d\mathbb{P} \times dt$ -a.e. Following the definition of \mathcal{P} just below (2.3), we denote by \mathcal{P}_t the collection of all such probability measures \mathbb{P} such that $v_s^\mathbb{P} \in U, d\mathbb{P} \times dt$ -a.e. on $\Omega \times [t, 1]$. Let

$$(4.2) \quad \mathcal{P}_{t,x} := \{ \mathbb{P} \in \mathcal{P}_t : \mathbb{P}[X_s = x, 0 \leq s \leq t] = 1 \}.$$

We notice that under probability $\mathbb{P} \in \mathcal{P}_{t,x}$, X is a semimartingale with $v_s^\mathbb{P} = 0$, $d\mathbb{P} \times dt$ -a.e. on $\Omega \times [0, t]$. The dynamic value function is defined for any $\lambda_1 \in C_b(\mathbb{R}^d)$ by

$$(4.3) \quad \lambda(t, x) := \inf_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^\mathbb{P} \left[\int_t^1 \ell(s, X_s, v_s^\mathbb{P}) ds + \lambda_1(X_1) \right].$$

As in the previous sections, we also introduce the corresponding probability measures on the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_1)$. For all $t \in [0, 1]$, we denote by $\overline{\mathcal{P}}_t$ the collection of all probability measures $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ under which X is a semimartingale characterized by (A, B) in $\overline{\Omega}$ and $\nu \in U$, $d\overline{\mathbb{P}} \times dt$ -a.e. on $\Omega \times [t, 1]$, where ν is defined above (3.5). For every $(t, x, a, b) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^d$, let

$$(4.4) \quad \overline{\mathcal{P}}_{t,x,a,b} := \{ \overline{\mathbb{P}} \in \overline{\mathcal{P}} : \overline{\mathbb{P}}[(X_s, A_s, B_s) = (x, a, b), 0 \leq s \leq t] = 1 \}.$$

By similar arguments as in Proposition 3.11, we have under Assumption 3.1 that

$$(4.5) \quad \lambda(t, x) = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}} \left[\int_t^1 \ell(s, X_s, v_s) ds + \lambda_1(X_1) \right]$$

for all $(a, b) \in \mathbb{R}^{d^2} \times \mathbb{R}^d$.

We would like to characterize the dynamic value function λ as the viscosity solution of a dynamic programming equation. The first step is as usual to establish the dynamic programming principle (DPP). We observe that a weak dynamic programming principle as introduced in Bouchard and Touzi [12] suffices to prove that λ is a viscosity solution of the corresponding dynamic programming equation. The main argument in [12] to establish the weak DPP is the conditioning and pasting techniques of the control process, which is convenient to use for control problems in a strong formulation, that is, when the measure space (Ω, \mathcal{F}) as well as the probability measure \mathbb{P} are fixed a priori. However, we cannot use their techniques since our problem is in weak formulation, where the controlled process is fixed as a canonical process and the controls are given as probability measures on the canonical space.

We will prove the standard dynamic programming principle. For a simpler problem (bounded convex controls set U and bounded cost functions, etc.), a DPP is shown (implicitly) in Haussmann [22]. El Karoui, Nguyen and JeanBlanc [19] considered a relaxed optimal control problem and provided a scheme of proof without all details. Our approach is to adapt the idea of [19] in our context and to provide all details for their scheme of proof.

PROPOSITION 4.1. *Let Assumptions 3.1, 3.2, 3.3 hold true. Then, for all $\overline{\mathbb{F}}$ -stopping time τ with values in $[t, 1]$, and all $(a, b) \in \mathbb{R}^{d^2+d}$,*

$$\lambda(t, x) = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, v_s) ds + \lambda(\tau, X_\tau) \right].$$

The proof is reported in Section 4.1.1. The dynamic programming equation is the infinitesimal version of the above dynamic programming principle. Let

$$(4.6) \quad H(t, x, p, \Gamma) := \inf_{(a,b) \in U} \left[b \cdot p + \frac{1}{2} a \cdot \Gamma + \ell(t, x, a, b) \right]$$

for all $(p, \Gamma) \in \mathbb{R}^d \times \mathcal{S}_d$.

THEOREM 4.2. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume further that λ is locally bounded and H is continuous. Then, λ is a viscosity solution of the dynamic programming equation*

$$(4.7) \quad -\partial_t \lambda(t, x) - H(t, x, D\lambda, D^2\lambda) = 0,$$

with terminal condition $\lambda(1, x) = \lambda_1(x)$.

The proof is very similar to that of Corollary 5.1 in [12]; we report it in the Appendix for completeness.

REMARK 4.3. We first observe that H is concave in (p, Γ) as infimum of a family of affine functions. Moreover, under Assumption 3.3, ℓ is positive and $u \mapsto \ell(t, x, u)$ has growth larger than $|u|^p$ for $p > 1$; it follows that H is finite valued and hence continuous in (p, Γ) for every fixed $(t, x) \in [0, 1] \times \mathbb{R}^d$. If we assume further that $(t, x) \mapsto \ell(t, x, u)$ is uniformly continuous uniformly in u , then clearly H is continuous in (t, x, p, Γ) .

REMARK 4.4. The following are two sets of sufficient conditions to ensure the local boundedness of λ in (4.3).

- (i) Suppose $0 \in U$, and let $\mathbb{P} \in \mathcal{P}_t$ be such that $v_s^\mathbb{P} = 0$, $d\mathbb{P} \times dt$ -a.e. Then, $\lambda(t, x) \leq |\lambda_1|_\infty + \int_t^1 \ell(s, x, 0) ds$ and, hence, λ is locally bounded.
- (ii) Suppose that there are constants $C > 0$ and $(a_0, b_0) \in U$ such that $\ell(t, x, a_0, b_0) \leq Ce^{C|x|}$, for all $(t, x) \in [0, 1] \times \mathbb{R}^d$. By considering $\mathbb{P} \in \mathcal{P}_t$ induced by the process $Y = (Y_s)_{t \leq s \leq 1}$ with $Y_s := x + b_0(s - t) + a_0^{1/2}(W_s - W_t)$, it follows that $\lambda(t, x) \leq |\lambda_1|_\infty + \mathbb{E}[Ce^{C \max_{t \leq s \leq 1} |Y_s|}] < \infty$.

4.1.1. Proof of the dynamic programming principle. We first prove that the dynamic value function λ is measurable and we can choose “in a measurable way” a family of probabilities $(\mathbb{Q}_{t,x,a,b})_{(t,x,a,b) \in [0,1] \times \mathbb{R}^{2d+d^2}}$ which achieves (or achieves with ε error) the infimum in (4.5). The main argument is Theorem A.1 cited in the Appendix which follows directly from the measurable selection theorem.

Let λ^* be the upper semicontinuous envelope of the function λ , and

$$\tilde{\mathcal{P}}_{t,x,a,b} := \left\{ \bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b} : \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_t^1 \ell(s, X_s, v_s) ds + \lambda_1(X_1) \right] \leq \lambda^*(t, x) \right\},$$

$$\tilde{\mathcal{P}} := \{(t, x, a, b, \bar{\mathbb{P}}) : \bar{\mathbb{P}} \in \tilde{\mathcal{P}}_{t,x,a,b}\}.$$

In the following statement, for the Borel σ -field $\mathcal{B}([0, 1] \times \mathbb{R}^{2d+d^2})$ of $[0, 1] \times \mathbb{R}^{2d+d^2}$ with an arbitrary probability measure μ on it, we denote by $\mathcal{B}^\mu([0, 1] \times \mathbb{R}^{2d+d^2})$ its σ -field completed by μ .

LEMMA 4.5. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume that λ is locally bounded. Then, for any probability measure μ on $([0, 1] \times \mathbb{R}^{2d+d^2}, \mathcal{B}([0, 1] \times \mathbb{R}^{2d+d^2}))$,*

- (i) *the function $(t, x, a, b) \mapsto \lambda(t, x)$ is $\mathcal{B}^\mu([0, 1] \times \mathbb{R}^{2d+d^2})$ -measurable,*
- (ii) *for any $\varepsilon > 0$, there is a family of probability $(\tilde{\mathbb{Q}}_{t,x,a,b}^\varepsilon)_{(t,x,a,b) \in [0,1] \times \mathbb{R}^{2d+d^2}}$ in $\tilde{\mathcal{P}}$ such that $(t, x, a, b) \mapsto \tilde{\mathbb{Q}}_{t,x,a,b}^\varepsilon$ is a measurable map from $[0, 1] \times \mathbb{R}^{2d+d^2}$ to $\mathbf{M}(\bar{\Omega})$ and*

$$\mathbb{E}^{\tilde{\mathbb{Q}}_{t,x,a,b}^\varepsilon} \left[\int_t^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \leq \lambda(t, x) + \varepsilon, \quad \mu\text{-a.s.}$$

PROOF. By Lemma 3.9, the map $\bar{\mathbb{P}} \mapsto \mathbb{E}^{\bar{\mathbb{P}}}[\int_t^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1)]$ is lower semicontinuous, and therefore measurable. Moreover, $\tilde{\mathcal{P}}_{t,x,a,b}$ is nonempty for every $(t, x, a, b) \in [0, 1] \times \mathbb{R}^{2d+d^2}$. Finally, by using the same arguments as in the proof of Lemma 3.13, we see that $\tilde{\mathcal{P}}$ is a closed subset of $[0, 1] \times \mathbb{R}^{2d+d^2} \times \mathbf{M}(\bar{\Omega})$. Then, both items of the lemma follow from Theorem A.1. \square

We next prove the stability properties of probability measures under condition- ing and concatenations at stopping times, which will be the key-ingredients for the proof of the dynamic programming principle.

