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ON STRATONOVICH AND SKOROHOD STOCHASTIC CALCULUS FOR GAUSSIAN PROCESSES

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In this article, we derive a Stratonovich and Skorohod-type change of variables formula for a multidimensional Gaussian process with low Hölder regularity γ (typically $\gamma \le 1/4$). To this aim, we combine tools from rough paths theory and stochastic analysis.

1. Introduction. Starting from the seminal paper [7], the stochastic calculus for Gaussian processes has been thoroughly studied during the last decade, fractional Brownian motion being the main example of application of the general results. The literature on the topic includes the case of Volterra processes corresponding to a fBm with Hurst parameter $H > 1/4$ (see [1, 12]), with some extensions to the whole range $H \in (0, 1)$, as in [2, 6, 11]. It should be noticed that all those contributions concern the case of real valued processes, this feature being an important aspect of the computations.

In a parallel and somewhat different way, the rough path analysis opens the possibility of a pathwise type stochastic calculus for general (including Gaussian) stochastic processes. Let us recall that this theory, initiated by Lyons in [21] (see also [9, 13, 19] for introductions to the topic), states that if a γ-Hölder process x allows to define sufficient number of iterated integrals, then:

(1) One gets a Stratonovich-type change of variable for $f(x)$ when f is smooth enough.

(2) Differential equations driven by x can be reasonably defined and solved.

In particular, the rough path method is still the only way to solve differential equations driven by Gaussian processes with Hölder regularity exponent less than $1/2$, except for some very particular (e.g., Brownian, linear or one-dimensional) situations.

More specifically, the rough path theory relies on the following set of assumptions:

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HYPOTHESIS 1.1. *Let $\gamma \in (0,1)$ and $x:[0,T] \to \mathbb{R}^d$ be a γ-Hölder process. Consider also the nth order simplex $\mathcal{S}_{n,T} = \{(u_1,\dots,u_n): 0 \le u_1 < \cdots < u_n \le T\}$ on $[0,T]$. The process x is supposed to generate a rough path, which can be understood as a stack $\{\mathbf{x}^{\mathbf{n}}; n \le \lfloor 1/\gamma \rfloor\}$ of functions of two variables satisfying the following three properties*:

(1) *Regularity*: *Each component of $\mathbf{x}^{\mathbf{n}}$ is $n\gamma$-Hölder continuous* [*in the sense of the Hölder norm introduced in* (2.2)] *for all $n \le \lfloor 1/\gamma \rfloor$, and $\mathbf{x}^{\mathbf{1}}_{st} = x_t - x_s$.*

(2) *Multiplicativity*: *Letting $(\delta \mathbf{x}^{\mathbf{n}})_{sut} := \mathbf{x}^{\mathbf{n}}_{st} - \mathbf{x}^{\mathbf{n}}_{su} - \mathbf{x}^{\mathbf{n}}_{ut}$ for $(s,u,t) \in \mathcal{S}_{3,T}$, one requires*

$$(\delta \mathbf{x}^{\mathbf{n}})_{sut}(i_1,\dots,i_n) = \sum_{n_1=1}^{n-1} \mathbf{x}^{\mathbf{n_1}}_{su}(i_1,\dots,i_{n_1}) \mathbf{x}^{\mathbf{n}-\mathbf{n_1}}_{ut}(i_{n_1+1},\dots,i_n). \tag{1.1}$$

(3) *Geometricity*: *For any n,m such that $n+m \le \lfloor 1/\gamma \rfloor$ and $(s,t) \in \mathcal{S}_{2,T}$, we have*

$$\mathbf{x}^{\mathbf{n}}_{st}(i_1,\dots,i_n)\mathbf{x}^{\mathbf{m}}_{st}(j_1,\dots,j_m) = \sum_{\bar{k} \in Sh(\bar{\imath},\bar{\jmath})} \mathbf{x}^{\mathbf{n}+\mathbf{m}}_{st}(k_1,\dots,k_{n+m}), \tag{1.2}$$

where, for two tuples $\bar{\imath}, \bar{\jmath}$, $\Sigma_{(\bar{\imath},\bar{\jmath})}$ stands for the set of permutations of the indices contained in $(\bar{\imath},\bar{\jmath})$, and $Sh(\bar{\imath},\bar{\jmath})$ is a subset of $\Sigma_{(\bar{\imath},\bar{\jmath})}$ defined by

$$Sh(\bar{\imath},\bar{\jmath}) = \{\sigma \in \Sigma_{(\bar{\imath},\bar{\jmath})}; \sigma \text{ does not change the orderings of } \bar{\imath} \text{ and } \bar{\jmath}\}.$$

With this set of abstract assumptions in hand, one can define integrals like $\int f(x)\,dx$ in a natural way (as recalled later in the article), and more generally set up the basis of a differential calculus with respect to x. Notice that according to Lyons's terminology [19], the family $\{\mathbf{x}^{\mathbf{n}}; n \le \lfloor 1/\gamma \rfloor\}$ is said to be a weakly geometric rough path above x.

Without any surprise, some substantial efforts have been made in recent years to construct rough paths above a wide class of Gaussian processes, from which emerges the case of fractional Brownian motion. Let us recall that a fractional Brownian motion B with Hurst parameter $H \in (0,1)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is a d-dimensional centered Gaussian process. Its law is thus characterized by its covariance function, which is given by

$$\mathbf{E}[B_t(i)B_s(i)] = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\mathbf{1}_{(i=j)}, \qquad s,t \in \mathbb{R}_+. \tag{1.3}$$

The variance of the increments of B is then given by

$$\mathbf{E}[(B_t(i) - B_s(i))^2] = (t-s)^{2H}, \qquad (s,t) \in \mathcal{S}_{2,T}, i = 1,\dots,d,$$

and this implies that almost surely the trajectories of the fBm are γ-Hölder continuous for any $\gamma < H$. Furthermore, for $H = 1/2$, fBm coincides with the usual Brownian motion, converting the family $\{B = B^H; H \in (0,1)\}$ into the most natural generalization of this classical process. This is why B can be considered as one of the canonical examples of application of the abstract rough path theory.

Until very recently, the rough path constructions for fBm were based on pathwise-type approximations of B, as in [4, 23, 28]. Namely, these references all use an approximation of B by a regularization B^{ε}, consider the associated (Riemann) iterated integrals $\mathbf{B}^{\mathbf{n},\varepsilon}$ and show their convergence, yielding the existence of a geometric rough path above B. These approximations all fail for $H \le 1/4$. Indeed, the oscillations of B are then too heavy to define even $\mathbf{B}^{\mathbf{2}}$, following this kind of argument, as illustrated by [5]. Nevertheless, the article [20] asserts that a rough path exists above any γ-Hölder function, and recent progress [26, 28] show that different concrete rough paths above fBm (and more general processes) can be exhibited, even if those rough paths do not correspond to a regularization of the process at stake.

Summarizing what has been said up to now, there are (at least) two ways to handle stochastic calculus for Gaussian processes: (i) Stochastic analysis tools, mainly leading to a Skorohod-type integral and (ii) rough paths analysis, based on the pathwise convergence of some Riemann sums, giving rise to a Stratonovich-type integral. Though some efforts have been made in [3] to relate the two approaches (essentially for a fBm with Hurst parameter $H > 1/4$), the current article proposes to delve deeper in this direction. Namely, we shall address the following issues:

(1) We recall that, starting from a given rough path of order N above a d-dimensional process x, one can derive a Stratonovich change of variables of the form

$$f(x_t) - f(x_s) = \sum_{i=1}^{d} \int_s^t \partial_i f(x_u)\, d\mathbf{x}_u(i) := \mathcal{J}_{st}\big(\nabla f(x_u)\, d\mathbf{x}_u\big) \tag{1.4}$$

for any $f \in C^{N+1}(\mathbb{R}^d; \mathbb{R})$, and where $\partial_i f$ stands for $\partial f/\partial x_i$. This formula is at the core of rough paths theory, and is explained at large, for example, in [9]. Furthermore, it is well-known that the following representation for the integral $\mathcal{J}_{st}(\nabla f(x_u)\, d\mathbf{x}_u)$ holds true: consider a family of partitions $\Pi_{st} = \{s = t_0, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to 0. Then, denoting by $N = \lfloor \frac{1}{\gamma} \rfloor$,

$$\begin{aligned} \mathcal{J}_{st}\big(\nabla f(x_u)\, d\mathbf{x}_u\big) = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \sum_{k=0}^{N-1} \bigg(& \frac{1}{k!} \partial^{k+1}_{i_k,\dots,i_1 i} f(x_{t_q}) \\ & \times \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_k) \cdots \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_1) \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i) \bigg). \end{aligned} \tag{1.5}$$

These modified Riemann sums will also be essential in the analysis of Skorohod-type integrals.

(2) We then specialize our considerations to a Gaussian setting, and use Malliavin calculus tools (in particular some elaborations of [2, 6]). Namely, supposing that x is a Gaussian process, plus mild additional assumptions on its covariance function, we are able to prove the following assertions:

(i) Consider a $C^2(\mathbb{R}^d;\mathbb{R})$ function f with exponential growth, and $0 \le s < t \le T$. Then the function $u \mapsto \mathbf{1}_{[s,t)}(u)\nabla f(x_u)$ lies in the domain of an extension of the divergence operator (in the Malliavin calculus sense) called $\delta^\diamond$.

(ii) The following Skorohod-type formula holds true:

$$f(x_t) - f(x_s) = \delta^\diamond\big(\mathbf{1}_{[s,t)}\nabla f(x)\big) + \frac{1}{2}\int_s^t \Delta f(x_u) R'_u \, du, \tag{1.6}$$

where Δ stands for the Laplace operator, $u \mapsto R_u := \mathbf{E}[|x_u(1)|^2]$ is assumed to be a differentiable function, and R' stands for its derivative.

It should be emphasized here that formula (1.6) is obtained by means of stochastic analysis methods only, independently of the Hölder regularity of x. Otherwise stated, as in many instances of Gaussian analysis, pathwise regularity can be replaced by a regularity on the underlying Wiener space. When both regularity of the paths and regularity of the underlying Wiener space are satisfied, we obtain the relation between the Stratonovich-type integral and the extended divergence operator.

Let us mention at this point the recent work [18] that considers problems similar to ours. In that article, the authors define also an extended divergence-type operator for Gaussian processes (in the one-dimensional case only) with very irregular covariance and study its relation with a Stratonovich-type integral. For the definition of the extended divergence, some conditions on the distributional derivatives of the covariance function R are imposed, one of them being that $\partial^2_{st} R_{st}$ satisfies that $\bar{\mu}(ds, dt) := \partial^2_{st} R_{st}(t-s)$ (i.e., well defined) is the difference of two Radon measures. Our conditions on R are of a different nature, and we suppose more regularity, but only for the first partial derivative of R and the variance function. On the other hand, the definition of the Stratonovich-type integral in [18] is obtained through a regularization approach instead of rough paths theory. As a consequence, some additional regularity conditions on the Gaussian process have to be imposed, while we just rely on the existence of a rough path above x.

(3) Finally, one can relate the two stochastic integrals introduced so far by means of modified Wick–Riemann sums. Indeed, we shall show that the integral $\delta^\diamond(\mathbf{1}_{[s,t)}\nabla f(x))$ introduced at relation (1.6) can also be expressed as

$$\begin{aligned} &\delta^\diamond\big(\mathbf{1}_{[s,t)}\nabla f(x)\big) \\ &\quad = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1}\sum_{k=0}^{N-1}\bigg(\frac{1}{k!}\partial^{k+1}_{i_k,\ldots,i_1 i} f(x_{t_q}) \\ &\qquad\qquad \diamond \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_k) \diamond \cdots \diamond \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_1) \diamond \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i)\bigg), \end{aligned} \tag{1.7}$$

where the (almost sure) limit is still taken along a family of partitions $\Pi_{st} = \{s = t_0, \ldots, t_n = t\}$ of $[s,t]$ whose mesh tends to 0, and where $\diamond$ stands for the usual

Wick product of Gaussian analysis. This result can be seen as the main contribution of our paper, and is obtained by a combination of rough paths and stochastic analysis methods. Specifically, we have mentioned that the modified Riemann sums in (1.5) can be proved to be convergent by means of rough paths analysis. Our main additional technical task will thus consist of computing the correction terms between those Riemann sums and the Wick–Riemann sums which appear in (1.7). This is the aim of the general Proposition 4.7 on Wick products, which has an interest in its own right, and is the key ingredient of our proof. It is worth mentioning at this point that Wick products are usually introduced within the landmark of white noise analysis. We rather rely here on the introduction given in [16], using the framework of Gaussian spaces. Let us also mention that Riemann–Wick sums have been used in [8] to study Skorohod stochastic calculus with respect to (one-dimensional) fBm for H greater than $1/2$, the case of $1/4 < H \leq 1/2$ being treated in [25]. We go beyond these cases in Theorem 4.8, and will go back to the link between our formulas and the one produced in [25] at Section 4.3.

In conclusion, this article is devoted to showing that Stratonovich and Skorohod stochastic calculus are possible for a wide range of Gaussian processes. A link between the integrals corresponding to those stochastic calculus is made through the introduction of Riemann–Wick modified sums. On the other hand, the reader might have noticed that the integrands considered in our stochastic integrals are restricted to processes of the form $\nabla f(x)$. The symmetries of this kind of integrand simplify the analysis of the Stratonovich–Skorohod corrections, reducing all the calculations to corrections involving $\mathbf{x^1}$ only. An extension to more general integrands would obviously require a lot more in terms of Wick-type computations, especially for the terms involving $\mathbf{x^k}$ for $k \geq 2$, and is deferred to a subsequent publication.

Here is how our paper is organized: Section 2 recalls some basic elements of rough paths theory which will be useful in the sequel. We obtain a Skorohod change of variable with Malliavin calculus tools only in Section 3. Finally, the representation of this Skorohod integral by Wick–Riemann sums is performed in Section 4.

2. Some elements of rough paths theory. We recall here the minimal amount of rough paths considerations allowing to obtain a Itô–Stratonovich-type change of variables for a general $\mathbb{R}^d$-valued path. These preliminaries will be presented using terminology taken from the so-called algebraic integration theory, which is a variant of the rough paths theory introduced in [13]. We also refer to [14] for a detailed introduction to the topic.

2.1. *Increments.* The extended pathwise integration introduced in [13] is based on the notion of *increments*, together with an elementary operator δ acting on them. The algebraic structure they generate is described in [13, 14], but here we present directly the definitions of interest for us, for sake of conciseness. First of all, for an arbitrary real number $T > 0$, a vector space V and an integer $k \geq 1$, we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0, T]^k \to V$ such that $g_{t_1,\dots,t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function is called a $(k-1)$-*increment*.

