

# Stochastic differential equations with Sobolev drifts and driven by $\alpha$ -stable processes

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**Abstract.** In this article we prove the pathwise uniqueness for stochastic differential equations in  $\mathbb{R}^d$  with time-dependent Sobolev drifts, and driven by symmetric  $\alpha$ -stable processes provided that  $\alpha \in (1, 2)$  and its spectral measure is non-degenerate. In particular, the drift is allowed to have jump discontinuity when  $\alpha \in (\frac{2d}{d+1}, 2)$ . Our proof is based on some estimates of Krylov's type for purely discontinuous semimartingales.

**Résumé.** Dans cet article nous prouvons l'existence et l'unicité d'équations différentielles stochastiques dans  $\mathbb{R}^d$  avec terme de dérive dépendant du temps dans un espace de Sobolev et dirigées par un processus de Lévy  $\alpha$ -stable symétrique avec  $\alpha \in (1, 2)$  et de mesure spectrale non-dégénérée.

En particulier, le terme de dérive peut avoir des discontinuités de saut quand  $\alpha \in (\frac{2d}{d+1}, 2)$ . Notre preuve est basée sur des estimations de type Krylov pour des semimartingales purement discontinues.

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## 1. Introduction and main result

Consider the following SDE driven by a symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$ :

$$dX_t = b_t(X_t) dt + dL_t, \quad X_0 = x, \quad (1.1)$$

where  $b: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a measurable function,  $(L_t)_{t \geq 0}$  is a  $d$ -dimensional symmetric  $\alpha$ -stable process defined on some filtered probability space  $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ . The aim of this paper is to study the pathwise uniqueness of SDE (1.1) with  $b$  in fractional Sobolev spaces.

Let us first briefly recall some well-known results in this direction. When  $L_t$  is a standard  $d$ -dimensional Brownian motion, Veretennikov [27] firstly proved the existence of a unique strong solution for SDE (1.1) with bounded measurable  $b$ . In [17], Krylov and Röckner relaxed the boundedness assumptions on  $b$  to the following integrability assumptions:

$$\int_0^T \left( \int_{\mathbb{R}^d} |b_t(x)|^p dx \right)^{q/p} dt < +\infty, \quad \forall T > 0, \quad (1.2)$$

provided that

$$\frac{d}{p} + \frac{2}{q} < 1. \quad (1.3)$$

Recently, in [28] we extended Krylov and Röckner’s result to the case of non-constant Sobolev diffusion coefficients and meanwhile, obtained the stochastic homeomorphism flow property of solutions and the strong Feller property. Other generalizations in various directions can be found in [9–12,14], etc.

In the case of symmetric  $\alpha$ -stable processes, the pathwise uniqueness for SDE (1.1) with irregular drift is far from being complete. When  $d = 1$ ,  $\alpha \in [1, 2)$  and  $b$  is time-independent and bounded continuous, Tanaka, Tsuchiya and Watanabe [25] proved the pathwise uniqueness of solutions to SDE (1.1). When  $d > 1$ ,  $\alpha \in [1, 2)$ , the spectral measure of  $L_t$  is non-degenerate, and  $b$  is time-independent and bounded Hölder continuous, where the Hölder index  $\beta$  satisfies

$$\beta > 1 - \frac{\alpha}{2},$$

Priola [21] recently proved the pathwise uniqueness and the stochastic homeomorphism flow property of solutions to SDE (1.1). When  $d = 1$ ,  $\alpha \in (1, 2)$  and  $b$  is only bounded measurable, Kurenok [18] obtained the existence of weak solutions for SDE (1.1) by proving an estimate of Krylov’s type: for any  $T > 0$ ,

$$\mathbb{E} \int_0^T f_t(X_t) dt \leq C \|f\|_{L^2([0,T] \times \mathbb{R})}. \tag{1.4}$$

When  $d > 1$ ,  $\alpha \in (1, 2)$  and  $b$  is time-independent and belongs to some Kato’s class, Chen, Kim and Song [7], Theorem 2.5, proved the existence of martingale solutions (equivalently weak solutions) in terms of Feller semigroup (cf. [6]). On the other hand, there are many works devoted to the study of weak uniqueness (i.e., the well-posedness of martingale problems) for SDEs with jumps. This is refereed to the survey paper of Bass [5]. Moreover, in the one dimensional case, the pathwise uniqueness for SDEs with Hölder continuous diffusion coefficients has been extensively studied (see the recent works of Fournier [13] and Li and Mytnik [20]). However, to the author’s knowledge, there are few results about the pathwise uniqueness for multidimensional SDE (1.1) with discontinuous drifts.

Before stating our main result, we recall some facts about symmetric  $\alpha$ -stable processes. Let  $(L_t)_{t \geq 0}$  be a  $d$ -dimensional symmetric  $\alpha$ -stable process. By Lévy–Khinchin’s formula, its characteristic function is given by (cf. [23])

$$\mathbb{E} e^{i\langle \xi, L_t \rangle} = e^{-t\psi(\xi)},$$

where

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, x \rangle} + i\langle \xi, x \rangle 1_{|x| \leq 1}) \nu(dx),$$

and the Lévy measure  $\nu$  with  $\nu(\{0\}) = 0$  is given by

$$\nu(U) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{1_U(r\theta)}{r^{d+\alpha}} dr \mu(d\theta), \quad U \in \mathcal{B}(\mathbb{R}^d), \tag{1.5}$$

where  $\mathcal{B}(\mathbb{R}^d)$  is the Borel  $\sigma$ -field of Euclidean space  $\mathbb{R}^d$ ,  $\mu$  is a symmetric finite measure on the unit sphere  $\mathbb{S}^{d-1} := \{\theta \in \mathbb{R}^d: |\theta| = 1\}$ , called *spectral measure* of stable process  $L_t$ . By an elementary calculation, we have

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \langle \xi, x \rangle) \nu(dx) = c_\alpha \int_{\mathbb{S}^{d-1}} |\langle \xi, \theta \rangle|^\alpha \mu(d\theta).$$

In particular, if  $\mu$  is the uniform distribution on  $\mathbb{S}^{d-1}$ , then  $\psi(\xi) = c_\alpha |\xi|^\alpha$ , here  $c_\alpha$  may be different. Throughout this paper, we make the following assumption:

**(H $^\alpha$ ):** For some  $\alpha \in (0, 2)$  and constant  $C_\alpha > 0$ ,

$$\psi(\xi) \geq C_\alpha |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \tag{1.6}$$

We remark that the above condition is equivalent that the support of spectral measure  $\mu$  is not contained in a proper linear subspace of  $\mathbb{R}^d$  (cf. [21], p. 4).

We now introduce the class of local strong solutions for SDE (1.1). Let  $\tau$  be any  $(\mathcal{F}_t)$ -stopping time. For  $x \in \mathbb{R}^d$ , let  $\mathcal{S}_b^\tau(x)$  be the class of all  $\mathbb{R}^d$ -valued  $(\mathcal{F}_t)$ -adapted càdlàg stochastic process  $X_t$  on  $[0, \tau)$  satisfying

$$P \left\{ \omega: \int_0^T |b_s(X_s(\omega))| ds < +\infty, \forall T \in [0, \tau(\omega)) \right\} = 1,$$

and such that

$$X_t = x + \int_0^t b_s(X_s) ds + L_t, \quad \forall t \in [0, \tau), \text{ a.s.}$$

The main result of the present paper is:

**Theorem 1.1.** Assume that  $(\mathbf{H}^\alpha)$  holds with  $\alpha \in (1, 2)$ , and  $b: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfies that for some  $\beta \in (1 - \frac{\alpha}{2}, 1)$ ,  $p > \frac{2d}{\alpha}$  and any  $T, R > 0$ ,

$$\sup_{t \in [0, T]} \int_{B_R} \int_{B_R} \frac{|b_t(x) - b_t(y)|^p}{|x - y|^{d+\beta p}} dx dy < +\infty \tag{1.7}$$

and

$$\sup_{(t,x) \in [0, T] \times B_R} |b_t(x)| < +\infty, \tag{1.8}$$

where  $B_R = \{x \in \mathbb{R}^d: |x| \leq R\}$ . Then, for any  $x \in \mathbb{R}^d$ , there exists an  $(\mathcal{F}_t)$ -stopping time  $\zeta(x)$  (called explosion time) and a unique strong solution  $X_t \in \mathcal{S}_b^{\zeta(x)}(x)$  to SDE (1.1) with

$$\lim_{t \uparrow \zeta(x)} X_t(x) = +\infty \quad \text{a.s. on } \{\zeta(x) < +\infty\}. \tag{1.9}$$

**Remark 1.2.** Let  $\mathcal{O}$  be a bounded smooth domain in  $\mathbb{R}^d$ . It is well-known that for any  $\beta \in (0, 1)$  and  $p \in (1, \frac{1}{\beta})$  (cf. [1]),

$$1_{\mathcal{O}} \in \mathbb{W}_p^\beta, \tag{1.10}$$

where  $\mathbb{W}_p^\beta$  is the fractional Sobolev space defined by (2.3) below. Hence, if  $\alpha \in (\frac{2d}{d+1}, 2)$ , then one can choose

$$\beta \in \left(1 - \frac{\alpha}{2}, \frac{\alpha}{2d}\right), \quad p \in \left(\frac{2d}{\alpha}, \frac{1}{\beta}\right)$$

so that Theorem 1.1 can be used to uniquely solve the following discontinuous SDE:

$$dX_t = [b^{(1)}1_{\mathcal{O}} + b^{(2)}1_{\mathcal{O}^c}](X_t) dt + dL_t, \quad X_0 = x,$$

where  $b^{(i)}, i = 1, 2$ , are two bounded and locally Hölder continuous functions with Hölder index greater than  $\beta$ . In one dimensional case, if  $\alpha \in (1, 2)$ , it is well-known that regularity condition (1.7) can be dropped (cf. [25], p. 82, Remark 1). The key point in this case is that the weak uniqueness is equivalent to the pathwise uniqueness. However, in the case of  $d \geq 2$ , it is still open that whether SDE (1.1) has a unique strong solution when  $b$  is only bounded measurable.

For proving this theorem, as in [17,28,30], we mainly study the following partial integro-differential equation (abbreviated as PIDE) by using some interpolation techniques:

$$\partial_t u = \mathcal{L}_0 u + b^i \partial_i u + f, \quad u_0(x) = 0,$$

where  $\mathcal{L}_0$  is the generator of Lévy process  $(L_t)_{t \geq 0}$  given by

$$\mathcal{L}_0 u(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x) - 1_{|z| \leq 1} z^i \partial_i u(x)) \nu(dz) = \lim_{\varepsilon \downarrow 0} \int_{|z| > \varepsilon} (u(x+z) - u(x)) \nu(dz), \tag{1.11}$$

where the second equality is due to the symmetry of  $\nu$ . Here and below, we use the convention that the repeated index will be summed automatically. However, we need to firstly extend Krylov’s estimate (1.4) to the multidimensional case. As in [18], we shall investigate the following semi-linear PIDE:

$$\partial_t u = \mathcal{L}_0 u + \kappa |\nabla u| + f, \quad u_0(x) = 0,$$

where  $\kappa > 0$  and  $\nabla$  is the gradient operator with respect to the spatial variable  $x$ . We want to emphasize that Fourier’s transform used in [18,19] seems only work for one-dimensional case.

Our method for studying the above two PIDEs is based on the semigroup argument. For this aim, we shall derive some smoothing and asymptotic properties about the Markov semigroup associated with  $\mathcal{L}_0$ . In particular, the interpolation techniques will be used frequently. This will be done in Section 2. In Section 3, partly following Kurenok’s idea, we shall prove two Krylov’s estimates for multidimensional purely discontinuous semimartingales. In Section 4, we prove our main result by using a transformation to remove the drift as adopted in [12,21,30].

