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INVARIANT MONOTONE COUPLING NEED NOT EXIST[1]

BY PÉTER MESTER

Indiana University

We show by example that there is a Cayley graph, having two invariant random subgraphs X and Y, such that there exists a monotone coupling between them in the sense that $X \subset Y$, although no such coupling can be invariant. Here, "invariant" means that the distribution is invariant under group multiplications.

1. Introduction. There are several models when one is selecting a random subset of vertices or edges of a given graph $G = G(V, E)$ according to some distribution. Formally these are 2^V-valued random objects where V is the vertex set of G (which can be replaced by E, the set of edges). We can look at this as a $\{0, 1\}$-labeling of the vertices; then it is natural to allow more general label sets Λ replacing $\{0, 1\} = 2$.

We are interested in particular in Cayley graphs, and in this case, most naturally occurring examples have an extra common feature: invariance. This means that their distribution is invariant under the group multiplication of the base graph. More precisely, if G is a right Cayley graph of the group Γ, then the random object $\mathcal{R}$ is *invariant* if for for any finite $\{v_1, \dots, v_n\} \subset V$ and $\gamma \in \Gamma$, the distribution of $(\mathcal{R}(\gamma v_1), \dots, \mathcal{R}(\gamma v_n))$ does not depend on γ. Note that in this case $V = \Gamma$, so it may seem confusing to use different notation. The reason is that many concepts we define naturally generalize to the case of a graph with a transitive group of automorphisms acting on the vertices, and in general these are distinct notions. The abundance of invariant processes on Cayley graphs motivates an investigation of them in general. This was done, for example, in [2].

In this context, our result is a counterexample. To explain it, we first need to recall the notion of coupling.

DEFINITION 1.1. If $\mathcal{S}_1, \mathcal{S}_2$ are random objects taking values in Δ_1, Δ_2, respectively, then a *coupling* of them is a random pair $(\tilde{S}_1, \tilde{S}_2)$ taking values in $\Delta_1 \times \Delta_2$ such that for $i \in \{1, 2\}$, $\tilde{S}_i$ has the same distribution as $\mathcal{S}_i$.

Intuitively this means that we manage to produce the two objects using the same random source, so that pointwise comparison makes sense. Proofs using coupling arguments are usually very conceptual and fit well with probabilistic intuition.

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REMARK 1.2. If $\mathcal{R}_i$ is a random Λ_i-labeling for $i \in \{1, 2\}$, then a coupling of them is a $\Lambda_1 \times \Lambda_2$-labeling, and we say that this is an *invariant coupling* if this labeling is invariant in the above sense.

A very simple instance of this is the simultaneous coupling of all Bernoulli(E, p) percolations corresponding to possible parameters $p \in [0, 1]$ which we briefly recall. A Bernoulli(E, p)-percolation is obtained by putting i.i.d. $\{0, 1\}$-labels on the edges, where for a given edge e, its label is 1 with probability p, and 0 otherwise. Note that replacing the edge set E with the set of vertices V in the above definition is well defined, and we denote this process by Bernoulli(V, p). There is a strong intuition that "the bigger p is, the bigger the subgraph with label 1." We can make this intuition have a precise formal meaning as follows: Put first i.i.d. uniform (from $[0, 1]$) labels on the edges, which we call U. Then for each p, define a $\{0, 1\}$-label U_p so that if an edge has U label $U(e)$, its U_p label is 1 if $U(e) \leq p$, and 0 otherwise. Clearly, as a distribution, U_p is nothing but a Bernoulli(E, p)-percolation, and for $p \leq p^+$ we have $U_p \subset U_{p^+}$.

This is an example of what is called monotone coupling. For the definition assume that the label set Λ is partially ordered by $\lesssim$.

DEFINITION 1.3. If X and Y are random Λ-labelings of the same graph, then we say that a coupling $(\tilde{X}, \tilde{Y})$ of X and Y is a *monotone coupling* if $\tilde{X} \lesssim \tilde{Y}$ almost surely.

The next two examples we mention are related to open questions which motivates the question we are going to ask.