The operator δ alluded to above acts on general increments, but we only need its definition on $\mathcal{C}_1$ and $\mathcal{C}_2$ for our purposes. Indeed, let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $s, u, t \in [0, T]$, we have

$$(\delta g)_{st} = g_t - g_s \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{2.1}$$

Analytic type assumptions are also essential in the sequel, and we measure the size of 1-increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$, let

$$\|f\|_\mu = \sup_{s,t \in [0,T]} \frac{|f_{st}|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V); \|f\|_\mu < \infty\}. \tag{2.2}$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = \|\delta g\|_\mu, \tag{2.3}$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ if and only if $\|g\|_\mu$ is finite. Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will generally work on spaces of the type

$$\mathcal{C}_{1,a}^\mu(V) = \{g : [0, T] \to V; g_0 = a, \|g\|_\mu < \infty\} \tag{2.4}$$

for a given $a \in V$, on which $\|g\|_\mu$ defines a distance in the usual way.

2.2. *Itô–Stratonovich formula.* One of the important byproducts of the rough paths theory is that it allows us to integrate functions of the form $\varphi(x)$ with respect to the path x. We refer to the aforementioned references [9, 13, 19] for more details on the topic, but let us stress informally the following facts:

- Whenever $x \in \mathcal{C}_1^\gamma$ gives raise to a rough path of order $N = \lfloor 1/\gamma \rfloor$, the integral of $\varphi(x)$ with respect to x is defined for any function $\varphi \in C^N(\mathbb{R}^d; \mathbb{R}^d)$. For $0 \leq s < t \leq T$, the integral from s to t is denoted by $\mathcal{J}_{st}(\varphi(x)\, d\mathbf{x})$ or $\int_s^t \langle g(x_u), d\mathbf{x}_u \rangle_{\mathbb{R}^d}$.
- The integral $\mathcal{J}_{st}(\varphi(x)\, d\mathbf{x})$ coincides with the usual Lebesgue–Stieljes integral of $\varphi(x)$ with respect to x whenever x is a smooth function.
- One should be aware of the fact that the quantity $\mathcal{J}_{st}(\varphi(x)\, d\mathbf{x})$ depends on the whole stack $\{\mathbf{x}^\mathbf{n}; n \leq \lfloor 1/\gamma \rfloor\}$. It should thus be understood as a function of the rough path $\mathbf{x}$ above x.
- Some suitable converging Riemann sums can be related to $\mathcal{J}_{st}(\varphi(x)\, d\mathbf{x})$. They will be recalled in Theorem 2.1 for the special case $g = \nabla f$ with $f \in C^{N+1}(\mathbb{R}^d; \mathbb{R})$.

The generalized integration theory induced by rough paths analysis has many consequences in terms of differential calculus with respect to an irregular path x. In this article, we shall focus on an important aspect, namely the change of variable formula for $f(x)$. This formula can then be read as follows (see, e.g., [9], Chapter 3 and 10):

THEOREM 2.1. *For a given $\gamma > 0$ with $\lfloor 1/\gamma \rfloor = N$, let x be a process satisfying the regularity, multiplicative and geometric assumptions of Hypothesis 1.1. Let f be a $C^{N+1}(\mathbb{R}^d;\mathbb{R})$ function. Then*

$$[\delta(f(x))]_{st} = \mathcal{J}_{st}(\nabla f(x)\,d\mathbf{x}) = \int_s^t \langle \nabla f(x_u),\, d\mathbf{x}_u\rangle_{\mathbb{R}^d}, \tag{2.5}$$

where the integral above has to be understood in the rough paths sense. Moreover,

$$\begin{aligned}
&\mathcal{J}_{st}(\nabla f(x)\,d\mathbf{x}) \\
&\quad = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \Bigg[\partial_i f(x_{t_q}) \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i) + \sum_{k=1}^{N-1} \partial^{k+1}_{i_k,\dots,i_1 i} f(x_{t_q}) \mathbf{x}^{\mathbf{k+1}}_{t_q t_{q+1}}(i_k,\dots,i_1,i) \Bigg] \\
&\quad = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \Bigg[\partial_i f(x_{t_q}) \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i) \\
&\qquad\qquad + \sum_{k=1}^{N-1} \frac{1}{k!} \partial^{k+1}_{i_k,\dots,i_1 i} f(x_{t_q}) \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_k) \cdots \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i_1) \mathbf{x}^{\mathbf{1}}_{t_q t_{q+1}}(i) \Bigg]
\end{aligned} \tag{2.6}$$

for any $0 \le s < t \le T$, where the limit is taken over all partitions $\Pi_{st} = \{s = t_0, \dots, t_n = t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

3. Skorohod-type formula via Malliavin calculus. We take now a completely different direction in our considerations: the pointwise point of view, which had been adopted previously, is abandoned in this section, and we try to construct an integral with respect to a (Gaussian) process x by means of stochastic analysis tools. We then prove that for any $0 \le s < t \le T$, the function $\mathbf{1}_{[s,t)} \nabla f(x)$ is in the domain of an extended divergence operator with respect to x, and prove an associated Skorohod-type formula. As we shall see, this mainly stems from an extension of [2] to the d-dimensional case, which is allowed thanks to the symmetries of $\nabla f(x)$.

3.1. *Preliminaries on Gaussian processes.* From now on, we specialize our setting to a centered Gaussian process $x = (x(1), \dots, x(d))$ with i.i.d. coordinates, and covariance function

$$R_{st} := \mathbf{E}[x_s(1) x_t(1)] \quad \text{and} \quad R_t := \mathbf{E}[|x_t(1)|^2] = R_{tt}, \qquad s,t \in [0,T]. \tag{3.1}$$

We will add later some hypotheses on these functions. We can also assume that $x_0(j) = 0$.

The Gaussian integration theory is based on a completion [in $L^2(\Omega)$] of elementary integrals with respect to x, which can be summarized as follows (see [24] for more details): consider the space of d-dimensional elementary functions

$$\mathcal{S} = \left\{ f = (f_1, \dots, f_d);\, f_j = \sum_{i=0}^{n_j - 1} a_i^j \mathbf{1}_{[t_i^j, t_{i+1}^j)}, 0 = t_0 < t_1^j < \dots < t_{n_j}^j = T \right.$$
$$\left. \text{for } j = 1, \dots, d \right\}.$$

For any element f in $\mathcal{S}$, we define the integral of first order of f with respect to x as

$$I_1(f) := \sum_{j=1}^{d} \sum_{i=0}^{n_j - 1} a_i^j \left(x_{t_{i+1}^j}(j) - x_{t_i^j}(j) \right).$$

For $\theta : [0, T] \to \mathbb{R}$, and $j \in \{1, \dots, d\}$, denote by $\theta^{[j]}$ the function with values in $\mathbb{R}^d$ having all the coordinates equal to zero, except the jth coordinate which is equal to θ. It is readily seen that

$$\mathbf{E}\left[I_1\left(\mathbf{1}_{[0,s)}^{[j]}\right) I_1\left(\mathbf{1}_{[0,t)}^{[k]}\right) \right] = \mathbf{1}_{(j=k)} R_{st}.$$

So, we can define for some indicator functions of $\mathcal{S}$, the following symmetric and semi-definite form:

$$\left\langle \mathbf{1}_{[0,s)}^{[j]}, \mathbf{1}_{[0,t)}^{[k]} \right\rangle_{\mathcal{S}} = \mathbf{1}_{(j=k)} R_{st},$$

and we can extend it to all elements of $\mathcal{S}$ by linearity. If we identify two functions f and g in $\mathcal{S}$ when $\langle f - g, f - g \rangle_{\mathcal{S}} = 0$, then $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ becomes an inner product on $\mathcal{S}$ (actually, on the quotient space obtained by this identification). Therefore, for f and g in $\mathcal{S}$, we have that

$$\mathbf{E}\left[I_1(f) I_1(g) \right] = \langle f, g \rangle_{\mathcal{S}}$$

and I_1 defines an isometric map from $\mathcal{S}$, endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ into a subspace of $L^2(\Omega)$. This map can be extended in the standard way to an isometric map, denoted also as I_1, from a real Hilbert space that we will denote by $\mathcal{H}$ into a closed subspace of $L^2(\Omega)$. From now on, denote the inner product of this extended isometry by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. We will assume that $\mathcal{H}$ is a separable Hilbert space (which is satisfied whenever R_{st} is continuous).

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of $\mathcal{H}$, and let $\hat{\otimes}$ denote the symmetric tensor product. Then

$$f_n = \sum_{\text{finite}} f_{i_1, \dots, i_n} e_{i_1} \hat{\otimes} \cdots \hat{\otimes} e_{i_n}, \qquad f_{i_1, \dots, i_n} \in \mathbb{R} \tag{3.2}$$

is an element of $\mathcal{H}^{\hat{\otimes} n}$ with the Hilbert norm

$$\|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}} = \sum_{\text{finite}} |f_{i_1,\dots,i_n}|^2. \tag{3.3}$$

Moreover, $\mathcal{H}^{\hat{\otimes} n}$ is the completion of all the elements like (3.2) with respect to the norm (3.3).

For an element $f_n \in \mathcal{H}^{\hat{\otimes} n}$, the multiple Itô integral of order n is well defined. First, any element of the form given by (3.2) can be rewritten as

$$f_n = \sum_{\text{finite}} f_{j_1,\dots,j_m} e_{j_1}^{\hat{\otimes} k_1} \hat{\otimes} \cdots \hat{\otimes} e_{j_m}^{\hat{\otimes} k_m}, \tag{3.4}$$

where the $j_1, \dots, j_m$ are different and $k_1 + \cdots + k_m = n$. Then, if $f_n \in \mathcal{H}^{\hat{\otimes} n}$ is given under the form (3.4), define its multiple integral as

$$I_n(f_n) = \sum_{\text{finite}} f_{j_1,\dots,j_m} H_{k_1}\big(I_1(e_{j_1})\big) \cdots H_{k_m}\big(I_1(e_{j_m})\big), \tag{3.5}$$

where H_k denotes the kth normalized Hermite polynomial given by

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2} = \sum_{j \le k/2} \frac{(-1)^j k!}{2^j j!(k-2j)!} x^{k-2j}.$$

It holds that the multiple integrals of different order are orthogonal and that

$$\mathbf{E}\big|I_n(f_n)\big|^2 = n! \|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}}.$$

This last isometric property allows us to extend the multiple integral for a general $f_n \in \mathcal{H}^{\hat{\otimes} n}$ by $L^2(\Omega)$ convergence (notice once again that this kind of closure is different in spirit from the pathwise convergences considered at Section 2). Finally, one can define the integral of $f_n \in \mathcal{H}^{\otimes n}$ by putting $I_n(f_n) := I_n(\tilde{f}_n)$, where $\tilde{f}_n \in \mathcal{H}^{\hat{\otimes} n}$ denotes the symmetrized version of f_n. Moreover, the chaos expansion theorem states that any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is the σ-field generated by x, can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n) \qquad \text{with } \mathbf{E}\big[F^2\big] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}}. \tag{3.6}$$

We will introduce now the (iterated) derivative and divergence operators of the Malliavin calculus. We denote by $\mathcal{C}_p^\infty(\mathbb{R}^n)$ the set of infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all its partial derivatives have polynomial growth. Let $\mathbf{S}$ denote the class of smooth random variables of the form

$$F = f\big(I_1(h_1), \dots, I_1(h_n)\big), \tag{3.7}$$

where $f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n$ are in $\mathcal{H}$, and $n \geq 1$. The derivative of a smooth random variable $F \in \mathbf{S}$ of the form (3.7) is the $\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \partial_i f\big(I_1(h_1), \dots, I_1(h_n)\big)h_i, \tag{3.8}$$

where ∂_i denotes as usual $\frac{\partial}{\partial x_i}$. One can also define for $h \in \mathcal{H}$ and $F \in \mathbf{S}$ the derivative of F in the direction of h as $D_h F = \langle F, h\rangle_{\mathcal{H}}$.

The iteration of the operator D is defined in such a way that for a smooth random variable $F \in \mathbf{S}$ the iterated derivative $D^k F$ is a random variable with values in $\mathcal{H}^{\otimes k}$. We also consider for $h^k \in \mathcal{H}^{\otimes k}$ the kth derivative of F in the direction of h^k, defined as

$$D_{h^k}^k F = \langle D^k F, h^k\rangle_{\mathcal{H}^{\otimes k}}.$$

Let us fix now a notation for the domain of the iterated derivative D^k: for every $p \geq 1$ and any natural number $k \geq 1$ we introduce the seminorm on $\mathbf{S}$ given by

$$\|F\|_{k,p} = \left[\mathbf{E}\big(|F|^p\big) + \sum_{j=1}^k \mathbf{E}\big(\|D^j F\|_{\mathcal{H}^{\otimes j}}^p\big)\right]^{1/p}.$$

It is well-known that the operator D^k is closable from $\mathbf{S}$ into $L^p(\Omega; \mathcal{H}^{\otimes k})$. We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathbf{S}$ with respect to the norm $\|\cdot\|_{k,p}$. We will also refer the space $\mathbb{D}^{k,2}$ as the domain of the operator D^k and denote it by $\operatorname{Dom} D^k$. If F has the chaotic representation (3.6), we have that

$$\mathbf{E}\big(\|D^k F\|_{\mathcal{H}^{\otimes k}}^2\big) = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)n!\|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2$$

and a useful characterization of $\operatorname{Dom} D^k$ is the following: $F \in \operatorname{Dom} D^k$ if and only if

$$\sum_{n=1}^\infty n^k n! \|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2 < \infty.$$

We will denote by $\delta^\diamond$ the adjoint of the operator D (this operator is also referred as the *divergence operator*) and more generally, we denote by $\delta^{\diamond k}$ the adjoint of D^k. The operator $\delta^{\diamond k}$ is closed and its domain, denoted by $\operatorname{Dom} \delta^{\diamond k}$, is the set of $\mathcal{H}^{\otimes k}$-valued square integrable random variables $u \in L^2(\Omega; \mathcal{H}^{\otimes k})$ such that

$$\big|\mathbf{E}\big(\langle D^k F, u\rangle_{\mathcal{H}^{\otimes k}}\big)\big| \leq C\|F\|_2$$

for all $F \in \operatorname{Dom} D^k$, where C is some constant depending on u. Moreover, for $u \in \operatorname{Dom} \delta^{\diamond k}$, $\delta^{\diamond k}(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$\mathbf{E}\big(F\delta^{\diamond k}(u)\big) = \mathbf{E}\big(\langle D^k F, u\rangle_{\mathcal{H}^{\otimes k}}\big) \tag{3.9}$$

for any $F \in \mathrm{Dom}\, D^k$. For $u \in \mathrm{Dom}\, \delta^\diamond$, the random variable $\delta^\diamond(u)$ is usually called *Skorohod integral* of u, because it coincides with the usual integral of u with respect to x for a large class of elementary processes u; see [24] for further details.

