Initial measures for the stochastic heat equation

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Abstract. We consider a family of nonlinear stochastic heat equations of the form $\partial_t u = \mathcal{L}u + \sigma(u)\dot{W}$, where $\dot{W}$ denotes space–time white noise, $\mathcal{L}$ the generator of a symmetric Lévy process on $\mathbf{R}$, and σ is Lipschitz continuous and zero at 0. We show that this stochastic PDE has a random-field solution for every finite initial measure u_0. Tight a priori bounds on the moments of the solution are also obtained.

In the particular case that $\mathcal{L}f = cf''$ for some $c > 0$, we prove that if u_0 is a finite measure of compact support, then the solution is with probability one a bounded function for all times $t > 0$.

Résumé. Nous considérons une famille d'équations de la chaleur stochastique de la forme $\partial_t u = \mathcal{L}u + \sigma(u)\dot{W}$, où $\dot{W}$ est un bruit-blanc espace–temps, $\mathcal{L}$ est le générateur d'un processus de Lévy symétrique sur $\mathbf{R}$, et σ est une fonction lipschizienne s'annulant en 0. Nous montrons que cette équation aux dérivées partielles stochastique a une solution de type champ aléatoire pour toute mesure initiale finie u_0. Nous obtenons également des bornes a priori sur les moments de la solution.

Dans le cas particulier où $\mathcal{L}f = cf''$ pour un $c > 0$, nous montrons que si u_0 est une mesure finie à support compact, la solution est presque sûrement une fonction bornée pour tout $t > 0$.

1. Introduction

Consider the stochastic heat equation

$$\frac{\partial}{\partial t}u_t(x) = \frac{\varkappa}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma\big(u_t(x)\big)\dot{W}_t(x), \tag{1.1}$$

where $\varkappa > 0$ is a constant, $\sigma : \mathbf{R} \to \mathbf{R}$ is a Lipschitz function that satisfies

$$\sigma(0) = 0, \tag{1.2}$$

and $\dot{W}$ denotes space–time white noise. In other words, $\dot{W}$ is a mean-zero generalized Gaussian random field [17], Ch. 2, §2.4, with covariance measure $\text{Cov}(\dot{W}_t(x), \dot{W}_s(y)) := \delta_0(x-y)\delta_0(t-s)$ for all $s, t \geq 0$ and $x, y \in \mathbf{R}$.

The solution to (1.1) represents the density of heat in an idealized thin metal rod that is placed in a homogeneous medium, the white noise represents a nonlinear source/sink of heat, and the constant $\varkappa/2 > 0$ – the so-called *viscosity coefficient* – denotes the viscosity of the medium. It is well known that (1.1) has a random-field solution if, for example, the initial heat profile u_0 is a bounded and measurable function [23], Ch. 3.

Now suppose that $u_0 : \mathbf{R} \to \mathbf{R}_+$ is in fact bounded uniformly away from zero, as well as infinity; i.e., that $0 < \inf u_0 \le \sup u_0 < \infty$. We have shown recently [10] that, in that case, $x \mapsto u_t(x)$ is a.s. unbounded for all $t > 0$ under various conditions on σ. In particular, if $\sigma(x) = cx$ for a constant $c > 0$ – this is the so called *parabolic Anderson model* [7] – then our results [10] imply that

$$0 < \limsup_{|x| \to \infty} \frac{\log u_t(x)}{(\log |x|)^{2/3}} < \infty \quad \text{a.s.} \tag{1.3}$$

Eq. (1.3) holds, for instance, for the parabolic Anderson model in the important [flat] case that $u_0(x) \equiv 1$.

Another well-studied case is the parabolic Anderson model when $u_0 = \delta_0$ is point mass at 0 [the narrow-wedge case]. This case arises in the study of directed random polymers [20]. Gérard Ben Arous, Ivan Corwin, and Jeremy Quastel have independently asked us whether (1.3) continues to hold in that case (private communications). One of the goals of the present articles is to prove that the answer to this question is "no." In fact, we have the following much more general fact, which is a corollary to the development of this paper.

Theorem 1.1. *If $\sigma(0) = 0$ and u_0 is a finite measure of compact support, then $\sup_{x \in \mathbf{R}} u_t(x) = \sup_{x \in \mathbf{R}} |u_t(x)| < \infty$ a.s. for all $t > 0$.*

2. Some background material

We begin by recalling some well-known facts; also, we use this opportunity to set forth some notation that will be used consistently in the sequel.

2.1. *White noise*

Throughout let $W := \{W_t(x)\}_{t \ge 0, x \in \mathbf{R}}$ denote a two-parameter Brownian sheet indexed by $\mathbf{R}_+ \times \mathbf{R}$; that is, W is a two-parameter mean-zero Gaussian process with covariance

$$\mathrm{Cov}\big(W_t(x), W_s(y)\big) = \min(s, t) \min\big(|x|, |y|\big) \mathbf{1}_{(0, \infty)}(xy) \tag{2.1}$$

for all $s, t \ge 0$ and $x, y \in \mathbf{R}$. The space–time mixed derivative of $W_t(x)$ is denoted by $\dot{W}_t(x) := \partial^2 W_t(x)/(\partial t \, \partial x)$ and is called *space–time white noise*. Space–time white noise is a generalized Gaussian random field with mean zero and covariance measure $\mathrm{Cov}(\dot{W}_t(x), \dot{W}_s(y)) = \delta_0(x - y)\delta_0(t - s)$.

2.2. *Lévy processes*

Let $X := \{X_t\}_{t \ge 0}$ denote a symmetric Lévy process on $\mathbf{R}$. That is, $t \mapsto X_t$ is [almost surely] a right-continuous random function with left limits at every $t > 0$ whose increments are independent, identically distributed and symmetric. It is well known that X is a strong Markov process; see Jacob [19] for this and all of the analytic theory of Lévy processes that we will require here and throughout. We denote the infinitesimal generator of X by $\mathcal{L}$. According to the Lévy–Khintchine formula, the law of the process X is characterized by its *characteristic exponent*; that is a function $\Psi : \mathbf{R} \to \mathbf{C}$ that is determined via the identity $\mathrm{E} \exp(i\xi \cdot X_t) = \exp(-t\Psi(\xi))$, valid for all $t \ge 0$ and $\xi \in \mathbf{R}$. Elementary arguments show that, because X is assumed to be symmetric, the characteristic exponent Ψ is a nonnegative – in particular real valued – symmetric function. For reasons that will become apparent later on, we will be interested only in symmetric Lévy processes that satisfy the following:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t\Psi(\xi)} \, \mathrm{d}\xi < \infty \quad \text{for all } t > 0. \tag{2.2}$$

In such a case, the inversion formula for Fourier transforms applies and tells us that X has transition densities $p_t(x)$ that can be defined by

$$p_t(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix \cdot \xi - t\Psi(\xi)} \, d\xi \quad (t > 0, x \in \mathbf{R}). \tag{2.3}$$

Note that the function $(t, x) \mapsto p_t(x)$ is continuous uniformly on $(\eta, \infty) \times \mathbf{R}$ for every $\eta > 0$.

Let us note two important consequences of the preceding formula for transition densities:

1. $p_t(x) \le p_t(0)$ for all $t > 0$ and $x \in \mathbf{R}$; and
2. $t \mapsto p_t(0)$ is nonincreasing.

We will appeal to the these properties without further mention.

Throughout we assume also that the transition densities of the Lévy process X satisfy the following regularity condition:

$$\Theta := \sup_{t > 0} \left[\frac{p_{t/2}(0)}{p_t(0)} \right] < \infty. \tag{2.4}$$

Because $p_t(0) \ge p_t(x)$ and $\int_{-\infty}^{\infty} p_t(x) \, dx = 1$, it follows $p_t(0) > 0$ and hence Θ is well defined [though it could in principle be infinity when X is a general symmetric Lévy process].

Let us mention one example very quickly before we move on.

Example 2.1. *Let X denote a one-dimensional standard Brownian motion. Then, X is a symmetric Lévy process with transition densities given by $p_t(x) := (2\pi t)^{-1/2} \exp\{-x^2/(2t)\}$ for $t > 0$ and $x \in \mathbf{R}$. In this case, we may note also that $\mathcal{L}f = (1/2)f''$, $\Psi(\xi) = \|\xi\|^2/2$, and $\Theta = \sqrt{2}$.*

3. The main result

Our main goal is to study the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + \sigma\big(u_t(x)\big) \dot{W}_t(x) \quad \text{for } t > 0, x \in \mathbf{R}, \tag{3.1}$$

where:

1. $\mathcal{L}$ is the generator of a symmetric Lévy process $\{X_t\}_{t \ge 0}$ that satisfies (2.2);
2. $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous with Lipschitz constant Lip_σ; and
3. $\sigma(0) = 0$.

