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SLE CURVES AND NATURAL PARAMETRIZATION

BY GREGORY F. LAWLER[1] AND WANG ZHOU[2]

University of Chicago and National University of Singapore

Developing the theory of two-sided radial and chordal *SLE*, we prove that the natural parametrization on SLE_κ curves is well defined for all $\kappa < 8$. Our proof uses a two-interior-point local martingale.

1. Introduction.

1.1. *Background and motivation.* Suppose that x_j, $j = 1, 2, \dots$, are independent and identically distributed random vectors in $\mathbb{Z}^2$ with probabilities

$$\mathbb{P}\{x_j = \mathrm{e}\} = 1/4, \qquad |\mathrm{e}| = 1.$$

It is well known that the scaled simple random walk in $\mathbb{R}^2$,

$$B_t^{(n)} = n^{-1/2} \sum_{j=1}^{[nt]} x_j, \qquad 0 \le t \le 1,$$

converges to a standard two-dimensional Brownian motion B_t, $0 \le t \le 1$, as $n \to \infty$. In the scaled walk, each step is traversed in the same amount of time. When passing to the scaling limit, this parameter t becomes the natural parametrization of Brownian motion. We can write the scaling factor as $n^{-1/d}$, where $d = 2$ is the fractal dimension of the Brownian paths. The natural parametrization is a d-dimensional measure.

One variant of simple random walk is the loop erased random walk first appeared in [8]. Its definition is as follows. Consider any finite or recurrent connected graph G, one vertex a and a set of vertices V. Loop-erased random walk (LERW) from a to V is a random simple curve joining a to V obtained by erasing the loops in chronological order from a simple random walk started at a and stopped upon hitting V. One can ask whether or not there is a corresponding result for LERW where the scaling factor is $n^{-1/d}$ and d is the fractal dimension of the paths. Schramm [13] introduced a process, now called the Schramm–Loewner evolution (SLE_κ), as a candidate for the scaling limit and gave a strong argument why

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SLE_2 should be the scaling limit of LERW. In order to use the Loewner equation, he used a capacity parametrization which is not the parametrization one would obtain by taking the limit above. In [10], it was proved that the scaling limit of planar LERW *in the capacity parametrization* is SLE_2. However, it is still open whether or not, one can take a limit as above. There are a number of other models that are known to converge to SLE_κ in the scaling limit using the capacity parametrization: critical site-percolation on the triangular lattice [16], the level lines of the discrete Gaussian free field [14], the interfaces of the random cluster model associated with the Ising model [17].

A start to taking limits as above is to define the natural parametrization for *SLE*. Possible definitions and constructions for a natural parametrization were suggested in [11]. As well as giving conjectures, one definition was proposed in terms of the Doob–Meyer decomposition of a path. We review this construction in Section 1.3. Although they conjectured that this definition is valid for all $\kappa < 8$, they were only able to establish the result for $\kappa < \kappa_0 = 4(7 - \sqrt{33})$. The technical problem came from difficult second moment estimates for the reverse Loewner flow.

In this paper, we prove that the definition in [11] is valid for $\kappa < 8$. Instead of using the reverse Loewner flow, we use a difficult estimate of Beffara [1] on the forward Loewner flow to establish the necessary uniform integrability to apply the Doob–Meyer theorem. Beffara's estimate was the key step in his proof of the Hausdorff dimension of SLE_κ curves. This estimate has recently been improved [6] and used to establish a multi-point Green's function for SLE_κ. We use this Green's function to give an appropriate two-interior-point local martingale. By establishing a correlation inequality for this Green's function, we are able to give a relatively simple proof of the existence of the natural parametrization.

1.2. *Notation.* In this subsection, we set up the notation for SLE_κ. For more background, see, for example, [2, 4, 7, 9, 18].

Throughout this paper, we let $\kappa < 8$ and $a = 2/\kappa > 1/4$, and we allow all constants to depend on κ. We let $d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}$ be the Hausdorff dimension of the paths. We parametrize the maps so that

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \tag{1}$$

where $U_t = -B_t$ is a standard Brownian motion. It can be shown [10, 12] that a.s. g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ for every $t \geq 0$ and $\gamma(t) := g_t^{-1}(U_t)$ is a continuous curve which is the *SLE* path. The domain of definition of g_t is the unbounded connected component H_t of $\mathbb{H} \setminus \gamma[0, t]$. We shall denote by K_t the closure of the complement of H_t in $\mathbb{H}$. If $z \in \overline{\mathbb{H}} \setminus \{0\}$, let

$$Z_t(z) = X_t(z) + iY_t(z) = g_t(z) + B_t.$$

Then the Loewner equation can be written as

$$dZ_t(z) = \frac{a}{Z_t(z)}\,dt + dB_t,$$

$$dX_t(z) = \frac{aX_t(z)}{|Z_t(z)|^2}\,dt + dB_t, \qquad \partial_t Y_t(z) = -\frac{aY_t}{|Z_t|^2}.$$

The Loewner equation is valid up to the time

$$T_z = \sup\{t : Y_t(z) > 0\}.$$

Let

$$\Upsilon_t(z) = \frac{Y_t(z)}{|g_t'(z)|}, \qquad \theta_t(z) = \arg Z_t(z), \qquad S_t(z) = \sin\theta_t(z) = \frac{Y_t(z)}{|Z_t(z)|}.$$

It is not difficult to see that $\Upsilon_t(z)$ is $1/2$ times the conformal radius of H_t with respect to z, by which we mean that if $f : \mathbb{D} \to H_t$ is a conformal transformation with $f(0) = z$, then $|f'(0)| = 2\Upsilon_t(z)$. Using the Schwarz lemma and the Koebe $(1/4)$-theorem, we can see that

$$\Upsilon_t(z) \asymp_2 \operatorname{dist}\bigl(z, \mathbb{R} \cup \gamma(0,t]\bigr),$$

where $\asymp_2$ means that both sides are bounded above by 2 times the other side. Using the Loewner equation (1), we see that

$$\partial_t \Upsilon_t(z) = -\Upsilon_t(z)\frac{2aY_t(z)^2}{|Z_t(z)|^4}. \tag{2}$$

In particular, $\Upsilon_t(z)$ decreases with t and hence we can define

$$\Upsilon(z) = \lim_{t \to T_z-} \Upsilon_t(z),$$

which satisfies

$$\Upsilon(z) \asymp_2 \operatorname{dist}[z, \mathbb{R} \cup \gamma(0,\infty)].$$

Using Itô's formula, we can see that

$$d\theta_t(z) = \frac{(1-2a)X_t(z)Y_t(z)}{|Z_t(z)|^4}\,dt - \frac{Y_t(z)}{|Z_t(z)|^2}\,dB_t. \tag{3}$$

The *Green's function* (for SLE_κ from 0 to ∞ in $\mathbb{H}$) is defined by

$$G(z) = y^{d-2}[\sin\arg z]^{4a-1} = y^{1/(4a)+4a-2}|z|^{1-4a},$$

where $z = x + iy = |z|e^{i\theta}$. Itô's formula shows that

$$M_t(z) = |g_t'(z)|^{2-d}G(Z_t(z)) = \Upsilon_t(z)^{d-2}S_t(z)^{4a-1}, \qquad t < T_z, \tag{4}$$

is a local martingale satisfying

$$dM_t(z) = \frac{(1-4a)X_t(z)}{|Z_t(z)|^2}M_t(z)\,dB_t.$$

More generally, if D is a simply connected domain, $z \in D$, and w_1, w_2 are distinct points in ∂D, we can define $\Upsilon_D(z)$, $S_D(z; w_1, w_2)$ using the following scaling rules: if $f : D \to f(D)$ is a conformal transformation, then

$$\Upsilon_{f(D)}(f(z)) = |f'(z)| \Upsilon_D(z), \qquad S_{f(D)}(f(z); f(w_1), f(w_2)) = S_D(z; w_1, w_2).$$

The Green's function $G_D(z; w_1, w_2)$ is defined by

$$G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}$$

and satisfies the scaling rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)). \tag{5}$$

Under this definition, the local martingale in (4) can be rewritten as

$$M_t(z) = G_{H_t}(z; \gamma(t), \infty).$$

The following easy lemma is useful for estimating $S_D(z; w_1, w_2)$.

LEMMA 1.1. *Suppose D is a simply connected domain, $w_1, w_2 \in \partial D$, and $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = w_1$, $f(\infty) = w_2$. Let $\partial_+ = f[(0, \infty)]$, $\partial_- = f[(-\infty, 0)]$. If $z \in D$, let*

$$q = q_D(z; w_1, w_2) = \min\{h_D(z, \partial_+), h_D(z, \partial_-)\},$$

where h_D denotes harmonic measure. Then

$$2q \le S_D(z; w_1, w_2) \le \pi q. \tag{6}$$

PROOF. We first note that q, h_D, S_D are conformal invariants, so it suffices to prove the result for $D = \mathbb{H}$, $w_1 = 0$, $w_2 = \infty$ and by symmetry we may assume that $\theta := \arg z \le \pi/2$. By explicit calculation, we can see that $\theta = \pi q$, and hence (6) follows from the estimate

$$\frac{2}{\pi} \le \frac{\sin x}{x} \le 1, \qquad 0 < x \le \frac{\pi}{2}. \qquad \square$$

Using Girsanov's theorem, it can be shown that

$$\lim_{\epsilon \to 0+} \epsilon^{d-2} \mathbb{P}\{\Upsilon_\infty(z) \le \epsilon\} = c_* G(z), \qquad c_* = 2\left[\int_0^\pi \sin^{4a} x \, dx\right]^{-1}.$$

A proof of this is given in [7], but we include a self-contained proof in this paper (Proposition 2.3) which also estimates the error term. It follows that if D is a simply connected domain, $z \in D$; γ is an SLE_κ curve connecting distinct boundary points $w_1, w_2 \in \partial D$, and D_∞ denotes the component of $D \setminus \gamma$ containing z, then

$$\mathbb{P}\{\Upsilon_{D_\infty}(z) \le \epsilon\} \sim c_* \epsilon^{2-d} G_D(z; w_1, w_2), \qquad \epsilon \to 0+.$$

1.3. *Review of natural parametrization.* We will briefly review the construction in [11]. The starting point is the following proposition.