3.2. *An operator associated to* $\mathbf{x}$. In this section, we will consider a d-dimensional continuous process satisfying the following set of assumptions:

HYPOTHESIS 3.1. *The process $x = (x(1), \dots, x(d))$ is a centered Gaussian process with i.i.d. coordinates and null at* 0. *Letting R_{st} and R_t be defined as in* (3.1), *we suppose that those two functions are continuous and the following two conditions hold*:

(1) *The variance function $R_t := R_{tt}$ is differentiable at any point $t \in (0, T)$ and satisfies that*

$$\int_0^T |R_t'|\, dt < \infty. \tag{3.10}$$

(2) *For each $t \in [0, T]$, the function $R_{\cdot t}$ is absolutely continuous. Moreover, the first partial derivative $\partial_s R_{st}$ of R_{st} is well defined a.e. on $[0, T]^2$ and verifies*

$$\int_0^T \int_0^T |\partial_s R_{sy}|\, ds\, dy < \infty. \tag{3.11}$$

We will try now to identify a useful operator for our future Gaussian computations.

Let $\varphi = (\varphi(1), \dots, \varphi(d)) \in (\mathcal{D}_T)^d$, where $\mathcal{D}_T$ is the space of $\mathcal{C}^\infty$ functions with compact support contained in $(0, T)$. We have (see, e.g., [17], where the 1-dimensional case is considered) that $\varphi \in \mathcal{H}$ and that

$$I_1(\varphi) = -\int_0^T \langle x_s, \varphi_s' \rangle\, ds,$$

where $\langle \cdot, \cdot \rangle$ denotes the ordinary Euclidean product in $\mathbb{R}^d$. Moreover, $(\mathcal{D}_T)^d$ is a dense subset of $\mathcal{H}$. From now on, we use also the notation $x(f)$ for $I_1(f)$.

Given a function $h : [0, T] \to \mathbb{R}$, recall that $h^{[j]}$ denotes the function with values in $\mathbb{R}^d$ in which all the coordinates except the jth one are equal to 0 and the jth coordinate equals to h. Therefore, for $\beta \in \mathcal{D}_T$ and $0 \le a < b \le T$, we have that

$$\begin{aligned} \langle \mathbf{1}_{[a,b)}^{[j]}, \beta^{[j]} \rangle_{\mathcal{H}} &= \mathbf{E}\big[I_1(\mathbf{1}_{[a,b)}^{[l]}) I_1(\beta^{[j]})\big] \\ &= -\mathbf{E}\left[\big(x_b(l) - x_a(l)\big) \int_0^T \langle x_t, [\beta^{[j]}]_t' \rangle\, dt\right] \\ &= -\mathbf{1}_{(j=l)} \int_0^T (R_{bt} - R_{at}) \beta_t'\, dt \qquad (3.12) \end{aligned}$$

$$= -\mathbf{1}_{(j=l)} \int_0^T \left(\int_a^b \partial_s R_{st}\, ds \right) \beta'_t\, dt$$
$$= -\mathbf{1}_{(j=l)} \int_a^b \left(\int_0^T \partial_s R_{sy} \beta'_y\, dy \right) ds.$$

We will consider the first iterated integral appearing on the right-hand side of (3.12) as a linear operator defined on $\mathcal{D}_T$. That is, we consider for $s \in [0, T]$ and $\beta \in \mathcal{D}_T$, the following function:

$$\mathbf{A}\beta(s) := -\int_0^T \partial_s R_{sy} \beta'_y\, dy.$$

We will suppose from now on that the following hypothesis holds.

HYPOTHESIS 3.2. *For any $\beta \in \mathcal{D}_T$, $\mathbf{A}\beta \in L^2([0, T])$.*

REMARK 3.3. Condition (3.11) on R_{st}, stated in Hypothesis 3.1, implies that $\mathbf{A}\beta$ belongs to $L^1([0, T])$ whenever $\beta \in \mathcal{D}_T$. We have imposed the additional condition $A\beta \in L^2([0, T])$ in order to guarantee the integrability of many terms appearing in the sequel. Although one can weaken Hypothesis 3.2, this would complicate some of the next statements. We have thus chosen to impose it for the sake of simplicity.

EXAMPLE 3.4. It is easy to verify that the covariance function R_{sy} of the fractional Brownian motion satisfies Hypothesis 3.1. It also satisfies Hypothesis 3.2. In fact,

$$\begin{aligned} \mathbf{A}\beta(s) &= -\int_0^T \partial s R_{sy} \beta'(y)\, dy \\ &= -\int_0^T H\big(s^{2H-1} - |s-y|^{2H-1} \operatorname{sign}(s-y)\big) \beta'(y)\, dy \\ &= \int_0^T H\big(|s-y|^{2H-1} \operatorname{sign}(s-y)\big) \beta'(y)\, dy, \end{aligned}$$

because $\beta \in \mathcal{D}_T$. And from this, it is easily seen that $\mathbf{A}\beta \in L^\infty([0, T])$ if $\beta \in \mathcal{D}_T$.

EXAMPLE 3.5. Consider now the subfractional Brownian motion whose covariance function is given by

$$R_{st} = s^{2H} + t^{2H} - \tfrac{1}{2}\big((s+t)^{2H} + |t-s|^{2H}\big),$$

with $H \in (0, 1)$. Arguments similar to those used in the above example allow us to see that this covariance verifies Hypotheses 3.1 and 3.2.

EXAMPLE 3.6. The fractional Brownian motion is, in some sense, a canonical example of the class of processes with stationary increments. Namely, for a centered process with stationary increments and null at zero, the covariance function is expressed as

$$R_{st} = \tfrac{1}{2}\big(R(s) + R(t) - R(|t-s|)\big),$$

[we use here the notation $R(\cdot)$ for $R_{\cdot}$]. In this particular case, we can show that relation (3.10) yields the other relations in 3.1 and that we also have $\mathbf{A}\beta \in L^{\infty}([0,T])$. Indeed, condition (2) of Hypothesis 3.1 is a consequence of the fact that, for any $y \in [0,T]$,

$$\partial_s R(s,y) = \tfrac{1}{2}\big(R'(s) - \operatorname{sign}(s-y) R'(|s-y|)\big)$$

is well defined for any $s \in (0,T) \setminus \{y\}$ and satisfies

$$\int_0^T \big|\partial_s R(s,y)\big|\, ds < \infty.$$

This implies, in particular, that $R_{\cdot y}$ is absolutely continuous. Moreover $\int_0^T \int_0^T |\partial_s R(s,y)|\, ds\, dy$ is finite. On the other hand, for $s > 0$,

$$\begin{aligned}
\mathbf{A}\beta(s) &= -\int_0^T \partial_s R_{sy} \beta'(y)\, dy \\
&= -\int_0^T \big(R'(s) - R'(|s-y|)\operatorname{sign}(s-y)\big)\beta'(y)\, dy \\
&= \int_0^T R'(|s-y|)\operatorname{sign}(s-y)\beta'(y)\, dy,
\end{aligned}$$

and thus $\mathbf{A}\beta$ is clearly bounded by a constant, due to (3.10).

EXAMPLE 3.7. Now let R_{sy} be the covariance function of the bi-fractional Brownian motion with parameters $H, K \in (0,1)$, so

$$R_{sy} = \frac{1}{2^K}\big((s^{2H} + y^{2H})^K - |s-y|^{2HK}\big).$$

It is straightforward to verify that R_{sy} satisfies Hypothesis 3.1. In order to see that R also satisfies Hypothesis 3.2 when $H > 1/4$, we compute

$$\partial_s R_{sy} = \frac{2HK}{2^K}\big((s^{2H} + y^{2H})^{K-1} s^{2H-1} - |s-y|^{2HK-1}\operatorname{sign}(s-y)\big).$$

If β has the support contained in $[t_\beta, T]$, with $t_\beta > 0$, then

$$\begin{aligned}
\mathbf{A}\beta(s) &= -\int_{t_\beta}^T \big|\partial s R_{sy} \beta'(y)\big|\, dy \\
&\le C\bigg[\int_{t_\beta}^T (s^{2H} + y^{2H})^{K-1} s^{2H-1}\, dy + \int_0^T |s-y|^{2HK-1}\, dy\bigg] \\
&\le C\big(s^{2H-1} + s^{2HK} + (T-s)^{2HK}\big),
\end{aligned}$$

from which it is easily verified that the covariance function of the bi-fractional Brownian motion satisfies Hypothesis 3.2 if $H > 1/4$. Moreover, one can check that the condition $H > 1/4$ is also necessary in order to have that $\mathbf{A}(\beta) \in L^2([0,T])$ for any $\beta \in \mathcal{D}_T$. In fact, one readily sees that $\mathbf{A}(\beta) \in L^2([0,T])$ if and only if

$$U_\beta(s) := \int_0^T \left(y^{2H} + s^{2H}\right)^{K-1} s^{2H-1} \beta'(y)\, dy \in L^2\big([0,T]\big).$$

Suppose now that β is a positive function ($\beta \geq 0$ and not null), and by the integration by parts formula, we have that for $s > 0$,

$$\begin{aligned} U_\beta(s) &= (1-K) s^{2H-1} \int_0^T \left(y^{2H} + s^{2H}\right)^{K-2} y^{2H-1} \beta(y)\, dy \\ &\geq (1-K) C_\beta s^{2H-1} \int_{a_\beta}^{b_\beta} \left(y^{2H} + T^{2H}\right)^{K-2} y^{2H-1}\, dy = C_{H,K,\beta} s^{2H-1}, \end{aligned}$$

where we have used that there exists a strictly positive constant C_β and an interval $[a_\beta, b_\beta] \subset (0,T)$ such that $\beta \geq C_\beta$ on $[a_\beta, b_\beta]$.

Let us now relate our operator A to the inner product in $\mathcal{H}$: equation (3.12) tells us that for any elementary function $g = (g(1), \ldots, g(d)) \in \mathcal{S}$ and $\beta \in \mathcal{D}_T$, we have that

$$\langle \beta^{[j]}, g(l)^{[l]} \rangle_{\mathcal{H}} = \mathbf{1}_{(j=l)} \int_0^T g_s(l) \mathbf{A}\beta(s)\, ds.$$

Since for $\varphi \in (\mathcal{D}_T)^d$, $g \in \mathcal{S}$, we have $\varphi = \sum_{j=1}^d \varphi(j)^{[j]}$ and $g = \sum_{l=1}^d g(l)^{[l]}$, we obtain that

$$\langle \varphi, g \rangle_{\mathcal{H}} = \sum_{j=1}^d \int_0^T g_s(j) \mathbf{A}\varphi(j)(s)\, ds = \int_0^T \langle g_s, \mathbf{A}\varphi(s) \rangle\, ds, \tag{3.13}$$

where we use the notation $\mathbf{A}\varphi = (\mathbf{A}\varphi(1), \ldots, \mathbf{A}\varphi(d))$. Extending this last relation by continuity, the following useful representation for the inner product in $\mathcal{H}$ is readily obtained:

LEMMA 3.8. *For any $g \in \mathcal{H} \cap (L^2([0,T]))^d$ and $\varphi \in (\mathcal{D}_T)^d$, one can write*

$$\langle \varphi, g \rangle_{\mathcal{H}} = \int_0^T \langle g_s, \mathbf{A}\varphi(s) \rangle\, ds, \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^d$.

Going back to our example 3.4, notice that expression (3.14) is similar to the following one pointed out in [2] for the one-dimensional fractional Brownian motion with Hurst parameter $H < 1/2$:

$$\langle \varphi, g \rangle_{\mathcal{H}} = c_H^2 \int_0^T g(s) \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(s)\, ds,$$

where $\alpha = \frac{1}{2} - H$; $\mathbf{D}_+^\alpha$ and $\mathbf{D}_-^\alpha$ are the Marchaud fractional derivatives (see [27] for more details about these objects), and c_H is a certain positive constant.

3.3. *Extended divergence operator.* Let us take up here the notation of Section 3.1. Having noticed that fBm gives rise to an operator $\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha$ which is a particular case of our operator $\mathbf{A}$ (see Example 3.13), one can naturally try to define an extension of the operator $\delta^\diamond$ using similar arguments to those of [2]. The idea is to consider first $u \in \operatorname{Dom} \delta^\diamond \cap (L^2(\Omega \times [0,T]))^d$ and $F = H_n(x(\varphi))$ where H_n is the nth normalized Hermite polynomial, and $\varphi \in (\mathcal{D}_T)^d$. Since $(L^2(\Omega \times [0,T]))^d \equiv L^2(\Omega; L^2([0,T]; \mathbb{R}^d))$ and $\operatorname{Dom} \delta^\diamond \subset L^2(\Omega; \mathcal{H})$, we have that $u \in (L^2(\Omega \times [0,T]))^d \cap \mathcal{H}$ almost surely. Moreover, $DH_{n-1}(x(\varphi)) = H_{n-1}(x(\varphi))\varphi \in (\mathcal{D}_T)^d$, a.s. So, using (3.14), the usual duality relationship between D and $\delta^\diamond$ can be written in the following way:

$$
\begin{aligned}
\mathbf{E}[\delta^\diamond(u) H_n(x(\varphi))] &= \mathbf{E}[\langle u, DH_n(x(\varphi))\rangle_{\mathcal{H}}] = \mathbf{E}[H_{n-1}(x(\varphi))\langle u, \varphi\rangle_{\mathcal{H}}] \\
&= \mathbf{E}\left[H_{n-1}(x(\varphi)) \int_0^T \langle u_s, \mathbf{A}\varphi(s)\rangle\, ds\right] \qquad (3.15)\\
&= \int_0^T \langle \mathbf{E}[H_{n-1}(x(\varphi)) u_s], \mathbf{A}\varphi(s)\rangle\, ds,
\end{aligned}
$$

and this motivates the following definition.

DEFINITION 3.9. We say that $u \in \operatorname{Dom}^* \delta^\diamond$ if $u \in (L^2(\Omega \times [0,T]))^d$ and there exists an element of $L^2(\Omega)$, that will be denoted by $\delta^\diamond(u)$, such that for any $\varphi \in (\mathcal{D}_T)^d$ and any $n \ge 0$, the following is satisfied:

$$
\mathbf{E}[\delta^\diamond(u) H_n(x(\varphi))] = \int_0^T \langle \mathbf{E}[H_{n-1}(x(\varphi)) u_s], \mathbf{A}\varphi(s)\rangle\, ds. \qquad (3.16)
$$

REMARK 3.10. Since the linear span of the set $\{H_n(\varphi) : n \ge 0, \varphi \in (\mathcal{D}_T)^d\}$ is dense in $L^2(\Omega)$, the element $\delta^\diamond(u)$, if it exists, is uniquely defined.

REMARK 3.11. One can easily see from our definition of the extended divergence that it is a closed operator in the following sense: if $\{u^k\}_{k\in\mathbb{N}} \subset \operatorname{Dom}^* \delta^\diamond$ and satisfies (1) $u^k \to u$ in $(L^2(\Omega \times [0,T]))^d$ and (2) $\delta^\diamond(u^k) \to X$ in $L^2(\Omega)$, then $u \in \operatorname{Dom}^* \delta^\diamond$ and $\delta^\diamond(u) = X$.

We show in the following proposition that the extended operator $\delta^\diamond$ defined above is actually an extension of the divergence operator of the Malliavin calculus.

