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DETERMINANTAL POINT PROCESSES WITH J-HERMITIAN CORRELATION KERNELS

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Let X be a locally compact Polish space and let m be a reference Radon measure on X. Let Γ_X denote the configuration space over X, that is, the space of all locally finite subsets of X. A point process on X is a probability measure on Γ_X. A point process μ is called determinantal if its correlation functions have the form $k^{(n)}(x_1,\dots,x_n)=\det[K(x_i,x_j)]_{i,j=1,\dots,n}$. The function $K(x,y)$ is called the correlation kernel of the determinantal point process μ. Assume that the space X is split into two parts: $X=X_1\sqcup X_2$. A kernel $K(x,y)$ is called J-Hermitian if it is Hermitian on $X_1\times X_1$ and $X_2\times X_2$, and $K(x,y)=-\overline{K(y,x)}$ for $x\in X_1$ and $y\in X_2$. We derive a necessary and sufficient condition of existence of a determinantal point process with a J-Hermitian correlation kernel $K(x,y)$.

1. Introduction and preliminaries.

1.1. *Macchi–Soshnikov theorem.* Let X be a locally compact Polish space, let $\mathcal{B}(X)$ be the Borel σ-algebra on X, and let $\mathcal{B}_0(X)$ denote the collection of all sets from $\mathcal{B}(X)$ which are pre-compact. The configuration space over X is defined as the set of all locally finite subsets of X:

$$\Gamma:=\Gamma_X:=\{\gamma\subset X \mid \text{for all } \Delta\in\mathcal{B}_0(X)\ |\gamma\cap\Delta|<\infty\}.$$

Here, for a set Λ, $|\Lambda|$ denotes its capacity. Elements $\gamma\in\Gamma$ are called configurations. The space Γ can be endowed with the vague topology, that is, the weakest topology on Γ with respect to which all maps $\Gamma\ni\gamma\mapsto\sum_{x\in\gamma}f(x)$, $f\in C_0(X)$, are continuous. Here $C_0(X)$ is the space of all continuous real-valued functions on X with compact support. The configuration space Γ equipped with the vague topology is a Polish space. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ. A probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ is called a point process on X. For more detail, see, for example, [9, 11, 13, 16].

A point process μ can be described with the help of correlation functions, if they exist. Let m be a reference Radon measure on $(X,\mathcal{B}(X))$. The nth correlation function of μ ($n\in\mathbb{N}$) is an $m^{\otimes n}$-a.e. nonnegative measurable symmetric

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function $k_\mu^{(n)}(x_1,\dots,x_n)$ on X^n such that, for any measurable symmetric function $f^{(n)}: X^n \to [0,\infty]$,

$$\begin{aligned}(1)\qquad &\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\mu(d\gamma)\\ &\quad = \frac{1}{n!}\int_{X^n} f^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)m(dx_1)\cdots m(dx_n).\end{aligned}$$

Under a mild condition on the growth of correlation functions as $n\to\infty$, they determine a point process uniquely [13].

A point process μ is called determinantal if there exists a function $K(x,y)$ on X^2, called the *correlation kernel*, such that

$$(2)\qquad k_\mu^{(n)}(x_1,\dots,x_n) = \det\bigl[K(x_i,x_j)\bigr]_{i,j=1}^n, \qquad n\in\mathbb{N};$$

see, for example, [21]. The integral operator K in $L^2(X,m)$ which has integral kernel $K(x,y)$ is called the *correlation operator of* μ.

Assume that the correlation operator K is self-adjoint and bounded on the (real or complex) Hilbert space $L^2(X,m)$. In particular, the integral kernel $K(x,y)$ is Hermitian (symmetric in the real case). If the correlation functions $(k_\mu^{(n)})_{n\in\mathbb{N}}$ in (2) are *pointwisely* nonnegative, then $K(x,y)$ is a positive definite kernel. Hence, if additionally the function $K(x,y)$ is continuous (it being possible to weaken the latter condition), then the operator K must be nonnegative ($K\ge 0$).

A bounded linear operator K on $L^2(X,m)$ is called a locally trace-class operator if, for each $\Delta\in\mathcal{B}_0(X)$, the operator $K^\Delta := P^\Delta K P^\Delta$ is trace-class. Here P^Δ denotes the operator of multiplication by χ_Δ, the indicator function of the set Δ. [Thus, P^Δ is the orthogonal projection of $L^2(X,m)$ onto $L^2(\Delta,m)$.] If the operator K is self-adjoint and nonnegative, then we can and will choose its integral kernel, $K(x,y)$, so that

$$\operatorname{Tr} K^\Delta = \int_\Delta K(x,x)m(dx) \qquad \text{for each } \Delta\in\mathcal{B}_0(X);$$

see [21] and [10]. By (1) and (2), for each $\Delta\in\mathcal{B}_0(X)$,

$$\int_\Gamma |\gamma\cap\Delta|\mu(d\gamma) = \int_\Delta K(x,x)m(dx).$$

Hence, in order that the correlation functions of μ be finite, we must indeed assume that the operator K is locally trace-class.

The following theorem, which is due to Macchi [15] and Soshnikov [21], plays a fundamental role in the theory of point processes.

THEOREM 1 (Macchi–Soshnikov). *Let K be a self-adjoint, nonnegative, locally trace-class, bounded linear operator on $L^2(X,m)$. Then the integral kernel $K(x,y)$ of the operator K is the correlation kernel of a determinantal point process if and only if* $0\le K\le 1$.

Note that, in the above theorem, the condition of boundedness of the operator K is not essential. One may instead initially assume that K is a Hermitian, nonnegative, locally trace-class operator which is defined on a proper domain in $L^2(X,m)$.

Determinantal point processes with Hermitian correlation kernels occur in various fields of mathematics and physics; see, for example, the review paper [21] and Chapter 4 in [1].

1.2. *Complementation principle* (*particle-hole duality*). Assume that the underlying space X is split into two disjoint parts: $X = X_1 \sqcup X_2$. Hence, we get $L^2(X,m) = L^2(X_1,m) \oplus L^2(X_2,m)$. For $i = 1, 2$, let P_i denote the orthogonal projection of $L^2(X,m)$ onto $L^2(X_i,m)$. Let us define a bounded linear operator J on $L^2(X,m)$ by $J := P_1 - P_2$. Following, for example, [2], we define an (indefinite) J-scalar product on $L^2(X,m)$ by

$$[f,g] := (Jf,g) = (P_1 f, P_1 g) - (P_2 f, P_2 g), \qquad f, g \in L^2(X,m).$$

Here $(\cdot,\cdot)$ denotes the usual scalar product in $L^2(X,m)$. A bounded linear operator K on $L^2(X,m)$ is called J-self-adjoint if $[Kf,g] = [f,Kg]$ for all $f, g \in L^2(X,m)$. An integral kernel $K(x,y)$ of a J-self-adjoint, integral operator K is called J-Hermitian. More precisely, $K(x,y)$ is J-Hermitian if $K(x,y) = \overline{K(y,x)}$ if $x, y \in X_1$ or $x, y \in X_2$, and $K(x,y) = -\overline{K(y,x)}$ if $x \in X_1$, $y \in X_2$.

For a bounded linear operator K on $L^2(X,m)$, we denote

$$\widehat{K} := K P_1 + (1 - K) P_2. \tag{3}$$

As is easily seen, K is J-self-adjoint if and only if $\widehat{K}$ is self-adjoint.

Assume now that the underlying space X is discrete, that is, X is a countable set, and as a topological space X it totally disconnected. Thus, a configuration γ in X is an arbitrary subset of X. Let m be the counting measure on X: $m(\{x\}) = 1$ for each $x \in X$. Any linear operator K in $L^2(X,m)$ may be identified with its matrix $[K(x,y)]_{x,y\in X}$ [$K(x,y)$ being the integral kernel of K in this case].

Let μ be a point process on X. By (1),

$$k_\mu^{(n)}(x_1,\dots,x_n) = \mu\bigl(\gamma \in \Gamma : \{x_1,\dots,x_n\} \subset \gamma\bigr)$$

for distinct points $x_1,\dots,x_n \in X$, otherwise $k_\mu^{(n)}(x_1,\dots,x_n) = 0$. In particular, the correlation functions uniquely identify the corresponding point process.

Following [4], we will now present a complementation principle (a particle-hole duality) for determinantal point processes. (This observation is referred by the authors of [4] to a private communication by S. Kerov.) Assume, as above, that the underlying space X is divided into two disjoint parts: $X = X_1 \sqcup X_2$. Consider the mapping $I : \Gamma \to \Gamma$ defined by

$$I\gamma := \widehat{\gamma} := (\gamma \cap X_1) \cup (X_2 \setminus \gamma).$$

Thus, on the X_1 part of the space, the configuration $\widehat{\gamma}$ coincides with γ, while on the X_2 part the configuration $\widehat{\gamma}$ consists of all points from X_2 which do not belong to γ (holes). The mapping I is clearly an involution, that is, I^2 is the identity mapping. For a point process μ on X, we denote by $\widehat{\mu}$ the push-forward of μ under I.

PROPOSITION 1 ([4]). *Let μ be an arbitrary determinantal point process on a discrete space $X = X_1 \sqcup X_2$, with a correlation kernel $K(x, y)$. Then $\widehat{\mu}$ is the determinantal point process on X with the correlation kernel $\widehat{K}(x, y)$, the integral kernel of the operator $\widehat{K}$ defined by* (3).

Combining the Macchi–Soshnikov theorem with Proposition 1, we immediately get the following:

PROPOSITION 2. *Let $X = X_1 \sqcup X_2$ be a discrete space and let m be a counting measure on X. Let K be a bounded linear operator on $L^2(X, m)$ and let K be J-self-adjoint. Then $K(x, y)$ is the correlation kernel of a determinantal point process on X if and only if* $0 \le \widehat{K} \le 1$.

1.3. *Formulation of the problem and the main result.* In the case of a discrete underlying space X, determinantal point processes with J-Hermitian correlation kernels occurred in Borodin and Olshanski's studies on harmonic analysis of both the infinite symmetric group and the infinite-dimensional unitary group; see, for example, [5–8, 17] and the references therein. The paper [7], page 1332, also contains references to some earlier works of mathematical physicists on solvable models of systems with positive and negative charged particles. In these papers, one finds further examples of determinantal point processes with J-Hermitian correlation kernels.

Furthermore, in their studies, Borodin and Olshanski derived three classes of determinantal point processes with J-Hermitian correlation kernels in the case where the underlying space X is *continuous*: the Whittaker kernel [6] ($X = \mathbb{R}_- \sqcup \mathbb{R}_+$), its scaling limit—the matrix tail kernel [17] ($X = \mathbb{R} \sqcup \mathbb{R}$), and the continuous hypergeometric kernel [7] [$X = (-\frac{1}{2}, \frac{1}{2}) \sqcup \{x \in \mathbb{R} : |x| > \frac{1}{2}\}$]. It is important to note that, in all these examples, the self-adjoint operator $\widehat{K}$ appears to be an orthogonal projection. This follows from Proposition 5.1 in [8] and the respective results of [6, 17] (see also [8], Proposition 6.6) and [7]. (It should be, however, noted that, in the case of a continuous hypergeometric kernel, the corresponding projection property was proved only under an additional assumption; see the last two paragraphs of Section 10 in [7].)

The aim of the present paper is to *derive*, *in the case of a general underlying space X*, *a necessary and sufficient condition of existence of a determinantal point process with a J-Hermitian correlation kernel*. This problem was formulated to the author by Grigori Olshanski. I am extremely grateful to him for this and for many useful discussions and suggestions.

Our main result may be stated as follows. (We will omit a technical detail related to the choice of an integral kernel of the operator K.)

Main result. Assume that K is a J-self-adjoint bounded linear operator on $L^2(X, m)$. Assume that the operators $P_1 K P_1$ and $P_2 K P_2$ are nonnegative. Assume that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ such that $\Delta_1 \subset X_1$ and $\Delta_2 \subset X_2$, the operators K^{Δ_i} $(i = 1, 2)$ are trace-class, while $P^{\Delta_2} K P^{\Delta_1}$ is a Hilbert–Schmidt operator. Then the integral kernel $K(x, y)$ of the operator K is the correlation kernel of a determinantal point process if and only if $0 \le \widehat{K} \le 1$.

Let us make two remarks regarding the conditions of the main result. First, we note that, if the correlation operator K of μ is J-self-adjoint, then the restrictions of the point process μ to X_1 and X_2 are determinantal point processes on X_1 and X_2 with self-adjoint, correlation operators $P_1 K P_1$ and $P_2 K P_2$, respectively. Therefore, we assume that the latter operators are nonnegative.