PROPOSITION A ([11]). *Suppose that there exists a parametrization for SLE_κ in $\mathbb{H}$ satisfying the domain Markov property and the conformal invariance assumption. For a fixed Lebesgue measurable subset $S \subset \mathbb{H}$, let $\Theta_t(S)$ denote the process that gives the amount of time in this parametrization spent in S before time t (in the half-plane capacity parametrization), and suppose further that $\Theta_t(S)$ is $\mathcal{F}_t$ adapted for all such S. If $\mathbb{E}\Theta_\infty(D)$ is finite for all bounded domains D, then it must be the case that (up to multiplicative constant)*

$$\mathbb{E}\Theta_\infty(D) = \int_D G(z)\, dA(z),$$

where dA denotes integration with respect to area, and more generally,

$$\mathbb{E}[\Theta_\infty(D) - \Theta_t(D)|\mathcal{F}_t] = \int_D M_t(z)\, dA(z).$$

Let $\mathcal{D}$ denote the set of bounded domains $D \subset \mathbb{H}$ with $\operatorname{dist}(\mathbb{R}, D) > 0$. Write

$$\mathcal{D} = \bigcup_{m=1}^{\infty} \mathcal{D}_m,$$

where $\mathcal{D}_m$ denotes the set of domains D with

$$D \subset \{x + iy : |x| < m, 1/m < y < m\}.$$

For any process $\Theta_t(D)$ with finite expectations, by Proposition A, one has

$$\Psi_t(D) = \mathbb{E}[\Theta_\infty(D)|\mathcal{F}_t] - \Theta_t(D), \tag{7}$$

where $\Psi_t(D) := \int_D M_t(z)\, dA(z)$, which is a supermartingale in t because $M_t(z)$ is a nonnegative local martingale. It is also not difficult to prove that $\Psi_t(D)$ is in fact continuous as a function of t by its definition. Assuming the conclusion of Proposition A, the first term on the right-hand side of (7) is a martingale and the map $t \mapsto \Theta_t(D)$ is increasing. Inspired by the continuous case of the standard Doob–Meyer theorem [3]: any continuous supermartingale can be written uniquely as the sum of a continuous adapted decreasing process with initial value zero and a continuous local martingale, Lawler and Shieffeld in [11] introduce the following definition.

DEFINITION ([11]).

- If $D \in \mathcal{D}$, then the natural parametrization $\Theta_t(D)$ is the unique continuous, increasing process such that

  $$\Psi_t(D) + \Theta_t(D)$$

  is a martingale (assuming such a process exists).

- If $\Theta_t(D)$ exists for each $D \in \mathcal{D}$, the natural parametrization in $\mathbb{H}$ is given by

$$\Theta_t = \lim_{m \to \infty} \Theta_t(D_m),$$

where $D_m = \{x + iy : |x| < m, 1/m < y < m\}$.

The main result in that paper is the following.

THEOREM B ([11]). *If* $\kappa < \kappa_0 := 4(7 - \sqrt{33})$, *there is an adapted, increasing, continuous process* $\Theta_t(D)$ *with* $\Theta_0(D) = 0$ *such that*

$$\Psi_t(D) + \Theta_t(D)$$

is a martingale. Moreover, with probability one for all t

$$\Theta_t(D) = \lim_{n \to \infty} \sum_{j \le t2^n} \int_{\mathbb{H}} |\hat{f}'_{(j-1)/2^n}(z)|^d \phi(z 2^{n/2}) G(z) \times 1\{\hat{f}_{(j-1)/2^n}(z) \in D\} \, dA(z), \tag{8}$$

where $\phi(z)$ *is defined by* $\mathbb{E}[M_1(z)] = M_0(z)(1 - \phi(z))$ *and* $\hat{f}_s(z) = g_s^{-1}(z + U_s)$.

As for the proof, they start by discretizing time and finding an approximation for $\Theta_t(D)$. This time discretization is the first step in proving the Doob–Meyer decomposition for any supermartingale. The second step is to take the limit. For this purpose, they use the reverse-time flow for the Loewner equation to derive uniform second moment estimates for the approximations when $\kappa < \kappa_0$. This estimate is the most difficult one in all their derivation. Then they can take a limit both in L^2 and with probability one.

1.4. *Multi-point Green's function.* As the main step in proving the Hausdorff dimension of the *SLE* curve, Beffara [1] proved the following lemma.

LEMMA 1.2. *Suppose* D *is a bounded subdomain of* $\mathbb{H}$ *with* $\operatorname{dist}(D, \mathbb{R}) > 0$. *Then there exists* $c_D < \infty$ *such that if* $z, w \in D$ *and* $\epsilon > 0$,

$$\mathbb{P}\{\Upsilon_\infty(z) \le \epsilon, \Upsilon_\infty(w) \le \epsilon\} \le c_D \epsilon^{2(2-d)} |z - w|^{d-2}.$$

Recently, Lawler and Werness [6] extended Beffara's argument to show that

$$\mathbb{P}\{\Upsilon_\infty(z) \le \epsilon, \Upsilon_\infty(w) \le \delta\} \le c_D \epsilon^{2-d} \delta^{2-d} |z - w|^{d-2}. \tag{9}$$

Building on this, they show that there is a multi-point Green's function $G(z, w)$ such that

$$\lim_{\epsilon, \delta \to 0+} \epsilon^{d-2} \delta^{d-2} \mathbb{P}\{\Upsilon_\infty(z) \le \epsilon, \Upsilon_\infty(w) \le \delta\} = c_*^2 G(z, w). \tag{10}$$

Although a closed form of the function $G(z, w)$ is not given, it is shown that

$$G(z, w) = G(z)G(w)[F(z, w) + F(w, z)], \tag{11}$$

where

$$F(z, w) = \frac{\mathbb{E}_z^*[|g_T'(w)|^{2-d} G(Z_T(w))]}{G(w)},$$

$\mathbb{E}_z^*$ denotes expectation with respect to two-sided radial SLE_κ through z (see Section 2 for definitions) and $T = T_z = \inf\{t : \gamma(t) = z\}$. Roughly speaking, $G(z, w)$ represents the probability of going through z and w, and $G(z)G(w)F(z, w)$ represents the probability of going first through z and then through w. Using (5), we can write

$$G(w)F(z, w) = \mathbb{E}_z^*[G_{D_T}(w; z, \infty)].$$

From the definition, we can see that if $r > 0$,

$$G(z, w) = r^{2(2-d)} G(rz, rw), \qquad F(z, w) = F(rz, rw).$$

More generally, if D is a simply connected domain with boundary points z_1, z_2, we can define $F_D(z, w; z_1, z_2)$ by conformal invariance.

Let

$$\begin{aligned} M_t(z, w) &= |g_t'(z)|^{2-d} |g_t'(w)|^{2-d} G(Z_t(z), Z_t(w)) \\ &= \frac{M_t(z)M_t(w)G(Z_t(z), Z_t(w))}{G(Z_t(z))G(Z_t(w))}. \end{aligned} \tag{12}$$

This is the so-called two-interior-point local martingale. A similar two-boundary-point local martingale appears in [15]. Using (10), we can see if $\epsilon > 0$ and T_ϵ is the first time such that $\Upsilon_{T_\epsilon}(z) \le \epsilon$ or $\Upsilon_{T_\epsilon}(w) \le \epsilon$, then $\mathbb{E}[M_{T_\epsilon}(z, w)] = M_0(z, w) = G(z, w)$. Hence, by Fatou's lemma, for every stopping time T,

$$G(z, w) \ge \mathbb{E}[M_T(z, w)]. \tag{13}$$

For our main result, we will need the following two estimates about $G(z, w)$. The first follows immediately from (9) and (10); establishing the second is the main technical work in this paper.

LEMMA 1.3.

- *Suppose D is a bounded subdomain of $\mathbb{H}$ with* $\mathrm{dist}(D, \mathbb{R}) > 0$. *Then there exists $c_D < \infty$ such that if $z, w \in D$,*

$$G(z, w) \le c_D |z - w|^{d-2}.$$

- *There exists $c > 0$ such that for all $z, w \in \mathbb{H}$,*

$$G(z, w) \ge cG(z)G(w). \tag{14}$$

Note that (14) is equivalent to saying that there exists c such that

$$F(z,w)+F(w,z)\ge c.$$

We remark that

$$\inf_{z,w} F(z,w)=0.$$

Indeed, one can check that

$$\lim_{y\to\infty} F(iy,i/y)=0.$$

The basic idea is that if y is large then the chance that the *SLE* path goes through yi *and then through* i/y is much smaller than the probability of going through i/y and then through iy.