In the remainder of this paper, the letter  $C$  with or without subscripts will denote a positive constant whose value may change in different occasions.

### 2. Preliminaries

For  $p \geq 1$ , the norm in  $L^p$ -space  $L^p(\mathbb{R}^d)$  is denoted by  $\|\cdot\|_p$ . For  $\beta \geq 0$  and  $p \geq 1$ , let  $\mathbb{H}_p^\beta := (I - \Delta)^{-\beta/2}(L^p(\mathbb{R}^d))$  be the Bessel potential space with the norm

$$\|f\|_{\beta,p} := \|(I - \Delta)^{\beta/2} f\|_p.$$

Notice that for  $\beta = m \in \mathbb{N}$ , an equivalent norm of  $\mathbb{H}_p^\beta$  is given by (cf. [26], p. 177)

$$\|f\|_{m,p} = \|f\|_p + \|\nabla^m f\|_p.$$

By Sobolev’s embedding theorem, if  $\beta - \frac{d}{p} > 0$  is not an integer, then (cf. [26], p. 206, (16))

$$\mathbb{H}_p^\beta \hookrightarrow C^{\beta-d/p}(\mathbb{R}^d), \tag{2.1}$$

where for  $\gamma > 0$ ,  $C^\gamma(\mathbb{R}^d)$  is the usual Hölder space with the norm:

$$\|f\|_{C^\gamma} := \sum_{k=0}^{[\gamma]} \sup_{x \in \mathbb{R}^d} |\nabla^k f(x)| + \sup_{x \neq y} \frac{|\nabla^{[\gamma]} f(x) - \nabla^{[\gamma]} f(y)|}{|x - y|^{\gamma - [\gamma]}},$$

where  $[\gamma] := \max\{m \in \mathbb{N} \cup \{0\} : m \leq \gamma\}$  is the integer part of  $\gamma$ .

Let  $A$  and  $B$  be an interpolation pair of Banach spaces. For  $\theta \in [0, 1]$ , we use  $[A, B]_\theta$  to denote the complex interpolation space between  $A$  and  $B$  (cf. [26]). We have the following relation (cf. [26], p. 185, (11)): for  $p > 1$ ,  $\beta_1 \neq \beta_2$  and  $\theta \in (0, 1)$ ,

$$[\mathbb{H}_p^{\beta_1}, \mathbb{H}_p^{\beta_2}]_\theta = \mathbb{H}_p^{\beta_1 + \theta(\beta_2 - \beta_1)}. \tag{2.2}$$

On the other hand, for  $0 < \beta \neq \text{integer}$ , the fractional Sobolev space  $\mathbb{W}_p^\beta$  is defined by (cf. [26], p. 190, (15))

$$\|f\|_{\tilde{\beta},p} := \|f\|_p + \sum_{k=0}^{[\beta]} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla^k f(x) - \nabla^k f(y)|^p}{|x - y|^{d + (\beta - [ \beta ] ) p}} dx dy \right)^{1/p} < +\infty. \tag{2.3}$$

For  $\beta = 0, 1, 2, \dots$ , we set  $\mathbb{W}_p^\beta := \mathbb{H}_p^\beta$ . The relation between  $\mathbb{H}_p^\beta$  and  $\mathbb{W}_p^\beta$  is given as follows (cf. [26], p. 180, (9)): for any  $\beta > 0$ ,  $\varepsilon \in (0, \beta)$  and  $p > 1$ ,

$$\mathbb{H}_p^{\beta+\varepsilon} \hookrightarrow \mathbb{W}_p^\beta \hookrightarrow \mathbb{H}_p^{\beta-\varepsilon}. \tag{2.4}$$

We recall the following complex interpolation theorem (cf. [26], p. 59, Theorem (a)).

**Theorem 2.1.** *Let  $A_i \subset B_i, i = 0, 1$ , be Banach spaces. Let  $\mathcal{T} : A_i \rightarrow B_i, i = 0, 1$ , be bounded linear operators. For  $\theta \in [0, 1]$ , we have*

$$\|\mathcal{T}\|_{A_\theta \rightarrow B_\theta} \leq \|\mathcal{T}\|_{A_0 \rightarrow B_0}^{1-\theta} \|\mathcal{T}\|_{A_1 \rightarrow B_1}^\theta,$$

where  $A_\theta := [A_0, A_1]_\theta, B_\theta := [B_0, B_1]_\theta$  and  $\|\mathcal{T}\|_{A_\theta \rightarrow B_\theta}$  denotes the operator norm of  $\mathcal{T}$  mapping  $A_\theta$  to  $B_\theta$ .

Let  $f$  be a locally integrable function on  $\mathbb{R}^d$ . The Hardy–Littlewood maximal function is defined by

$$\mathcal{M}f(x) := \sup_{0 < r < \infty} \frac{1}{|B_r|} \int_{B_r} |f(x + y)| \, dy,$$

where  $B_r := \{x \in \mathbb{R}^d : |x| < r\}$ . The following well-known results can be found in [8,22,29] and [24], p. 5, Theorem 1.

**Lemma 2.2.** (i) *For  $f \in \mathbb{W}_1^1$ , there exists a constant  $C_d > 0$  and a Lebesgue zero set  $E$  such that for all  $x, y \notin E$ ,*

$$|f(x) - f(y)| \leq C_d |x - y| (\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y)). \tag{2.5}$$

(ii) *For  $p > 1$ , there exists a constant  $C_{d,p} > 0$  such that for all  $f \in L^p(\mathbb{R}^d)$ ,*

$$\|\mathcal{M}f\|_p \leq C_{d,p} \|f\|_p. \tag{2.6}$$

For fixed  $z \in \mathbb{R}^d$ , define the shift operator

$$\mathcal{T}_z f(x) := f(x + z) - f(x).$$

We have the following useful estimate.

**Lemma 2.3.** *For  $p > 1$  and  $\gamma \in [1, 2]$ , there exists a constant  $C = C(p, \gamma, d) > 0$  such that for any  $f \in \mathbb{H}_p^\gamma$ ,*

$$\|\mathcal{T}_z f\|_{1,p} \leq C |z|^{\gamma-1} \|f\|_{\gamma,p}. \tag{2.7}$$

**Proof.** By (2.5), we have for Lebesgue almost all  $x \in \mathbb{R}^d$ ,

$$|\mathcal{T}_z f(x)| \leq C |z| (\mathcal{M}|\nabla f|(x + z) + \mathcal{M}|\nabla f|(x)),$$

and so, by (2.6),

$$\|\mathcal{T}_z f\|_p \leq C |z| \cdot \|\mathcal{M}|\nabla f|\|_p \leq C |z| \cdot \|\nabla f\|_p \leq C |z| \cdot \|f\|_{1,p}.$$

On the other hand, it is clear that for any  $\beta > 0$ ,

$$\|\mathcal{T}_z f\|_{\beta,p} \leq 2 \|f\|_{\beta,p}.$$

By Theorem 2.1 and (2.2), for  $\theta \in (0, 1)$ , we immediately have

$$\|\mathcal{T}_z f\|_{\theta\beta,p} \leq C |z|^{1-\theta} \|f\|_{1+\theta(\beta-1),p},$$

which gives the desired result by letting  $\theta = 2 - \gamma$  and  $\beta = \frac{1}{2-\gamma}$ . □

We now recall the following well-known properties about the symmetric  $\alpha$ -stable process  $(L_t)_{t \geq 0}$  (cf. [23], Theorem 25.3, and [21], Lemma 3.1).

**Proposition 2.4.** *Let  $\mu_t$  be the law of  $\alpha$ -stable process  $L_t$ .*

- (i) *Scaling property: For any  $\lambda > 0$ ,  $(L_t)_{t \geq 0}$  and  $(\lambda^{-1/\alpha} L_{\lambda t})_{t \geq 0}$  have the same finite dimensional law. In particular, for any  $t > 0$  and  $A \in \mathcal{B}(\mathbb{R}^d)$ ,  $\mu_t(A) = \mu_1(t^{-1/\alpha} A)$ .*
- (ii) *Existence of smooth density: For any  $t > 0$ ,  $\mu_t$  has a smooth density  $p_t$  with respect to the Lebesgue measure, which is given by*

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, \xi \rangle} e^{-t\psi(\xi)} d\xi.$$

Moreover,  $p_t(x) = p_t(-x)$  and for any  $k \in \mathbb{N}$ ,  $\nabla^k p_t \in L^1(\mathbb{R}^d)$ .

- (iii) *Moments: For any  $t > 0$ , if  $\beta < \alpha$ , then  $\mathbb{E}|L_t|^\beta < +\infty$ ; if  $\beta \geq \alpha$ , then  $\mathbb{E}|L_t|^\beta = \infty$ .*

The Markov semigroup associated with the Lévy process  $(L_t)_{t \geq 0}$  is given by

$$\mathcal{T}_t f(x) = \mathbb{E}(f(L_t + x)) = \int_{\mathbb{R}^d} p_t(z - x) f(z) dz = \int_{\mathbb{R}^d} p_t(x - z) f(z) dz. \tag{2.8}$$

We have:

**Lemma 2.5.** (i) *For any  $\alpha \in (0, 2)$ ,  $p > 1$  and  $\beta, \gamma \geq 0$ , we have for all  $f \in \mathbb{H}_p^\beta$ ,*

$$\|\mathcal{T}_t f\|_{\beta+\gamma, p} \leq C t^{-\gamma/\alpha} \|f\|_{\beta, p}. \tag{2.9}$$

- (ii) *For any  $\alpha \in (1, 2)$ ,  $\theta \in [0, 1]$  and  $p > 1$ , there exists a constant  $C = C(d, p, \theta) > 0$  such that for all  $f \in \mathbb{H}_p^\theta$ ,*

$$\|\mathcal{T}_t f - f\|_p \leq C t^{\theta/\alpha} \|f\|_{\theta, p}. \tag{2.10}$$

**Proof.** (i) Let  $f \in C_0^\infty(\mathbb{R}^d)$ . For any  $k, m \in \mathbb{N}$ , by the scaling property, we have

$$\nabla^{k+m} \mathcal{T}_t f(x) = t^{-(d+k)/\alpha} \int_{\mathbb{R}^d} (\nabla^k p_1)(t^{-1/\alpha}(x - z)) \nabla^m f(z) dz.$$

Hence,

$$\|\nabla^{k+m} \mathcal{T}_t f\|_p \leq t^{-k/\alpha} \|\nabla^m f\|_p \int_{\mathbb{R}^d} |\nabla^k p_1|(x) dx.$$

Since  $C_0^\infty(\mathbb{R}^d)$  is dense in  $\mathbb{H}_p^m$ , we further have for any  $f \in \mathbb{H}_p^m$ ,

$$\|\nabla^{k+m} \mathcal{T}_t f\|_p \leq t^{-k/\alpha} \|f\|_{m, p} \int_{\mathbb{R}^d} |\nabla^k p_1|(x) dx.$$

On the other hand, it is clear that

$$\|\mathcal{T}_t f\|_p \leq \|f\|_p.$$

By Theorem 2.1, we obtain (2.9).