Second, choose any $\Delta \in \mathcal{B}_0(X)$ such that $m(\Delta_i) > 0$, where $\Delta_i := \Delta \cap X_i$, $i = 1, 2$. Then, since the operator K is not self-adjoint, the assumption in the main result is weaker than the requirement that the operator K^{Δ} be trace-class. In fact, K being locally trace-class seems to be a rather unnatural assumption for J-self-adjoint operators. This, of course, leads us to some additional difficulties in the proof.

Clearly, Proposition 2 is the special case of our main result in the case where the underlying space X is discrete. The drastic difference between the discrete and the continuous cases is that the mapping $\gamma \mapsto \widehat{\gamma}$ has no analog in the case of a continuous space X. Furthermore, if the space X is not discrete, the self-adjoint operator $\widehat{K}$ is not even an integral operator, so it cannot be a correlation operator of a determinantal point process.

To prove the main result, we follow the strategy of dealing with determinantal point processes through the corresponding Fredholm determinants (compare with [15, 19, 21]), or rather the extension of Fredholm determinant as proposed in [4].

Combining the main result and Proposition 5.1 in [8], we also derive a method of constructing a big class of determinantal point processes with J-self-adjoint correlation operators K such that the corresponding operators $\widehat{K}$ are orthogonal projections. This class includes the above mentioned examples of determinantal point processes obtained by Borodin and Olshanski.

The paper is organized as follows. In Section 2 we prove a couple of results related to the mentioned extension of the Fredholm determinant. In Section 3 we prove a series of auxiliary statements regarding J-self-adjoint operators and their extended Fredholm determinants. Finally, in Section 4 we formulate and prove the main results of the paper.

2. An extension of the Fredholm determinant. We first recall the classical definition of a Fredholm determinant; see, for example, [20] for further detail. Let

H be a complex, separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We denote by $\mathscr{L}(H)$ the space of all bounded linear operators on H. An operator $A\in\mathscr{L}(H)$ is called a trace-class operator if $\|A\|_1=\operatorname{Tr}(|A|)<\infty$, where $|A|=(A^*A)^{1/2}$. The set of all trace-class operators in H will be denoted by $\mathscr{L}_1(H)$. The trace of an operator $A\in\mathscr{L}_1(H)$ is given by $\operatorname{Tr}(A)=\sum_{n=1}^\infty(Ae_n,e_n)$, where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of H. $\operatorname{Tr}(A)$ is independent of the choice of a basis. Note also that $|\operatorname{Tr}(A)|\le\operatorname{Tr}(|A|)$. For any $A\in\mathscr{L}_1(H)$ and $B\in\mathscr{L}(H)$, we have $AB, BA\in\mathscr{L}_1(H)$ with

$$\max\big\{\|AB\|_1,\|BA\|_1\big\}\le\|A\|_1\|B\|,$$

where $\|B\|$ denotes the usual operator norm of B. In the latter case, we have

$$\operatorname{Tr}(AB)=\operatorname{Tr}(BA). \tag{4}$$

Denote by $\wedge^n(H)$ the nth antisymmetric tensor power of the Hilbert space H, which is a closed subspace of $H^{\otimes n}$, the nth tensor power of H. For any $A\in\mathscr{L}(H)$, the operator $A^{\otimes n}$ in $H^{\otimes n}$ acts invariantly on $\wedge^n(H)$ and we denote by $\wedge^n(A)$ the restriction of $A^{\otimes n}$ to $\wedge^n(H)$. If $A\in\mathscr{L}_1(H)$, then $\wedge^n(A)\in\mathscr{L}_1(\wedge^n(H))$ and

$$\big\|\wedge^n(A)\big\|_1\le\frac{1}{n!}\|A\|_1^n. \tag{5}$$

The Fredholm determinant is then defined by

$$\operatorname{Det}(1+A)=1+\sum_{n=1}^\infty\operatorname{Tr}\big(\wedge^n(A)\big). \tag{6}$$

The Fredholm determinant can be characterized as the unique function which is continuous in A with respect to the trace norm $\|A\|_1$ and which coincides with the usual determinant when A is a finite-dimensional operator.

One can extend the Fredholm determinant to a wider class on operators. Assume that we are given a splitting of H into two subspaces:

$$H=H_1\oplus H_2. \tag{7}$$

According to this splitting, we write an operator $A\in\mathscr{L}(H)$ in block form,

$$A=\begin{bmatrix}A_{11}&A_{21}\\A_{12}&A_{22}\end{bmatrix}, \tag{8}$$

where $A_{ij}:H_j\to H_i$, $i,j=1,2$. We define the even and odd parts of A as follows:

$$A_{\mathrm{even}}:=\begin{bmatrix}A_{11}&0\\0&A_{22}\end{bmatrix},\qquad A_{\mathrm{odd}}:=\begin{bmatrix}0&A_{21}\\A_{12}&0\end{bmatrix}.$$

We denote by $\mathscr{L}_{1|2}(H)$ the set of all operators $A\in\mathscr{L}(H)$ such that $A_{\mathrm{even}}\in\mathscr{L}_1(H)$ and $A_{\mathrm{odd}}\in\mathscr{L}_2(H)$. Here $\mathscr{L}_2(H)$ denotes the space of all Hilbert–Schmidt operators on H, equipped with the norm

$$\|A\|_2=\left(\sum_{n=1}^\infty\|Ae_n\|^2\right)^{1/2},$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H. Since $\mathscr{L}_1(H) \subset \mathscr{L}_2(H)$, one concludes that

$$\mathscr{L}_1(H) \subset \mathscr{L}_{1|2}(H) \subset \mathscr{L}_2(H).$$

We endow $\mathscr{L}_{1|2}(H)$ with the topology induced by the trace norm on the even part and by the Hilbert–Schmidt norm on the odd part.

PROPOSITION 3 ([4]). *The function $A \mapsto \operatorname{Det}(1 + A)$ admits a unique extension to $\mathscr{L}_{1|2}(H)$ which is continuous in the topology of $\mathscr{L}_{1|2}(H)$. This extension is given by the formula*

$$\operatorname{Det}(1 + A) = \operatorname{Det}\bigl((1 + A)e^{-A}\bigr) \cdot e^{\operatorname{Tr}(A_{\mathrm{even}})}. \tag{9}$$

REMARK 1. Note that, for each $A \in \mathscr{L}_2(H)$, $(1 + A)e^{-A} - 1 \in \mathscr{L}_1(H)$. Therefore, $\operatorname{Det}((1 + A)e^{-A})$ is a classical Fredholm determinant.

REMARK 2. It should be noted that a possibility of extension of the Fredholm determinant to $\mathscr{L}_{1|2}(H)$ was already known to Berezin in the 1960s; see [3], page 8.

We will now give another useful representation of $\operatorname{Det}(1+A)$ for $A \in \mathscr{L}_{1|2}(H)$.

PROPOSITION 4. *Let $A \in \mathscr{L}_{1|2}(H)$ have a block form (8). Assume that $\|A_{11}\| < 1$. Then*

$$\operatorname{Det}(1 + A) = \operatorname{Det}(1 + A_{11}) \cdot \operatorname{Det}\bigl(1 + A_{22} - A_{21}(1 + A_{11})^{-1}A_{12}\bigr). \tag{10}$$

[*On the right-hand side of formula* (10), *both factors are classical Fredholm determinants*, *as both operators A_{11} and $A_{22} - A_{21}(1 + A_{11})^{-1}A_{12}$ belong to $\mathscr{L}_1(H)$*.]

REMARK 3. It should be stressed that the inequality $\|A_{11}\| < 1$ can be achieved by *every* operator in $\mathscr{L}_{1|2}(H)$. More generally, for each fixed $\varepsilon > 0$, we can always assume that $\|A_{11}\| < \varepsilon$. Indeed, assume $\|A_{11}\| \geq \varepsilon$. By the canonical decomposition of a compact (in particular, trace-class) operator (e.g., [20], Theorems 1.1 and 1.2), there exists an orthogonal splitting $H_1 = H_1' \oplus R$ such that the operator A_{11} acts invariantly in both subspaces H_1' and R, the subspace R is finite-dimensional, and the norm of the operator A_{11} in the space H_1' is strictly less than ε. Setting $H_2' := H_2 \oplus R$, we get a new orthogonal splitting $H = H_1' \oplus H_2'$. Write the operator A in the block form with respect to this new splitting of H. Since R is a finite-dimensional space, the even part of A in the new splitting is still a trace-class operator, while the odd part of A in the new splitting is still a Hilbert–Schmidt operator.

PROOF OF PROPOSITION 4. For $i = 1, 2$, let $\{P_i^{(n)}\}_{n=1}^{\infty}$ be an ascending sequence of finite-dimensional orthogonal projections in H_i such that $P_i^{(n)}$ strongly converges to the identity operator in H_i as $n \to \infty$. Set $P^{(n)} := P_1^{(n)} + P_2^{(n)}$, $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $A^{(n)} := P^{(n)} A P^{(n)}$ is a finite-dimensional operator in H, and

$$\|A^{(n)} - A\|_{1|2} \to 0 \qquad \text{as } n \to \infty.$$

Hence, by Proposition 3,

$$\operatorname{Det}(1 + A) = \lim_{n\to\infty} \operatorname{Det}\bigl(1 + A^{(n)}\bigr). \tag{11}$$

In the block form,

$$A^{(n)} = \begin{bmatrix} A_{11}^{(n)} & A_{21}^{(n)} \\ A_{12}^{(n)} & A_{22}^{(n)} \end{bmatrix}, \tag{12}$$

where $A_{ij}^{(n)} = P_i^{(n)} A_{ij} P_j^{(n)}$, $i, j = 1, 2$. For each $n \in \mathbb{N}$, the operator $A^{(n)}$ is finite-dimensional, hence, $\operatorname{Det}(1 + A^{(n)})$ is a classical Fredholm determinant. Therefore,

$$\operatorname{Det}\bigl(1 + A^{(n)}\bigr) = \operatorname{Det} \begin{bmatrix} 1 + A_{11}^{(n)} & +A_{21}^{(n)} \\ A_{12}^{(n)} & 1 + A_{22}^{(n)} \end{bmatrix}; \tag{13}$$

the latter (in fact, usual) determinant refers to the finite-dimensional Hilbert space $P^{(n)} H$. Since $\|A_{11}\| < 1$, we have $\|A_{11}^{(n)}\| < 1$ for all n. Hence, $1 + A_{11}^{(n)}$ is invertible in $P_{11}^{(n)} H$. Employing the well-known formula for the determinant of a block matrix, we get from (11) and (13)

$$\operatorname{Det}(1 + A) = \lim_{n\to\infty} \operatorname{Det}\bigl(1 + A_{11}^{(n)}\bigr) \cdot \operatorname{Det}\bigl(1 + A_{22}^{(n)} - A_{21}^{(n)}\bigl(1 + A_{11}^{(n)}\bigr)^{-1} A_{12}^{(n)}\bigr). \tag{14}$$

We state that

$$\|A_{11}^{(n)} - A_{11}\|_1 \to 0, \qquad \|A_{22}^{(n)} - A_{22}\|_1 \to 0, \tag{15}$$

$$\|A_{21}^{(n)}\bigl(1 - A_{11}^{(n)}\bigr)^{-1} A_{12}^{(n)} - A_{21}(1 - A_{11})^{-1} A_{12}\|_1 \to 0 \tag{16}$$

as $n \to \infty$. Formula (15) is evident. In view of the formula

$$\|BC\|_1 \le \|B\|_2 \|C\|_2, \qquad B, C \in \mathscr{L}_2(H)$$

(see, e.g., [20], Theorem 2.8), the proof of (16) is routine, so we skip it. Thus, (10) follows from (14)–(16). □

We will now derive an analog of formula (6) for $A \in \mathscr{L}_{1|2}(H)$. As follows from the proof of [19], Theorem 2.4, for each $A \in \mathscr{L}_1(H)$, we have

$$\operatorname{Tr}\bigl(\wedge^n(A)\bigr) = \frac{1}{n!} \sum_{\xi \in S_n} \operatorname{sign}(\xi) \prod_{\eta \in \operatorname{Cycle}(\xi)} \operatorname{Tr}\bigl(A^{|\eta|}\bigr). \tag{17}$$

Here S_n denotes the set of all permutations of $1, \dots, n$, the product in (17) is over all cycles η in permutation ξ, and $|\eta|$ denotes the length of cycle η. For $A \in \mathscr{L}_1(H)$, we clearly have $\operatorname{Tr}(A) = \operatorname{Tr}(A_{\text{even}})$. We further note that, for each $A \in \mathscr{L}_2(H)$, we have $A^k \in \mathscr{L}_1(H)$ for $k \geq 2$. Thus, for each $A \in \mathscr{L}_{1|2}(H)$, we set $C_n(A)$ to be equal to the right-hand side of (17) in which we set

$$\operatorname{Tr}(A) := \operatorname{Tr}(A_{\text{even}}), \qquad A \in \mathscr{L}_{1|2}(H).$$

Hence, $C_n(A)$ is well defined for each $A \in \mathscr{L}_{1|2}(H)$, and $C_n(A) = \operatorname{Tr}(\wedge^n(A))$ for each $A \in \mathscr{L}_1(H)$.

