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CONVERGENCE TO THE EQUILIBRIA FOR SELF-STABILIZING PROCESSES IN DOUBLE-WELL LANDSCAPE[1]

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We investigate the convergence of McKean–Vlasov diffusions in a nonconvex landscape. These processes are linked to nonlinear partial differential equations. According to our previous results, there are at least three stationary measures under simple assumptions. Hence, the convergence problem is not classical like in the convex case. By using the method in Benedetto et al. [*J. Statist. Phys.* **91** (1998) 1261–1271] about the monotonicity of the free-energy, and combining this with a complete description of the set of the stationary measures, we prove the global convergence of the self-stabilizing processes.

Introduction. We investigate the weak convergence in long-time of the following so-called self-stabilizing process:

$$\text{(I)} \qquad \begin{cases} X_t = X_0 + \sqrt{\varepsilon} B_t - \int_0^t V'(X_s)\,ds - \int_0^t F' * u_s^\varepsilon(X_s)\,ds, \\ u_s^\varepsilon = \mathcal{L}(X_s). \end{cases}$$

Here, $*$ denotes the convolution. Since the own law of the process intervenes in the drift, this equation is nonlinear, in the sense of McKean. We note that X_t depends on ε. We do not write ε for simplifying the reading.

The motion of the process is generated by three concurrent forces. The first one is the derivative of a potential V—the confining potential. The second influence is a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. It allows the particle to move upwards the potential V. The third term—the so-called self-stabilizing term—represents the attraction between all the others trajectories. Indeed, we remark: $F' * u_s^\varepsilon(X_s(\omega_0)) = \int_{\omega\in\Omega} F'(X_s(\omega_0) - X_s(\omega))\,d\mathbb{P}(\omega)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying measurable space.

This kind of processes were introduced by McKean, see [25] or [24]. Here, we will make some smoothness assumptions on the interaction potential F. Let just note that it is possible to consider nonsmooth F. If F is the Heaviside step function

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and $V := 0$, (I) is the Burgers equation; see [29]. If $F := \delta_0$, and without confining potential, it is the Oelschläger equation, studied in [27].

The particle X_t which verifies (I) can be seen as one particle in a continuous mean-field system of an infinite number of particles. The mean-field system that we will consider is a random dynamical system like

$$
\text{(II)} \qquad \begin{cases} dX_t^1 = \sqrt{\varepsilon}\, dB_t^1 - V'(X_t^1)\, dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^N F'(X_t^1 - X_t^j)\, dt, \\ \vdots \\ dX_t^i = \sqrt{\varepsilon}\, dB_t^i - V'(X_t^i)\, dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^N F'(X_t^i - X_t^j)\, dt, \\ \vdots \\ dX_t^N = \sqrt{\varepsilon}\, dB_t^N - V'(X_t^N)\, dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^N F'(X_t^N - X_t^j)\, dt, \end{cases}
$$

where the N Brownian motions $(B_t^i)_{t\in\mathbb{R}_+}$ are independents. Mean-field systems are the subject of a rich literature; see [13] for the large deviations for $N \to +\infty$ and [26] under weak assumptions on V and F. For applications, see [10] for social interactions or [11] for the stochastic partial differential equations.

The link between the self-stabilizing process and the mean-field system when N goes to $+\infty$ is called the propagation of chaos; see [30] under Lipschitz properties; [4] if V is a constant; [22] or [23] when both potentials are convex; [3] for a more precise result; [7, 12] or [13] for a sharp estimate; [9] for a uniform result in time in the nonuniformly convex case.

Equation (II) can be rewritten in the following way:

$$
\text{(II)} \qquad d\mathcal{X}_t = \sqrt{\varepsilon}\mathcal{B}_t - N\nabla\Upsilon^N(\mathcal{X}_t)\, dt,
$$

where the ith coordinate of $\mathcal{X}_t$ (resp., $\mathcal{B}_t$) is X_t^i (resp., B_t^i) and

$$
\Upsilon^N(\mathcal{X}) := \frac{1}{N}\sum_{j=1}^N V(\mathcal{X}_j) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F(\mathcal{X}_i - \mathcal{X}_j)
$$

for all $\mathcal{X} \in \mathbb{R}^N$. As noted in [33], the potential Υ^N converges toward a functional Υ acting on the measures. A perturbation (proportional to ε) of Υ will play the central role in the article.

As observed in [13], the empirical law of the mean-field system can be seen as a perturbation of the law of the diffusion (I). Consequently, the long-time behavior of $\mathcal{L}(X_t)$ that we study in this paper provides some consequences on the exit time for the particle system (II).

Also, the convergence plays an important role in the exit problem for the self-stabilizing process since the exit time is strongly linked to the drift according to the

Kramers law (see [14] or [16]) which converges toward a homogeneous function if the law of the process converges toward a stationary measure.

Let us recall briefly some of the previous results on diffusions like (I). The existence problem has been investigated by two different methods. The first one consists in the application of a fixed point theorem; see [4, 9, 25] or [16] in the nonconvex case. The other consists of a propagation of chaos; see, for example, [26]. Moreover, it has been proved in Theorem 2.13 in [16] that there is a unique strong solution.

In [25], the author proved—by using Weyl's lemma—that the law of the unique strong solution du_t^ε admits a $\mathcal{C}^\infty$-continuous density u_t^ε with respect to the Lebesgue measure for all $t > 0$. Furthermore, this density satisfies a nonlinear partial differential equation of the following type:

$$\text{(III)} \qquad \frac{\partial}{\partial t}u_t^\varepsilon(x) = \frac{\partial}{\partial x}\left\{\frac{\varepsilon}{2}\frac{\partial}{\partial x}u_t^\varepsilon(x) + u_t^\varepsilon(x)\big(V'(x) + F' * u_t^\varepsilon(x)\big)\right\}.$$

It is then possible to study equations like (III) by probabilistic methods which involve diffusions (I) or (II); see [9, 15, 23]. Reciprocally, equation (III) is a useful tool for characterizing the stationary measure(s) and the long-time behavior; see [4, 5, 31, 32, 34]. In [17], in the nonconvex case, by using (III), it has been proved that the diffusion (I) admits at least three stationary measures under assumptions easy to verify. One is symmetric, and the two others are not. Moreover, Theorem 3.2 in [17] states the thirdness of the stationary measures if V'' is convex and F' is linear. This nonuniqueness prevents the long-time behavior from being as intuitive as in the case of unique stationary measure.

The work in [18] and [19] provides some estimates of the small-noise asymptotic of these three stationary measures. In particular, the convergence toward Dirac measures and its rate of convergence have been investigated. This will be one of the two main tools for obtaining the convergence.

Convergence for (I) is not a new subject. In [5], if V is identically equal to 0, the authors proved the convergence toward the stationary measure by using an ultracontractivity property, a Poincaré inequality and a comparison lemma for stochastic processes. The ultracontractivity property still holds if V is not convex by using the results in [21]. It is possible to conserve the Poincaré inequality by using the theorem of Muckenhoupt (see [1]) instead of the Bakry–Emery theorem. But, the comparison lemma needs some convexity properties. However, it is possible to apply these results if the initial law is symmetric in the synchronized case ($V''(0) + F''(0) \geq 0$); see Theorem 7.10 in [33].

Another method consists of using the propagation of chaos in order to derive the convergence of the self-stabilizing process from the one of the mean-field system. However, we shall use it independently of the time and the classical result which is on a finite interval of time is not sufficiently strong. Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [9] and obtained the convergence in the convex case, including the nonuniformly strictly convex case. See also [23]. Nevertheless, according to Proposition 5.17 and Remark 5.18 in [33], it is impossible to find a general result of uniform propagation of chaos. In the synchronized case, if the initial law is symmetric, it is possible to find such a uniform propagation of chaos; see Theorems 7.11 and 7.12 in [33].

The method that we will use in this paper is based on the one of [6]. See also [2, 20, 22, 23, 31] for the convex case. In the nonconvex case, Carrillo, McCann and Villani provide the convergence in [8] under two restrictions: the center of mass is fixed and $V''(0) + F''(0) > 0$ (that means it is the synchronized case).

However, by combining the results in [17–19] with the work of [6] (and the more rigorous proofs in [8] about the free-energy), we will be able to prove the convergence in a more general setting. The principal tool of the paper is the monotonicity of the free-energy along the trajectories of (III).

First, we introduce the following functional:

$$\Upsilon(u) := \int_{\mathbb{R}} V(x)\,du(x) + \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y)\,du(x)\,du(y). \tag{IV}$$

This quantity appears intuitively as the limit of the potential in (II) for $N \to +\infty$. We consider now the free-energy of the self-stabilizing process (I),

$$\Upsilon_\varepsilon(u) := \frac{\varepsilon}{2}\int_{\mathbb{R}} u(x)\log\bigl(u(x)\bigr)\,dx + \Upsilon(u)$$

for all measures du which are absolutely continuous with respect to the Lebesgue measure. We can note that du_t^ε satisfies this hypothesis for all $t > 0$.

The paper is organized as follows. After presenting the assumptions, we will state the first results, in particular, the convergence of a subsequence $(u_{t_k}^\varepsilon)_k$. This subconvergence will be used for improving the results about the thirdness of the stationary measures. Then, we will give the main statement which is the convergence toward a stationary measure, briefly discuss the assumptions of the theorem and give the proof. Subsequently, we will study the basins of attraction by two different methods and prove that these basins are not reduced to a single point. Finally, we postpone four results in the annex, including Proposition A.2 which extends the classical higher-bound for the moments of the self-stabilizing processes.

Assumptions. We assume the following properties on the confining potential V (see Figure 1):

(V-1) V is an even polynomial function with $\deg(V) =: 2m \geq 4$.

(V-2) The equation $V'(x) = 0$ admits exactly three solutions: a, $-a$ and 0 with $a > 0$. Furthermore, $V''(a) > 0$ and $V''(0) < 0$. Then, the bottoms of the wells are located in $x = a$ and $x = -a$.

(V-3) $V(x) \geq C_4x^4 - C_2x^2$ for all $x \in \mathbb{R}$ with $C_2, C_4 > 0$.

(V-4) $\lim_{x\to\pm\infty} V''(x) = +\infty$ and $V''(x) > 0$ for all $x \geq a$.

(V-5) V'' is convex.

(V-6) Initialization: $V(0) = 0$.

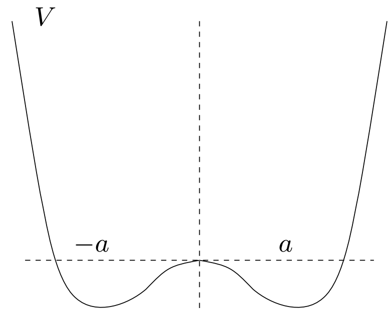

FIG. 1. *Potential V.*

Let us remark that the positivity of V'' on $[-a;a]^c$ [in hypothesis (V-4)] is an immediate consequence of (V-1) and (V-5). The simplest and most studied example is $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$. Also, we would like to stress that weaker assumptions could be considered, but all the mathematical difficulties are present in the polynomial case, and it allows us to avoid some technical and tedious computations. Let us present now the assumptions on the interaction potential F:

(F-1) F is an even polynomial function with $\deg(F) =: 2n \geq 2$.
(F-2) F and F'' are convex.
(F-3) Initialization: $F(0) = 0$.

Under these assumptions, we know by [17] that (I) admits at least one symmetric stationary measure. And, if $\sum_{p=0}^{2n-2} \frac{|F^{(p+2)}(a)|}{p!} a^p < F''(0) + V''(a)$, there are at least two asymmetric stationary measures: u_+^ε and u_-^ε. Furthermore, we know by [18] that there is a unique nonnegative real x_0 such that $V'(x_0) + \frac{1}{2}F'(2x_0) = 0$ and $V''(x_0) + \frac{F''(0)+F''(2x_0)}{2} > 0$. The same paper provides that u_0^ε converges weakly toward $\frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$ and $u_\pm^\varepsilon$ converges weakly toward $\delta_{\pm a}$ in the small-noise limit.

We present now the assumptions on the initial law du_0:

(ES) The $8q^2$th moment of the measure du_0 is finite with $q := \max\{m, n\}$.
(FE) The probability measure du_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. And, the entropy $\int_{\mathbb{R}} u_0 \log(u_0)$ is finite.

Under (ES), (I) admits a unique strong solution. Indeed, the assumptions of Theorem 2.13 in [16] are satisfied: V' and F' are locally Lipschitz, F' is odd, F' grows polynomially, V' is continuously differentiable, and there exists a compact K such that V'' is uniformly negative on K^c. Moreover, we have the following inequality:

$$\max_{1\leq j\leq 8q^2} \sup_{t\in\mathbb{R}_+} \mathbb{E}\left[|X_t|^j\right] \leq M_0. \qquad \text{(V)}$$

We deduce immediately that the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$ is tight. The assumption (FE) ensures that the initial free-energy is finite. In the following, we shall use occasionally one of the following three additional properties concerning the two potentials V and F and the initial law du_0:

(LIN) F' is linear.
(SYN) $V''(0) + F''(0) > 0$.
(FM) For all $N \in \mathbb{N}$, we have $\int_{\mathbb{R}} |x|^N \, du_0(x) < +\infty$.

In the following, three important properties linked to the enumeration of the stationary measures for the self-stabilizing process (I) will be helpful for proving the convergence:

(M3) The process (I) admits exactly three stationary measures. One is symmetric: u_0^ε and the other ones are asymmetric: u_+^ε and u_-^ε. Furthermore, $\Upsilon_\varepsilon(u_+^\varepsilon) = \Upsilon_\varepsilon(u_-^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$.

(M3)$'$ There exists $M > 0$ such that the diffusion (I) admits exactly three stationary measures with free-energy less than M. Furthermore, we have $\Upsilon_\varepsilon(u_+^\varepsilon) = \Upsilon_\varepsilon(u_-^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon) \le M$; u_0^ε is symmetric, and u_+^ε and u_-^ε are asymmetric.

(0M1) The process (I) admits only one symmetric stationary measure u_0^ε.

In the following, we will give some simple conditions such that (M3), (M3)$'$ or (0M1) are true.

Finally, we recall assumption (H) introduced in [18]:

(H) A family of measures $(v^\varepsilon)_\varepsilon$ verifies assumption (H) if the family of positive reals $(\int_{\mathbb{R}} x^{2n} v^\varepsilon(dx))_{\varepsilon>0}$ is bounded.

The aim of the weaker assumption (M3)$'$ is to obtain the convergence even if there exists a family of stationary measures which does not verify the assumption (H).