(ii) First, we assume that  $f \in \mathbb{H}_p^1$ . By (2.5), we have for Lebesgue almost all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} |\mathcal{T}_t f(x) - f(x)| &\leq \int_{\mathbb{R}^d} |f(x+y) - f(x)| \cdot p_t(y) \, dy \\ &\leq C \int_{\mathbb{R}^d} (\mathcal{M}|\nabla f|(x+y) + \mathcal{M}|\nabla f|(x)) \cdot |y| \cdot p_t(y) \, dy, \end{aligned}$$

and so, by (2.6) and the scaling property,

$$\|\mathcal{T}_t f - f\|_p \leq C \|\mathcal{M}|\nabla f|\|_p \int_{\mathbb{R}^d} |y| \cdot p_t(y) \, dy \leq C \|\nabla f\|_p \mathbb{E}|L_t| = Ct^{1/\alpha} \|\nabla f\|_p \mathbb{E}|L_1|.$$

Estimate (2.10) follows by (iii) of Proposition 2.4 and Theorem 2.1 again. □

We also need the following simple result for proving the uniqueness.

**Lemma 2.6.** *Let  $(Z_t)_{t \geq 0}$  be a locally bounded and  $(\mathcal{F}_t)$ -adapted process and  $(A_t)_{t \geq 0}$  a continuous real valued non-decreasing  $(\mathcal{F}_t)$ -adapted process with  $A_0 = 0$ . Assume that for any stopping time  $\eta$  and  $t \geq 0$ ,*

$$\mathbb{E}|Z_{t \wedge \eta}| \leq \mathbb{E} \int_0^{t \wedge \eta} |Z_s| \, dA_s.$$

Then  $Z_t = 0$  a.s. for all  $t \geq 0$ .

**Proof.** By replacing  $A_t$  by  $A_t + t$ , one may assume that  $t \mapsto A_t$  is strictly increasing. For  $t \geq 0$ , define the stopping time

$$\tau_t := \inf\{s \geq 0: A_s \geq t\}.$$

It is clear that  $\tau_t$  is the inverse of  $t \mapsto A_t$ . Fix  $T > 0$ . By the assumption and the change of variables, we have

$$\mathbb{E}|Z_{T \wedge \tau_t}| \leq \mathbb{E} \int_0^{T \wedge \tau_t} |Z_s| \, dA_s \leq \mathbb{E} \int_0^{\tau_t} |Z_{T \wedge s}| \, dA_s = \int_0^t \mathbb{E}|Z_{T \wedge \tau_s}| \, ds.$$

By Gronwall's lemma, we obtain  $Z_{T \wedge \tau_t} = 0$ . Letting  $t \rightarrow \infty$  gives the conclusion. □

### 3. Krylov's estimates for purely discontinuous semimartingales

Let  $(L_t)_{t \geq 0}$  be a symmetric  $\alpha$ -stable process. The associated Poisson random measure is defined by

$$N((0, t] \times U) := \sum_{s \in (0, t]} 1_U(L_s - L_{s-}), \quad U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), t > 0.$$

The compensated Poisson random measure is given by  $\tilde{N}((0, t] \times U) = N((0, t] \times U) - t\nu(U)$ . By Lévy-Itô's decomposition, one may write (cf. [23])

$$L_t = \int_0^t \int_{|x| \leq 1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x| > 1} x N(ds, dx). \tag{3.1}$$

Let  $X_t$  a purely discontinuous semimartingale with the form

$$X_t = X_0 + \int_0^t \xi_s \, ds + L_t, \tag{3.2}$$

where  $X_0 \in \mathcal{F}_0$  and  $(\xi_t)_{t \geq 0}$  is a measurable and  $(\mathcal{F}_t)$ -adapted  $\mathbb{R}^d$ -valued process. Let  $u$  be a bounded smooth function on  $\mathbb{R}_+ \times \mathbb{R}^d$ . By Itô's formula (cf. [3]), we have

$$u_t(X_t) = u_0(X_0) + \int_0^t ([\partial_s u_s + \mathcal{L}_0 u_s](X_s) + \xi_s^i \partial_i u_s(X_s)) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u_s(X_{s-} + y) - u_s(X_{s-})] \tilde{N}(ds, dy).$$

In this section, we prove two estimates of Krylov's type for the above  $X_t$ . Let us first prove the following simple Krylov's estimate, which will be used in Section 4 to prove the existence of weak solutions for SDE (1.1) with singular drift  $b$ .

**Theorem 3.1.** *Suppose that  $\alpha \in (1, 2)$ ,  $p > \frac{d}{\alpha-1}$  and  $q > \frac{p\alpha}{p(\alpha-1)-d}$ . Then, for any  $T_0 > 0$ , there exist a constant  $C = C(T_0, d, \alpha, p, q) > 0$  such that for any  $(\mathcal{F}_t)$ -stopping time  $\tau$ , and  $0 \leq S \leq T \leq T_0$ , and all  $f \in L^q([S, T]; L^p(\mathbb{R}^d))$ ,*

$$\mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s(X_s) ds \middle| \mathcal{F}_S \right) \leq C \left( 1 + \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} |\xi_s| ds \middle| \mathcal{F}_S \right) \right) \|f\|_{L^q([S, T]; L^p(\mathbb{R}^d))}. \tag{3.3}$$

**Proof.** Let us first assume that  $f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$  and define

$$u_t(x) = \int_0^t \mathcal{T}_{t-s} f_s(x) ds,$$

where  $\mathcal{T}_t$  is defined by (2.8). By Lemma 2.5, it is easy to see that  $u_t(x) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$  and solves the following PIDE:

$$\partial_t u_t(x) = \mathcal{L}_0 u_t(x) + f_t(x).$$

Choosing  $\gamma \in (1 + \frac{d}{p}, \alpha - \frac{q}{q})$ , by (2.9) and Hölder's inequality, we have

$$\begin{aligned} \|u_t\|_{\gamma, p} &\leq \int_0^t \|\mathcal{T}_{t-s} f_s\|_{\gamma, p} ds \leq C \int_0^t (t-s)^{-\gamma/\alpha} \|f_s\|_p ds \\ &\leq C \left( \int_0^t (t-s)^{-q^* \gamma/\alpha} ds \right)^{1/q^*} \|f\|_{L^q(\mathbb{R}_+; L^p)} \leq C \|f\|_{L^q(\mathbb{R}_+; L^p)}, \end{aligned} \tag{3.4}$$

where  $q^* = q/(q-1)$ .

Fix  $T_0 > 0$  and an  $(\mathcal{F}_t)$ -stopping time  $\tau$ . Using Itô's formula for  $u_{T_0-t}(X_t)$  and by Doob's optimal theorem, we have

$$\begin{aligned} &\mathbb{E}(u_{T_0-T \wedge \tau}(X_{T \wedge \tau}) | \mathcal{F}_S) - u_{T_0-S \wedge \tau}(X_{S \wedge \tau}) \\ &= \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} ([\partial_s u_{T_0-s} + \mathcal{L}_0 u_{T_0-s}](X_s) + \xi_s^i \partial_i u_{T_0-s}(X_s)) ds \middle| \mathcal{F}_S \right) \\ &\leq \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} (-f_s(X_s) + |\xi_s| \cdot |\nabla u_{T_0-s}|(X_s)) ds \middle| \mathcal{F}_S \right), \end{aligned}$$

which yields by (3.4) and (2.1) that

$$\begin{aligned} \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s(X_s) ds \middle| \mathcal{F}_S \right) &\leq 2 \sup_{s,x} |u_s(x)| + \sup_{s,x} |\nabla u_s|(x) \cdot \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} |\xi_s| ds \middle| \mathcal{F}_S \right) \\ &\leq C \|f\|_{L^q(\mathbb{R}_+; L^p)} \left( 1 + \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} |\xi_s| ds \middle| \mathcal{F}_S \right) \right), \end{aligned}$$



where we have used  $p(\gamma - 1) > d$ . By a standard density argument, we obtain (3.3) for general  $f \in L^q([0, T]; L^p(\mathbb{R}^d))$ .  $\square$

In one-dimensional case, as in [19], we even have:

**Theorem 3.2.** *Let  $X_t$  take the following form:*

$$X_t = X_0 + \int_0^t \xi_s ds + \int_0^t h_s dL_s,$$

where  $h_s$  is a bounded predictable process. Suppose that  $\alpha \in (1, 2)$  and  $p > \frac{1}{\alpha-1}$ . Then, for any  $T_0 > 0$ , there exist a constant  $C = C(T_0, \alpha, p, q) > 0$  such that for any  $(\mathcal{F}_t)$ -stopping time  $\tau$ , and  $0 \leq S \leq T \leq T_0$ , and all  $f \in L^p(\mathbb{R}^d)$ ,

$$\mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} |h_s|^\alpha f(X_s) ds \mid \mathcal{F}_S \right) \leq C \left( 1 + \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} (|\xi_s| + |h_s|^\alpha) ds \mid \mathcal{F}_S \right) \right) \|f\|_p. \tag{3.5}$$

**Proof.** Fix  $T_0 > 0$ . For  $f \in C_0^\infty(\mathbb{R}^d)$ , let us define

$$u_{T_0}(x) := \int_0^{T_0} \mathcal{T}_{T_0-s} f(x) ds.$$

It is easy to see that

$$\mathcal{L}_0 u_{T_0}(x) = \mathcal{T}_{T_0} f(x) - f(x). \tag{3.6}$$

Using Itô's formula for  $u_{T_0}(X_t)$  (cf. [4], Proposition 2.1), one finds that

$$\mathbb{E}(u_{T_0}(X_{T \wedge \tau}) \mid \mathcal{F}_S) = u_{T_0}(X_{S \wedge \tau}) + \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} (|h_s|^\alpha \mathcal{L}_0 u_{T_0}(X_s) + \xi_s u'_{T_0}(X_s)) ds \mid \mathcal{F}_S \right),$$

which together with (3.6) yields that

$$\begin{aligned} \left( \int_{S \wedge \tau}^{T \wedge \tau} |h_s|^\alpha f(X_s) ds \mid \mathcal{F}_S \right) &\leq 2 \|u_{T_0}\|_\infty + \|u'_{T_0}\|_\infty \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} \xi_s ds \mid \mathcal{F}_S \right) + \|\mathcal{T}_{T_0} f\|_\infty \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} |h_s|^\alpha ds \mid \mathcal{F}_S \right) \\ &\leq C \left( 1 + \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} (|\xi_s| + |h_s|^\alpha) ds \mid \mathcal{F}_S \right) \right) \|f\|_p, \end{aligned}$$

where we have used  $p(\alpha - 1) > d$ , (2.9) and (2.1). By a standard density argument, we obtain (3.5) for general  $f \in L^p(\mathbb{R}^d)$ .  $\square$

In the above two theorems, the requirement of  $p > \frac{d}{\alpha-1}$  is too strong to prove Theorem 1.1. It is clear that this is caused by directly controlling the  $\infty$ -norm of  $\nabla u_s(x)$  by Sobolev embedding theorem. In what follows, we shall relax it to  $p > \frac{d}{\alpha}$ . The price to pay is that we need to assume that  $\xi_t$  is a bounded  $(\mathcal{F}_t)$ -adapted process. Nevertheless, Theorem 3.1 can be used to prove the existence of weak solutions for SDE (1.1) with globally integrable drift.

We now start by solving the following semi-linear PIDE:

$$\partial_t u = \mathcal{L}_0 u + \kappa |\nabla u| + f, \quad u_0 \equiv 0, t \geq 0, \tag{3.7}$$

where  $\kappa > 0$ ,  $\mathcal{L}_0$  is the generator of Lévy process  $(L_t)_{t \geq 0}$  given by (1.11), and  $f$  is a locally integrable function on  $\mathbb{R}_+ \times \mathbb{R}^d$ .

We first give the following definition of generalized solutions to PIDE (3.7).