We first recall some results from Stroock and Varadhan [35] and define some notation:

- For $0 \leq t \leq 1$, let $\bar{\mathcal{F}}_{t,1} := \sigma((X_s, A_s, B_s) : t \leq s \leq 1)$, and let $\bar{\mathbb{P}}$ be a probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}_{t,1})$ with $\bar{\mathbb{P}}[(X_t, A_t, B_t) = \eta_t] = 1$ for some $\eta \in C([0, t], \mathbb{R}^{2d+d^2})$. Then, there is a unique probability measure $\delta_\eta \otimes_t \bar{\mathbb{P}}$ on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$ such that $\delta_\eta \otimes_t \bar{\mathbb{P}}[(X_s, A_s, B_s) = \eta_s, 0 \leq s \leq t] = 1$ and $\delta_\eta \otimes_t \bar{\mathbb{P}}[A] = \bar{\mathbb{P}}[A]$ for all $A \in \bar{\mathcal{F}}_{t,1}$. In addition, if $\bar{\mathbb{P}}$ is also a probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$, under which a process M defined on $\bar{\Omega}$ is a $\bar{\mathbb{F}}$ -martingale after time t , then M is still a $\bar{\mathbb{F}}$ -martingale after time t in probability space $(\bar{\Omega}, \bar{\mathcal{F}}_1, \eta \otimes_t \bar{\mathbb{P}})$. In particular, for $t \in [0, 1]$, a constant $c_0 \in \mathbb{R}^{2d+d^2}$ and $\bar{\mathbb{P}}$ satisfying $\bar{\mathbb{P}}[(X_t, A_t, B_t) = c_0] = 1$, we denote $\delta_{c_0} \otimes_t \bar{\mathbb{P}} := \delta_{\eta^{c_0}} \otimes_t \bar{\mathbb{P}}$, where $\eta_s^{c_0} = c_0, s \in [0, t]$.
- Let $\bar{\mathbb{Q}}$ be a probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$ and τ a $\bar{\mathbb{F}}$ -stopping time. Then, there is a family of probability measures $(\bar{\mathbb{Q}}_\omega)_{\omega \in \bar{\Omega}}$ such that $\omega \mapsto \bar{\mathbb{Q}}_\omega$ is $\bar{\mathcal{F}}_\tau$ -measurable, for every $E \in \bar{\mathcal{F}}_1$, $\bar{\mathbb{Q}}[E|\bar{\mathcal{F}}_\tau](\omega) = \bar{\mathbb{Q}}_\omega[E]$ for $\bar{\mathbb{Q}}$ -almost every $\omega \in \bar{\Omega}$ and, finally, $\bar{\mathbb{Q}}_\omega[(X_t, A_t, B_t) = \omega_t : t \leq \tau(\omega)] = 1$, for all $\omega \in \bar{\Omega}$. This

is Theorem 1.3.4 of [35], and $(\bar{Q}_\omega)_{\omega \in \bar{\Omega}}$ is called the regular conditional probability distribution (r.c.p.d.)

LEMMA 4.6. *Let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$, τ be an $\bar{\mathbb{F}}$ -stopping time taking value in $[t, 1]$, and $(\bar{Q}_\omega)_{\omega \in \bar{\Omega}}$ be a r.c.p.d. of $\bar{\mathbb{P}}|\bar{\mathcal{F}}_\tau$. Then there is a $\bar{\mathbb{P}}$ -null set $N \in \bar{\mathcal{F}}_\tau$ such that $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega \in \bar{\mathcal{P}}_{\tau(\omega), \omega_{\tau(\omega)}}$ for all $\omega \notin N$.*

PROOF. Since $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$, it follows from Theorem II.2.21 of Jacod and Shiryaev [25] that

$$(X_s - B_s)_{t \leq s \leq 1}, \quad ((X_s - B_s)^2 - A_s)_{t \leq s \leq 1}$$

are all local martingales after time t . Then it follows from Theorem 1.2.10 of Stroock and Varadhan [35] together with a localization technique that there is a $\bar{\mathbb{P}}$ -null set $N_1 \in \bar{\mathcal{F}}_\tau$ such that they are still local martingales after time $\tau(\omega)$ both under \bar{Q}_ω and $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega$, for all $\omega \notin N_1$. It is clear, moreover, that $\nu \in U, d\bar{Q}_\omega \times dt$ -a.e. on $\bar{\Omega} \times [\tau(\omega), 1]$ for $\bar{\mathbb{P}}$ -a.e. $\omega \in \bar{\Omega}$. Then there is a $\bar{\mathbb{P}}$ -null set $N \in \bar{\mathcal{F}}_\tau$ such that $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{Q}_\omega \in \bar{\mathcal{P}}_{\tau(\omega), \omega_{\tau(\omega)}}$ for every $\omega \notin N$. \square

LEMMA 4.7. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume that λ is locally bounded. Let $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$, $\tau \geq t$ a $\bar{\mathbb{F}}$ -stopping time, and $(\bar{Q}_\omega)_{\omega \in \bar{\Omega}}$ a family of probability measures such that $\bar{Q}_\omega \in \bar{\mathcal{P}}_{\tau(\omega), \omega_{\tau(\omega)}}$ and $\omega \mapsto \bar{Q}_\omega$ is $\bar{\mathcal{F}}_\tau$ -measurable. Then there is a unique probability measure, denoted by $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$, in $\bar{\mathcal{P}}_{t,x,a,b}$, such that $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot = \bar{\mathbb{P}}$ on $\bar{\mathcal{F}}_\tau$, and*

$$(4.8) \quad (\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega)_{\omega \in \bar{\Omega}} \quad \text{is a r.c.p.d. of } \bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot | \bar{\mathcal{F}}_\tau.$$

PROOF. The existence and uniqueness of the probability measure $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$ on $(\bar{\Omega}, \bar{\mathcal{F}}_1)$, satisfying (4.8), follows from Theorem 6.1.2 of [35]. It remains to prove that $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot \in \bar{\mathcal{P}}_{t,x,a,b}$.

Since $\bar{Q}_\omega \in \bar{\mathcal{P}}_{\tau(\omega), \omega_{\tau(\omega)}}$, X is a $\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega$ -semimartingale after time $\tau(\omega)$, characterized by (A, B) . Then, the processes $X - B$ and $(X - B)^2 - A$ are local martingales under $\delta_\omega \otimes_{\tau(\omega)} \bar{Q}_\omega$ after time $\tau(\omega)$. By Theorem 1.2.10 of [35] together with a localization argument, they are still local martingales under $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{Q}_\cdot$. Hence, the required result follows from Theorem II.2.21 of [25]. \square

We have now collected all the ingredients for the proof of the dynamic programming principle.

PROOF OF PROPOSITION 4.1. Let τ be an $\bar{\mathbb{F}}$ -stopping time taking value in $[t, 1]$. We proceed in two steps:

(1) For $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$, we denote by $(\bar{\mathbb{Q}}_\omega)_{\omega \in \bar{\Omega}}$ a family of regular conditional probability distribution of $\bar{\mathbb{P}}|\bar{\mathcal{F}}_\tau$, and $\bar{\mathbb{P}}_\tau^\omega := \delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega$. By the representation (4.5) of λ , together with the tower property of conditional expectations, we see that

$$\begin{aligned}
 &\lambda(t, x) \\
 &= \inf_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \int_\tau^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \\
 (4.9) \quad &= \inf_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \mathbb{E}^{\bar{\mathbb{P}}_\tau^\omega} \left\{ \int_\tau^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right\} \right] \\
 &\geq \inf_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \lambda(\tau, X_\tau) \right],
 \end{aligned}$$

where the last inequality follows from the fact that $\bar{\mathbb{P}}_\tau^\omega \in \bar{\mathcal{P}}_{\tau(\omega), \omega_{\tau(\omega)}}$ by Lemma 4.6.

(2) For $\varepsilon > 0$, let $(\bar{\mathbb{Q}}_{t,x,a,b}^\varepsilon)_{[0,1] \times \mathbb{R}^{2d+d^2}}$ be the family defined in Lemma 4.5, and denote $\bar{\mathbb{Q}}_\omega^\varepsilon := \bar{\mathbb{Q}}_{\tau(\omega), \omega_{\tau(\omega)}}^\varepsilon$. Then $\omega \mapsto \bar{\mathbb{Q}}_\omega^\varepsilon$ is $\bar{\mathcal{F}}_\tau$ -measurable. Moreover, for all $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$, we may construct by Lemmas 4.5 and 4.7 $\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}} \in \bar{\mathcal{P}}_{t,x,a,b}$ such that

$$\begin{aligned}
 &\mathbb{E}^{\bar{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}} \left[\int_t^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \\
 &\leq \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \lambda(\tau, X_\tau) \right] + \varepsilon.
 \end{aligned}$$

By the arbitrariness of $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_{t,x,a,b}$ and $\varepsilon > 0$, together with the representation (4.5) of λ , this implies that the reverse inequality to (4.9) holds true, and the proof is complete. \square

We conclude this section by the following:

PROOF OF LEMMA 3.5. By the same arguments as in Lemma 4.5, we can easily deduce that λ_0 is $\mathcal{B}^{\mu_0}(\mathbb{R}^d)$ -measurable, and we just need to prove that

$$\mu_0(\lambda_0) = \inf_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0)} \mathbb{E}^{\bar{\mathbb{P}}} \left[\int_0^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right].$$

Given a probability measure $\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0)$, we can get a family of conditional probabilities $(\bar{\mathbb{Q}}_\omega)_{\omega \in \bar{\Omega}}$ such that $\bar{\mathbb{Q}}_\omega \in \bar{\mathcal{P}}_{0,\omega_0}$, which implies that

$$\mathbb{E}^{\bar{\mathbb{P}}} \left[\int_0^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \geq \mu_0(\lambda_0) \quad \forall \bar{\mathbb{P}} \in \bar{\mathcal{P}}(\mu_0).$$

On the other hand, for every $\varepsilon > 0$ and $\mu_0 \in \mathbf{M}(\mathbb{R}^d)$, we can select a measurable family of $(\bar{Q}_x^\varepsilon \in \bar{\mathcal{P}}_{0,x,0,0})_{x \in \mathbb{R}^d}$ such that

$$\mathbb{E}^{\bar{Q}_x^\varepsilon} \left[\int_0^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \leq \lambda_0(x) + \varepsilon, \quad \mu_0\text{-a.s.},$$

and then construct a probability measure $\mu_0 \otimes_0 \bar{Q}^\varepsilon \in \bar{\mathcal{P}}(\mu_0)$ by concatenation such that

$$\mathbb{E}^{\mu_0 \otimes_0 \bar{Q}^\varepsilon} \left[\int_0^1 \ell(s, X_s, \nu_s) ds + \lambda_1(X_1) \right] \leq \mu_0(\lambda_0) + \varepsilon \quad \forall \varepsilon > 0,$$

which completes the proof. \square

4.2. *Bounded domain approximation under bounded characteristics.* The main purpose of this section is to show that when U is bounded, then λ_0 in (4.1) is Lipschitz, and we may construct a convenient approximation of λ_0 by restricting the space domain to bounded domains. These properties induce a first approximation for the minimum transportation cost $V(\mu_0, \mu_1)$, which serves as a preparation for the numerical approximation in Section 5. Let us assume the following conditions.