COROLLARY 1.4. *If D is a bounded subdomain of $\mathbb{H}$ with* $\operatorname{dist}(D,\mathbb{R})>0$, *then there exists $c_D<\infty$ such that if $z,w\in D$, and T is a stopping time*,

$$\mathbb{E}[M_T(z)M_T(w)]\le c_D|z-w|^{d-2}. \tag{15}$$

PROOF. Using (12), (13), and Lemma 1.3, we get

$$\mathbb{E}[M_T(z)M_T(w)]\le c\mathbb{E}[M_T(z,w)]\le cG(z,w)\le c_D|z-w|^{d-2}. \qquad \square$$

1.5. *The main theorems*. As in Lawler and Sheffield, we will prove that there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D)=0$ such that $\Psi_t(D)+\Theta_t(D)$ is a martingale. The basic idea of the proof is the same. Here, we show how (15) yields the uniform integrability (class $\mathfrak{D}$) needed to establish the existence of the martingale.

THEOREM 1. *If $0<\kappa<8$, there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D)=0$ such that*

$$\Psi_t(D)+\Theta_t(D)$$

is a martingale. Moreover, let

$$\Theta_{t,n}(D)=\sum_{j\le t2^n}\int_{\mathbb{H}}|\hat f'_{(j-1)/2^n}(z)|^d\phi(z2^{n/2})G(z)1\{\hat f_{(j-1)/2^n}(z)\in D\}\,dA(z),$$

then for any stopping time T,

$$\lim_{n\to\infty}\mathbb{E}[|\Theta_{T,n}(D)-\Theta_T(D)|]=0.$$

REMARK. From Theorem B, we know that $\mathbb{E}M_1(z)=M_0(z)(1-\phi(z))<M_0(z)$. So $M_t(z)$ is a local martingale and a supermartingale which is not a proper martingale. This implies that $\Psi_t(D)$ is not a proper martingale too. Since the theorem establishes that $\Psi_t(D)+\Theta_t(D)$ is actually a martingale, $\Theta_t(D)$ is nontrivial. In other words, it is not identically zero.

Before discussing how to prove Theorem 1, let us briefly review Doob–Meyer decomposition for supermartingales of class $\mathfrak{D}$. First, we assume that Ψ_t is a supermartingale with respect to a filtration $\mathcal{F}_t$, defined on the interval $[0, \infty)$. We also suppose that $\mathcal{F}_t$ satisfies the usual conditions. The supermartingale $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ is said to be of *class* $\mathfrak{D}$ if the family $\{\Psi_T : T$ is an almost surely finite stopping time$\}$ is uniformly integrable. The following result is about Doob–Meyer decomposition for supermartingales of class $\mathfrak{D}$. One can refer to Section 1.4 of [5] for more details.

THEOREM C (Doob–Meyer decomposition [5]). *Let $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ be a continuous supermartingale of class $\mathfrak{D}$. Then there exists a continuous predictable, nondecreasing process $\{A_t, \mathcal{F}_t, t \geq 0\}$, such that $A_0 = 0$, A_∞ is integrable, and*

$$\Psi_t = \mathcal{M}_t - A_t,$$

where $\mathcal{M}_t := \mathbb{E}[A_\infty + \Psi_\infty | \mathcal{F}_t]$ is a martingale. This decomposition is unique up to indistinguishability, that is, if $\{\mathcal{M}'_t, \mathcal{F}_t, t \geq 0\}$ and $\{A'_t, \mathcal{F}_t, t \geq 0\}$ are a martingale and a predictable, nondecreasing process satisfying the above properties, respectively, then

$$\mathbb{P}\{\mathcal{M}_t = \mathcal{M}'_t, A_t = A'_t, \forall t \geq 0\} = 1.$$

In order to prove Theorem 1, we need one result from Section 20 of Chapter VII in [3] about Doob–Meyer decompositions of an increasing sequence of positive supermartingales.

THEOREM D ([3]). *Let $\{\Psi^n_t, \mathcal{F}_t, t \geq 0\}$ be a sequence of positive supermartingales, whose limit $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ belongs to class $\mathfrak{D}$ and is regular. Let A^n_t and A denote the nondecreasing processes associated with Ψ^n_t and Ψ_t, respectively. Then for any stopping time T,*

$$\lim_{n \to \infty} \mathbb{E}[|A^n_T - A_T|] = 0.$$

PROOF OF THEOREM 1 GIVEN LEMMA 1.3. From Theorem C, in order to show the existence of $\Theta_t(D)$ with the desired properties, we need to find a $c(D) < \infty$ such that for every stopping time T,

$$\mathbb{E}[\Psi^2_T(D)] \leq c(D). \tag{16}$$

From (15), we have

$$\begin{aligned} \mathbb{E}[\Psi^2_T(D)] &= \int_D \int_D \mathbb{E}[M_T(z_1) M_T(z_2)] \, dA(z_1) \, dA(z_2) \\ &\leq c_D \int_D \int_D |z_1 - z_2|^{d-2} \, dA(z_1) \, dA(z_2) \leq c(D). \end{aligned}$$

This gives the first result. Now we turn to the second result. For all t, we set

$$\Psi_t^n(D) = \mathbb{E}\big[\Psi_{(i+1)2^{-n}}(D)|\mathcal{F}_t\big] \qquad \text{if } i2^{-n} \le t < (i+1)2^{-n}.$$

Hence, $\{\Psi_t^n(D), \mathcal{F}_t, t \ge 0\}$ is a positive supermartingale bounded above by $\Psi_t(D)$, which increases to $\Psi_t(D)$ as $n \to \infty$. Let A_t^n be the nondecreasing process associated with $\Psi_t^n(D)$. Then for $i2^{-n} \le t < (i+1)2^{-n}$,

$$A_t^n = \sum_{0 \le k < i} \mathbb{E}\big[\Psi_{k2^{-n}}(D) - \Psi_{(k+1)2^{-n}}(D)|\mathcal{F}_{k2^{-n}}\big].$$

Actually, by the change of variables and the scaling rule of ϕ, we have

$$A_t^n = \Theta_{t,n}(D).$$

By Theorem D, we can complete the proof. $\square$

REMARK. Although we have used the existence of the multi-point Green's function $G(z, w)$ as established in [6], we could have proven our result here without its existence. In fact, an earlier draft of our paper derived the theorem from Beffara's estimate replacing (14) with

$$\mathbb{P}\{\Upsilon(z) \le \epsilon, \Upsilon(w) \le \delta\} \ge c\mathbb{P}\{\Upsilon(z) \le \epsilon\}\mathbb{P}\{\Upsilon(w) \le \delta\}.$$

However, the argument is cleaner when written in terms of $G(z, w)$ so we use it here.

The next theorem shows that our natural parametrization can be seen as conformal Minkowski measure (although is not the same definition as the conformal Minkowski content defined in [11]).

THEOREM 2. *Let* $\gamma^\epsilon(0, t] = \{z \in \mathbb{H} : \sup_{0 \le s \le t} M_s(z) \ge \epsilon^{d-2}\}$. *If* $0 < \kappa < 8$, *then for any stopping time* T,

$$\lim_{\epsilon \to 0} \mathbb{E}\big[\big|\epsilon^{d-2} A\big(D \cap \gamma^\epsilon(0, T]\big) - \Theta_T(D)\big|\big] = 0.$$

PROOF OF THEOREM 2 GIVEN LEMMA 1.3. Note that

$$\epsilon^{d-2} A\big(D \cap \gamma^\epsilon(0, t]\big) = \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0, t]\}\, dA(z),$$

where $\tilde{\tau}_\epsilon(z) = \inf\{t \ge 0 : M_t(z) \ge \epsilon^{d-2}\}$. The above integral appears in the identity

$$\begin{aligned}
&\int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0, t]\}\, dA(z) \\
&\quad = \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z)\, dA(z) - \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0, t]\}\, dA(z).
\end{aligned}$$

This is actually the Doob–Meyer decomposition of the supermatingale $\int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0, t]\}\, dA(z)$. By Theorem D and (16), we can complete the proof. $\square$

1.6. *Outline of the paper.* In Section 2, we will develop the theory of two-sided radial *SLE* in order to prove Proposition 3.1. Although this was discussed somewhat in [7], our treatment here is self-contained. One main goal is the refined "one-point" estimate in Proposition 2.3. The first step of the proof goes back to [12] in which the radial parametrization was used to establish a weaker form of the estimate. Here we use Girsanov's theorem to reduce the question to the rate of convergence to equilibrium of a simple one-dimensional diffusion. The sharp estimate is used in [6] to improve Beffara's estimate and show existence of the multi-point Green's function. When considering radial SLE_κ going through points $z = x + iy$ with $|x| \gg y$, it is useful to compare this to a process headed to x. We call the appropriate one-dimensional process "two-sided chordal SLE_κ" and give some of its properties.

The only thing we need to prove Theorem 1 is the estimate (14) which will be handled in Section 3.

2. Two-sided radial *SLE*.

One can see that with probability one $M_{T_z-}(z) = 0$; indeed, with probability one $\Upsilon(z) > 0$ and $S_{T_z-}(z) = 0$. This local martingale blows up on the event of probability zero that $z \in \gamma(0, \infty)$. To be more precise, let

$$\tau_\epsilon(z) = \inf\{t : \Upsilon_t(z) \le \epsilon\}.$$

Then $M_{t\wedge\tau_\epsilon(z)}(z)$ is a martingale. For convenience, we will fix z and write $M_t, X_t, Y_t, \tau_\epsilon, \dots$ for $M_t(z), X_t(z), Y_t(z), \tau_\epsilon(z), \dots$.