As regards the initial data, we will assume here and throughout that

$$u_0 \text{ is a nonrandom, finite Borel measure on } \mathbf{R}. \tag{3.2}$$

We recall from Walsh [23] that a solution to (3.1) a *mild solution* if it solves the following random integral equation:

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbf{R}} p_{t-s}(y - x) \sigma\big(u_s(y)\big) W(ds \, dy). \tag{3.3}$$

It is known that a mild solution to (3.1) exists, under the above assumptions, provided that u_0 is a bounded and measurable, nonrandom function [23], Ch. 3. We will soon see that the same fact remains to hold under the less restrictive condition (3.2). Since we will never need another notion of a solution to (1.1), from now on we will mean "mild solution" when we refer to a "solution" to (1.1).

The best-studied special case of the random heat equation (3.1) is when $\mathcal{L}f = \nu f''$ is a constant multiple of the Laplacian. In that case, Eq. (3.1) arises for several reasons that include its connections to the stochastic Burgers'

equation (see Gyöngy and Nualart [18]), the parabolic Anderson model (see Carmona and Molchanov [7]) and the KPZ equation (see Kardar, Parisi and Zhang [21]).

One can think of the solution $u_t(x)$ to (3.1) as the expected density of particles, at place $x \in \mathbf{R}$ and time $t > 0$, for a system of interacting branching random walks in continuous time: The particles move as independent Lévy processes on $\mathbf{R}$; and the particles move through an independent external random environment that is space–time white noise $\dot{W}$. The mutual interactions of the particles occur through a possibly-nonlinear birthing mechanism σ. The special case $\mathcal{L}f = \nu f''$ deals with the case that the mentioned particles move as independent Brownian motions.

The most special example of (3.1) is when $\sigma(x) \equiv 0$; that is the linear heat [Kolmogorov] equation for $\mathcal{L}$, whose [weak] solution is $u_t(x) = (p_t * u_0)(x)$. It is a simple exercise in harmonic analysis that when $\sigma(x) \equiv 0$, the solution to (3.1) exists, is unique, and is a bounded function for all time $t > 0$. Indeed,

$$(p_t * u_0)(x) = \int_{\mathbf{R}} p_t(y - x) u_0(\mathrm{d}y) \leq p_t(0) \int_{\mathbf{R}} u_0(\mathrm{d}y) = p_t(0) u_0(\mathbf{R}) < \infty. \tag{3.4}$$

Hence, $\sup_{x \in \mathbf{R}}(p_t * u_0)(x)$ is finite, as was asserted.

Consider the case where the characteristic exponent Ψ of our Lévy process X satisfies the following condition: For some [hence all] $\beta > 0$,

$$\Upsilon(\beta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\beta + 2\Psi(\xi)} < \infty. \tag{3.5}$$

It is well known that if, in addition, u_0 is a bounded and measurable *function*, then (3.1) has a solution that is a.s. unique among all possible "natural" candidates. This statement follows easily from the theory of Dalang [11], for instance. Moreover, Dalang's theory shows also that (3.5) is necessary as well as sufficient for the existence of a random-field solution to (3.1) when σ is a constant function. This is why we assume (3.5) per force. The technical condition (2.2) in fact follows as a consequence of (3.5); see Foondun et al. [16], Lemma 8.1.

Dalang's method proves also the following without any extra effort; for close variations on this, see also Foondun and Khoshnevisan [15]:

Theorem 3.1. *Suppose u_0 is a random field, independent of the white noise $\dot{W}$, such that $\sup_{x \in \mathbf{R}} \mathrm{E}(|u_0(x)|^k) < \infty$ for some $k \in [2, \infty)$. Then (3.1) has a mild solution $\{u_t(x)\}_{t>0, x \in \mathbf{R}}$ that solves the random integral equation (3.3). Furthermore, $\{u_t(x)\}_{t>0, x \in \mathbf{R}}$ is a.s.-unique in the class of all predictable random fields $\{v_t(x)\}_{t>0, x \in \mathbf{R}}$ that satisfy:*

$$\sup_{t \in (0, T)} \sup_{x \in \mathbf{R}} \mathrm{E}\left(|v_t(x)|^k\right) < \infty \quad \text{for all } T > 0. \tag{3.6}$$

Finally, the random field $(t, x) \mapsto u_t(x)$ is continuous in probability.

We will not describe the proof, since all of the requisite ideas are already in the paper [11]. However, we mention that the reference to "predictable" assumes tacitly that the Brownian filtration of Walsh [23] has been augmented with the sigma-algebra generated by the random field $\{u_0(x)\}_{x \in \mathbf{R}}$. The mentioned stochastic integrals are also as defined in [23]. We will need the following variation of a theorem of Foondun and Khoshnevisan [15] also:

Theorem 3.2 (Foondun and Khoshnevisan [15]). *Suppose u_0 is a random field, independent of the white noise $\dot{W}$, such that $\sup_{x \in \mathbf{R}} \mathrm{E}(|u_0(x)|^k) < \infty$ for every $k \in [2, \infty)$. Then the mild solution $\{u_t(x)\}_{t>0, x \in \mathbf{R}}$ to (3.1) satisfies the following: For all $\varepsilon > 0$ there exists a finite and positive constant C_ε such that for all $t > 0$ and $k \in [2, \infty)$,*

$$\sup_{x \in \mathbf{R}} \mathrm{E}\left(|u_t(x)|^k\right) \leq C_\varepsilon^k \mathrm{e}^{(1+\varepsilon)\gamma(k)t}, \tag{3.7}$$

where $\gamma(k)$ is defined by:

$$\gamma(k) := \inf\left\{\beta > 0 : \Upsilon(2\beta/k) < \frac{1}{4k \mathrm{Lip}_\sigma^2}\right\}. \tag{3.8}$$

Once again, we omit the proof, since it follows closely the ideas of the paper [15] without making novel alterations.

The existence and uniqueness of the solution to (3.1) under a measure-valued initial condition has been studied earlier in various papers. For example, Bertini and Cancrini [1] obtain moment formulas for the parabolic Anderson model [that is, $\sigma(x) = cx$] in the special case that $\mathcal{L}f = (\varkappa/2)f''$ and $u_0 = \delta_0$ is the Dirac point mass at zero. They also give sense to what a "solution" might mean. The most-recent word on this topic can be found in Borodin and Corwin [2]. Weak solutions to the fully-nonlinear equation (3.1) have been studied in Conus and Khoshnevisan [9]. An independent work in preparation by Chen and Dalang [8] establishes, in the framework of Walsh [23], the existence of random field solutions to (3.1) in the case where $\mathcal{L}f = f''$, and derives very precise information about the moments of the solution.

We are now ready to state one of the main results of this paper, which extends the previous two results to the case when the initial data is a nonrandom, finite Borel measure.

Theorem 3.3. *If $\Theta < \infty$ and* (3.2) *holds, then* (3.1) *has a mild solution u that satisfies the following for all real numbers $x \in \mathbf{R}$, $\varepsilon, t > 0$, and $k \in [2, \infty)$: There exists a positive and finite constant $C_\varepsilon := C_\varepsilon(\Theta)$ – depending only on ε and Θ – such that*

$$\mathrm{E}\big(|u_t(x)|^k\big) \leq C_\varepsilon^k \mathrm{e}^{(1+\varepsilon)\gamma(k)t}\big\{1 + p_t(0)(p_t * u_0)(x)\big\}^{k/2}, \tag{3.9}$$

where

$$\gamma(k) := \inf\bigg\{\beta > 0 : \Upsilon(2\beta/k) < \frac{1}{4k\mathrm{Lip}_\sigma^2}\bigg\}. \tag{3.10}$$

Moreover, the solution is almost-surely unique among all predictable random fields v that solve (3.1) *and satisfy*

$$\sup_{t>0}\sup_{x \in \mathbf{R}}\left[\frac{\mathrm{E}(|v_t(x)|^2)}{\mathrm{e}^{(1+\varepsilon)\gamma(2)t}\{1 \vee p_t(0)(p_t * u_0)(x)\}}\right] < \infty \quad \textit{for some } \varepsilon > 0. \tag{3.11}$$

From this we shall see that, in the particular case where $\mathcal{L}$ is a multiple of the Laplacian, the solution remains bounded for every finite time $t > 0$, as long as the finite initial measure u_0 has compact support. This verifies Theorem 1.1.

4. An example

Consider (3.1) where $\mathcal{L} = -\varkappa(-\Delta)^{\alpha/2}$ is the fractional Laplacian of index $\alpha \in (0, 2]$, where $\varkappa > 0$ is a viscosity parameter. The operator $\mathcal{L}$ is the generator of a symmetric stable-α Lévy process with $\Psi(\xi) \propto \varkappa|\xi|^\alpha$, where the constant of proportionality does not depend on $(\varkappa, \xi)$. It is possible to check directly that $\Upsilon(1) < \infty$ if and only if $\alpha > 1$. Let us restrict attention to the case that $\alpha \in (1, 2]$, and recall that we consider only the case that u_0 is a finite measure.