The first is the case of wired and free uniform spanning forest measures (WUSF and FUSF, resp.); see [3]. These processes both can be considered as natural generalizations of the uniform spanning tree (easily defined on finite graphs) to infinite graphs. It is known that there is a monotone coupling where the free one dominates the wired one. However, in general, it is still open if there is an invariant monotone coupling.

There are partial results which show that for certain classes of graphs there indeed exists an invariant monotone coupling between the FUSF and WUSF. For example, Lewis Bowen [4] showed it for Cayley graphs of residually amenable groups, while recently Russell Lyons and Andreas Thom (personal communication, [8]) showed it for the Cayley graphs of so-called *sofic* groups.

The second example is random walk in random environment. In [1], Aldous and Lyons considered a continuous time nearest-neighbor random walk RW(t, μ), with jumps governed by Poisson clocks on the edges with rates given by a distribution μ. The walks start at the origin o of the Cayley graph, and we are interested in how different environments affect the return probabilities. In [1] they showed that if for two random environments μ_1, μ_2 (different clock frequencies in this case) there exists a monotone coupling $\mu_1 \le \mu_2$, which is itself invariant, then

$$\mathbf{E}_{\mu_1}\bigl(\mathbf{P}\bigl(\mathrm{RW}(t, \mu_1) = o\bigr)\bigr) \ge \mathbf{E}_{\mu_2}\bigl(\mathbf{P}\bigl(\mathrm{RW}(t, \mu_2) = o\bigr)\bigr).$$

We may ask what happens if we drop the condition for the coupling being invariant. Is it enough, for example, that the marginals are invariant? Note that in [1] they actually dealt with so-called *unimodular* processes, but this condition always holds for invariant processes on Cayley graph (this fact is the mass transport principle which we prove later).

Schramm and Lyons asked (unpublished, [7]) the following; note that a positive answer would immediately settle the above problems (note also that a more general question was asked in [1], as Question 2.4):

QUESTION 1.4. *Let X and Y be invariant subgraphs of a Cayley graph Γ, so that there exists a monotone coupling between them. Does it follow that there exists a monotone coupling between them which is also invariant?*

It is known that the answer to the above question is "yes" if the Cayley graph is amenable; see Proposition 8.6. in [1]. In this paper we show by an example that in full generality the answer is "no."

The Cayley graph we use is $T_3 \square C_n$ for n large enough. Here T_3 is the 3-regular tree, and C_n is the cycle of length n, and, in general, for graphs G and H their *Cartesian product* $G \square H$ is the graph with vertex set $V(G \square H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) are connected in $G \square H$ if and only if either $u_1 = v_1$ and u_2 is adjacent with v_2 in H, or $u_2 = v_2$ and u_1 is adjacent with v_1 in G. It is easy to see that if G_1, G_2 are Cayley graphs of Γ_1, Γ_2, respectively, then $G_1 \square G_2$ is a Cayley graph of $\Gamma_1 \times \Gamma_2$.

Note that T_3 is a Cayley graph of $\mathbb{Z}_2^{*3} := \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ (here $H * K$ is the *free product* of H and K), and C_n is a Cayley-graph of $\mathbb{Z}_n$, so $T_3 \square C_n$ is a Cayley graph of $\mathbb{Z}_2^{*3} \times \mathbb{Z}_n$.

For simplicity we make an assumption about n which may not be optimal. See Remark 1.7 at the end of this section for an explanation.

THEOREM 1.5. *If $n \ge 376$, then there exist two invariant random $\{0, 1\}$-labelings X and Y of $T_3 \square C_n$ so that there is a coupling $(\tilde{X}, \tilde{Y})$ of them for which $\tilde{X} \le \tilde{Y}$ holds, but no such coupling can be invariant.*

The proof will be more succinct if we first show a similar result with labels different from $\{0, 1\}$. In this case the (partially ordered) label set will be the power set $\mathcal{P}(S)$ of some finite set S. Note also that in this case we can use a tree as the underlying Cayley graph:

LEMMA 1.6. *If $n \geq 376$ and $|S| = n$, then there exist invariant $\mathcal{P}(S)$-labelings $\mathcal{X}$ and $\mathcal{Y}$ of T_3 so that there is a coupling $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ of them for which $\tilde{\mathcal{X}} \subset \tilde{\mathcal{Y}}$ holds, but no such coupling can be invariant.*

Although the examples themselves might be artificial, they will have some "nice" properties as well. So if we want to add some extra conditions to Question 1.4 to get an affirmative answer, then we know for sure that these nice properties will not work (at least not alone). For a discussion of these, see the end of the last section.