PROPOSITION 5. *For each $A \in \mathscr{L}_{1|2}(H)$, we have*

$$\operatorname{Det}(1 + A) = 1 + \sum_{n=1}^{\infty} C_n(A). \tag{18}$$

PROOF. We know that formula (18) holds for all $A \in \mathscr{L}_1(H)$. Next, for each $A \in \mathscr{L}_2(H)$,

$$\|A^k\|_1 \leq \|A\|^{k-2}\|A^2\|_1 \leq \|A\|^{k-2}\|A\|_2^2 \leq \|A\|_2^k, \qquad k \geq 2.$$

Hence, by the definition of $C_n(A)$,

$$\bigl|C_n(A)\bigr| \leq \|A\|_{1|2}^n, \tag{19}$$

where

$$\|A\|_{1|2} := \max\{\|A\|_2, \|A_{\text{even}}\|_1\}.$$

[Note that $\|\cdot\|_{1|2}$ is a norm on $\mathscr{L}_{1|2}(H)$ which determines its topology.] Hence, if $\|A\|_{1|2} < 1$, the series on the right-hand side of (18) converges absolutely. We fix any $A \in \mathscr{L}_{1|2}(H)$ with $\|A\|_{1|2} < 1$. For $i = 1, 2$, let $\{P_i^{(k)}\}_{k=1}^{\infty}$ be an ascending sequence of finite-dimensional orthogonal projections as in the proof of Proposition 4. Then, for each $k \in \mathbb{N}$, $A^{(k)} := P^{(k)} A P^{(k)}$ is a finite-dimensional operator in H, and

$$\|A^{(k)} - A\|_{1|2} \to 0 \qquad \text{as } k \to \infty.$$

Hence, by (19) and the dominated convergence theorem,

$$\operatorname{Det}(1 + A^{(k)}) = 1 + \sum_{n=1}^{\infty} C_n(A^{(k)}) \to 1 + \sum_{n=1}^{\infty} C_n(A).$$

Therefore, by Proposition 3, formula (18) holds in this case.

Now we fix an arbitrary $A \in \mathscr{L}_{1|2}(H)$. Then, by (19), the function

$$z \mapsto 1 + \sum_{n=1}^{\infty} C_n(zA) = 1 + \sum_{n=1}^{\infty} z^n C_n(A)$$

is analytic on the set $\{z \in \mathbb{C} : |z| < \|A\|_{1|2}^{-1}\}$. Thus, by the uniqueness of analytic continuation, to prove the proposition, it suffices to show that the function

$$\mathbb{C} \ni z \mapsto \mathrm{Det}(1 + zA)$$

is entire. But this can be easily deduced from Proposition 4 and Remark 3. □

Let us now assume that $H = L^2(X, m)$, where X is a locally compact Polish space and m is a Radon measure on $(X, \mathcal{B}(X))$. We fix any $X_1, X_2 \in \mathcal{B}(X)$ such that $X = X_1 \sqcup X_2$. By setting $H_i := L^2(X_i, m)$, $i = 1, 2$, we get a splitting (7) of H.

PROPOSITION 6. *Let $K \in \mathscr{L}_{1|2}(L^2(X, m))$ be an integral operator with integral kernel $K(x, y)$ such that $\int_X |K(x, x)| m(dx) < \infty$ and*

$$\mathrm{Tr}(K_{\mathrm{even}}) = \int_X K(x, x) m(dx). \tag{20}$$

Then

$$\mathrm{Det}(1 + K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} \det\big[K(x_i, x_j)\big]_{i,j=1,\dots,n} m(dx_1) \cdots m(dx_n). \tag{21}$$

PROOF. For each $l = 2, 3, \dots,$ we have

$$\mathrm{Tr}\big(K^l\big) = \int_{X^l} K(x_1, x_2) K(x_2, x_3) \cdots K(x_l, x_1) m(dx_1) \cdots m(dx_l). \tag{22}$$

Note that the integral in (22) is independent of the choice of a version of the integral kernel of K. Hence, by the definition of $C_n(K)$ and formulas (20) and (22), we conclude that

$$C_n(K) = \frac{1}{n!} \int_{X^n} \det\big[K(x_i, x_j)\big]_{i,j=1,\dots,n} m(dx_1) \cdots m(dx_n).$$

Now formula (21) follows from Proposition 5. □

3. *J*-self-adjoint operators. We again assume that a Hilbert space H is split into two subspaces; see (7). According to this splitting, we write a vector $f \in H$ as $f = (f_2, f_2)$ and an operator $A \in \mathscr{L}(H)$ in the block form (8). Denote by P_1 and P_2 the orthogonal projections of the Hilbert space H onto H_1 and H_2, respectively. Setting $J := P_1 - P_2$, we introduce an (indefinite) J-scalar product on H by

$$[f, g] := (Jf, g) = (f_1, g_1) - (f_2, g_2), \qquad f, g \in H.$$

An operator $A \in \mathscr{L}(H)$ is called self-adjoint in the indefinite scalar product $[\cdot, \cdot]$, or J-self-adjoint, if

$$[Af, g] = [f, Ag], \qquad f, g \in H;$$

see, for example, [2]. In terms of the block form (8), an operator $A \in \mathscr{L}(H)$ is J-self-adjoint if and only if

$$A_{11}^* = A_{11}, \qquad A_{22}^* = A_{22}, \qquad A_{21}^* = -A_{12}. \tag{23}$$

REMARK 4. Assume A is a usual matrix which has a block form (8). If the blocks of A satisfy (23), then we will call A a J-Hermitian matrix.

For any $A \in \mathscr{L}(H)$, we denote by $\widehat{A}$ the operator from $\mathscr{L}(H)$ given by

$$\widehat{A} := AP_1 + (1 - A)P_2 \tag{24}$$

or, equivalently, in the block form,

$$\widehat{A} = \begin{bmatrix} A_{11} & A_{21} \\ -A_{12} & 1 - A_{22} \end{bmatrix}.$$

Clearly, if the operator A is self-adjoint, then $\widehat{A}$ is J-self-adjoint, while if A is J-self-adjoint, then $\widehat{A}$ is self-adjoint. Also $\widehat{\widehat{A}} = A$.

We will use below the following results.

LEMMA 1. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint. Then $\|A\| = \|\widehat{A} - P_2\|$.*

PROOF. We have $A = \widehat{A}P_1 + (1 - \widehat{A})P_2$, hence,

$$A^* = P_1\widehat{A} + P_2(1 - \widehat{A}).$$

Denote by B_{A^*} the quadratic form on H with generator A^*. For any $f, g \in H$,

$$\begin{aligned} B_{A^*}(f, g) &= (A^* f, g) \\ &= (\widehat{A}_{11} f_1, g_1) + (\widehat{A}_{12} f_2, g_1) \\ &\quad + ((1 - \widehat{A}_{22}) f_2, g_2) + (-\widehat{A}_{21} f_1, g_2). \end{aligned}$$

Denote $\tilde{g} = (g_1, -g_2) = (\tilde{g}_1, \tilde{g}_2)$. Then

$$\begin{aligned} B_{A^*}(f, g) &= (\widehat{A}_{11} f_1, \tilde{g}_1)_H + (\widehat{A}_{12} f_2, \tilde{g}_1)_H + ((\widehat{A}_{22} - 1) f_2, \tilde{g}_2)_H + (\widehat{A}_{21} f_1, \tilde{g}_2) \\ &= (\widehat{A} f, \tilde{g}) - (f_2, \tilde{g}) \\ &= ((\widehat{A} - P_2) f, \tilde{g}) \\ &= B_{\widehat{A} - P_2}(f, \tilde{g}). \end{aligned}$$

From here

$$\|\widehat{A} - P_2\| = \|B_{\widehat{A} - P_2}\| = \|B_{A^*}\| = \|A^*\| = \|A\|. \qquad \square$$

PROPOSITION 7. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint and assume that $0 \le \widehat{A} \le 1$. Then $\|A\| \le 1$.*

PROOF. By Lemma 1, it suffices to show that $\|\widehat{A}-P_2\|\le 1$. Note that $\widehat{A}-P_2$ is self-adjoint. For each $f\in H$,

$$\big((\widehat{A}-P_2)f,f\big)=(\widehat{A}f,f)-(f_2,f_2)\le(\widehat{A}f,f)\le(f,f).$$

Hence, $\widehat{A}-P_2\le 1$. Next,

$$\big((\widehat{A}-P_2)f,f\big)=(\widehat{A}f,f)-(f_2,f_2)\ge-(f_2,f_2)\ge-(f,f),$$

and so $\widehat{A}-P_2\ge -1$. Thus, $-1\le\widehat{A}-P_2\le 1$, which implies the statement. □

PROPOSITION 8. *Let $A\in\mathscr{L}(H)$ be J-self-adjoint and assume that $0\le\widehat{A}\le 1$. Then $\|A\|=1$ if and only if $\|A_{\mathrm{even}}\|=1$.*

PROOF. By Lemma 1, it suffices to prove that $\|\widehat{A}-P_2\|=1$ if and only if $\|A_{\mathrm{even}}\|=1$. Let us first assume that $\|\widehat{A}-P_2\|=1$.

Since $0\le\widehat{A}\le 1$, we have $0\le\widehat{A}_{11}\le 1$ and $0\le\widehat{A}_{22}\le 1$. Hence, $0\le A_{11}\le 1$ and $0\le A_{22}\le 1$, and so $0\le A_{\mathrm{even}}\le 1$, which in turn implies that $\|A_{\mathrm{even}}\|\le 1$. We have to consider two cases.

Case 1. $-1\in\sigma(\widehat{A}-P_2)$. [Here, $\sigma(B)$ denotes the spectrum of an operator $B\in\mathscr{L}(H)$.] Then there exists a sequence $(f^{(n)})_{n=1}^\infty$ in H such that $\|f^{(n)}\|=1$ and

$$\big((\widehat{A}-P_2)f^{(n)},f^{(n)}\big)\to -1.$$

Since $(\widehat{A}f^{(n)},f^{(n)})\ge 0$ and $(P_2f^{(n)},f^{(n)})\le 1$, we get

$$\big(\widehat{A}f^{(n)},f^{(n)}\big)\to 0,\qquad \|f_2^{(n)}\|\to 1.$$

Hence, $f_1^{(n)}\to 0$. From here

$$\big(\widehat{A}_{11}f_1^{(n)},f_1^{(n)}\big)+\big(\widehat{A}_{21}f_1^{(n)},f_2^{(n)}\big)+\big(\widehat{A}_{12}f_2^{(n)},f_1^{(n)}\big)\to 0.$$

Thus,

$$\big(\widehat{A}_{22}f_2^{(n)},f_2^{(n)}\big)\to 0.$$

Hence,

$$\left(\widehat{A}_{22}\frac{f_2^{(n)}}{\|f_2^{(n)}\|},\frac{f_2^{(n)}}{\|f_2^{(n)}\|}\right)\to 0.$$

Hence, $0\in\sigma(\widehat{A}_{22})$, and so $1\in\sigma(1-\widehat{A}_{22})=\sigma(A_{22})$.

Case 2. $1\in\sigma(\widehat{A}-P_2)$. Then there exists a sequence $(f^{(n)})_{n=1}^\infty$ in H such that $\|f^{(n)}\|=1$ and

$$\big((\widehat{A}-P_2)f^{(n)},f^{(n)}\big)\to 1.$$

Since $(\widehat{A} f^{(n)}, f^{(n)}) \le 1$ and $(P_2 f^{(n)}, f^{(n)}) \ge 0$, we get

$$(\widehat{A} f^{(n)}, f^{(n)}) \to 1, \qquad \|f_2^{(n)}\| \to 0.$$

From here, analogously to the above, we conclude that $1 \in \sigma(\widehat{A}_{11}) = \sigma(A_{11})$.

Thus, in both cases, we get $\|A_{\mathrm{even}}\| = 1$. By inverting the arguments, we conclude the inverse statement. □

PROPOSITION 9. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint and let $A \in \mathscr{L}_{1|2}(H)$. Assume that $\|A\| < 1$ and $A_{11} \ge 0$. Then* $\mathrm{Det}(1 - A) > 0$.