**Definition 3.3.** For  $p \geq 1$ , a function  $u \in C([0, \infty); \mathbb{H}_p^1)$  is called a generalized solution of (3.7), if for all test function  $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^d)$ , it holds that

$$-\int_0^\infty \int_{\mathbb{R}^d} u \partial_t \varphi = \int_0^\infty \int_{\mathbb{R}^d} u \mathcal{L}_0^* \varphi + \int_0^\infty \int_{\mathbb{R}^d} (\kappa |\nabla u| + f) \varphi,$$

where  $\mathcal{L}_0^*$  is the adjoint operator of  $\mathcal{L}_0$  given by

$$\mathcal{L}_0^* \varphi(x) = \int_{\mathbb{R}^d} (\varphi(x - z) - \varphi(x) + 1_{|z| \leq 1} z^i \partial_i \varphi(x)) \nu(dz).$$

**Remark 3.4.** If we extend  $u$  and  $f$  to  $\mathbb{R}$  by setting  $u_t = f_t \equiv 0$  for  $t \leq 0$ , then for all  $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$ ,

$$-\int_{\mathbb{R}^{d+1}} u \partial_t \varphi = \int_{\mathbb{R}^{d+1}} u \mathcal{L}_0^* \varphi + \int_{\mathbb{R}^{d+1}} (\kappa |\nabla u| + f) \varphi.$$

Since the Lévy measure  $\nu$  is symmetric,  $\mathcal{L}_0^*$  is in fact the same as  $\mathcal{L}_0$ .

The following proposition is now standard. For the reader’s convenience, a proof is provided here.

**Proposition 3.5.** For  $p \geq 1$ , let  $u \in C([0, \infty); \mathbb{H}_p^1)$  and  $f \in L_{loc}^1([0, \infty); L^p(\mathbb{R}^d))$ . The following three statements are equivalent:

- (i)  $u$  is a generalized solution of (3.7).
- (ii) For any  $\phi \in C_b^\infty(\mathbb{R}^d)$ , it holds that for all  $t \geq 0$ ,

$$\int_{\mathbb{R}^d} u_t \phi = \int_0^t \int_{\mathbb{R}^d} u_s \mathcal{L}_0^* \phi + \int_0^t \int_{\mathbb{R}^d} (\kappa |\nabla u_s| + f_s) \phi.$$

- (iii)  $u$  satisfies the following integral equation:

$$u_t(x) = \int_0^t \mathcal{T}_{t-s}(\kappa |\nabla u_s| + f_s)(x) ds, \quad \forall t \geq 0.$$

**Proof.** For simplicity of notation, we write  $F_s(x) := \kappa |\nabla u_s| + f_s$ . By the assumptions, we have

$$F \in L_{loc}^1([0, \infty); L^p(\mathbb{R}^d)).$$

(i)  $\Rightarrow$  (ii). Let  $\phi \in C_b^\infty(\mathbb{R}^d)$ . Let  $\rho \in C_0^\infty(\mathbb{R})$  with support in  $[0, 1]$  and  $\int_0^1 \rho(r) dr = 1$ . Fix  $t > 0$  and define for  $\varepsilon > 0$ ,

$$\chi_\varepsilon(s) := \int_{(s-t)/\varepsilon}^\infty \rho(r) dr.$$

It is easy to see that  $\chi_\varepsilon \in C_0^\infty([0, \infty))$  converges to  $1_{[0,t]}$  for any  $s \geq 0$ . Let  $\zeta_R(x) \in C_0^\infty(\mathbb{R}^d)$  converge to 1 pointwisely. Taking  $\varphi_s(x) := \chi_\varepsilon(s) \zeta_R(x) \phi(x)$  in the definition of generalized solutions, we have

$$\varepsilon^{-1} \int_0^\infty \int_{\mathbb{R}^d} u \rho((\cdot - t)/\varepsilon) \zeta_R \phi = \int_0^\infty \int_{\mathbb{R}^d} \chi_\varepsilon u \mathcal{L}_0^* (\zeta_R \phi) + \int_0^\infty \int_{\mathbb{R}^d} F \chi_\varepsilon \zeta_R \phi.$$

Letting  $\varepsilon \rightarrow 0$  and  $R \rightarrow \infty$  and using the dominated convergence theorem, we immediately have (ii).

(ii)  $\Rightarrow$  (i) and (iii). Fix  $t > 0$ . By the chain rule, one sees that for any  $h \in C^1([0, t])$ ,

$$\int_{\mathbb{R}^d} u_t h_t \phi = \int_0^t \int_{\mathbb{R}^d} u_s h'_s \phi + \int_0^t \int_{\mathbb{R}^d} u_s h_s \mathcal{L}_0^* \phi + \int_0^t \int_{\mathbb{R}^d} F_s h_s \phi.$$

By the linearity and a density argument, we obtain (i) and also have for all  $\varphi \in C^1([0, t]; C_b^\infty(\mathbb{R}^d))$ ,

$$\int_{\mathbb{R}^d} u_t \varphi_t = \int_0^t \int_{\mathbb{R}^d} u_s \partial_s \varphi_s + \int_0^t \int_{\mathbb{R}^d} u_s \mathcal{L}_0^* \varphi_s + \int_0^t \int_{\mathbb{R}^d} F_s \varphi_s.$$

Now for any  $\phi \in C_b^\infty(\mathbb{R}^d)$ , taking  $\varphi_s(x) = \mathcal{T}_{t-s}\phi(x) \in C^1([0, t]; C_b^\infty(\mathbb{R}^d))$  in the above equality, we obtain

$$\int_{\mathbb{R}^d} u_t \phi = - \int_0^t \int_{\mathbb{R}^d} u_s \mathcal{L}_0 \phi_s + \int_0^t \int_{\mathbb{R}^d} u_s \mathcal{L}_0^* \phi_s + \int_0^t \int_{\mathbb{R}^d} F_s \mathcal{T}_{t-s} \phi = \int_0^t \int_{\mathbb{R}^d} \mathcal{T}_{t-s} F_s \phi,$$

where we have used the symmetry of  $\mathcal{T}_t$ . Thus, (iii) follows by the arbitrariness of  $\phi$ .

(iii)  $\Rightarrow$  (ii). By Fubini's theorem, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} u_s \mathcal{L}_0^* \phi \, ds &= \int_0^t \int_{\mathbb{R}^d} \left( \int_0^s \mathcal{T}_{s-r} F_r \, dr \right) \mathcal{L}_0^* \phi \, ds = \int_0^t \int_{\mathbb{R}^d} \left( \int_r^t \mathcal{L}_0^* \mathcal{T}_{s-r} \phi \, ds \right) F_r \, dr \\ &= \int_0^t \int_{\mathbb{R}^d} (\mathcal{T}_{t-r} \phi - \phi) F_r \, dr = \int_{\mathbb{R}^d} u_t \phi - \int_0^t \int_{\mathbb{R}^d} F_r \phi \, dr. \end{aligned}$$

(ii) now follows. □

We have the following existence-uniqueness result about the generalized solution of Eq. (3.7).

**Theorem 3.6.** For  $p > 1$ ,  $\alpha \in (1, 2)$ ,  $\gamma \in [1, \alpha)$  and  $q > \frac{\alpha}{\alpha-\gamma}$ , assume that  $f \in L_{loc}^q(\mathbb{R}_+; L^p(\mathbb{R}^d))$ . Then, there exists a unique generalized solution  $u \in C([0, \infty); \mathbb{H}_p^\gamma)$  to PIDE (3.7). Moreover,

$$\|u_t\|_{\gamma, p} \leq C_t \|f\|_{L^q([0, t]; L^p)}, \quad \forall t \geq 0, \quad (3.8)$$

where  $C_t \geq 0$  is a continuous increasing function of  $t$  with  $C_t = O(t^{1-\gamma/\alpha-1/q})$  as  $t \rightarrow 0$ .

**Proof.** Let  $u^{(0)} \equiv 0$ . For  $n \in \mathbb{N}$ , define  $u^{(n)}$  recursively by

$$u_t^{(n)}(x) = \int_0^t \mathcal{T}_{t-s} (\kappa |\nabla u_s^{(n-1)}| + f_s)(x) \, ds, \quad \forall t \geq 0. \quad (3.9)$$

By (i) of Lemma 2.5 and Hölder's inequality, we have for  $q > \frac{\alpha}{\alpha-\gamma}$ ,

$$\begin{aligned} \|u_t^{(n)}\|_{\gamma, p} &\leq C \int_0^t (t-s)^{-\gamma/\alpha} (\kappa \|\nabla u_s^{(n-1)}\|_p + \|f_s\|_p) \, ds \\ &\leq C \left( \int_0^t (t-s)^{-q\gamma/((q-1)\alpha)} \, ds \right)^{(q-1)/q} \left( \int_0^t (\kappa^q \|\nabla u_s^{(n-1)}\|_p^q + \|f_s\|_p^q) \, ds \right)^{1/q}, \end{aligned}$$

which yields that

$$\begin{aligned} \|u_t^{(n)}\|_{\gamma, p}^q &\leq C t^{(q(\alpha-\gamma)-\alpha)/\alpha} \left( \int_0^t \|u_s^{(n-1)}\|_{1, p}^q \, ds + \int_0^t \|f_s\|_p^q \, ds \right) \\ &\leq C t^{(q(\alpha-\gamma)-\alpha)/\alpha} \left( \int_0^t \|u_s^{(n-1)}\|_{\gamma, p}^q \, ds + \int_0^t \|f_s\|_p^q \, ds \right). \end{aligned}$$

By Gronwall's inequality, we obtain that for all  $t \geq 0$ ,

$$\sup_{n \in \mathbb{N}} \|u_t^{(n)}\|_{\gamma, p}^q \leq C_t \int_0^t \|f_s\|_p^q \, ds. \quad (3.10)$$

Next, fixing  $T > 0$ , we want to prove the Hölder continuity of mapping  $[0, T] \ni t \mapsto u_t^{(n)} \in \mathbb{H}_p^\gamma$ . For  $T \geq t > t' \geq 0$ , we have

$$u_t^{(n)} - u_{t'}^{(n)} = \int_0^{t'} (\mathcal{T}_{t-s} - \mathcal{T}_{t'-s})(\kappa |\nabla u_s^{(n-1)}| + f_s) \, ds + \int_{t'}^t \mathcal{T}_{t-s}(\kappa |\nabla u_s^{(n-1)}| + f_s) \, ds =: I_1(t, t') + I_2(t, t').$$

For  $I_1(t, t')$ , using the semigroup property of  $\mathcal{T}_t$ , we further have

$$I_1(t, t') = \int_0^{t'} \mathcal{T}_{(t'-s)/2}(\mathcal{T}_{t-t'} - I)\mathcal{T}_{(t'-s)/2}(\kappa |\nabla u_s^{(n-1)}| + f_s) \, ds.$$

Hence, by Lemma 2.5, (3.10) and Hölder's inequality again, for  $\delta \in (0, \alpha - \gamma - \frac{\alpha}{q})$ , we have

$$\begin{aligned} \|I_1(t, t')\|_{\gamma,p} &\leq C \int_0^{t'} (t' - s)^{-\gamma/\alpha} \|(\mathcal{T}_{t-t'} - I)\mathcal{T}_{(t'-s)/2}(\kappa |\nabla u_s^{(n-1)}| + f_s)\|_p \, ds \\ &\leq C \int_0^{t'} (t' - s)^{-\gamma/\alpha} (t - t')^{\delta/\alpha} \|\mathcal{T}_{(t'-s)/2}(\kappa |\nabla u_s^{(n-1)}| + f_s)\|_{\delta,p} \, ds \\ &\leq C (t - t')^{\delta/\alpha} \int_0^{t'} (t' - s)^{-(\gamma+\delta)/\alpha} (\|\nabla u_s^{(n-1)}\|_p + \|f_s\|_p) \, ds \\ &\leq C (t - t')^{\delta/\alpha} \left( \int_0^{t'} (t' - s)^{((-\gamma+\delta)/\alpha)(q/(q-1))} \, ds \right)^{1-1/q} \left( \int_0^{t'} (\|\nabla u_s^{(n-1)}\|_p^q + \|f_s\|_p^q) \, ds \right)^{1/q} \\ &\leq C (t - t')^{\delta/\alpha} \|f\|_{L^q([0,T];L^p(\mathbb{R}^d))}. \end{aligned} \tag{3.11}$$

For  $I_2(t, t')$ , we similarly have

$$\begin{aligned} \|I_2(t, t')\|_{\gamma,p} &\leq C \int_{t'}^t (t - s)^{-\gamma/\alpha} (\|\nabla u_s^{(n-1)}\|_p + \|f_s\|_p) \, ds \\ &\leq C \left( \int_{t'}^t (t - s)^{(-\gamma/\alpha)(q/(q-1))} \, ds \right)^{1-1/q} \left( \int_{t'}^t (\|\nabla u_s^{(n-1)}\|_p^q + \|f_s\|_p^q) \, ds \right)^{1/q} \\ &\leq C_T (t - t')^{1-\gamma/\alpha-1/q} \|f\|_{L^q([0,T];L^p)}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain the desired Hölder continuity.