We summarize some conventions we use: When $\mathcal{S}$ is a random object and μ is its distribution, we often will just express this by saying that $\mathcal{S}$ is a *copy* of μ, and in a similar way with a further abuse of notation, if $\mathcal{T}$ is a random object with the same distribution as $\mathcal{S}$, we will also say that $\mathcal{S}$ is a copy of $\mathcal{T}$. We also note that one way to specify a probability measure is to describe a random object with the given measure as distribution. We will do it without further comments.

If the graph G is understood, $V(G)$ will be its set of vertices and $E(G)$ its set of edges. We use right Cayley graphs and then left multiplications are graph automorphisms.

REMARK 1.7. The condition that $n \geq 376$ we made in Theorem 1.5 was meant to ensure the following: If S is a finite set of cardinality n, and $\alpha_1, \alpha_2, \dots, \alpha_{20}$ are i.i.d. uniform elements of S, and $\beta_1, \dots, \beta_9$ are also i.i.d. uniform elements of S (we emphasize that we make no extra assumption on the joint distribution of the full family $\alpha_1, \dots, \alpha_{20}, \beta_1, \dots, \beta_9$), then with probability strictly greater than $\frac{1}{2}$ the random elements $\alpha_1, \dots, \alpha_{20}$ are all distinct, and the random elements $\beta_1, \dots, \beta_9$ are all distinct as well (but it may happen that some $\beta_i = \alpha_j$). It is easy to see that if $(1 - \prod_{i=1}^{19}(1 - \frac{i}{n})) + (1 - \prod_{i=1}^{8}(1 - \frac{i}{n})) < \frac{1}{2}$ (which is true for $n \geq 376$), then this holds.

2. The mass-transport principle and ends. This section owes a lot to the exposition in [6]. An effective tool in showing that there is no invariant random process on a Cayley graph satisfying a certain requirement is the so-called mass-transport principle. Recall that Λ is the label set, which will always be finite in our case, and Γ is the group to which the Cayley graph is associated. The "space of configurations" $\Omega := \Lambda^V$ will be naturally equipped with the product σ-algebra. Assume that $\mathcal{R}$ is a probability measure on Ω. Note that Γ acts on Ω: for $\omega \in \Omega$, $\gamma \in \Gamma$ and $v \in V$, let $\gamma\omega$ be the element of Ω for which $\gamma\omega(v) = \omega(\gamma^{-1}(v))$.

Let $F: V \times V \times \Omega \to [0, \infty]$ be a diagonally invariant measurable function [meaning that $F(x, y, \omega) = F(\gamma x, \gamma y, \gamma\omega)$ for all $\gamma \in \Gamma$]. The quantity $F(x, y, \omega)$ is often called *the mass sent by x to y* or *the mass received by y from x*, and then F is thought to describe a "mass transport" among the vertices which may depend on some randomness created by $\mathcal{R}$. The mass-transport principle says that if $\mathcal{R}$ is invariant, then for the identity $o \in V$ the expected overall mass o receives is the same as the expected overall mass it sends out. Now we formalize and prove this:

THEOREM 2.1. *If $\mathcal{R}$ and F are as above, $\mathcal{R}$ is invariant, $f(x,y) := \mathbf{E}_{\mathcal{R}}F(x,y,*)$, then*

$$\sum_{x\in V} f(o,x) = \sum_{x\in V} f(x,o).$$

To prove it, first observe that the invariance of $\mathcal{R}$ implies that f is also diagonally invariant. This implies that $f(o,x) = f(x^{-1}o, x^{-1}x) = f(x^{-1}, o)$, and this finishes the proof since inversion is a bijection.

This means that in order to show that a random process with a given property cannot be invariant, it is enough to show that the property in question allows us to define a mass transport contradicting the above equality. We emphasize that it is important here that we mean invariant processes on a Cayley graph and not just on a graph which has a transitive group of automorphism. The notion of an "end" in a tree, which we are about to define, will also lead to an example where the obvious generalization of the mass transport principle to an arbitrary transitive graph fails.