A computation shows that $\Upsilon(\beta) \propto \varkappa^{-1/\alpha}\beta^{-(\alpha-1)/\alpha}$ uniformly for all $\beta > 0$. Moreover, $p_t(x)$ is a fundamental solution of the heat equation for $\mathcal{L}$, and $\Theta = 2^{1/\alpha}$ since $p_t(0) \propto (\varkappa t)^{-1/\alpha}$ uniformly for all $t > 0$. Theorem 3.3 then tells us that (3.1) has a unique mild solution which satisfies the following for all real numbers $t > 0$ and $k \in [2, \infty)$:

$$\sup_{x \in \mathbf{R}}\mathrm{E}\big(|u_t(x)|^k\big) \leq C_1^k\big(1 + (\varkappa t)^{-1}\big)^{k/\alpha}\exp\left(C_2\frac{tk^{(2\alpha-1)/(\alpha-1)}}{\varkappa^{1/(\alpha-1)}}\right), \tag{4.1}$$

where C_1 and C_2 are positive and finite constants that do not depend on $(t, k, \varkappa)$. In other words, the large-t behavior of the kth moment of the solution is, as in [15], the same as it would be had u_0 been a bounded measurable function; that is,

$$\limsup_{t \to \infty}\frac{1}{t}\log\sup_{x \in \mathbf{R}}\|u_t(x)\|_k \leq \mathrm{const} \cdot \big(k^\alpha/\varkappa\big)^{1/(\alpha-1)}. \tag{4.2}$$

However, we also observe the small-t estimate,

$$\limsup_{t \downarrow 0} t^{1/\alpha} \sup_{x \in \mathbf{R}} \|u_t(x)\|_k \le \text{const} \cdot \varkappa^{-1/\alpha}, \tag{4.3}$$

which is a new property. Moreover, the preceding estimate is tight. Indeed, it is not hard to see that

$$\|u_t(x)\|_k \ge \|u_t(x)\|_2 \ge (p_t * u_0)(x). \tag{4.4}$$

Therefore, in the case that u_0 is a *positive-definite* finite measure,

$$\sup_{x \in \mathbf{R}} \|u_t(x)\|_k \ge (p_t * u_0)(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\text{const} \cdot \varkappa t |\xi|^\alpha} \hat{u}_0(\xi) \, d\xi$$

$$= \frac{1}{2\pi (t\varkappa)^{1/\alpha}} \int_{-\infty}^{\infty} e^{-\text{const} \cdot |\xi|^\alpha} \hat{u}_0\left(\frac{\xi}{(t\varkappa)^{1/\alpha}}\right) d\xi. \tag{4.5}$$

The second inequality follows from applying Parseval's identity to $(p_{t+\varepsilon} * u_0)(x)$ and then letting $\varepsilon \downarrow 0$ using Fatou's lemma. Another application of Fatou's lemma then shows that

$$\liminf_{t \downarrow 0} t^{1/\alpha} \sup_{x \in \mathbf{R}} \|u_t(x)\|_k \ge \text{const} \cdot \varkappa^{-1/\alpha}, \tag{4.6}$$

as long as u_0 is a positive-definite finite measure such that $\underline{\lim}_{|z| \to \infty} \hat{u}_0(z) > 0$; that is, as long as the conclusion of the Riemann–Lebesgue lemma does *not* apply to u_0. Thus, (4.3) is tight, as was claimed. There are many examples of such measure u_0. For instance, we can choose $u_0 = a\delta_0 + \mu$, where $a > 0$ and μ is any given positive-definite finite Borel measure on $\mathbf{R}$.

5. Preliminaries

5.1. *Some inequalities*

We recall from Foondun and Khoshnevsian [15] that

$$\Upsilon(\beta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\Psi(\xi)} = \int_0^{\infty} e^{-\beta t} \|p_t\|_{L^2(\mathbf{R})}^2 \, dt. \tag{5.1}$$

This is merely a consequence of Plancherel's theorem. Because X is symmetric we can describe Υ in terms of the resolvent of X. To this end define

$$\overline{\Upsilon}(\beta) := \int_0^{\infty} e^{-\beta t} p_t(0) \, dt. \tag{5.2}$$

This is the resolvent density, at zero, of the Lévy process X. Because of symmetry,

$$\|p_t\|_{L^2(\mathbf{R})}^2 = (p_t * p_t)(0) = p_{2t}(0). \tag{5.3}$$

Therefore, it follows that $\Upsilon(\beta) = \frac{1}{2}\overline{\Upsilon}(\beta/2)$. In particular, Dalang's condition [11], (26), thm. 2, $\Upsilon(1) < \infty$ is equivalent to the condition that $\overline{\Upsilon}(\beta) < \infty$ for some, hence all, $\beta > 0$.

We close this subsection with some convolution estimates.

Lemma 5.1. *For all $t > 0$,*

$$p_t(0) \cdot \int_0^t p_r(0) \, dr \le \int_0^t p_{t-s}(0) p_s(0) \, ds \le 2\Theta \cdot p_t(0) \cdot \int_0^t p_r(0) \, dr. \tag{5.4}$$

As it turns out, the preceding simple-looking result is the key to our analysis of existence and uniqueness, because it tells us that

$$\int_0^t \frac{p_{t-s}(0)p_s(0)}{p_t(0)}\,\mathrm{d}s \downarrow 0 \quad \text{as } t \downarrow 0, \tag{5.5}$$

at sharp rate $\int_0^t p_r(0)\,\mathrm{d}r$.

Proof of Lemma 5.1. The first inequality holds simply because $p_s(0) \geq p_t(0)$ for all $s \in (0, t)$. For the second one, we split $\int_0^t p_{t-s}(0)p_s(0)\,\mathrm{d}s$ into two parts: $\int_0^{t/2}$ and $\int_{t/2}^t$. Note that $p_{t-s}(0) \leq p_{t/2}(0) \leq \Theta p_t(0)$ when $s \in (0, t/2)$; and $p_s(0) \leq p_{t/2}(0) \leq \Theta p_t(0)$ when $s \in (t/2, t)$. The lemma follows from these observations. $\qquad\square$

Let $\circledast$ denote space–time convolution; that is,

$$(f \circledast g)_t(x) := \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, f_{t-s}(x - y)g_s(y), \tag{5.6}$$

whenever $f, g : (0, \infty) \times \mathbf{R} \to \mathbf{R}_+$ are both measurable.

Lemma 5.2. *For all $t > 0$, $x \in \mathbf{R}$, and $n \geq 1$,*

$$\big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\big)_t(x) \leq \left(2\Theta \int_0^t p_s(0)\,\mathrm{d}s\right)^{n-1} \cdot p_t(0)p_t(x). \tag{5.7}$$

Proof. The result holds trivially when $n = 1$. Let us suppose that (5.7) is valid for $n = m$; we prove that (5.7) is valid also for $n = m + 1$. Note that

$$\begin{aligned}
\mathcal{T}_{m+1} &:= \big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{m+1 \text{ times}}\big)_t(x) \\
&= \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, \big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{m \text{ times}}\big)_{t-s}(y) p_s^2(x - y) \\
&\leq \int_0^t \mathrm{d}s \left(2\Theta \int_0^{t-s} p_r(0)\,\mathrm{d}r\right)^{m-1} p_{t-s}(0) \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}(y) p_s^2(x - y) \\
&\leq \left(2\Theta \int_0^t p_r(0)\,\mathrm{d}r\right)^{m-1} \int_0^t \mathrm{d}s\, p_{t-s}(0) \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}(y) p_s^2(x - y). \tag{5.8}
\end{aligned}$$

Since $p_s^2(x - y) \leq p_s(0)p_s(x - y)$, the Chapman–Kolmogorov equation implies that

$$\mathcal{T}_{m+1} \leq \left(2\Theta \int_0^t p_r(0)\,\mathrm{d}r\right)^{m-1} \cdot \int_0^t p_{t-s}(0)p_s(0)\,\mathrm{d}s \cdot p_t(x), \tag{5.9}$$

and the result follows from this, Lemma 5.1, and induction. $\qquad\square$

Lemma 5.3. *For all $t > 0$, $x \in \mathbf{R}$, and $n \geq 1$,*

$$\begin{aligned}
&\Big(\big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\big) \circledast (p_\bullet * u_0)^2\Big)_t(x) \\
&\qquad \leq u_0(\mathbf{R}) \left(2\Theta \int_0^t p_s(0)\,\mathrm{d}s\right)^n \cdot p_t(0)(p_t * u_0)(x). \tag{5.10}
\end{aligned}$$