Two-sided radial SLE_κ through z is the process obtained from the Girsanov transformation by weighting by the local martingale M_t. We should really call this two-sided radial SLE_κ from 0 to ∞ going through z stopped when it reaches z, but we will just say two-sided radial SLE_κ through z. This could also be called SLE_κ conditioned to go through z (and stopped at T_z) although this is a conditioning on an event of probability zero. Using (4) and the Girsanov theorem, we see that for each ϵ

$$dB_t = \frac{(1-4a)X_t}{|Z_t|^2}\,dt + dW_t, \qquad t \le \tau_\epsilon, \tag{17}$$

where W_t is a standard Brownian motion in the weighted measure $\mathbb{P}^*$ which can be defined by saying that if V is an event depending only on $\{B_t : 0 \le t \le \tau_\epsilon\}$,

$$\mathbb{P}^*(V) = \mathbb{P}_z^*(V) = G(z)^{-1}\mathbb{E}[M_{\tau_\epsilon} 1_V].$$

We write $\mathbb{E}^*$ for expectations with respect to $\mathbb{P}^*$. There is an implicit z dependence in $\mathbb{P}^*$ and $\mathbb{E}^*$; when we need to make this explicit, we write $\mathbb{P}_z^*, \mathbb{E}_z^*$. It is not hard to show (see Section 2.1) that for every $\epsilon > 0$, $\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1$. Since the SDE in (17) has no ϵ dependence, we can let $0 \le t < T_z$. Note that

$$dX_t = \frac{(1-3a)X_t}{|Z_t|^2}\,dt + dW_t,$$

$$-U_t = B_t = X_t - a\int_0^t \frac{X_s}{|Z_s|^2}\,ds.$$

2.1. *Radial parametrization.* When studying the behavior of an SLE_κ curve near an interior point z, it is useful to reparametrize the curve so that $\log \Upsilon_t(z)$ decays linearly. This is the parametrization generally used for radial *SLE* and for this reason we call it the *radial parametrization*. We will study the radial parametrization in this subsection. We denote this time change as a function $\sigma(t)$ and we write $\hat{X}_t = X_{\sigma(t)}$, $\hat{Y}_t = Y_{\sigma(t)}$, etc.

We define σ by asserting that

$$\hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-2at}.$$

Using (2), we see that

$$-2a\hat{\Upsilon}_t = \partial_t[\hat{\Upsilon}_t] = -\hat{\Upsilon}_t \frac{2a\hat{Y}_t^2}{|\hat{Z}_t|^4}[\partial_t \sigma(t)]$$

and hence

$$\partial_t \sigma(t) = \frac{|\hat{Z}_t|^4}{\hat{Y}_t^2}.$$

From (3), we see that $\hat{\theta}_t := \theta_{\sigma(t)}$ satisfies

$$d\hat{\theta}_t = 2a \cot \hat{\theta}_t \, dt + d\hat{W}_t, \tag{18}$$

where $\hat{W}_t$ is a standard Brownian motion with respect to the weighted measure $\mathbb{P}^*$. Since $a > 1/4$, the process (18) stays in $(0, \pi)$ for all times. This implies that for all $\epsilon > 0$,

$$\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1. \tag{19}$$

2.1.1. *The corresponding SDE.* The analysis of two-sided radial *SLE* relies on detailed properties of (18) which fortunately are not very difficult to obtain. In this subsection, we focus on this SDE and set $r = 2a > 1/2$. Constants in this section may depend on r. We consider the equation

$$dX_t = r \cot X_t \, dt + dB_t, \qquad X_0 \in (0, \pi). \tag{20}$$

In studying this equation, it is useful to note that this is the equation that one gets if one starts with a Browian motion X_t and then weights paths locally by $\sin^r X_t$. To be more precise, if X_t is a standard Brownian motion, then

$$M_t = [\sin X_t]^r \exp\left\{-\frac{r(r-1)}{2}\int_0^t \frac{ds}{\sin^2 X_s} + \frac{1}{2}r^2 t\right\}$$

is a local martingale satisfying

$$dM_t = r[\cot X_t] M_t \, dX_t.$$

If we weight by the local martingale, then X_t satisfies (20) where B_t is a Brownian motion in the weighted measure. In fact, since the weighted process stays in $(0, \pi)$, one can see that M_t is actually a martingale.

The equation (20) is closely related to the Bessel equation

$$dX_t = \frac{r}{X_t}\,dt + dB_t. \tag{21}$$

This equation is obtained by starting with a Brownian motion X_t and weighting locally by X_t^r, that is, by the martingale

$$N_t = X_t^r \exp\left\{-\frac{r(r-1)}{2}\int_0^t \frac{ds}{X_s^2}\right\},$$

which satisfies

$$dN_t = r\frac{1}{X_t}N_t\,dX_t.$$

The next lemma gives a precise statement about the relationship.

LEMMA 2.1. *There exists $c < \infty$ such that the following is true. Suppose μ_1 is the probability measure on paths given by*

$$X_t, \qquad 0 \le t \le 1 \wedge T,$$

where X_t satisfies (20) *with $X_0 = x$ and*

$$T = \inf\{t : X_t \ge \pi/2\}.$$

Let μ_2 be the analogous probability measure using (21). *Then*

$$\frac{1}{c} \le \frac{d\mu_1}{d\mu_2} \le c. \tag{22}$$

PROOF. From the explicit forms of M_t, N_t, one can see there exists c such that

$$c^{-1}N_t \le M_t \le cN_t, \qquad 0 \le t \le T \wedge 1. \qquad \square$$

By comparison with the Bessel process, we see that the process satisfying (20) never leaves $(0, \pi)$. The invariant density for the process is

$$h(x) = C_{2r}\sin^{2r} x, \qquad C_{2r}^{-1} = \int_0^\pi \sin^{2r} y\,dy.$$

(This can be derived in a number of ways. It essentially follows because the invariant density for Brownian motion weighted by a function F is proportional to F^2.) In particular,

$$\int_0^\pi h(x)[\sin x]^{1-2r}\,dx = 2C_{2r}. \tag{23}$$

If $t \geq 0$, $x \in (0, \pi)$, we define

$$\psi(t, x) = \mathbb{E}^x[(\sin X_t)^{1-2r}],$$

where X_t satisfies (20). Note that $\psi(t, x) = \psi(t, \pi - x)$. Let

$$\tilde{\psi}(t, x) = \mathbb{E}^x[X_t^{1-2r}],$$

where X_t satisfies (21). Using the previous lemma, we can see there exist c_1, c_2 such that

$$c_1 \psi(t, x) \leq \tilde{\psi}(t, x) \leq c_2 \psi(t, x), \qquad 0 < x \leq \pi/2, 0 \leq t \leq 1. \tag{24}$$

Let

$$v(t, x) = \left[x \vee \sqrt{t \wedge 1}\right]^{1-4a}.$$

The next lemma collects the facts about the SDE that we will need.

LEMMA 2.2. *There exists $c < \infty$ such that the following is true for all $x \in (0, \pi/2]$.*

- *If $t \geq 1$,*

$$|\psi(t, x) - 2C_{2r}| \leq c e^{-(r+1/2)t}. \tag{25}$$

- *If $t \geq 0$,*

$$c^{-1} v(t, x) \leq \psi(t, x) \leq c v(t, x). \tag{26}$$

- *For every $\epsilon > 0$, there is a $\delta > 0$ such that if $t > 0$, $0 < x < \pi$, and $I \subset \mathbb{R}$ with*

$$\mathbb{P}^x\{X_t \in I\} \geq \epsilon,$$

then

$$\mathbb{E}^x[(\sin X_t)^{1-2r}; X_t \in I] \geq \delta v(t, x). \tag{27}$$

PROOF. The fact that

$$\lim_{t \to \infty} \psi(t, x) = 2C_{2r}$$

follows from (23). To get the error estimate, one needs the next eigenvalue. Suppose $x_0 \leq \pi/2$ and let T be the first time that the process reaches $\pi/2$. A simple application of Itô's formula shows that

$$M_t = \cos(X_t) e^{(r+1/2)t},$$

is a martingale. Using this one can show that

$$\mathbb{P}^{x_0}\{T \geq t\} \leq c e^{-(r+1/2)t}$$

with a constant independent of the starting point and then a coupling argument can be used to show that if we start two processes, one at x_0 and the other in the invariant density, then the probability that they do not couple by time t is $O(e^{-(r+1/2)t})$.

Note that (26) for $t \geq 1$ follows immediately from (25), so it suffices to prove it for $t \leq 1$.

A coupling argument shows that if $x \leq x_1$, then $\mathbb{P}^x\{X_t \leq y\} \geq \mathbb{P}^{x_1}\{X_t \leq y\}$. Therefore, if $X_0 = x$ and $h = C_{2r}(\sin x)^{2r}$ denotes the invariant density,

$$\mathbb{P}^x\{X_t \leq y\} \leq \frac{\int_0^x \mathbb{P}^s\{X_t \leq y\} h(s)\, ds}{\int_0^x h(s)\, ds} \leq \frac{\int_0^y h(s)\, ds}{\int_0^x h(s)\, ds} \leq c(y/x)^{2r+1}.$$

By integrating, we see for all t,

$$\mathbb{E}^x[(\sin X_t)^{1-2r}] \leq cx^{1-2r} + \mathbb{E}^x[(\sin X_t)^{1-2r}; X_t \leq x] \leq cx^{1-2r}.$$

This gives the upper bound for $t \leq x^2$; for the lower bound for these t, we need only note that there is a positive probability that the process starting at x stays within distance $x/2$ of x up to time x^2.