From now until Section 5, the base graph is always T_3, the 3-regular tree. If v is a vertex, then $J(v)$ will denote the set of edges for which v is one of the endpoints.

A "ray" is a one-sided infinite path (i.e., a sequence of vertices $v_0, \dots, v_n, \dots$ so that there is no repetition and v_i and v_{i+1} are adjacent). We call two rays equivalent if their symmetric difference is finite. An equivalence class is then called an *end*. If we fix an end ξ, then for any vertex v there is a unique ray $v = v_0^\xi, v_1^\xi, \dots, v_n^\xi, \dots$ so that the ray starts at v and belongs to the equivalence class ξ. Let the unique edge joining v with v_1^ξ be $e_{v\to\xi}$, and let us denote $J(v) - \{e_{v\to\xi}\}$ as $J^\xi(v)$. Observe that for distinct vertices v_1, v_2, we have

$$J^\xi(v_1) \cap J^\xi(v_2) = \varnothing.$$

This will be important in constructing a monotone coupling of our processes, and it also implies that an end cannot be determined using invariant processes. The intuition is simple: given an end $\xi(\omega)$ (which is "somehow determined" by a configuration ω) a vertex v could send mass 1 to each of the two vertices that are the other endpoints of the two edges in $J^{\xi(\omega)}(v)$. In this way the overall mass sent out is 2, while the overall mass received is 1. To make this precise in a general setting, we have to deal with measurability issues related to how a configuration ω determines an end $\xi(\omega)$, but this is not needed for our purposes. While it is not important for our later work, note that if we put extra edges into T_3 by connecting every vertex v with v_2^ξ, then we get a transitive graph where the obvious generalization of the mass transport principle fails.

3. The fixed-end trick. As we have indicated, an end cannot be determined using invariant processes in a tree, and Lalley (unpublished, [5]) proposed a way to exploit this fact to settle Question 1.4. Here we present a simpler version of the idea; see the last paragraph in this section for the original one. Given an end ξ in

T_3, we shall define a $\{0,1\}^2$-labeling $(\mathrm{X}^\xi, \mathrm{Y}^\xi)$ so that its components, X^ξ and Y^ξ, are invariant, and $(\mathrm{X}^\xi, \mathrm{Y}^\xi)$ is a monotone coupling of them, that is,

$$\mathrm{X}^\xi \le \mathrm{Y}^\xi.$$

Let $\{\eta(e)\}_{e\in E}$ be a Bernoulli$(E, \frac{1}{2})$ label. For a vertex v, let

$$\mathrm{X}^\xi(v) := \max\{\eta(e); e \in J^\xi(v)\},$$

while

$$\mathrm{Y}^\xi(v) := \max\{\eta(e); e \in J(v)\}.$$

It is clear that Y^ξ itself is an invariant labeling.

However, X^ξ is also invariant since the family $\mathrm{X}^\xi(v)_{v\in V}$ is actually i.i.d.! This is because of the observation from the last section that $J^\xi(v_1)$ and $J^\xi(v_2)$ are disjoint for $v_1 \neq v_2$. So X^ξ itself is actually Bernoulli$(V, \frac{3}{4})$. Since the monotone coupling of these processes was defined using an end (a noninvariant step), it is reasonable that maybe these processes already witness Theorem 1.5.

However, the construction below—which is due to Peres (unpublished, [9])—shows that there exists an invariant monotone coupling between X^ξ and Y^ξ.

PROPOSITION 3.1. *Let $\{\eta(e)\}_{e\in E}$ be as above. For each vertex v with $J(v) =: \{e_1(v), e_2(v), e_3(v)\}$, define $\hat{\mathrm{X}}(v) := 0$ if and only if $\{\eta(e_1) = \eta(e_2) = \eta(e_3)\}$, and $\hat{\mathrm{X}}(v) := 1$ otherwise.*

Then $(\hat{\mathrm{X}}, \mathrm{Y}^\xi)$ is an invariant and monotone coupling of X^ξ and Y^ξ.