ASSUMPTION 4.8. The control set U is compact, and ℓ is Lipschitz-continuous in x uniformly in (t, u) .

ASSUMPTION 4.9. $\int_{\mathbb{R}^d} |x|(\mu_0 + \mu_1)(dx) < \infty$.

REMARK 4.10. We suppose that U is compact for two main reasons. First, the uniqueness of viscosity solution of the HJB (4.7) relies on the comparison principle, for which the boundedness of U is generally necessary. Further, to construct a convergent (monotone) numerical scheme for a stochastic control problem, it is also generally necessary to suppose that the diffusion functions are bounded (see also Section 5.1 for more discussions).

4.2.1. *The unconstrained control problem in the bounded domain.* Denote

$$(4.10) \quad M := \sup_{(t,x,u) \in [0,1] \times \mathbb{R}^d \times U} (|u| + |\ell(t, 0, u)| + |\nabla_x \ell(t, x, u)|),$$

where $\nabla_x \ell(t, x, u)$ is the gradient of ℓ with respect to x which exists a.e. under Assumption 4.8. Let $O_R := (-R, R)^d \subset \mathbb{R}^d$ for every $R > 0$, a stopping time τ_R can be defined as the first exit time of the canonical process X from O_R ,

$$\tau_R := \inf\{t : X_t \notin O_R\},$$

and define for all bounded functions $\lambda_1 \in C_b(\mathbb{R}^d)$,

$$(4.11) \quad \lambda^R(t, x) := \inf_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}} \left[\int_t^{\tau_R \wedge 1} \ell(s, X_s, \nu_s^{\mathbb{P}}) ds + \lambda_1(X_{\tau_R \wedge 1}) \right].$$

LEMMA 4.11. *Suppose that λ_1 is K -Lipschitz satisfying $\lambda_1(0) = 0$ and Assumption 4.8 holds true. Then λ and λ^R are Lipschitz-continuous, and there is a constant C depending on M such that*

$$|\lambda(t, 0)| + |\lambda^R(t, 0)| + |\nabla_x \lambda(t, x)| + |\nabla_x \lambda^R(t, x)| \leq C(1 + K)$$

for all $(t, x) \in [0, 1] \times \mathbb{R}^d$.

PROOF. We only provide the estimates for λ ; those for λ_R follow from the same arguments. First, by Assumption 4.8 together with the fact that λ_1 is K -Lipschitz and $\lambda_1(0) = 0$, for every $\mathbb{P} \in \mathcal{P}_{t,0}$,

$$\mathbb{E}^{\mathbb{P}} \left[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}}) ds + \lambda_1(X_1) \right] \leq M + (M + K) \sup_{t \leq s \leq 1} \mathbb{E}^{\mathbb{P}} |X_s|.$$

Recall that X is a continuous semimartingale under \mathbb{P} whose finite variation part and quadratic variation of the martingale part are both bounded by a constant M . Separating the two parts and using Cauchy–Schwarz’s inequality, it follows that $\mathbb{E}^{\mathbb{P}} |X_s| \leq M + \sqrt{M}, \forall t \leq s \leq 1$, and then $|\lambda(t, 0)| \leq M + (M + K)(M + \sqrt{M})$.

We next prove that λ is Lipschitz and provide the corresponding estimate. Observe that $\mathcal{P}_{t,y} = \{\mathbb{P} := \tilde{\mathbb{P}} \circ (X + y - x)^{-1} : \tilde{\mathbb{P}} \in \mathcal{P}_{t,x}\}$. Then

$$\begin{aligned} & |\lambda(t, x) - \lambda(t, y)| \\ & \leq \sup_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}} \left| \int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}}) - \ell(s, X_s + y - x, \nu_s^{\mathbb{P}}) ds \right. \\ & \qquad \qquad \qquad \left. + \lambda_1(X_1) - \lambda_1(X_1 + y - x) \right| \\ & \leq (M + K)|y - x| \end{aligned}$$

by the Lipschitz property of ℓ and λ in x . \square

Denoting $\lambda_0^R := \lambda^R(0, \cdot)$, in the special case where U is a singleton, equation (4.14) degenerates to the heat equation. Barles, Daher and Romano [2] proved that the error $\lambda - \lambda^R$ satisfies a large deviation estimate as $R \rightarrow \infty$. The next result extends this estimate to our context.

LEMMA 4.12. *Letting Assumption 4.8 hold true, we denote $|x| := \max_{i=1}^d |x_i|$ for $x \in \mathbb{R}^d$ and choose $R > 2M$. Then, there is a constant C such that for all $K > 0$, all K -Lipschitz function λ_1 and $|x| \leq R - M$,*

$$|\lambda^R - \lambda|(t, x) \leq C(1 + K)e^{-(R-M-|x|)^2/2M}.$$

PROOF. (1) For arbitrary $(t, x) \in [0, 1] \times \mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_{t,x}$, we denote $Y^i := \sup_{0 \leq s \leq 1} |X_s^i|$, where X^i is the i th component of the canonical process X . By

the Dubins–Schwarz time-change theorem (see, e.g., Theorem 4.6, Chapter 3 of Karatzas and Shreve [26]), we may represent the continuous local martingale part of X^i as a time-changed Brownian motion W . Since the characteristics of X are bounded by M , we see that

$$\begin{aligned}
 S^i(R) &:= \mathbb{P}[Y^i \geq R] \leq \mathbb{P}\left[\sup_{0 \leq t \leq M} |W_t| \geq R - |x_i| - M\right] \\
 (4.12) \qquad &\leq 2\mathbb{P}\left[\sup_{0 \leq t \leq M} W_t \geq R - |x_i| - M\right] \\
 &= 4(1 - \mathbf{N}(R_{|x_i|}^M)),
 \end{aligned}$$

where $R_{|x_i|}^M := (R - M - |x_i|)/\sqrt{M}$, \mathbf{N} is the cumulative distribution function of the standard normal distribution $N(0, 1)$, and the last equality follows from the reflection principle of the Brownian motion. Then by integration by parts as well as (4.12),

$$\begin{aligned}
 \mathbb{E}^{\mathbb{P}}[Y^i \mathbf{1}_{Y^i \geq R}] &= RS^i(R) + \int_R^\infty S^i(z) dz \\
 &\leq 4 \int_R^\infty \frac{1}{\sqrt{M}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - M - |x_i|)^2}{2M}\right) z dz \\
 &= 4(|x_i| + M)(1 - \mathbf{N}(R_{|x_i|}^M)) + \frac{4\sqrt{M}}{\sqrt{2\pi}} \exp\left(-\frac{(R_{|x_i|}^M)^2}{2}\right).
 \end{aligned}$$

We further remark that for any $R > 0$,

$$(1 - \mathbf{N}(R)) = \int_R^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \leq \frac{1}{R} \int_R^\infty \frac{1}{\sqrt{2\pi}} t e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} \frac{1}{R} e^{-R^2/2}.$$

(2) By definitions of λ, λ^R , it follows that for all (t, x) such that $|x| \leq R - M$,

$$\begin{aligned}
 (4.13) \qquad |\lambda - \lambda^R|(t, x) &\leq \sup_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}} \left[\int_{\tau_R \wedge 1}^1 |\ell(s, X_s, v_s^{\mathbb{P}})| ds + |\lambda_1(X_{\tau_R \wedge 1}) - \lambda_1(X_1)| \right] \\
 &\leq \sup_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}} \left[\left(M + \sqrt{d}KR + (M + K) \sup_{t \leq s \leq 1} |X_s| \right) \mathbf{1}_{\tau_R < 1} \right] \\
 &\leq \sup_{\mathbb{P} \in \mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}} \left[\sum_{i=1}^d (M + \sqrt{d}KR + \sqrt{d}(M + K)Y_i) \mathbf{1}_{Y_i \geq R} \right] \\
 &\leq C(1 + K)e^{-(R_{|x|}^M)^2/2}
 \end{aligned}$$

for some constant C depending on M and d . This completes the proof. \square

With the estimate in Lemma 4.11, we have the following result.

THEOREM 4.13. *Suppose that Assumptions 3.1, 3.2, 3.3 hold true and H given by (4.6) is continuous. Then the function λ^R in (4.11) is the unique viscosity solution of equation*

$$(4.14) \quad -\partial_t \lambda^R(t, x) - H(t, x, D\lambda^R, D^2\lambda^R) = 0, \quad (t, x) \in [0, 1) \times O_R,$$

with boundary conditions

$$(4.15) \quad \lambda^R(t, x) = \lambda_1(x) \quad \text{for all } (t, x) \in ([0, 1) \times \partial O_R) \cup (\{1\} \times O_R),$$

where ∂O_R denotes the boundary of O_R .