PROOF. Since $\|A\| < 1$, we get $\|A_{11}\| < 1$. Hence, by formula (10),

$$\begin{aligned} \mathrm{Det}(1 - A) &= \mathrm{Det}(1 - A_{11}) \cdot \mathrm{Det}\big(1 - A_{22} - A_{21}(1 - A_{11})^{-1} A_{12}\big) \\ &= \mathrm{Det}(1 - A_{11}) \cdot \mathrm{Det}\big(1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}\big). \end{aligned} \tag{25}$$

Note that both operators $-A_{11}$ and $-A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}$ are trace-class and self-adjoint. Since $\|A_{11}\| < 1$, we get $\mathrm{Det}(1 - A_{11}) > 0$. Further, $\|A_{22}\| < 1$ and, hence, there exists $\varepsilon > 0$ such that $1 - A_{22} \ge \varepsilon 1$. Clearly, since $A_{11} \ge 0$,

$$A_{12}^*(1 - A_{11})^{-1} A_{12} \ge 0,$$

which implies

$$1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12} \ge \varepsilon 1.$$

From here

$$\mathrm{Det}\big(1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}\big) > 0,$$

and the proposition is proven. □

PROPOSITION 10. *Let $A \in \mathscr{L}_{1|2}(H)$ and let A be J-self-adjoint. Let $0 \le \widehat{A} \le 1$ and let $\|A\| < 1$. Let $L := A(1 - A)^{-1}$. Then L is J-self-adjoint, $L \in \mathscr{L}_{1|2}(H)$, and $L_{11} \ge 0$, $L_{22} \ge 0$.*

PROOF. We have $L = A + \sum_{n=2}^{\infty} A^n$, and

$$\sum_{n=2}^{\infty} \|A^n\|_1 \le \|A\|_2^2 \sum_{n=0}^{\infty} \|A\|^n < \infty.$$

Hence, $\sum_{n=2}^{\infty} A^n \in \mathscr{L}_1(H)$, so $L \in \mathscr{L}_{1|2}(H)$.

Let us show that the operator L is J-self-adjoint. For any $f, g \in H$, we have

$$(Lf, g) = \sum_{n=1}^{\infty} (A^n f, g) = \sum_{n=1}^{\infty} \big(f, (A^*)^n g\big) = \sum_{n=1}^{\infty} \big(f, (A_{11} - A_{21} - A_{12} + A_{22})^n g\big).$$

Denoting $A'_{11} := A_{11}$, $A'_{22} := A_{22}$, $A'_{12} := -A_{12}$, $A'_{21} := -A_{21}$, we get

$$(Lf, g) = \sum_{n=1}^{\infty} \sum_{\substack{i_k, j_k=1,2\\ k=1,\dots,n}} (f, A'_{i_1 j_1} A'_{i_2 j_2} \cdots A'_{i_n j_n} g). \tag{26}$$

Assume that $f = f_1 \in H_1$, $g = g_1 \in H_1$. Then, in the latter sum, the terms, in which the number of the A'_{12} operators is not equal to the number of the A'_{21} operators, are equal to zero. Hence,

$$\begin{aligned}(L_{11} f_1, g_1) = (L f_1, g_1) &= \sum_{n=1}^{\infty} \sum_{\substack{i_k, j_k=1,2\\ k=1,\dots,n}} (f_1, A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} g_1)\\ &= \sum_{n=1}^{\infty} (f_1, A^n g_1) = (f_1, L g_1) = (f_1, L_{11} g_1).\end{aligned} \tag{27}$$

Thus, $L_{11}^* = L_{11}$. Analogously, $L_{22}^* = L_{22}$.

In the case where $f = f_1 \in H_1$ and $g = g_2 \in H_2$, those terms in the sum in (26), in which the number of the A'_{21} operators is not equal to the number of the A'_{12} operators plus one, are equal to zero. Hence, similar to (27), we get

$$(L_{21} f_1, g_2) = (f_1, -L_{12} g_2),$$

so $L_{21}^* = -L_{12}$. Thus, L is J-self-adjoint.

Next, we will show that $L_{11} \ge 0$. Analogously to the proofs of Propositions 4 and 5, we define operators $A^{(n)}$, $n \in \mathbb{N}$. Thus, each $A^{(n)}$ is J-self-adjoint and

$$\|A^{(n)}\| \le \|A\| < 1. \tag{28}$$

Let $\widehat{A}^{(n)}$ denote the corresponding transformation of the operator $A^{(n)}$ in the Hilbert space $P^{(n)} H$. Recalling representation (12) of $A^{(n)}$, we thus get

$$\widehat{A}^{(n)} = \begin{bmatrix} A_{11}^{(n)} & A_{21}^{(n)} \\ -A_{12}^{(n)} & P_2^{(n)} - A_{22}^{(n)} \end{bmatrix} = P^{(n)} \begin{bmatrix} A_{11} & A_{21} \\ -A_{12} & 1 - A_{22} \end{bmatrix} P^{(n)} = P^{(n)} \widehat{A} P^{(n)}.$$

Since $0 \le \widehat{A} \le 1$, we therefore conclude that $0 \le \widehat{A}^{(n)} \le 1$. In particular, $A_{11}^{(n)} \ge 0$.

We may assume that the dimension of the Hilbert space $P^{(n)} H$ is n. Choose an orthonormal basis $(e^{(i)})_{i=1,\dots,n}$ of $P^{(n)} H$ such that $e^{(i)} \in P_1^{(n)} H$, $i = 1, \dots, k$, and $e^{(i)} \in P_2^{(n)} H$, $i = k+1, \dots, n$. In terms of this orthonormal basis, we may treat the operator $A^{(n)}$ in $P^{(n)} H$ as an $n \times n$ J-Hermitian matrix $[A_{ij}^{(n)}]_{i,j=1,\dots,n}$. Let

$$X^{(n)} := \{1, 2, \dots, n\}, \qquad X_1^{(n)} := \{1, 2, \dots, k\}, \qquad X_2^{(n)} := \{k+1, k+2, \dots, n\},$$

so that $X^{(n)} = X_1^{(n)} \sqcup X_2^{(n)}$. In view of Proposition 2, there exists a determinantal point process $\mu^{(n)}$ on $\Gamma_{X^{(n)}}$ with correlation kernel

$$K^{(n)}(i, j) := A_{ij}^{(n)}, \qquad i, j = 1, \dots, n.$$

By Proposition 9, we have $\det(1 - A^{(n)}) > 0$. Let

$$L^{(n)} := A^{(n)}\big(1 - A^{(n)}\big)^{-1}.$$

We define a possibly signed measure $\rho^{(n)}$ on the configuration space $\Gamma_{X^{(n)}}$ by setting

$$\rho^{(n)}\big(\{\varnothing\}\big) := \mathrm{Det}\big(1 - A^{(n)}\big)$$

and for each nonempty configuration $\{i_1, i_2, \dots, i_m\} \in \Gamma_{X^{(n)}}$,

$$\rho^{(n)}\big(\{i_1, i_2, \dots, i_m\}\big) := \det\big(1 - A^{(n)}\big) \cdot \det\big(L^{(n)}(i_u, i_v)\big)_{u,v=1,\dots,m}.$$

Analogously to the proof of Theorem 2 below, we may show that $\rho^{(n)} = \mu^{(n)}$. Hence, for each nonempty configuration $\{i_1, i_2, \dots, i_m\} \in \Gamma_{X^{(n)}}$,

$$\det\big(L^{(n)}(i_u, i_v)\big)_{u,v=1,\dots,m} \ge 0.$$

In particular, for any nonempty configuration $\{i_1, i_2, \dots, i_m\} \in \Gamma_{X_1^{(n)}}$,

$$\det\big(L_{11}^{(n)}(i_u, i_v)\big)_{u,v=1,\dots,m} \ge 0.$$

Hence, by the Sylvester criterion, $L_{11}^{(n)} \ge 0$, and so

$$\big(L_{11}^{(n)} f_1, f_1\big) \ge 0, \qquad f_1 \in H_1. \tag{29}$$

By (28) and the dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty}\big(L_{11}^{(n)} f_1, f_1\big) &= \lim_{n\to\infty}\big(L^{(n)} f_1, f_1\big) \\
&= \lim_{n\to\infty} \sum_{l=1}^{\infty}\big((A^{(n)})^l f_1, f_1\big) \\
&= \sum_{l=1}^{\infty} \lim_{n\to\infty}\big((A^{(n)})^l f_1, f_1\big) \\
&= \sum_{l=1}^{\infty}\big(A^l f_1, f_1\big) = (L f_1, f_1) = (L_{11} f_1, f_1).
\end{aligned} \tag{30}$$

Thus, by (29) and (30), $(L_{11} f_1, f_1) \ge 0$ for all $f_1 \in H_1$. Analogously, we get $L_{22} \ge 0$. □

The following statement about J-Hermitian matrices was proven in [17].

PROPOSITION 11 ([17]). *Assume that A is a J-Hermitian matrix and assume that its diagonal blocks, A_{11}, A_{22}, are nonnegative definite. Then* $\det(A) \ge 0$.

REMARK 5. Note that the arguments we used in the proof of Proposition 9 are similar to the arguments Olshanski [17] used to prove Proposition 11.

From now on we will again assume that $H=L^2(X,m)$, where X is a locally compact Polish space, m is a Radon measure on $(X,\mathcal{B}(X))$, and $X_1,X_2\in\mathcal{B}(X)$ are such that $X=X_1\sqcup X_2$. We also set $H_i:=L^2(X_i,m)$, $i=1,2$. We further define

$$\mathcal{B}(X_i):=\{\Lambda\in\mathcal{B}(X)\mid \Lambda\subset X_i\}$$

and, analogously, $\mathcal{B}_0(X_i)$, for $i=1,2$.

For $\Delta\in\mathcal{B}_0(X)$, we denote by P^Δ the orthogonal projection of $L^2(X,m)$ onto $L^2(\Delta,m)$, that is, the operator of multiplication by χ_Δ. For an operator $K\in\mathscr{L}(L^2(X,m))$, we denote $K^\Delta:=P^\Delta KP^\Delta$. We will say that an operator $K\in\mathscr{L}(L^2(X,m))$ is locally trace-class on X_1 and X_2 if, for each $\Delta\in\mathcal{B}_0(X_i)$, $i=1,2$, we have $K^\Delta\in\mathscr{L}_1(L^2(X,m))$.

PROPOSITION 12. *Let $K\in\mathscr{L}(L^2(X,m))$ be J-self-adjoint and a locally trace-class operator on X_1 and X_2, and let $0\le\widehat{K}\le 1$. Then, for each $\Delta\in\mathcal{B}_0(X)$, $K^\Delta\in\mathscr{L}_{1|2}(L^2(X,m))$.*

PROOF. For each $\Delta_1\in\mathcal{B}_0(X_1)$, we have

$$P^{\Delta_1}\widehat{K}P^{\Delta_1}=K^{\Delta_1}\in\mathscr{L}_1\big(L^2(X,m)\big). \tag{31}$$

Since $\widehat{K}\ge 0$, we get $P^{\Delta_1}\widehat{K}P^{\Delta_1}\ge 0$. Hence, by (31), $\sqrt{\widehat{K}}P^{\Delta_1}\in\mathscr{L}_2(L^2(X,m))$. Next, for each $\Delta_2\in\mathcal{B}_0(X_2)$,

$$P^{\Delta_2}(1-\widehat{K})P^{\Delta_2}=K^{\Delta_2}\in\mathscr{L}_1\big(L^2(X,m)\big).$$

Hence, analogously to the above, $\sqrt{1-\widehat{K}}P^{\Delta_2}\in\mathscr{L}_2(L^2(X,m))$. From here

$$KP^{\Delta_1}=\widehat{K}P^{\Delta_1}=\sqrt{\widehat{K}}\sqrt{\widehat{K}}P^{\Delta_1}\in\mathscr{L}_2\big(L^2(X,m)\big),$$

$$KP^{\Delta_2}=(1-\widehat{K})P^{\Delta_2}=\sqrt{1-\widehat{K}}\sqrt{1-\widehat{K}}P^{\Delta_2}\in\mathscr{L}_2\big(L^2(X,m)\big).$$

Therefore, $K(P^{\Delta_1}+P^{\Delta_2})\in\mathscr{L}_2(L^2(X,m))$. Thus, for each $\Delta\in\mathcal{B}_0(X)$, $KP^\Delta\in\mathscr{L}_2(L^2(X,m))$, and so $K^\Delta\in\mathscr{L}_2(L^2(X,m))$.