PROOF. It is clear that $\hat{\mathrm{X}} \le \mathrm{Y}^\xi$ and the coupling $(\hat{\mathrm{X}}, \mathrm{Y}^\xi)$ is clearly invariant. What we need to show is that the above defined $\hat{\mathrm{X}}$ is a Bernoulli$(V, \frac{3}{4})$ vertex labeling (i.e., a copy of X^ξ) so $(\hat{\mathrm{X}}, \mathrm{Y}^\xi)$ is a coupling of X^ξ and Y^ξ.

First let us introduce a notation: if V_1, V_2 are finite disjoint sets of vertices, then let $\mathrm{S}[V_1, V_2] := \{\hat{\mathrm{X}} \restriction V_1 = 1, \hat{\mathrm{X}} \restriction V_2 = 0\}$. We show that $\hat{\mathrm{X}}$ is Bernoulli$(V, \frac{3}{4})$ directly by proving that $\mathbf{P}(\mathrm{S}[V_1, V_2]) = (\frac{3}{4})^{|V_1|}(\frac{1}{4})^{|V_2|}$.

The proof goes by induction on $|V_1 \cup V_2|$. The statement holds when $|V_1 \cup V_2| = 1$.

To proceed, consider the subgraph spanned by $V_1 \cup V_2$. This is a forest, so it has some vertex t which is either a leaf or an isolated point (i.e., t has at most one neighbor in $V_1 \cup V_2$). Let e_1, e_2, e_3 be the edges emanating from t, and assume that the other endpoints of e_1, e_2 are *not* in $V_1 \cup V_2$.

A key observation is that "flipping" the η labels of each edge leaves the $\hat{\mathrm{X}}$ labels unchanged. That means that for any pair of finite disjoint vertex sets W_1, W_2 and *any* edge e the event $\{\eta(e) = 1\}$ [and similarly $\{\eta(e) = 0\}$] cuts $\mathrm{S}[W_1, W_2]$ exactly in half: $\mathbf{P}(\{\eta(e) = 1\} \cap \mathrm{S}[(W_1, W_2]) = P(\{\eta(e) = 0\} \cap \mathrm{S}[W_1, W_2]) = \frac{1}{2}P(\mathrm{S}[W_1, W_2])$.

Using the η-flipping observation it is enough to show that $\mathbf{P}(\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2]) = (\frac{1}{2})(\frac{3}{4})^{|V_1|}(\frac{1}{4})^{|V_2|}$.

Consider first the case where $t \in V_1$. In that case $\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2] = \{\hat{\mathrm{X}}(t) = 1\} \cap \{\eta(e_3) = 1\} \cap \mathrm{S}[V_1 - \{t\}, V_2] = \{$at least one of $\eta(e_1)$ and $\eta(e_2)$ is $0\} \cap \{\eta(e_3) = 1\} \cap \mathrm{S}[V_1 - \{t\}, V_2]$. The point of this is that $\{$at least one of $\eta(e_1)$ and $\eta(e_2)$ is $0\}$ and $\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1 - \{t\}, V_2]$ are independent, and by induction and the η-flipping observation, we know the probability of the latter; it is $(\frac{1}{2})(\frac{3}{4})^{|V_1|-1}(\frac{1}{4})^{|V_2|}$. Combining this with the fact that $\mathbf{P}\{$at least one of $\eta(e_1)$ and $\eta(e_2)$ is $0\} = \frac{3}{4}$ gives us $\mathbf{P}(\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2]) = (\frac{1}{2})(\frac{3}{4})^{|V_1|}(\frac{1}{4})^{|V_2|}$.

If $t \in V_2$ we have $\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2] = \{\hat{\mathrm{X}}(t) = 0\} \cap \{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2 - \{t\}] = \{\eta(e_1) = \eta(e_2) = 1\} \cap \{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2 - \{t\}]$. The independence of $\{\eta(e_1) = \eta(e_2) = 1\}$ and $\{\eta(e_3) = 1\} \cap \mathrm{S}[V_1, V_2 - \{t\}]$ combined with what we know by induction proves the claim again. $\square$

Although the above processes could be coupled in an invariant way, it is clear that the idea leaves us a lot of freedom to use other partially ordered sets and other monotone operations (instead of taking maxima, we could take the sum, e.g., which was Lalley's original suggestion). But it seems that other examples are difficult to analyze from the point of view of Question 1.4. With our next construction, however, it will be very succinct why a monotone coupling cannot be invariant.