For concluding the Introduction, we write the statement of the main theorem:

THEOREM. *Let du_0 be a probability measure which verifies* (FE) *and* (FM). *Under* (M3), *u_t^ε converges weakly toward a stationary measure.*

1. First results. This section is devoted to present the tools that we will use for proving the main result of the paper. Furthermore, we provide some new results about the thirdness of the stationary measures for the self-stabilizing processes.

We introduce the following functional:

$$\Upsilon_\varepsilon^-(u) := \frac{\varepsilon}{2} \int_{\mathbb{R}} u(x) \log\bigl(u(x)\bigr) \mathbb{1}_{\{u(x)<1\}} \, dx + \int_{\mathbb{R}} V(x) u(x) \, dx.$$

This new functional is linked to the free-energy Υ_ε. The interaction part and the positive contribution of the entropy term $\int_{\mathbb{R}} u \log(u)$ have been removed. Let us consider a measure u which verifies the previous assumptions. Due to the nonnegativity of the functions F and $x \mapsto \frac{\varepsilon}{2} u(x) \log(u(x)) \mathbb{1}_{\{u(x)\ge 1\}}$, we obtain directly the inequality $\Upsilon_\varepsilon(u) \ge \Upsilon_\varepsilon^-(u)$.

In the following, we will need two particular functions [the free-energy of the system and a function η_t such that $\frac{d}{dt} u_t^\varepsilon(x) = \frac{d}{dx} \eta_t(x)$].

DEFINITION 1.1. For all $t \in \mathbb{R}_+$, we introduce the functions

$$\xi(t) := \Upsilon_\varepsilon(u_t^\varepsilon) \quad \text{and} \quad \eta_t := \frac{\varepsilon}{2}\frac{\partial}{\partial x}u_t^\varepsilon + u_t^\varepsilon(V' + F' * u_t^\varepsilon).$$

According to (III), we remark that if η_t is identically equal to 0, then u_t^ε is a stationary measure for (I).

We recall the following well-known entropy dissipation:

PROPOSITION 1.2. *Let du_0 be a probability measure which verifies* (FE) *and* (ES). *Then, for all $t, s \geq 0$, we have*

$$\xi(t+s) \leq \xi(t) \leq \xi(0) < +\infty.$$

Furthermore, ξ is derivable, and we have

$$\xi'(t) \leq -\int_{\mathbb{R}} \frac{\eta_t^2}{u_t^\varepsilon}.$$

See [8] for a proof.

1.1. *Preliminaries.* Let us introduce the functional space

$$\mathcal{M}_{8q^2} := \left\{ f \in \mathcal{C}_0^2(\mathbb{R}, \mathbb{R}_+) \Big| \int_{\mathbb{R}} f(x)\,dx = 1 \right\}.$$

We can remark that $u_t^\varepsilon \in \mathcal{M}_{8q^2}$ for all $t > 0$; see [25]. The first tool is the Proposition 1.2 [i.e., to say the fact that the free-energy is decreasing along the orbits of (III)]. The second one is its lower-bound.

LEMMA 1.3. *There exists $\Xi_\varepsilon \in \mathbb{R}$ such that* $\inf_{u\in\mathcal{M}_{8q^2}} \Upsilon_\varepsilon(u) \geq \Xi_\varepsilon$.

PROOF. Let us recall $\Upsilon_\varepsilon(u) \geq \Upsilon_\varepsilon^-(u)$. It suffices then to prove the inequality $\inf_{u\in\mathcal{M}_{8q^2}} \Upsilon_\varepsilon^-(u) \geq \Xi_\varepsilon$. We proceed as in the first part of the proof of Theorem 2.1 in [6]. We show that we can minorate the negative part of the entropy by a function of the second moment. Then a growth condition of V will provide the result.

We split the negative part of the entropy into two integrals,

$$-\int_{\mathbb{R}} u(x)\log(u(x))\mathbb{1}_{\{u(x)<1\}}\,dx = -I_+ - I_-$$

with

$$I_+ := \int_{\mathbb{R}} u(x)\log(u(x))\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}\,dx$$

and

$$I_- := \int_{\mathbb{R}} u(x)\log(u(x))\mathbb{1}_{\{u(x)\leq e^{-|x|}\}}\,dx.$$

By definition of I_+, we have the following estimate:

$$I_+ \geq \int_{\mathbb{R}} u(x)\log\bigl(e^{-|x|}\bigr)\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}\,dx$$
$$\geq -\int_{\mathbb{R}} |x|u(x)\mathbb{1}_{\{e^{-|x|}<u(x)<1\}}\,dx$$
$$\geq -\int_{\mathbb{R}} |x|u(x)\,dx \geq -\frac{1}{2}-\frac{1}{2}\int_{\mathbb{R}} x^2u(x)\,dx.$$

By putting $\gamma(x) := \sqrt{x}\log(x)\mathbb{1}_{\{x<1\}}$, a simple computation provides $\gamma(x) \geq -2e^{-1}$ for all $x<1$. We deduce

$$I_- = \int_{\mathbb{R}} \sqrt{u(x)}\gamma\bigl(u(x)\bigr)\mathbb{1}_{\{u(x)\leq e^{-|x|}\}}\,dx \geq -2e^{-1}\int_{\mathbb{R}} e^{-|x|/2}\,dx = -8e^{-1}.$$

Consequently, it yields

$$-\int_{\mathbb{R}} u(x)\log\bigl(u(x)\bigr)\mathbb{1}_{\{u(x)<1\}}\,dx \leq \frac{1}{2}\int_{\mathbb{R}} x^2u(x)\,dx+\frac{1}{2}+8e^{-1}.$$

This implies

$$\Upsilon_\varepsilon^-(u) \geq -\frac{\varepsilon}{4}-4\varepsilon e^{-1}+\int_{\mathbb{R}}\left(V(x)-\frac{\varepsilon}{4}x^2\right)u(x)\,dx. \tag{1.1}$$

By hypothesis, there exist $C_2, C_4>0$ such that $V(x)\geq C_4x^4-C_2x^2$ so the function $x\mapsto V(x)-\frac{\varepsilon}{4}x^2$ is lower-bounded by a negative constant. This achieves the proof. □

Let us note that the unique assumption we used is $\lim_{x\to\pm\infty} V''(x)=+\infty$.

LEMMA 1.4. *Let du_0 be a probability measure which satisfies the assumptions* (FE) *and* (ES). *Then, there exists $L_0\in\mathbb{R}$ such that $\Upsilon_\varepsilon(u_t^\varepsilon)$ converges toward L_0 as time goes to infinity.*

PROOF. The assumption (FE) implies $\xi(0)=\Upsilon_\varepsilon(u_0)<\infty$. As ξ is nonincreasing by Lemma 1.2 and lower-bounded by a constant Ξ_ε according to Lemma 1.3, we deduce that the function ξ converges toward a real L_0. □

LEMMA 1.5. *If and only if $\xi'(t)=0$, the following is true*: *u_t^ε is a stationary measure u^ε.*

PROOF. If u_t^ε is a stationary measure u^ε, then $\xi(t)=\Upsilon_\varepsilon(u_t^\varepsilon)=\Upsilon_\varepsilon(u^\varepsilon)$ is a constant. This provides $\xi'(t)=0$.

Reciprocally, if $\xi'(t)=0$, Proposition 1.2 implies

$$\int_{\mathbb{R}}\frac{\eta_t^2}{u_t^\varepsilon}=0.$$

We deduce $\eta_t(x)=0$ for all $x\in\mathbb{R}$. This means that u_t^ε is a stationary measure. □

1.2. *Subconvergence.*

THEOREM 1.6. *Let du_0 be a probability measure which satisfies the assumptions* (FE) *and* (ES). *Then there exists a stationary measure u^ε and a sequence $(t_k)_k$ which converges toward infinity such that $u^\varepsilon_{t_k}$ converges weakly toward u^ε.*

PROOF. *Plan*: First, we use the convergence of $\int_t^\infty \xi'(s)\,ds$ toward 0 when t goes to infinity, and we deduce the existence of a sequence $(t_k)_k$ such that $\xi'(t_k)$ tends toward 0 when k goes to infinity. Then, we extract a subsequence of $(t_k)_k$ for obtaining an adherence value. By using a test function, we prove that this adherence value is a stationary measure.

Step 1. Lemma 1.4 implies that $\int_t^\infty \xi'(s)\,ds$ collapses at infinity. According to Proposition 1.2, the sign of ξ' is a constant, so we deduce the existence of an increasing sequence $(t_k)_{k\in\mathbb{N}}$ which goes to infinity such that $\xi'(t_k) \longrightarrow 0$.

Step 2. The uniform boundedness of the first $8q^2$ moments with respect to the time allows us to use Prohorov's theorem: we can extract a subsequence [we continue to write it $(t_k)_k$ for simplifying] such that $u^\varepsilon_{t_k}$ converges weakly toward a probability measure u^ε.

Step 3. We consider now a function $\varphi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \cap \mathcal{L}_2(u^\varepsilon)$ with compact support, and we estimate the following quantity:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \varphi(x)\left\{\frac{\varepsilon}{2}\frac{\partial}{\partial x}u^\varepsilon_{t_k}(x) + u^\varepsilon_{t_k}(x)\left[V'(x) + \left(F' * u^\varepsilon_{t_k}\right)(x)\right]\right\}dx\right| \\
&\quad = \left|\int_{\mathbb{R}} \varphi(x)\eta_{t_k}(x)\,dx\right| = \left|\int_{\mathbb{R}} \varphi(x)\sqrt{u^\varepsilon_{t_k}(x)}\frac{|\eta_{t_k}(x)|}{\sqrt{u^\varepsilon_{t_k}(x)}}\,dx\right| \\
&\quad \le \left(\int_{\mathbb{R}} \varphi(x)^2 u^\varepsilon_{t_k}(x)\,dx\right)^{1/2} \times \left(\int_{\mathbb{R}} \frac{1}{u^\varepsilon_{t_k}(x)}\left(\eta_{t_k}(x)\right)^2 dx\right)^{1/2} \\
&\quad \le \sqrt{-\xi'(t_k)}\sqrt{\int_{\mathbb{R}} \varphi(x)^2 u^\varepsilon_{t_k}(x)} \longrightarrow 0
\end{aligned}$$

when k goes to infinity; by using the Cauchy–Schwarz inequality, the hypothesis about the sequence $(t_k)_k$, and the weak convergence of $u^\varepsilon_{t_k}$ toward u^ε. The support of φ is compact, so we can apply an integration by part to the integral $\int_{\mathbb{R}} \varphi(x)\frac{\partial}{\partial x}u^\varepsilon_{t_k}(x)\,dx$. Hence, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(x)\left\{\frac{\varepsilon}{2}\frac{\partial}{\partial x}u^\varepsilon_{t_k}(x) + u^\varepsilon_{t_k}(x)\left[V'(x) + F' * u^\varepsilon_{t_k}(x)\right]\right\}dx \\
&\quad = \int_{\mathbb{R}} \varphi(x)\left[V'(x) + F' * u^\varepsilon_{t_k}(x)\right]u^\varepsilon_{t_k}(x)\,dx - \int_{\mathbb{R}} \frac{\varepsilon}{2}\varphi'(x)u^\varepsilon_{t_k}(x)\,dx.
\end{aligned}$$

The weak convergence of $u^\varepsilon_{t_k}$ to u^ε implies that the previous term tends toward $\int_{\mathbb{R}} \varphi(V' + F' * u^\varepsilon)u^\varepsilon - \int_{\mathbb{R}} \frac{\varepsilon}{2}\varphi' u^\varepsilon$ when k goes to ∞. It has already been proven that $\int_{\mathbb{R}} \varphi\{\frac{\varepsilon}{2}\frac{\partial}{\partial x}u^{\varepsilon}_{t_k} + u^{\varepsilon}_{t_k}(V' + F' * u^{\varepsilon}_{t_k})\}$ is collapsing when k goes to ∞. We deduce the following statement:

$$\int_{\mathbb{R}} \varphi(V' + F' * u^{\varepsilon})u^{\varepsilon} - \int_{\mathbb{R}} \frac{\varepsilon}{2}\varphi' u^{\varepsilon} = 0. \tag{1.2}$$

Step 4. This means that u^{ε} is a weak solution of the equation

$$\frac{\varepsilon}{2}\frac{\partial}{\partial x}u(x) + [V'(x) + F' * u(x)]u(x) = 0.$$

Now, we consider a smooth function $\widetilde{\varphi}$ with compact support $[a, b]$. We put

$$\varphi(x) := \exp\left\{\frac{2}{\varepsilon}[V(x) + F * u^{\varepsilon}(x)]\right\}\widetilde{\varphi}'(x).$$

φ is also a smooth function with compact support. Indeed, the application $x \mapsto F * u^{\varepsilon}(x)$ is a polynomial function parametrized by the moments of u^{ε}, and these moments are bounded. Equality (1.2) becomes

$$\int_{\mathbb{R}} \widetilde{\varphi}''(x)\exp\left\{\frac{2}{\varepsilon}[V(x) + F * u^{\varepsilon}(x)]\right\}u^{\varepsilon}(x)\,dx = 0.$$

By applying Weyl's lemma, we deduce that $\exp[\frac{2}{\varepsilon}(V + F * u^{\varepsilon})]u^{\varepsilon}$ is a smooth function. Moreover, its second derivative is equal to 0. Then, there exists $A, B \in \mathbb{R}$ such that

$$u^{\varepsilon}(x) = (Ax + B)\exp\left[-\frac{2}{\varepsilon}(V(x) + F * u^{\varepsilon}(x))\right]$$

for all $x \in \mathbb{R}$. If $A \neq 0$, it yields $u^{\varepsilon}(-Ax) < 0$ for x big enough. This is impossible. Consequently, $u^{\varepsilon}(x) = Z^{-1}\exp[-\frac{2}{\varepsilon}(V(x) + F * u^{\varepsilon}(x))]$. This means that u^{ε} is a stationary measure. □

DEFINITION 1.7. From now on, we call $\mathcal{A}$ the set of the adherence values of the family $(u^{\varepsilon}_t)_{t\in\mathbb{R}_+}$.

PROPOSITION 1.8. *With the assumptions and the notation of Theorem* 1.6, *we have the following limit*:

$$L_0 := \lim_{t\longrightarrow+\infty} \Upsilon_{\varepsilon}(u^{\varepsilon}_t) = \Upsilon_{\varepsilon}(u^{\varepsilon}).$$

PROOF. The convergence from the quantity $\int_{\mathbb{R}} V u^{\varepsilon}_{t_k} + \frac{1}{2}\int_{\mathbb{R}}(F * u^{\varepsilon}_{t_k})u^{\varepsilon}_{t_k}$ toward $\int_{\mathbb{R}} V u^{\varepsilon} + \frac{1}{2}\int_{\mathbb{R}}(F * u^{\varepsilon})u^{\varepsilon}$ is a consequence of Theorem 1.6. So we focus on the entropy term.