For the remainder, we consider $x^2 \leq t \leq 1$. The estimates are more easily done for the Bessel process using scaling. Since $t \leq 1$, it suffices by (24) to prove that

$$\tilde{\psi}(t, x) \asymp t^{1/2-r}, \qquad x^2 \leq t \leq 1, 0 < x \leq \pi/2,$$

where the implicit constants in the $\asymp$ notation are uniform over t, x. Suppose X_t satisfies (21) with $X_0 = x$. Then X_t has the same distribution as xY_{t/x^2}, where Y_t satisfies (21) with $Y_0 = 1$. We consider Y_t for $1 \leq t \leq x^{-2}$.

To study this equation, it is convenient to consider $Z_t = e^{-t/2} Y_{e^t}$. Note that

$$\begin{aligned} dZ_t &= \left[-e^{-t/2} Y_{e^t} + e^{-t/2} \frac{re^t}{Y_{e^t}}\right] dt + e^{-t/2}\, dB_{e^{t/2}} \\ &= \left[-Z_t + \frac{r}{Z_t}\right] dt + d\hat{B}_t, \end{aligned}$$

where $\hat{B}_t$ is a standard Brownian motion. This is a positive recurrent SDE with invariant density proportional to

$$x^{2r} e^{-x^2}.$$

We consider this with the initial condition $Z_{-\infty} = \infty$. (One can show this makes sense by writing the equation for $R_t = 1/Z_t$ and showing that this can be well defined with $R_{-\infty} = 0$.) By time 0, the process is within a constant multiple of the invariant density. All we will need from this are the following easily derived facts.

$$\mathbb{E}[Z_t^{1-2r}] \leq c, \qquad t \geq 0.$$

Secondly, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $t \geq 0$ and

$$\mathbb{P}\{Z_t \geq K\} \geq \epsilon,$$

then

$$\mathbb{E}[Z_t^{1-2r}; Z_t \geq K] \geq \delta. \qquad \square$$

2.1.2. *The one-point estimate.* The results of the previous subsection will be used with $r = 2a > 1/2$. We use the radial parametrization to prove the next proposition.

PROPOSITION 2.3. *If $z = x + iy$ and $\theta = \arg z$, then for $\epsilon \le y$,*

$$\mathbb{P}\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z)\psi(t, \theta),$$

where t satisfies $\epsilon = ye^{-2ta}$, that is,

$$\mathbb{P}\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z)\psi\left(\frac{\log(y/\epsilon)}{2a}, \theta\right).$$

It follows that for $1/2 \le \epsilon \le 1$,

$$\mathbb{P}\{\tau_{\epsilon y} < \infty\} \asymp \theta^{4a-1}\left[\theta \vee \sqrt{1-\epsilon}\right]^{1-4a} = \min\left\{1, \left(\theta/\sqrt{1-\epsilon}\right)^{4a-1}\right\}.$$

For $\epsilon \le 1/2$,

$$\mathbb{P}\{\tau_{\epsilon y} < \infty\} \asymp \epsilon^{2-d}\theta^{4a-1}.$$

PROOF OF PROPOSITION 2.3. Let $\tau = \tau_\epsilon$. Since $M_{t\wedge\tau}$ is a martingale, we have

$$G(z) = \mathbb{E}[M_{t\wedge\tau}] = \mathbb{E}[M_t; t < \tau] + \mathbb{E}[M_\tau; \tau \le t].$$

We can let $t \to \infty$, and from (19) and the dominated convergence theorem, we can deduce that

$$G(z) = \mathbb{E}[M_\tau; \tau < \infty].$$

Also,

$$\begin{aligned}\mathbb{P}\{\tau < \infty\} &= \epsilon^{2-d}\mathbb{E}[\Upsilon_\tau^{d-2}; \tau < \infty] = \epsilon^{2-d}\mathbb{E}[M_\tau S_\tau^{1-4a}; \tau < \infty] \\ &= \epsilon^{2-d} G(z)\mathbb{E}^*[S_\tau^{1-4a}].\end{aligned}$$

From Girsanov, we see that

$$\mathbb{E}^*[S_\tau^{1-4a}] = \psi(t, \theta). \qquad \square$$

The following is a corollary of this and Lemma 2.2.

PROPOSITION 2.4. *For $\epsilon \le y$,*

$$\mathbb{P}\{\tau_\epsilon < \infty\} = c_* G(z)\epsilon^{2-d}\left[1 + O\left([\epsilon/y]^{1+1/(4a)}\right)\right]. \tag{28}$$

In particular, there exist finite constants c and c' such that for all z and all ϵ,

$$\mathbb{P}\{\tau_\epsilon < \infty\} \le c[\epsilon/\operatorname{Im}(z)]^{2-d} \tag{29}$$

and

$$\mathbb{P}\{\tau_\epsilon < \infty\} \ge c' G(z)\epsilon^{2-d}. \tag{30}$$

2.2. *Two-sided chordal SLE_κ.* When $z = x + iy$ with $|x| \gg y$, then the radial parametrization is not so useful, because the path can get close to z without significantly decreasing the radial parametrization. Fortunately, one can study such processes by studying another process which we call *two-sided chordal SLE_κ through x*. It is SLE_κ weighted locally by X_t^{1-4a} (as compared to $|Z_t|^{1-4a}$ for two-sided radial).

Let us fix $z = x + iy$ and for convenience we assume $x > 0$. We write $\tilde{X}_t = Z_t(x) = g_t(x) - U_t$ which satisfies

$$d\tilde{X}_t = \frac{a}{\tilde{X}_t}\, dt + dB_t.$$

Itô's formula shows that

$$d\tilde{X}_t^{1-4a} = X_t^{1-4a}\left[\frac{a(4a-1)}{X_t^2}\, dt + \frac{1-4a}{X_t}\, dB_t\right]$$

and hence

$$N_t = \tilde{X}_t^{1-4a} \exp\left\{-a(4a-1)\int_0^t \frac{ds}{\tilde{X}_s^2}\right\} = \tilde{X}_t^{1-4a} g_t'(x)^{4a-1}$$

is a local martingale satisfying

$$dN_t = \frac{1-4a}{\tilde{X}_t} M_t\, dB_t.$$

Hence,

$$dB_t = \frac{1-4a}{\tilde{X}_t}\, dt + d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian motion in the weighted measure. In particular,

$$d\tilde{X}_t = \frac{1-3a}{\tilde{X}_t}\, dt + d\tilde{W}_t.$$

From this, we see that the terminal time T_x is finite with probability one in the weighted measure. Proposition 2.6 shows that $\gamma(T_x) = x$. (The proof is similar to the proof that SLE_κ hits points for $\kappa \geq 8$.) We precede this with a standard lemma about Bessel process whose proof we omit.

LEMMA 2.5. *Suppose $r < 1/2$ and X_t satisfies the Bessel equation*

$$dX_t = \frac{r}{X_t}\, dt + dB_t, \qquad X_0 = 1.$$

Let $T = \inf\{t : X_t = 0\}$. Then with probability one,

$$T < \infty, \qquad \int_0^T \frac{dt}{X_t} < \infty, \qquad \int_0^T \frac{dt}{X_t^2} = \infty.$$

Moreover, for every $\delta > 0$,

$$\mathbb{P}\left\{0 \le X_t \le 1+\delta, 0 \le t \le \delta; T \le \delta; \int_0^T \frac{dt}{X_t} \le \delta\right\} > 0. \tag{31}$$

If $B_0 = 0$, then

$$-B_t = X_0 - X_t + \int_0^t \frac{r}{X_s}\, ds = 1 - X_t + \int_0^t \frac{r}{X_s}\, ds$$

and hence on the event described in (31)

$$-\delta \le -B_t \le 1+\delta, \qquad 0 \le t \le T.$$

PROPOSITION 2.6. *Suppose $0 < x < x_1$ and let $R_t = Z_t(x_1) = g_t(x_1) - U_t$. Then with probability one, if γ is two-sided chordal to x with terminal time $T = T_x$,*

$$R_T > 0.$$

PROOF. Since $1 - 3a < \frac{1}{2}$, Lemma 2.5 implies that with probability one $T < \infty$ and

$$\int_0^T \frac{dt}{\tilde{X}_t} < \infty, \tag{32}$$

$$\int_0^T \frac{dt}{\tilde{X}_t^2} = \infty. \tag{33}$$

Let $q(x, y)$ denote the probability that $R_T > 0$ given $\tilde{X}_0 = x$, $R_0 = y$. By Bessel scaling, $q(x, y) = q(1, y/x)$. We claim that $q(1, r) \to 1$ as $r \to \infty$. Indeed, we have

$$\begin{aligned}\log R_T = \log[R_T - \tilde{X}_T] &= \log(y - x) + \int_0^T \partial_t[\log(R_t - \tilde{X}_t)]\, dt \\ &= \log(y - x) - \int_0^T \frac{a\, dt}{\tilde{X}_t R_t}.\end{aligned}$$

From this and (32), we can deduce this fact. From this and the strong Markov property, it follows that on the event

$$\sup_{0 \le t < T} \frac{R_t - \tilde{X}_t}{\tilde{X}_t} = \infty, \tag{34}$$

we must have $R_T > 0$.