4. Set valued labels on T_3.

In this section, we describe an example that will prove Lemma 1.6.

Let S be a finite set with $|S| = n \geq 376$, and let $\mathcal{P}(S)$ denote its power set. We will use $\mathcal{P}(S)$ as a label set with inclusion as a partial order. The two invariant $\mathcal{P}(S)$-labelings $\mathcal{Y}_S$ and $\mathcal{X}_S$ of the vertices of T_3 are defined as follows (for the rest of this section we drop the subscript S but in the next section we use it again).

To construct $\mathcal{Y}$, we first label the edges of T_3 with independent uniform elements from S. Let us call this labeling λ. Then for a vertex v, let $\mathcal{Y}(v) := \bigcup_{e \in J(v)} \{\lambda(e)\}$.

To construct $\mathcal{X}$, we first define its marginal ν on the vertices. To get a copy of ν first pick a uniform $(x_1, x_2) \in S \times S$, and then take $\{x_1\} \cup \{x_2\}$. Finally let $\{\mathcal{X}(v)\}_{v \in V(T_3)}$ be a labeling of the vertices with i.i.d. copies of ν.

REMARK 4.1. Observe that if $\hat{\mathcal{Y}}$ is any copy of $\mathcal{Y}$, then the following is true: if v_0 is any vertex with neighbors v_1, v_2, v_3, then any $s \in \hat{\mathcal{Y}}(v_0)$ is also contained in at least one of the $\hat{\mathcal{Y}}(v_i)$'s for $i \in \{1, 2, 3\}$.

By fixing an end ξ, we can present a monotone coupling of $\mathcal{X}$ and $\mathcal{Y}$ just as in Lalley's example. Get the copy of $\mathcal{Y}$ in the exact same way as above using λ as a source, but also use this λ to get the copy of $\mathcal{X}$ as $\mathcal{X}^{\xi}(v) := \bigcup_{e \in J^{\xi}(v)} \{\lambda(e)\}$. Then clearly $\mathcal{X}^{\xi}(v) \subset \mathcal{Y}(v)$ holds for all v, and $\mathcal{X}^{\xi}$ is indeed a copy of $\mathcal{X}$ [recall that the disjointness of the different $J^{\xi}(v)$'s guarantees the independence for different vertices and the marginals are clearly the same].

However there cannot be any invariant monotone coupling as we will show now (which together with the previous paragraph proves Lemma 1.6).

PROPOSITION 4.2. *There exists no coupling of $\mathcal{X}$ and $\mathcal{Y}$ which is both invariant and monotone.*

PROOF. Let $(\mathcal{X}^*, \mathcal{Y}^*)$ be any monotone coupling of $\mathcal{X}$ and $\mathcal{Y}$. We will show that using this monotone coupling, we can define a mass transport F which contradicts the mass transport principle, showing that the coupling cannot be invariant. To define the mass transport we have to say for every pair (v_0, v) of vertices and every possible configuration ω [defined in terms of $(\mathcal{X}^*, \mathcal{Y}^*)$] the value $F(v_0, v, \omega) \in [0, \infty]$. The dependence on ω will be through an event $E(v_0)$ which we define now.

DEFINITION 4.3. First, let v_1, v_2, v_3 be the neighbors of v_0 and $v_4, \dots, v_9$ be the vertices at graph distance 2 from v_0 (in any order).

We say that $E_1(v_0)$ holds if for each $1 \le i, j \le 3, i \neq j$, we have $|\mathcal{Y}^*(v_j)| = 3$, $\mathcal{Y}^*(v_j) \cap \mathcal{Y}^*(v_i) = \varnothing$.

We say that $E_2(v_0)$ holds if for each $0 \le i, j \le 9, i \neq j$, we have $|\mathcal{X}^*(v_i)| = 2$ and $\mathcal{X}^*(v_i) \cap \mathcal{X}^*(v_j) = \varnothing$.