First of all, we aim to prove that $(u^{\varepsilon}_{t_k})_k$ is uniformly bounded in the space $W^{1,1}$. For doing this, we will bound the integral on $\mathbb{R}$ of $\frac{\partial}{\partial x}u^{\varepsilon}_{t_k}(x)$. The triangular inequality provides

$$\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}u^{\varepsilon}_{t_k}\right| \leq \frac{2}{\varepsilon}\int_{\mathbb{R}}|\eta_{t_k}| + \frac{2}{\varepsilon}\int_{\mathbb{R}}|V' + F' * u^{\varepsilon}_{t_k}|u^{\varepsilon}_{t_k},$$

where $t \mapsto \eta_t$ is defined in Definition 1.1. By using (V) and the growth property of V' and F', it yields

$$\int_{\mathbb{R}} \left| V'(x) + F' * u_{t_k}^{\varepsilon}(x) \right| u_{t_k}^{\varepsilon}(x)\, dx \le C_1 \int_{\mathbb{R}} \left(1 + |x^{2q}|\right) u_{t_k}^{\varepsilon}(x)\, dx \le C_2,$$

where C_2 is a constant. By using the Cauchy–Schwarz inequality, like in the proof of Theorem 1.6, we obtain

$$\int_{\mathbb{R}} \left| \eta_{t_k}(x) \right| dx \le \sqrt{-\xi'(t_k)}.$$

The quantity $\sqrt{-\xi'(t_k)}$ tends toward 0, so it is bounded. Finally, it leads to

$$\int_{\mathbb{R}} \left| \frac{\partial}{\partial x} u_{t_k}^{\varepsilon}(x) \right| dx \le C_3,$$

where C_3 is a constant. Consequently, $u_{t_k}^{\varepsilon}(x) \le u_{t_k}^{\varepsilon}(0) + C$ for all $x \in \mathbb{R}$. And, since the sequence $(u_{t_k}^{\varepsilon}(0))_k$ converges, it is bounded, so there exists a constant C_4 such that $u_{t_k}^{\varepsilon}(x) \le C_4$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. It is then easy to prove the convergence of $\int_{\mathbb{R}} u_{t_k}^{\varepsilon}(x) \log(u_{t_k}^{\varepsilon}(x))\, dx$ toward $\int_{\mathbb{R}} u^{\varepsilon}(x) \log(u^{\varepsilon}(x))\, dx$.

Indeed, the application $x \mapsto u_{t_k}^{\varepsilon}(x) \log(u_{t_k}^{\varepsilon}(x))$ is lower-bounded, uniformly with respect to k. We can then apply the Lebesgue theorem which provides the convergence—when k goes to infinity—of $\int_{\mathbb{R}} u_{t_k}^{\varepsilon}(x) \log(u_{t_k}^{\varepsilon}(x)) \mathbb{1}_{\{|x| \le R\}}\, dx$ toward $\int_{\mathbb{R}} u^{\varepsilon}(x) \log(u^{\varepsilon}(x)) \mathbb{1}_{\{|x| \le R\}}\, dx$ for all $R \ge 0$. The other integral is split into two terms. The first one is

$$\begin{aligned}\int_{\mathbb{R}} u_{t_k}^{\varepsilon}(x) \log\left(u_{t_k}^{\varepsilon}(x)\right) \mathbb{1}_{\{|x| > R; u_{t_k}^{\varepsilon}(x) \ge 1\}}\, dx &\le \log(C) u_{t_k}^{\varepsilon}\left([-R; R]^c\right) \\ &\le \frac{\log(C) M_0}{R^2}.\end{aligned}$$

The second term is bounded as in the proof of Lemma 1.3:

$$\begin{aligned}&- \int_{\mathbb{R}} u_{t_k}^{\varepsilon}(x) \log\left(u_{t_k}^{\varepsilon}(x)\right) \mathbb{1}_{\{|x| > R; u_{t_k}^{\varepsilon}(x) < 1\}}\, dx \\ &\quad \le \int_{[-R;R]^c} \left\{ |x| u_{t_k}^{\varepsilon}(x) - \gamma\left(u_{t_k}^{\varepsilon}(x)\right) e^{-|x|/2} \right\} dx \le \frac{M_0}{R} + 4e^{-R/2}.\end{aligned}$$

Consequently, $\Upsilon_{\varepsilon}(u_{t_k}^{\varepsilon})$ converges toward $\Upsilon_{\varepsilon}(u^{\varepsilon})$, then $\Upsilon_{\varepsilon}(u_t^{\varepsilon})$ converges toward $\Upsilon_{\varepsilon}(u^{\varepsilon})$ since the free-energy is monotonous.

By taking R big enough and then k big enough, we can make the following quantity arbitrarily small: $|\int u_{t_k}^{\varepsilon} \log(u_{t_k}^{\varepsilon}) - \int u^{\varepsilon} \log(u^{\varepsilon})|$. $\square$

1.3. *Consequences*. When V is symmetric, Proposition 3.1 (resp., Theorem 4.6) in [17] states the existence of at least three stationary measures for ε small enough if F' is linear [resp., if $\sum_{p=0}^{\infty} \frac{|F^{(p+2)}(a)|}{p!} a^p < F''(0) + V''(a)$]. Theorem 1.6 permits to extend these results.

COROLLARY 1.9. *For ε small enough, process* (I) *admits at least three stationary measures: one is symmetric* (u_0^ε), *and two are asymmetric* (u_+^ε *and* u_-^ε). *Moreover, for sufficiently small* ε, $\Upsilon_\varepsilon(u_+^\varepsilon) = \Upsilon_\varepsilon(u_-^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$.

PROOF. We know by Theorem 4.5 in [17] that there exists a symmetric stationary measure u_0^ε. Theorem 5.4 in [18] implies the weak convergence of u_0^ε toward $\frac{1}{2}(\delta_{x_0} + \delta_{-x_0})$ in the small-noise limit where $x_0 \in [0; a[$ is the unique solution of

$$\begin{cases} V'(x_0) + \dfrac{1}{2}F'(2x_0) = 0, \\ V''(x_0) + \dfrac{F''(0)}{2} + \dfrac{F''(2x_0)}{2} \geq 0. \end{cases}$$

Lemma A.3 provides

$$\lim_{\varepsilon\to 0} \Upsilon_\varepsilon(u_0^\varepsilon) = V(x_0) + \tfrac{1}{4}F(2x_0) \quad \text{and} \quad \lim_{\varepsilon\to 0} \Upsilon_\varepsilon(v_+^\varepsilon) = V(a)$$

with

$$v_+^\varepsilon(x) := Z^{-1}\exp\left[-\frac{2}{\varepsilon}\big(V(x) + F(x-a)\big)\right].$$

We note that $V(x_0) + \frac{1}{4}F(2x_0) > V(a)$. Consequently, for ε small enough, we have $\Upsilon_\varepsilon(v_+^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$.

We consider now process (I) starting by $u_0 := v_+^\varepsilon$. This is possible because the $8q^2$th moment of v_+^ε is finite. Theorem 1.6 implies the existence of a sequence $(t_k)_k$ which goes to infinity such that $u_{t_k}^\varepsilon$ converges weakly toward a stationary measure u^ε satisfying $\Upsilon_\varepsilon(u^\varepsilon) \leq \Upsilon_\varepsilon(u_0) = \Upsilon_\varepsilon(v_+^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$. So $u^\varepsilon \neq u_0^\varepsilon$. We immediately deduce the existence of at least two stationary measures.

If $V''(0) + F''(0) \neq 0$, we know by Theorem 1.6 in [19] that there exists a unique symmetric stationary measure for ε small enough. Hence u^ε is not symmetric.

Let us assume now that $V''(0) + F''(0) = 0$. By (1.1), and by the definition of $\Upsilon_\varepsilon^-(u)$, we have

$$\Upsilon_\varepsilon(u) \geq -\frac{\varepsilon}{4} - 4\varepsilon e^{-1} + \int_{\mathbb{R}} \left\{V(x) - \frac{\varepsilon x^2}{4}\right\} u(x)\,dx + \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y)u(x)u(y)\,dx\,dy$$

for all $u \in \mathcal{M}_{8q^2}$. Since F'' is convex, $x \mapsto F(x) - \frac{F''(0)}{2}x^2$ is nonnegative. It yields

$$\Upsilon_\varepsilon(u) \geq -\frac{\varepsilon}{4} - 4\varepsilon e^{-1} + \int_{\mathbb{R}} \left\{V(x) + \frac{F''(0)}{2}x^2 - \frac{\varepsilon x^2}{4}\right\} u(x)\,dx - \frac{F''(0)}{2}\left(\int_{\mathbb{R}} x u(x)\,dx\right)^2.$$

We deduce the following inequality for all the probability measures satisfying $\int_{\mathbb{R}} xu(x)\,dx = 0$:

$$\Upsilon_\varepsilon(u) \geq -\frac{\varepsilon}{4} - 4\varepsilon e^{-1} + \int_{\mathbb{R}} \left\{ V(x) + \frac{F''(0)}{2} x^2 - \frac{\varepsilon x^2}{4} \right\} u(x)\,dx.$$

In particular, this holds for the symmetric measures. Then, for ε small enough, $\Upsilon_\varepsilon(u) > \frac{V(a)}{2}$ for all the symmetric measures. However, $\Upsilon_\varepsilon(v_+^\varepsilon) < \frac{V(a)}{2}$ [then $\Upsilon_\varepsilon(u^\varepsilon) < \frac{V(a)}{2}$] for ε small enough.

Consequently, the process admits at least one asymmetric stationary measure that we call u_+^ε. The measure $u_-^\varepsilon(x) := u_+^\varepsilon(-x)$ is invariant too. By construction of u_+^ε and u_-^ε, $\Upsilon_\varepsilon(u_+^\varepsilon) = \Upsilon_\varepsilon^-(u_-^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$. □

REMARK 1.10. By a similar method, we could also prove the existence of at least one stationary measure in the asymmetric-landscape case.

We know by Theorem 3.2 in [17] that if V'' is convex and if F' is linear, there are exactly three stationary measures for ε small enough. We present a more general setting. In view of the convergence, we will prove that the number of relevant stationary measures is exactly three even if it is a priori possible to imagine the existence of at least four such measures.

THEOREM 1.11. *We assume* $F''(0) + V''(0) > 0$. *Then, for all* $M > 0$, *there exists* $\varepsilon(M) > 0$ *such that for all* $\varepsilon \leq \varepsilon(M)$, *the number of measures* u *satisfying the two following conditions is exactly three*:

(1) *u is a stationary measure for the diffusion* (I).
(2) $\Upsilon_\varepsilon(u) \leq M$.

Moreover, if $\deg(V) = 2m > 2n = \deg(F)$, *diffusion* (I) *admits exactly three stationary measures for* ε *small enough.*

PROOF. *Plan*. We will begin to prove the second statement (when $m > n$). For doing this, we will use Corollary 1.9 and the results in [18, 19]. Then, we will prove the first statement by using the second one and a minoration of the free-energy for a sequence of stationary measures which does not verify (H).

Step 1. Corollary 1.9 implies the existence of $\varepsilon_0 > 0$ such that process (I) admits at least three stationary measures (one is symmetric, and two are asymmetric) if $\varepsilon < \varepsilon_0$: u_+^ε, u_-^ε and u_0^ε.

Step 2. First, we assume that $\deg(V) > \deg(F)$.

Step 2.1. Proposition 3.1 in [18] implies that each family of stationary measures for the self-stabilizing process (I) verifies condition (H). It has also been shown that under (H), we can extract a subsequence which converges weakly from any family of stationary measures $(u^\varepsilon)_{\varepsilon>0}$ of the diffusion (I).

Step 2.2. Since $F''(0) + V''(0) > 0$, there are three possible limiting values: δ_0, δ_a and δ_{-a} according to Proposition 3.7 and Remark 3.8 in [18].

Step 2.3. As $F''(0) + V''(0) > 0$ and V'' and F'' are convex, there is a unique symmetric stationary measure for ε small enough by Theorem 1.6 in [19]. Also, Theorem 1.6 in [19] implies there are exactly two asymmetric stationary measures for ε small enough. This achieves the proof of the statement.

Step 3. Now, we will prove the first statement. First, if $m > n$, by applying the second statement, the result is obvious. We assume now $m \le n$. Let $M > 0$. All the previous results still hold if we restrict the study to the families of stationary measures which verify condition (H). Consequently, it is sufficient to show the following results in order to achieve the proof of the theorem:

(1) $\sup\{\Upsilon_\varepsilon(u_0^\varepsilon); \Upsilon_\varepsilon(u_+^\varepsilon); \Upsilon_\varepsilon(u_-^\varepsilon)\} < M$ for ε small enough.

(2) If $(u^{\varepsilon_k})_k$ is a sequence of stationary measures, $\int_{\mathbb{R}} x^{2n} u^{\varepsilon_k}(x)\,dx \to \infty$ implies $\Upsilon_{\varepsilon_k}(u^{\varepsilon_k}) \to \infty$.

Step 3.1. Lemma A.3 tells us that $\Upsilon_\varepsilon(u_0^\varepsilon)$ [resp., $\Upsilon_\varepsilon(u_+^\varepsilon) = \Upsilon_\varepsilon(u_-^\varepsilon)$] tends toward 0 [resp., $V(a) < 0$] when ε goes to 0. Hence, the first point is obvious.

Step 3.2. We will prove the second point. We recall lower-bound (1.1),

$$\Upsilon_\varepsilon^-(u) \ge -\frac{\varepsilon}{4} - 4\varepsilon e^{-1} + \int_{\mathbb{R}} \left(V(x) - \frac{\varepsilon}{4} x^2 \right) u(x)\,dx.$$

As $V(x) \ge C_4 x^4 - C_2 x^2$ and $\Upsilon_\varepsilon^-(u) \le \Upsilon_\varepsilon(u)$ for all smooth u, we obtain

$$\Upsilon_\varepsilon(u) \ge \int_{\mathbb{R}} x^2 u(x)\,dx - C,$$

where C is a constant. It is now sufficient to prove that $\int_{\mathbb{R}} x^{2n} u^{\varepsilon_k}(x)\,dx \to \infty$ implies $\int_{\mathbb{R}} x^2 u^{\varepsilon_k}(x)\,dx \to \infty$. We will not write the index k for simplifying the reading. We proceed areductio ad absurdum by assuming the existence of a sequence $(u^\varepsilon)_\varepsilon$ which verifies $\int_{\mathbb{R}} x^{2n} u^\varepsilon(x)\,dx \to \infty$ and $\int_{\mathbb{R}} x^2 u^\varepsilon(x)\,dx \to C_+ \in \mathbb{R}_+$.