If

$$L_t = \log \frac{R_t - \tilde{X}_t}{\tilde{X}_t},$$

then

$$dL_t = \left[-\frac{a}{\tilde{X}_t R_t} - \frac{1/2 - 3a}{\tilde{X}_t^2} \right] dt - \frac{1}{\tilde{X}_t} d\tilde{W}_t.$$

Under a suitable time change $\hat{L}_t = L_{\sigma(t)}$, this can be written as

$$d\hat{L}_t = \left[-\frac{a\hat{X}_t}{\hat{R}_t} - \left(\frac{1}{2} - 3a \right) \right] dt + d\hat{B}_t$$

for a standard Brownian motion $\hat{B}_t$. By (33), in this time change it takes infinite time for $\hat{X}_t$ to reach zero, that is, $\sigma(\infty) = T$. Since $\hat{X}_t \le \hat{R}_t$, the drift term is bounded below by

$$-a - (\tfrac{1}{2} - 3a) > 0$$

and hence $\hat{L}_t \to \infty$ as $t \to \infty$. In particular, (34) holds. $\square$

If $0 < x < x_1 < x_2$, then the random variable

$$\Delta = \Delta(x, x_1, x_2) = \max_{0 \le t \le T_x} \frac{Z_t(x_2)}{Z_t(x_1)}$$

is well defined and satisfies $1 < \Delta < \infty$. Moreover, Bessel scaling implies that the distribution of $\Delta(x, x_1, x_2)$ is the same as that of $\Delta(rx, rx_1, rx_2)$. The next lemma gives uniform bounds on the distribution in terms of $(x_2 - x)/(x_1 - x)$.

LEMMA 2.7. *For every $\rho > 0$ there exists $C < \infty$ such that the following is true. Suppose $x < x_1 < x_2$ with $x_1 - x \ge \rho(x_2 - x)$. Then with probability at least $1 - \rho$, $\Delta(x, x_1, x_2) \le C$.*

PROOF. We fix ρ and allow constants to depend on ρ. By scaling and monotonicity, we may assume $x = 1$ and $x_1 - x = \rho(x_2 - x)$. Hence, we write $x_1 = 1 + s\rho$, $x_2 = 1 + s$ where $s > 0$ (our estimates must be uniform over s). If s is very large, then since $T_x < \infty$, with probability one, one can find a K such that with probability at least $1 - \rho$ for all $\tilde{x} \ge K\rho$.

$$\frac{Z_0(\tilde{x})}{2} \le Z_t(\tilde{x}) \le 2Z_0(\tilde{x}), \qquad 0 \le t \le T_x,$$

and hence with probability at least $1-\rho$, $\Delta(1, 1+s\rho, 1+s) \le 4$. For $1/2 \le s \le K$, we can bound

$$\Delta(1, 1 + s\rho, 1 + s) \le \Delta\left(1, 1 + \frac{\rho}{2}, 1 + K\right).$$

For the remainder, we assume that $s \le 1/2$. Let

$$\xi = \inf\{t : |\gamma(t) - 1| = 2s\}.$$

By distortion estimates, we see that

$$g_t'(x) \asymp g_t'(x_1) \asymp g_t'(x_2), \qquad 0 \le t \le \xi,$$

with the implicit constants uniform over s. Since $g_t'(x')$ is an increasing function of x', this implies

$$Z_t(x_1) \asymp Z_t(x_2), \qquad 0 \le t \le \xi,$$

which gives uniform estimates (with probability one) on

$$\max_{0 \le t \le \xi} \frac{Z_t(x_2)}{Z_t(x_1)}.$$

Also, using the 1/4-theorem with distortion estimates,

$$Z_\xi(x) \asymp Z_\xi(x_1) - Z_\xi(x) \asymp Z_\xi(x_2) - Z_\xi(x)$$

with again the estimates uniform over s. The conditional distribution of

$$\max_{\xi \le t \le T_x} \frac{Z_t(x_2)}{Z_t(x_1)}$$

given $\mathcal{F}_\xi$ is the distribution of $\Delta(Z_\xi(x), \mathbb{Z}_\xi(x_1), Z_\xi(x_2))$. Hence, we reduce this to the case where x, x_1, x_2 are comparable which we have already stated and handled. $\square$

LEMMA 2.8. *For every $\rho > 0$, there is a $u > 0$ such that the following holds. Suppose $x > 0$ and γ is two-sided chordal SLE$_\kappa$ to x. Let $0 < \epsilon \le x$ and*

$$\xi = \inf\{t : |\gamma(t) - x| = \epsilon\}.$$

Define ψ by

$$\gamma(\xi) = x + \epsilon e^{i\psi}.$$

Then with probability at least $1 - \rho$, $\psi \ge u$.

PROOF. As $\psi \to 0$, the extremal distance between $(x, x + \epsilon/2)$ and $(x + 2\epsilon, \infty)$ in H_ξ tends to ∞. By conformal invariance of extremal distance, if ψ is very close to zero, then we can see that $Z_\xi(x + 2\epsilon) - Z_\xi(x) \gg Z_\xi(x + \frac{\epsilon}{2}) - Z_\xi(x)$. The previous lemma bounds the probability that this happens. $\square$

2.2.1. *Comparison with two-sided radial.* In this section, we fix $x > 0$ $\epsilon < x/2$. If $z \in \mathbb{H}$, we write $Z_t = Z_t(z) = X_t + iY_t$, $\tilde{X}_t = Z_t(x)$ as above. Recall that two-sided radial to z is the process obtained by weighting with respect to the local martingale

$$M_t = |Z_t|^{1-4a} (\Upsilon_t / y)^{1/(4a)-1} (Y_t / y)^{4a-1}.$$

Here we have included a constant $y^{2-4a-1/(4a)}$ to the usual local martingale, but this does not affect the weighting. Under this choice of M_t, $M_0 = |z|^{1-4a}$. Two-sided chordal is obtained by weighting by the local martingale

$$N_t = |\tilde{X}_t|^{1-4a} g_t'(x)^{4a-1}.$$

Let

$$\sigma = \inf\{t : |\gamma(t) - x| = 2\epsilon\}.$$

Distortion estimates and the 1/4-theorem imply that for $0 \le t \le \sigma$ and $|z - x| \le \epsilon$,

$$g_t'(x) \asymp |g_t'(z)|, \qquad \Upsilon_t \asymp \Upsilon_0 = y, \qquad Y_t \asymp y|g_t'(z)|, \qquad \tilde{X}_t \asymp |Z_t|$$

and hence

$$M_t \asymp N_t, \qquad 0 \le t \le \sigma,$$

where the implicit constants are uniform over z. We have just proved the following.

PROPOSITION 2.9. *There exists $c < \infty$, such that for every $z = x + iy$ with the property $|z - x| \le \epsilon$, if μ denotes the measure on paths*

$$\gamma(t), \qquad 0 \le t \le \sigma,$$

given by two-sided radial SLE$_\kappa$ to z and ν denotes the analogous measure on paths using two-sided chordal to x, then

$$c^{-1} \le \frac{d\mu}{d\nu} \le c.$$

The following is a corollary of this proposition and Lemma 2.8.

LEMMA 2.10. *For every $\rho > 0$, there is a $u > 0$ such that the following is true. Suppose $x > 0$ and $\delta < x$. Suppose $z \in \mathbb{H}$ with $|z - x| \le \delta/2$. Let γ be a two-sided radial SLE$_\kappa$ path through z, and let*

$$\sigma = \inf\{t : |\gamma(t) - x| = \delta\}.$$

Then,

$$\mathbb{P}_z^*\{S_\xi(z) \ge u\} \ge 1 - \rho.$$

3. Proof of (14). In Section 3.1, we spend a relatively long time deriving an estimate that states roughly that there is a positive probability that a two-sided radial path stays with a small distance of an "L"-shape. The proof uses two well-known ideas:

- If $f : [0, 1] \to \mathbb{R}$ is a continuous function with $f(0) = 0$ and $\epsilon > 0$, and U_t is a standard Brownian motion, then the probability that $\|f - U\|_\infty < \epsilon$ is positive, where $\|\cdot\|_\infty$ denotes the supremum norm.

- If the driving functions (using the Loewner equation) of two curves are close in the supremum norm, then the curves are close in the Hausdorff metric. This follows in a straightforward manner from the Loewner equation. (It is possible that the curves are not close in the supremum norm on curves, but this is not important for us.)

It takes a little time to write out the details because our driving functions are those for two-sided radial and we need some uniformity in the estimates. Readers who are convinced that this can be done can skip the proof of Proposition 3.1. Proposition 3.2 is a simple deterministic estimate which uses Proposition 3.1 to establish (14) for z, w far apart.

It remains to prove (14) for z, w close and this is the goal of Section 3.2. Indeed, this does not seem that it should be difficult, since the events of getting close to z and getting close to w should be very positively correlated. We give the argument in two cases: when $|z-w|$ is much smaller than $\mathrm{Im}(z)$ and when z, w are close to each other and also close to the boundary.

3.1. *Probability of L-shapes.* If $z = x + iy$, let L_z denote the "L"-shape

$$L_z = [0, x] \cup [x, x + iy], \qquad x \geq 0,$$
$$L_z = [x, 0] \cup [x, x + iy], \qquad x \leq 0,$$

and if $\rho > 0$,

$$L_{z,\rho} = \{z' \in \overline{\mathbb{H}} : \mathrm{dist}(z', L_z) \leq \rho|z|\}.$$

The goal of this section is to prove the following proposition.

PROPOSITION 3.1. *For every $\rho > 0$, there is a $u > 0$ such that for all $z = x + iy \in \mathbb{H}$,*

$$\mathbb{P}^*_z\{\gamma[0, T_z] \subset L_{z,\rho}\} \geq u.$$

The important thing is to show that we can choose u uniformly over z. Before discussing the proof of this proposition, let us show how it can be used to prove (14) for z and w sufficiently spread apart. We use the following deterministic estimate.