Now, as (3.12), we can make the following estimation:

$$\begin{aligned} \|u_t^{(n)} - u_t^{(m)}\|_{\gamma,p} &\leq C \int_0^t (t - s)^{-\gamma/\alpha} (\|\nabla u_s^{(n-1)}\|_p - \|\nabla u_s^{(m-1)}\|_p) \, ds \\ &\leq C t^{1-\gamma/\alpha-1/q} \left( \int_0^t \|\nabla(u_s^{(n-1)} - u_s^{(m-1)})\|_p^q \, ds \right)^{1/q}, \end{aligned}$$

which then gives that

$$\|u_t^{(n)} - u_t^{(m)}\|_{\gamma,p}^q \leq C t^{(q(\alpha-\gamma)-\alpha)/\alpha} \int_0^t \|u_s^{(n-1)} - u_s^{(m-1)}\|_{\gamma,p}^q \, ds,$$

where  $C$  is independent of  $n, m$  and  $t$ . Using (3.10) and Fatou's lemma, we find that

$$\begin{aligned} \overline{\lim}_{n,m \rightarrow \infty} \sup_{s \in [0,t]} \|u_s^{(n)} - u_s^{(m)}\|_{\gamma,p}^q &\leq C t^{(q(\alpha-\gamma)-\alpha)/\alpha} \int_0^t \overline{\lim}_{n,m \rightarrow \infty} \|u_s^{(n-1)} - u_s^{(m-1)}\|_{\gamma,p}^q \, ds \\ &\leq C t^{(q(\alpha-\gamma)-\alpha)/\alpha} \int_0^t \overline{\lim}_{n,m \rightarrow \infty} \sup_{r \in [0,s]} \|u_r^{(n-1)} - u_r^{(m-1)}\|_{\gamma,p}^q \, ds, \end{aligned}$$

and so, for any  $t > 0$ ,

$$\overline{\lim}_{n,m \rightarrow \infty} \sup_{s \in [0,t]} \|u_s^{(n)} - u_s^{(m)}\|_{\gamma,p}^q = 0.$$

Thus, there exists a  $u \in C([0, \infty); \mathbb{H}_p^\gamma)$  such that for any  $t > 0$ ,

$$\lim_{n \rightarrow \infty} \sup_{s \in [0,t]} \|u_s^{(n)} - u_s\|_{\gamma,p} = 0.$$

Taking limits for both sides of (3.9), we obtain the existence of a generalized solution, and (3.8) is direct from (3.10).

As for the uniqueness, it follows from a similar calculation. The proof is complete.  $\square$

Let us now prove our second Krylov’s estimate.

**Theorem 3.7.** *Suppose that  $\alpha \in (1, 2)$ ,  $p > \frac{d}{\alpha} \vee 1$  and  $q > \frac{p\alpha}{p\alpha-d}$ . Let  $(\xi_t)_{t \geq 0}$  be a measurable and  $(\mathcal{F}_t)$ -adapted process bounded by  $\kappa$ , and let  $X_t$  have the form (3.2). Fix an  $(\mathcal{F}_t)$ -stopping time  $\tau$ . Then for any  $T_0 > 0$ , there exist a constant  $C = C(T_0, \kappa, d, \alpha, p, q) > 0$  such that any  $(\mathcal{F}_t)$ -stopping time  $\tau$ , and  $0 \leq S \leq T \leq T_0$ , and all  $f \in L^q([S, T]; L^p(\mathbb{R}^d))$ ,*

$$\mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s(X_s) ds \middle| \mathcal{F}_S \right) \leq C \|f\|_{L^q([S,T]; L^p(\mathbb{R}^d))}. \tag{3.13}$$

**Proof.** Let us first assume that  $f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ . Choose  $\gamma \in (\frac{d}{p}, \alpha - \frac{\alpha}{q})$ . Let  $u \in C([0, \infty); \mathbb{H}_p^\gamma)$  be the unique solution of PIDE (3.7). Fix  $T_0 > 0$ , and let  $v_t(x) = u_{T_0-t}(x)$ . It is easy to see that  $v_t$  is a generalized solution of the following PIDE:

$$\partial_t v + \mathcal{L}_0 v + \kappa |\nabla v| + f = 0, \quad v_{T_0} \equiv 0. \tag{3.14}$$

Let  $\rho$  be a smooth non-negative function in  $\mathbb{R}^{d+1}$  with support in  $\{(s, x) \in \mathbb{R}^{d+1} : |s| + |x| \leq 1\}$  and  $\int_{\mathbb{R}^{d+1}} \rho = 1$ . For  $\varepsilon > 0$ , set

$$\rho_\varepsilon(s, x) = \varepsilon^{-(d+1)} \rho(\varepsilon^{-1}s, \varepsilon^{-1}x)$$

and

$$v^{(\varepsilon)} = v * \rho_\varepsilon, \quad f^{(\varepsilon)} = f * \rho_\varepsilon.$$

Taking convolutions for both sides of (3.14), we obtain

$$\partial_t v^{(\varepsilon)} + \mathcal{L}_0 v^{(\varepsilon)} + \kappa |\nabla v^{(\varepsilon)}| + f^{(\varepsilon)} \leq (\partial_t v + \mathcal{L}_0 v + \kappa |\nabla v| + f) * \rho_\varepsilon = 0.$$

Here we have used Remark 3.4.

Using Itô’s formula for  $v_t^{(\varepsilon)}(X_t)$ , we get

$$\begin{aligned} \mathbb{E}(v_{T \wedge \tau}^{(\varepsilon)}(X_{T \wedge \tau}) | \mathcal{F}_S) - v_{S \wedge \tau}^{(\varepsilon)}(X_{S \wedge \tau}) &= \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} [(\partial_s v_s^{(\varepsilon)} + \mathcal{L}_0 v_s^{(\varepsilon)})](X_s) + \xi_s^i \partial_i v_s^{(\varepsilon)}(X_s) ds \middle| \mathcal{F}_S \right) \\ &\leq \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} [\partial_s v_s^{(\varepsilon)} + \mathcal{L}_0 v_s^{(\varepsilon)} + \kappa |\nabla v_s^{(\varepsilon)}|](X_s) ds \middle| \mathcal{F}_S \right) \\ &\leq -\mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s^{(\varepsilon)}(X_s) ds \middle| \mathcal{F}_S \right), \end{aligned}$$

which yields by (3.8) and (2.1) that

$$\begin{aligned} \mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s^{(\varepsilon)}(X_s) \, ds \middle| \mathcal{F}_S \right) &\leq 2 \sup_{(s,x) \in [0, T_0] \times \mathbb{R}^d} |v_s^{(\varepsilon)}(x)| \leq 2 \sup_{(t,x) \in [0, T_0] \times \mathbb{R}^d} |v_t(x)| \\ &\leq 2 \sup_{(t,x) \in [0, T_0] \times \mathbb{R}^d} |u_t(x)| \leq C \|f\|_{L^q([0, T_0]; L^p)}. \end{aligned}$$

Taking limits  $\varepsilon \rightarrow 0$ , by the dominated convergence theorem, we have

$$\mathbb{E} \left( \int_{S \wedge \tau}^{T \wedge \tau} f_s(X_s) \, ds \middle| \mathcal{F}_S \right) \leq C \|f\|_{L^q([0, T_0]; L^p)}.$$

By a standard density argument, we obtain (3.13) for general  $f \in L^q([S, T]; L^p(\mathbb{R}^d))$ . □

#### 4. Weak solutions for SDE (1.1) with globally integrable drift

In this section, we use Theorem 3.1 to prove the following existence of weak solutions for SDE (1.1).

**Theorem 4.1.** *Suppose that  $\alpha \in (1, 2)$ ,  $\gamma \in (1, \alpha)$ ,  $p > \frac{d}{\gamma-1}$  and  $q > \frac{\alpha}{\alpha-\gamma}$ . Then for any  $b \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^d)) + L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d))$  and  $x_0 \in \mathbb{R}^d$ , there exists a weak solution to SDE (1.1). More precisely, there exists a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  and two càdlàg stochastic processes  $\tilde{X}_t$  and  $\tilde{L}_t$  defined on it such that  $\tilde{L}_t$  is a symmetric  $\alpha$ -stable process with respect to the completed filtration  $\tilde{\mathcal{F}}_t := \sigma^{\tilde{P}}\{\tilde{X}_s, \tilde{L}_s, s \leq t\}$  and*

$$\tilde{X}_t = x_0 + \int_0^t b(s, \tilde{X}_s) \, ds + \tilde{L}_t, \quad \forall t \geq 0.$$

**Proof.** Our proof is adapted from the proof of [16], p. 87, Theorem 1. Let  $b = b_1 + b_2$  with  $b_1 \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$  and  $b_2 \in L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d))$ . Let  $b_i^{(n)}(t, x) = (b_i(t, \cdot) * \rho_n)(x)$  be the mollifying approximation of  $b_i$ ,  $i = 1, 2$ . It is easy to see that for some  $\ell_t^{(n)} \in L^1_{loc}(\mathbb{R}_+)$ ,

$$|b^{(n)}(t, x) - b^{(n)}(t, y)| \leq \ell_t^{(n)} |x - y|, \quad \forall x, y \in \mathbb{R}^d.$$

Thus, the following SDE can be uniquely solved and the solution is denoted by  $X_t^{(n)}$ ,

$$X_t^{(n)} = x_0 + \int_0^t b^{(n)}(s, X_s^{(n)}) \, ds + L_t.$$

*Claim 1:* For some  $\delta > 1$ , we have

$$\sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T |b^{(n)}(s, X_s^{(n)})|^\delta \, ds < +\infty, \quad \forall T > 0. \tag{4.1}$$

In fact, choosing  $\delta > 1$  and  $p' \in (\frac{d}{\gamma-1}, p)$ ,  $q' \in (\frac{\alpha}{\alpha-\gamma}, q)$  such that  $p'\delta = p$  and  $q'\delta = q$ , by (3.3) and Young's inequality, we have

$$\begin{aligned} \mathbb{E} \int_0^T |b_2^{(n)}(s, X_s^{(n)})|^\delta \, ds &\leq C_T \left( 1 + \mathbb{E} \int_0^T |b^{(n)}(s, X_s^{(n)})| \, ds \right) \| |b_2^{(n)}|^\delta \|_{L^{q'}([0, T]; L^{p'}(\mathbb{R}^d))} \\ &\leq C_T \left( 1 + \|b_1\|_{L^\infty([0, T] \times \mathbb{R}^d)} + \mathbb{E} \int_0^T |b_2^{(n)}(s, X_s^{(n)})| \, ds \right) \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}^\delta \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \mathbb{E} \int_0^T |b_2^{(n)}(s, X_s^{(n)})|^\delta ds + C_T \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}^{\delta^2/(\delta-1)} \\ &\quad + C_T (1 + \|b_1\|_{L^\infty([0, T] \times \mathbb{R}^d)}) \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}^\delta, \end{aligned}$$

which then implies (4.1).