PROPOSITION 3.12. *The domain* $\operatorname{Dom}^* \delta^\diamond$ *is an extension of* $\operatorname{Dom} \delta^\diamond$ *in the following sense*:

$$
\operatorname{Dom} \delta^\diamond \cap (L^2(\Omega \times [0,T]))^d = \operatorname{Dom}^* \delta^\diamond \cap L^2(\Omega; \mathcal{H}).
$$

Furthermore, the extended operator $\delta^\diamond$ *restricted to* $\mathrm{Dom}\,\delta^\diamond \cap (L^2(\Omega \times [0,T]))^d$ *coincides with the standard divergence operator.*

PROOF. If $u \in \mathrm{Dom}\,\delta^\diamond \cap (L^2(\Omega \times [0,T]))^d$, then $u \in (L^2([0,T]))^d \cap \mathcal{H}$ almost surely. Thus (3.14) can be applied to u, and (3.15) holds true for $\delta^\diamond(u)$ (the standard divergence operator). This proves that $\mathrm{Dom}\,\delta^\diamond \cap (L^2(\Omega \times [0,T]))^d \subset \mathrm{Dom}^*\delta^\diamond \cap L^2(\Omega;\mathcal{H})$ and that $\delta^\diamond$ is an extension of the standard divergence operator on $\mathrm{Dom}\,\delta^\diamond \cap (L^2(\Omega \times [0,T]))^d$.

To see the other inclusion, take $u \in \mathrm{Dom}^*\,\delta^\diamond \cap L^2(\Omega;\mathcal{H})$. By our Definition 3.9 of $\mathrm{Dom}^*\,\delta^\diamond$, u belongs also to $(L^2(\Omega \times [0,T]))^d$. We will show that $u \in \mathrm{Dom}\,\delta^\diamond$. First, we will prove that the element $\delta^\diamond(u)$ defined by equality (3.16) satisfies, for any $\varphi \in (\mathcal{D}_T)^d$ and any $n \geq 0$, that

$$\mathbf{E}\big[\delta^\diamond(u) H_n\big(x(\varphi)\big)\big] = \mathbf{E}\big[\big\langle u, DH_n\big(x(\varphi)\big)\big\rangle_{\mathcal{H}}\big]. \tag{3.17}$$

Indeed, since $u \in (L^2([0,T]))^d \cap \mathcal{H}$ a.s. by assumption, we can apply again identity (3.14), and so

$$\big\langle u, DH_n\big(x(\varphi)\big)\big\rangle_{\mathcal{H}} = H_{n-1}\big(x(\varphi)\big)\int_0^T \langle u_s, \mathbf{A}\varphi(s)\rangle\,ds.$$

Hence, using Fubini's theorem and (3.16) we end up with

$$\begin{aligned}\mathbf{E}\big\langle u, DH_n\big(x(\varphi)\big)\big\rangle_{\mathcal{H}} &= \int_0^T \big\langle \mathbf{E}\big[H_{n-1}\big(x(\varphi)\big)u_s\big], \mathbf{A}\varphi(s)\big\rangle\,ds \\ &= \mathbf{E}\big[\delta^\diamond(u) H_n\big(x(\varphi)\big)\big],\end{aligned}$$

which is exactly (3.17).

By using density arguments [the linear space generated by the elements of the form $H_n(x(\varphi))$, with $\varphi \in (\mathcal{D}_T)^d$, $n \geq 0$, is dense in $\mathrm{Dom}\,\mathrm{D}$] we obtain that

$$\mathbf{E}\big[\langle u, DF\rangle_{\mathcal{H}}\big] = \mathbf{E}\big[\delta^\diamond(u)F\big]$$

for any $F \in \mathrm{Dom}\,\mathrm{D}$, and this finishes the proof. □

EXAMPLE 3.13. Go back to our fBm Example 3.4, and let us compare the extended divergence operator introduced above with the one defined in [2]. First of all, we must point out that in [2], the (standard) divergence operator is presented in a more general setting than ours: the divergence can belong to any $L^p(\Omega)$, for $p > 1$. In our paper, we will only consider this divergence over L^2 spaces for sake of conciseness.

According to the computations carried out in [15] (see identity (5.30) of that work), for any element ψ in the space of test functions $\mathcal{D}_T$, one has

$$c_H^2 \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \psi(s) = \int_0^T H|s-y|^{2H-1}\,\mathrm{sign}(s-y)\psi'(y)\,dy.$$

On the other hand, we have already seen in example 3.4 that $\mathbf{A}\psi(s) = \int_0^T H|s - y|^{2H-1} \operatorname{sign}(s-y)\psi'(y)\,dy$. That is, on $\mathcal{D}_T$, we have the following identity of operators: $c_H^2 \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha = \mathbf{A}$. Moreover, these operators can be extended (and coincide) by density arguments to $I_-^\alpha(\mathcal{E}_H)$; see [2] for the definition of this space. Finally, in this case, $\mathcal{H} = I_-^\alpha(L^2([0,T])$ is a subset of $L^2([0,T])$. Using these observations, it is readily checked that the extended divergence operator defined above coincides with the extended divergence given in [2], restricted to L^2 spaces.

3.4. *Change of variable formula for Skorohod integrals*. We can now turn to the main aim of this section, namely the proof of a change of variable formula for $f(x)$ based on our extended divergence operator $\delta^\diamond$. Interestingly enough, this will be achieved under some nonrestrictive exponential growth conditions on f.

DEFINITION 3.14. We will say that a function $f:\mathbb{R}^d \to \mathbb{R}$ satisfies the growth condition (GC) if there exist positive constants C and λ such that

$$\lambda < \frac{1}{4d \max_{t\in[0,T]} R_t} \quad \text{and} \quad |f(x)| \le C e^{\lambda |x|^2} \qquad \text{for all } x \in \mathbb{R}^d. \tag{3.18}$$

Notice that $\max_{t\in[0,T]} R_t = \max_{t\in[0,T]} E[|x_t|^2]$. Thus the growth condition above implies that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |f(x_t)|^r\Big] \le C^r \mathbf{E}\big(e^{r\lambda \max_{t\in[0,T]} |x_t|^2}\big),$$

and this last expectation is finite (see, e.g. [22], Corollary 5.4.6) if and only if

$$r\lambda < \frac{1}{2\max_{t\in[0,T]} E(|x_r|^2)} = \frac{1}{2d \max_{t\in[0,T]} R_t}.$$

So, if condition (3.18) is satisfied, there exists $r > 2$ such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |f(x_t)|^r\Big] < \infty. \tag{3.19}$$

With these preliminaries in hand, we first state a Skorohod-type change of variable formula for a very regular function f.

PROPOSITION 3.15. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that f and all its derivatives satisfy the growth condition* (GC) (*with possibly different λ's and C's*). *Then, for any $0 \le s < t \le T$,*

$$\mathbf{1}_{[s,t)}(\cdot)\nabla f(x_\cdot) \in \operatorname{Dom}^* \delta^\diamond$$

and

$$\delta^\diamond[\mathbf{1}_{[s,t)}(\cdot)\nabla f(x_\cdot)] = f(x_t) - f(x_s) - \frac{1}{2}\int_s^t \Delta f(x_\rho) R'_\rho \, d\rho.$$

PROOF. Since f and all its derivatives satisfy growth condition (GC), the process $\mathbf{1}_{[s,t)}\nabla f(x)$ is an element of $(L^2(\Omega\times[0,T]))^d$, and we also have

$$f(x_t)-f(x_s)-\frac{1}{2}\int_s^t \Delta f(x_\rho)R'_\rho\,d\rho\in L^2(\Omega).$$

So, we only need to show that for any $n\geq 0$ and any $\varphi\in(\mathcal{D}_T)^d$, the following equality is satisfied:

$$\begin{aligned}&\mathbf{E}\biggl[\biggl(f(x_t)-f(x_s)-\frac{1}{2}\int_s^t \Delta f(x_\rho)R'_\rho\,d\rho\biggr)H_n\bigl(x(\varphi)\bigr)\biggr]\\&\qquad=\int_s^t\langle\mathbf{E}\bigl[H_{n-1}\bigl(x(\varphi)\bigr)\nabla f(x_\rho)\bigr],\mathbf{A}\varphi(\rho)\rangle\,d\rho.\end{aligned}\tag{3.20}$$

The proof of this fact is similar to that of [2], Lemma 4.3, although some technical complications arise from the fact that here we deal with the multidimensional case.

Consider thus the Gaussian kernel

$$p(\sigma,y)=(2\pi\sigma)^{-d/2}\exp\biggl(-\frac{1}{2}\frac{|y|^2}{\sigma}\biggr)\qquad\text{for }\sigma>0,\ y\in\mathbb{R}^d.\tag{3.21}$$

It is a well-known fact that $\partial_\sigma p=\frac{1}{2}\Delta p$. Moreover, we have that $\mathbf{E}[g(x_t)]=\int_{\mathbb{R}^d}p(R_t,y)g(y)\,dy$ for any regular function $g:\mathbb{R}^d\to\mathbb{R}$ such that g and all its derivatives satisfy (GC). Using these identities, we can perform the following computations:

$$\begin{aligned}\frac{d}{dt}\mathbf{E}\bigl[g(x_t)\bigr]&=\frac{d}{dt}\int_{\mathbb{R}^d}p(R_t,y)g(y)\,dy=\int_{\mathbb{R}^d}\frac{\partial}{\partial\sigma}p(R_t,y)R'_tg(y)\,dy\\&=\frac{1}{2}R'_t\int_{\mathbb{R}^d}\Delta p(R_t,y)g(y)\,dy=\frac{1}{2}R'_t\int_{\mathbb{R}^d}p(R_t,y)\Delta g(y)\,dy\\&=\frac{1}{2}R'_t\mathbf{E}\bigl[\Delta g(x_t)\bigr].\end{aligned}\tag{3.22}$$

This shows that the function $\frac{d}{dt}\mathbf{E}[g(x_t)]$ is defined in all $t\in(0,T)$ and is integrable on $[0,T]$. As a consequence, $\mathbf{E}[g(x_t)]$ is absolutely continuous. Using this fact and identity (3.22), we can now prove (3.20) when $n=0$. Indeed, observe that in this case $H_0(x)\equiv 1$ and, by definition, $H_{-1}(x)\equiv 0$. Hence, the right-hand side of (3.20) is equal to 0 while the left-hand side gives

$$\begin{aligned}&\mathbf{E}\biggl[f(x_t)-f(x_s)-\frac{1}{2}\int_s^t \Delta f(x_\rho)R'_\rho\,d\rho\biggr]\\&\qquad=\int_s^t\frac{d}{d\rho}\mathbf{E}\bigl[f(x_\rho)\bigr]\,d\rho-\frac{1}{2}\int_s^t\mathbf{E}\bigl[\Delta f(x_\rho)\bigr]R'_\rho\,d\rho,\end{aligned}$$

and this last quantity vanishes due to (3.22).

Let now $n \ge 1$. Define for $j \in \{1, \dots, d\}$ and $t \in [0, T]$,

$$G_\varphi^j(\rho) = \int_0^\rho \mathbf{A}\varphi(j)(s)\,ds = \langle \mathbf{1}_{[0,\rho)}^{[j]}, \varphi(j)^{[j]} \rangle_{\mathcal{H}}. \tag{3.23}$$

Clearly, G_φ^j is absolutely continuous, and $(G_\varphi^j)' = \mathbf{A}\varphi(j)$ (a.e). Moreover, for a regular function g satisfying (GC) together with all its derivatives and for any multiindex $(j_1, \dots, j_n) \in \{1, \dots, d\}^n$, we have

$$\begin{aligned} &\frac{d}{d\rho}\bigl(\mathbf{E}[g(x_\rho)] G_\varphi^{j_1}(\rho) \cdots G_\varphi^{j_n}(\rho)\bigr) \\ &\qquad = \frac{1}{2}\mathbf{E}[\Delta g(x_\rho)] R'_\rho G_\varphi^{j_1}(\rho) \cdots G_\varphi^{j_n}(\rho) \\ &\qquad\quad + \mathbf{E}[g(x_\rho)] \sum_{r=1}^n \mathbf{A}\varphi(j_r)(\rho) \Bigl[\prod_{l:l\neq r} G_\varphi^{j_l}(\rho)\Bigr], \end{aligned} \tag{3.24}$$

where we have used (3.22).

Set $M_t^{j_1,\dots,j_n} \equiv \mathbf{E}[\partial^n_{j_1,\dots,j_n} f(x_\rho)] \prod_{l=1}^n G_\varphi^{j_l}(\rho)$ for $\rho \in [s,t]$, where we are writing $\partial^n_{j_1,\dots,j_n} f$ for $\frac{\partial^n}{\partial y_{j_1} \cdots \partial y_{j_n}} f$. By integrating (3.24) from s to t and taking $g = \partial^n_{j_1,\dots,j_n} f$, we obtain that

$$\begin{aligned} M_t^{j_1,\dots,j_n} - M_s^{j_1,\dots,j_n} &= \frac{1}{2}\int_s^t \biggl(\mathbf{E}[\Delta\partial^n_{j_1,\dots,j_n} f(x_\rho)] R'_\rho \prod_{l=1}^n G_\varphi^{j_l}(\rho)\biggr) d\rho \\ &\quad + \int_s^t \biggl(\mathbf{E}[\partial^n_{j_1,\dots,j_n} f(x_\rho)] \biggl(\sum_{r=1}^n \mathbf{A}\varphi(j_r)(\rho) \prod_{l:l\neq r} G_\varphi^{j_l}(\rho)\biggr)\biggr) d\rho. \end{aligned}$$

Summing these expressions over all the multiindices $(j_1, \dots, j_n)$ belonging to $\{1, \dots, d\}^n$ and owing to the fact that

$$\begin{aligned} &\sum_{j_1,\dots,j_n} \int_s^t \mathbf{E}[\partial^n_{j_1,\dots,j_n} f(x_\rho)] \biggl(\sum_{r=1}^n \mathbf{A}\varphi(j_r) \prod_{l:l\neq r} G_\varphi^{j_l}(\rho)\biggr) d\rho \\ &\qquad = n \sum_{j_1,\dots,j_n} \int_s^t \mathbf{E}[\partial^n_{j_1,\dots,j_n} f(x_\rho)] \prod_{l=1}^{n-1} G_\varphi^{j_l}(\rho) \mathbf{A}\varphi(j_n)(\rho)\,d\rho, \end{aligned}$$

we end up with an expression of the form

$$\begin{aligned} &\sum_{j_1,\dots,j_n} [M_t^{j_1,\dots,j_n} - M_s^{j_1,\dots,j_n}] \\ &\qquad = \frac{1}{2} \sum_{j_1,\dots,j_n} \int_s^t \mathbf{E}[\Delta\partial^n_{j_1,\dots,j_n} f(x_\rho)] R'_\rho \prod_{l=1}^n G_\varphi^{j_l}(\rho)\,d\rho \\ &\qquad\quad + n \sum_{j_1,\dots,j_n} \int_s^t \mathbf{E}[\partial^n_{j_1,\dots,j_n} f(x_\rho)] \prod_{l=1}^{n-1} G_\varphi^{j_l}(\rho) \mathbf{A}\varphi(j_n)(\rho)\,d\rho. \end{aligned} \tag{3.25}$$

It should be observed at this point that, as in identity (2.6), the symmetries of the partial derivatives of f play a crucial role in the proof of the current proposition. This symmetry property appears precisely in the computations above.