PROPOSITION 3.2. *For every $0 < \rho \leq 1/4$ there exists $c < \infty$ such that the following holds. Suppose $z \in \mathbb{H}$, $w \in \mathbb{H} \setminus L_{z,2\rho}$, and suppose $\gamma : [0, T] \to \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0$, $\gamma(T) = z$ and $\gamma[0, T] \subset L_{z,\rho}$. Let $g = g_T$ be the corresponding conformal transformation and let $Z = g(w) - g(z)$. Then*

$$G(Z) \geq cG(w),$$
$$c^{-1} \leq |g'(w)| \leq c.$$

PROOF. By scaling it suffices to consider $|z| = 1$ which we assume. The estimate is easy for $|w| \ge 2$ so we assume $w \in \mathbb{H} \setminus L_{z,2\rho}$ with $|w| \le 2$. We fix ρ and allow (implicit or explicit) constants to depend on ρ. Let $J = \{w' : \mathrm{Im}(w') \ge 3\}$. Using conformal invariance, one can see that $\mathrm{Im}[g(w)]$ is comparable to the probability that a Brownian motion starting at w reaches J without hitting $\mathbb{R} \cup \gamma[0, T]$ and this is not smaller than the the probability of reaching J before hitting $\mathbb{R} \cup L_{z,\rho}$. It is not difficult to see that this is bounded below by a constant times $\mathrm{Im}(w)$ and hence

$$c\,\mathrm{Im}[w] \le \mathrm{Im}[g(w)] \le \mathrm{Im}[w].$$

Using (6), we also see that

$$S_{H_T}(w) \asymp \mathrm{Im}[g(w)] \asymp \mathrm{Im}[w] \asymp S_0(w)$$

(remember that the implicit constants depend on ρ). Also, $\mathrm{dist}(w, \mathbb{R} \cup L_{z,\rho}) \asymp \mathrm{Im}(w)$, and hence by the Koebe-1/4 theorem,

$$|g'(w)| \asymp 1.$$

Therefore, $G(Z) \asymp G(w)$. $\square$

COROLLARY 3.3. *For every $\rho > 0$, there exists $c > 0$ such that if $z \in \mathbb{H}$ and $w \in \mathbb{H} \setminus L_{z,\rho}$, then $F(z, w) \ge c$.*

PROOF. By Propositions 3.1 and 3.2, there exists $u = u(\rho) > 0$ such that

$$\mathbb{P}^*_z\{G_{H_T}(w; z, \infty) \ge uG(w)\} \ge u.$$

Therefore,

$$\mathbb{E}^*_z[G_{H_T}(w; z, \infty)] \ge u^2 G(w). \qquad \square$$

The proof of Proposition 3.1 combines a probabilistic estimate for the driving function and a deterministic estimate using the Loewner equation. Assume $z = x + iy$, $x \ge 0$, $y > 0$. Note that one obtains the hull $[x, x + iy]$ by solving (1) with $U_t \equiv x$, $0 \le t \le y^2/(2a)$. For every $\delta > 0$, let $E_{z,\delta}$ denote the event

$$E_{z,\delta} = \left\{T_z \le \frac{y^2}{2a} + \delta; -\delta \le U_t \le x + \delta, 0 \le t \le \delta; |U_t - x| \le \delta, \delta \le t \le T_z\right\}.$$

LEMMA 3.4. *For every $\rho > 0$, there exists $\delta > 0$ such that the following holds. Suppose $z = x + iy$ with $0 \le x \le 1$, $0 < y \le 1$. Then on the event $E_{z,\delta}$,*

$$\gamma[0, T_z] \subset L_{z,\rho}.$$

PROOF. This is a straightforward deterministic estimate using (1). One first shows that if δ is sufficiently small, then $|g_\delta(w) - w| \le \rho/100$ for $\operatorname{dist}(w, [0, x]) \ge \rho$. For $\delta \le t \le T_z$, we compare g_t to the corresponding function $\tilde{g}_t$ obtained with $\tilde{U}_t \equiv x$. These estimates are standard (see, e.g., [9], Proposition 4.47), and we omit the details. $\square$

Therefore, Proposition 3.1 reduces to the following probabilistic estimate on the driving function.

LEMMA 3.5. *For every $\delta > 0$, there exists $u > 0$ such that if $z = x + iy$ with $0 \le x \le 1, 0 < y \le 1$, then*

$$\mathbb{P}_z^*[E_{z,\delta}] > u.$$

We will use the next lemma in the proof of Lemma 3.5. The lemma may seem to follow immediately from $T_z < \infty$, but it is important to establish uniformity in z.

LEMMA 3.6. *For every $\epsilon > 0$, there exists $r < \infty$ such that if $z = x + iy \in \mathbb{H}$,*

$$\mathbb{P}_z^*\{T_z \le r|z|^2; |U_t| \le r|z|, 0 \le t \le T_z\} \ge 1 - \epsilon. \tag{35}$$

PROOF. By scaling and symmetry, it suffices to prove the result for $|z| = 1$, $x \ge 0$. For fixed z, the result is immediate from the fact that $\mathbb{P}_z^*\{T_z < \infty\} = 1$. It is not difficult to extend this as follows: for every $u > 0$ and every $\epsilon > 0$, there exists r such that if $|z| = 1$ and $S_0(z) \ge u$, then

$$\mathbb{P}_z^*\{T_z \le r; |U_t| \le r, 0 \le t \le T_z\} \ge 1 - \epsilon. \tag{36}$$

For $y < 1/100$, let $\mathbb{P}_1^*$ denote probabilities for two-sided chordal SLE_κ to 1. Again, since $\mathbb{P}_1^*\{T_1 < \infty\} = 1$, it is easy to see that for every r, there exists $\epsilon > 0$ such that

$$\mathbb{P}_1^*\{T_1 \le r; |U_t| \le r, 0 \le t \le T_1\} \ge 1 - \epsilon. \tag{37}$$

Suppose $z = x + iy \in \mathbb{H}$ with $|z - 1| = \delta/2 < 1/4$. Let

$$\sigma = \inf\{t : |\gamma(t) - 1| = \delta\}$$

and define ξ by $\gamma(\sigma) = 1 + \delta e^{i\xi}$. From (37) and Proposition 2.9, we see that there exists c_1 such that

$$\mathbb{P}_z^*\{\sigma \le r; |U_t| \le r, 0 \le t \le \sigma\} \ge 1 - c_1\epsilon.$$

Using Lemma 2.10, we see that there exists $u > 0$, such that

$$\mathbb{P}_z^*\{\sigma \le r; |U_t| \le r, 0 \le t \le \sigma; S_\sigma(z) \ge u\} \ge 1 - 2c_1\epsilon.$$

Using the form of the Poisson kernel in the upper half plane, we can see that $|Z_\sigma(z)| \le c_2\delta$. By using (36), we see that there exists $\tilde{r}$ such that

$$\mathbb{P}_z^*\{\sigma \le r; |U_t| \le r, 0 \le t \le \sigma; S_\sigma(z) \ge u; T_z - \sigma \le \tilde{r}\delta^2; \\ |U_t - U_\sigma| \le \tilde{r}\delta, \sigma \le t \le T_z\} \\ \ge 1 - 3c_1\epsilon. \qquad \square$$

PROOF OF LEMMA 3.5. By scaling and symmetry, we may assume that $z = x + iy$ with $|z| = 1$ and $x \ge 0$, $y > 0$. We will show that there exists $c < \infty$ such that for each ρ, there exists $q(\rho) > 0$ such that for all z,

$$\mathbb{P}_z^*[E_{z,c\sqrt{\rho}}] \ge q(\rho).$$

If suffices to consider $0 < \rho \le 1/1000$. We will consider two cases: $y \le 10\rho$ and $y > 10\rho$. In this proof, constants $c_1, c_2, \dots$ are independent of ρ, but constants $\delta, q_1, q_2, \dots$ may depend on ρ.

First assume $y \le 10\rho \le 1/100$, and hence $3/4 < x \le 1$. Let

$$\eta = \eta_{\rho,z} = \inf\{t : \mathrm{Re}[\gamma(t)] = x - 4\rho\}.$$

For every $\delta > 0$, consider the event $V_\delta = V_{\delta,\rho,z}$ given by

$$V_\delta = \{\eta \le \delta; -\delta \le U_t \le 1 + \delta, 0 \le t \le \eta\}.$$

Using the deterministic estimate, Lemma 3.4, we can see that by choosing δ sufficiently small, then on the event V_δ, $\mathrm{Im}[\gamma(\eta)] \le \rho$. By choosing δ smaller if necessary, we assume $\delta < \rho$.