Step 3.2.1. By taking the notation of [18], we have the equality $u^\varepsilon(x) = Z^{-1} \exp[-\frac{2}{\varepsilon}(W_\varepsilon(x))]$ with

$$W_\varepsilon(x) := V(x) + F * u^\varepsilon(x) = \sum_{k=1}^{2n} \omega_k(\varepsilon) x^k,$$

$$\omega_k(\varepsilon) := \frac{1}{k!} \left\{ V^{(k)}(0) + (-1)^k \sum_{j \ge k/2}^{2n} \frac{F^{(2j)}(0)}{(2j-k)!} m_{2j-k}(\varepsilon) \right\} \quad \text{and}$$

$$m_l(\varepsilon) := \int_{\mathbb{R}} x^l u^\varepsilon(x)\,dx \qquad \forall l \in \mathbb{N}.$$

We introduce $\omega(\varepsilon) := \sup\{|\omega_k(\varepsilon)|^{1/(2n-k)}; 1 \le k \le 2n\}$.

Step 3.2.2. We note that $\omega_{2n}(\varepsilon) = \frac{V^{(2n)}(0) + F^{(2n)}(0)}{(2n)!} > 0$. Then, $\omega(\varepsilon)$ is uniformly lower-bounded. Consequently, we can divide by $\omega(\varepsilon)$.

Step 3.2.3. The change of variable $x := \omega(\varepsilon)y$ provides

$$\frac{m_{2l}(\varepsilon)}{\omega(\varepsilon)^{2l}} = \frac{\int_{\mathbb{R}} y^{2l} \exp[-(2/\widehat{\varepsilon})\widehat{W}^{\varepsilon}(y)]\,dy}{\int_{\mathbb{R}} \exp[-2/(\widehat{\varepsilon})\widehat{W}^{\varepsilon}(y)]\,dy} \qquad \text{with } \widehat{W}^{\varepsilon}(x) := \sum_{k=1}^{2n} \frac{\omega_k(\varepsilon)}{\omega(\varepsilon)^{2n-k}} x^k$$

for all $l \in \mathbb{N}$, with $\widehat{\varepsilon} := \frac{\varepsilon}{\omega(\varepsilon)^{2n}}$.

Step 3.2.4. The $2n$ sequences $(\frac{\omega_k(\varepsilon)}{\omega(\varepsilon)^{2n-k}})_\varepsilon$ are bounded so we can extract a subsequence of ε (we continue to write ε for simplifying) such that $\frac{\omega_k(\varepsilon)}{\omega(\varepsilon)^{2n-k}}$ converges toward $\widehat{\omega}_k$ when $\varepsilon \to 0$. We put $\widehat{W}(x) := \sum_{k=1}^{2n} \widehat{\omega}_k x^k$. We call $A_1, \dots, A_r$ the $r \geq 1$ location(s) of the global minimum of $\widehat{W}$.

Step 3.2.5. By applying the result of Lemma A.4, we can extract a subsequence (and we continue to denote it by ε) such that $\frac{\int_{\mathbb{R}} y^{2l} \exp[-2/\widehat{\varepsilon}\widehat{W}^{\varepsilon}(y)]\,dy}{\int_{\mathbb{R}} \exp[-2/\widehat{\varepsilon}\widehat{W}^{\varepsilon}(y)]\,dy}$ converges toward $\sum_{j=1}^{r} p_j A_j^{2l}$ where $p_1 + \cdots + p_r = 1$ and $p_j \geq 0$.

Step 3.2.6. If $(\omega(\varepsilon))_\varepsilon$ is bounded, since the quantity $\sum_{j=1}^{r} p_j A_j^{2n}$ is finite, we deduce that $(m_{2n}(\varepsilon))_\varepsilon$ is bounded too. Since $m_{2n}(\varepsilon)$ tends toward infinity when ε goes to 0, we deduce that $(\omega(\varepsilon))_\varepsilon$ converges toward infinity. As $m_2(\varepsilon)$ is bounded, the quantity $\frac{m_2(\varepsilon)}{\omega(\varepsilon)^2}$ vanishes when ε goes to 0. This means $\sum_{j=1}^{r} p_j A_j^2 = 0$ which implies $A_j = 0$ for all $1 \leq j \leq r$. Then $\sum_{j=1}^{r} p_j A_j^{2n} = 0$. Consequently, $m_{2n}(\varepsilon) = o\{\omega(\varepsilon)^{2n}\}$. The Jensen inequality provides $m_k(\varepsilon) = o\{\omega(\varepsilon)^k\}$.

Step 3.2.7. We recall the definition of $\omega_k(\varepsilon)$,

$$\omega_k(\varepsilon) = \frac{1}{k!}\left\{V^{(k)}(0) + (-1)^k \sum_{j \geq k/2}^{2n} \frac{F^{(2j)}(0)}{(2j-k)!} m_{2j-k}(\varepsilon)\right\}.$$

We deduce $\omega_k(\varepsilon) = O\{m_{2n-k}(\varepsilon)\} = o\{\omega(\varepsilon)^{2n-k}\}$. So

$$\omega(\varepsilon) = \sup\{|\omega_k(\varepsilon)|^{1/(2n-k)}; 1 \leq k \leq 2n\} = o\{\omega(\varepsilon)\}.$$

This is a contradiction which achieves the proof. □

This theorem means that—even if the diffusion (I) admits more than three stationary measures—there are only three stationary measures which play a role in the convergence. Indeed, if we take a measure u_0 with a finite free-energy, we know that for ε small enough, there are only three (maybe fewer) stationary measures which can be adherence value of the family $(u_t^\varepsilon)_{t \in \mathbb{R}_+}$.

The assumption (LIN) implies (M3) (and (M3)′ because it is weaker) and (0M1) for ε small enough. The condition (SYN) implies (M3)′ and (0M1) for ε small enough. Furthermore, if $\deg(V) > \deg(F)$, (SYN) implies (M3) when ε is less than a threshold.

This description of the stationary measures permits us to obtain the principal result, that is to say, the long-time convergence of the process.

2. Global convergence.

2.1. *Statement of the theorem.* We write the main result of the paper:

THEOREM 2.1. *Let du_0 be a probability measure which verifies* (FE) *and* (FM). *Under* (M3), u_t^ε *converges weakly toward a stationary measure.*

The proof is postponed in Section 2.3. First, we will discuss briefly the assumptions.

2.2. *Remarks on the assumptions.*

du_0 is absolutely continuous with respect to the Lebesgue measure. We shall use Theorem 1.6 and prove that the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$ admits a unique adherence value. This theorem requires that the initial law is absolutely continuous with respect to the Lebesgue measure. However, it is possible to relax this hypothesis by using the following result (see Lemma 2.1 in [17] for a proof):

Let du_0 be a probability measure which verifies $\int_{\mathbb{R}} x^{8q^2}\, du_0(x) < +\infty$. Then, for all $t > 0$, the probability du_t^ε is absolutely continuous with respect to the Lebesgue measure.

Consequently, it is sufficient to apply Theorem 2.1 to the probability measure u_1^ε since there is a unique solution to the nonlinear equation (I).

The entropy of du_0 is finite. An essential point of the proof is the convergence of the free-energy. To be sure of this, we assume that it is finite at time 0. The assumption about the moments implies $\Upsilon_\varepsilon(u_t^\varepsilon) < +\infty$ if and only if $\int_{\mathbb{R}} u_t^\varepsilon \log(u_t^\varepsilon) < +\infty$.

If V was convex, a little adaptation of the theorem in [28] (taking into account the fact that the drift is not homogeneous here) would provide the nonoptimal following inequality:

$$\Upsilon_\varepsilon(u_t^\varepsilon) \le \frac{1}{2t} \inf\{\sqrt{\mathbb{E}|X-Y|^2}; \mathcal{L}(X) = u_t^\varepsilon; \mathcal{L}(Y) = v_t^\varepsilon\}$$

with

$$v_t^\varepsilon(x) := Z^{-1} \exp\left[-\frac{2}{\varepsilon}\big(V(x) + F * u_t^\varepsilon(x)\big)\right]$$

for all $t > 0$. The second moment of u_t^ε is upper-bounded uniformly with respect to t. By using the convexity of V and F, we can prove the same thing for v_t^ε. Consequently, since $t > 0$, the free-energy is finite so the entropy is finite. However, in this paper, we deal with nonconvex landscape, so we will not relax this hypothesis.

All the moments are finite. Theorem 1.6 tells us that we can extract a sequence from the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$ such that it converges toward a stationary measure. The last step in order to obtain the convergence is the uniqueness of the limiting value. The most difficult part will be to prove this uniqueness when the symmetric stationary measure u_0^ε is an adherence value and the only one of these adherence values to be stationary. To do this, we will consider a function like this one:

$$\Phi(u) := \int_{\mathbb{R}} \varphi(x) u(x)\, dx,$$

where φ is an odd and smooth function with compact support such that $\varphi(x) = x^{2l+1}$ for all x in a compact subset of $\mathbb{R}$. Then, we will prove—by proceeding a *reductio ad absurdum*—that there exists an integer l such that $\Phi(u_0^\varepsilon) \neq \Phi(u_\infty^\varepsilon)$, where u_∞^ε would be another limiting value which is a stationary measure. This inequality will allow us to construct a stationary measure u^ε such that $\Phi(u^\varepsilon) \notin \{\Phi(u_0^\varepsilon); \Phi(u_+^\varepsilon); \Phi(u_-^\varepsilon)\}$. This implies the existence of a stationary measure which does not belong to $\{u_0^\varepsilon; u_+^\varepsilon; u_-^\varepsilon\}$. Under (M3), it is impossible.

We make the integration with an "almost-polynomial" function because we need the square of the derivative of such function to be uniformly bounded with respect to the time.

However, it is possible to relax the condition (FM). Indeed, according to Proposition A.2, if we assume that $\int_{\mathbb{R}} x^{8q^2}\, du_0(x) < +\infty$ (the condition used for the existence of a strong solution), we have

$$\int_{\mathbb{R}} x^{2l} u_t^\varepsilon(x)\, dx < +\infty \qquad \forall t > 0, l \in \mathbb{N}.$$

Hypothesis (M3). As written before, the key for proving the uniqueness of the adherence value is to proceed a reductio ad absurdum and then to construct a stationary measure u^ε such that $\Phi(u^\varepsilon)$ takes a forbidden value [a value different from $\Phi(u_0^\varepsilon)$, $\Phi(u_+^\varepsilon)$ and $\Phi(u_-^\varepsilon)$].

But, it is possible to deal with a weaker hypothesis. Indeed, by considering an initial law with finite free-energy and since the free-energy is decreasing, it is impossible for u_t^ε to converge toward a stationary measure with a higher energy. Consequently, we can consider (M3)$'$ instead of (M3).

All of these remarks allow us to obtain the following result:

THEOREM 2.2. *Let du_0 be a probability measure with finite entropy. If V and F are polynomial functions such that $F''(0) + V''(0) > 0$, u_t^ε converges weakly toward a stationary measure for ε small enough.*

2.3. *Proof of the theorem.* In order to obtain the statement of Theorem 2.1, we will provide two lemmas and one proposition about the free-energy. The lemmas state that a probability measure which verifies simple properties and with a level of energy is necessary a stationary measure for the self-stabilizing process (I). The third one allows us to confine all the adherence values under a level of energy.

LEMMA 2.3. *Under* (M3), *if u is a probability measure which satisfies* (FE) *and* (ES), *the inequality $\Upsilon_\varepsilon(u) \leq \Upsilon_\varepsilon(u_\pm^\varepsilon)$ implies $u \in \{u_+^\varepsilon; u_-^\varepsilon\}$.*

PROOF. Let u be such a measure. We consider the process (I) starting by the initial law $u_0 := u$. Theorem 1.6 implies that there exists a stationary measure u^ε such that $\Upsilon_\varepsilon(u_t^\varepsilon)$ converges toward $\Upsilon_\varepsilon(u^\varepsilon)$.

However, according to Propositions 1.2 and 1.8,

$$\Upsilon_\varepsilon(u^\varepsilon) = \lim_{t\to+\infty} \Upsilon_\varepsilon(u_t^\varepsilon) \leq \Upsilon_\varepsilon(u_t^\varepsilon) \leq \Upsilon_\varepsilon(u) \leq \Upsilon_\varepsilon(u_\pm^\varepsilon).$$

Condition (M3) provides $u^\varepsilon \in \{u_+^\varepsilon; u_-^\varepsilon; u_0^\varepsilon\}$. But, $\Upsilon_\varepsilon(u^\varepsilon) \leq \Upsilon_\varepsilon(u_\pm^\varepsilon) < \Upsilon_\varepsilon(u_0^\varepsilon)$ so $u^\varepsilon \in \{u_+^\varepsilon; u_-^\varepsilon\}$. Without loss of generality, we will assume $u^\varepsilon = u_+^\varepsilon$.

Consequently, the function ξ (see Definition 1.1) is constant. We deduce that $\xi'(t) = 0$ for all $t \geq 0$. Lemma 1.5 implies that u_t^ε is a stationary measure. This means that $u = u_0 = u^\varepsilon = u_+^\varepsilon$. $\square$

We have a similar result with the symmetric measures:

LEMMA 2.4. *Under* (0M1), *if u is a symmetric probability measure satisfying* (FE) *and* (ES), *$\Upsilon_\varepsilon(u) \leq \Upsilon_\varepsilon(u_0^\varepsilon)$ implies $u = u_0^\varepsilon$.*

The key-argument is the following: if the initial law is symmetric, then the law at time t is still symmetric. The proof is similar to the previous one, so it is left to the reader's attention.