Let  $\mathbb{D}$  be the space of all càdlàg functions from  $\mathbb{R}_+$  to  $\mathbb{R}^d$ , which is endowed with the Skorohod topology so that  $\mathbb{D}$  is a Polish space. Set

$$H_t^{(n)} := \int_0^t b^{(n)}(s, X_s^{(n)}) ds.$$

Using Claim 1, it is easy to check that the following Aldous' tightness criterions [2] (see also [15], Theorem 15.55) hold:

$$\lim_{N \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} P\left(\sup_{t \in [0, T]} |H_t^{(n)}| \geq N\right) = 0, \quad \forall T > 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{\tau \in \mathcal{S}_T} P(|H_\tau^{(n)} - H_{\tau+\varepsilon}^{(n)}| \geq a) = 0, \quad \forall T, a > 0,$$

where  $\mathcal{S}_T$  denotes all the bounded stopping times with bound  $T$ . Thus, the law of  $t \mapsto H_t^{(n)}$  in  $\mathbb{D}$  is tight, and so does  $(H^{(n)}, L)$ . By Prohorov's theorem, there exists a subsequence still denoted by  $n$  such that the law of  $(H^{(n)}, L)$  in  $\mathbb{D} \times \mathbb{D}$  weakly converges, which then implies that the law of  $(X^{(n)}, L)$  weakly converges. By Skorohod's representation theorem, there is a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  and the  $\mathbb{D} \times \mathbb{D}$ -valued random variables  $(\tilde{X}^{(n)}, \tilde{L}^{(n)})$  and  $(\tilde{X}, \tilde{L})$  such that

- (i)  $(\tilde{X}^{(n)}, \tilde{L}^{(n)})$  has the same law as  $(X^{(n)}, L)$  in  $\mathbb{D} \times \mathbb{D}$ .
- (ii)  $(\tilde{X}^{(n)}, \tilde{L}^{(n)})$  converges to  $(\tilde{X}, \tilde{L})$ ,  $\tilde{P}$ -almost surely.

In particular,  $\tilde{L}$  is still a symmetric  $\alpha$ -stable process and

$$\tilde{X}_t^{(n)} = x_0 + \int_0^t b^{(n)}(s, \tilde{X}_s^{(n)}) ds + \tilde{L}_t^{(n)}.$$

*Claim 2:* For any non-negative measurable function  $f$  and  $T > 0$ , we have

$$\tilde{\mathbb{E}} \int_0^T f_s(\tilde{X}_s) ds \leq C_T \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))},$$

where  $\tilde{\mathbb{E}}$  denotes the expectation with respect to the probability measure  $\tilde{P}$ .

Let  $f \in C_0([0, T] \times \mathbb{R}^d)$ . By the dominated convergence theorem, we have

$$\tilde{\mathbb{E}} \int_0^T f_s(\tilde{X}_s) ds = \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \int_0^T f_s(\tilde{X}_s^{(n)}) ds = \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T f_s(X_s^{(n)}) ds \leq C \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))},$$

where in the last step we have used (3.3) and (4.1). For general  $f$ , it follows by the monotone class theorem.

The proof will be finished if one can show the following claim:

*Claim 3:* For any  $T > 0$ , we have

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left( \int_0^T |b_i^{(n)}(s, \tilde{X}_s^{(n)}) - b_i(s, \tilde{X}_s)| ds \right) = 0, \quad i = 1, 2. \tag{4.2}$$

Let  $\chi_R(x)$  be a smooth non-negative function on  $\mathbb{R}^d$  with  $\chi_R(x) = 1$  for  $|x| \leq R$  and  $\chi_R(x) = 0$  for  $|x| > R + 1$ . Then for any  $n, m \in \mathbb{N}$ ,

$$\begin{aligned} \tilde{\mathbb{E}}\left(\int_0^T |b_1^{(n)}(s, \tilde{X}_s^{(n)}) - b_1(s, \tilde{X}_s)| ds\right) &\leq \tilde{\mathbb{E}}\left(\int_0^T |b_1^{(n)}(s, \tilde{X}_s^{(n)}) - b_1^{(m)}(s, \tilde{X}_s^{(n)})| ds\right) \\ &\quad + \tilde{\mathbb{E}}\left(\int_0^T |b_1^{(m)}(s, \tilde{X}_s^{(n)}) - b_1^{(m)}(s, \tilde{X}_s)| ds\right) \\ &\quad + \tilde{\mathbb{E}}\left(\int_0^T |b_1^{(m)}(s, \tilde{X}_s) - b_1(s, \tilde{X}_s)| ds\right) \\ &=: I_1^{(n,m)} + I_2^{(n,m)} + I_3^{(n,m)}. \end{aligned} \tag{4.3}$$

For fixed  $m$ , by the above (ii) and the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} I_2^{(n,m)} = 0.$$

For  $I_1^{(n,m)}$ , by Claim 1, we have

$$\begin{aligned} I_1^{(n,m)} &\leq \|b_1\|_{L^\infty([0,T]; L^\infty(\mathbb{R}^d))} \tilde{\mathbb{E}}\left(\int_0^T |1 - \chi_R(\tilde{X}_s^{(n)})| ds\right) + \tilde{\mathbb{E}}\left(\int_0^T [\chi_R |b_1^{(n)} - b_1^{(m)}|](s, \tilde{X}_s^{(n)}) ds\right) \\ &\leq \frac{C}{R} \int_0^T \mathbb{E}|X_s^{(n)}| ds + \mathbb{E}\left(\int_0^T [\chi_R |b_1^{(n)} - b_1^{(m)}|](s, X_s^{(n)}) ds\right) \\ &\leq \frac{C}{R} + C \|\chi_R |b_1^{(n)} - b_1^{(m)}|\|_{L^q([0,T]; L^p)}. \end{aligned}$$

Similarly, by Claim 2, we have

$$I_3^{(n,m)} \leq \frac{C}{R} + C \|\chi_R |b_1^{(m)} - b_1|\|_{L^q([0,T]; L^p)}.$$

Taking limits for both sides of (4.3) in order:  $n \rightarrow \infty, m \rightarrow \infty$  and  $R \rightarrow \infty$ , we obtain (4.2) for  $i = 1$ . It is similar to prove (4.2) for  $i = 2$ . The whole proof is complete. □

**Remark 4.2.** When  $b$  is time-independent and the Lévy measure  $\nu(d\xi) = \frac{C_\alpha}{|\xi|^{d+\alpha}} d\xi$ , Theorem 4.1 has been proven by Chen, Kim and Song [7], Theorem 2.5, by different argument.

**5. Proof of Theorem 1.1**

We now consider the following linear PIDE for  $\lambda > 0$ :

$$\partial_t u = (\mathcal{L}_0 - \lambda)u + b^i \partial_i u + f, \quad u_0 \equiv 0. \tag{5.1}$$

As in the previous section, one may define the notion of generalized solutions and has:

**Theorem 5.1.** Let  $\alpha \in (1, 2)$  and  $\gamma \in (1, \alpha)$ . Assume that for some  $p > \frac{d}{\gamma}$  and  $0 \leq \beta \in (1 - \gamma + \frac{d}{p}, 1)$ ,

$$b \in L^\infty_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^d) \cap \mathbb{W}_p^\beta), \quad f \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{W}_p^\beta).$$

Then there exists a unique generalized solution  $u = u^\lambda \in C(\mathbb{R}_+; \mathbb{H}_p^{\gamma+\beta})$  to PIDE (5.1). Moreover, for some  $\delta > 0$  and any  $\lambda \geq 1$ ,

$$\|u_t^\lambda\|_{\gamma+\beta, p} \leq C_t \lambda^{-\delta} \|f\|_{L^\infty([0,t]; \mathbb{W}_p^\beta)}, \quad \forall t \geq 0, \tag{5.2}$$



where  $C_t > 0$  is an increasing function of  $t$  with  $\lim_{t \downarrow 0} C_t = 0$ .

**Proof.** As in the proof of Theorem 3.6, we only need to prove the a priori estimate (5.2). Let  $u$  satisfy the following integral equation:

$$u_t(x) = \int_0^t e^{-\lambda(t-s)} \mathcal{T}_{t-s}(b_s^i \partial_i u_s + f_s)(x) ds, \quad \forall t \geq 0.$$

Let  $\varepsilon \in (0, \alpha - \gamma)$  and  $q > \frac{\alpha}{\alpha - \gamma - \varepsilon}$ . By Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned} \|u_t\|_{\gamma+\beta,p} &\leq C \int_0^t e^{-\lambda(t-s)} (t-s)^{(-\gamma+\varepsilon)/\alpha} (\|b_s^i \partial_i u_s\|_{\beta-\varepsilon,p} + \|f_s\|_{\beta-\varepsilon,p}) ds \\ &\stackrel{(2.4)}{\leq} C \left( \int_0^t e^{-\lambda q(t-s)} (t-s)^{-q(\gamma+\varepsilon)/((q-1)\alpha)} ds \right)^{(q-1)/q} \left( \int_0^t (\|b_s^i \partial_i u_s\|_{\beta,p}^q + \|f_s\|_{\beta,p}^q) ds \right)^{1/q} \\ &\leq C \lambda^{(\gamma+\varepsilon)/\alpha-1+1/q} \left( \int_0^\infty e^{-s} s^{-q(\gamma+\varepsilon)/((q-1)\alpha)} ds \right)^{(q-1)/q} \left( \int_0^t (\|b_s^i \partial_i u_s\|_{\beta,p}^q + \|f_s\|_{\beta,p}^q) ds \right)^{1/q}. \end{aligned}$$

In view of  $(\gamma + \beta - 1)p > d$  and  $\gamma > 1$ , we have

$$\begin{aligned} \|b_s^i \partial_i u_s\|_{\beta,p} &\stackrel{(2.3)}{\leq} \|b_s\|_\infty \|\nabla u_s\|_p + \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(b_s^i \partial_i u_s)(x) - (b_s^i \partial_i u_s)(y)|^p}{|x-y|^{d+\beta p}} dx dy \right)^{1/p} \\ &\leq \|b_s\|_\infty \|\nabla u_s\|_p + \|b_s\|_\infty \|\nabla u_s\|_{\beta,p} + \|b_s\|_{\beta,p} \|\nabla u_s\|_\infty \\ &\stackrel{(2.1)}{\leq} \|b_s\|_\infty \|u_s\|_{1,p} + \|b_s\|_\infty \|u_s\|_{1+\beta,p} + C \|b_s\|_{\beta,p} \|u_s\|_{\gamma+\beta,p} \\ &\stackrel{(2.4)}{\leq} C (\|b_s\|_\infty + \|b_s\|_{\beta,p}) \|u_s\|_{\gamma+\beta,p}. \end{aligned}$$

Hence,

$$\|u_t\|_{\gamma+\beta,p}^q \leq C \lambda^{q(\gamma+\varepsilon)/\alpha-q+1} \left( \|b\|_{L^\infty([0,t]; L^\infty \cap \mathbb{W}_p^\beta)}^q \int_0^t \|u_s\|_{\gamma+\beta,p}^q ds + t \|f\|_{L^\infty([0,t]; \mathbb{W}_p^\beta)}^q \right).$$