PROOF. First, it follows by the same arguments as in Theorem 4.2 that λ^R is a viscosity solution of (4.14) with boundary condition (4.15). The uniqueness follows by the comparison principle of (4.14), (4.15), which holds clearly true from discussions in Example 3.6 of Crandall et al. [13]. \square

4.2.2. Approximation of the transportation cost value. In the bounded characteristics case, we can give a first approximation of the minimum transportation cost. Nevertheless, a complete resolution needs a numerical approximation which will be provided in Section 5. Let us fix the two probability measures μ_0 and μ_1 , and simplify the notation $V(\mu_0, \mu_1)$ [resp., $\mathcal{V}(\mu_0, \mu_1)$] to V (resp., \mathcal{V}).

First, under Assumptions 3.1, 3.2, 3.3, 3.7 and 4.8, it follows by our duality result of Theorem 3.6 together with Theorem 3.8 that

$$(4.16) \quad \begin{aligned} V = \mathcal{V} &:= \sup_{\lambda_1 \in C_b(\mathbb{R}^d)} (\mu_0(\lambda_0) - \mu_1(\lambda_1)) \\ &= \overline{\mathcal{V}} := \sup_{\lambda_1 \in C_b^\infty(\mathbb{R}^d)} (\mu_0(\lambda_0) - \mu_1(\lambda_1)), \end{aligned}$$

where λ_0 is defined in (3.2).

Let Lip_K^0 denote the collection of all bounded K -Lipschitz-continuous functions $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ with $\phi(0) = 0$, and denote $\text{Lip}^0 := \bigcup_{K>0} \text{Lip}_K^0$. Since $v(\lambda_1 + c) = v(\lambda_1) + c$ for any $\lambda_1 \in C_b(\mathbb{R}^d)$ and $c \in \mathbb{R}$, we deduce from (4.16) that

$$V = \sup_{\lambda_1 \in \text{Lip}^0} v(\lambda_1) \quad \text{where } v(\lambda_1) := \mu_0(\lambda_0) - \mu_1(\lambda_1).$$

As a first approximation, we introduce the function

$$(4.17) \quad V^K := \sup_{\lambda_1 \in \text{Lip}_K^0} v(\lambda_1).$$

Under Assumptions 4.8 and 4.9, it is clear that $V^K < \infty, \forall K > 0$ by Lemma 4.11. Then, it is immediate that

$$(4.18) \quad (V^K)_{K>0} \text{ is increasing and } V^K \rightarrow V \text{ as } K \rightarrow \infty.$$

Letting λ^R be defined in (4.11) for every $R > 0$, denote

$$(4.19) \quad \begin{aligned} V^{K,R} &:= \sup_{\lambda_1 \in \text{Lip}_0^K} v^R(\lambda_1) \quad \text{and} \\ v^R(\lambda_1) &:= \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_1(\lambda_1 \mathbf{1}_{O_R}). \end{aligned}$$

Then the second approximation is on variable R .

PROPOSITION 4.14. *Let Assumptions 4.8 and 4.9 hold true, then for all $K > 0$,*

$$(4.20) \quad \begin{aligned} &|V^{K,R} - V^K| \\ &\leq C(1 + K) \left(e^{-R^2/8M+R/2} + \int_{O_{R/2}^c} (1 + |x|)(\mu_0 + \mu_1)(dx) \right). \end{aligned}$$

PROOF. By their definitions in (4.17) and (4.19), we have

$$\begin{aligned} &|V^{K,R} - V^K| \\ &= \left| \sup_{\lambda_1 \in \text{Lip}_K^0} \{ \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_1(\lambda_1 \mathbf{1}_{O_R}) \} - \sup_{\lambda_1 \in \text{Lip}_K^0} \{ \mu_0(\lambda_0) - \mu_1(\lambda_1) \} \right| \\ &\leq \sup_{\lambda_1 \in \text{Lip}_K^0} | \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_0(\lambda_0) | + K \int_{O_R^c} |x| \mu_1(dx). \end{aligned}$$

Now for all $\lambda_1 \in \text{Lip}_K^0$, we estimate from Lemmas 4.11 and 4.12 that

$$\begin{aligned} &| \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_0(\lambda_0) | \\ &\leq \mu_0(|\lambda_0^R - \lambda_0| \mathbf{1}_{O_{R/2}}) + \mu_0((|\lambda_0^R| + |\lambda_0|) \mathbf{1}_{(O_{R/2})^c}) \\ &\leq C(1 + K) \left(\int_{O_{R/2}} e^{-(R_{|x|}^M)^2/2} \mu_0(dx) + \int_{(O_{R/2})^c} (1 + |x|) \mu_0(dx) \right). \end{aligned}$$

Observing that $(R_{|x|}^M)^2 \geq R^2/4M - R + M$ on $O_{R/2}$, this implies that

$$\begin{aligned} &| \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_0(\lambda_0) | \\ &\leq C(1 + K) \left(e^{-R^2/8M+R/2} + \int_{(O_{R/2})^c} (1 + |x|) \mu_0(dx) \right), \end{aligned}$$

and the required estimate follows. \square

5. Numerical approximation. Throughout this section, we consider the Markovian context where $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$ under bounded characteristics. Our objective is to provide an implementable numerical algorithm to compute

$V^{K,R}$ in (4.19), which is itself an approximation of the minimum transportation cost V in (4.16).

Although there are many numerical methods for nonlinear PDEs, our problem concerns the maximization over the solutions of a class of nonlinear PDEs. To the best of our knowledge, it is not addressed in the previous literature. In Bonnans and Tan [10], a similar but more specific problem is considered. Their set U allowing for unbounded diffusions is out of the scope of this paper. However, by using the specific structure of their problem, their key observation is to convert the unconstrained control problem into an optimal stopping problem for which they propose a numerical approximation scheme. Our numerical approximation is slightly different, as we avoid the issue of singular stochastic control by restricting to bounded controls, but uses their gradient algorithm for the minimization over the choice of Lagrange multipliers λ_1 .

In the following, we shall first give an overview of the numerical methods for nonlinear PDEs in Section 5.1. Then by constructing the finite difference scheme for nonlinear PDE (4.14), we get a discrete optimization problem in Section 5.2 which is an approximation of $V^{K,R}$. We then provide a gradient algorithm for the resolution of the discrete optimization problem in Section 5.3. Finally, we implement our numerical algorithm to test its efficiency in Section 5.4.

In the remaining part of this paper, we restrict the discussion to the one-dimensional case

$$d = 1 \quad \text{so that } O_R = (-R, R).$$

5.1. *Overview of numerical methods for nonlinear PDEs.* There are several numerical schemes for nonlinear PDEs of the form (4.7), for example, the finite difference scheme, semi-Lagrangian scheme and Monte-Carlo schemes. General convergence is usually deduced by the monotone convergence technique of Barles and Souganidis [4] or the controlled Markov-chain method of Kushner and Dupuis [27]. Both methods demand the monotonicity of the scheme, which implies that in practice we should assume the boundedness of drift and diffusion functions [see, e.g., the CFL condition (5.3) below]. To derive a convergence rate, we usually apply Krylov's perturbation method; see, for example, Barles and Jakobsen [3].

For the finite difference scheme, the monotonicity is guaranteed by the CFL condition [see, e.g., (5.3) below] in the one-dimensional case $d = 1$. However, in the general d -dimensional case, it is usually hard to construct a monotone scheme. Kushner and Dupuis [27] suggested a construction when all covariance matrices are diagonal dominated. Bonnans et al. [8, 11] investigated this issue and provided an implementable but sophisticated algorithm in the two-dimensional case. Debraant and Jakobsen [15] proposed recently a semi-Lagrangian scheme for nonlinear equations of the form (4.7). However, to be implemented, it still needs to discretize the space and then to use an interpolation technique. Therefore, it can be viewed as a kind of finite difference scheme.

In the high-dimensional case, it is generally preferred to use Monte-Carlo schemes. For linear and semilinear parabolic PDEs, the Monte-Carlo methods are usually induced by the Feynman–Kac formula and backward stochastic differential equations (BSDEs). This scheme is then generalized by Fahim, Touzi and Warin [20] for fully nonlinear PDEs. The idea is to approximate the derivatives of the value function arising in the PDE by conditional expectations, which can then be estimated by simulation-regression methods. However, the Monte-Carlo method is not convenient to be used here since for every terminal condition λ_1 , one needs to simulate many paths of a stochastic differential equation and then to solve the PDE by regression method, which makes the computation too costly.

For our problem in (4.19), we finally choose to use the finite difference scheme for the resolution of λ_0^R since it is easy to be constructed explicitly as a monotone scheme under explicit conditions in our context.