We will see now how to obtain the desired identity (3.20) from (3.25). Indeed, it is a well-known fact (see [24] again) that $H_{n-1}(x(\varphi))\varphi \in \mathrm{Dom}\,\delta^\diamond$ and $\delta^\diamond[H_{n-1}(x(\varphi))\varphi] = nH_n(x(\varphi))$. Using these last two facts, the duality relationship between D and $\delta^\diamond$ and the definition (3.23) of G_j^φ we have that, for g satisfying (GC) as well as its derivatives,

$$\begin{aligned}
\mathbf{E}\big[H_n\big(x(\varphi)\big)g(x_t)\big] &= \frac{1}{n}\mathbf{E}\big\langle H_{n-1}\big(x(\varphi)\big)\varphi, Dg(x_t)\big\rangle_{\mathcal{H}} \\
&= \frac{1}{n}\mathbf{E}\big\langle H_{n-1}\big(x(\varphi)\big)\varphi, \mathbf{1}_{[0,t)}\nabla g(x_t)\big\rangle_{\mathcal{H}} \\
&= \frac{1}{n}\mathbf{E}\left[H_{n-1}\big(x(\varphi)\big)\sum_{j=1}^{d}\int_0^t \partial_j g(x_t)\mathbf{A}\varphi(j)(\rho)\,d\rho\right] \\
&= \frac{1}{n}\sum_{j=1}^{d}\mathbf{E}\big[H_{n-1}\big(x(\varphi)\big)\partial_j g(x_t)\big]G_\varphi^j(t).
\end{aligned}$$

Iterating this procedure n times, one ends up with the identity

$$\begin{aligned}
&\mathbf{E}\big[H_n\big(x(\varphi)\big)g(x_t)\big] \\
&\quad = \frac{1}{n!}\sum_{j_1,\dots j_n}\mathbf{E}\big[\partial^n_{j_1,\dots,j_n}g(x_t)\big]G_\varphi^{j_1}(t)\cdots G_\varphi^{j_n}(t).
\end{aligned}$$

As an application of this general calculation, we can deduce the following equalities:

$$\mathbf{E}\big[H_n\big(x(\varphi)\big)f(x_t)\big] = \frac{1}{n!}\sum_{j_1,\dots j_n}\mathbf{E}\big[\partial^n_{j_1,\dots,j_n}f(x_t)\big]G_\varphi^{j_1}(t)\cdots G_\varphi^{j_n}(t), \tag{3.26}$$

$$\mathbf{E}\big[H_n\big(x(\varphi)\big)f(x_s)\big] = \frac{1}{n!}\sum_{j_1,\dots j_n}\mathbf{E}\big[\partial^n_{j_1,\dots,j_n}f(x_s)\big]G_\varphi^{j_1}(s)\cdots G_\varphi^{j_n}(s), \tag{3.27}$$

$$\mathbf{E}\big[H_n\big(x(\varphi)\big)\Delta f(x_\rho)\big] = \frac{1}{n!}\sum_{j_1,\dots j_n}\mathbf{E}\big[\partial^n_{j_1,\dots,j_n}\Delta f(x_\rho)\big]G_\varphi^{j_1}(\rho)\cdots G_\varphi^{j_n}(\rho) \tag{3.28}$$

and

$$\begin{aligned}
&\mathbf{E}\big[H_{n-1}\big(x(\varphi)\big)\partial_j f(x_\rho)\big] \\
&\quad = \frac{1}{(n-1)!}\sum_{j_1,\dots j_{n-1}}\mathbf{E}\big[\partial_{j_1,\dots,j_{n-1}j}f(x_\rho)\big]G_\varphi^{j_1}(\rho)\cdots G_\varphi^{j_{n-1}}(\rho).
\end{aligned}\tag{3.29}$$

Substituting now (3.26)–(3.29) in (3.25), we obtain

$$\begin{aligned}
&n!\mathbf{E}\big[H_n\big(x(\varphi)\big)f(x_t)\big]-n!\mathbf{E}\big[H_n\big(x(\varphi)\big)f(x_s)\big]\\
&\quad=\frac{1}{2}n!\int_s^t \mathbf{E}\big[H_n\big(x(\varphi)\big)\Delta f(x_\rho)\big]R'_\rho\,d\rho\\
&\qquad+n(n-1)!\int_s^t\sum_{j_n=1}^d \mathbf{E}\big[H_{n-1}\big(x(\varphi)\big)\partial_{j_n}f(x_\rho)\big]\mathbf{A}\varphi(j_n)(\rho)\,d\rho,
\end{aligned}$$

and this is actually equality (3.20). The proof is now complete. □

Since the Skorohod divergence operator is closable, we can now generalize our change of variable formula:

THEOREM 3.16. *The conclusions of Proposition 3.15 still hold true whenever f is an element of $\mathcal{C}^2(\mathbb{R}^d)$ such that f and its partial derivatives up to second order verify the growth condition* (GC).

PROOF. Let λ be the constant appearing in the growth condition (GC). Given $k>2\lambda$, denote by $p_k(y)=p(\frac{1}{k},y)$ the Gaussian kernel defined in (3.21), and introduce $f_k(y)=(f*p_k)(y)$ where, as usual, $*$ denotes the convolution product.

We first claim that there exist $k_0\in\mathbb{N}$, $C'>0$ and λ' satisfying $\lambda<\lambda'<\frac{1}{4d\max_{t\in[0,T]}R_t}$, such that

$$\sup_{k\ge k_0}\big|f_k(y)\big|\le C'e^{\lambda'|y|^2}. \tag{3.30}$$

Indeed, condition (GC) easily yields

$$\begin{aligned}
\big|f_k(y)\big|&\le\left(\frac{k}{2\pi}\right)^{d/2}\int_{\mathbb{R}^d}\big|f(y-z)\big|e^{-k|z|^2/2}\,dz\\
&\le\left(\frac{k}{2\pi}\right)^{d/2}C\int_{\mathbb{R}^d}e^{\lambda|y-z|^2}e^{-k|z|^2/2}\,dz\\
&=C\prod_{i=1}^d\left(\sqrt{\frac{k}{2\pi}}\int_{\mathbb{R}}e^{\lambda(y_i-z_i)^2}e^{-kz_i^2/2}\,dz_i\right).
\end{aligned}$$

On the other hand,

$$\sqrt{\frac{k}{2\pi}}\int_{\mathbb{R}}e^{\lambda(y_i-z_i)^2}e^{-kz_i^2/2}\,dz_i=\sqrt{\frac{k}{k-2\lambda}}\exp\left\{\left(\frac{\lambda k}{k-2\lambda}\right)y_i^2\right\},$$

and $\lim_{k\to\infty}\frac{\lambda k}{k-2\lambda}=\lambda$. Hence, given $\lambda'\in(\lambda,\frac{1}{4d\max_{t\in[0,T]}R_t})$, there exists $k_0\in\mathbb{N}$ such that for any $k\ge k_0$, the following inequalities are satisfied:

$$\lambda<\frac{\lambda k}{k-2\lambda}<\lambda'<\frac{1}{4d\max_{t\in[0,T]}R_t}.$$

Our claim (3.30) is now easily deduced.

Notice that (3.30) means that for $k \geq k_0$, f_k also satisfies the growth condition (GC) (with C' and λ' substituting C and λ, resp.). Moreover, we have that

$$\mathbf{E}\Big[\sup_{\rho\in[0,T]}\sup_{k\geq k_0}\big|f_k(x_\rho)\big|^2\Big] < \infty.$$

Thanks to this inequality, as well as similar ones involving the derivatives of f, one can easily see that:

(1) $f_k(x_s) \to f(x_s)$ and $f_k(x_t) \to f(x_t)$ in $L^2(\Omega)$,
(2) $\int_s^t \Delta f_k(x_\rho) R'_\rho\, d\rho \to \int_s^t \Delta f(x_\rho) R'_\rho\, d\rho$ in $L^2(\Omega)$ and
(3) $\mathbf{1}_{[s,t)} \nabla f_k(x_\cdot) \to \mathbf{1}_{[s,t)} \nabla f(x_\cdot)$ in $(L^2(\Omega \times [0, T]))^d$.

The result is finally obtained by applying Proposition 3.15 and the closeness of the extended operator $\delta^\diamond$ alluded to at Remark 3.11. □

4. Representation of the Skorohod integral. Up to now, we have given two unrelated change of variable formulas for $f(x)$: one based on pathwise considerations (Theorem 2.1) and the other one by means of Malliavin calculus (Theorem 3.16). We propose now to make a link between the two formulas and integrals by means of Riemann sums.

Namely, let x be a process generating a rough path of order N. We have seen in equation (2.6) that the Stratonovich integral $\mathcal{J}_{st}(\nabla f(x)\, d\mathbf{x})$ is given by $\lim_{|\Pi_{st}|\to 0} S^{\Pi_{st}}$, where

$$S^{\Pi_{st}} := \sum_{q=0}^{n-1}\sum_{k=1}^{N} \frac{1}{k!} \partial^k_{i_k,\dots,i_1} f(x_{t_q}) \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_1)\mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_2)\cdots \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_k).$$

In a Gaussian setting, it is thus natural to think that a natural candidate for the Skorohod integral $\delta^\diamond(\nabla f(x))$ is also given by $\lim_{|\Pi_{st}|\to 0} S^{\Pi_{st},\diamond}$, with

$$S^{\Pi_{st},\diamond} := \sum_{q=0}^{n}\sum_{k=1}^{N} \frac{1}{k!} \partial^k_{i_k,\dots,i_1} f(x_{t_q}) \diamond \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_1) \diamond \cdots \diamond \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_k), \tag{4.1}$$

where $\diamond$ denotes the Wick product. We shall see that this is indeed the case, with the following strategy:

(i) One should thus first check that $\lim_{\Pi_{st}} S^{\Pi_{st},\diamond}$ exists. In order to check this convergence, we shall use extensively Wick calculus, in order to write

$$\begin{aligned}&\partial^k_{i_k,\dots,i_1} f(x_{t_q}) \diamond \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_1) \diamond \cdots \diamond \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_k)\\ &\quad = \partial^k_{i_k,\dots,i_1} f(x_{t_q}) \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_1)\cdots \mathbf{x}^{\mathbf{1}}_{t_q,t_{q+1}}(i_k) + \rho_{t_q,t_{q+1}},\end{aligned}$$

where ρ is a certain correction increment which can be computed explicitly. Plugging this relation into (4.1), we obtain

$$S^{\Pi_{st},\diamond} = S^{\Pi_{st}} + \sum_{q=0}^{n-1} \rho_{t_q,t_{q+1}}. \tag{4.2}$$

(ii) Manipulating the exact expression of the remainder ρ, we will be able to prove that $\lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \rho^1_{t_q,t_{q+1}} = -\frac{1}{2}\int_s^t \Delta f(x_v) R'_v \, dv$. Hence, going back to (4.2) and invoking the fact that $S^{\Pi_{st}}$ converges to $\mathcal{J}_{st}(\nabla f(x)\, d\mathbf{x})$, we obtain

$$\begin{aligned}\lim_{|\Pi_{st}|\to 0} S^{\Pi_{st},\diamond} &= \mathcal{J}_{st}\big(\nabla f(x)\, d\mathbf{x}\big) - \frac{1}{2}\int_s^t \Delta f(x_v) R'_v \, dv \\ &= \big[\delta f(x)\big]_{st} - \frac{1}{2}\int_s^t \Delta f(x_v) R'_v \, dv.\end{aligned}$$

This gives both the convergence of $S^{\Pi_{st},\diamond}$ and an Itô–Skorohod formula of the form

$$\big[\delta f(x)\big]_{st} = \lim_{|\Pi_{st}|\to 0} S^{\Pi_{st},\diamond} + \frac{1}{2}\int_s^t \Delta f(x_v) R'_v \, dv.$$

(iii) Putting together this last equality and Theorem 3.16, it can be deduced that under Hypotheses 3.1 and 3.2, the limit of $S^{\Pi_{st},\diamond}$ coincides with the Skorohod integral $\delta^\diamond(\nabla f(x))$. This gives our link relating the Stratonovich integral $\mathcal{J}(\nabla f(x)\, d\mathbf{x})$ and the Skorohod integral $\delta^\diamond(\nabla f(x))$.

This relatively straightforward strategy being set, we turn now to the technical details of its realization. To this end, the main issue is obviously the computation of the corrections between Wick and ordinary products in sums like $S^{\Pi_{st},\diamond}$. We thus start by recalling some basic facts of Wick computations.

4.1. *Notions of Wick calculus.* We present here the notions of Wick calculus needed later on, basically following [16]. We also use extensively the notation introduced in Section 3.1.

One way to introduce Wick products on a Wiener space is to impose the relation

$$I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \hat{\otimes} g_m)$$

for any $f_n \in \mathcal{H}^{\hat{\otimes} n}$ and $g_m \in \mathcal{H}^{\hat{\otimes} m}$, where the multiple integrals $I_n(f_n)$ and $I_m(g_m)$ are defined by (3.5). If $F = \sum_{n=1}^{N_1} I_n(f_n)$ and $G = \sum_{m=1}^{N_2} I_m(g_m)$, we define $F \diamond G$ by

$$F \diamond G = \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} I_{n+m}(f_n \hat{\otimes} g_m).$$

By a limit argument, we can then extend the Wick product to more general random variables; see [16] for further details. In this paper, we will take the limits in the $L^2(\Omega)$ topology.

For $f \in \mathcal{H}$, we define its exponential vector $\mathcal{E}(f)$ by

$$\mathcal{E}(f) := e^{\diamond I_1(f)} = \exp\left(I_1(f) - \frac{\|f\|_{\mathcal{H}}^2}{2}\right) = \exp\left(I_1(f) - \frac{1}{2}\mathbf{E}\big(I_1(f)\big)^2\right)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} I_n\big(f^{\otimes n}\big) = \sum_{n=0}^{\infty} \frac{1}{n!} I_1(f)^{\diamond n}.$$

In a similar way we can define the complex exponential vector of f by

$$e^{\diamond \imath I_1(f)} = \exp\left(\imath I_1(f) + \frac{\|f\|_{\mathcal{H}}^2}{2}\right) = \sum_{n=0}^{\infty} \frac{\imath^n}{n!} I_1(f)^{\diamond n}, \tag{4.3}$$

where $\imath$ denotes the imaginary unity. With these notation in hand, an important property of a Wick product is the following relation: for any two elements f and g in $\mathcal{H}$, we have

$$\mathcal{E}(f) \diamond \mathcal{E}(g) = \mathcal{E}(f+g), \tag{4.4}$$

an analogous property for the complex exponential vector being also satisfied.

We now state a result which is a generalization of [16], Proposition 4.8.