Proof. For every nonnegative function f, (5.6) and Lemma 5.2 together imply that

$$\Big(\Big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\Big) \circledast f\Big)_t(x)$$

$$= \int_0^t \mathrm{d}s \int_{\mathbf{R}} \mathrm{d}y \, \Big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\Big)_s(y) f_{t-s}(x-y)$$

$$\leq \int_0^t \mathrm{d}s \, \Big(2\Theta \int_0^s p_r(0) \, \mathrm{d}r\Big)^{n-1} p_s(0) \int_{\mathbf{R}} \mathrm{d}y \, p_s(y) f_{t-s}(x-y)$$

$$\leq \Big(2\Theta \int_0^t p_r(0) \, \mathrm{d}r\Big)^{n-1} \int_0^t \mathrm{d}s \, p_s(0)(p_s * f_{t-s})(x). \tag{5.11}$$

We set $f_t(x) := p_t(0)(p_t * u_0)(x)$ and appeal the Chapman–Kolmogorov property $[p_s * p_{t-s} = p_t]$ in order to obtain the following:

$$\Big(\Big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\Big) \circledast (p_\bullet * u_0)^2\Big)_t(x)$$

$$\leq u_0(\mathbf{R})\Big(\Big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}}\Big) \circledast p_\bullet(0)(p_\bullet * u_0)\Big)_t(x)$$

$$\leq u_0(\mathbf{R})\Big(2\Theta \int_0^t p_r(0) \, \mathrm{d}r\Big)^{n-1} \int_0^t p_s(0) p_{t-s}(0) \, \mathrm{d}s \cdot (p_t * u_0)(x). \tag{5.12}$$

An application of Lemma 5.1 completes the proof. $\qquad\square$

6. Finite-horizon estimates

We first define a sequence $\{u^{(n)}\}_{n\in\mathbf{N}}$ of random fields by: $u_t^{(0)}(x) := 0$ for all $t > 0$ and $x \in \mathbf{R}$. Then, for every $n \geq 0$, we set

$$u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\big(u_s^{(n)}\big) W(\mathrm{d}s \, \mathrm{d}y). \tag{6.1}$$

Thus, $u^{(n)}$ denotes simply the nth stage of a Picard iteration approximation to a reasonable candidate for a solution to (3.1).

Proposition 6.1 below will show that the random variables $\{u_t^{(n)}(x)\}_{n\in\mathbf{N}}$ are well-defined with values in $L^k(\mathrm{P})$, for $x \in \mathbf{R}$ and all times t that are "reasonably small."

For each $a > 0$, let

$$\mathfrak{g}(a) := \inf\Big\{t > 0: \int_0^t p_r(0) \, \mathrm{d}r \geq a\Big\}, \tag{6.2}$$

where $\inf \varnothing := \infty$. Clearly, $\mathfrak{g}(a) > 0$ when $a > 0$.

Proposition 6.1. *For all integers $n \geq 0$, real numbers $k \in [2, \infty)$ and $x \in \mathbf{R}$,*

$$\big\|u_t^{(n+1)}(x)\big\|_k \leq 4\big[1 \vee k^{1/2}\mathrm{Lip}_\sigma\big]\sqrt{u_0(\mathbf{R}) p_t(0)(p_t * u_0)(x)} \tag{6.3}$$

for $0 < t \leq \mathfrak{g}((32\Theta[1 \vee k\mathrm{Lip}_\sigma^2])^{-1})$.

Proof. Define

$$\mu_t^{(n,k)}(x) := \big\|u_t^{(n)}(x)\big\|_k^2. \tag{6.4}$$

Clearly, $\mu_t^{(0,k)}(x) \equiv 0$. Then, we have

$$\sqrt{\mu_t^{(n+1,k)}(x)}$$

$$\leq (p_t * u_0)(x) + \bigg\|\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\big(u_s^{(n)}(y)\big)W(\mathrm{d}s\,\mathrm{d}y)\bigg\|_k$$

$$\leq (p_t * u_0)(x) + \bigg(4k\cdot\int_0^t\mathrm{d}s\int_{-\infty}^\infty\mathrm{d}y\, p_{t-s}^2(y-x)\big\|\sigma\big(u_s^{(n)}(y)\big)\big\|_k^2\bigg)^{1/2}$$

$$\leq (p_t * u_0)(x) + \bigg(4k\mathrm{Lip}_\sigma^2\cdot\int_0^t\mathrm{d}s\int_{-\infty}^\infty\mathrm{d}y\, p_{t-s}^2(y-x)\mu_s^{(n,k)}(y)\bigg)^{1/2}$$

$$= (p_t * u_0)(x) + \big(4k\mathrm{Lip}_\sigma^2\cdot\big(p^2\circledast\mu^{(n,k)}\big)_t(x)\big)^{1/2}. \tag{6.5}$$

Notice, that (3.4) implies that $|(p_t * u_0)(x)|^2 \leq u_0(\mathbf{R})p_t(0)(p_t * u_0)(x)$. Now, define $C_k := 8(1 \vee k\mathrm{Lip}_\sigma^2)$, and note that

$$\mu_t^{(n+1,k)}(x) \leq 2\big|(p_t * u_0)(x)\big|^2 + 8k\mathrm{Lip}_\sigma^2\cdot\big(p^2\circledast\mu^{(n,k)}\big)_t(x)$$

$$\leq C_k\big[\big|(p_t * u_0)(x)\big|^2 + \big(p^2\circledast\mu^{(n,k)}\big)_t(x)\big]$$

$$\vdots$$

$$\leq \sum_{j=0}^n C_k^{j+1}\big(\big(\underbrace{p^2\circledast\cdots\circledast p^2}_{j\text{ times}}\big)\circledast(p_\bullet * u_0)^2\big)_t(x). \tag{6.6}$$

We apply Lemma 5.3 to find that

$$\mu_t^{(n+1,k)}(x) \leq C_k u_0(\mathbf{R})p_t(0)(p_t * u_0)(x)\cdot\sum_{j=0}^n\bigg(2C_k\Theta\cdot\int_0^t p_s(0)\,\mathrm{d}s\bigg)^j$$

$$\leq 2C_k u_0(\mathbf{R})p_t(0)(p_t * u_0)(x), \tag{6.7}$$

provided that $t \leq \mathfrak{g}((4C_k\Theta)^{-1})$. This is another way to state the lemma. □

Proposition 6.2. *If $0 < t \leq \mathfrak{g}((16k\Theta\mathrm{Lip}_\sigma^2)^{-1})$, then for all integers $n \geq 0$ and for all $x \in \mathbf{R}$,*

$$\big\|u_t^{(n+1)}(x) - u_t^{(n)}(x)\big\|_k^2 \leq 2^{-n}u_0(\mathbf{R})p_t(0)(p_t * u_0)(x). \tag{6.8}$$

Proof. Define for all $n \geq 0$, $t > 0$, and $x \in \mathbf{R}$,

$$D_t^{(n+1,k)}(x) := \mathrm{E}\big(\big|u_t^{(n+1)}(x) - u_t^{(n)}(x)\big|^k\big)^{2/k} = \big\|u_t^{(n+1)}(x) - u_t^{(n)}(x)\big\|_k^2. \tag{6.9}$$

Clearly,

$$D_t^{(n+1,k)}(x) \leq 4k\int_0^t\mathrm{d}s\int_{-\infty}^\infty\mathrm{d}y\, p_{t-s}^2(y-x)\mathrm{E}\big(\big|\sigma\big(u_s^{(n)}(y)\big) - \sigma\big(u_s^{(n-1)}(y)\big)\big|^k\big)^{2/k}$$

$$\leq 4k\mathrm{Lip}_\sigma^2\cdot\int_0^t\mathrm{d}s\int_{-\infty}^\infty\mathrm{d}y\, p_{t-s}^2(y-x)D_s^{(n,k)}(y)$$

$$= 4k\text{Lip}_\sigma^2 \cdot \left(p^2 \circledast D^{(n,k)}\right)_t(x)$$

$$\leq \cdots \leq \left(4k\text{Lip}_\sigma^2\right)^n \cdot \Big(\underbrace{p^2 \circledast \cdots \circledast p^2}_{n \text{ times}} \circledast D^{(1,k)}\Big)_t(x). \tag{6.10}$$

Because $D_s^{(1,k)}(y) = |(p_s * u_0)(y)|^2$ for all $k \in [2, \infty)$, Lemma 5.3 yields

$$D_t^{(n+1,k)}(x) \leq u_0(\mathbf{R}) \left(8k\text{Lip}_\sigma^2 \Theta \int_0^t p_r(0)\, dr\right)^n p_t(0)(p_t * u_0)(x), \tag{6.11}$$

and hence the result follows. $\qquad\square$

Let us conclude this section by making a few remarks about the predictability of the Picard iterates $u^{(1)}, u^{(2)}, \ldots$. [We thank Dr. Le Chen and Professor Robert Dalang for correctly pointing out to us that this issue requires an explanation.] In the case that $\mathcal{L}f = f''$, a detailed proof can be found in Chen and Dalang [8].