By Gronwall's inequality, we obtain (5.2) with  $\delta = q - 1 - \frac{q(\gamma+\varepsilon)}{\alpha} > 0$ . □

Below, we assume that  $b \in L_{loc}^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d) \cap \mathbb{W}_p^\beta)$  with

$$\beta \in \left(1 - \frac{\alpha}{2}, 1\right), \quad p > \frac{2d}{\alpha}, \tag{5.3}$$

and fix

$$\gamma \in \left( \left(1 + \frac{\alpha}{2} - \beta\right) \vee 1, \alpha \right).$$

As in [12,21], let  $u^\ell$  solve the following PIDE:

$$\partial_t u^\ell = (\mathcal{L}_0 - \lambda)u^\ell + b^i \partial_i u^\ell + b^\ell, \quad u_0^\ell(x) = 0, \ell = 1, \dots, d.$$

Fix  $T > 0$  and set

$$\mathbf{v}_t(x) := (u_{T-t}^1(x), \dots, u_{T-t}^d(x)).$$

Then  $\mathbf{v}_t(x)$  solves the following PIDE:

$$\partial_t \mathbf{v} + (\mathcal{L}_0 - \lambda)\mathbf{v} + b^i \partial_i \mathbf{v} + b = 0, \quad \mathbf{v}_T(x) = 0. \tag{5.4}$$

Since  $(\gamma + \beta - 1)p > d$ , by (2.1) and (5.2), one can choose  $\lambda$  sufficiently large such that

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} |\nabla \mathbf{v}_t(x)| \leq \sup_{t \in [0, T]} C \|\mathbf{v}_t\|_{\gamma+\beta, p} \leq C_T \lambda^{-\delta} \|b\|_{L^\infty([0, T]; \mathbb{W}_p^\beta)} \leq \frac{1}{2}. \tag{5.5}$$

Let us define

$$\Phi_t(x) = x + \mathbf{v}_t(x).$$

Since for each  $t \in [0, T]$ ,

$$\frac{1}{2}|x - y| \leq |\Phi_t(x) - \Phi_t(y)| \leq \frac{3}{2}|x - y|,$$

$x \mapsto \Phi_t(x)$  is a diffeomorphism and

$$|\nabla \Phi_t(x)| \leq \frac{3}{2}, \quad |\nabla \Phi_t^{-1}(x)| \leq 2. \tag{5.6}$$

**Lemma 5.2.** *Let  $\Phi_t(x)$  be defined as above. Fix an  $(\mathcal{F}_t)$ -stopping time  $\tau$  and let  $X_t \in \mathcal{I}_b^x$  be a local solution of SDE (1.1). Then  $Y_t = \Phi_t(X_t)$  solves the following SDE on  $[0, T \wedge \tau]$ :*

$$Y_t = \Phi_0(x) + \int_0^t \tilde{b}_s(Y_s) ds + \int_0^t \int_{|z| \leq 1} g_s(Y_{s-}, z) \tilde{N}(ds, dz) + \int_0^t \int_{|z| > 1} g_s(Y_{s-}, z) N(ds, dz), \tag{5.7}$$

where

$$\tilde{b}_s(y) := \lambda \mathbf{v}_s(\Phi_s^{-1}(y)) - \int_{|z| > 1} [\mathbf{v}_s(\Phi_s^{-1}(y) + z) - \mathbf{v}_s(\Phi_s^{-1}(y))] \nu(dz) \tag{5.8}$$

and

$$g_s(y, z) := \Phi_s(\Phi_s^{-1}(y) + z) - y. \tag{5.9}$$

**Proof.** Set

$$\mathbf{v}_t^\varepsilon(x) := (\mathbf{v} * \rho_\varepsilon)(t, x), \quad \Phi_t^\varepsilon(x) = x + \mathbf{v}_t^\varepsilon(x).$$

By Itô's formula, we have for all  $t \in [0, T \wedge \tau]$ ,

$$\begin{aligned} \Phi_t^\varepsilon(X_t) &= \Phi_0^\varepsilon(X_0) + \int_0^t [\partial_s \Phi_s^\varepsilon(X_s) + (b_s^i \partial_i \Phi_s^\varepsilon)(X_s)] ds \\ &\quad + \int_0^t \int_{|z| \leq 1} [\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-}) - z^i \partial_i \Phi_s^\varepsilon(X_{s-})] \nu(dz) ds \\ &\quad + \int_0^t \int_{|z| \leq 1} [\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-})] \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{|z| > 1} [\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-})] N(ds, dz) \\ &=: \Phi_0^\varepsilon(X_0) + I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t) + I_4^\varepsilon(t). \end{aligned}$$

We want to take limits for the above equality. First of all, for  $I_4^\varepsilon(t)$ , by the dominated convergence theorem, we have

$$\begin{aligned} I_4^\varepsilon(t) &= \int_0^t \int_{|z|>1} z N(ds, dz) + \int_0^t \int_{|z|>1} [\mathbf{v}_s^\varepsilon(X_{s-} + z) - \mathbf{v}_s^\varepsilon(X_{s-})] N(ds, dz) \\ &\rightarrow \int_0^t \int_{|z|>1} z N(ds, dz) + \int_0^t \int_{|z|>1} [\mathbf{v}_s(X_{s-} + z) - \mathbf{v}_s(X_{s-})] N(ds, dz) \\ &= \int_0^t \int_{|z|>1} [\Phi_s(X_{s-} + z) - \Phi_s(X_{s-})] N(ds, dz) = \int_0^t \int_{|z|>1} g_s(Y_{s-}, z) N(ds, dz), \end{aligned}$$

and for  $I_3^\varepsilon(t)$ ,

$$\begin{aligned} &\mathbb{E} \left| \int_0^{t \wedge \tau} \int_{|z| \leq 1} [\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-}) - \Phi_s(X_{s-} + z) + \Phi_s(X_{s-})] \tilde{N}(ds, dz) \right|^2 \\ &= \mathbb{E} \int_0^{t \wedge \tau} \int_{|z| \leq 1} |\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-}) - \Phi_s(X_{s-} + z) + \Phi_s(X_{s-})|^2 \nu(dz) ds \rightarrow 0, \end{aligned}$$

where we have used that for some  $C$  independent of  $\varepsilon$ ,

$$|\Phi_s^\varepsilon(X_{s-} + z) - \Phi_s^\varepsilon(X_{s-})| \leq C|z|^2.$$

Noting that

$$\partial_s \Phi^\varepsilon = \partial_s \mathbf{v}^\varepsilon = -(\mathcal{L}_0 - \lambda) \mathbf{v}^\varepsilon - (b^i \partial_i \mathbf{v}) * \rho_\varepsilon - b * \rho_\varepsilon = -(\mathcal{L}_0 - \lambda) \mathbf{v}^\varepsilon - (b^i \partial_i \Phi) * \rho_\varepsilon,$$

we have

$$\begin{aligned} I_1^\varepsilon(t) + I_2^\varepsilon(t) &= \lambda \int_0^t \mathbf{v}_s^\varepsilon(X_s) ds - \int_0^t \int_{|z|>1} [\mathbf{v}_s^\varepsilon(X_s + z) - \mathbf{v}_s^\varepsilon(X_s)] \nu(dz) ds \\ &\quad + \int_0^t [(b_s^i \partial_i \Phi_s^\varepsilon)(X_s) - ((b^i \partial_i \Phi) * \rho_\varepsilon)(s, X_s)] ds. \end{aligned}$$

By the dominated convergence theorem, the first two terms converge to

$$\lambda \int_0^t \mathbf{v}_s(X_s) ds - \int_0^t \int_{|z|>1} [\mathbf{v}_s(X_s + z) - \mathbf{v}_s(X_s)] \nu(dz) ds = \int_0^t \tilde{b}_s(Y_s) ds.$$

Using Krylov's estimate (3.13), we have

$$\begin{aligned} &\mathbb{E} \int_0^{t \wedge \tau} |(b_s^i \partial_i \Phi_s^\varepsilon)(X_s) - ((b^i \partial_i \Phi) * \rho_\varepsilon)(s, X_s)| ds \\ &\leq C \int_0^t \left( \int_{\mathbb{R}^d} |(b_s^i \partial_i \Phi_s^\varepsilon)(x) - ((b^i \partial_i \Phi) * \rho_\varepsilon)(s, x)|^p dx \right)^{q/p} ds \rightarrow 0, \end{aligned}$$

where  $q > \frac{\alpha}{\alpha-1}$ . Combining the above calculations, we obtain that  $Y_t$  solves (5.7). □

We are now in a position to give:

**Proof of Theorem 1.1.** We first assume that for some  $\beta, p$  satisfying (5.3),

$$b \in L_{loc}^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d) \cap \mathbb{W}_p^\beta).$$

The existence of weak solutions has been obtained in Theorem 4.1. Below, we concentrate on the proof of the pathwise uniqueness.

Fix an  $(\mathcal{F}_t)$ -stopping time  $\tau$  and let  $X_t, \hat{X}_t \in \mathcal{G}_b^\tau(x)$  be two solutions of SDE (1.1). Fixing  $T > 0$ , we want to prove that

$$Y_t := \Phi_t(X_t) = \Phi_t(\hat{X}_t) =: \hat{Y}_t, \quad \forall t \in [0, T \wedge \tau).$$

Define  $\sigma_0 \equiv 0$  and for  $n \in \mathbb{N}$ ,

$$\sigma_n := \inf\{t \geq \sigma_{n-1}: |L_t - L_{t-}| > 1\}.$$

Set

$$\sigma_n^T = \sigma_n \wedge T \wedge \tau.$$

Recall (3.1) and

$$\int_0^t \int_{|z|>1} g_s(Y_{s-}, z) N(ds, dz) = \sum_{s \in (0, t]} g_s(Y_{s-}, L_s - L_{s-}) \cdot 1_{|L_s - L_{s-}|>1}.$$

By Lemma 5.2,  $Z_t := Y_t - \hat{Y}_t$  satisfies the following equation on random interval  $[\sigma_n^T, \sigma_{n+1}^T)$ :

$$Z_t = Z_{\sigma_n^T} + \int_{\sigma_n^T}^t [\tilde{b}_s(Y_s) - \tilde{b}_s(\hat{Y}_s)] ds + \int_{\sigma_n^T}^t \int_{|z| \leq 1} [g_s(Y_{s-}, z) - g_s(\hat{Y}_{s-}, z)] \tilde{N}(ds, dz). \tag{5.10}$$

Let us first prove that

$$Z_t = 0 \quad \text{a.s. on } [0, \sigma_1^T).$$

Note that by (5.8), (5.5) and (5.6),

$$|\tilde{b}_s(y) - \tilde{b}_s(y')| \leq C|y - y'|, \tag{5.11}$$

and by (2.5),

$$\begin{aligned} |g_s(y, z) - g_s(y', z)| &= |\Phi_s(\Phi_s^{-1}(y) + z) - \Phi_s(\Phi_s^{-1}(y)) - \Phi_s(\Phi_s^{-1}(y') + z) + \Phi_s(\Phi_s^{-1}(y'))| \\ &= |\mathbf{v}_s(\Phi_s^{-1}(y) + z) - \mathbf{v}_s(\Phi_s^{-1}(y)) - \mathbf{v}_s(\Phi_s^{-1}(y') + z) + \mathbf{v}_s(\Phi_s^{-1}(y'))| \\ &= |(\mathcal{T}_z \mathbf{v}_s)(\Phi_s^{-1}(y)) - (\mathcal{T}_z \mathbf{v}_s)(\Phi_s^{-1}(y'))| \\ &\leq C|\Phi_s^{-1}(y) - \Phi_s^{-1}(y')| \cdot (\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(y)) + \mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(y'))) \\ &\leq C|y - y'| \cdot (\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(y)) + \mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(y'))). \end{aligned} \tag{5.12}$$