Before making the convergence, we need a last result on the adherence values: *the free-energy of a limiting value is less than the limit value of the free-energy.*

PROPOSITION 2.5. *We assume that u_∞^ε is an adherence value of the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$. We call $L_0 := \lim_{t\to+\infty} \Upsilon_\varepsilon(u_t^\varepsilon)$. Then $\Upsilon_\varepsilon(u_\infty^\varepsilon) \leq L_0$.*

PROOF. As u_∞^ε is an adherence value of the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$, there exists an increasing sequence $(t_k)_k$ which goes to infinity such that $u_{t_k}^\varepsilon$ converges weakly toward u_∞^ε. We remark

$$\Upsilon(u_{t_k}^\varepsilon) = V(a) + \int_{\mathbb{R}} (V(x) - V(a)) u_{t_k}^\varepsilon(x)\,dx + \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y) u_{t_k}^\varepsilon(x) u_{t_k}^\varepsilon(y)\,dx\,dy,$$

where the functional Υ is defined in (IV). As $V(x) - V(a) \geq 0$ for all $x \in \mathbb{R}$, the Fatou lemma implies $\Upsilon(u_\infty^\varepsilon) \leq \liminf_{k\to\infty} \Upsilon(u_{t_k}^\varepsilon)$. In the same way,

$$\int_{\mathbb{R}} u_\infty^\varepsilon(x) \log(u_\infty^\varepsilon(x)) \mathbb{1}_{\{u_\infty^\varepsilon(x)\geq 1\}}\,dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}} u_{t_k}^\varepsilon(x) \log(u_{t_k}^\varepsilon(x)) \mathbb{1}_{\{u_{t_k}^\varepsilon(x)\geq 1\}}\,dx.$$

Let $R>0$. By putting $\gamma_k^-(x):=u_{t_k}^\varepsilon(x)\log(u_{t_k}^\varepsilon(x))\mathbb{1}_{\{u_{t_k}^\varepsilon(x)<1\}}\mathbb{1}_{\{|x|\le R\}}$, we note that $|\gamma_k^-(x)|\le e^{-1}\mathbb{1}_{\{|x|\le R\}}$ for all $x\in\mathbb{R}$ and $k\in\mathbb{N}$. We can apply the Lebesgue theorem,

$$\int_{\mathbb{R}} u_\infty^\varepsilon(x)\log(u_\infty^\varepsilon(x))\mathbb{1}_{\{u_\infty^\varepsilon(x)\ge 1\}}\mathbb{1}_{\{|x|\le R\}}=\lim_{k\to\infty}\int_{\mathbb{R}}\gamma_k^-(x)\,dx.$$

We put $\gamma_k^+(x):=u_{t_k}^\varepsilon(x)\log(u_{t_k}^\varepsilon(x))\mathbb{1}_{\{u_{t_k}^\varepsilon(x)<1\}}\mathbb{1}_{\{|x|>R\}}$. By proceeding as in the proof of Lemma 1.3, we have

$$\begin{aligned}-\gamma_k^+(x)&=-u_{t_k}^\varepsilon(x)\log(u_{t_k}^\varepsilon(x))\mathbb{1}_{\{e^{-|x|}\le u_{t_k}^\varepsilon(x)<1\}}\mathbb{1}_{\{|x|>R\}}\\&\quad-u_{t_k}^\varepsilon(x)\log(u_{t_k}^\varepsilon(x))\mathbb{1}_{\{u_{t_k}^\varepsilon(x)<e^{-|x|}\}}\mathbb{1}_{\{|x|>R\}}\\&\le|x|u_{t_k}^\varepsilon(x)\mathbb{1}_{\{|x|>R\}}+2e^{-1}e^{-|x|/2}\mathbb{1}_{\{|x|>R\}}.\end{aligned}$$

Consequently, it leads to the lower-bound

$$\int_{\mathbb{R}} u_{t_k}^\varepsilon(x)\log(u_{t_k}^\varepsilon(x))\mathbb{1}_{\{u_{t_k}^\varepsilon(x)<1\}}\mathbb{1}_{\{|x|>R\}}\,dx\ge-\frac{M_0}{R}-8e^{-1}e^{-R/2},$$

where M_0 is defined in (V).

By introducing $\widehat{\Upsilon}_\varepsilon(u):=\Upsilon_\varepsilon(u)-\frac{\varepsilon}{2}\int_{\mathbb{R}}u(x)\log(u(x))\mathbb{1}_{\{u(x)<1\}}\mathbb{1}_{\{|x|>R\}}\,dx$, we obtain

$$\begin{aligned}\Upsilon_\varepsilon(u_\infty^\varepsilon)\le\widehat{\Upsilon}_\varepsilon(u_\infty^\varepsilon)&\le\liminf_{k\to\infty}\widehat{\Upsilon}_\varepsilon(u_{t_k}^\varepsilon)\\&\le\liminf_{k\to\infty}\Upsilon_\varepsilon(u_{t_k}^\varepsilon)+\frac{M_0\varepsilon}{2R}+4e^{-1}\exp\left(-\frac{R}{2}\right)\varepsilon\\&\le L_0+\frac{M_0\varepsilon}{2R}+4e^{-1}\varepsilon\exp\left(-\frac{R}{2}\right)\end{aligned}$$

for all $R>0$. Consequently, $\Upsilon_\varepsilon(u_\infty^\varepsilon)\le L_0$. $\square$

PROOF OF THE THEOREM. *Plan*: The first step of the proof consists of the application of the Prohorov theorem since the family of measure is tight. We shall prove the uniqueness of the adherence value. We will proceed a reductio ad absurdum. The previous results provide $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}\ne\varnothing$ where $\mathcal{A}$ is introduced in Definition 1.7. We will then study all the possible cases, and we will prove that all of these cases imply contradictions. If $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_+^\varepsilon\}$ and $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_-^\varepsilon\}$ imply contradiction since u_+^ε and u_-^ε are the unique minimizers of the free-energy. The cases $u_0^\varepsilon\in\mathcal{A}$ and $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_+^\varepsilon;u_-^\varepsilon\}$ contradict (M3).

Step 1. Inequality (V) implies that the family of probability measures $\{u_t^\varepsilon;t\in\mathbb{R}_+\}$ is tight. Prohorov's theorem allows us to conclude that each extracted sequence of this family is relatively compact with respect to the weak convergence.

So, in order to prove the statement of the theorem, it is sufficient to prove that this family admits exactly one adherence value. We proceed a reductio ad absurdum. We assume in the following that the family admits at least two adherence values.

Step 2. As condition (M3) is true, there are exactly three stationary measures: u_0^ε, u_+^ε and u_-^ε. By Theorem 1.6, we know that $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}\neq\emptyset$. We split this step into three cases:

- $u_0^\varepsilon\in\mathcal{A}$.
- $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_+^\varepsilon;u_-^\varepsilon\}$.
- $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_+^\varepsilon\}$.

By symmetry, we will not deal with the case $\mathcal{A}\cap\{u_0^\varepsilon;u_+^\varepsilon;u_-^\varepsilon\}=\{u_-^\varepsilon\}$.

Step 2.1. We will prove that the first case, $u_0^\varepsilon\in\mathcal{A}$, is impossible. It will be the core of the proof.

Step 2.1.1. Let u_∞^ε be an other adherence value of the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$. Proposition 2.5 tells us $\Upsilon_\varepsilon(u_\infty^\varepsilon)\leq\Upsilon_\varepsilon(u_0^\varepsilon)$. Since $u_\infty^\varepsilon\neq u_0^\varepsilon$, Lemma 2.4 implies that the law u_∞^ε is not symmetric. We deduce that there exists $l\in\mathbb{N}$ such that $\int_{\mathbb{R}}x^{2l+1}u_\infty^\varepsilon(x)\,dx\neq0$. Let $R>0$. We introduce the function

$$\begin{aligned}\varphi(x):=&\;x^{2l+1}\mathbb{1}_{[-R;R]}(x)\\&+x^{2l+1}\mathbb{1}_{[R;R+1]}(x)Z^{-1}\int_x^{R+1}\exp\left[-\frac{1}{(y-R)^2}-\frac{1}{(y-R-1)^2}\right]dy\\&+x^{2l+1}\mathbb{1}_{[-R-1;-R]}(x)Z^{-1}\int_{-R-1}^{x}\exp\left[-\frac{1}{(y+R)^2}-\frac{1}{(y+R+1)^2}\right]dy\end{aligned}$$

with

$$Z:=\int_0^1\exp\left[-\frac{1}{z^2}-\frac{1}{(z-1)^2}\right]dz.$$

By construction, φ is an odd function, so $\int_{\mathbb{R}}\varphi(x)u_0^\varepsilon(x)\,dx=0$. Furthermore, $|\varphi(x)|\leq|x|^{2l+1}$. By using the triangular inequality and (FM), we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\varphi(x)u_\infty^\varepsilon(x)\,dx\right|&\geq\left|\int_{\mathbb{R}}x^{2l+1}u_\infty^\varepsilon(x)\,dx\right|-\int_{[-R;R]^c}|x|^{2l+1}u_\infty^\varepsilon(x)\,dx\\&\geq\left|\int_{\mathbb{R}}x^{2l+1}u_\infty^\varepsilon(x)\,dx\right|-\frac{1}{R^3}C_0,\end{aligned}$$

where $C_0:=\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}}|x|^{2l+4}u_t^\varepsilon(x)\,dx<+\infty$. Since $\int_{\mathbb{R}}x^{2l+1}u_\infty^\varepsilon(x)\,dx\neq0$, we deduce that $\int_{\mathbb{R}}\varphi(x)u_\infty^\varepsilon(x)\,dx\neq0$ for R big enough. Consequently, we obtain the existence of a smooth function φ with compact support such that

$$0=\int_{\mathbb{R}}\varphi(x)u_0^\varepsilon(x)\,dx<\int_{\mathbb{R}}\varphi(x)u_\infty^\varepsilon(x)\,dx.$$

Moreover, we can verify that $\varphi'(x)^2\leq C(R)x^{4l+2}$ for all $x\in\mathbb{R}$. This implies $\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}}\varphi'(x)^2u_t^\varepsilon(x)\,dx<+\infty$.

Step 2.1.2. Let $\kappa > 0$ such that $|\int_{\mathbb{R}} \varphi(x) u_+^\varepsilon(x)\,dx| > 3\kappa$. By definition of $\mathcal{A}$, there exist two increasing sequences $(t_k^{(1)})_k$ [resp., $(t_k^{(2)})_k$] which go to infinity such that $u^\varepsilon_{t_k^{(1)}}$ [resp., $u^\varepsilon_{t_k^{(2)}}$] converges weakly toward u_0^ε (resp., u_∞^ε). We deduce that there exist two increasing sequences $(r_k)_k$ and $(s_k)_k$ such that $\int_{\mathbb{R}} \varphi(x) u_{r_k}^\varepsilon(x)\,dx = \kappa$ and $\int_{\mathbb{R}} \varphi(x) u_{s_k}^\varepsilon(x)\,dx = 2\kappa$. Then, for all $k \in \mathbb{N}$, we put $\widehat{r_k} := \sup\{t \in [0; s_k] \,|\, \int_{\mathbb{R}} \varphi(x) u_t^\varepsilon(x)\,dx = \kappa\}$, and then we define $\widehat{s_k} := \inf\{s \in [\widehat{r_k}; s_k] \,|\, \int_{\mathbb{R}} \varphi(x) u_s^\varepsilon(x)\,dx = 2\kappa\}$. For simplifying, we write r_k (resp., s_k) instead of $\widehat{r_k}$ (resp., $\widehat{s_k}$). And we have

$$\kappa = \int_{\mathbb{R}} \varphi(x) u_{r_k}^\varepsilon(x)\,dx \le \int_{\mathbb{R}} \varphi(x) u_t^\varepsilon(x)\,dx \le \int_{\mathbb{R}} \varphi(x) u_{s_k}^\varepsilon(x)\,dx = 2\kappa$$

for all $t \in [r_k; s_k]$.

Step 2.1.3. By applying Proposition A.1, we deduce that there exists an increasing sequence $(q_k)_k$ going to $+\infty$ such that $(u_{q_k}^\varepsilon)_k$ converges weakly toward a stationary measure u^ε verifying $\int_{\mathbb{R}} \varphi(x) u^\varepsilon(x)\,dx \in [\kappa; 2\kappa]$. Since we have the inequality $|\int_{\mathbb{R}} \varphi(x) u_+^\varepsilon(x)\,dx| = |\int_{\mathbb{R}} \varphi(x) u_-^\varepsilon(x)\,dx| > 3\kappa$, we deduce $u^\varepsilon = u_0^\varepsilon$. This is impossible since $\int_{\mathbb{R}} \varphi(x) u_0^\varepsilon(x)\,dx = 0 \notin [\kappa; 2\kappa]$.

Step 2.2. We deal now with the third case, $\mathcal{A} \cap \{u_0^\varepsilon; u_+^\varepsilon; u_-^\varepsilon\} = \{u_+^\varepsilon; u_-^\varepsilon\}$.

Step 2.2.1. By definition of u_+^ε and u_-^ε, these measures are not symmetric. Consequently, there exists $l \in \mathbb{N}$ such that $\int_{\mathbb{R}} x^{2l+1} u_+^\varepsilon(x)\,dx \neq 0$. As $u_-^\varepsilon(x) = u_+^\varepsilon(-x)$, by proceeding as in *Step* 2.1, we deduce that there exists an increasing sequence $(q_k)_{k\in\mathbb{N}}$ which goes to ∞ such that $u_{q_k}^\varepsilon$ converges weakly toward a stationary measure u^ε which verifies $\int_{\mathbb{R}} \varphi u^\varepsilon \in [\kappa; 2\kappa]$ where φ is a smooth function with compact support such that $\int_{\mathbb{R}} \varphi u_\pm^\varepsilon \notin [\kappa; 2\kappa]$. We deduce that $u^\varepsilon = u_0^\varepsilon$ which contradicts $u_0^\varepsilon \notin \mathcal{A}$.

Step 2.3. We consider now the last case, $\mathcal{A} \cap \{u_0^\varepsilon; u_+^\varepsilon; u_-^\varepsilon\} = \{u_+^\varepsilon\}$. Proposition 1.8 implies that $\Upsilon_\varepsilon(u_t^\varepsilon)$ converges toward $\Upsilon_\varepsilon(u_+^\varepsilon)$. Let u_∞^ε be a limit value of the family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$ which is not u_+^ε. By Proposition 2.5, we know that $\Upsilon_\varepsilon(u_\infty^\varepsilon) \le \Upsilon_\varepsilon(u_+^\varepsilon) = \lim_{t\longrightarrow+\infty} \Upsilon_\varepsilon(u_t^\varepsilon)$. Then, Lemma 2.3 implies $u_\infty^\varepsilon = u_-^\varepsilon \notin \mathcal{A}$.

CONCLUSION. The family $(u_t^\varepsilon)_{t\in\mathbb{R}_+}$ admits only one adherence value with respect to the weak convergence. So u_t^ε converges weakly toward a stationary measure which achieves the proof. □

3. Basins of attraction.

Now we shall provide some conditions in order to precise the limit.

3.1. *Domain of u_0^ε.*

THEOREM 3.1. *Let du_0 be a symmetric probability measure which verifies* (FE) *and* (ES). *We assume that $V''(0) + F''(0) \neq 0$. Then, for ε small enough u_t^ε converges weakly toward u_0^ε.*

PROOF. $V''(0) + F''(0) \neq 0$, and both functions V'' and F'' are convex. Theorem 1.6 in [19] implies the existence and the uniqueness of a symmetric stationary measure u_0^ε for ε small enough.

