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THE FUNDAMENTAL THEOREM OF ASSET PRICING, THE HEDGING PROBLEM AND MAXIMAL CLAIMS IN FINANCIAL MARKETS WITH SHORT SALES PROHIBITIONS

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This paper consists of two parts. In the first part we prove the fundamental theorem of asset pricing under short sales prohibitions in continuous-time financial models where asset prices are driven by nonnegative, locally bounded semimartingales. A key step in this proof is an extension of a well-known result of Ansel and Stricker. In the second part we study the hedging problem in these models and connect it to a properly defined property of "maximality" of contingent claims.

1. Introduction. The practice of short selling is alleged to magnify the decline of asset prices. As a result, short sales bans and restrictions have been commonly used as a regulatory measure to stabilize prices during downturns in the economy. The most notable recent examples are: (i) in August of 2011, the European Securities and Markets Authority curtailed short sales in France, Belgium, Italy and Spain in an effort to stop the tailspin in the markets caused by the European debt crisis (see [15]); (ii) in September of 2008, after the burst of the housing bubble, the U.S. Securities and Exchange Commission (SEC) prohibited short selling for 797 financial companies in an effort to stabilize those companies (see [2, 3]); (iii) at the same time, in September of 2008, the U.K. Financial Services Authority (FSA) prohibited short selling for 32 financial companies (see [2, 3]).

Short sales prohibitions, however, are seen not only after the burst of a price bubble or during times of financial stress. In certain cases, the inability to short sell is inherent to the specific market. There are over 150 stock markets worldwide, many of which are in the third world. In most of the third world emerging markets the practice of short selling is not allowed; see [4, 5]. Additionally, in markets such as commodity markets and the housing market, primary securities such as mortgages cannot be sold short because they cannot be borrowed.

This paper aims to understand the consequences of short sales prohibition in semimartingale financial models. The fundamental theorem of asset pricing establishes the equivalence between the absence of arbitrage, a key concept in mathematical finance, and the existence of a probability measure under which the asset

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prices in the market have a characteristic behavior. In Section 3, we prove the fundamental theorem of asset pricing in continuous time financial models with short sales prohibition where prices are driven by locally bounded semimartingales. This extends related results by Jouini and Kallal in [23], Schürger in [34], Frittelli in [18], Pham and Touzi in [30], Napp in [29] and more recently by Karatzas and Kardaras in [26] to the framework of the seminal work of Delbaen and Schachermayer in [9].

Additionally, the hedging problem of contingent claims in markets with convex portfolio constraints where prices are driven by diffusions and discrete processes has been extensively studied; see [6], Chapter 5 of [27] and Chapter 9 of [17]. In Section 4, inspired by the works of Jacka in [19] and Ansel and Stricker in [1], and using ideas from [16], we extend some of these classical results to more general semimartingale financial models. We also reveal an interesting financial connection to the concept of maximal claims, first introduced by Delbaen and Schachermayer in [9] and [10].

2. The set-up.

2.1. *The financial market.* We focus our analysis on a finite time trading horizon $[0, T]$ and assume that there are N risky assets trading in the market. We suppose, as in the seminal work of Delbaen and Schachermayer in [9], that the price processes of the N risky assets are nonnegative locally bounded P-semimartingales over a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual hypotheses. We let $S := (S^i)_{1\leq i\leq N}$ be the $\mathbb{R}^N$-valued stochastic process representing the prices of the risky assets. We assume without loss of generality that the spot interest rates are constant and equal to 0, that is, the price processes are already discounted. We also assume that the risky assets have no cash flows associated to them, and there are no transaction costs.

The probability measure P denotes our reference probability measure. We suppose that $\mathcal{F}_0$ is P-trivial and $\mathcal{F}_T = \mathcal{F}$. Hence, all random variables measurable with respect to $\mathcal{F}_0$ are P-almost surely constant and there is no additional source of randomness on the probability space other than the one specified by the filtration $\mathbb{F}$. As usual, we identify random variables that are equal P-almost surely. If X is a semimartingale over this stochastic basis, we denote by $L(X)$ the space of predictable processes integrable with respect to X. Given $H \in L(X)$, $H \cdot X$ denotes the stochastic integral of H with respect to X; see page 165 of [32]. If $t \in [0, T]$, we let $\Delta X_t = X_t - X_{t-}$ be the jump of X at time t, with the convention that $X_{0-} = 0$. If τ is a stopping time, $X^\tau := X_{\cdot\wedge\tau}$ denotes the process X stopped at τ. Given two semimartingales X, Y we denote by $[X, Y]$ the quadratic covariation of X and Y; see page 66 of [32]. Given a probability measure Q equivalent to P, denoted by $Q \sim P$, we let $L^0(Q)$, $L^0_+(Q)$, $L^\infty(Q)$, $L^\infty_+(Q)$ and $L^1(Q)$ be the spaces of equivalent classes of real-valued random variables, nonnegative random variables, Q-essentially bounded random variables, nonnegative

Q-essentially bounded random variables and Q-integrable random variables, respectively. For a measure $Q \sim P$ and a random variable f bounded from below, we let $E^Q[f]$, $E^Q[f|\mathcal{F}_t]$ be the expectation with respect to Q and the conditional expectation with respect to Q given $\mathcal{F}_t$, respectively. Finally, $\mathcal{H}^1(Q)$ denotes the space of martingales X such that $E^Q[[X, X]_T^{1/2}] < \infty$.