We wish to demonstrate that if Z is a predictable random field and satisfies the integrability condition

$$\int_0^t ds \int_{-\infty}^\infty dy \left[p_{t-s}(y-x)\right]^2 \mathrm{E}\big(|Z_s(y)|^2\big) < \infty, \tag{6.12}$$

then $(t, x) \mapsto \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x) Z_s(y) W(ds\, dy)$ defines a predictable random field. Thanks to the construction of space–time stochastic integrals, due to Walsh [23], it suffices to consider only the case that Z is an "elementary random field" [23], (2.5), p. 292, and this reduces our problem to one about showing that the Gaussian field

$$(t, x) \mapsto \Gamma_t(x) := \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x) W(ds\, dy) \tag{6.13}$$

is itself a predictable random field. It is not hard to verify the following "stochastic Fubini theorem":

$$(\Gamma_t * \varphi)(x) = \int_{(0,t)\times\mathbf{R}} (p_{t-s} * \varphi)(y) W(ds\, dy) \quad \text{a.s.}, \tag{6.14}$$

valid for every nonrandom rapidly-decreasing test function $\varphi : \mathbf{R} \to \mathbf{R}$. A variant of this can be found in Walsh [23], Theorem 2.6, p. 296; the present formulation can be proved in a similar way.

Standard facts about Gaussian random fields and Lévy processes imply that $(t, x) \mapsto (\Gamma_t * \varphi)(x)$ is continuous a.s. [up to a modification], and this implies that $(t, x) \mapsto (\Gamma_t * \varphi)(x)$ is a predictable random field. Finally, let φ_ε denote the probability density function of a mean-zero normal distribution on $\mathbf{R}$ with variance ε. Then, one can check directly that

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [0,T]} \sup_{x \in \mathbf{R}} \mathrm{E}\big(|(\Gamma_t * \varphi_\varepsilon)(x) - \Gamma_t(x)|^2\big) = 0 \tag{6.15}$$

for all $T \in (0, \infty)$. In accord with the Walsh theory [23], this is sufficient for the predictability of the Gaussian random field Γ.

7. Proof of Theorem 3.3

We prove Theorem 3.3 in two parts: First we show that there exists a solution up to time

$$\mathfrak{T} := \mathfrak{g}\big((64\Theta[1 \vee \text{Lip}_\sigma^2])^{-1}\big). \tag{7.1}$$

From there on, it is easy to produce an all-time solution, starting from time $t = \mathfrak{T}$.

Let $\{u^{(n)}\}_{n=0}^\infty$ be the described Picard iterates defined in (6.1). Since

$$0 < \mathfrak{T} \leq \mathfrak{g}\big((32\Theta\text{Lip}_\sigma^2)^{-1}\big), \tag{7.2}$$

Proposition 6.2 implies that the sequence of random variables $\{u_t^{(n)}(x)\}_{n\in\mathbf{N}}$ converge in $L^2(\Omega)$ for every $0 < t \leq \mathfrak{T}$ and $x \in \mathbf{R}$.[3]

Define $U_t(x) := \lim_{n\to\infty} u_t^{(n)}(x)$, where the limit is taken in $L^2(\Omega)$. By default, $\{U_t(x); x \in \mathbf{R}, t \in (0, \mathfrak{T}]\}$ is a predictable random field such that

$$\lim_{n\to\infty} \mathrm{E}\big(\big|u_t^{(n)}(x) - U_t(x)\big|^k\big) = 0 \quad \text{for all } x \in \mathbf{R}, \tag{7.3}$$

provided that $t \in (0, \mathfrak{T}_k]$, where $\mathfrak{T}_k := \mathfrak{g}((32k\Theta[1 \vee \mathrm{Lip}_\sigma^2])^{-1})$. (Notice that $\mathfrak{T}_2 = \mathfrak{T}$.) Moreover, Proposition 6.1 tells us that

$$\big\|U_t(x)\big\|_k \leq 4k^{1/2}[1 \vee \mathrm{Lip}_\sigma]\sqrt{u_0(\mathbf{R})\,p_t(0)(p_t * u_0)(x)} \tag{7.4}$$

for all $x \in \mathbf{R}$ and $t \in (0, \mathfrak{T}_k]$. Finally, these remarks readily imply that

$$U_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y - x)\sigma\big(U_s(y)\big)W(\mathrm{d}s\,\mathrm{d}y) \tag{7.5}$$

for all $x \in \mathbf{R}$ and $t \in (0, \mathfrak{T}]$. In other words, U is a mild solution to (3.1) up to the nonrandom time $\mathfrak{T}$.

Next we define a space–time white noise $\dot{\mathcal{W}}$ by defining its Wiener integrals as follows: For all $h \in L^2(\mathbf{R}_+ \times \mathbf{R})$,

$$\int_{\mathbf{R}_+\times\mathbf{R}} h_s(y)\mathcal{W}(\mathrm{d}s\,\mathrm{d}y) := \int_{(\mathfrak{T},\infty)\times\mathbf{R}} h_{s-\mathfrak{T}}(y)W(\mathrm{d}s\,\mathrm{d}y). \tag{7.6}$$

In other words, $\dot{\mathcal{W}}$ is obtained from $\dot{W}$ by shifting the time $\mathfrak{T}$ steps. Induction reveals that every $\{u_t^{(n)}(x); x \in \mathbf{R}, t \in (0, \mathfrak{T}]\}$ is independent of the space–time white noise $\dot{\mathcal{W}}$. Therefore, so is the short-time solution $\{U_t(x); x \in \mathbf{R}, t \in (0, \mathfrak{T}]\}$.

Next let $V := \{V_t(x)\}_{x\in\mathbf{R},t>0}$ denote the mild solution to the stochastic heat equation

$$\frac{\partial}{\partial t}V_t(x) = (\mathcal{L}V_t)(x) + \sigma\big(V_t(x)\big)\dot{\mathcal{W}}_t(x), \tag{7.7}$$

subject to $V_0(x) = U_\mathfrak{T}(x)$. Since $U_\mathfrak{T}$ is independent of the noise $\dot{\mathcal{W}}$, the preceding has a unique solution, thanks to Dalang's theorem (Theorem 3.1). And since $\sup_{x\in\mathbf{R}} \|U_\mathfrak{T}(x)\|_2 < \infty$ for all $\varepsilon > 0$, there exists $D_\varepsilon \in (0,\infty)$ such that for all $t > 0$, and $x \in \mathbf{R}$,

$$\mathrm{E}\big(\big|V_t(x)\big|^2\big) \leq D_\varepsilon^2 \mathrm{e}^{(1+\varepsilon)\gamma(2)t}, \tag{7.8}$$

thanks to Theorem 3.2. Finally, we define for all $x \in \mathbf{R}$,

$$u_t(x) := \begin{cases} U_t(x) & \text{if } t \in (0, \mathfrak{T}], \\ V_{t-\mathfrak{T}}(x) & \text{if } t > \mathfrak{T}. \end{cases} \tag{7.9}$$

Then it is easy to see that the random field u is predictable, and is a mild solution to (3.1) for all $t > 0$, subject to initial measure being u_0. Uniqueness is a standard consequence of (3.9).

Let us now consider $k > 2$ and follow the same argument as above, but use $\mathfrak{T}_k$ instead of $\mathfrak{T}$; i.e., we run the solution U up to time $\mathfrak{T}_k$ only and then keep going with the classical technique of Dalang. Then, a similar argument leads us to a moment estimate of order k for $u_t(x)$, thanks to another application of Theorem 3.2. The solution obtained from $\mathfrak{T}_k$ is the same as the one obtained when stopping at $\mathfrak{T}$ by the uniqueness result.

We pause to state an immediate corollary of the proof of Theorem 3.3, as it might be of some independent interest. In words, the following shows that if σ has truly-linear growth and $\Theta < \infty$, then the solution to (3.1) has nontrivial moment Liapounov exponents.

---

[3]As is customary, Ω denotes the underlying sample space on which random variables are defined, and $L^2(\Omega)$ denotes the collection of all random vaiables on Ω that have two finite moments.

Corollary 7.1. *Suppose* $\Theta < \infty$, $L_\sigma := \inf_{x \in \mathbf{R}} |\sigma(x)/x| > 0$, *and* u_0 *is a finite Borel measure on* $\mathbf{R}$. *Then*,

$$0 < \limsup_{t \to \infty} \frac{1}{t} \log E\big(|u_t(x)|^k\big) < \infty \tag{7.10}$$

for all $k \in [2, \infty)$.

Proof. Let $\{V_t(x)\}_{t>0, x \in \mathbf{R}}$ denote the post-$\mathfrak{T}_k$ process used in the proof of Theorem 3.3. It suffices to prove that

$$0 < \limsup_{t \to \infty} \frac{1}{t} \log E\big(|V_t(x)|^k\big) < \infty \tag{7.11}$$

for all $k \in [2, \infty)$. This follows from [15]. $\qquad\square$

The proof of Theorem 3.3 is based on the idea that one can solve (3.1) up to time $\mathfrak{T}$, using the method of the present paper; and then from time $\mathfrak{T}$ on we paste the more usual solution, shifted by time $\mathfrak{T}$ time steps, in order to obtain a global solution to (3.1). But in fact since the pre-$\mathfrak{T}$ and the post-$\mathfrak{T}$ solutions are unique [a.s.], we could replace $\mathfrak{T}$ by any other time η (not necessarily one of the $\mathfrak{T}_k$) before it as well. The following merely enunciates these observations in the form of a proposition. The proof follows from the fact that the sequence $\mathfrak{T}_k$ goes to 0 as k increases. We omit the details. However, we state this simple result explicitly, as it will be central to our proof of Theorem 1.1.