By our assumption, for each $\Delta\in\mathcal{B}_0(X)$,

$$\begin{aligned}K^\Delta_{\text{even}}&=P^\Delta K_{\text{even}}P^\Delta=P^{\Delta_1}K_{11}P^{\Delta_1}+P^{\Delta_2}K_{22}P^{\Delta_2}\\&=K^{\Delta_1}+K^{\Delta_2}\in\mathscr{L}_1\big(L^2(X,m)\big).\end{aligned}$$

(Here $\Delta_i:=\Delta\cap X_i$, $i=1,2$.) Thus, $K^\Delta\in\mathscr{L}_{1|2}(L^2(X,m))$. □

PROPOSITION 13. *Let $K\in\mathscr{L}(L^2(X,m))$ be J-self-adjoint, let $K^\Delta\in\mathscr{L}_{1|2}(L^2(X,m))$ for each $\Delta\in\mathcal{B}_0(X)$, and let $K_{11}\ge 0$, $K_{22}\ge 0$. Then K is an integral operator and its integral kernel $K(x,y)$ can be chosen so that*:

(i) *The kernel* $K(x,y)$ *is* J*-Hermitian*.

(ii) *For* $i=1,2$ *and any* $x_1,\dots,x_n\in X_i$ $(n\in\mathbb{N})$, *the matrix*

$$\big[K(x_i,x_j)\big]_{i,j=1,\dots,n}$$

is nonnegative definite.

(iii) *For each* $\Delta\in\mathcal{B}_0(X)$,

$$\operatorname{Tr}\big(K^{\Delta}_{\mathrm{even}}\big)=\int_{\Delta}K(x,x)m(dx). \tag{32}$$

PROOF. For any $\Delta_1\in\mathcal{B}_0(X_1)$ and $\Delta_2\in\mathcal{B}_0(X_2)$, $P^{\Delta_2}KP^{\Delta_1}$ is a Hilbert–Schmidt operator, hence an integral operator. Therefore, we can choose an integral kernel of K_{21}, which is a function $K_{21}(x,y)$ on $X_2\times X_1$. We now define an integral kernel $K_{12}(x,y)$ of the operator K_{12} by setting $K_{12}(x,y):=-\overline{K_{21}(y,x)}$ for $(x,y)\in X_1\times X_2$. Next, the operators K_{11} and K_{22} are nonnegative, locally trace-class operators. Hence, we can choose their integral kernels according to [10], Lemma A.3; see also [14], Section 3. By combining the integral kernels $K_{ij}(x,y)$, $i,j=1,2$, we obtain an integral kernel $K(x,y)$ of K with needed properties. □

From now on, for an operator K as in Proposition 13, we will always assume that its integral kernel satisfies statements (i)–(iii) of this proposition.

We denote by $B_0(X)$ the space of all measurable bounded real-valued functions on X with compact support. For each $\varphi\in B_0(X)$, we preserve the notation φ for the bounded linear operator of multiplication by φ in $L^2(X,m)$.

PROPOSITION 14. *Let* $K\in\mathscr{L}(L^2(X,m))$ *be* J*-self-adjoint*, *let* $K_{11}\geq 0$, $K_{22}\geq 0$, *and let* $K^{\Delta}\in\mathscr{L}_{1|2}(L^2(X,m))$ *for each* $\Delta\in\mathcal{B}_0(X)$. *Fix any* $\Delta\in\mathcal{B}_0(X)$ *and any* $\varphi\in B_0(X)$ *which vanishes outside* Δ. *Then* $K^{\Delta}\varphi,\operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}\in\mathscr{L}_{1|2}(L^2(X,m))$ *and*

$$\begin{aligned}\operatorname{Det}\big(1+K^{\Delta}\varphi\big)&=\operatorname{Det}\big(1+\operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}\big)\\&=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{X^n}\varphi(x_1)\cdots\varphi(x_n)\\&\qquad\times\det\big[K(x_i,x_j)\big]_{i,j=1,\dots,n}m(dx_1)\cdots m(dx_n).\end{aligned}\tag{33}$$

PROOF. Since $K^{\Delta}\in\mathscr{L}_2(L^2(X,m))$, $K^{\Delta}\varphi\in\mathscr{L}_2(L^2(X,m))$. Since $K^{\Delta}_{\mathrm{even}}\in\mathscr{L}_1(L^2(X,m))$,

$$\big(K^{\Delta}\varphi\big)_{\mathrm{even}}=K^{\Delta}_{\mathrm{even}}\varphi\in\mathscr{L}_1\big(L^2(X,m)\big).$$

Thus, $K^{\Delta}\varphi\in\mathscr{L}_{1|2}(L^2(X,m))$.

Denote $\psi_1:=\operatorname{sgn}(\varphi)\sqrt{|\varphi|}$ and $\psi_2:=\sqrt{|\varphi|}$, $\psi_1,\psi_2\in B_0(X)$. Since ψ_1 and ψ_2 vanish outside Δ, we get

$$\psi_1K\psi_2=\psi_1K^{\Delta}\psi_2$$

and, analogously to the above, we conclude that $\psi_1 K \psi_2 \in \mathscr{L}_{1|2}(L^2(X,m))$.

Since $K^\Delta_{\mathrm{even}} \in \mathscr{L}_1(L^2(X,m))$ and since $\psi_1, \psi_2 \in \mathscr{L}(L^2(X,m))$, by (4),

$$\begin{aligned}\mathrm{Tr}\big((K^\Delta\varphi)_{\mathrm{even}}\big) &= \mathrm{Tr}\big(K^\Delta_{\mathrm{even}}\psi_2\psi_1\big) = \mathrm{Tr}\big(\psi_1 K^\Delta_{\mathrm{even}}\psi_2\big)\\ &= \mathrm{Tr}(\psi_1 K_{\mathrm{even}}\psi_2) = \mathrm{Tr}\big((\psi_1 K\psi_2)_{\mathrm{even}}\big).\end{aligned} \tag{34}$$

Next, for $l = 2, 3, \ldots,$

$$\begin{aligned}\mathrm{Tr}\big((\psi_1 K\psi_2)^l\big) &= \mathrm{Tr}\big(\psi_1 K^\Delta\varphi K^\Delta\varphi\cdots K^\Delta\varphi K^\Delta\psi_2\big)\\ &= \mathrm{Tr}\big(K^\Delta\varphi K^\Delta\varphi\cdots K^\Delta\varphi K^\Delta\psi_2\psi_1\big) = \mathrm{Tr}\big((K^\Delta\varphi)^l\big).\end{aligned} \tag{35}$$

By (34) and (35), $C_n(K^\Delta\varphi) = C_n(\psi_1 K\psi_2)$ for each $n \in \mathbb{N}$, hence, formula (33) holds.

Next, we note that the integral kernel $K^\Delta(x,y)$ of the operator K^Δ is the restriction of $K(x,y)$ to Δ^2. Clearly, the integral kernel of $K^\Delta\varphi$ is $K^\Delta(x,y)\varphi(y)$. Using (32), it is not hard to show that

$$\mathrm{Tr}\big((K^\Delta\varphi)_{\mathrm{even}}\big) = \int_X K^\Delta(x,x)\varphi(x)m(dx).$$

Hence, by Proposition 6,

$$\begin{aligned}&\mathrm{Det}\big(1 + K^\Delta\varphi\big)\\ &\quad= 1 + \sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} \det\big[K^\Delta(x_i,x_j)\varphi(x_j)\big]_{i,j=1,\ldots,n} m(dx_1)\cdots m(dx_n)\\ &\quad= 1 + \sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} \varphi(x_1)\cdots\varphi(x_n)\\ &\qquad\qquad\times \det\big[K^\Delta(x_i,x_j)\big]_{i,j=1,\ldots,n} m(dx_1)\cdots m(dx_n)\\ &\quad= 1 + \sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} \varphi(x_1)\cdots\varphi(x_n)\\ &\qquad\qquad\times \det\big[K(x_i,x_j)\big]_{i,j=1,\ldots,n} m(dx_1)\cdots m(dx_n).\end{aligned}$$

□

4. Main results. We again assume that X is a locally compact Polish space and m is a Radon measure on $(X, \mathcal{B}(X))$. We will also assume that m takes a positive value on each open nonempty set in X. Let $\Gamma = \Gamma_X$ be the configuration space over X. Let μ be a point process on X, that is, a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. Assume that μ satisfies

$$\int_\Gamma C^{|\gamma\cap\Delta|}\mu(d\gamma) \qquad \text{for all } \Delta \in \mathcal{B}_0(X) \text{ and all } C > 0. \tag{36}$$

Then the Bogoliubov functional of μ is defined as

$$B_\mu(\varphi) := \int_\Gamma \prod_{x\in\gamma}\bigl(1+\varphi(x)\bigr)\mu(d\gamma), \qquad \varphi \in B_0(X). \tag{37}$$

Note that, since the function φ has compact support, only a finite number of terms in the product $\prod_{x\in\gamma}(1+\varphi(x))$ are not equal to one. Note also that the integrability of the function $\prod_{x\in\gamma}(1+\varphi(x))$ for each $\varphi \in B_0(X)$ is equivalent to condition (36). If a point process μ has correlation functions $(k_\mu^{(n)})_{n=1}^\infty$ [see (1)], then condition (36) is also equivalent to

$$\sum_{n=1}^\infty \frac{C^n}{n!}\int_{\Delta^n} k_\mu^{(n)}(x_1,\dots,x_n)m(dx_1)\cdots m(dx_n) < \infty$$
$$\text{for all } \Delta \in \mathcal{B}_0(X) \text{ and all } C > 0,$$

and the Bogoliubov functional of μ is given by

$$B_\mu(\varphi) = 1 + \sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} \varphi(x_1)\cdots\varphi(x_n)k_\mu^{(n)}(x_1,\dots,x_n)m(dx_1)\cdots m(dx_n) \tag{38}$$

for each $\varphi \in B_0(X)$. The Bogoliubov functional of μ uniquely determines this point process. For more detail about the Bogoliubov functional see, for example, [12].

Let us now briefly recall some known facts about configuration spaces and point processes; see, for example, [9, 16] for further details. The σ-algebra $\mathcal{B}(\Gamma)$ coincides with the minimal σ-algebra on Γ with respect to which all mappings of the form $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda|$ with $\Lambda \in \mathcal{B}_0(X)$ are measurable. For a fixed set $\Delta \in \mathcal{B}(X)$, we denote by $\mathcal{B}_\Delta(\Gamma)$ the minimal σ-algebra on Γ with respect to which all mappings of the form $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda|$ with $\Lambda \in \mathcal{B}_0(X)$, $\Lambda \subset \Delta$, are measurable. In particular, $\mathcal{B}_\Delta(\Gamma)$ is a sub-σ-algebra of $\mathcal{B}(\Gamma)$. The σ-algebras $\mathcal{B}(\Gamma_\Delta)$ and $\mathcal{B}_\Delta(\Gamma)$ can be identified in the sense that, for each $A \in \mathcal{B}(\Gamma_\Delta)$, $\{\gamma \in \Gamma \mid \gamma \cap \Delta \in A\} \in \mathcal{B}_\Delta(\Gamma)$ and each set from $\mathcal{B}_\Delta(\Gamma)$ has a unique such representation. Hence, the restriction of a point process μ on X to the σ-algebra $\mathcal{B}_\Delta(\Gamma)$—denoted by μ_Δ—can be identified with a point process on Δ, that is, with a probability measure on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$.

Let Δ be a compact subset of X. Then the configuration space Γ_Δ consists of all finite configurations in Δ, that is, $\Gamma_\Delta = \bigsqcup_{n=0}^\infty \Gamma_\Delta^{(n)}$, where $\Gamma_\Delta^{(0)} := \{\varnothing\}$ and for $n \in \mathbb{N}$, $\Gamma_\Delta^{(n)}$ consists of all n-point configurations in Δ. Denote

$$\widetilde{\Delta}^n := \bigl\{(x_1,\dots,x_n) \in \Delta^n \mid x_i \neq x_j \text{ if } i \neq j\bigr\}.$$

Let $\mathcal{B}(\Gamma_\Delta^{(n)})$ denote the image of the σ-algebra $\mathcal{B}(\widetilde{\Delta}^n)$ under the mapping

$$\widetilde{\Delta}^n \ni (x_1,\dots,x_n) \mapsto \{x_1,\dots,x_n\} \in \Gamma_\Delta^{(n)}.$$

Then $\mathcal{B}(\Gamma_\Delta)$ is the minimal σ-algebra on Γ_Δ which contains all $\mathcal{B}(\Gamma_\Delta^{(n)})$, $n \in \mathbb{N}$. A point process μ on X has local densities in Δ if, for each $n \in \mathbb{N}$, there exists a nonnegative measurable symmetric function $d_\mu^{(n)}[\Delta](x_1, \dots, x_n)$ on $\widetilde{\Delta}^n$ such that

$$\int_{\Gamma_\Delta^{(n)}} f^{(n)}(\gamma)\mu_\Delta(d\gamma) = \frac{1}{n!}\int_{\widetilde{\Delta}^n} f^{(n)}(\{x_1, \dots, x_n\})\, d_\mu^{(n)}[\Delta](x_1, \dots, x_n) m(dx_1)\cdots m(dx_n)$$

for each measurable function $f^{(n)}: \Gamma_\Delta^{(n)} \to [0, \infty)$. We also denote $d_\mu^{(0)}[\Delta] := \mu_\Delta(\{\varnothing\})$. In the case where $X = \Delta$ (so that X is a compact Polish space), we will write $d_\mu^{(n)}$ instead of $d_\mu^{(n)}[\Delta]$.