PROPOSITION 4.1. *Let $F \in \operatorname{Dom} D^k$ and $g \in \mathcal{H}^{\otimes k}$. Then*:

(1) *$F \diamond I_k(g)$ is well defined in $L^2(\Omega)$*;
(2) *$Fg \in \operatorname{Dom} \delta^{\diamond k}$*;
(3) *$F \diamond I_k(g) = \delta^{\diamond k}(Fg)$.*

PROOF. Let $F \in \operatorname{Dom} D^k$. This implies that F admits the chaos decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$, with

$$\sum_{n=0}^{\infty} n^k n! \|f_n\|_{\mathcal{H}^{\hat{\otimes} n}}^2 < \infty. \tag{4.5}$$

Define then $F_N = \sum_{n=0}^{N} I_n(f_n)$. Consider also $g \in \mathcal{H}^{\otimes k}$. In order to check (1), we shall see that the limit in $L^2(\Omega)$ of $F_N \diamond I_k(g)$ exists, as $N \to \infty$. But $F_N \diamond I_k(g) = \sum_{n=0}^{N} I_{n+k}(f_n \hat{\otimes} g)$, and the limit in $L^2(\Omega)$ of this expression exists if and only if

$$\sum_{n=0}^{\infty} (n+k)! \|f_n \hat{\otimes} g\|_{\mathcal{H}^{\hat{\otimes} n+k}}^2 < \infty.$$

This last condition is clearly satisfied, thanks to (4.5).

Now we will prove our claims (2) and (3) for F with a finite chaos decomposition and $g = g_1 \otimes \cdots \otimes g_k$, with $g_i \in \mathcal{H}$, for $i = 1, \ldots, k$. More precisely, we will see that for any $G \in \mathbf{S}$ the following relationship holds:

$$\mathbf{E}\big[\big(F \diamond I_k(g)\big)G\big] = \mathbf{E}\big[\langle Fg, D^k G\rangle_{\mathcal{H}^{\otimes k}}\big]. \tag{4.6}$$

This will be done by an induction argument. For $k=1$, this is a consequence of [16], Proposition 4.8, since $F \diamond I_1(g) = \delta^\diamond(Fg)$. Suppose now that (4.6) is satisfied for $k=K$. Therefore,

$$\begin{aligned}\mathbf{E}[[F \diamond I_{K+1}(g)]G] &= \mathbf{E}[(F \diamond I_1(g_1) \diamond I_k(g_2 \otimes \cdots \otimes g_{K+1}))G] \\ &= \mathbf{E}[\langle (F \diamond I_1(g_1))g_2 \otimes \cdots \otimes g_{K+1}, D^K G\rangle_{\mathcal{H}^{\otimes K}}],\end{aligned}$$

where in the last equality, we have used that $F \diamond I_1(g_1)$ has a finite chaos expansion and the induction hypothesis. The last expression can be rewritten as

$$\mathbf{E}[(F \diamond I_1(g_1)) D^K_{g_2 \otimes \cdots \otimes g_{K+1}} G].$$

Since $D^K_{g_2 \otimes \cdots \otimes g_{K+1}} G \in \mathbf{S}$, we can apply the case $k=1$ to the above expression, and we obtain

$$\begin{aligned}\mathbf{E}[[F \diamond I_{K+1}(g)]G] &= \mathbf{E}[\langle F g_1, D^K_{g_2 \otimes \cdots \otimes g_{K+1}} G\rangle_{\mathcal{H}}] \\ &= \mathbf{E}[F D^1_{g_1}(D^K_{g_2 \otimes \cdots \otimes g_{K+1}} G)] = \mathbf{E}[F D^{K+1}_{g_1 \otimes g_2 \otimes \cdots \otimes g_{K+1}} G] \\ &= \mathbf{E}[\langle F g_1 \otimes \cdots \otimes g_{K+1}, D^{K+1} G\rangle_{\mathcal{H}^{\otimes K+1}}],\end{aligned}$$

which finishes our induction procedure. Thus, (4.6) is satisfied for F with a finite chaos expansion and g a tensor product of elements of $\mathcal{H}$.

To extend the result to a general $F \in \mathrm{Dom}(D^k)$ and $g \in \mathcal{H}^{\otimes k}$, we first consider the case $F \in \mathrm{Dom}(D^k)$ and $g = g_1 \otimes \cdots \otimes g_k$. In this situation, identity (4.6) is a consequence of the fact that this relationship holds for $F_N = \sum_{n=0}^N I_n(f_n)$ defined above and of the part (1) of the proposition. Finally, for a general $g \in \mathcal{H}^{\otimes k}$, using the fact that both sides of (4.6) are linear in g, we can generalize this identity to g belonging to the linear span of elements of the form $g_1 \otimes \cdots \otimes g_k$, which is a dense subspace of $\mathcal{H}^{\otimes k}$. So if $g \in \mathcal{H}^{\otimes k}$ and $\{g^M\}_{M \in \mathbb{N}}$ is a sequence of elements of this linear span of tensor products such that $g^M \to g$ in $\mathcal{H}^{\otimes k}$, one can easily see [by using (4.5)] that

$$F \diamond I_k(g^M) \to F \diamond I_k(g)$$

as $M \to \infty$ in $L^2(\Omega)$. Since

$$E[(F \diamond I_k(g^M))G] = E[\langle F g^M, D^k G\rangle_{\mathcal{H}^{\otimes k}}],$$

the proof is completed by a limiting argument. □

4.2. *One-dimensional case.* In order to simplify a little our presentation, we first show the identification of $\delta^\diamond(\nabla f(x))$ with $\lim_{|\Pi_{st}| \to 0} S^{\Pi_{st},\diamond}$ when $d=1$, that is, when x is a one-dimensional Gaussian process satisfying Hypothesis 3.1. The first step in this direction is a general formula for Wick products of the form $G(X) \diamond Y^{\diamond p}$, where X and Y are elements of the first chaos; see Section 3.1 for a definition. Notice that the proof of this proposition is deferred to the Appendix for sake of clarity.

PROPOSITION 4.2. *Let $g, h \in \mathcal{H}$ and let $G : \mathbb{R} \to \mathbb{R}$ be differentiable up to order p such that all its derivatives $G^{(j)}$ are elements of $L^r(\mu_g)$ for any $j = 0, \dots, p$ and for some $r > 2$, with $\mu_g = \mathcal{N}(0, \|g\|_{\mathcal{H}}^2)$. Define $X = I_1(g)$ and $Y = I_1(h)$. Then the Wick product $G(X) \diamond Y^{\diamond p}$ can be expressed in terms of ordinary products as*

$$
\begin{aligned}
G(X) \diamond Y^{\diamond p} = G(X)Y^p + \sum_{0<l+2m\le p} & \frac{(-1)^{m+l} p!}{2^m m!(p-2m-l)!} \\
& \times G^{(l)}(X)[\mathbf{E}(XY)]^l [\mathbf{E}(Y^2)]^m Y^{p-2m-l}.
\end{aligned} \tag{4.7}
$$

EXAMPLE 4.3. In order to illustrate the kind of correction terms we obtain, let us write formula (4.7) for $p = 1, 2, 3$.

$$
\begin{aligned}
G(X) \diamond Y &= G(X)Y - G'(X)\mathbf{E}(XY), \\
G(X) \diamond Y^{\diamond 2} &= G(X)Y^2 - G(X)\mathbf{E}(Y^2) \\
&\quad - 2G'(X)\mathbf{E}(XY)Y + G''(X)[\mathbf{E}(XY)]^2, \\
G(X) \diamond Y^{\diamond 3} &= G(X)Y^3 - 3G(X)\mathbf{E}(Y^2)Y + 3G'(X)\mathbf{E}(XY)\mathbf{E}(Y^2) \\
&\quad - 3G'(X)\mathbf{E}(XY)Y^2 + 3G''(X)[\mathbf{E}(XY)]^2 Y \\
&\quad - G'''(X)[\mathbf{E}(XY)]^3.
\end{aligned}
$$

We are now ready to state our representation of the Skorohod integral by Riemann–Wick sums:

THEOREM 4.4. *Let x be a 1-dimensional centered Gaussian process with covariance function fulfilling Hypotheses 3.1 and 3.2, and assume that x also satisfies Hypotheses 1.1. Let f be a function in $C^{2N}(\mathbb{R})$ such that $f^{(k)}$ verifies the growth condition* (GC) *for $k = 1, \dots, 2N$. Then the Skorohod integral $\delta^\diamond(\mathbf{1}_{[s,t)} f'(x))$ (whose existence is ensured by Theorem 3.16) can be represented as* a.s.—$\lim_{\Pi_{st}\to 0} S^{\Pi_{st},\diamond}$, *where $S^{\Pi_{st},\diamond}$ is defined by*

$$
S^{\Pi_{st},\diamond} = \sum_{i=0}^{n-1} \sum_{k=1}^{N} \frac{1}{k!} f^{(k)}(x_{t_i}) \diamond (\mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}})^{\diamond k}.
$$

Moreover, we have

$$
\delta^\diamond(\mathbf{1}_{[s,t)} f'(x)) = \int_s^t f'(x_\rho)\, d\mathbf{x}_\rho - \int_s^t f'(x_\rho) R'_\rho \, d\rho. \tag{4.8}
$$

PROOF. As mentioned at the beginning of the section, our main task is to compute $S^{\Pi_{st},\diamond}$ in terms of ordinary products. This will be achieved by applying

Proposition 4.2 to each term in the above sum, with $G = f$, $X = x_{t_i}$ and $Y = \mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}} = x_{t_{i+1}} - x_{t_i}$.

To this end, notice first that the integrability conditions on f required at Proposition 4.2 are fulfilled as soon as condition (GC) (see Definition 3.14) is met. Fix then $i \in \{0, \dots, n-1\}$, recall that we set $X = x_{t_i}$ and $Y = \mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}$ and consider the quantity $\mathcal{S}^i_k := \frac{1}{k!} f^{(k)}(x_{t_i}) \diamond (\mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}})^{\diamond, k}$. A direct application of Proposition 4.2 yields

$$\sum_{k=1}^{N} \mathcal{S}^i_k = \sum_{k=1}^{N} \sum_{l+2m \le k} \frac{(-1)^{l+m}}{2^m m! l! (k-2m-l)!} \times f^{(k+l)}(X) \big[\mathbf{E}(XY)\big]^l \big[\mathbf{E}(Y^2)\big]^m Y^{k-2m-l}.$$

Making a substitution $q = k + l$ and $l + m = u$, this expression can be simplified into

$$\begin{aligned} \sum_{k=1}^{N} \mathcal{S}^i_k &= \sum_{q=1}^{2N} f^{(q)}(X) \sum_{l+m \le q/2} \frac{(-1)^{l+m}}{2^m m! l! (q-2l-2m)!} \\ &\qquad \times \big[\mathbf{E}(XY)\big]^l \big[\mathbf{E}(Y^2)\big]^m Y^{q-2l-2m} \\ &= \sum_{q=1}^{2N} f^{(q)}(X) \sum_{u=0}^{[q/2]} \frac{(-1)^u}{(q-2u)!} \sum_{l+m=u} \frac{1}{m! l!} \big[\mathbf{E}(XY)\big]^l \left[\frac{\mathbf{E}(Y^2)}{2}\right]^m Y^{q-2u} \\ &= \sum_{q=1}^{2N} f^{(q)}(X) \sum_{u=0}^{[q/2]} \frac{(-1)^u}{(q-2u)! u!} \left[\mathbf{E}(XY) + \frac{\mathbf{E}(Y^2)}{2}\right]^u Y^{q-2u}. \end{aligned} \tag{4.9}$$

Moreover, recalling again that $X = x_{t_i}$ and $Y = x_{t_{i+1}} - x_{t_i}$, it is easily seen that

$$\mathbf{E}(XY) + \frac{\mathbf{E}(Y^2)}{2} = \frac{1}{2}\big[\mathbf{E}(x^2_{t_{i+1}}) - \mathbf{E}(x^2_{t_i})\big] = \frac{1}{2} \delta R_{t_i t_{i+1}}, \tag{4.10}$$

where we recall that $\delta R_{t_i t_{i+1}}$ stands for $R_{t_{i+1}} - R_{t_i}$. Therefore, summing now over $i \in \{i, \dots, n-1\}$, we get

$$S^{\Pi_{st}, \diamond} = \sum_{i=0}^{n-1} \sum_{k=1}^{N} \mathcal{S}^i_k = \sum_{q=1}^{2N} \sum_{u=0}^{[q/2]} \frac{(-1)^u}{(q-2u)! u! 2^u} \sum_{i=0}^{n-1} \mathcal{T}^{q,u}_i, \tag{4.11}$$

where the quantity $\mathcal{T}^{q,u}_i$ is defined by

$$\mathcal{T}^{q,u}_i = f^{(q)}(x_{t_i}) (\delta R_{t_i t_{i+1}})^u \big(\mathbf{x}^{\mathbf{1}}_{t_i, t_{i+1}}\big)^{q-2u}. \tag{4.12}$$

We now separate the study into different cases.

Case 1: If $u=1$ and $q-2u=0$ (namely $q=2$), then

$$\sum_{i=0}^{n-1} \mathcal{T}_i^{q,u} = -\frac{1}{2}\sum_{i=0}^{n-1} f''(x_{t_i})[R_{t_{i+1}} - R_{t_i}]$$
$$= -\frac{1}{2}\int_s^t \left(\sum_{i=0}^{n-1} f''(x_{t_i})\mathbf{1}_{[t_i,t_{i+1})}(\rho)\right) R'_\rho \, d\rho,$$

where in the last equality we resort to the fact that R_ρ is absolutely continuous; see Hypothesis 3.1. From this expression, by a dominated convergence argument one easily gets $\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathcal{T}_i^{q,u} = -\frac{1}{2}\int_s^t f''(x_\rho)R'_\rho\,d\rho$.

Case 2: If $u\geq 2$ or $u=1$, $q-2u\geq 1$, then $\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathcal{T}_i^{q,u}=0$. Indeed, recalling definition (4.12) of $\mathcal{T}_i^{q,u}$, we observe that

$$\sum_{i=0}^{n-1}\mathcal{T}_i^{q,u} \leq \max_{0\leq i\leq n-1}\{|\mathbf{x}^{\mathbf{1}}_{t_it_{i+1}}|^{q-2u}, |\delta R_{t_it_{i+1}}|^{u-1}\}$$
$$\times \int_s^t \left(\sum_{i=0}^{n-1}|f^{(q)}(x_{t_i})|\mathbf{1}_{[t_i,t_{i+1})}(\rho)\right)|R'(\rho)|\,d\rho.$$

On the right-hand side of the above inequality, it is now easily seen that

$$\lim_{n\to\infty}\max_{0\leq i\leq n-1}\{|\mathbf{x}^{\mathbf{1}}_{t_it_{i+1}}|^{q-2u}, |\delta R_{t_it_{i+1}}|^{u-1}\}=0,$$

while the integral term remains bounded by $C\int_s^t |R'_\rho|\,d\rho$, which is bounded by assumption. This completes the proof of our claim.