5.2. *A finite differences approximation.* Let $(l, r) \in \mathbb{N}^2$ and $h = (\Delta t, \Delta x) \in (\mathbb{R}^+)^2$ be such that $l\Delta t = 1$ and $r\Delta x = R$. Denote $x_i := i\Delta x$, $t_k := k\Delta t$ and define the discrete grids:

$$\begin{aligned} \mathcal{N} &:= \{x_i : i \in \mathbb{Z}\}, & \mathcal{N}_R &:= \mathcal{N} \cap (-R, R), \\ \mathcal{M}_{T,R} &:= \{(t_k, x_i) : (k, i) \in \mathbb{Z}^+ \times \mathbb{Z}\} \cap ([0, 1] \times (-R, R)). \end{aligned}$$

The terminal set and boundary set as well as the interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\begin{aligned} \partial_T \mathcal{M}_{T,R} &:= \{(1, x_i) : x_i \in \mathcal{N}_R\}, & \partial_R \mathcal{M}_{T,R} &:= \{(t_k, \pm R) : k = 0, \dots, l\}, \\ \overset{\circ}{\mathcal{M}}_{T,R} &:= \mathcal{M}_{T,R} \setminus (\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}). \end{aligned}$$

We shall use the finite differences method to solve the dynamic programming equation (4.14), (4.15) on the grid $\mathcal{M}_{T,R}$. For a function w defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivatives of w :

$$D^\pm w(t_k, x_i) := \frac{w(t_k, x_{i\pm 1}) - w(t_k, x_i)}{\Delta x}$$

and

$$(bD)w := b^+ D^+ w + b^- D^- w \quad \text{for } b \in \mathbb{R},$$

where $b^+ := \max(0, b)$, $b^- := \max(0, -b)$; and

$$D^2 w(t_k, x_i) := \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1}))}{\Delta x^2}.$$

We now define the function $\hat{\lambda}^{h,R}$ (or $\hat{\lambda}^{h,R,\hat{\lambda}_1}$ to emphasize its dependence on the boundary condition $\hat{\lambda}_1$) on the grid $\mathcal{M}_{T,R}$ by the following explicit finite differences approximation of the dynamic programming equation (4.14):

$$\hat{\lambda}^{h,R}(t_k, x_i) = \hat{\lambda}_1(x_i) \quad \text{on } \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R},$$

and on $\mathring{\mathcal{M}}_{T,R}$,

$$(5.1) \quad \hat{\lambda}^{h,R}(t_k, x_i) = \left(\hat{\lambda}^{h,R} + \Delta t \inf_{u=(a,b) \in U} \left\{ \ell(\cdot, u) + (bD)\hat{\lambda}^{h,R} + \frac{1}{2}aD^2\hat{\lambda}^{h,R} \right\} \right)(t_{k+1}, x_i).$$

We then introduce the following natural approximation of v^R :

$$(5.2) \quad \hat{v}_h^R(\hat{\lambda}_1) := \mu_0(\text{lin}^R[\hat{\lambda}_0^{h,R}]) - \mu_1(\text{lin}^R[\hat{\lambda}_1]) \quad \text{where } \hat{\lambda}_0^{h,R} := \hat{\lambda}^{h,R}(0, \cdot),$$

and for all functions ϕ defined on the grid \mathcal{N}_R we denote by $\text{lin}^R[\phi]$ the linear interpolation of ϕ extended by zero outside $[-R, R]$.

We shall also assume that the discretization parameters $h = (\Delta t, \Delta x)$ satisfy the CFL condition

$$(5.3) \quad \Delta t \left(\frac{|b|}{\Delta x} + \frac{|a|}{\Delta x^2} \right) \leq 1 \quad \text{for all } (a, b) \in U.$$

Then the scheme (5.1) is L^∞ -monotone, so that the convergence of the scheme is guaranteed by the monotonic scheme method of Barles and Souganidis [4]. For our next result, we assume that the following error estimate holds.

ASSUMPTION 5.1. There are positive constants $L_{K,R}, \rho_1, \rho_2$ which are independent of $h = (\Delta t, \Delta x)$, such that

$$\mu_0(|\text{lin}^R[\hat{\lambda}_0^{h,R}] - \lambda_0 \mathbf{1}_{[-R,R]}|) \leq L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2})$$

for all $\lambda_1 \in \text{Lip}_0^K$ and $\hat{\lambda}_1 = \lambda_1|_{\mathcal{N}_R}$.

Let $\text{Lip}_0^{K,R}$ be the collection of all functions on the grid \mathcal{N}_R defined as restrictions of functions in Lip_0^K :

$$(5.4) \quad \text{Lip}_0^{K,R} := \{ \hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \text{ for some } \lambda_1 \in \text{Lip}_0^K \}.$$

The above approximation of the dynamic value function λ suggests the following natural approximation of the minimal transportation cost value:

$$(5.5) \quad \begin{aligned} V_h^{K,R} &:= \sup_{\hat{\lambda}_1 \in \text{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1) \\ &= \sup_{\hat{\lambda}_1 \in \text{Lip}_0^{K,R}} \mu_0(\text{lin}^R[\hat{\lambda}_0^{h,R}]) - \mu_1(\text{lin}^R[\hat{\lambda}_1]). \end{aligned}$$

REMARK 5.2. Under Assumption 4.8 and the additional condition that ℓ is uniformly $\frac{1}{2}$ -Hölder in t with constant M , then in spirit of the analysis in Barles and Jakobsen [3], Assumption 5.1 holds true with $\rho_1 = \frac{1}{10}$, $\rho_2 = \frac{1}{5}$ and $L_{K,R} = C(1 + K + KR)$ with some constant C depending on M . This rate is not the best, but to the best of our knowledge, it is the best rate which has been proved.

THEOREM 5.3. *Let Assumption 5.1 be true, then with the constants $L_{K,R}$, ρ_1 , ρ_2 introduced in Assumption 5.1, we have*

$$|V_h^{K,R} - V^{K,R}| \leq L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K \Delta x.$$

PROOF. First, given $\lambda_1 \in \text{Lip}_0^K$, we take $\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \in \text{Lip}_0^{K,R}$, then clearly $|\text{lin}^R[\hat{\lambda}_1] - \lambda_1|_{L^\infty([-R,R])} \leq K \Delta x$, and it follows from Assumption 5.1 and (4.19) as well as (5.2) that $v^R(\lambda_1) \leq \hat{v}_h^R(\hat{\lambda}_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K \Delta x$. Hence,

$$V^{K,R} \leq V_h^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K \Delta x.$$

Next, given $\hat{\lambda} \in \text{Lip}_0^{K,R}$, let $\lambda_1 := \text{lin}[\hat{\lambda}_1] \in \text{Lip}_0^K$ be the linear interpolation of $\hat{\lambda}_1$. It follows from Assumption 5.1 that $\hat{v}_h^R(\hat{\lambda}_1) \leq v^R(\lambda_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2})$ and, therefore,

$$V_h^{K,R} \leq V^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}). \quad \square$$

5.3. Gradient projection algorithm. In this section we suggest a numerical scheme to approximate $V_h^{K,R} = \sup_{\hat{\lambda}_1 \in \text{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1)$ in (5.5). The crucial observation for our methodology is the following. By $B(\mathcal{N}_R)$, we denote the set of all bounded functions on \mathcal{N}_R .

PROPOSITION 5.4. *Under the CFL condition (5.3), the function $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ is concave on $B(\mathcal{N}_R)$.*

PROOF. Letting $\bar{u} = (\bar{u}_{k,i})_{0 \leq k < l, -r < i < r}$, with $\bar{u}_{k,i} = (\bar{a}_{k,i}, \bar{b}_{k,i}) \in U$, we introduce $\bar{\lambda}^{h,\bar{u},\hat{\lambda}_1}$ (or just $\bar{\lambda}^{h,\bar{u}}$ if there is no risk of ambiguity) as the unique solution of the discrete linear system on $\mathcal{M}_{T,R}$ with a given $\hat{\lambda}_1$:

$$\bar{\lambda}^{h,\bar{u}}(t_k, x_i) = \hat{\lambda}_1(x_i) \quad \text{for } (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R},$$

and on $\overset{\circ}{\mathcal{M}}_{T,R}$

$$(5.6) \quad \begin{aligned} & \bar{\lambda}^{h,\bar{u}}(t_k, x_i) \\ &= (\bar{\lambda}^{h,\bar{u}} + \Delta t(\ell(\cdot, \bar{u}_{k,i}) + (\bar{b}_{k,i} D) \bar{\lambda}^{h,\bar{u}} + \bar{a}_{k,i} D^2 \bar{\lambda}^{h,\bar{u}}))(t_{k+1}, x_i). \end{aligned}$$

Let $\bar{\lambda}_0^{h,\bar{u}} := \bar{\lambda}^{h,\bar{u}}(0, \cdot)$, and define

$$\bar{v}_h^{R,\bar{u}}(\hat{\lambda}_1) := \mu_0(\text{lin}^R[\bar{\lambda}_0^{h,\bar{u}}]) - \mu_1(\text{lin}^R[\hat{\lambda}_1]).$$

We claim that

$$(5.7) \quad \hat{v}_h^R(\hat{\lambda}_1) = \inf_{\bar{u} \in U^{l(2r-1)}} \bar{v}_h^{R,\bar{u}}(\hat{\lambda}_1).$$

Indeed, under the CFL condition (5.3), the finite difference scheme (5.1) as well as (5.6) are both L^∞ -monotone in the sense of Barles and Souganidis [4]. Moreover, the linear interpolation $\hat{\lambda}_0 \mapsto \text{lin}^R[\hat{\lambda}_0]$ is also monotone. Then taking infimum step by step in (5.1) and (5.5) is equivalent to taking infimum globally in (5.7).

Finally, the concavity of $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ follows from its representation as the infimum of linear maps in (5.7). \square

By the previous proposition, $V_h^{K,R}$ consists in the maximization of a concave function, and a natural scheme to approximate it is the gradient projection algorithm.

REMARK 5.5. Since U is compact by Assumption 4.8, then for every function $\hat{\lambda}_1$, we have the optimal control $\hat{u}(\hat{\lambda}_1) = (\hat{u}_{k,i}(\hat{\lambda}_1))_{0 \leq k < l, -r < i < r}$ such that

$$(5.8) \quad \hat{\lambda}_0^{h,R} = \bar{\lambda}_0^{h,\hat{u}(\hat{\lambda}_1)} \quad \text{and} \quad \hat{v}_h^R(\hat{\lambda}_1) = \bar{v}_h^{R,\hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1).$$

Now we are ready to give the gradient projection algorithm for $V_h^{K,R}$ in (5.5). Given a function $\varphi \in B(\mathcal{N}_R)$, we denote by $P_{\text{Lip}_0^{K,R}}(\varphi)$ the projection of φ on $\text{Lip}_0^{K,R}$, where $\text{Lip}_0^{K,R} \subset B(\mathcal{N}_R)$ is defined in (5.4). Of course, the projection depends on the choice of the norm equipping $B(\mathcal{N})$ which in turn has serious consequences on the numerics. We shall discuss this important issue later.