THEOREM 2. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint. Let K be a locally trace-class operator on X_1 and X_2, and let $0 \le \widehat{K} \le 1$. Then there exists a unique point process μ on X which has correlation functions*

$$k_\mu^{(n)}(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1,\dots,n}. \tag{39}$$

The Bogoliubov functional of μ is given by

$$B_\mu(\varphi) = \operatorname{Det}(1 + \operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}), \qquad \varphi \in B_0(X). \tag{40}$$

If additionally $\|K\| < 1$, then for each $\Delta \in \mathcal{B}_0(X)$, the point process μ has local densities in Δ:

$$\begin{aligned} d_\mu^{(0)}[\Delta] &= \operatorname{Det}(1 - K^\Delta), \\ d_\mu^{(n)}[\Delta](x_1, \dots, x_n) &= \operatorname{Det}(1 - K^\Delta)\det[L[\Delta](x_i, x_j)]_{i,j=1,\dots,n}, \end{aligned} \tag{41}$$

where $L[\Delta] := K^\Delta(1 - K^\Delta)^{-1}$.

PROOF. By Proposition 7, $\|K\| \le 1$. We first assume that $\|K\| < 1$. We fix any compact $\Delta \subset X$. By Proposition 12, $K^\Delta \in \mathscr{L}_{1|2}(L^2(X, m))$, hence, $K^\Delta \in \mathscr{L}_{1|2}(L^2(\Delta, m))$. Clearly, K^Δ is J-self-adjoint. Setting $\Delta_i := \Delta \cap X_i$, $i = 1, 2$, we get

$$\begin{aligned} P^\Delta \widehat{K} P^\Delta &= P^\Delta(KP_1 + (1 - K)P_2)P^\Delta \\ &= K^\Delta P^{\Delta_1} + (1 - K^\Delta)P^{\Delta_2} = \widehat{K}^\Delta, \end{aligned} \tag{42}$$

where the latter operator is understood as the transformation (24) of the operator K^Δ in the Hilbert space $L^2(\Delta, m) = L^2(\Delta_1, m) \oplus L^2(\Delta_2, m)$. As $0 \le \widehat{K} \le 1$, we conclude from (42) that $0 \le \widehat{K}^\Delta \le 1$. Since $\|K\| < 1$, we have $\|K^\Delta\| < 1$. Hence, by Proposition 9, $\operatorname{Det}(1 - K^\Delta) > 0$.

Furthermore, by Proposition 10, the operator $L[\Delta]$ is J-self-adjoint and

$$L[\Delta] \in \mathscr{L}_{1|2}(L^2(\Delta, m)), \qquad L[\Delta]_{11} \ge 0, \qquad L[\Delta]_{22} \ge 0.$$

Hence, we can choose an integral kernel $L[\Delta](x,y)$ of the operator $L[\Delta]$ according to Proposition 13. Therefore, for any $x_1,\dots,x_n\in\Delta_1$, $x_{n+1},\dots,x_{n+m}\in\Delta_2$, the matrix $[L[\Delta](x_i,x_j)]_{i,j=1,\dots,n+m}$ is J-Hermitian and the diagonal blocks

$$\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n},\qquad \big[L[\Delta](x_i,x_j)\big]_{i,j=n+1,\dots,n+m}$$

are nonnegative definite. Hence, by Proposition 11,

$$\det\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n+m}\ge 0.$$

Therefore, for each $n\in\mathbb{N}$, the function

$$\widetilde{\Delta}^n\ni(x_1,\dots,x_n)\mapsto\det\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n}$$

is symmetric and takes nonnegative values.

Hence, we can define a positive measure μ_Δ on $(\Gamma_\Delta,\mathcal{B}(\Gamma_\Delta))$ for which

$$\begin{aligned} d^{(0)}_{\mu_\Delta}&=\operatorname{Det}(1-K^\Delta),\\ d^{(n)}_{\mu_\Delta}(x_1,\dots,x_n)&=\operatorname{Det}(1-K^\Delta)\det\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n},\qquad n\in\mathbb{N}.\end{aligned}\tag{43}$$

Note that

$$\det\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n}=0\qquad\text{for all }(x_1,\dots,x_n)\in\Delta^n\setminus\widetilde{\Delta}_n,\qquad n\in\mathbb{N}.$$

Hence, the Bogoliubov functional of μ_Δ is given by

$$\begin{aligned} B_{\mu_\Delta}(\varphi)&=\operatorname{Det}(1-K^\Delta)\\ &\quad\times\Bigg(1+\sum_{n=1}^\infty\frac{1}{n!}\int_{\Delta^n}\big(1+\varphi(x_1)\big)\cdots\big(1+\varphi(x_n)\big)\\ &\qquad\times\det\big[L[\Delta](x_i,x_j)\big]_{i,j=1,\dots,n}m(dx_1)\cdots m(dx_n)\Bigg),\\ &\qquad\varphi\in B(\Delta).\end{aligned}\tag{44}$$

Here $B(\Delta)$ denotes the set of all bounded measurable functions on Δ. It follows from Proposition 14 and (44) that

$$B_{\mu_\Delta}(\varphi)=\operatorname{Det}(1-K^\Delta)\operatorname{Det}\big(1+L[\Delta](1+\varphi)\big),\qquad\varphi\in B(\Delta).\tag{45}$$

Hence, by [4], Corollary A.3, and Proposition 14,

$$\begin{aligned} B_{\mu_\Delta}(\varphi)&=\operatorname{Det}(1-K^\Delta)\big(1+L[\Delta](1+\varphi)\big)\\ &=\operatorname{Det}\big(1+K^\Delta\varphi\big)\\ &=\operatorname{Det}\big(1+\operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}\big),\qquad\varphi\in B(\Delta).\end{aligned}\tag{46}$$

Now we take any sequence of compact subsets of X, $\{\Delta_n\}_{n=1}^\infty$, such that

$$\Delta_1\subset\Delta_2\subset\cdots,\qquad\bigcup_{n=1}^\infty\Delta_n=X.$$

By (46), the probability measures μ_{Δ_n} on $(\Gamma, \mathcal{B}_{\Delta_n}(\Gamma))$ form a consistent family of probability measures. Therefore, by the Kolmogorov theorem, there exists a unique probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ such that the restriction of μ to each $\mathcal{B}_{\Delta_n}(\Gamma)$ coincides with μ_{Δ_n}. By (46), the Bogoliubov functional of μ is given by (40), while the statement about the local densities of μ follows from (43). The determinantal form of the correlation functions of μ—formula (39)—follows from (38), (40) and Proposition 14.

Let us now consider the case where $\|K\| = 1$. For each $\varepsilon \in (0, 1)$, set $K_\varepsilon := \varepsilon K$. Hence, $\|K_\varepsilon\| < 1$. We have

$$\widehat{K}_\varepsilon = \varepsilon K P_1 + (1 - \varepsilon K) P_2 = \varepsilon \widehat{K} + (1 - \varepsilon) P_2. \tag{47}$$

Since $\widehat{K} \geq 0$ and $P_2 \geq 0$, we get $\widehat{K}_\varepsilon \geq 0$, and since $\widehat{K} \leq 1$ and $P_2 \leq 1$, we get $\widehat{K}_\varepsilon \leq 1$. Hence, by the proved above, there exists a point process μ_ε which has correlation functions

$$k_{\mu_\varepsilon}^{(n)}(x_1, \dots, x_n) = \varepsilon^n \det\bigl[K(x_i, x_j)\bigr]_{i,j=1,\dots,n}. \tag{48}$$

Hence, the corresponding correlation measure is $\star$-positive definite in the sense of [11]; see also [14]. By taking the limit as $\varepsilon \to 0$, we therefore conclude that the functions

$$k_{\mu}^{(n)}(x_1, \dots, x_n) := \det\bigl[K(x_i, x_j)\bigr]_{i,j=1,\dots,n}, \qquad n \in \mathbb{N}, \tag{49}$$

determine a $\star$-positive definite correlation measure. By Proposition 14, for each $\Delta \in \mathcal{B}_0(X)$ and $C > 0$,

$$1 + \sum_{n=1}^{\infty} \frac{C^n}{n!} \int_{\Delta^n} k_\mu^{(n)}(x_1, \dots, x_n) m(dx_1) \cdots m(dx_n) = \mathrm{Det}\bigl(1 + C K^\Delta\bigr) < \infty.$$

Hence, by [14], Corollary 1, we conclude that there exists a unique probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ which has correlation functions (49). By Proposition 14 and formula (38), the Bogoliubov functional of μ is given by (40). $\square$

The following corollary easily follows from Theorem 2 and Proposition 5.1 in [8] and its proof.

COROLLARY 1. *Let $G : L^2(X_1, m) \to L^2(X_2, m)$ be a bounded linear operator such that, for any $\Delta_1 \in \mathcal{B}_0(X_1)$ and $\Delta_2 \in \mathcal{B}_0(X_2)$, the operators $G P^{\Delta_1}$ and $P^{\Delta_2} G$ are Hilbert–Schmidt. Let an operator $L \in \mathscr{L}(L^2(X, m))$ be defined by*

$$L := \begin{bmatrix} 0 & G \\ -G^* & 0 \end{bmatrix}.$$

Then operator $1 + L$ is invertible, and we set $K := L(1 + L)^{-1}$. We further have the following:

(i) *The operator K is J-self-adjoint.*

(ii) *The operator K is locally trace-class on X_1 and X_2.*

(iii) *The operator $\widehat{K}$ is the orthogonal projection of $L^2(X,m)$ onto the subspace*

$$\{h \oplus Gh \mid h \in L^2(X_1,m)\}.$$

Thus, by Theorem 2, there exists a unique determiminantal point process with correlation kernel $K(x,y)$.

REMARK 6. As we mentioned in Section 1, the Whittaker kernel [6], the matrix tail kernel [17] and the continuous hypergeometric kernel [7] have their L operators as in Corollary 1, and so their $\widehat{K}$ operators are orthogonal projections.

PROOF OF COROLLARY 1. That the operator $1+L$ is invertible is shown in [8], Section 5. Statement (iii) is just [8], Proposition 5.1. By the proof of Proposition 5.1 [8], the operator L has the following block form:

$$\begin{aligned} K_{11} &= G^*G\big(1+G^*G\big)^{-1}, \\ K_{22} &= GG^*\big(1+GG^*\big)^{-1}, \\ K_{21} &= G\big(1+G^*G\big)^{-1}, \\ K_{21} &= -G^*\big(1+G^*G\big)^{-1}. \end{aligned}$$

Hence, statement (i) obviously follows. So we only need to prove statement (ii). To this end, we fix any $\Delta_1 \in \mathcal{B}_0(X_1)$ and $\Delta_2 \in \mathcal{B}_0(X_2)$. By the assumption of the corollary, $P^{\Delta_2}G$ is a Hilbert–Schmidt operator. Therefore,

$$P^{\Delta_2}K_{21}P^{\Delta_1} = P^{\Delta_2}G\big(1+G^*G\big)^{-1}P^{\Delta_1}$$

is a Hilbert–Schmidt operator, hence so is the operator $P^{\Delta_1}K_{12}P^{\Delta_2}$. Again by the assumption of the corollary, GP^{Δ_1} is a Hilbert–Schmidt operator, hence,

$$(GP^{\Delta_1})^*(GP^{\Delta_1}) = P^{\Delta_1}G^*GP^{\Delta_1}$$

is a trace-class operator. Let $\{e^{(n)}\}_{n\geq 1}$ be an orthonormal basis in $L^2(\Delta_1,m)$. Then, by the spectral theorem,

$$\begin{aligned} \sum_{n\geq 1}\big(K_{11}e^{(n)}, e^{(n)}\big)_{L^2(\Delta_1,m)} &= \sum_{n\geq 1}\big(G^*G\big(1+G^*G\big)^{-1}e^{(n)}, e^{(n)}\big)_{L^2(\Delta_1,m)} \\ &\leq \sum_{n\geq 1}\big(G^*Ge^{(n)}, e^{(n)}\big)_{L^2(\Delta_1,m)} < \infty. \end{aligned}$$

Therefore, the operator $P^{\Delta_1}K_{11}P^{\Delta_1}$ is trace-class. Analogously, we may also show that the operator $P^{\Delta_2}K_{22}P^{\Delta_2}$ is trace-class. Thus, statement (ii) is proven. □

COROLLARY 2. *Let an operator $K \in \mathscr{L}(L^2(X,m))$ be J-self-adjoint and locally trace-class on X_1 and X_2. Let $0 \le \widehat{K} \le 1$ and let $\|K\| = 1$. Let μ be the corresponding determinantal point process. Assume that $\Delta \in \mathcal{B}_0(X)$ is such that $\|K^\Delta\| = 1$. Then*

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta) = 0,$$

that is, the μ probability of the event that there are no particles in Δ is equal to zero.