Theorem 1.6 provides the existence of a stationary measure u^ε and an increasing sequence $(t_k)_k$ which goes to ∞ such that $u_{t_k}^\varepsilon$ converges weakly toward u^ε and $\Upsilon_\varepsilon(u_t^\varepsilon)$ converges toward $\Upsilon_\varepsilon(u^\varepsilon)$. As u_t^ε is symmetric for all $t \geq 0$, we deduce $u^\varepsilon = u_0^\varepsilon$, the unique symmetric stationary measure.

We proceed a reductio ad absurdum by assuming the existence of another sequence $(s_k)_k$ which goes to ∞ such that $u_{s_k}^\varepsilon$ does not converge toward u_0^ε. The uniform boundedness of the second moment with respect to the time permits to extract a subsequence [we continue to write $(s_k)_k$ for simplifying] such that $u_{s_k}^\varepsilon$ converges weakly toward $u_\infty^\varepsilon \neq u_0^\varepsilon$. Proposition 2.5 implies $\Upsilon_\varepsilon(u_\infty^\varepsilon) \leq \Upsilon_\varepsilon(u_0^\varepsilon)$. Lemma 2.4 implies $u_\infty^\varepsilon = u_0^\varepsilon$. This is absurd. □

REMARK 3.2. We assume $V''(0) + F''(0) \neq 0$ in order to have a unique symmetric stationary measure for ε small enough, that is to say (0M1). We can extend to the case $V''(0) + F''(0) = 0$ by using auniform propagation of chaos; see Theorem 6.5 in [33]. We can also assume that $n = 2$ which means $\deg(F) = 4$ by Section 4.2 in [17].

REMARK 3.3. In the previous theorem, if we assumed (FM) instead of (ES), we could have directly applied Theorem 2.1.

3.2. *Domain of* $u_\pm^\varepsilon$. The principal tool of the previous theorem is the stability of a subset (all the symmetric measures with a finite $8q^2$-moment). If we could find an invariant subset which contains u_+^ε, but neither u_0^ε nor u_-^ε, we could apply the same method than previously.

Instead of this, we will consider an inequality linked to the free-energy and we will exhibit a simple subset included in the domain of attraction of u_+^ε. Let us first introduce the following hyperplan:

$$\mathcal{H} := \left\{ u \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}_+) \,\middle|\, \int_{\mathbb{R}} x^{8q^2} u(x)\,dx < \infty \text{ and } \int_{\mathbb{R}} x u(x)\,dx = 0 \right\}.$$

THEOREM 3.4. *Let du_0 be a probability measure which verifies* (FE) *and* (FM). *We assume also*

$$\Upsilon_\varepsilon(u_0) < \inf_{u \in \mathcal{H}} \Upsilon_\varepsilon(u) \quad \textit{and} \quad \int_{\mathbb{R}} x u_0(x)\,dx > 0.$$

Under (M3), *u_t^ε converges weakly toward u_+^ε.*

PROOF. We know by Theorem 2.1 that there exists a stationary measure u^ε such that $(u_t^\varepsilon)_t$ converges weakly toward u^ε. And, by Proposition 1.8, $\Upsilon_\varepsilon(u_t^\varepsilon)$ converges toward $\Upsilon_\varepsilon(u^\varepsilon)$.

Step 1. As $\int_{\mathbb{R}} x u_0^\varepsilon(x)\,dx = 0$ and $\int_{\mathbb{R}} x^{8q^2} u_0^\varepsilon(x)\,dx < +\infty$, we have

$$\Upsilon_\varepsilon\big(u_0^\varepsilon\big) \ge \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) > \Upsilon_\varepsilon(u_0).$$

We deduce $u^\varepsilon \neq u_0^\varepsilon$ since $t \mapsto \xi(t) = \Upsilon_\varepsilon(u_t^\varepsilon)$ is nonincreasing.

Step 2. We proceed now a reductio ad absurdum by assuming $u^\varepsilon = u_-^\varepsilon$. There exists $t_0 > 0$ such that $\int_{\mathbb{R}} x u_{t_0}^\varepsilon(x)\,dx = 0$. Consequently,

$$\Upsilon_\varepsilon\big(u_{t_0}^\varepsilon\big) \ge \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) > \Upsilon_\varepsilon(u_0),$$

which contradicts the fact that ξ is nonincreasing.

Step 3. Assumption (M3) implies the weak convergence toward u_+^ε. □

We use now Theorem 3.4 in some particular cases.

THEOREM 3.5. *Let du_0 be a probability measure which verifies* (FE) *and* (FM). *We assume also*

$$\Upsilon(u_0) < V(x_0) + \frac{1}{4}F(2x_0) \quad \textit{and} \quad \int_{\mathbb{R}} x u_0(x)\,dx > 0,$$

where x_0 is defined in the Introduction. Under either conditions (LIN) *or* (SYN), *u_t^ε converges weakly toward u_+^ε for ε small enough.*

PROOF. *Step* 1. Theorem 3.2 in [17] and Theorem 1.11 imply condition (M3) under (LIN) or (SYN).

Step 2. Lemma A.3 provides the limit $\lim_{\varepsilon\longrightarrow 0} \Upsilon_\varepsilon(u_0^\varepsilon) = V(x_0) + \frac{1}{4}F(2x_0)$. Then, we deduce

$$\lim_{\varepsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) \le V(x_0) + \frac{1}{4}F(2x_0). \tag{3.1}$$

Step 3. We prove now that $V(x_0) + \frac{1}{4}F(2x_0) = \lim_{\varepsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u)$. Indeed, if u is a probability measure such that $\int_{\mathbb{R}} x u(x)\,dx = 0$, it verifies the following inequality:

$$\begin{aligned}\Upsilon_\varepsilon(u) &\ge \Upsilon_\varepsilon^-(u) + \frac{F''(0)}{4}\iint_{\mathbb{R}^2}(x-y)^2 u(x)u(y)\,dx\,dy\\ &\ge \Upsilon_\varepsilon^-(u) + \frac{F''(0)}{2}\int_{\mathbb{R}} x^2 u(x)\,dx.\end{aligned}$$

By using (1.1), it yields

$$\Upsilon_\varepsilon(u) \ge -\frac{\varepsilon}{4} - \frac{4\varepsilon}{\exp(1)} + \int_{\mathbb{R}}\left\{V(x) + \frac{F''(0)}{2}x^2 - \frac{\varepsilon x^2}{4}\right\}u(x)\,dx. \tag{3.2}$$

We split now the study depending on whether we use conditions (LIN) or (SYN):

(LIN) If F' is linear, $\frac{F''(0)}{2}x^2 = \frac{1}{4}F(2x)$. So the minimum of $x \mapsto V(x) + \frac{1}{4}F(2x)$ is $V(x_0) + \frac{1}{4}F(2x_0)$. We can easily prove that

$$\min_{x\in\mathbb{R}}\left(V(x) + \frac{F''(0)}{2}x^2 - \frac{\varepsilon}{4}x^2\right) = V(x_0) + \frac{1}{4}F(2x_0) + o(1).$$

Consequently,

$$\Upsilon_\varepsilon(u) \geq -\frac{\varepsilon}{4} - \frac{4\varepsilon}{\exp(1)} + V(x_0) + \frac{1}{4}F(2x_0) + o(1)$$

for all $u \in \mathcal{H}$. Then, $\lim_{\varepsilon\longrightarrow 0} \min_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) \geq V(x_0) + \frac{1}{4}F(2x_0)$. Inequality (3.1) provides $\lim_{\varepsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) = V(x_0) + \frac{1}{4}F(2x_0)$.

(SYN) Since $V''(0) + F''(0) > 0$, (3.2) implies $\Upsilon_\varepsilon(u) \geq -\frac{\varepsilon}{4} - \frac{4\varepsilon}{\exp(1)}$ for all $u \in \mathcal{H}$ if ε is less than $2(V''(0) + F''(0))$. We deduce that $\lim_{\varepsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) \geq 0$. However, as $V''(0) + F''(0) > 0$, Theorem 5.4 in [18] implies $x_0 = 0$ so $V(x_0) + \frac{1}{4}F(2x_0) = 0$. Inequality (3.1) provides the following limit: $\lim_{\varepsilon\longrightarrow 0} \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u) = 0 = V(x_0) + \frac{1}{4}F(2x_0)$.

Step 4. Consequently, $\Upsilon_\varepsilon(u_0) < \inf_{u\in\mathcal{H}} \Upsilon_\varepsilon(u)$ for ε small enough. Then, we apply Theorem 3.4. □

REMARK 3.6. We can replace $\int_{\mathbb{R}} xu_0(x)\,dx > 0$ by $\int_{\mathbb{R}} xu_0(x)\,dx < 0$ in Theorems 3.4 and 3.5; then the same results hold with u_-^ε instead of u_+^ε.

APPENDIX: USEFUL TECHNICAL RESULTS

In this annex, we present some results used previously in the proofs of the main theorems.

Proposition A.1 allows us to ensure that even if the free-energy does not reach its global minimum on the stationary measure u_0^ε, if the unique symmetric stationary measure is an adherence value, then it is unique.

Proposition A.2 is a general result on the self-stabilizing processes. Indeed, it is well known that du_t^ε is absolutely continuous with respect to the Lebesgue measure for all $t > 0$. Proposition A.2 extends this instantaneous regularization to the finiteness of all the moments.

Lemma A.3 consists in asymptotic computation of the free-energy in the small-noise limit for some useful measures. Lemma A.4 use a Laplace method for making a tedious computation which is necessary for avoiding to assume that each family of stationary measures verify condition (H).

We present now the essential proposition for proving Theorem 2.1.

PROPOSITION A.1. *Let du_0 be a probability measure which verifies* (FE) *and* (FM). *We assume the existence of two polynomial functions $\mathcal{P}$ and $\mathcal{Q}$, a smooth* *function* φ *with compact support such that* $|\varphi(x)| \leq \mathcal{P}(x)$ *and* $|\varphi'(x)|^2 \leq \mathcal{Q}(x)$, $\kappa > 0$ *and two sequences* $(r_k)_k$ *and* $(s_k)_k$ *which go to* ∞ *such that for all* $r_k \leq t \leq s_k < r_{k+1}$,

$$\kappa = \int_{\mathbb{R}} \varphi(x) u_{r_k}^{\varepsilon}(x)\, dx \leq \int_{\mathbb{R}} \varphi(x) u_t^{\varepsilon}(x)\, dx \leq \int_{\mathbb{R}} \varphi(x) u_{s_k}^{\varepsilon}(x)\, dx = 2\kappa.$$

Then, there exists a stationary measure u^{ε} *which verifies* $\int_{\mathbb{R}} \varphi(x) u^{\varepsilon}(x)\, dx \in [\kappa; 2\kappa]$ *and an increasing sequence* $(q_k)_k$ *which goes to* ∞ *such that* $u_{q_k}^{\varepsilon}$ *converges weakly toward* u^{ε}.

PROOF. *Step* 1. We will prove that $\liminf_{k\longrightarrow+\infty}(s_k - r_k) > 0$. We introduce the function

$$\Phi(t) := \int_{\mathbb{R}} \varphi(x) u_t^{\varepsilon}(x)\, dx.$$

This function is well defined since $|\varphi|$ is bounded by a polynomial function. The derivation of Φ, the use of equation (III) and an integration by parts lead to

$$\begin{aligned}\Phi'(t) &= -\int_{\mathbb{R}} \varphi'(x) \left\{ \frac{\varepsilon}{2} \frac{\partial}{\partial x} u_t^{\varepsilon}(x) + u_t^{\varepsilon}(x) \big(V'(x) + F' * u_t^{\varepsilon}(x) \big) \right\} dx \\ &= -\int_{\mathbb{R}} \varphi'(x) \eta_t(x)\, dx.\end{aligned}$$

The Cauchy–Schwarz inequality implies

$$\big|\Phi'(t)\big| \leq \sqrt{\big|\xi'(t)\big|} \sqrt{\int_{\mathbb{R}} \big(\varphi'(x)\big)^2 u_t^{\varepsilon}(x)\, dx},$$

where we recall that $\xi(t) = \Upsilon_{\varepsilon}(u_t^{\varepsilon})$. The function $(\varphi')^2$ is bounded by a polynomial function, and $\int_{\mathbb{R}} x^{2N} u_t^{\varepsilon}(x)$ is uniformly bounded with respect to $t \in \mathbb{R}_+$ for all $N \in \mathbb{N}$. So, there exists $C > 0$ such that $\int_{\mathbb{R}} (\varphi'(x))^2 u_t^{\varepsilon}(x)\, dx \leq C^2$ for all $t \in \mathbb{R}_+$. We deduce

$$\big|\Phi'(t)\big| \leq C \sqrt{\big|\xi'(t)\big|}. \tag{A.1}$$

By definition of the two sequences $(r_k)_k$ and $(s_k)_k$, we have

$$\Phi(s_k) - \Phi(r_k) = \kappa.$$

Combining this identity with (A.1), it yields

$$C \int_{r_k}^{s_k} \sqrt{\big|\xi'(t)\big|}\, dt \geq \kappa.$$

We apply the Cauchy–Schwarz inequality, and we obtain

$$C\sqrt{s_k - r_k}\sqrt{\xi(r_k) - \xi(s_k)} \geq \kappa$$

since ξ is nonincreasing; see Proposition 1.2. Moreover, $\xi(t)$ converges as t goes to ∞; see Lemma 1.4. It implies the convergence of $\xi(r_k) - \xi(s_k)$ toward 0 when k goes to $+\infty$. Consequently, $s_k - r_k$ converges toward $+\infty$ so $\liminf_{k\longrightarrow+\infty} s_k - r_k > 0$.

Step 2. By Lemma 1.4, $\Upsilon_\varepsilon(u_t^\varepsilon) - \Upsilon_\varepsilon(u^\varepsilon) = \int_t^\infty \xi'(s)\,ds$ converges toward 0. As ξ' is nonpositive, we deduce that $\sum_{k=N}^\infty \int_{r_k}^{s_k} \xi'(s)\,ds$ converges also toward 0 when N goes to $+\infty$. As $\liminf_{k\longrightarrow+\infty} s_k - r_k > 0$, we deduce that there exists an increasing sequence $q_k \in [r_k; s_k]$ which goes to ∞ and such that $\xi'(q_k)$ converges toward 0 when k goes to ∞. Furthermore, $\int_{\mathbb{R}} \varphi(x) u_{q_k}^\varepsilon(x)\,dx \in [\kappa; 2\kappa]$ for all $k \in \mathbb{N}$.