Letting $\gamma := (\gamma_n)_{n \geq 0}$ be a sequence of positive constants, we propose the following algorithm:

ALGORITHM 1. *To solve problem (5.5):*

- (1) Let $\hat{\lambda}_1^0 := 0$.
- (2) Given $\hat{\lambda}_1^n$, compute the super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1^n)$ of $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ at $\hat{\lambda}_1^n$.
- (3) Let $\hat{\lambda}_1^{n+1} = P_{\text{Lip}_0^{K,R}}(\hat{\lambda}_1^n + \gamma_n \nabla \hat{v}_h^R(\hat{\lambda}_1^n))$.
- (4) Go back to step 2.

In the following, we shall discuss the computation of super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1)$, the projection $P_{\text{Lip}_0^{K,R}}$ as well as the convergence of the above gradient projection algorithm.

5.3.1. *Super-gradient.* Let $\hat{\lambda}_1 \in B(\mathcal{N}_R)$ be fixed. Then, by Remark 5.5, we may find an optimal control $\hat{u}(\hat{\lambda}_1) = (\hat{u}_{k,i}(\hat{\lambda}_1))_{0 \leq k < l, -r \leq i \leq r}$, where $\hat{u}_{k,i}(\hat{\lambda}_1) = (\hat{a}_{k,i}(\hat{\lambda}_1), \hat{b}_{k,i}(\hat{\lambda}_1)) \in U$, for system (5.7). We then denote by g^j the unique solution of the following linear system on $\mathcal{M}_{T,R}$, for every $-r \leq j \leq r$:

$$g^j(t_k, x_i) = \delta_{i,j}, \quad \text{for } (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R},$$

and on $\overset{\circ}{\mathcal{M}}_{T,R}$,

$$(5.9) \quad g^j(t_k, x_i) = (g^j + \Delta t((\hat{b}_{k,i}(\hat{\lambda}_1)D)g^j + \hat{a}_{k,i}(\hat{\lambda}_1)D^2g^j))(t_{k+1}, x_i).$$

Denote $g_0^j := g^j(0, \cdot)$ and δ_j a function on \mathcal{N}_R defined by $\delta_j(x_i) := \delta_{i,j}$.

PROPOSITION 5.6. *Let CFL condition (5.3) hold true, then the vector*

$$(5.10) \quad \nabla \hat{v}_h^R(\hat{\lambda}_1) := (\mu_0(\text{lin}^R[g_0^j]) - \mu_1(\text{lin}^R[\delta_j]))_{-r \leq j \leq r}$$

is a super-gradient of $\varphi \in B(\mathcal{N}_R) \mapsto \hat{v}_h^R(\varphi) \in \mathbb{R}$ at $\hat{\lambda}_1$.

PROOF. Consider the system (5.6) introduced in the proof of Proposition 5.4. Under the CFL condition (5.3), by (5.7), we have for every perturbation $\Delta \hat{\lambda}_1 \in B(\mathcal{N}_R)$,

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta \hat{\lambda}_1) = \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1 + \Delta \hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta \hat{\lambda}_1) \leq \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta \hat{\lambda}_1),$$

which implies that

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta \hat{\lambda}_1) - \hat{v}_h^R(\hat{\lambda}_1) \leq \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta \hat{\lambda}_1) - \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1).$$

We next observe that for fixed $\hat{\lambda}_1$, the function $\varphi \mapsto \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\varphi)$ is linear, and it follows that

$$(5.11) \quad (\bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \delta_j) - \bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1))_{-r \leq j \leq r}$$

is a super-gradient of $\varphi \mapsto \hat{v}_h^R(\varphi)$ at $\hat{\lambda}_1$. Finally, by (5.6) and (5.9), $g^j(t_k, x_i) = \bar{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}(t_k, x_i) - \bar{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1}(t_k, x_i)$, where $\bar{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}$ is the solution of (5.6) with boundary condition $\hat{\lambda}_1 + \delta_j$. By the definition of $\bar{v}_h^{R, \hat{u}(\hat{\lambda}_1)}$ in (5.7), it follows that the super-gradient (5.11) is equivalent to $\nabla \hat{v}_h^R(\hat{\lambda}_1)$ defined in (5.10). \square

5.3.2. *Projection.* To compute the projection $P_{\text{Lip}_0^{K,R}}(\varphi)$, $\forall \varphi \in B(\mathcal{N}_R)$, we need to equip $B(\mathcal{N}_R)$ with a specific norm. In order to obtain a simple projection algorithm, we shall introduce an invertible linear map between $B(\mathcal{N}_R)$ and \mathbb{R}^{2r+1} , then equip on $B(\mathcal{N}_R)$ the norm induced by the classical L^2 -norm on \mathbb{R}^{2r+1} .

Let us define the invertible linear map \mathcal{T}_R from $B(\mathcal{N}_R)$ to \mathbb{R}^{2r+1} as

$$\psi_i = \mathcal{T}_R(\varphi)_i := \begin{cases} \varphi(x_{i+1}) - \varphi(x_i), & i = 1, \dots, r, \\ \varphi(0), & i = 0, \\ \varphi(x_{i-1}) - \varphi(x_i), & i = -1, \dots, -r, \end{cases}$$

and define the norm $|\cdot|_R$ on $B(\mathcal{N}_R)$ (easily be verified) by

$$|\varphi|_R := |\mathcal{T}_R(\varphi)|_{L^2(\mathbb{R}^{2r+1})} \quad \forall \varphi \in B(\mathcal{N}_R).$$

Notice that

$$\begin{aligned} \mathcal{T}_R \text{Lip}_0^{K,R} &:= \{\psi = \mathcal{T}_R \varphi : \varphi \in \text{Lip}_0^{K,R}\} \\ &= \{\psi = (\psi_i)_{-r \leq i \leq r} \in [-K \Delta x, K \Delta x]^{2r+1} : \psi_0 = 0\}. \end{aligned}$$

Then the projection $P_{\text{Lip}_0^{K,R}}$ from $B(\mathcal{N}_R)$ to $\text{Lip}_0^{K,R}$ under norm $|\cdot|_R$ is equivalent to the projection $P_{\mathcal{T}_R \text{Lip}_0^{K,R}}$ from \mathbb{R}^{2r+1} to $\mathcal{T}_R \text{Lip}_0^{K,R}$ under the L^2 -norm, which is simply written as

$$(P_{\mathcal{T}_R \text{Lip}_0^{K,R}}(\psi))_i = \begin{cases} 0, & \text{if } i = 0, \\ (K \Delta x) \wedge \psi_i \vee (-K \Delta x), & \text{otherwise.} \end{cases}$$

5.3.3. *Convergence rate.* Now, let us give a convergence rate for the above gradient projection algorithm. In preparation, we first provide an estimate for the norm of super-gradients $\nabla \hat{v}_h^R$.

PROPOSITION 5.7. *Suppose that CFL condition (5.3) holds true, then $|\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \leq 2|\varphi_1 - \varphi_2|_\infty$ for every $\varphi_1, \varphi_2 \in B(\mathcal{N}_R)$. In particular, the super-gradient $\nabla \hat{v}_h^R$ satisfies*

$$(5.12) \quad |\nabla \hat{v}_h^R(\hat{\lambda}_1)|_R \leq 2\sqrt{\frac{R}{\Delta x} + 1}, \quad \text{for all } \hat{\lambda}_1 \in B(\mathcal{N}).$$

PROOF. Under the CFL condition, the scheme (5.1) is L^∞ -monotone, then $|\hat{\lambda}_0^{h,R,\varphi_1} - \hat{\lambda}_0^{h,R,\varphi_2}|_\infty \leq |\varphi_1 - \varphi_2|_\infty$, and it follows from the definition of \hat{v}_h^R in (5.2) that

$$(5.13) \quad |\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \leq 2|\varphi_1 - \varphi_2|_\infty.$$

Next, by the Cauchy–Schwarz inequality,

$$\begin{aligned} |\varphi_1 - \varphi_2|_\infty &\leq \max\left(\sum_{i=0}^r |\mathcal{T}_R(\varphi_1 - \varphi_2)_i|, \sum_{i=0}^{-r} |\mathcal{T}_R(\varphi_1 - \varphi_2)_i|\right) \\ &\leq \sqrt{r + 1} |\varphi_1 - \varphi_2|_R. \end{aligned}$$

Together with (5.13), this implies that (5.12) holds for every super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1)$. \square

Let us finish this section by providing a convergence rate for our gradient projection algorithm. Denote

$$\Pi := \max_{\varphi_1, \varphi_2 \in \text{Lip}_0^{K,R}} |\varphi_1 - \varphi_2|_R^2 \leq 2r(K \Delta x)^2 \leq 2K^2 R \Delta x,$$

and it follows from Section 5.3.1 of Ben-Tal and Nemirovski [5] that

$$\begin{aligned}
 (5.14) \quad 0 \leq V_h^{K,R} - \max_{n \leq N} \hat{v}_h^R(\hat{\lambda}_1^n) &\leq \frac{\Pi + \sum_{n=1}^N \gamma_n^2 |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R^2}{\sum_{n=1}^N \gamma_n} \\
 &\leq \frac{2K^2 R \Delta x + 4(R/\Delta x + 1) \sum_{n=1}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}.
 \end{aligned}$$

We have several choices for the series $\gamma = (\gamma_n)_{n \geq 1}$:

- Divergent series: $\gamma_n \geq 0$, $\sum_{n=1}^\infty \gamma_n = +\infty$ and $\sum_{n=1}^\infty \gamma_n^2 < +\infty$, then the right-hand side of (5.14) converges to 0 as $N \rightarrow \infty$.
- Optimal stepsizes: $\gamma_n = \frac{\sqrt{2\Pi}}{|\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R \sqrt{n}}$, [5] shows that

$$\begin{aligned}
 V_h^{K,R} - \max_{n \leq N} \hat{v}_h^R(\hat{\lambda}_1^n) &\leq C_1 \frac{(\max_{1 \leq n \leq N} |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R) \cdot \sqrt{2\Pi}}{\sqrt{N}} \\
 &\leq C \frac{K(R + \sqrt{R \Delta x})}{\sqrt{N}}
 \end{aligned}$$

for some constant C independent of $K, R, \Delta t, \Delta x$ and N .