Proposition 7.2. *Choose and fix some* $\eta \in (0, \mathfrak{T})$, *and let us define the predictable random field* $\{\bar{V}_t(x)\}_{t>0, x \in \mathbf{R}}$ *exactly as we defined* $\{V_t(x)\}_{t>0, x \in \mathbf{R}}$, *except with* $\mathfrak{T}$ *replaced everywhere by* η. *Finally define* $\bar{u}_t(x)$ *as we did* $u_t(x)$, *except we replace* $(U, V, \mathfrak{T})$ *by* $(U, \bar{V}, \eta)$; *that is*,

$$\bar{u}_t(x) := \begin{cases} U_t(x) & \text{if } t \in (0, \eta], \\ \bar{V}_{t-\eta}(x) & \text{if } t > \eta. \end{cases} \tag{7.12}$$

Then, the random field $\bar{u}$ *is a modification of the random field* u.

8. Stability and positivity

Let u denote the solution to (3.1), as defined in Theorem 3.3, starting from a finite Borel measure u_0. We have seen that $(p_t * u_0)(x)$ is finite for all $t > 0$ and $x \in \mathbf{R}$ fixed. Also, for $\varepsilon > 0$, let $U^{(\varepsilon)}$ denote the solution to (3.1), starting from the [bounded and measurable] initial function $p_\varepsilon * u_0$.

Proposition 8.1 (Stability). *For every* $t, \varepsilon > 0$, $u_t, U_t^{(\varepsilon)} \in L^2(\Omega \times \mathbf{R})$. *Moreover, the following bound is valid for all* β *such that* $\Upsilon(\beta) \geq (2\mathrm{Lip}_\sigma^2)^{-1}$:

$$\int_0^\infty e^{-\beta t}\, dt \int_{-\infty}^\infty dx\, E\big(|u_t(x) - U_t^{(\varepsilon)}(x)|^2\big)$$

$$\leq \frac{[u_0(\mathbf{R})]^2}{\pi} \int_{-\infty}^\infty \frac{(1 - e^{-\varepsilon \Psi(\xi)})^2}{\beta + 2\Psi(\xi)}\, d\xi. \tag{8.1}$$

In particular, the left-hand side tends to zero as $\varepsilon \downarrow 0$.

Proof. Let $u_t^{(n)}(x)$ be the nth Picard iterate, defined in (6.1). Then,

$$\big\|u_t^{(n+1)}(x)\big\|_2^2$$

$$\leq \big|(p_t * u_0)(x)\big|^2 + \mathrm{Lip}_\sigma^2 \cdot \int_0^t ds \int_{-\infty}^\infty dy\, p_{t-s}^2(y - x) \big\|u_s^{(n)}(y)\big\|_2^2. \tag{8.2}$$

We integrate $[\mathrm{d}x]$ to find that

$$
\mathrm{E}\big(\big\|u_t^{(n+1)}\big\|_{L^2(\mathbf{R})}^2\big)
$$

$$
\leq \|p_t * u_0\|_{L^2(\mathbf{R})}^2 + \mathrm{Lip}_\sigma^2 \cdot \int_0^t \|p_{t-s}\|_{L^2(\mathbf{R})}^2 \cdot \mathrm{E}\big(\big\|u_s^{(n)}\big\|_{L^2(\mathbf{R})}^2\big)\,\mathrm{d}s
$$

$$
= \|p_t * u_0\|_{L^2(\mathbf{R})}^2 + \mathrm{Lip}_\sigma^2 \cdot \int_0^t p_{2(t-s)}(0) \cdot \mathrm{E}\big(\big\|u_s^{(n)}\big\|_{L^2(\mathbf{R})}^2\big)\,\mathrm{d}s; \tag{8.3}
$$

see (5.3). Note that $|\hat{u}_0(\xi)| \leq u_0(\mathbf{R})$, whence

$$
\|p_t * u_0\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-2t\Psi(\xi)}\big|\hat{u}_0(\xi)\big|^2\,\mathrm{d}\xi \leq \big|u_0(\mathbf{R})\big|^2 \cdot p_{2t}(0), \tag{8.4}
$$

thanks to Plancherel's theorem. [One can construct an alternative proof of this inequality, using the semigroup property of p_t and the Young inequality.] Therefore,

$$
\mathrm{E}\big(\big\|u_t^{(n+1)}\big\|_{L^2(\mathbf{R})}^2\big)
$$

$$
\leq \big|u_0(\mathbf{R})\big|^2 \cdot p_{2t}(0) + \mathrm{Lip}_\sigma^2 \cdot \int_0^t p_{2(t-s)}(0) \cdot \mathrm{E}\big(\big\|u_s^{(n)}\big\|_{L^2(\mathbf{R})}^2\big)\,\mathrm{d}s. \tag{8.5}
$$

Define, for all predictable random fields f, the quantity

$$
\mathcal{K}_t(f) := \sup_{s \in (0,t)}\left[\frac{\mathrm{E}(\|f_s\|_{L^2(\mathbf{R})}^2)}{1 + p_{2s}(0)}\right], \tag{8.6}
$$

in order to find that

$$
\mathrm{E}\big(\big\|u_t^{(n+1)}\big\|_{L^2(\mathbf{R})}^2\big)
$$

$$
\leq \big|u_0(\mathbf{R})\big|^2 \cdot p_{2t}(0) + \mathrm{Lip}_\sigma^2 \cdot \mathcal{K}_t\big(u^{(n)}\big) \cdot \int_0^t p_{2(t-s)}(0)\big[1 + p_{2s}(0)\big]\,\mathrm{d}s
$$

$$
\leq \big|u_0(\mathbf{R})\big|^2 \cdot p_{2t}(0) + \frac{1}{2}\mathrm{Lip}_\sigma^2 \cdot \mathcal{K}_t\big(u^{(n)}\big) \cdot \left(\int_0^{2t} p_s(0)\,\mathrm{d}s + \int_0^{2t} p_{2t-s}(0)p_s(0)\,\mathrm{d}s\right)
$$

$$
\leq \big|u_0(\mathbf{R})\big|^2 \cdot p_{2t}(0) + \frac{1}{2}\mathrm{Lip}_\sigma^2 \cdot \mathcal{K}_t\big(u^{(n)}\big) \cdot \int_0^{2t} p_s(0)\,\mathrm{d}s \cdot \big(1 + 2\Theta p_{2t}(0)\big); \tag{8.7}
$$

see Lemma 5.1. Since $\Theta \geq 1$, this leads us to the following:

$$
\mathcal{K}_t\big(u^{(n+1)}\big) \leq \big|u_0(\mathbf{R})\big|^2 + \mathrm{Lip}_\sigma^2\Theta \cdot \mathcal{K}_t\big(u^{(n)}\big) \cdot \int_0^{2t} p_s(0)\,\mathrm{d}s. \tag{8.8}
$$

Recall $\mathfrak{T}$ from (7.1). If $t \in (0, \mathfrak{T}/2]$, then $\int_0^{2t} p_s(0)\,\mathrm{d}s$ is certainly bounded above by $(4\Theta\mathrm{Lip}_\sigma^2)^{-1}$, whence we have

$$
\mathcal{K}_t\big(u^{(n+1)}\big) \leq \big|u_0(\mathbf{R})\big|^2 + \frac{1}{2}\mathcal{K}_t\big(u^{(n)}\big) \leq \cdots \leq 2\big|u_0(\mathbf{R})\big|^2, \tag{8.9}
$$

since $\mathcal{K}_t(u^{(0)}) = 0$. Therefore,

$$
\mathrm{E}\big(\|u_t\|_{L^2(\mathbf{R})}^2\big) \leq 2\big|u_0(\mathbf{R})\big|^2\big(1 + p_{2t}(0)\big) \quad \text{for all } t \in (0, \mathfrak{T}/2]. \tag{8.10}
$$

One proves, similarly, that uniformly for all $\varepsilon > 0$,

$$\mathrm{E}\big(\big\|U_t^{(\varepsilon)}\big\|_{L^2(\mathbf{R})}^2\big) \le 2\big|u_0(\mathbf{R})\big|^2\big(1 + p_{2t}(0)\big) \quad \text{for all } t \in (0, \mathfrak{T}/2]. \tag{8.11}$$

By Proposition 7.2, the process $\{u_{t+\mathfrak{T}/2}\}_{t \ge 0}$ starts from $u_{\mathfrak{T}/2} \in L^2(\Omega \times \mathbf{R})$ and solves the shifted form of (3.1), and hence is in $L^2(\mathbf{R})$ for all time $t \ge \mathfrak{T}/2$ by Foondun and Khoshnevisan [14, Theorem 1.1]; for earlier developments along similar lines see Dalang and Mueller [12]. Similar remarks also apply to $\{U_t^{(\varepsilon)}(x)\}_{x \in \mathbf{R}, t>0}$.