Case 3: If $u=0$, then

$$\sum_{i=0}^{n-1}\mathcal{T}_i^{q,u} = \sum_{i=0}^{n-1}\sum_{q=1}^{N}\frac{1}{q!}f^{(q)}(x_{t_i})(\mathbf{x}^{\mathbf{1}}_{t_it_{i+1}})^q + \sum_{i=0}^{n-1}\sum_{q=N+1}^{2N}\frac{1}{q!}f^{q}(x_{t_i})(\mathbf{x}^{\mathbf{1}}_{t_it_{i+1}})^q.$$

Thus Theorem 2.1 asserts that the first sum above converges to $\int_s^t f'(x_u)\,d\mathbf{x}_u$, while it is easy to see that the second sum converges to 0, thanks to the regularity properties of x.

Plugging now the study of our 3 cases into equation (4.11), the proof of our theorem is easily completed. □

4.3. *Relationship with existing results.* Several results exist on the convergence of Riemann–Wick sums, among which emerges [25], dealing with a situation which is similar to ours in the case of a one-dimensional process.

In order to compare our results with those of [25], let us specialize our situation to the case of a dyadic partition of an interval $[s,t]$ with $s<t$ (the case of a general partition is handled in [25], but this restriction will be more convenient for our

purposes). Namely, for $n \geq 1$, we consider the partition $\Pi_{st}^n = \{t_k^n; 0 \leq k \leq 2^n\}$, where $t_k^n = s + k(t-s)/2^n$. For notational sake, we often write t_k instead of t_k^n. We shall also restrict our study to the case of a fBm B, though [25] deals with a rather general Gaussian process.

Let us first quote some results about weighted sums taken from [10, 12]:

PROPOSITION 4.5. *Let B be a one-dimensional fractional Brownian motion, whose covariance function is defined by* (1.3). *Let g be a $\mathcal{C}^4$ function satisfying Hypothesis* (GC) *together with all its derivatives.*

(i) *For $n \geq 1$, set*

$$V_n^{(2)}(g) = \sum_{k=0}^{2^n-1} g(B_{t_k})\big[(\mathbf{B}^{\mathbf{1}}_{t_k t_{k+1}})^2 - 2^{-2nH}\big].$$

Then if $1/4 < H < 3/4$, we have

$$\mathcal{L} - \lim_{n\to\infty} n^{2H-1/2} V_n^{(2)}(g) = \sigma_H \int_0^{t-s} g(B_s)\, dW_s, \tag{4.13}$$

where $\mathcal{L} - \lim$ stands for a convergence in law, σ_H is a positive constant depending only on H and W is a Brownian motion independent of B.

(ii) *For $n \geq 1$, set*

$$\tilde{V}_n^{(3)}(g) = \sum_{k=0}^{2^n-1} g(B_{t_k})(\mathbf{B}^{\mathbf{1}}_{t_k t_{k+1}})^3.$$

Then if $H < 1/2$, we have

$$L^2(\Omega) - \lim_{n\to\infty} n^{4H-1} \tilde{V}_n^{(3)}(g) = -\frac{3}{2}\int_0^{t-s} g'(B_s)\, ds.$$

We can now recall the main result of [25], to which we would like to compare our own computations:

PROPOSITION 4.6. *Let B be a one-dimensional fBm with Hurst parameter $1/4 < H \leq 1/2$ and f be a $\mathcal{C}^4$ function satisfying Hypothesis* (GC) *together with all its derivatives. For $0 \leq s < t \leq T$, consider the set of dyadic partitions $\{\Pi_{st}^n; n \geq 1\}$, and set*

$$\tilde{S}^{n,\diamond} = \sum_{k=0}^{2^n-1} f'(B_{t_k}) \diamond \mathbf{B}^{\mathbf{1}}_{t_k t_{k+1}}. \tag{4.14}$$

Then $\tilde{S}^{n,\diamond}$ converges in $L^2(\Omega)$ to $\delta^\diamond(\mathbf{1}_{[s,t)} f'(B))$ (which is the Skorohod integral introduced at Theorem 3.16).

PROOF. Our aim here is not to reproduce the proof contained in [25], but to give a version compatible with our formalism. We shall focus on the case $1/4 < H \le 1/3$, the other one being easier.

Consider first $0 \le u < v \le T$. According to Example 4.3, we have

$$\begin{aligned} f'(B_u) \diamond \mathbf{B}^{\mathbf{1}}_{uv} &= f'(B_u)\mathbf{B}^{\mathbf{1}}_{uv} - \tfrac{1}{2} f''(B_u)\mathbf{E}[B_u \mathbf{B}^{\mathbf{1}}_{uv}] \\ &= f'(B_u)\mathbf{B}^{\mathbf{1}}_{uv} - \tfrac{1}{2} f''(B_u)[v^{2H} - u^{2H}] + \tfrac{1}{2} f''(B_u)|t-s|^{2H}. \end{aligned}$$

In a rather artificial way, we shall recast this identity into

$$f'(B_u) \diamond \mathbf{B}^{\mathbf{1}}_{uv} = \sum_{j=1}^{3} \frac{1}{j!} f^{(j)}(B_u)(\mathbf{B}^{\mathbf{1}}_{uv})^j - R^1_{uv} - R^2_{uv}, \tag{4.15}$$

with

$$R^1_{uv} = \tfrac{1}{2} f''(B_u)[(\mathbf{B}^{\mathbf{1}}_{uv})^2 - |v-u|^{2H}] \quad \text{and} \quad R^2_{uv} = \tfrac{1}{6} f^{(3)}(B_u)(\mathbf{B}^{\mathbf{1}}_{uv})^3.$$

Plugging (4.15) into the definition of $\tilde{S}^{n,\diamond}$, we thus obtain

$$\begin{aligned} \tilde{S}^{n,\diamond} = S^{\Pi^n_{st}} &- \frac{1}{2} \sum_{k=0}^{2^n-1} f''(B_{t_k})[t^{2H}_{k+1} - t^{2H}_k] + \frac{1}{2} V^{(2)}_n(f'') \\ &+ \frac{1}{6} \tilde{V}^{(3)}_n(f^{(3)}). \end{aligned} \tag{4.16}$$

Now, invoking Proposition 4.5, it is readily checked that both $V^{(2)}_n(f'')$ and $\tilde{V}^{(3)}_n(f^{(3)})$ converge to 0 in $L^2(\Omega)$ as $n \to \infty$. Hence

$$L^2(\Omega) - \lim_{n\to\infty} \tilde{S}^{n,\diamond} = \mathcal{J}_{st}(f'(B)\,dB) - H \int_s^t f''(B_u) u^{2H-1}\,du,$$

which ends the proof. □

The aim of the computations above was to prove that the results of [25] do not contradict ours for $H > 1/4$. Note, however, the following points:

(i) Having a look at Proposition 4.6, one might think that the first order Riemann–Wick sums $\tilde{S}^{n,\diamond}$ are always convergent in $L^2(\Omega)$. However, when $H < 1/4$, relation (4.13) still holds true. This means that the term $V^{(2)}_n(f'')$ appearing in equation (4.16) is now divergent, due to the fact that $2H - 1/2 < 0$. The same kind of arguments also yield the divergence of $\tilde{V}^{(3)}_n(f^{(3)})$ in (4.16). It is thus reasonable to think that first order Riemann–Wick sums will be divergent for $H < 1/4$, justifying our higher order expansions.

(ii) In light of Proposition 4.6, it is, however, possible that expansions of lower order than ours are sufficient to guarantee the convergence of sums like $S^{\Pi_{st},\diamond}$

in Theorem 4.4. We haven't followed this line of investigation for sake of conciseness, but let us stress the fact that almost sure convergences of our Wick–Riemann sums are obtained in Theorems 4.4 and 4.8 (for any sequence of partitions whose mesh tends to 0), while only $L^2(\Omega)$ convergences are considered in Proposition 4.6.

(iii) It is also worth reminding that we aim at considering a general d-dimensional Gaussian process, while [12, 25] focus on 1-dimensional situations. It is an open question for us to know if the methods of the aforementioned papers could be easily adapted to a multidimensional process.

4.4. *Multidimensional case*. We shall now give the representation theorem for Skorohod's integral in the multidimensional case. Technically speaking, this will be an elaboration of the one-dimensional case, relying on tensorization and cumbersome notation.

We first need an analog of Proposition 4.2 in the multidimensional case, whose proof is also postponed to the Appendix. To this aim, let us introduce some additional notation: given $g_1, \dots, g_d \in \mathcal{H}$ define $\bar{g} = (g_1, \dots, g_d)$, and denote by $\mu_{\bar{g}}$ the law in $\mathbb{R}^d$ of the random vector $(I_1(g_1), \dots, I_1(g_d))$.

PROPOSITION 4.7. *Using the notation introduced above, let* $g_1, \dots, g_d, h_1, \dots, , h_d \in \mathcal{H}$. *Consider the random variables* $X_1 = I_1(g_1), \dots, X_d = I_1(g_d)$ *and* $Y_1 = I_1(h_1), \dots, Y_d = I_1(h_d)$. *Suppose that* $Y_1, \dots, Y_d$ *are independent and also that* X_j *and* Y_k *are independent for* $k \neq j$. *Let* $p = (p_1, \dots, p_d)$ *be a multiindex, and set* $|p| = \sum_{j=1}^d p_j$. *Assume that* $G \in \mathcal{C}^{|p|}(\mathbb{R}^d)$ *is such that* $\partial^\alpha G \in L^r(\mu_{\bar{g}})$ *for any multiindex* $\alpha = (\alpha_1, \dots, \alpha_d)$ *and for some* $r > 2$, *with* $\alpha_k \leq p_k$, $k = 1, \dots, d$. *Then* $G(X_1, \dots, X_d) \diamond Y_1^{\diamond p_1} \diamond \cdots \diamond Y_d^{\diamond p_d}$ *is well defined in* $L^2(\Omega)$, *and the following formula holds*:

$$
\begin{aligned}
& G(X_1, \dots, X_d) \diamond Y_1^{\diamond p_1} \diamond \cdots \diamond Y_d^{\diamond p_d} \\
&\quad = \sum_{l_1 + 2m_1 \leq p_1} \cdots \sum_{l_d + 2m_d \leq p_d} \partial^{l_1, \dots, l_d} G(X_1, \dots, X_d) \\
&\qquad \times \prod_{k=1}^d \biggl[\frac{(-1)^{(m_k + l_k)} p_k!}{2^{m_k} m_k! l_k! (p_k - 2m_k - l_k)!} \\
&\qquad\qquad \times \bigl(\mathbf{E}(X_k Y_k)\bigr)^{l_k} \bigl(\mathbf{E}(Y_k^2)\bigr)^{m_k} Y_k^{p_k - 2m_k - l_k} \biggr],
\end{aligned} \tag{4.17}
$$

where $\partial^{j_1, \dots, j_d}$ *denotes* $\frac{\partial^{j_1 + \cdots + j_d}}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$.

As in the one-dimensional case, the proposition above is the key ingredient in order to establish the following representation formula for Skorohod's integral:

THEOREM 4.8. *Let* x *be a* d-*dimensional centered Gaussian process with covariance function fulfilling Hypotheses* 3.1 *and* 3.2, *and assume that* x *also sat-*

isfies Hypothesis 1.1. Let f be a function in $C^{2N}(\mathbb{R}^d)$ such that $\partial_\alpha f$ verifies the growth condition (GC) *for any multiindex α such that $|\alpha| \le 2N$. Then the Skorohod integral $\delta^\diamond(\mathbf{1}_{[s,t)}\nabla f(x))$* (*whose existence is ensured by Theorem 3.16*) *can be represented as* a.s.—$\lim_{\Pi_{st}\to 0} S^{\Pi_{st},\diamond}$, *where $S^{\Pi_{st},\diamond}$ is defined by*

$$S^{\Pi_{st},\diamond} = \sum_{i=0}^{n-1}\sum_{k=1}^{N}\frac{1}{k!}\partial^k_{i_k,\dots,i_1} f(x_{t_i}) \diamond \mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}(i_1) \diamond \cdots \diamond \mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}(i_k).$$

PROOF. We mimic here the proof of Theorem 4.4: decompose first $S^{\Pi_{st},\diamond}$ into $\sum_{i=0}^{n-1}\sum_{u=0}^{N}\sum_{j_1+\cdots+j_d=u}\mathcal{S}^i_{j_1,\dots,j_d}$, where

$$\begin{aligned}\mathcal{S}^i_{j_1,\dots,j_d} = {}& \frac{1}{j_1!\cdots j_d!}\partial^u_{j_1,\dots,j_d} f\big(x_{t_i}(1),\dots,x_{t_i}(d)\big)\\ &\diamond \big(\mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}(1)\big)^{\diamond j_1} \diamond\cdots\diamond \big(\mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}(d)\big)^{\diamond j_d}.\end{aligned}$$

For a fixed $i \in \{0,\dots,n-1\}$ and $k \in \{0,\dots,d\}$, set now

$$X_k = x_{t_i}(k) \quad\text{and}\quad Y_k = \mathbf{x}^{\mathbf{1}}_{t_i t_{i+1}}(k).$$

As in the one-dimensional case, it is readily checked that if $\partial_\alpha f$ satisfies the growth condition (GC) for any $|\alpha| \le 2N$, then the integrability conditions of Proposition 4.7 are also fulfilled for $x_{t_i} = (x_{t_i}(1),\dots,x_{t_i}(d)) = (I_1(\mathbf{1}^{[1]}_{[0,t_i)}),\dots, I_1(\mathbf{1}^{[d]}_{[0,t_i)}))$. This allows us to write

$$\begin{aligned}&\sum_{u=1}^{N}\sum_{j_1+\cdots+j_d=u}\mathcal{S}^i_{j_1,\dots,j_d}\\ &\quad= \sum_{u=1}^{N}\sum_{j_1+\cdots+j_d=u}\sum_{l_1+2m_1\le j_1}\cdots\sum_{l_d+2m_d\le j_d}\partial_{l_1+j_1,\dots,l_d+j_d} f(X_1,\dots,X_d)\\ &\qquad\times\left(\prod_{k=1}^{d}\frac{(-1)^{(m_k+l_k)}}{2^{m_k}m_k!l_k!(j_k-2m_k-l_k)!}\big(\mathbf{E}(X_kY_k)\big)^{l_k}\big(\mathbf{E}(Y_k^2)\big)^{m_k}Y_k^{j_k-2m_k-l_k}\right).\end{aligned}$$

Making the substitution of $l_k + j_k = q_k$ for $k = 1, 2, \dots, d$, or $j_k = q_k - l_k$, the condition $l_k + 2m_k \le j_k$ can be written as $l_k + m_k = u_k$ with $0 \le u_k \le q_k/2$, and therefore the same kind of manipulations as in (4.9) yield

$$\begin{aligned}&\sum_{u=1}^{N}\sum_{j_1+\cdots+j_d=u}\mathcal{S}^i_{j_1,\dots,j_d}\\ &\quad= \sum_{1\le q_1+\cdots+q_d\le 2N}\partial_{q_1,\dots,q_d} f(X_1,\dots,X_d)\\ &\qquad\times\prod_{k=1}^{d}\left(\sum_{u_k=0}^{[q_k/2]}\frac{(-1)^{u_k}}{u_k!(q_k-2u_k)!}\left(\mathbf{E}(X_kY_k)+\frac{\mathbf{E}(Y_k^2)}{2}\right)^{u_k}Y_k^{q_k-2u_k}\right).\end{aligned}$$