2.2. *The trading strategies*. We fix $0 \leq d \leq N$ and assume that the first d risky assets can be sold short in an admissible fashion to be specified below and that the last $N - d$ risky assets cannot be sold short under any circumstances. This leads us to define the set of admissible strategies in the market as follows.

DEFINITION 2.1. A vector valued process $H = (H^1, \dots, H^N)$, where for $1 \leq i \leq N$ and $t \in [0, T]$, H_t^i denotes the number of shares of asset i held at time t, is called an *admissible trading strategy* if:

(i) $H \in L(S)$;
(ii) $H_0 = 0$;
(iii) $(H \cdot S) \geq -\alpha$ for some $\alpha > 0$;
(iv) $H^i \geq 0$ for all $i > d$.

We let $\mathcal{A}$ be the set of admissible trading strategies.

Hence, by condition (ii), we assume that the initial risky assets' holdings are always equal to 0 and therefore initial endowments are always in numéraire denomination. Condition (iii) above is usually called the *admissibility condition* and restricts the agents' strategies to those whose value is uniformly bounded from below over time. The only sources of friction in our market come from conditions (iii) and (iv) above. For every admissible strategy $H \in \mathcal{A}$ we define the optional process H^0 by

$$H^0 := (H \cdot S) - \sum_{i=1}^{N} H^i S^i. \tag{2.1}$$

If H^0 denotes the balance in the money market account, then the strategy $\overline{H} = (H^0, H)$ is *self-financing* with initial value 0.

2.3. *No arbitrage conditions*. In [9] and [12], Delbaen and Schachermayer considered the no arbitrage paradigm known as no free lunch with vanishing risk (NFLVR) and proved the fundamental theorem of asset pricing (FTAP) under this framework. Below we will redefine the (NFLVR) condition in our context.

Define the following cones in $L^0(P)$:

$$\mathcal{K} := \{(H \cdot S)_T : H \in \mathcal{A}\}, \tag{2.2}$$

$$\begin{aligned} \mathcal{C} &:= (\mathcal{K} - L_+^0(P)) \cap L^\infty(P) \\ &= \{g \in L^\infty(P) : g = f - h \text{ for some } f \in K \text{ and } h \in L_+^0(P)\}. \end{aligned} \tag{2.3}$$

The cone $\mathcal{K}$ corresponds to the cone of random variables that can be obtained as payoffs of admissible strategies with zero initial endowment. The cone $\mathcal{C}$ is the cone of random variables that are P-almost surely bounded and are dominated from above by an element of $\mathcal{K}$. These sets of random variables are cones and not subspaces of $L^0(P)$ due to conditions (iii) and (iv) in Definition 2.1. We define in our market the following "no arbitrage" type conditions.

DEFINITION 2.2. We say that the financial market satisfies the condition of *no arbitrage under short sales prohibition* (*NA-S*) if

$$\mathcal{C} \cap L_+^\infty(P) = \{0\}.$$

In order to prove the (FTAP), the condition of (NA-S) has to be modified.

DEFINITION 2.3. We say that the financial market satisfies the condition of *no free lunch with vanishing risk under short sales prohibition* (*NFLVR-S*) if

$$\overline{\mathcal{C}} \cap L_+^\infty(P) = \{0\},$$

where the closure above is taken with respect to the $\|\cdot\|_\infty$ norm on $L^\infty(P)$.

REMARK 2.4. Observe that (NFLVR-S) does not hold if and only if there exists a sequence $({}^nH)$ in $\mathcal{A}$, a sequence of bounded random variables (f_n) and a bounded random variable f measurable with respect to $\mathcal{F}$ such that $({}^nH \cdot S)_T \geq f_n$ for all n, f_n converges to f in $L^\infty(P)$, $P(f \geq 0) = 1$ and $P(f > 0) > 0$.

In the next section we prove the (FTAP) in our context. This theorem establishes a relationship between the (NFLVR-S) condition defined above and the existence of a measure, usually known as the risk neutral measure, under which the price processes behave in a particular way.