Define,

$$\mathcal{D}_t^{(\varepsilon)}(x) := \mathrm{E}\big(\big|u_t(x) - U_t^{(\varepsilon)}(x)\big|^2\big). \tag{8.12}$$

Since $p_t * (p_\varepsilon * u_0) = p_{t+\varepsilon} * u_0$,

$$\mathcal{D}_t^{(\varepsilon)}(x) \le \big|(p_t * u_0)(x) - (p_{t+\varepsilon} * u_0)(x)\big|^2 + \mathrm{Lip}_\sigma^2 \cdot \big(p^2 \circledast \mathcal{D}^{(\varepsilon)}\big)_t(x). \tag{8.13}$$

We integrate $[\mathrm{d}x]$ and apply the Plancherel theorem to find that

$$\begin{aligned}
\big\|\mathcal{D}_t^{(\varepsilon)}\big\|_{L^1(\mathbf{R})} &\le \frac{[u_0(\mathbf{R})]^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2t\Psi(\xi)}\big(1 - \mathrm{e}^{-\varepsilon\Psi(\xi)}\big)^2 \, \mathrm{d}\xi \\
&\quad + \mathrm{Lip}_\sigma^2 \cdot \int_0^t \|p_{t-s}\|_{L^2(\mathbf{R})}^2 \cdot \big\|\mathcal{D}_s^{(\varepsilon)}\big\|_{L^1(\mathbf{R})} \, \mathrm{d}s.
\end{aligned} \tag{8.14}$$

We integrate one more time $[\exp(-\beta t)\,\mathrm{d}t]$ in order to see that

$$\mathcal{E}_\beta^{(\varepsilon)} := \int_0^\infty \mathrm{e}^{-\beta t}\big\|\mathcal{D}_t^{(\varepsilon)}\big\|_{L^1(\mathbf{R})} \, \mathrm{d}t \tag{8.15}$$

satisfies

$$\mathcal{E}_\beta^{(\varepsilon)} \le \frac{[u_0(\mathbf{R})]^2}{2\pi} \int_{-\infty}^{\infty} \frac{(1 - \mathrm{e}^{-\varepsilon\Psi(\xi)})^2}{\beta + 2\Psi(\xi)} \, \mathrm{d}\xi + \mathrm{Lip}_\sigma^2 \Upsilon(\beta) \cdot \mathcal{E}_\beta^{(\varepsilon)}. \tag{8.16}$$

Pick β large enough that $\Upsilon(\beta) \le (2\mathrm{Lip}_\sigma^2)^{-1}$ to obtain the claimed inequality of the proposition. And since $\Upsilon(\beta) < \infty$, the final assertion about convergence to 0 follows from this inequality and the dominated convergence theorem. $\quad\square$

Proposition 8.2 (Positivity). *If $\sigma(0) = 0$ and $u_0(\mathbf{R}) > 0$, then $u_t(x) \ge 0$ a.s. for all $t > 0$ and $x \in \mathbf{R}$.*

Proof. Since u_0 is a finite measure, it follows that $U_0^{(\varepsilon)}(x) = (p_\varepsilon * u_0)(x) \le p_\varepsilon(0)u_0(\mathbf{R}) < \infty$, uniformly in $x \in \mathbf{R}$. According to Mueller's comparison principle, because $U_0^{(\varepsilon)}(x) \ge 0$, it follows that $U_t^{(\varepsilon)}(x) \ge 0$ a.s. for all $t > 0$ and $x \in \mathbf{R}$. [Mueller's comparison principle [22] was proved originally in the case that $\mathcal{L}$ is proportional to the Laplacian. This comparison principle can be shown to hold in the more general setting of the present paper as well, though we admit that this undertaking requires some effort when $\mathcal{L}$ is not proportional to the Laplacian.]

Thanks to Proposition 8.1, $\mathrm{P}\{u_t(x) \ge 0 \text{ for a.e. } t > 0 \text{ and } x \in \mathbf{R}\} = 1$. In particular, $\mathrm{P}\{u_t(x) \ge 0 \text{ for a.e. } t \ge \eta \text{ and } x \in \mathbf{R}\} = 1$ for all $\eta > 0$. This shows that

$$\mathrm{P}\big\{\bar{V}_t(x) \ge 0 \text{ for almost every } t > 0 \text{ and } x \in \mathbf{R}\big\} = 1, \tag{8.17}$$

where $\bar{V}$ was defined in Proposition 7.2. According to Dalang's theory (Theorem 3.1), $(t, x) \mapsto \bar{V}_t(x)$ is continuous in probability. Therefore, it follows that $\bar{V}_t(x) \ge 0$ a.s. for every $t > 0$ and $x \in \mathbf{R}$ [note the order of the quantifiers]. Therefore, a second application of Proposition 7.2 implies the proposition. $\quad\square$

9. Proof of Theorem 1.1

Throughout this section, we assume that $\sigma(0) = 0$. We simplify the notation somewhat by assuming, without a great loss in generality, that $\varkappa = 1$. In this way, $\mathcal{L}f = (1/2) f''$ is the generator of standard Brownian motion, and $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ satisfies (3.3) with

$$p_t(x) := \frac{\mathrm{e}^{-x^2/(2t)}}{(2\pi t)^{1/2}} \quad \text{for } x \in \mathbf{R} \text{ and } t > 0. \tag{9.1}$$

The proof of Theorem 1.1 uses the theory of the present paper, but also borrows heavily from the method of Foondun and Khoshnevisan [14].

Lemma 9.1. *Suppose u_0 is a finite measure that is supported in $[-K, K]$ for some $K > 0$. Then for all $t > 0$, $k \in [1, \infty)$, and $x \in \mathbf{R}$,*

$$\limsup_{|x|\to\infty} \frac{1}{x^2} \log \mathrm{E}\left(|u_t(x)|^k\right) < 0. \tag{9.2}$$

Proof. This is essentially the same result as [14], Lemma 3.3. We mention how to make the requisite changes to the proof of the said result in order to derive the present form of the lemma.

Since $u_t(x) \geq 0$ a.s. (Proposition 8.2), we obtain from (3.3) the following:

$$\mathrm{E}\left(|u_t(x)|\right) = (p_t * u_0)(x) = \int_{-K}^{K} \frac{\mathrm{e}^{-(x-y)^2/(2t)}}{(2\pi t)^{1/2}} u_0(\mathrm{d}y) \leq \mathrm{const} \cdot \mathrm{e}^{-x^2/(4t)}, \tag{9.3}$$

using the elementary inequality: $(x - y)^2 \geq (x^2/2) - K^2$, valid when $|y| \leq K$. And because the preceding constant does not depend on x, we have for all $k \in [2, \infty)$ and $c \in (0, \infty)$,

$$\mathrm{E}\left(|u_t(x)|^k\right) \leq c^k + \left\{\mathrm{E}\left(|u_t(x)|^{2k}\right)\right\}^{1/2} \cdot \left[\mathrm{P}\{u_t(x) > c\}\right]^{1/2}$$
$$\leq c^k + \mathrm{const} \cdot \left[\mathrm{P}\{u_t(x) > c\}\right]^{1/2}; \tag{9.4}$$

this follows readily from the estimate of Theorem 3.3. We emphasize that the "const" does not depend on (c, x). Owing to (9.3), this leads us to

$$\mathrm{E}\left(|u_t(x)|^k\right) \leq \left[c^k + \frac{\alpha}{\sqrt{c}}\mathrm{e}^{-x^2/(8t)}\right], \tag{9.5}$$

where $\alpha \in (0, \infty)$ does not depend on c. Therefore we may optimize over $c > 0$ in order to obtain $\limsup_{|x|\to\infty} x^{-2} \times \log \mathrm{E}(|u_t(x)|^k) \leq -k/(4(2k+1)t)$. The lemma follows readily from this. $\qquad\square$

Lemma 9.2. *For all $t > 0$ and $k \in [1, \infty)$,*

$$\sup_{j\in\mathbf{Z}} \sup_{j\leq x<x'\leq j+1} \mathrm{E}\left(\frac{|u_t(x) - u_t(x')|^{2k}}{|x - x'|^k}\right) < \infty. \tag{9.6}$$

Proof. It is not so easy to prove this result directly from (3.3), since the map $s \mapsto u_s(y)$ is singular near $s = 0$. Because $t > 0$ is fixed in the statement of our lemma, we may instead apply Proposition 7.2 in order to see that our lemma follows from the following.