5.4. Numerical examples. We finally implement the above algorithm in the context of an application in finance which consists in the determination of the optimal no-arbitrage bounds of exotic options.

As discussed in the Introduction, this problem has been solved by means of the Skorokhod Embedding Problem (SEP) in the context of some specific examples of derivative securities. However, the SEP approach is not suitable for numerical approximation. In Davis, Obloj and Raval [14], the authors consider a similar problem for the weighted variance swap option which can be included in our context. In contrast to our constraint $\mathbb{P} \circ X_1^{-1} = \mu_1$ in (2.5), they impose the constraint of the form $\mathbb{E}^{\mathbb{P}}[\phi_k(X_1)] = p_k, k = 1, \dots, n$ for some functions $(\phi_k)_{1 \leq k \leq n}$ and constants $(p_k)_{1 \leq k \leq n}$. Then, they convert their problem into a semi-infinite linear programming problem which can be solved numerically. We shall use some techniques in [14] to derive an explicit solution for some examples in order to compare with our numerical results.

5.4.1. A toy example. Suppose that $\ell(t, x, a, b) = a$, and $U := [a, \bar{a}] \times \{0\}$, then under every $\mathbb{P} \in \mathcal{P}$, the canonical process X is a martingale. Suppose that $\mathcal{P}(\mu_0, \mu_1)$ is nonempty, then it is clear that

$$\begin{aligned}
 V &= \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 \alpha_t^{\mathbb{P}} dt \right] \\
 &= \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} [X_1^2 - X_0^2] = \mu_1(\phi_0) - \mu_0(\phi_0),
 \end{aligned}$$

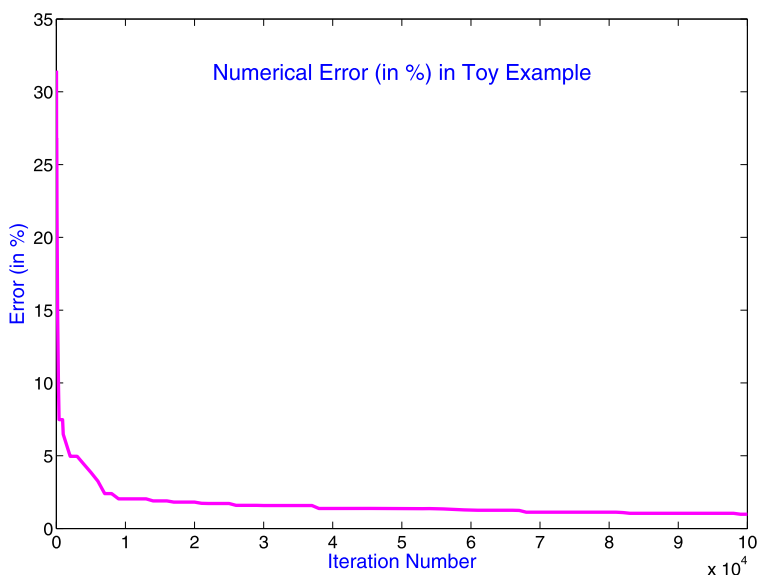


FIG. 1. Numerical Example 1 (toy example): $\mu_i = N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$, $K = 1.5$, $R = 1$, $\Delta x = 0.1$, $\Delta t = 0.025$. The computation time is 56.38 seconds for 10^5 iterations.

where $\phi_0(x) := x^2$. In our implemented example, we choose μ_i as normal distribution $N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$, $\underline{a} = 0$ and $\bar{a} = 0.1$. It follows by direct computation that $V = 0.03$. In our numerical test, for 10^5 iterations, the computation time is 56.38 seconds, and it gives a numerical solution 0.029705, which implies that the relative error is less than 1%; see Figure 1.

5.4.2. *The weighted variance swap contract.* Let $S = (S_t)_{t \geq 0}$ denote the price process of an underlying stock. We assume that S is a scalar positive continuous semimartingale. The variance swap contract is therefore defined by the payoff $\langle \log S \rangle_1$ at maturity 1, which is the quadratic variation of the process $(\log S_t)_{t \geq 0}$ at time 1.

Following Section 4 of [14], we shall consider an η -weighted variance swap, for some Lipschitz function $\eta: \mathbb{R} \rightarrow \mathbb{R}$. This is a derivative security defined by the payoff at maturity 1:

$$\int_0^1 \eta(\log S_t) d\langle \log S \rangle_t.$$

Under no additional information, any martingale measure \mathbb{P} (i.e., a probability measure under which the process S is a martingale) induces an admissible no-arbitrage price

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^1 \eta(\log(S_t)) d\langle \log S \rangle_t \right].$$

Following Galichon et al. [21], we assume that all European options maturing at time 1 with all possible strikes are liquids and available for trading, that is, $c_1(y) := \mathbb{E}[(S_1 - y)^+]$ is given for all $y \geq 0$. Then the marginal distribution of S_1 under \mathbb{P} is given by $\tilde{\mu}_1[y, \infty) = -\partial^- c_1(y)$. In other words, for every $\lambda_1 \in C_b(\mathbb{R})$, the derivative security with payoff $\lambda_1(S_1)$ at maturity 1 is available for trading (long or short) at the no-arbitrage price $\tilde{\mu}_1(\lambda_1)$. Under this additional information, a no-arbitrage lower bound of the η -weighted variance swap is given by

$$(5.15) \quad \sup_{\lambda_1 \in C_b(\mathbb{R})} \left\{ \inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^1 \eta(\log S_t) d(\log S)_t + \lambda_1(S_1) \right] - \tilde{\mu}_1(\lambda_1) \right\},$$

where the infimum is taken over all martingale measures for S .

This problem can be studied by our mass transportation problem. Suppose that $S_t = \exp(X_t)$, where X is the canonical process on the canonical space Ω . Suppose further that

$$U := \{(a, -\frac{1}{2}a) \in S_1 \times \mathbb{R} : a \in [a, \bar{a}]\},$$

with positive constants $a \leq \bar{a} < \infty$. Then under every $\mathbb{P} \in \mathcal{P}$ [defined below (2.3)], the process $S_t := \exp(X_t)$ is a positive continuous martingale. If we take the infimum in (5.15) over \mathcal{P} , it follows by our duality result (Theorem 3.6) that the bound (5.15) equals

$$V = \inf_{\mathbb{P} \in \mathcal{P}(\delta_{x_0}, \mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 \eta(X_t) \alpha_t^{\mathbb{P}} dt,$$

where $x_0 = \log S_0 \in \mathbb{R}$ and μ_1 is the distribution of $X_1 = \log S_1$ when $S_1 \sim \tilde{\mu}_1$ [or, equivalently, it is derived from $\tilde{\mu}_1$ by $\int_{\mathbb{R}} \varphi(x) \mu_1(dx) := \int_{\mathbb{R}} \varphi(\log y) \tilde{\mu}_1(dy)$, $\forall \varphi \in C_b(\mathbb{R})$]. Furthermore, by similar techniques as in Section 4 of Davis et al. [14], using Itô's formula, it follows that when $\mathcal{P}(\delta_{x_0}, \mu_1)$ is nonempty, we have

$$(5.16) \quad V = \inf_{\mathbb{P} \in \mathcal{P}(\delta_{x_0}, \mu_1)} \mathbb{E}^{\mathbb{P}} [\phi(X_1) - \phi(X_0)] = \mu_1(\phi) - \phi(x_0),$$

where ϕ is a solution to $\phi''(x) - \phi'(x) = 2\eta(x)$.

In our numerical experiments, we choose $a = 0, \bar{a} = 0.1, x_0 = 1, \mu_1$ is a normal distribution $N(1 - a/2, a)$ with $a = 0.04 \in [0, 0.1]$. Then $\mathcal{P}(\delta_{x_0}, \mu_1)$ is nonempty since the probability \mathbb{P} induced by the process $(1 - at/2 + \sqrt{a}W_t)_{0 \leq t \leq 1}$ (with Brownian motion W) belongs to it. In a first example, we choose $\eta_1(x) = 1$, then $\phi_1(x) := -2x + C_1 e^x + C_2$ is the solution to $\phi''(x) - \phi'(x) = 2\eta_1(x)$. It follows by direct computation that the value in (5.16) is given by $V_1 = 0.04$. Our numerical solution is 0.0395311 after 10^5 iterations, which takes 138.51 seconds. In a second example, we choose $\eta_2(x) = x$, then $\phi_2(x) := -x^2 - 2x - C_1 e^x + C_2$ is the solution to $\phi''(x) - \phi'(x) = 2\eta_2(x)$. It follows that the value in (5.16) is given by $V_2 = a - a^2/4 = 0.0396$. In our numerical test, the computation time is 142.23 seconds for 10^5 iterations and it gives the numerical solution 0.0391632; see Figure 2.