Finally, let $E(v_0) := E_1(v_0) \cap E_2(v_0)$.

Note the connection with the condition in Remark 1.7: the labels $\mathcal{X}(v_0), \dots, \mathcal{X}(v_9)$ can be identified with $\{\alpha_1, \alpha_2\}, \dots, \{\alpha_{19}, \alpha_{20}\}$, while the edge labels of those 9 edges which are relevant in the $\mathcal{Y}$ labels of v_0, v_1, v_2, v_3 can be identified with $\beta_1, \dots, \beta_9$. Then the condition we made on n ensures that $\mathbf{P}(E(v_0)) > \frac{1}{2}$.

Now we are ready to define the mass transport $F : V \times V \times \Omega \to [0, \infty]$. If $E(v_0)$ does not hold, then set $F(v_0, v, \omega) := 0$ for each vertex v. If $E(v_0)$ holds, then let $F(v_0, v, \omega) := 1$ if v_0 is a neighbor of v and $\mathcal{X}^*(v_0) \cap \mathcal{Y}^*(v) \neq \varnothing$, while in every other case, set $F(v_0, v, \omega) := 0$.

We claim that the expected mass the origin sends out is strictly greater than 1, while the mass it receives is not greater than 1 (even point-wise).

To prove this we show first that if $E(v_0)$ holds, then the mass v_0 sends out is *exactly* 2. This combined with the fact that $\mathbf{P}(E(v_0)) > \frac{1}{2}$ implies the first part of the claim. Let $\mathcal{X}^*(v_0) =: \{s_1, s_2\}$; observe that $E_2(v_0)$ implies $s_1 \neq s_2$. By monotonicity of the coupling, $\{s_1, s_2\} \subset \mathcal{Y}^*(v_0)$, so by Remark 4.1 there exist neighbors $v_0(s_1), v_0(s_2)$ of v_0 so that $\mathcal{Y}^*(v_0(s_i))$ contains s_i. Since $\mathcal{Y}^*(v_1), \mathcal{Y}^*(v_2), \mathcal{Y}^*(v_3)$ are pairwise disjoint sets [by $E_1(v_0)$], there can be at most two of them which nontrivially intersect $\{s_1, s_2\}$, and this implies $v_0(s_1) \neq v_0(s_2)$. By definition, $v_0(s_1)$ and $v_0(s_2)$ are exactly the vertices receiving nonzero mass from v_0.

To prove that the expected mass v_0 receives is at most 1, assume that v_0 receives nonzero mass from v_1 and v_2. First of all, v_1 sends out nonzero mass only if $E(v_1)$ [and in particular $E_2(v_1)$] holds. Since v_0 and v_2 are both within distance 2 from v_1, the event $E_2(v_1)$ implies that $\{a_1, a_2\} := \mathcal{X}^*(v_1)$, $\{b_1, b_2\} := \mathcal{X}^*(v_2)$, and $\{c_1, c_2\} := \mathcal{X}^*(v_0)$ are pairwise disjoint and each has size 2. By the condition for the mass transport, $\mathcal{Y}^*(v_0)$ contains one of the a_i's, one of the b_i's and—by the monotonicity of the coupling—$\{c_1, c_2\}$ as well. But this would mean that $\mathcal{Y}^*(v_0)$ has at least four distinct elements, which is impossible.

This mass transport violates the mass transport theorem, so no monotone coupling of $\mathcal{X}$ and $\mathcal{Y}$ can be invariant. $\square$

REMARK 4.4. Observe that an end ξ can be identified by the orientation on the edges given as follows: orient the edges in $J^{\xi}(v)$ *away* from v. Then the out-degree of a vertex is always 2 while the in-degree is always 1. The mass transport above has some similarity with this end: if for a vertex v we define $J^{(\mathcal{X}^*,\mathcal{Y}^*)}(v)$ to be the set of edges connecting v with vertices receiving nonzero mass from v and we orient the edges in $J^{(\mathcal{X}^*,\mathcal{Y}^*)}(v)$ away from v, then the out-degree of a vertex is either 0 or 2 and the in-degree is either 0 or 1.