Claim. *Suppose $\bar{V}$ solves* (3.1), *where $\bar{V}_0$ is a random field, independent of the noise, and $m_\nu := \sup_{x \in \mathbf{R}} \|\bar{V}_0(x)\|_\nu < \infty$ for all $\nu \in [2, \infty)$. Then for every fixed $t > 0$ and $k \in [1, \infty)$, there exists a positive and finite constant K such that*

$$\sup_{\substack{x, x' \in \mathbf{R}: \\ |x - x'| \le 1}} \frac{\|\bar{V}_t(x) - \bar{V}_t(x')\|_{2k}}{|x - x'|^{1/2}} \le K. \tag{9.7}$$

We prove Claim by adapting the method of proof of [14], Lemma 3.4, to the present setting.

First, we assert that for all fixed $t > 0$ and $k \in [1, \infty)$,

$$\left\|(p_t * \bar{V}_0)(x) - (p_t * \bar{V}_0)(x')\right\|_{2k} \le \text{const} \cdot |x - x'|, \tag{9.8}$$

where the constant is independent of x, x'. Indeed, according to the Minkowski inequality,

$$\begin{aligned}
&\left\|(p_t * \bar{V}_0)(x) - (p_t * \bar{V}_0)(x')\right\|_{2k} \\
&\le \int_{-\infty}^{\infty} \left\|\bar{V}_0(x)\right\|_{2k} \left|p_t(y - x) - p_t(y - x')\right| \mathrm{d}y \\
&\le m_{2k} \cdot \int_{-\infty}^{\infty} \left|p_t(y - |x - x'|) - p_t(y)\right| \mathrm{d}y.
\end{aligned} \tag{9.9}$$

We estimate the last integral by applying the fundamental theorem of calculus – using the fact that $p'_t(z) = -(z/t) p_t(z)$ – in order to deduce (9.8).

Next we observe that, as a consequence of (3.3), (9.8), and the BDG inequality [using the Carlen–Kree bound [6] on Davis's optimal constant [13] in the Burkholder–Davis–Gundy inequality [3–5]],

$$\begin{aligned}
&\left\|\bar{V}_t(x) - \bar{V}_t(x')\right\|_{2k} \\
&\le \text{const} \cdot |x - x'| \\
&\quad + \left(8k \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left\|\sigma(\bar{V}_s(y))\right\|_{2k}^2 \cdot \left|p_{t-s}(y - x) - p_{t-s}(y - x')\right|^2\right)^{1/2};
\end{aligned} \tag{9.10}$$

see [15] for details for deriving this sort of inequality. Since $|\sigma(z)/z| \le \text{Lip}_\sigma$, we are led to the bound

$$\begin{aligned}
&\left\|\bar{V}_t(x) - \bar{V}_t(x')\right\|_{2k} \\
&\le \text{const} \cdot |x - x'| \\
&\quad + \left(8k \text{Lip}_\sigma^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left\|\bar{V}_s(y)\right\|_{2k}^2 \cdot \left|p_{t-s}(y - x) - p_{t-s}(y - x')\right|^2\right)^{1/2}.
\end{aligned} \tag{9.11}$$

Theorem 3.2, applied to the present choice of $\mathcal{L}$, tells us that $\|\bar{V}_s(y)\|_{2k} \le \exp(csk^2)$ for a constant $c \in (1, \infty)$ that does not depend on any of the parameters except Lip_σ. Therefore, there exists a finite constant $C := C(\text{Lip}_\sigma) > 1$, such that

$$\begin{aligned}
&\left\|\bar{V}_t(x) - \bar{V}_t(x')\right\|_{2k} \\
&\le \text{const} \cdot |x - x'| + Ck^{1/2} e^{ck^2 t} \left(\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \left|p_s(y - x) - p_s(y - x')\right|^2\right)^{1/2}.
\end{aligned} \tag{9.12}$$

By Plancherel's formula,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|p_s(y - x) - p_s(y - x')\right|^2 \mathrm{d}y &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\xi^2 s/2} \left|e^{-i\xi x} - e^{-i\xi x'}\right|^2 \mathrm{d}\xi \\
&\le \frac{2}{\pi} \int_0^{\infty} e^{-\xi^2 s/2} \left(1 \wedge \xi |x - x'|\right)^2 \mathrm{d}\xi.
\end{aligned} \tag{9.13}$$

Consequently,

$$
\int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \left| p_s(y-x) - p_s(y-x') \right|^2
$$

$$
\leq \mathrm{e}^t \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \left| p_s(y-x) - p_s(y-x') \right|^2
$$

$$
\leq \frac{2\mathrm{e}^t}{\pi} \int_0^\infty \frac{(1 \wedge \xi|x-x'|)^2}{1+(\xi^2/2)}\,\mathrm{d}\xi. \tag{9.14}
$$

Let $\mathcal{I}$ denote the latter integral. For simplicity, let us denote $\delta = |x-x'|$. It suffices to prove that $\mathcal{I} \leq 3\delta$; this inequality implies (9.7), whence the lemma. In order to estimate $\mathcal{I}$ we write it as $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$, where $\mathcal{I}_1 := \int_0^1 (\cdots)\,\mathrm{d}\xi$, $\mathcal{I}_2 := \int_1^{1/\delta}(\cdots)\,\mathrm{d}\xi$, and $\mathcal{I}_3 := \int_{1/\delta}^\infty(\cdots)\,\mathrm{d}\xi$. Note that: $\mathcal{I}_1 \leq \delta^2 \int_0^1 \xi^2\,\mathrm{d}\xi = \delta^2/3$; $\mathcal{I}_2 \leq \delta^2 \int_1^{1/\delta}\mathrm{d}\xi = \delta - \delta^2$; and $\mathcal{I}_3 \leq 2\int_{1/\delta}^\infty \xi^{-2}\,\mathrm{d}\xi = 2\delta$. Therefore, $\mathcal{I} \leq 3\delta - (2/3)\delta^2 < 3\delta$, as asserted. $\qquad\square$

Our next result follows immediately from Lemma 9.2 and a quantitative form of the Kolmogorov continuity theorem. The proof is exactly the same as that of Ref. [14], Lemma 3.6, and is therefore omitted.

Lemma 9.3. *For all $t > 0$, $k \in [1, \infty)$, and $\varepsilon \in (0, 1)$,*

$$
\sup_{j \in \mathbf{Z}} \left\| \sup_{j \leq x < x' \leq j+1} \frac{|u_t(x) - u_t(x')|^2}{|x-x'|^{1-\varepsilon}} \right\|_{2k} < \infty. \tag{9.15}
$$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We follow carefully the arguments of Foondun and Khoshnevisan [14], (3.43) and on.
For all $j \geq 1$, we may write [14], (3.43),

$$
\sup_{\log j \leq x \leq \log(j+1)} \left| u_t(x) \right|^6 \leq 32\left(\left| u_t(\log j) \right|^6 + \left(\log(j+1) - \log j \right)^3 \Omega_j^3 \right), \tag{9.16}
$$

where

$$
\Omega_j := \sup_{\log j \leq x < x' \leq \log(j+1)} \frac{|u_t(x) - u_t(x')|^2}{|x-x'|^{1/2}}. \tag{9.17}
$$

Consequently,

$$
\mathrm{E}\left(\sup_{\log j \leq x \leq \log(j+1)} \left| u_t(x) \right|^6 \right)
$$

$$
\leq 32\left(\mathrm{E}\left(|u_t(\log j)|^6 \right) + \left(\log\left[1 + \frac{1}{j} \right] \right)^3 \mathrm{E}\left(\Omega_j^3 \right) \right). \tag{9.18}
$$

According to Lemma 9.1, $\mathrm{E}(|u_t(\log j)|^6) \leq \exp(-D(\log j)^2)$ for a positive and finite constant D that does not depend on j. Lemma 9.3 tells us that $\mathrm{E}(\Omega_j^3)$ is bounded uniformly in j. Since $\log(1 + j^{-1}) \leq j^{-1}$, we therefore have

$$
\mathrm{E}\left(\sup_{\log j \leq x \leq \log(j+1)} \left| u_t(x) \right|^6 \right) \leq \mathrm{const} \cdot \left(\mathrm{e}^{-D(\log j)^2} + j^{-3} \right), \tag{9.19}
$$

where "const" can be chosen independently of j. Because the preceding is summable [in j], it follows that $\sup_{x \geq 0} |u_t(x)| \in L^6(\Omega)$, whence $\sup_{x \geq 0} |u_t(x)| < \infty$ a.s. Similarly, $\sup_{x \leq 0} |u_t(x)| < \infty$ a.s. This completes the proof, since we know that $u_t(x) \geq 0$ a.s. for all $t > 0$ and $x \in \mathbf{R}$ (Proposition 8.2), and $x \mapsto u_t(x)$ is continuous (Lemma 9.3). $\qquad\square$

Acknowledgments

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