PROOF. By (40), for each $\Delta \in \mathcal{B}_0(X)$ and $z > 0$,

$$\begin{aligned}\int_\Gamma e^{-z|\gamma\cap\Delta|}\mu(d\gamma) &= \int_\Gamma \prod_{x\in\gamma}(1 + (e^{-z} - 1)\chi_\Delta)\mu(d\gamma)\\ &= \mathrm{Det}(1 - (1 - e^{-z})K^\Delta).\end{aligned}$$

Letting $z \to \infty$ and using the dominated convergence theorem, we get

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta).$$

Since $\|K^\Delta\| = 1$, by Proposition 8, at least one of the operators $K^{\Delta_1} = K_{11}^{\Delta_1}$, $K^{\Delta_2} = K_{22}^{\Delta_2}$ must have norm 1. (Here, as above, $\Delta_i = \Delta \cap X_i$, $i = 1, 2$.) Assume $\|K^{\Delta_1}\| = 1$ (the other case is analogous). As $\mathrm{Det}(1 - K^{\Delta_1})$ is a classical Fredholm determinant and the operator K^{Δ_1} is self-adjoint, we get $\mathrm{Det}(1 - K^{\Delta_1}) = 0$. Thus, we have $\mu_{\Delta_1}(\{\varnothing\}) = 0$, that is, the μ probability of the event that there are no particles in the set Δ_1 is equal to 0. From here the statement follows. $\square$

REMARK 7. Note that, for a determinantal point process μ with a J-self-adjoint correlation operator K, the restriction of μ to the σ-algebra $\mathcal{B}_{X_i}(\Gamma)$ ($i = 1, 2$) may be identified with the determinantal point process on X_i whose correlation operator is the self-adjoint operator K_{ii}.

We will now show that the conditions on a J-self-adjoint operator K in Theorem 2 are, in fact, necessary for a determinantal point process with correlation kernel $K(x, y)$ to exist.

THEOREM 3. *Let $K \in \mathscr{L}(L^2(X,m))$ be J-self-adjoint, let $K_{11} \ge 0$, $K_{22} \ge 0$, and let $K^\Delta \in \mathscr{L}_{1|2}(L^2(X,m))$ for each $\Delta \in \mathcal{B}_0(X)$. Let an integral kernel $K(x, y)$ of the operator K be chosen so that statements* (i)–(iii) *of Proposition* 13 *are satisfied. Then there exists a unique point process μ on X which has correlation functions* (39) *if and only if* $0 \le \widehat{K} \le 1$.

PROOF. We only have to prove that, if a point process μ exists, then $0 \le \widehat{K} \le 1$. We divide the proof into several steps.

(1) Fix any compact $\Delta \subset X$. By Proposition 14 and (38), the Bogoliubov functional of μ is given by (40). Hence, analogously to the proof of Corollary 2, we get

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta).$$

In particular,

$$\mathrm{Det}(1 - K^\Delta) \geq 0. \tag{50}$$

(2) From now on we will additionally assume that $\|K\| < 1$. Then $\|K^\Delta\| < 1$ and we set $L[\Delta] := K^\Delta(1 - K^\Delta)^{-1}$. Just as in the proof of Proposition 10, we derive that $L[\Delta]$ is J-self-adjoint and $L[\Delta] \in \mathscr{L}_{1|2}(L^2(X, m))$. To choose an integral kernel of the operator $L[\Delta]$, we represent it in the form

$$L[\Delta] = K^\Delta + K^\Delta L[\Delta].$$

As $L[\Delta] \in \mathscr{L}_2(L^2(\Delta, m))$, we first choose an arbitrary J-Hermitian integral kernel of this operator, which we denote by $\widetilde{L}[\Delta](x, y)$. Now we set

$$L[\Delta](x, y) := K(x, y) + \int_\Delta K(x, z)\widetilde{L}[\Delta](z, y)m(dy), \qquad x, y \in \Delta.$$

As is easily seen, this integral kernel satisfies

$$\begin{aligned} &\mathrm{Tr}\big(L[\Delta]^\Lambda_{\mathrm{even}}\big) \\ &\quad = \int_\Lambda L[\Delta](x, x)m(dx) \qquad \text{for each } \Lambda \in \mathcal{B}_0(X), \Lambda \subset \Delta. \end{aligned} \tag{51}$$

(3) By (51),

$$\mathrm{Tr}\big((L[\Delta](1 + \varphi))_{\mathrm{even}}\big) = \int_\Delta L[\Delta](x, x)\big(1 + \varphi(x)\big), \qquad \varphi \in B_0(X). \tag{52}$$

Since formula (45) clearly holds for the Boliubov functional of μ_Δ, using (52) and Proposition 6, we get, for each $\varphi \in B(\Delta)$,

$$\begin{aligned} B_{\mu_\Delta}(\varphi) = {}& \mathrm{Det}(1 - K^\Delta) \\ & \times \Bigg(1 + \sum_{n=1}^\infty \int_{\Delta^n} \big(1 + \varphi(x_1)\big) \cdots \big(1 + \varphi(x_n)\big) \\ & \qquad \times \det\big[L[\Delta](x_i, x_j)\big]_{i,j=1,\dots,n} m(dx_1) \cdots m(dx_n)\Bigg). \end{aligned}$$

This implies that the measure μ_Δ has densities (43). Hence, by (50), for $m^{\otimes n}$-a.a. $(x_1, \dots, x_n) \in \Delta^n$,

$$\det\big[L[\Delta](x_i, x_j)\big]_{i,j=1,\dots,n} \geq 0.$$

In particular, for $i=1,2$, for $m^{\otimes n}$-a.a. $(x_1,\dots,x_n)\in\Delta_i^n$,

$$\det\bigl[L[\Delta]_{ii}(x_i,x_j)\bigr]_{i,j=1,\dots,n}\ge 0. \tag{53}$$

Here $\Delta_i:=\Delta\cap X_i$, $i=1,2$.

(4) Following [17], Proposition 1.5, let us find a representation of $L[\Delta]_{11}$ in terms of the blocks of the operator K^Δ. Since $L[\Delta](1-K^\Delta)=K^\Delta$, we have

$$L[\Delta]_{11}\bigl(1-K_{11}^\Delta\bigr)-L[\Delta]_{12}K_{21}^\Delta=K_{11}^\Delta, \tag{54}$$

$$-L[\Delta]_{11}K_{12}^\Delta+L[\Delta]_{12}\bigl(1-K_{22}^\Delta\bigr)=K_{12}^\Delta. \tag{55}$$

From (55),

$$-L[\Delta]_{11}K_{12}^\Delta\bigl(1-K_{22}^\Delta\bigr)^{-1}+L[\Delta]_{12}=K_{12}^\Delta\bigl(1-K_{22}^\Delta\bigr)^{-1},$$

hence,

$$-L[\Delta]_{11}K_{12}^\Delta\bigl(1-K_{22}^\Delta\bigr)^{-1}K_{21}^\Delta+L[\Delta]_{12}K_{21}^\Delta=K_{12}^\Delta\bigl(1-K_{22}^\Delta\bigr)^{-1}K_{21}^\Delta.$$

Adding this to (54) yields

$$L[\Delta]_{11}\bigl(1-Q[\Delta]_{11}\bigr)=Q[\Delta]_{11},$$

where

$$\begin{aligned}Q[\Delta]_{11}&:=K_{11}^\Delta+K_{12}^\Delta\bigl(1-K_{22}^\Delta\bigr)^{-1}K_{21}^\Delta\\&=K_{11}^\Delta-\bigl(K_{21}^\Delta\bigr)^*\bigl(1-K_{22}^\Delta\bigr)^{-1}K_{21}^\Delta.\end{aligned} \tag{56}$$

Since the operator $1-K_{11}^\Delta$ is strictly positive and the operator $(K_{21}^\Delta)^*(1-K_{22}^\Delta)^{-1}K_{21}^\Delta$ is nonnegative, the operator $1-Q[\Delta]_{11}$ is strictly positive, hence invertible. Therefore,

$$L[\Delta]_{11}=Q[\Delta]_{11}\bigl(1-Q[\Delta]_{11}\bigr)^{-1}. \tag{57}$$

(5) By (56), the operator $Q[\Delta]_{11}$ is self-adjoint and trace-class. Since the operator $1-Q[\Delta]_{11}$ is strictly positive, we therefore get

$$\mathrm{Det}\bigl(1-Q[\Delta]_{11}\bigr)>0.$$

[Note that $\mathrm{Det}(1-Q[\Delta]_{11})$ is a usual Fredholm determinant.] Therefore, by (53), we can define a nonnegative, finite measure $\nu[\Delta_1]$ on $(\Gamma_{\Delta_1},\mathcal{B}(\Gamma_{\Delta_1}))$ whose local densities are

$$\begin{aligned}d_{\nu[\Delta_1]}^{(0)}&=\mathrm{Det}\bigl(1-Q[\Delta]_{11}\bigr),\\ d_{\nu[\Delta_1]}^{(n)}(x_1,\dots,x_n)&=\mathrm{Det}\bigl(1-Q[\Delta]_{11}\bigr)\det\bigl[L[\Delta]_{11}(x_i,x_j)\bigr]_{i,j=1,\dots,n},\quad n\in\mathbb{N}.\end{aligned} \tag{58}$$

Analogously to (43)–(46), we conclude from (58) that the Bogoliubov transform of the measure $\nu[\Delta_1]$ is given by

$$
\begin{aligned}
B_{\nu[\Delta_1]}(\varphi) &= \int_{\Gamma_{\Delta_1}} \prod_{x\in\gamma} (1+\varphi(x))\nu[\Delta_1](d\gamma) \\
&= \operatorname{Det}\bigl(1+\operatorname{sgn}(\varphi)\sqrt{|\varphi|}Q[\Delta]_{11}\sqrt{|\varphi|}\bigr), \qquad \varphi \in B_0(\Delta_1).
\end{aligned} \tag{59}
$$

Setting $\varphi \equiv 0$, we see that $\nu[\Delta_1](\Gamma_{\Delta_1}) = 1$, that is, $\nu[\Delta_1]$ is a point process in Δ_1.

(6) We can now choose an integral kernel of the operator $Q[\Delta]_{11}$ analogously to [10], Lemma A.3, and [14], Section 3. Indeed, since K_{21}^{Δ} is a Hilbert–Schmidt operator, $(K_{21}^{\Delta})^*(1-K_{22}^{\Delta})^{-1}K_{21}^{\Delta}$ is a nonnegative trace-class operator in $L^2(\Delta_1, m)$. The operator $((K_{21}^{\Delta})^*(1-K_{22}^{\Delta})^{-1}K_{21}^{\Delta})^{1/2}$ is Hilbert–Schmidt, hence an integral operator. We choose its integral kernel, denoted by $\theta(x, y)$, so that

$$
\begin{aligned}
&\theta(x,y) = \overline{\theta(y,x)} \qquad \text{for all } x, y \in \Delta_1, \\
&\theta(x,\cdot) \in L^2(\Delta_1, m) \qquad \text{for all } x \in \Delta_1.
\end{aligned}
$$

[Recall that $\int_{\Delta_1^2} |\theta(x,y)|^2 m(dx)m(dy) = \|((K_{21}^{\Delta})^*(1-K_{22}^{\Delta})^{-1}K_{21}^{\Delta})^{1/2}\|_2^2 < \infty$.] Now, we set an integral kernel of the operator $(K_{21}^{\Delta})^*(1-K_{22}^{\Delta})^{-1}K_{21}^{\Delta}$ to be

$$
\begin{aligned}
\bigl(K_{21}^{\Delta}\bigr)^*\bigl(1-K_{22}^{\Delta}\bigr)^{-1}K_{21}^{\Delta}(x,y) &:= \int_{\Delta_1} \theta(x,z)\theta(z,y)m(dz) \\
&= \bigl(\theta(x,\cdot), \theta(y,\cdot)\bigr)_{L^2(\Delta_1,m)}, \qquad x, y \in \Delta_1.
\end{aligned}
$$

We similarly construct an integral kernel of the operator K_{11}^{Δ}:

$$
K_{11}^{\Delta}(x,y) = \bigl(\eta(x,\cdot), \eta(y,\cdot)\bigr)_{L^2(\Delta_1,m)}, \qquad x, y \in \Delta_1.
$$