Furthermore, like in the proof of Theorem 4.4, we have $\mathbf{E}(X_k Y_k) + \frac{\mathbf{E}(Y_k^2)}{2} = \delta R_{t_i t_{i+1}}/2$. Summing over $i \in \{i, \dots, n-1\}$, we thus end up with

$$
\begin{aligned}
S^{\Pi_{st},\diamond} = {} & \sum_{i=0}^{n-1} \sum_{1 \le q_1 + \cdots + q_d \le N} \partial_{q_1,\dots,q_d} f\big(x_{t_i}(1), \dots, x_{t_i}(d)\big) \prod_{k=1}^{d} \frac{1}{(q_k)!} \big(\mathbf{x}^{\mathbf{1}}_{t_i,t_{i+1}}(k)\big)^{q_k} \\
& + \sum_{i=0}^{n-1} \sum_{N+1 \le q_1 + \cdots + q_d \le 2N} \partial_{q_1,\dots,q_d} f\big(x_{t_i}(1), \dots, x_{t_i}(d)\big) \\
& \times \prod_{k=1}^{d} \frac{1}{(q_k)!} \big(\mathbf{x}^{\mathbf{1}}_{t_i,t_{i+1}}(k)\big)^{q_k} \\
& - \frac{1}{2} \sum_{i=0}^{n-1} \sum_{k=1}^{d} \partial^2_{kk} f\big(x_{t_i}(1), \dots, x_{t_i}(d)\big)(R_{t_{i+1}} - R_{t_i}) + \tilde{\Theta}^{\Pi}_{st} \\
:= {} & \Theta_1^{\Pi_{st}} + \Theta_2^{\Pi_{st}} + \sum_{k=1}^{d} \Theta_{3,k}^{\Pi_{st}} + \tilde{\Theta}^{\Pi_{st}}.
\end{aligned}
$$

In the last sum, the variables Θ correspond to the 3 cases we have distinguished in the proof of Theorem 4.4: $\Theta_1^{\Pi_{st}}$ denotes the sums in which the u_k are equal to 0 and $1 \le q_1 + \cdots + q_d \le N$; $\Theta_2^{\Pi_{st}}$ are the terms with $u_k = 0$ and $N + 1 \le q_1 + \cdots + q_d \le 2N$; $\Theta_{3,k}^{\Pi_{st}}$ corresponds to the terms with $q_k = 2, q_j = 0$ for all $j \ne k$ and $u_k = 1$ (so that $u_j = 0$ if $j \ne k$). Finally, $\tilde{\Theta}^{\Pi_{st}}$ denotes the sums with either $u_1 + \cdots + u_d \ge 2$ or some $u_k = 1$ (and $u_j = 0$ for $j \ne k$) but $q_k \ge 3$. Referring again to the proof of Theorem 4.4, it is then easy to argue that $\tilde{\Theta}^{\Pi_{st}}$ and $\Theta_2^{\Pi_{st}}$ converge to 0, $\Theta_1^{\Pi_{st}}$ converges to $\int_s^t \langle \nabla f(x_\rho), d\mathbf{x}_\rho \rangle_{\mathbb{R}^d}$ and $\sum_{k=1}^d \Theta_{3,k}^{\Pi_{st}}$ converges to $-\frac{1}{2} \int_s^t \Delta f(x_\rho) R'_\rho \, d\rho$. $\square$

APPENDIX

In this Appendix, we prove Propositions 4.2 and 4.7. For this, we will need the following analytical lemma.

LEMMA A.1. *Let μ be a finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $G \in \mathcal{C}^p(\mathbb{R}^d)$ be such that $\partial^\alpha G \in L^r(\mu)$ for some $r \ge 1$ and any multiindex α such that $|\alpha| := \sum_{j=1}^d \alpha_j \le p$. Then, there exists a sequence $(G_n)_{n \in \mathbb{N}}$ such that*:

(1) *Each G_n is a trigonometric polynomial of several variables, that is, $G_n(x_1, \dots, x_d) = \sum_{\text{finite}} a^n_{l_1,\dots,l_d} e^{\imath \xi^n_{l_1} x_1 + \cdots + \imath \xi^n_{l_d} x_d}$, where $a^n_{l_1,\dots,l_d}$ and $\xi^n_{l_j}$ are real numbers.*

(2) *We have $\lim_{n\to\infty} \partial^\alpha G_n = \partial^\alpha G$ in $L^r(\mu)$ for any α such that $|\alpha| \le p$.*

PROOF. This lemma is folklore, but we haven't been able to find it in any standard text book. For this reason and for the sake of completeness, we give here the main ideas of its proof. First, given $G \in \mathcal{C}^p(\mathbb{R}^d)$, there exists a sequence of $\mathcal{C}^\infty(\mathbb{R}^d)$ functions with compact support that converge, jointly with their derivatives, to G in $L^r(\mu)$. So, one only needs to approximate a function $G \in \mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in a rectangle of $\mathbb{R}^d$, say K. Moreover, given $\varepsilon > 0$, we can suppose that $\mu(K^c) < \varepsilon$. For such a function, consider its Fourier partial sums on the rectangle K that converge uniformly, jointly with their derivatives to G and its derivatives. Since these partial sums are periodic functions with the same period, their sup-norm on all $\mathbb{R}^d$ is the same that the sup-norm on the compact K. With these ingredients, the result is easily obtained. $\square$

PROOF OF PROPOSITION 4.2. We start with $G(x) = e^{\imath\xi x}$, for an arbitrary $\xi \in \mathbb{R}$, which means that we wish to evaluate the Wick product $e^{\imath\xi X} \diamond Y^{\diamond p}$.

Recall that $X = I_1(g)$ and $Y = I_1(h)$. For $\xi, \eta \in \mathbb{R}$, consider the random variable

$$\begin{aligned} M(\xi,\eta) &= \exp\left(-\frac{\xi^2}{2}\mathbf{E}(X^2)\right)\mathcal{E}(\imath\xi g + \imath\eta h) \\ &= \exp\left(-\frac{\xi^2}{2}\mathbf{E}(X^2)\right)\mathcal{E}(\imath\xi g) \diamond \mathcal{E}(\imath\eta h) \\ &= \exp(\imath\xi X) \diamond \mathcal{E}(\imath\eta h) = \sum_{p=0}^{\infty} \frac{\imath^p \eta^p}{p!} \exp(\imath\xi X) \diamond Y^{\diamond p}, \end{aligned}$$

where we have invoked relation (4.4) for the second equality and relation (4.3) for the last one. It is thus obvious that $e^{\imath\xi X} \diamond Y^{\diamond p}$ can be expressed as

$$\frac{p!}{\imath^p} \times \text{the coefficient of } \eta^p \text{ in the expansion of } M(\xi,\eta).$$

We now proceed to the following expansion: we have

$$\begin{aligned} M(\xi,\eta) &= \exp\left\{\imath\xi X + \imath\eta Y + \frac{\eta^2}{2}\mathbf{E}(Y^2) + \xi\eta\mathbf{E}(XY)\right\} \\ &= e^{\imath\xi X} \sum_{k=0}^{\infty} \frac{(\imath\eta Y)^k}{k!} \sum_{m=0}^{\infty} \frac{\eta^{2m}}{2^m m!}[\mathbf{E}(Y^2)]^m \sum_{l=0}^{\infty} \frac{\xi^l \eta^l}{l!}[\mathbf{E}(XY)]^l. \end{aligned}$$

Hence, by computing the coefficient of η^p in the above expression, it is easily checked that

$$\begin{aligned} e^{\imath\xi X} \diamond Y^{\diamond p} = e^{\imath\xi X} Y^p + e^{\imath\xi X} \sum_{0<l+2m\le p} & \frac{\imath^{-2m-l}\xi^l p!}{2^m m! l! (p-2m-l)!} \\ & \times [\mathbf{E}(XY)]^l [\mathbf{E}(Y^2)]^m Y^{p-2m-l} \end{aligned}$$

$$= e^{\imath \xi X} Y^p + \sum_{0<l+2m\le p} \left(\frac{d^l}{dx^l} e^{\imath \xi X}\right) \frac{(-1)^{m+l} p!}{2^m m! l! (p-2m-l)!} \times [\mathbf{E}(XY)]^l [\mathbf{E}(Y^2)]^m Y^{p-2m-l},$$

which is the desired formula (4.7) for $G(x) = e^{\imath \xi x}$.

Let us now see how to extend this relation to a more general function G. By linearity, we first obtain the result for any trigonometric polynomial G. Now, let G be such that $G^{(j)} \in L^r(\mu_g)$ for any $j = 0, \dots, p$ and some $r > 2$. By Lemma A.1, there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of trigonometric polynomials such that

$$G_n^{(j)} \to G^{(j)} \qquad \text{in } L^r(\mu_g) \text{ for any } j = 0, \dots, p.$$

This implies that $G(X) \in \mathrm{Dom}(D^p)$ and that

$$D^j G(X) = G^{(j)}(X) g^{\otimes j} \qquad \text{for any } j = 0, \dots, p.$$

Indeed, $G_n(X) \in \mathbf{S}$ and $D^j G_n(X) = G_n^{(j)}(X) g^{\otimes j}$ for any $j = 0, \dots, p$. Moreover, since

$$E\big[\big|G_n^{(j)}(X) - G^{(j)}(X)\big|^r\big] = \|G_n^{(j)} - G^{(j)}\|_{L^r(\mu_g)}^r,$$

we have that

$$D^j G_n(X) \to G^{(j)}(X) g^{\otimes j} \qquad \text{in } L^r(\Omega; \mathcal{H}^{\otimes j}).$$

Using that the $D^{(j)}$ are closed operators, we obtain that $G(X) \in \mathrm{Dom}(D^p)$ and that $D^j G(X) = G^{(j)}(X) g^{\otimes j}$ for any $j = 0, \dots, p$. In particular, owing to Proposition 4.1 we have that

$$G(X) \diamond Y^{\diamond p} = G(X) \diamond I_p(h^{\otimes p}) = \delta^{\diamond p}(G(X) h^{\otimes p}). \tag{A.1}$$

Let us go back now to our approximating sequence $(G_n)_{n\in\mathbb{N}}$. It is readily checked that relation (A.1) also holds for any G_n. Thus, putting together the relation $G_n(X) \diamond I_p(h^{\otimes p}) = \delta^{\diamond p}(G_n(X) h^{\otimes p})$ with equation (4.7) for a trigonometric polynomial, we get that

$$\begin{aligned} \delta^{\diamond p}(G_n(X) g^{\otimes p}) &= G_n(X) Y^p \\ &\quad + \sum_{0<l+2m\le p} \frac{(-1)^{m+l} p!}{2^m m! l! (p-2m-l)!} \\ &\qquad \times G_n^{(l)}(X) [\mathbf{E}(XY)]^l [\mathbf{E}(Y^2)]^m Y^{p-2m-l}. \end{aligned} \tag{A.2}$$

Since $G_n^{(l)}(X) \to G^{(l)}(X)$ in $L^r(\Omega)$ with $r > 2$ and the Y^{k-2m-l} belong to all the $L^q(\Omega)$, the right-hand side of (A.2) converges in $L^2(\Omega)$, as $n \to \infty$, to

$$G(X) Y^p + \sum_{0<l+2m\le p} \frac{(-1)^{m+l} p!}{2^m m! l! (p-2m-l)!} G^{(l)}(X) [\mathbf{E}(XY)]^l [\mathbf{E}(Y^2)]^m Y^{p-2m-l}.$$

Finally we obtain the general case of equation (4.7) by taking limits in both sides of equation (A.2) and by resorting to the closeness of the operator $\delta^{\diamond p}$. $\square$

As in the previous section, the extension of Proposition 4.2 to the multidimensional case is now an elaboration of the previous computations relying on some notational technicalities.

PROOF OF PROPOSITION 4.7. As for Proposition 4.2, we first consider $G(x) = e^{\imath\langle \xi, x\rangle}$, where $x = (x_1, \dots, x_d)$ and $\xi = (\xi_1, \dots, \xi_d)$ are arbitrary vectors in $\mathbb{R}^d$. The extension of the formula to a G satisfying the general integrability conditions of our hypotheses is then obtained following the same approximation scheme as in the one-dimensional case, and is left to the reader for sake of conciseness.

In order to treat the case of $G(x) = e^{\imath\langle \xi, x\rangle}$, set

$$M(\xi, \eta) = \exp\bigl(\imath\langle \xi, X\rangle\bigr) \diamond \exp\left(\imath\langle \eta, Y\rangle + \frac{1}{2}\sum_{k=1}^{d} \eta_k Y_k^2\right).$$

Along the same lines as for Proposition 4.2, one can then identify $e^{\imath\langle \xi, X\rangle} \diamond Y_1^{\diamond p_1} \diamond \cdots \diamond Y_d^{\diamond p_d}$ with $\frac{p_1! \cdots p_d!}{\imath^{p_1+\cdots+p_d}} \times$ the coefficient of $\eta_1^{p_1} \cdots \eta_d^{p_d}$ in the expansion of $M(\xi, \eta)$. Moreover, thanks to relation (4.4), and invoking the fact that X_j and Y_k are independent for $k \neq j$, we get that

$$M(\xi, \eta) = \exp\left(\imath \sum_{k=1}^{d} \xi_k X_k + \imath \sum_{k=1}^{d} \eta_k Y_k + \frac{1}{2}\sum_{k=1}^{d} \eta_k^2 \mathbf{E}(Y_k^2) + \sum_{k=1}^{d} \xi_k \eta_k \mathbf{E}(X_k Y_k)\right).$$

Expanding now the exponential according to formula (4.3), we end up with

$$\begin{aligned} M(\xi, \eta) = \sum_{p_1, \dots, p_d = 0}^{\infty} \Biggl[\sum_{l_1 + 2m_1 \le p_1} \cdots \sum_{l_d + 2m_d \le p_d} \prod_{k=1}^{d} \Biggl\{ &(\imath\xi_k)^{l_k} e^{\imath \xi_k X_k} \\ &\times \frac{\imath^{(p_k - 2m_k - 2l_k)}}{2^{m_k} m_k! l_k! (p_k - 2m_k - l_k)!} \\ &\times \bigl(\mathbf{E}(X_k Y_k)\bigr)^{l_k} \bigl(\mathbf{E}(Y_k^2)\bigr)^{m_k} Y_k^{p_k - 2m_k - l_k} \Biggr\} \Biggr] \eta_1^{p_1} \cdots \eta_d^{p_d}. \end{aligned}$$

Taking into account the fact that

$$\prod_{k=1}^{d} (\imath\xi_k)^{l_k} e^{\imath \sum_{k=1}^{d} \xi_k X_k} = \partial^{l_1, \dots, l_d} e^{\imath \sum_{k=1}^{d} \xi_k X_k},$$

our formula (4.17) is now easily deduced, which ends the proof. $\square$

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