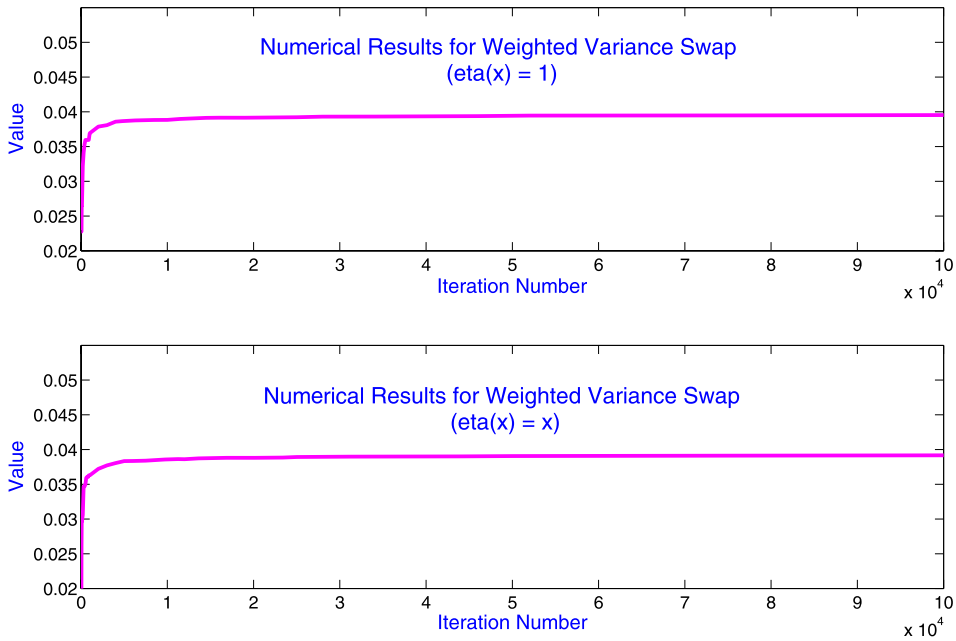


FIG. 2. Numerical Example 2 (weighted variance swap): $\sigma = 0.2, K = 1.5, R = 2, \Delta x = 0.1, \Delta t = 0.025$. For weight function $\eta_1(x) = 1$, the numerical solution is 0.0395311 after 10^5 iterations. For weight function $\eta_2(x) = x$, the numerical solution is 0.0391632 after 10^5 iterations.

APPENDIX

We first give a result which follows directly from the measurable selection theorem. Let E and F be two Polish spaces with their Borel σ -fields $\mathcal{E} := \mathcal{B}(E)$ and $\mathcal{F} := \mathcal{B}(F)$. $A \in \mathcal{E} \otimes \mathcal{F}$ is a measurable subset in the product space $E \times F$ satisfying that for every $x \in E$, there is $y \in F$ such that $(x, y) \in A$. Letting μ be a probability measure on (E, \mathcal{E}) , we denote by \mathcal{E}^μ the μ -completed σ -field of \mathcal{E} . Suppose that $f : A \rightarrow \mathbb{R} \cup \{\infty\}$ is $\mathcal{E} \otimes \mathcal{F}$ -measurable, and denote

$$(A.1) \quad g(x) := \inf\{f(x, y), (x, y) \in A\}.$$

THEOREM A.1. *The function g is \mathcal{E}^μ -measurable. Moreover, for every $\varepsilon > 0$, there is a \mathcal{E}^μ -measurable variable Y_ε such that for μ -a.e. $x \in E$, $(x, Y_\varepsilon(x)) \in A$ and*

$$(A.2) \quad f(x, Y_\varepsilon(x)) \leq (g(x) + \varepsilon)\mathbf{1}_{g(\omega) > -\infty} - \frac{1}{\varepsilon}\mathbf{1}_{g(x) = -\infty}.$$

REMARK A.2. Theorem A.1 is almost the same as Proposition 7.50 of Bertsekas and Shreve [6], and can be easily deduced from it. The key argument is the

measurable selection theorem. We also refer to Section 12.1 of Stroock and Varadhan [35], Chapter 7 of Bertsekas and Shreve [6] or Chapter 3 of Dellacherie and Meyer [16] for different versions of the measurable selection theorem.

We next report a theorem which provides the unique canonical decomposition of a continuous semimartingale under different filtrations. In particular, it follows that an Itô process has a diffusion representation, by taking the filtration generated by itself. This is in fact Theorem 7.17 of Liptser and Shirayev [28] in the 1-dimensional case or Theorem 4.3 of Wong [37] in the multidimensional case.

THEOREM A.3. *In a filtrated space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbb{P})$ (here Ω is not necessarily the canonical space), a process X is a continuous semimartingale with canonical decomposition:*

$$X_t = X_0 + B_t + M_t,$$

where $B_0 = M_0 = 0$, and $B = (B_t)_{0 \leq t \leq 1}$ is of finite variation and $M = (M_t)_{0 \leq t \leq 1}$ a local martingale. In addition, suppose that there are measurable and \mathbb{F} -adapted processes (α, β) such that

$$B_t = \int_0^t \beta_s ds, \quad \int_0^1 \mathbb{E}[|\beta_s|] ds < \infty \quad \text{and} \quad A_t := \langle M \rangle_t = \int_0^t \alpha_s ds.$$

Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leq t \leq 1}$ be the filtration generated by process X and $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \leq t \leq 1}$ be a filtration such that $\mathcal{F}_t^X \subseteq \bar{\mathcal{F}}_t \subseteq \mathcal{F}_t$. Then X is still a continuous semimartingale under $\bar{\mathbb{F}}$, whose canonical decomposition is given by

$$X_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}_t \quad \text{with} \quad \bar{A}_t := \langle \bar{M} \rangle_t = \int_0^t \bar{\alpha}_s ds,$$

where

$$\bar{\beta}_t = \mathbb{E}[\beta_t | \bar{\mathcal{F}}_t] \quad \text{and} \quad \bar{\alpha}_t = \alpha_t, d\mathbb{P} \times dt\text{-a.e.}$$

PROOF OF THEOREM 4.2. The characterization of the value function as viscosity solution to a dynamic programming equation is a natural result of the dynamic programming principle. Here, we give a proof, similar to that of Corollary 5.1 in [12], in our context.

(1) We first prove the subsolution property. Suppose that $(t_0, x_0) \in [0, 1) \times \mathbb{R}^d$ and $\phi \in C_c^\infty([0, 1) \times \mathbb{R}^d)$ is a smooth function such that

$$0 = (\lambda - \phi)(t_0, x_0) > (\lambda - \phi)(t, x) \quad \forall (t, x) \neq (t_0, x_0).$$

By adding $\varepsilon(|t - t_0|^2 + |x - x_0|^4)$ to $\phi(t, x)$, we can suppose that

$$(A.3) \quad \phi(t, x) \geq \lambda(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|^4)$$

without losing generality. Assume to the contrary that

$$-\partial_t \phi(t_0, x_0) - H(t_0, x_0, D_x \phi(t_0, x_0), D_{xx}^2 \phi(t_0, x_0)) > 0,$$

and we shall derive a contradiction. Indeed, by definition of H , there is $c > 0$ and $(a, b) \in U$ such that

$$-\partial_t \phi(t, x) - b \cdot D_x \phi(t, x) - \frac{1}{2} a \cdot D_{xx}^2 \phi(t, x) - \ell(t, x, a, b) > 0 \quad \forall (t, x) \in B_c(t_0, x_0),$$

where $B_c(t_0, x_0) := \{(t, x) \in [0, 1) \times \mathbb{R}^d : |(t, x) - (t_0, x_0)| \leq c\}$. Let $\tau := \inf\{t \geq t_0 : (t, X_t) \notin B_c(t_0, x_0)\} \wedge T$, then

$$\begin{aligned} \lambda(t_0, x_0) = \phi(t_0, x_0) &\geq \inf_{\mathbb{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \phi(\tau, X_\tau) \right] \\ &\geq \inf_{\mathbb{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\mathbb{P}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \lambda(\tau, X_\tau) \right] + \eta, \end{aligned}$$

where η is a positive constant by (A.3) and the definition of τ . This is a contradiction to Proposition 4.1.

(2) For the supersolution property, we assume to the contrary that there is $(t_0, x_0) \in [0, 1) \times \mathbb{R}^d$ and a smooth function ϕ satisfying

$$0 = (\lambda - \phi)(t_0, x_0) < (\lambda - \phi)(t, x) \quad \forall (t, x) \neq (t_0, x_0)$$

and

$$-\partial_t \phi(t_0, x_0) - H(t_0, x_0, D_x \phi(t_0, x_0), D_{xx}^2 \phi(t_0, x_0)) < 0.$$

We also suppose without losing generality that

$$(A.4) \quad \phi(t, x) \leq \lambda(t, x) - \varepsilon(|t - t_0|^2 + |x - x_0|^4).$$

By continuity of H , there is $c > 0$ such that for all $(t, x) \in B_c(t_0, x_0)$ and every $(a, b) \in U$,

$$-\partial_t \phi(t, x) - b \cdot D_x \phi(t, x) - \frac{1}{2} a \cdot D_{xx}^2 \phi(t, x) - \ell(t, x, a, b) < 0.$$

Let $\tau := \inf\{t \geq t_0 : (t, X_t) \notin B_c(t_0, x_0)\} \wedge T$, then

$$\begin{aligned} \lambda(t_0, x_0) = \phi(t_0, x_0) &\leq \inf_{\mathbb{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\mathbb{P}} \left[\phi(\tau, X_\tau) + \int_{t_0}^\tau \ell(s, X_s, \nu_s) ds \right] \\ &\leq \inf_{\mathbb{P} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\mathbb{P}} \left[\lambda(\tau, X_\tau) + \int_{t_0}^\tau \ell(s, X_s, \nu_s) ds \right] - \eta \end{aligned}$$

for some $\eta > 0$ by (A.4), which is a contradiction to Proposition 4.1. \square

Acknowledgments. We are grateful to J. Frédéric Bonnans, Nicole El Karoui and Alfred Galichon for fruitful discussions and two anonymous referees for helpful comments and suggestions.

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CMAP
ÉCOLE POLYTECHNIQUE, PARIS
91128, PALAISEAU
FRANCE
E-MAIL: xiaolu.tan@polytechnique.edu
nizar.touzi@polytechnique.edu