Hence, by virtue of (56), we may choose an integral kernel of the operator $Q[\Delta]_{11}$ as follows:

$$
Q[\Delta]_{11}(x,y) = \bigl(\eta(x,\cdot), \eta(y,\cdot)\bigr)_{L^2(\Delta_1,m)} - \bigl(\theta(x,\cdot), \theta(y,\cdot)\bigr)_{L^2(\Delta_1,m)}. \tag{60}
$$

As is easily seen, for each $\Delta \in \mathcal{B}_0(X)$, $\Lambda \subset \Delta_1$,

$$
\operatorname{Tr}\bigl(Q[\Delta]_{11}^{\Lambda}\bigr) = \int_{\Lambda} Q[\Delta]_{11}(x,x)m(dx).
$$

Now, analogously to Proposition 14, we get from (59)

$$
\begin{aligned}
B_{\nu[\Delta_1]}(\varphi) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Delta_1^n} &\varphi(x_1)\cdots\varphi(x_n) \\
&\times \det\bigl[Q[\Delta]_{11}(x_i,x_j)\bigr]_{i,j=1,\dots,n} m(dx_1)\cdots m(dx_n)
\end{aligned}
$$

for each $\varphi \in B_0(\Delta_1)$. Hence, the correlation functions of the point process $\nu[\Delta_1]$ are

$$
k_{\nu[\Delta_1]}^{(n)}(x_1,\dots,x_n) = \det\bigl[Q[\Delta]_{11}(x_i,x_j)\bigr]_{i,j=1,\dots,n}, \qquad n \in \mathbb{N}.
$$

Therefore, for each $n \in \mathbb{N}$,

$$\det\big[Q[\Delta]_{11}(x_i, x_j)\big]_{i,j=1,\dots,n} \ge 0 \qquad \text{for } m^{\otimes n}\text{-a.a. } (x_1, \dots, x_n) \in \Delta_1^n. \tag{61}$$

(7) Obviously, the following two mappings are measurable:

$$\Delta_1 \ni x \mapsto \eta(x, \cdot) \in L^2(\Delta_1, m), \qquad \Delta_1 \ni x \mapsto \theta(x, \cdot) \in L^2(\Delta_1, m).$$

Therefore, by Lusin's theorem (see, e.g., [18]), for each $\varepsilon > 0$, there exists a compact set $\Lambda_\varepsilon \subset \Delta_1$ such that $m(\Delta_1 \setminus \Lambda_\varepsilon) \le \varepsilon$ and the mappings

$$\Lambda_\varepsilon \ni x \mapsto \eta(x, \cdot) \in L^2(\Delta_1, m), \qquad \Lambda_\varepsilon \ni x \mapsto \theta(x, \cdot) \in L^2(\Delta_1, m)$$

are continuous. Therefore, by (60), the function

$$\Lambda_\varepsilon^2 \ni (x, y) \mapsto Q[\Delta]_{11}(x, y) \in \mathbb{C} \tag{62}$$

is continuous. Hence, by (61),

$$\det\big[Q[\Delta]_{11}(x_i, x_j)\big]_{i,j=1,\dots,n} \ge 0 \qquad \text{for all } (x_1, \dots, x_n) \in \Lambda_\varepsilon^n.$$

Thus, the continuous kernel (62) is positive definite, and therefore the operator $Q[\Delta]_{11}$ is nonnegative on $L^2(\Lambda_\varepsilon, m)$. By letting $\varepsilon \to 0$, we conclude that $Q[\Delta]_{11} \ge 0$ on $L^2(\Delta_1, m)$. Hence, by (56),

$$K_{11}^\Delta \ge (K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1} K_{21}^\Delta \qquad \text{on } L^2(\Delta_1, m). \tag{63}$$

(8) We denote by $\widehat{K}^\Delta$ the corresponding transformation of the operator K^Δ in the Hilbert space $L^2(\Delta, m) = L^2(\Delta_1, m) \oplus L^2(\Delta_2, m)$. Hence, $\widehat{K}^\Delta = P^\Delta \widehat{K} P^\Delta$ and

$$\widehat{K}^\Delta = \begin{bmatrix} K_{11}^\Delta & K_{21}^\Delta \\ (K_{21}^\Delta)^* & 1 - K_{22}^\Delta \end{bmatrix}.$$

By (63), for each $f = (f_1, f_2) \in L^2(\Delta, m)$,

$$\begin{aligned} (\widehat{K}^\Delta f, f) &= (K_{11}^\Delta f_1, f_1) + (K_{21}^\Delta f_1, f_2) \\ &\quad + ((K_{21}^\Delta)^* f_2, f_1) + ((1 - K_{22}^\Delta) f_2, f_2) \\ &\ge ((K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1} K_{21}^\Delta f_1, f_1) \\ &\quad + ((1 - K_{22}^\Delta) f_2, f_2) - 2|(K_{21}^\Delta f_1, f_2)| \\ &= ((1 - K_{22}^\Delta)^{-1} K_{21}^\Delta f_1, K_{21}^\Delta f_1) \\ &\quad + ((1 - K_{22}^\Delta) f_2, f_2) - 2|(K_{21}^\Delta f_1, f_2)|. \end{aligned} \tag{64}$$

Since K_{22}^Δ is a compact self-adjoint operator in $L^2(\Delta_2, m)$, we can choose an orthnormal basis of $L^2(\Delta_2, m)$ which consists of eigenvectors of the operator K_{22}^Δ, and we denote by λ_n the eigenvalue belonging to eigenvector e_n, $n \geq 1$. Clearly, $\lambda_n < 1$ for all n. Then, by (64),

$$\begin{aligned}(\widehat{K}^\Delta f, f) &\geq \sum_{n=1}^{\infty}(1-\lambda_n)^{-1}\big|(K_{21}^\Delta f_1, e_n)\big|^2 + \sum_{n=1}^{\infty}(1-\lambda_n)\big|(f_2, e_n)\big|^2 \\ &\qquad - \sum_{n=1}^{\infty} 2\big|(K_{21}^\Delta f_1, e_n)(f_2, e_n)\big| \\ &= \sum_{n=1}^{\infty}\big((1-\lambda_n)^{-1/2}\big|(K_{21}^\Delta f_1, e_n)\big| - (1-\lambda_n)^{1/2}\big|(f_2, e_n)\big|\big)^2 \geq 0.\end{aligned}$$

Thus, for each compact $\Delta \subset X$, the operator $\widehat{K}^\Delta = P_\Delta \widehat{K} P_\Delta$ is nonnegative. Hence, $\widehat{K} \geq 0$. Exchanging the role of the sets X_1 and X_2 and using instead of the operator K the operator $1 - K$, we therefore get $1 - \widehat{K} \geq 0$. Thus, $0 \leq \widehat{K} \leq 1$.

(9) We now assume that $\|K\| = 1$. Using the procedure of thinning of the point process μ (see, e.g., [9], Example 8.2(a)), we conclude that, for each $\varepsilon \in (0, 1)$, there exists a point process μ_ε which has correlation functions as in formula (48), that is, a determinantal point process corresponding to the operator $K_\varepsilon := \varepsilon K$. By the proved above $0 \leq \widehat{K}_\varepsilon \leq 1$. Hence, by (47),

$$0 \leq \varepsilon \widehat{K} + (1-\varepsilon)P_2 \leq 1.$$

Letting $\varepsilon \to 1$, we get $0 \leq \widehat{K} \leq 1$.

(10) Finally, we assume that $\|K\| > 1$ and we have to show that a determinantal point process does not exist in this case. Assume the contrary, that is, assume that there exists a determinantal point process with correlation kernel $K(x, y)$. Since $\|K\| > 1$, there exists a compact set $\Delta \subset X$ such that $\|K^\Delta\| > 1$. Analogously to part (9), using the procedure of thinning, we conclude that, for each $\varepsilon \in (0, 1)$, there exists a determinantal point process with correlation kernel $K_\varepsilon(x, y) := \varepsilon K(x, y)$. We choose $\varepsilon := \|K^\Delta\|^{-1}$, so that $\|K_\varepsilon^\Delta\| = 1$. We take the restriction of the corresponding probability measure to the σ-algebra $\mathcal{B}_\Delta(\Gamma)$, that is, a point process on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$. We denote this point process by $\mu_{\varepsilon,\Delta}$. By part (9), we have $0 \leq \widehat{K}_\varepsilon^\Delta \leq 1$. Then, by Corollary 2,

$$\mathrm{Det}(1 - K_\varepsilon^\Delta) = 0.$$

Next, following the idea of [21], Remark 4, we consider

$$\begin{aligned}\int_\Gamma (1-\varepsilon)^{|\gamma\cap\Delta|}\mu(d\gamma) &= \int_\Gamma \prod_{x\in\gamma}\big(1 - \varepsilon\chi_\Delta(x)\big)\mu(d\gamma) \\ &= \mathrm{Det}\big(1 - \varepsilon K^\Delta\big) = \mathrm{Det}\big(1 - K_\varepsilon^\Delta\big) = 0.\end{aligned}$$

On the other hand, $(1-\varepsilon)^{|\gamma\cap\Delta|} > 0$ for all $\gamma \in \Gamma$. Hence,

$$\int_\Gamma (1-\varepsilon)^{|\gamma\cap\Delta|}\mu(d\gamma) > 0,$$

which is a contradiction. $\square$

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REFERENCES

[1] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[2] AZIZOV, T. Y. and IOKHVIDOV, I. S. (1989). *Linear Operators in Spaces with an Indefinite Metric*. Wiley, Chichester. Translated from the Russian by E. R. Dawson. MR1033489

[3] BEREZIN, F. A. (1966). *The Method of Second Quantization*. *Pure and Applied Physics* **24**. Academic Press, New York. MR0208930

[4] BORODIN, A., OKOUNKOV, A. and OLSHANSKI, G. (2000). Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.* **13** 481–515 (electronic). MR1758751

[5] BORODIN, A. and OLSHANSKI, G. (1998). Point processes and the infinite symmetric group. *Math. Res. Lett.* **5** 799–816. MR1671191

[6] BORODIN, A. and OLSHANSKI, G. (2000). Distributions on partitions, point processes, and the hypergeometric kernel. *Comm. Math. Phys.* **211** 335–358. MR1754518

[7] BORODIN, A. and OLSHANSKI, G. (2005). Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes. *Ann. of Math.* (2) **161** 1319–1422. MR2180403

[8] BORODIN, A. and OLSHANSKI, G. (2005). Random partitions and the gamma kernel. *Adv. Math.* **194** 141–202. MR2141857

[9] DALEY, D. J. and VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*. Springer, New York. MR0950166

[10] GEORGII, H.-O. and YOO, H. J. (2005). Conditional intensity and Gibbsianness of determinantal point processes. *J. Stat. Phys.* **118** 55–84. MR2122549

[11] KONDRATIEV, Y. G. and KUNA, T. (2002). Harmonic analysis on configuration space. I. General theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** 201–233. MR1914839

[12] KONDRATIEV, Y. G., KUNA, T. and OLIVEIRA, M. J. (2006). Holomorphic Bogoliubov functionals for interacting particle systems in continuum. *J. Funct. Anal.* **238** 375–404. MR2253724

[13] LENARD, A. (1973). Correlation functions and the uniqueness of the state in classical statistical mechanics. *Comm. Math. Phys.* **30** 35–44. MR0323270

[14] LYTVYNOV, E. and MEI, L. (2007). On the correlation measure of a family of commuting Hermitian operators with applications to particle densities of the quasi-free representations of the CAR and CCR. *J. Funct. Anal.* **245** 62–88. MR2310803

[15] MACCHI, O. (1975). The coincidence approach to stochastic point processes. *Adv. in Appl. Probab.* **7** 83–122. MR0380979

[16] MATTHES, K., KERSTAN, J. and MECKE, J. (1978). *Infinitely Divisible Point Processes*. Wiley, Chichester. MR0517931

[17] OLSHANSKI, G. (1998). Point processes and the infinite symmetric group. Part V: Analysis of the matrix Whittaker kernel. Available at http://arxiv.org/abs/math/9810014.

[18] SCHWARTZ, L. (1967). *Analyse Mathématique*. *I*. Hermann, Paris. MR0226972

[19] SHIRAI, T. and TAKAHASHI, Y. (2003). Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J. Funct. Anal.* **205** 414–463. MR2018415

[20] SIMON, B. (1979). *Trace Ideals and Their Applications*. *London Mathematical Society Lecture Note Series* **35**. Cambridge Univ. Press, Cambridge. MR0541149

[21] SOSHNIKOV, A. (2000). Determinantal random point fields. *Russian Math*. *Surveys* **55** 923–975.

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