Since  $\mathbb{E}|X_t|^2 = +\infty$ , in order to take expectations for (5.10), we need to use stopping time to cut off it. For  $R > 0$ , define

$$\zeta_R := \inf\{t \geq 0: |X_t| \vee |\hat{X}_t| \geq R\}. \tag{5.13}$$

Let  $\eta$  be any  $(\mathcal{F}_t)$ -stopping time. By (5.10), (5.11) and (5.12), we have

$$\begin{aligned} \mathbb{E}|Z_{t \wedge \sigma_1^T \wedge \zeta_R \wedge \eta}|^2 &\leq C\mathbb{E} \int_0^{t \wedge \sigma_1^T \wedge \zeta_R \wedge \eta} \left( |Z_s|^2 + \int_{|z| \leq 1} |g_s(Y_{s-}, z) - g_s(\hat{Y}_{s-}, z)|^2 v(dz) \right) ds \\ &\leq C\mathbb{E} \int_0^{t \wedge \sigma_1^T \wedge \zeta_R \wedge \eta} |Z_{s-}|^2 d(s + A_s) \leq C\mathbb{E} \int_0^{t \wedge \eta} |Z_{s \wedge \sigma_1^T \wedge \zeta_R}|^2 d(s + A_{s \wedge \tau}), \end{aligned}$$

where

$$A_t := \int_0^t \int_{|z| \leq 1} (\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(Y_s)) + \mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\Phi_s^{-1}(\hat{Y}_s)))^2 \nu(dz) ds.$$

By Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}A_{t \wedge \tau} &= \int_{|z| \leq 1} \mathbb{E} \int_0^{t \wedge \tau} (\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(X_s) + \mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|(\hat{X}_s))^2 ds \nu(dz) \\ &\stackrel{(3.13)}{\leq} C \int_{|z| \leq 1} \sup_{s \in [0, t]} \|(\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|)^2\|_{p/2} \nu(dz) \\ &= C \int_{|z| \leq 1} \sup_{s \in [0, t]} \|\mathcal{M}|\nabla \mathcal{T}_z \mathbf{v}_s|\|_p^2 \nu(dz) \stackrel{(2.6)}{\leq} C \int_{|z| \leq 1} \sup_{s \in [0, t]} \|\mathcal{T}_z \mathbf{v}_s\|_{1,p}^2 \nu(dz) \\ &\stackrel{(2.7)}{\leq} C \sup_{s \in [0, t]} \|\mathbf{v}_s\|_{\gamma+\beta, p}^2 \int_{|z| \leq 1} |z|^{2(\gamma+\beta-1)} \nu(dz) < +\infty, \end{aligned}$$

where in the last step we have used (5.5),  $2(\gamma + \beta - 1) > \alpha$  and (1.5). Therefore,  $t \mapsto A_{t \wedge \tau}$  is a continuous  $(\mathcal{F}_t)$ -adapted increasing process. By Lemma 2.6, we obtain that for all  $t \geq 0$ ,

$$Z_{t \wedge \sigma_1^T \wedge \zeta_{R^-}} = 0, \quad \text{a.s.}$$

Letting  $R \rightarrow \infty$  yields that for all  $t \in [0, T \wedge \tau)$ ,

$$Z_{t \wedge \sigma_1^-} = Z_{t \wedge \sigma_1^T} = 0, \quad \text{a.s.}$$

Thus, if  $\sigma_1 < T \wedge \tau$ , then

$$Z_{\sigma_1} = Z_{\sigma_1^-} + [g_{\sigma_1}(Y_{\sigma_1^-}, L_{\sigma_1} - L_{\sigma_1^-}) - g_{\sigma_1}(\hat{Y}_{\sigma_1^-}, L_{\sigma_1} - L_{\sigma_1^-})] = 0.$$

Repeating the above calculations, we find that for all  $n \in \mathbb{N}$  and  $t \in [0, T \wedge \tau)$ ,

$$Z_{t \wedge \sigma_n^-} = 0 \quad \text{a.s.}$$

Letting  $n, T \rightarrow \infty$  produces that for all  $t \in [0, \tau)$ ,

$$Z_t = 0 \quad \Rightarrow \quad Y_t = \hat{Y}_t \quad \Rightarrow \quad X_t = \hat{X}_t \quad \text{a.s.}$$

Lastly, we assume that  $b$  satisfies (1.7) and (1.8). For  $n \in \mathbb{N}$ , let  $\chi_n \in C_0^\infty(\mathbb{R}^d)$  with  $\chi_n(x) = 1$  for  $|x| \leq n$  and  $\chi_n(x) = 0$  for  $|x| > n + 1$ . Define

$$b_t^{(n)}(x) = b_t(x) \chi_n(x).$$

Then  $b^{(n)} \in L_{loc}^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d) \times \mathbb{W}_p^\beta)$ . By the previous proof, for each  $x \in \mathbb{R}^d$ , there exists a unique strong solution  $X_t^{(n)} \in \mathcal{G}_{b^{(n)}}^\infty(x)$  to SDE (1.1) with drift  $b^{(n)}$ . For  $n \geq k$ , define

$$\tau_{n,k}(x, \omega) := \inf\{t \geq 0: |X_t^{(n)}(\omega, x)| \geq k\}.$$

It is easy to see that

$$X_t^{(n)}(x), X_t^{(k)}(x) \in \mathcal{G}_{b^{(k)}}^{\tau_{n,k}(x)}(x).$$

Since the local uniqueness has been proven, we have

$$P\{\omega: X_t^{(n)}(\omega, x) = X_t^{(k)}(\omega, x), \forall t \in [0, \tau_{n,k}(x, \omega)]\} = 1,$$

which implies that for  $n \geq k$ ,

$$\tau_{k,k}(x) \leq \tau_{n,k}(x) \leq \tau_{n,n}(x), \quad \text{a.s.}$$

Hence, if we let  $\zeta_k(x) := \tau_{k,k}(x)$ , then  $\zeta_k(x)$  is an increasing sequence of  $(\mathcal{F}_t)$ -stopping times and for  $n \geq k$ ,

$$P\{\omega: X_t^{(n)}(x, \omega) = X_t^{(k)}(x, \omega), \forall t \in [0, \zeta_k(x, \omega)]\} = 1.$$

Now, for each  $k \in \mathbb{N}$ , we can define  $X_t(x, \omega) = X_t^{(k)}(x, \omega)$  for  $t < \zeta_k(x, \omega)$  and  $\zeta(x) = \lim_{k \rightarrow \infty} \zeta_k(x)$ . It is clear that  $X_t(x) \in \mathcal{S}_b^{\zeta(x)}(x)$  and (1.9) holds.  $\square$

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## References

- [1] H. Airault and X. Zhang. Smoothness of indicator functions of some sets in Wiener spaces. *J. Math. Pures Appl.* **79** (2000) 515–523. [MR1759438](#)
- [2] D. Aldous. Stopping times and tightness. *Ann. Probab.* **6** (1978) 335–340. [MR0474446](#)
- [3] D. Applebaum. *Lévy Processes and Stochastic Calculus. Cambridge Studies in Advance Mathematics* **93**. Cambridge Univ. Press, Cambridge, UK, 2004. [MR2072890](#)
- [4] R. Bass. Stochastic differential equations driven by symmetric stable processes. In *Seminaire de Probabilités, XXXVI* 302–313. *Lecture Notes in Math.* **1801**. Springer, Berlin, 2003. [MR1971592](#)
- [5] R. Bass. Stochastic differential equations with jumps. *Probab. Surv.* **1** (2004) 1–19. [MR2095564](#)
- [6] K. Bogdan and T. Jakubowski. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.* **271** (2007) 179–198. [MR2283957](#)
- [7] Z. Chen, P. Kim and R. Song. Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation. Available at <http://arxiv.org/abs/1011.3273>.
- [8] G. Crippa and C. De Lellis. Estimates and regularity results for the DiPerna–Lions flow. *J. Reine Angew. Math.* **616** (2008) 15–46. [MR2369485](#)
- [9] G. Da Prato and F. Flandoli. Pathwise uniqueness for a class of SDE in Hilbert spaces and applications. *J. Funct. Anal.* **259** (2010) 243–267. [MR2610386](#)
- [10] A. M. Davie. Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN* (2007) Art. ID rnm124. [MR2377011](#)
- [11] E. Fedrizzi and F. Flandoli. Pathwise uniqueness and continuous dependence for SDEs with nonregular drift. Available at [arXiv:1004.3485v1](http://arxiv.org/abs/1004.3485v1).
- [12] F. Flandoli, M. Gubinelli and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (2010) 1–53. [MR2593276](#)
- [13] N. Fournier. On pathwise uniqueness for stochastic differential equations driven by stable Lévy processes. Available at <http://arxiv.org/abs/1011.0532>.
- [14] I. Gyöngy and T. Martinez. On stochastic differential equations with locally unbounded drift. *Czechoslovak Math. J.* **51** (2001) 763–783. [MR1864041](#)
- [15] S. He, J. Wang and J. Yan. *Semimartingale Theory and Stochastic Calculus*. Science Press and CRC Press, Beijing, 1992. [MR1219534](#)
- [16] N. V. Krylov. *Controlled Diffusion Processes. Applications of Mathematics* **14**. Springer, New York, Berlin, 1980. Translated from the Russian by A. B. Aries. [MR0601776](#)
- [17] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** (2005) 154–196. [MR2117951](#)
- [18] V. P. Kurenok. Stochastic equations with time-dependent drifts driven by Lévy processes. *J. Theoret. Probab.* **20** (2007) 859–869. [MR2359059](#)
- [19] V. P. Kurenok. A note on  $L_2$ -estimates for stable integrals with drift. *Trans. Amer. Math. Soc.* **360** (2008) 925–938. [MR2346477](#)
- [20] Z. Li and L. Mytnik. Strong solutions for stochastic differential equations with jumps. Available at <http://arxiv.org/abs/0910.0950>.
- [21] E. Priola. Pathwise uniqueness for singular SDEs driven by stable processes. Available at <http://arxiv.org/abs/1005.4237>.
- [22] J. Ren and X. Zhang. Limit theorems for stochastic differential equations with discontinuous coefficients. *SIAM J. Math. Anal.* **43** (2011) 302–321. [MR2765692](#)
- [23] K. I. Sato. *Lévy Processes and Infinite Divisible Distributions*. Cambridge Univ. Press, Cambridge, 1999. [MR1739520](#)
- [24] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton, NJ, 1970. [MR0290095](#)
- [25] H. Tanaka, M. Tsuchiya and S. Watanabe. Perturbation of drift-type for Lévy processes. *J. Math. Kyoto Univ.* **14** (1974) 73–92. [MR0368146](#)

- [26] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Amsterdam, 1978. [MR0503903](#)
- [27] A. J. Veretennikov. On the strong solutions of stochastic differential equations. *Theory Probab. Appl.* **24** (1979) 354–366.
- [28] X. Zhang. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.* **16** (2011) 1096–1116. [MR2820071](#)
- [29] X. Zhang. Well-posedness and large deviation for degenerate SDEs with Sobolev coefficients. Available at <http://arxiv.org/abs/1002.4297>.
- [30] A. K. Zvonkin. A transformation of the phase space of a diffusion process that removes the drift. *Mat. Sb.* **93** (1974) 129–149. [MR0336813](#)