5. The $\{0, 1\}$-labels on $T_3\square C_n$. Now we prove Theorem 1.5. The Cayley graph we use is $T_3\square C_n$ defined in the Introduction. Recall that $T_3\square C_n$ is a Cayley graph of $\mathbb{Z}_2^{*3} \times \mathbb{Z}_n$. If we want to check the invariance of a process defined on $T_3\square C_n$, it is enough to check invariance under group multiplication from $\mathbb{Z}_2^{*3}$ and $\mathbb{Z}_n$ since the direct components generate the full group.

The processes we define can be considered as very faithful copying of the previous processes. In the previous section, the set S whose subsets were used as labels was not important besides its cardinality $|S|$. Now it will be convenient to choose it to be $S := V(C_n)$. If $\mathcal{Z}$ is any $\mathcal{P}(S)$-labeling of the vertices of T_3, then let $\mathrm{lift}(\mathcal{Z})$ be the following $\{0, 1\}$-labeling of $T_3\square C_n$: for a vertex $(u, v) \in V(T_3\square C_n)$, let $\mathrm{lift}(\mathcal{Z})(u, v) := 1$ if $v \in \mathcal{Z}(u)$, otherwise let $\mathrm{lift}(\mathcal{Z})(u, v) := 0$. Note that this function lift from the $\mathcal{P}(S)$-labelings of T_3 to the $\{0, 1\}$-labelings of $T_3\square C_n$ is invertible.

Consider the previously defined processes $\mathcal{X}_S, \mathcal{Y}_S$. Let $X := \mathrm{lift}(\mathcal{X}_S)$ and $Y := \mathrm{lift}(\mathcal{Y}_S)$.

PROPOSITION 5.1. *The above defined X and Y witness the truth of Theorem 1.5.*

PROOF. First, the invariance of X and Y under group multiplication from $\mathbb{Z}_2^{*3}$ follows from the fact that $\mathcal{X}_S$ and $\mathcal{Y}_S$ were invariant on T_3, and the invariance under $\mathbb{Z}_n$ follows from the fact that for a fixed vertex v_0, the distribution of $\mathcal{X}_S$ ($\mathcal{Y}_S$) is invariant under any permutation of S.

Second, there exists a monotone coupling of X, Y since if $(\mathcal{X}_S^*, \mathcal{Y}_S^*)$ is any monotone coupling of $\mathcal{X}_S, \mathcal{Y}_S$, then $(\text{lift}(\mathcal{X}_S^*), \text{lift}(\mathcal{Y}_S^*))$ is clearly a monotone coupling of X and Y.

Third, if (X^*, Y^*) was an invariant monotone coupling, then $(\text{lift}^{-1}(\mathcal{X}^*), \text{lift}^{-1}(\mathcal{Y}^*))$ would have been an invariant coupling of $\mathcal{X}_S, \mathcal{Y}_S$, which is impossible as we have seen. $\square$

It would be nice to have some natural condition on random subgraphs under which the answer to Question 1.4 would be affirmative. Here we point out two conditions which are ruled out by our example.

A random subgraph Z is said to be *k-dependent* if for vertex sets $S_1, S_2, \dots, S_m$ whose pairwise distances are all at least k, the random objects $F_i := Z \upharpoonright S_i$, $1 \le i \le m$, are independent. Our example is k-dependent for large enough k (depending on the cycle size n). So assuming k-dependence is certainly not enough.

With slight modifications, we can exclude other conditions as well. Observe that the mass transport we used would still work [in the sense that $E(v)$ would have probability greater than $\frac{1}{2}$] if we "perturbed" our processes with a Bernoulli(V, ε) process for $\varepsilon > 0$ small enough [meaning that we change the original labels on those vertices where Bernoulli(V, ε) turns out to be 1]. A random subgraph Z is said to have *uniform finite energy* if there exists an $\varepsilon \in (0, 1)$ so that for a vertex v we have $\varepsilon < \mathbf{P}(Z(v) = 1 | Z \upharpoonright V - \{v\}) < 1 - \varepsilon$. By using this idea of perturbing the labels we see that assuming that the process has uniform finite energy is not enough either.

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DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405-5701
USA
E-MAIL: pmester@indiana.edu