Step 3. By proceeding similarly as in the proof of Theorem 1.6, we extract a subsequence of $(q_k)_k$ (we continue to write it q_k for simplifying the reading) such that $u_{q_k}^\varepsilon$ converges weakly toward a stationary measure u^ε. Moreover, u^ε verifies $\int_{\mathbb{R}} \varphi(x) u^\varepsilon(x)\,dx \in [\kappa; 2\kappa]$. □

We provide now a result which allows us to obtain the statements of the main theorem (Theorem 2.1) with a weaker condition:

PROPOSITION A.2. *Let du_0 be a probability measure which verifies* (FE) *and* (ES). *Then, for all $t > 0$, du_t^ε satisfies* (FM).

PROOF. *Step* 1. If du_0 verifies (FM), then du_t^ε satisfies (FM) for all $t > 0$; see Theorem 2.13 in [16]. We assume now that du_0 does not satisfy (FM). Let us introduce $l_0 := \min\{l \geq 0 | \mathbb{E}[X_0^{2l}]\} = +\infty$. We know that $\mathbb{E}[X_t^{2l_0-2}] < +\infty$ for all $t \geq 0$.

Step 2. Let $t_0 > 0$. We proceed a reductio ad absurdum by assuming that $\mathbb{E}[X_{t_0}^{2l_0}] = +\infty$. This implies directly $\mathbb{E}[X_t^{2l_0}] = +\infty$ for all $t \in [0, t_0]$. We recall that $2m$ (resp., $2n$) is the degree of the confining (resp., interaction) potential V (resp., F). Also, $q := \max\{m; n\}$. For all $t \in [0, t_0]$, the application $x \mapsto F' * u_t^\varepsilon(x)$ is a polynomial function with parameters $m_1(t), \dots, m_{2n-1}(t)$, where $m_j(t)$ is the jth moment of the law du_t^ε. We recall inequality (V),

$$\sup_{1\leq j\leq 8q^2} \sup_{t\in[0,t_0]} m_j(t) \leq M_0.$$

Consequently, the application $x \mapsto V'(x) + F' * u_t^\varepsilon(x)$ is a polynomial function with degree $2q - 1$. Furthermore, the principal term does not depend of the moments of the law du_t^ε, so we can write

$$V'(x) + F' * u_t^\varepsilon(x) = \kappa_{2q-1} x^{2q-1} + \mathcal{P}_t(x),$$

where $\kappa_{2q-1} \in \mathbb{R}_+^*$ is a constant, and $\mathcal{P}_t$ is a polynomial function with degree at most $2q - 2$. Moreover, $\mathcal{P}_t$ is parametrized by the $2n$ first moments only. Let $l \in \mathbb{N}$. We introduce the function $\mathcal{Q}_t(x) := 2l x^{2q-1} \mathcal{P}_t(x) - l(2l-1)\varepsilon x^{2l-2}$. As $\mathcal{Q}_t$ is a polynomial function of degree less than $2l + 2q - 3$, we have the following inequality:

$$2l\kappa_{2q-1} x^{2l+2q-2} + \mathcal{Q}_t(x) \geq C_l \bigl(x^{2l+2q-2} - 1\bigr), \tag{A.2}$$

where C_l is a positive constant. The application of Ito formula provides

$$dX_t^{2l} = 2lX_t^{2l-1}\sqrt{\varepsilon}\,dB_t - \big[2l\kappa_{2q-1}X_t^{2l+2q-2} + \mathcal{Q}_t(X_t)\big]\,dt.$$

After integration, we obtain

$$\begin{aligned} X_{t_0}^{2l} &= X_0^{2l} + 2l\sqrt{\varepsilon}\int_0^{t_0} X_t^{2l-1}\,dB_t - \int_0^{t_0}\big[2l\kappa_{2q-1}X_t^{2l+2q-2} + \mathcal{Q}_t(X_t)\big]\,dt \\ &\le X_0^{2l} + 2l\sqrt{\varepsilon}\int_0^{t_0} X_t^{2l-1}\,dB_t - \int_0^{t_0} C_l\big(X_t^{2l+2q-2} - 1\big)\,dt \end{aligned}$$

after using (A.2). We choose $l := l_0 + 1 - q$, and then we take the expectation. We obtain

$$0 \le \mathbb{E}\big[X_{t_0}^{2l_0+2-2q}\big] \le C_1 - C_2\int_0^{t_0}\mathbb{E}\big[X_t^{2l_0}\big]\,dt,$$

where C_1 and C_2 are positive constants. Since $\mathbb{E}[X_t^{2l_0}] = +\infty$ for all $t \in [0; t_0]$, this contradicts the inequality $0 \le \mathbb{E}[X_{t_0}^{2l_0+2-2q}]$. Consequently, for all $t_0 > 0$: $\mathbb{E}[X_{t_0}^{2l_0}] < +\infty$.

Step 3. Let $T > 0$ and $l_1 \in \mathbb{N}$ such that $l_1 \ge l_0$ where the integer l_0 is defined as previously: $l_0 := \min\{l \ge 0 | \mathbb{E}[X_0^{2l}] = +\infty\}$. If $l_1 = l_0$, the application of *Step* 2 leads to $\mathbb{E}[X_T^{2l_1}] < +\infty$. If $l_1 > l_0$, we put $t_i := \frac{i}{l_1+1-l_0}T$ for all $1 \le i \le l_1 + 1 - l_0$. We apply *Step* 2 to t_1, and we deduce $\mathbb{E}[X_{t_1}^{2l_0}] < +\infty$. By recurrence, we deduce $\mathbb{E}[X_{t_i}^{2l_0+2i}] < +\infty$ for all $1 \le i \le l_1 + 1 - l_0$, in particular $\mathbb{E}[X_{t_{l_0-l_1}}^{2l_0+2(l_1-l_0)}] < +\infty$ that means $\mathbb{E}[X_T^{2l_1}] < +\infty$. This inequality holds for all $l_1 \ge l_0$, so the probability measure du_T^ε satisfies (FM). □

In order to obtain the thirdness of the stationary measure (or a weaker result, see Theorem 1.11), we need to compute the small-noise limit of the free-energy for the stationary measures u_+^ε, u_-^ε and u_0^ε.

LEMMA A.3. *Let ε_0 such that there exist three families of stationary measures $(u_+^\varepsilon)_{\varepsilon\in]0;\varepsilon_0]}$, $(u_-^\varepsilon)_{\varepsilon\in]0;\varepsilon_0]}$ and $(u_0^\varepsilon)_{\varepsilon\in]0;\varepsilon_0]}$ which verify*

$$\lim_{\varepsilon\to 0} u_\pm^\varepsilon = \delta_{\pm a} \qquad \text{and} \qquad \lim_{\varepsilon\to 0} u_0^\varepsilon = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0},$$

where x_0 is defined in the Introduction. Then, we have the following limits:

$$\lim_{\varepsilon\to 0}\Upsilon_\varepsilon\big(u_\pm^\varepsilon\big) = V(a) \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\Upsilon_\varepsilon\big(u_0^\varepsilon\big) = V(x_0) + \frac{1}{4}F(2x_0).$$

Plus, by considering the measure $v_+^\varepsilon(x) := Z^{-1}\exp[-\frac{2}{\varepsilon}(V(x) + F(x-a))]$, we have

$$\lim_{\varepsilon\to 0}\Upsilon_\varepsilon\big(v_+^\varepsilon\big) = V(a).$$

PROOF. *Step* 1. We begin to prove the result for u_0^ε.

Step 1.1. We can write $u_0^\varepsilon(x) = Z^{-1}\exp[-\frac{2}{\varepsilon}(V(x) + F * u_0^\varepsilon(x))]$ since it is a stationary measure. Hence

$$\Upsilon_\varepsilon(u_0^\varepsilon) = -\frac{\varepsilon}{2}\log\biggl(\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx\biggr) - \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y)u_0^\varepsilon(x)u_0^\varepsilon(y)\,dx\,dy.$$

It has been proved in [19] [Theorem 1.2 if $V''(0) + F''(0) > 0$, Theorem 1.4 if $V''(0) + F''(0) = 0$ and Theorem 1.3 if $V''(0) + F''(0) < 0$ applied with $f_{2l}(x) := x^{2l}$] that the $2l$th moment of u_0^ε tends toward x_0^{2l} for all $l \in \mathbb{N}$. Since F is a polynomial function, we deduce the convergence of $\iint_{\mathbb{R}^2} F(x-y)u_0^\varepsilon(x)u_0^\varepsilon(y)\,dx\,dy$ toward $\frac{F(2x_0)}{2}$.

Step 1.2. If $V''(0) + F''(0) \neq 0$, we can apply Lemma A.4 in [19] to $f(x) := 1$ and $U_\varepsilon(x) := V(x) + F * u_0^\varepsilon(x)$. This provides

$$\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx = C_\varepsilon\exp\biggl[-\frac{2}{\varepsilon}\biggl(V(x_0) + \frac{F(2x_0)}{2}\biggr)\biggr],$$

where the constant C_ε verifies $\varepsilon\log(C_\varepsilon) \longrightarrow 0$ in the small-noise limit. We deduce

$$-\frac{\varepsilon}{2}\log\biggl(\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx\biggr) \longrightarrow V(x_0) + \frac{F(2x_0)}{2}$$

when ε collapses. Consequently, it leads to the following limit:

$$\Upsilon_\varepsilon(u_0^\varepsilon) \longrightarrow V(x_0) + \tfrac{1}{4}F(2x_0).$$

Step 1.3. We assume now $V''(0) + F''(0) = 0$. Then $x_0 = 0$ according to Proposition 3.7 and Remark 3.8 in [18]. Propositions 3.5 and 3.6 in [19] imply

$$0 < \liminf_{\varepsilon\to 0}\varepsilon^{1/(2m_0)}\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx$$

and

$$\limsup_{\varepsilon\to 0}\varepsilon^{1/(2m_0)}\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx < +\infty,$$

where $m_0 \in \mathbb{N}^*$ depends only on V and F. We deduce

$$-\frac{\varepsilon}{2}\log\biggl(\int_{\mathbb{R}}\exp\biggl[-\frac{2}{\varepsilon}\bigl(V(x) + F * u_0^\varepsilon(x)\bigr)\biggr]dx\biggr) \longrightarrow 0$$

when ε collapses. Consequently, we obtain the following limit:

$$\Upsilon_\varepsilon(u_0^\varepsilon) \longrightarrow 0 = V(x_0) + \tfrac{1}{4}F(2x_0).$$

Step 2. We prove now the result for u_+^ε (the proof is similar for u_-^ε).

Step 2.1. We can write $u_+^\varepsilon(x) = Z^{-1}\exp[-\frac{2}{\varepsilon}(V(x) + F * u_+^\varepsilon(x))]$ since it is a stationary measure. Hence

$$\Upsilon_\varepsilon(u_+^\varepsilon) = -\frac{\varepsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\varepsilon}\big(V(x) + F * u_+^\varepsilon(x)\big)\right]dx\right) - \frac{1}{2}\iint_{\mathbb{R}^2} F(x-y)u_+^\varepsilon(x)u_+^\varepsilon(y)\,dx\,dy.$$

It has been proved in [19] (Theorem 1.5 applied with $f_l(x) := x^l$) that the lth moment of u_+^ε tends toward a^l for all $l \in \mathbb{N}$. Since F is a polynomial function, we obtain the convergence of $\iint_{\mathbb{R}^2} F(x-y)u_+^\varepsilon(x)u_+^\varepsilon(y)\,dx\,dy$ toward 0.

Step 2.2. Since the second derivative of the application $x \mapsto V(x) + F(x-a)$ in a is positive, we can apply Lemma A.4 in [19] to $f(x) := 1$ and $U_\varepsilon(x) := V(x) + F * u_+^\varepsilon(x)$ [after noting that $U_\varepsilon^{(i)}(x)$ tends toward $V^{(i)}(x) + F^{(i)}(x-a)$ uniformly on each compact for all $i \in \mathbb{N}$]. This provides

$$\int_{\mathbb{R}}\exp\left[-\frac{2}{\varepsilon}\big(V(x) + F * u_+^\varepsilon(x)\big)\right]dx = C_\varepsilon \exp\left[-\frac{2}{\varepsilon}V(a)\right],$$

where the constant C_ε verifies $\varepsilon\log(C_\varepsilon) \longrightarrow 0$ in the small-noise limit. We deduce

$$-\frac{\varepsilon}{2}\log\left(\int_{\mathbb{R}}\exp\left[-\frac{2}{\varepsilon}\big(V(x) + F * u_+^\varepsilon(x)\big)\right]dx\right) \longrightarrow V(a)$$

when $\varepsilon \longrightarrow 0$. Consequently, the following limit holds:

$$\Upsilon_\varepsilon(u_0^\varepsilon) \longrightarrow V(a).$$

Step 3. We proceed similarly for v_+^ε. □

We provide here a useful asymptotic result linked to the Laplace method.

LEMMA A.4. *Let U_k and $U \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ such that for all $i \in \mathbb{N}$, $U_k^{(i)}$ converges toward $U(i)$ uniformly on each compact subset when k goes to $+\infty$. Let $(\varepsilon_k)_k$ be a sequence which converges toward 0 as k goes to $+\infty$. If U has r global minimum locations $A_1 < \cdots < A_r$ and if there exist $R > 0$ and k_c such that $U_k(x) > x^2$ for all $|x| > R$ and $k > k_c$, then, for k big enough, we have*:

(1) *U_k has exactly one global minimum location $A_j^{(k)}$ on each interval I_j, where I_j represents the Voronoï cells corresponding to the central points A_j, with $1 \le j \le r$.*

(2) *$A_j^{(k)}$ tends toward A_j when k goes to $+\infty$.*

Furthermore, for all $N \in \mathbb{N}$, there exist $p_1, \dots, p_r$ which verify $p_1 + \cdots + p_r = 1$ and $p_i \ge 0$ for all $1 \le i \le r$ such that we can extract a subsequence $\psi(k)$ which satisfies

$$\lim_{k\to+\infty}\frac{\int_{\mathbb{R}} x^l \exp[-2/\varepsilon_{\psi(k)}U_{\psi(k)}]\,dx}{\int_{\mathbb{R}}\exp[-2/\varepsilon_{\psi(k)}U_{\psi(k)}]\,dx} = \sum_{j=1}^r p_j A_j^l$$

for all $1 \leq l \leq N$.

PROOF. (1) The first point of the lemma is exactly the one of Lemma A.4 in [19].

(2) Since $U_k(x) \geq x^2$ for $|x| \geq R$ and $k > k_c$, we can confine each $A_j^{(k)}$ in a compact subset. Then, the uniform convergence on all the compact subset implies the convergence of $A_j^{(k)}$ toward A_j when k goes to $+\infty$.