Using Lemma 2.5, we can see that $\mathbb{P}[V_\delta] \ge q_1 > 0$. There is a curve of length at most 11ρ in H_η connecting z and $\gamma(\eta)$. Hence, using the Beurling estimate, there exists c_1 such that

$$|Z_\eta(z)| \le c_1\sqrt{\rho}.$$

[Actually, we can get an estimate of $O(\rho)$, but the estimate above suffices for our purposes.] Using Lemma 3.6, we can say there exists c_2 such that

$$\mathbb{P}\Big\{T - \eta \le c_2\sqrt{\rho}, \sup_{\eta \le t \le T} |U_t - U_\eta| \le c_2\sqrt{\rho} \Big| V_\delta\Big\} \ge \frac{1}{2}.$$

Therefore, with probability at least $q_1/2$,

$$T \le (c_2 + 1)\sqrt{\rho}, \qquad -(1 + c_2)\sqrt{\rho} \le U_t \le 1 + (1 + c_2)\sqrt{\rho}.$$

We now assume $y \ge 10\rho$. Let

$$\eta = \eta_{\rho,z} = \inf\{t : \mathrm{Im}[\gamma(t)] = y - 4\rho\}.$$

Let W_t denote a standard Brownian motion and consider the event $E = E_{\delta,x}$ that

$$|W_t - (tx/\delta)| < \delta, \qquad 0 \le t \le \delta,$$
$$|W_t - x| < \delta, \qquad \delta \le t \le 1/a.$$

Using standard estimates for Brownian motion (including the Cameron–Martin formula), it is standard to show that for every $\delta > 0$ there exists $u_1 > 0$ such that for all $0 \le x \le 1$, $\mathbb{P}(E) \ge u_1$. If we let $U_t = W_t$, then by choosing δ sufficiently small, we see that $\mathbb{P}[E] \ge u_1$.

We claim that there exists $c_3 > 0$ such that on the event E,

$$S_\eta(z) \ge c_3.$$

To show this, we consider the path $\gamma[0,\eta]$. Let γ^+ be the part of the path mapped to $[U_\eta, \infty)$ under g_η and let γ^- be the part mapped to $(-\infty, U_\eta)$. Using the fact that $\gamma[0,\eta] \subset L_\rho$; $\mathrm{Im}[\gamma(\eta)] = y - 4\rho$ and $\mathrm{Im}[\gamma(t)| < y - 4\rho, t < \eta$, we can see geometrically that there is a positive probability u_2 such that a Brownian motion starting at z exists H_η at γ^+ with probability at least c_2' and at γ^- with probability at least c_2'. This combined with (6) gives the lower bound on $S_\eta(z)$. Since $\Upsilon_t(z)$ decreases with t, we get a lower bound on $M_\eta(z)$. Therefore, there exists $q_2 > 0$ such that

$$\mathbb{P}_z^*\{\gamma[0,\eta] \subset L_{z,\rho}\} \ge q_2.$$

[The reader may note that we have used the trivial bound $\Upsilon_\eta(z) \le 1$. In fact, $\Upsilon_\eta(z) \asymp \rho$, so we can improve the last estimate but we do not need to. For ρ small, it is much more likely for two-sided radial *SLE* to follow the L-shape to z then for usual *SLE*.]

On the event E, there is a curve connecting $\gamma(\eta)$ to z in H_η of length $O(\rho)$. Using the Beurling estimate, we can see that there exists c_4 such that $|Z_\eta(z)| \le c_4\sqrt{\rho}$. The proof proceeds as in the previous case. $\square$

3.2. *Remainder of proof.* To finish the proof, we need to consider z, w that are close. In this case, we will take a stopping time σ such that z, w are not so close in the domain H_σ. The next lemma is easy, but it is useful to state it.

LEMMA 3.7. *There exists $c > 0$ such that the following is true. Suppose z, $w \in \mathbb{H}$, $u \ge 0$ and σ is a stopping time for SLE_κ such that*

$$|\gamma(t) - z| \ge 3|z - w|, \qquad 0 \le t \le \sigma,$$

and such that

$$\mathbb{P}_z^*\{F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \ge u\} \ge u. \tag{38}$$

Then

$$F(z, w) \ge cu^3.$$

REMARK. Implicit in the assumptions is $|z| \geq 3|z-w|$. We do not assume that the disk of radius $|z-w|$ about z is contained in $\mathbb{H}$. In particular, $\mathrm{Im}(z), \mathrm{Im}(w)$ can be small and very different.

PROOF OF LEMMA 3.7. Let $r = |z-w|$ and let $\mathcal{B}$ denote the open disk of radius $2r$ centered at z. By assumption $0 \notin \mathcal{B}$. It is not necessarily the case that $\mathcal{B} \subset \mathbb{H}$; however, for $t \leq \sigma$, the conformal map g_t can be extended to $\mathcal{B}$ by Schwarz reflection.

Let $E = E_u$ denote the event that $F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \geq u$. Using the Beurling estimate, we can see that $M_{t\wedge\sigma}(z), M_{t\wedge\sigma}(w)$ are bounded martingales and hence

$$G(z) = M_0(z) = \mathbb{E}[M_\sigma(z)], \qquad G(w) = M_0(w) = \mathbb{E}[M_\sigma(w)].$$

Also, by definition of $\mathbb{E}_z^*$,

$$\begin{aligned}\mathbb{E}_z^*[M_\sigma(z)1_E] &= \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{M_0(z)} = \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z)]} \geq u\frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z)1_E]}\\ &\geq u\mathbb{E}[M_\sigma(z)1_E] \geq u^2 G(z).\end{aligned}$$

The first inequality uses (38). Using the distortion theorem on $\mathcal{B}$, we can see that $|g_\sigma'(z)| \asymp |g_\sigma'(w)|$. We also claim that

$$\frac{S_\sigma(z)}{S_\sigma(w)} \asymp \frac{\mathrm{Im}(z)}{\mathrm{Im}(w)}. \tag{39}$$

To see this, consider the first time that a Brownian motion starting at z, w reaches $\mathbb{R} \cup \partial\mathcal{B}$. If $p(z), p(w)$ denotes the probabilities that the process hits $\partial\mathcal{B} \cap \mathbb{H}$ before leaving $\mathbb{H}$, then standard estimates (gambler's ruin estimate) show that $p(z)/p(w) \asymp \mathrm{Im}(z)/\mathrm{Im}(w)$. Also, the conditional distributions given that one hits $\partial\mathcal{B}$ are mutually absolutely continuous (here we use either a boundary Harnack principle or the explicit form of the Poisson kernel in a half disk). Given this and (6), we can conclude (39).

Therefore, using (5), we see that

$$\frac{M_\sigma(z)}{M_\sigma(w)} = \frac{G_{H_\sigma}(z; \gamma(\sigma), \infty)}{G_{H_\sigma}(w; \gamma(\sigma), \infty)} \asymp \frac{G(z)}{G(w)}$$

and hence,

$$\mathbb{E}_z^*[M_\sigma(w)1_E] \geq cu^2 G(w).$$

Also,

$$\mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty)1_E | \mathcal{F}_\sigma] = 1_E u G_{H_\sigma}(w; \gamma(\sigma), \infty) = 1_E u M_\sigma(w).$$

Taking expectations, we get

$$\mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty)] \geq \mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty)1_E] \geq cu^3 G(w),$$

which implies $F(z, w) \geq cu^3$. $\square$

PROOF OF (14). By symmetry and scaling, it suffices to consider $1 = |z| \leq |w|$ with $\mathrm{Re}(z) \geq 0$. If $|w| > 1.01$, then $w \notin L_{z,1/100}$ and hence Corollary 3.3 implies that $F(z, w) \geq c > 0$.

Similarly, if $\mathrm{Im}(z) > \mathrm{Im}(w) + (1/100)$, then $z \notin L_{w,1/100}$, and Corollary 3.3 implies that $F(w, z) \geq c$. Using similar facts about real parts and interchanging z, w, it suffices to consider $z = x + iy$, $w = \tilde{x} + i\tilde{y}$ with $1 \leq |z|, |w| \leq 1.01$, $x \geq 0$ and

$$y \leq \tilde{y} \leq y + \frac{1}{100}, \qquad |x - \tilde{x}| \leq \frac{1}{100}.$$

We now split into two cases: $\tilde{y} \geq 1/10$ and $\tilde{y} < 1/10$.

For $\tilde{y} \geq 1/10$, let $\tau = \inf\{t : \Upsilon_t(z) = 10|z - w|\}$, $T = T_z$. Using Lemma 2.2, we see that there exists $u > 0$ such that

$$\mathbb{P}_z^*\{S_\tau(z) \geq 1/4\} \geq u.$$

Using distortion estimates and Corollary 3.3, we can see that there exists $c > 0$ such that on the event that $S_\tau(z) \geq 1/4$,

$$F_{H_\tau}(z, w; \gamma(\tau), \infty) \geq c.$$

We can now apply Lemma 3.7.

If $\tilde{y} < 1/10$ and $|z - w| \leq \tilde{y}/20$, we can do similarly as above, interchanging the roles of z and w.

For the remainder, we assume that $\tilde{y} < 1/10$ and $|z - w| \geq \tilde{y}/20$. Note that $x, \tilde{x} > 9/10$. Let

$$r = \max\{\tilde{y}, |z - w|\} < 1/10.$$

Let $\tau = \inf\{t : |\gamma(t) - \tilde{x}| = 4r\}$. If we write

$$\gamma(\tau) = \tilde{x} + 4re^{i\xi},$$

then by Lemma 2.10 there exists u such that

$$\mathbb{P}_z^*\{\xi > u\} \geq u.$$

On this event, distortion estimates and Corollary 3.3 imply that $F_{H_\tau}(z, w; \gamma(\tau), \infty) \geq c$ for some c (depending on ξ). We can now apply Lemma 3.7. $\square$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
5734 S. UNIVERSITY AVENUE
CHICAGO, ILLINOIS 60637
USA
E-MAIL: lawler@math.uchicago.edu

DEPARTMENT OF STATISTICS
AND APPLIED PROBABILITY
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE 117546
SINGAPORE
E-MAIL: stazw@nus.edu.sg