(3) Let $\rho > 0$ arbitrarily small such that $[A_j - \rho, A_j + \rho] \subset I_j$. For obvious reasons, we can extract a subsequence such that

$$\frac{\int_{A_i-\rho}^{A_i+\rho} \exp[-2/\varepsilon_{\psi(k)} U_{\psi(k)}(x)]\,dx}{\sum_{j=1}^{r} \int_{A_j-\rho}^{A_j+\rho} \exp[-2/\varepsilon_{\psi(k)} U_{\psi(k)}(x)]\,dx} \longrightarrow \lambda_i(\rho)$$

with $\lambda_i(\rho) \geq 0$ for all $1 \leq i \leq r$ and $\sum_{j=1}^{r} \lambda_j(\rho) = 1$.

We can note that the generation of the sequence $\psi(k)$ depends on the choice of ρ. Consequently, in the following, we can take ρ arbitrarily small, then $\varepsilon_{\psi(k)}$ arbitrarily small.

As the r families $(\lambda_j(\rho))_{\rho>0}$ are bounded, we can extract a subsequence $(\rho_p)_p$ such that $\lambda_j(\rho_p)$ tends toward λ_j when p goes to $+\infty$. Furthermore, $\lambda_j \geq 0$ for all $1 \leq j \leq r$ and $\sum_{j=1}^{r} \lambda_j = 1$. For simplifying, we will write ρ (resp., k) instead of ρ_p [resp., $\psi(k)$].

We introduce the function $\zeta_l^{(k)}(x) := x^l \exp[-\frac{2}{\varepsilon_k} U_{\varepsilon_k}(x)]$ for all $l \in \mathbb{N}$. By using classical analysis' inequality, we obtain

$$\left| \frac{\int_{\mathbb{R}} \zeta_l^{(k)}(x)\,dx}{\int_{\mathbb{R}} \zeta_0^{(k)}(x)\,dx} - \sum_{j=1}^{r} \lambda_j A_j^l \right| \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 \tag{A.3}$$

with

$$\mathcal{T}_1(\rho) := \left| \sum_{j=1}^{r} (\lambda_j - \lambda_j(\rho)) A_j^l \right|, \qquad \mathcal{T}_2(\rho, R) := \rho l R^{l-1},$$

$$\mathcal{T}_3(\rho, k) := \sum_{j=1}^{r} \frac{\int_{I_j \cap [A_j-\rho, A_j+\rho]^c} \zeta_l^{(k)}(x)\,dx}{\int_{\mathbb{R}} \zeta_0^{(k)}(x)\,dx}, \qquad \mathcal{T}_4(R, k) := 2\frac{\int_R^{+\infty} \zeta_l^{(k)}(x)\,dx}{\int_{\mathbb{R}} \zeta_0^{(k)}(x)\,dx}$$

and

$$\mathcal{T}_5(\rho, R, k) := \sum_{j=1}^{r} |A_j|^l \left| \frac{\int_{A_j-\rho}^{A_j+\rho} \zeta_0^{(k)}(x)\,dx}{\int_{\mathbb{R}} \zeta_0^{(k)}(x)\,dx} - \lambda_j(\rho) \right| \leq \left(\sum_{j=1}^{r} |A_j|^l \right) (\mathcal{T}_3 + \mathcal{T}_4).$$

Let $\tau > 0$ arbitrarily small. We take $R \geq 2$ such that

$$\max_{z \in [A_1-1; A_1+1]} U(z) + 2 < \frac{R^2}{2}.$$

3.1. The convergence of $\lambda_j(\rho)$ toward λ_j implies the existence of $\rho_0 > 0$ such that for all $\rho < \rho_0$, we have

$$\mathcal{T}_1(\rho) \leq \frac{\tau}{5} \tag{A.4}$$

for all $1 \leq l \leq N$.

3.2. By taking $\rho < \min\{\rho_0; \min_{1\leq l\leq N} \frac{\tau}{5lR^{l-1}}\}$, we deduce

$$\mathcal{T}_2(\rho, R) \leq \frac{\tau}{5} \tag{A.5}$$

for all $1 \leq l \leq N$.

3.3. We will prove that the third term tends toward 0. It is sufficient to prove the following convergence:

$$\frac{\int_{I_j\cap[A_j-\rho,A_j+\rho]^c} \zeta_l^{(k)}(x)\,dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)\,dx} \longrightarrow 0$$

for all $1 \leq j \leq r$. Since $I_j \subset [-R, R]$, we have

$$\frac{\int_{I_j\cap[A_j-\rho,A_j+\rho]^c} \zeta_l^{(k)}(x)\,dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)\,dx} \leq R^l \frac{\int_{I_j\cap[A_j-\rho,A_j+\rho]^c} \zeta_0^{(k)}(x)\,dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)\,dx}.$$

Let us prove the convergence toward 0 of the right-hand term:

$$\begin{aligned}
&\frac{\int_{I_j\cap[A_j-\rho,A_j+\rho]^c} \zeta_0^{(k)}(x)\,dx}{\int_{[A_j-\rho,A_j+\rho]} \zeta_0^{(k)}(x)\,dx} \\
&\quad\leq R^{l+1} \frac{\sup\{\zeta_0^{(k)}(z); z \in I_j \cap [A_j-\rho, A_j+\rho]^c\}}{\int_{A_j-\rho/2}^{A_j+\rho/2} \zeta_0^{(k)}(x)\,dx} \\
&\quad\leq \frac{R^{l+1}}{\rho} \frac{\sup\{\zeta_0^{(k)}(z); z \in I_j \cap [A_j-\rho, A_j+\rho]^c\}}{\inf\{\zeta_0^{(k)}(z); z \in [A_j-\rho/2, A_j+\rho/2]\}} \\
&\quad\leq \frac{R^{l+1}}{\rho} \exp\biggl\{-\frac{2}{\varepsilon_k}\Bigl[\inf_{z\in I_j\cap[A_j-\rho,A_j+\rho]^c} U_k(z) - \sup_{z\in[A_j-\rho/2,A_j+\rho/2]} U_k(z)\Bigr]\biggr\}.
\end{aligned}$$

Let $\rho_1 > 0$ such that for all $\rho < \rho_1$, we have

$$\min_{1\leq j\leq r}\Bigl\{\inf_{z\in I_j\cap[A_j-\rho,A_j+\rho]^c} U(z) - \sup_{z\in[A_j-\rho/2,A_j+\rho/2]} U(z)\Bigr\} \geq \delta > 0.$$

We take $\rho < \min\{\rho_0, \rho_1, \min_{1\leq l\leq N} \frac{\tau}{5lR^{l-1}}\}$. As U_k converges uniformly toward U on all the compact subset, we deduce that for $k \geq k_0$, we have

$$\mathcal{T}_3(\rho, k) \leq \frac{\tau}{5(1 + \max_{1\leq l\leq N} \sum_{j=1}^r |A_j|^l)} \tag{A.6}$$

for all $1 \leq l \leq N$.

3.4. By using the growth property on U_k then the change of variable $x := \sqrt{\varepsilon_k}y$, it yields

$$\int_R^{+\infty} \zeta_l^{(k)}(x) \leq \int_R^{+\infty} x^l \exp\left[-\frac{2}{\varepsilon_k}x^2\right] dx \leq C(l)e^{-R^2/\varepsilon_k}\varepsilon_k^{(l+1)/2},$$

where $C(l)$ is a constant. We recall the assumption $\max_{z\in[A_1-1;A_1+1]} U(z) + 2 < \frac{R^2}{2}$. Since U_k converges toward U uniformly on each compact subset, we have $\max_{z\in[A_1-1;A_1+1]} U_k(z) + 1 < \frac{R^2}{2}$ for $k \geq k_1$ (independently of ρ). Consequently,

$$\begin{aligned}\mathcal{T}_4(R,k) &\leq \frac{2C(l)\varepsilon_k^{(l+1)/2}\exp[-2/\varepsilon_k(\max_{z\in[A_1-1;A_1+1]} U_k(z)+1)]}{\int_{A_1-1}^{A_1+1}\exp[-2/\varepsilon_k U_k(z)]\,dx} \\ &\leq C(l)\varepsilon_k^{(l+1)/2}\exp\left[-\frac{2}{\varepsilon_k}\right].\end{aligned}$$

For $k \geq k_2$, we have the inequality

$$\varepsilon_k^{(l+1)/2}\exp\left[-\frac{2}{\varepsilon_k}\right] \leq \frac{\tau}{5\max_{1\leq l\leq N} C(l)\times(1+\max_{1\leq l\leq N}\sum_{j=1}^r |A_j|^l)}.$$

By taking $k \geq \max\{k_0, k_1, k_2\}$, we obtain

$$\mathcal{T}_4(R,k) \leq \frac{\tau}{5(1+\max_{1\leq l\leq N}\sum_{j=1}^r |A_j|^l)} \tag{A.7}$$

for all $1 \leq l \leq N$.

3.5. By taking $\rho < \min\{\rho_0, \rho_1, \frac{\tau}{5lR^{l-1}}\}$ and $k \geq \max\{k_0, k_1, k_2\}$, inequalities (A.3)–(A.6) and (A.7) provide

$$\left|\frac{\int_{\mathbb{R}}\zeta_l^{(k)}(x)\,dx}{\int_{\mathbb{R}}\zeta_0^{(k)}(x)\,dx} - \sum_{j=1}^r \lambda_j A_j^l\right| < \tau$$

for all $1 \leq l \leq N$. This achieves the proof. □

REMARK A.5. This lemma seems weaker than Lemma A.4 in [19]. However, here, we do not assume that the second derivative of U is positive in all the global minimum locations.

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REFERENCES

[1] ANÉ, C., BLACHÈRE, S., CHAFAÏ, D., FOUGÈRES, P., GENTIL, I., MALRIEU, F., ROBERTO, C. and SCHEFFER, G. (2000). *Sur les Inégalités de Sobolev Logarithmiques*. *Panoramas et Synthèses* [*Panoramas and Syntheses*] **10**. Société Mathématique de France, Paris. MR1845806

[2] ARNOLD, A., MARKOWICH, P., TOSCANI, G. and UNTERREITER, A. (2001). On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations. *Comm. Partial Differential Equations* **26** 43–100. MR1842428

[3] BEN AROUS, G. and ZEITOUNI, O. (1999). Increasing propagation of chaos for mean field models. *Ann. Inst. Henri Poincaré Probab. Stat.* **35** 85–102. MR1669916

[4] BENACHOUR, S., ROYNETTE, B., TALAY, D. and VALLOIS, P. (1998). Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.* **75** 173–201. MR1632193

[5] BENACHOUR, S., ROYNETTE, B. and VALLOIS, P. (1998). Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.* **75** 203–224. MR1632197

[6] BENEDETTO, D., CAGLIOTI, E., CARRILLO, J. A. and PULVIRENTI, M. (1998). A non-Maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91** 979–990. MR1637274

[7] BOLLEY, F., GUILLIN, A. and VILLANI, C. (2007). Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields* **137** 541–593. MR2280433

[8] CARRILLO, J. A., MCCANN, R. J. and VILLANI, C. (2003). Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.* **19** 971–1018. MR2053570

[9] CATTIAUX, P., GUILLIN, A. and MALRIEU, F. (2008). Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** 19–40. MR2357669

[10] COLLET, F., DAI PRA, P. and SARTORI, E. (2010). A simple mean field model for social interactions: Dynamics, fluctuations, criticality. *J. Stat. Phys.* **139** 820–858. MR2639892

[11] CRISAN, D. and XIONG, J. (2010). Approximate McKean–Vlasov representations for a class of SPDEs. *Stochastics* **82** 53–68. MR2677539

[12] DAI PRA, P. and DEN HOLLANDER, F. (1996). McKean–Vlasov limit for interacting random processes in random media. *J. Stat. Phys.* **84** 735–772. MR1400186

[13] DAWSON, D. A. and GÄRTNER, J. (1987). Large deviations from the McKean–Vlasov limit for weakly interacting diffusions. *Stochastics* **20** 247–308. MR0885876

[14] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications*. *Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. MR2571413

[15] FUNAKI, T. (1984). A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete* **67** 331–348. MR0762085

[16] HERRMANN, S., IMKELLER, P. and PEITHMANN, D. (2008). Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.* **18** 1379–1423. MR2434175

[17] HERRMANN, S. and TUGAUT, J. (2010). Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Process. Appl.* **120** 1215–1246. MR2639745

[18] HERRMANN, S. and TUGAUT, J. (2010). Stationary measures for self-stabilizing processes: Asymptotic analysis in the small noise limit. *Electron. J. Probab.* **15** 2087–2116. MR2745727

[19] HERRMANN, S. and TUGAUT, J. (2012). Self-stabilizing processes: Uniqueness problem for stationary measures and convergence rate in the small-noise limit. *ESAIM Probab. Stat.* **16** 277–305. MR2956575

[20] HOLLEY, R. and STROOCK, D. (1987). Logarithmic Sobolev inequalities and stochastic Ising models. *J. Stat. Phys.* **46** 1159–1194. MR0893137

[21] KAVIAN, O., KERKYACHARIAN, G. and ROYNETTE, B. (1993). Quelques remarques sur l'ultracontractivité. *J. Funct. Anal.* **111** 155–196. MR1200640

[22] MALRIEU, F. (2001). Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.* **95** 109–132. MR1847094

[23] MALRIEU, F. (2003). Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.* **13** 540–560. MR1970276

[24] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595

[25] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ.*, 1967) 41–57. Air Force Office Sci. Res., Arlington, VA. MR0233437

[26] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299

[27] OELSCHLÄGER, K. (1985). A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete* **69** 279–322. MR0779460

[28] OTTO, F. and VILLANI, C. (2001). Comment on "Hypercontractivity of Hamilton–Jacobi equations," by S. G. Bobkov, I. Gentil and M. Ledoux. *J. Math. Pures Appl.* (9) **80** 697–700. MR1846021

[29] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[30] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*–1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185

[31] TAMURA, Y. (1984). On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **31** 195–221. MR0743525

[32] TAMURA, Y. (1987). Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** 443–484. MR0914029

[33] TUGAUT, J. (2010). Processus autostabilisants dans un paysage multi-puits. Ph.D. thesis, Univ. Henri Poincaré, Nancy. Available at http://tel.archives-ouvertes.fr/tel-00573044/fr/.

[34] VERETENNIKOV, A. Y. (2006). On ergodic measures for McKean–Vlasov stochastic equations. In *Monte Carlo and Quasi-Monte Carlo Methods* 2004 471–486. Springer, Berlin. MR2208726

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