3. The fundamental theorem of asset pricing. The results presented in this section are a combination of the results obtained by Frittelli in [18] for simple predictable strategies in markets under convex constraints, and the extension of the classical theorem of Delbaen and Schachermayer (see [9]) to markets with convex cone constraints established by Kabanov in [24]. The characterization of (NFLVR-S) is in accordance with the (FTAP) as proven in [23] by Jouini and Kallal, who assumed that S_t is square integrable under P for all times t and considered simple predictable strategies.

3.1. *The set of risk neutral measures.* We first define our set of *risk neutral measures*.

DEFINITION 3.1. We let $\mathcal{M}_{\sup}(S)$ be the set of probability measures Q on $(\Omega, \mathcal{F})$ such that:

(i) $Q \sim P$ and

(ii) for $1 \le i \le d$, S^i is a Q-local martingale and, for $d < i \le N$, S^i is a Q-supermartingale.

We will call the set $\mathcal{M}_{\sup}(S)$ the set of *risk neutral measures* or *equivalent supermartingale measures* (*ESMM*).

The following proposition plays a crucial role in the analysis below.

PROPOSITION 3.2. *Let $\mathcal{C}$ be as in* (2.3). *Then*

$$\mathcal{M}_{\sup}(S) = \Big\{ Q \sim P : \sup_{f \in \mathcal{C}} E^Q[f] = 0 \Big\}.$$

To prove this proposition we need the following results.

LEMMA 3.3. *Suppose that Q is a probability measure on $(\Omega, \mathcal{F})$. Let V be an $\mathbb{R}^N$-valued Q-semimartingale such that V^i is Q-local supermartingale for $i > d$, and V^i is a Q-local martingale for $i \le d$. Let H be an $\mathbb{R}^N$-valued bounded predictable process, such that $H^i \ge 0$ for $i > d$. Then $(H \cdot V)$ is a Q-local supermartingale.*

PROOF. Without loss of generality we can assume that, under Q, V^i is a supermartingale for $i > d$. Suppose that for $i > d$, $V^i = M^i - A^i$ is the Doob–Meyer decomposition of the Q-supermartingale V^i, with M^i a Q-local martingale and A^i a predictable nondecreasing process such that $A^i_0 = 0$. Let $M^i = V^i$ and $A^i = 0$ for $i \le d$. Then $V = M - A$, with $M = (M^1, \dots, M^N)$ and $A = (A^1, \dots, A^N)$, is the canonical decomposition of the special vector valued semimartingale V under Q. Since H is bounded, $(H \cdot V)$ is a Q-special semimartingale, $H \in L(M) \cap L(A)$, $(H \cdot V) = (H \cdot M) - (H \cdot A)$ and $(H \cdot M)$ is a Q-local martingale; see Proposition 2 in [20]. Additionally, since $H^i \ge 0$ for $i > d$ we have that $(H \cdot A)$ is a nondecreasing process starting at 0. We conclude then that $(H \cdot V)$ is a Q-local supermartingale. □

The following lemma is a known result of stochastic analysis which we present here for completion.

LEMMA 3.4. *Suppose that H is a bounded predictable process, and $X \in \mathcal{H}^1(Q)$ is a real-valued martingale. Then $H \cdot X$ is also in $\mathcal{H}^1(Q)$. In particular, $H \cdot X$ is a Q-martingale.*

PROOF. The argument to prove this result is analogous to the one used in the proof of Emery's inequality (see Theorem V-3 in [32]) and we do not include its proof in this paper. □

The next proposition is a key step in the extension of the (FTAP) to markets with short sales prohibition and prices driven by arbitrary locally bounded semimartingales. It extends a well-known result by Ansel and Stricker; see Proposition 3.3 in [1].

PROPOSITION 3.5. *Let $Q \in \mathcal{M}_{\sup}(S)$ and $H \in L(S)$ be such that $H^i \geq 0$ for $i > d$. Then $H \cdot S$ is a Q-local supermartingale if and only if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ that increases Q-almost surely to ∞ and a sequence of nonpositive random variables Θ_n in $L^1(Q)$ such that $\Delta(H \cdot S)^{T_n} \geq \Theta_n$ for all n.*

PROOF. ($\Leftarrow$) It is enough to show that for all n, $(H \cdot S)^{T_n}$ is a Q-local supermartingale. Hence, without loss of generality we can assume that

$$\Delta(H \cdot S) \geq \Theta$$

with $\Theta \in L^1(Q)$ a nonpositive random variable. By Proposition 3 in [20], if we define

$$U_t = \sum_{s \leq t} 1_{\{|\Delta S_s|>1 \text{ or } |\Delta(H\cdot S)_s|>1\}} \Delta S_s,$$

there exist a Q-local martingale N and a predictable process of finite variation B such that $H \in L(N) \cap L(B+U)$, $Y := S - U$ is a Q-special semimartingale with bounded jumps and canonical decomposition $Y = N + B$ and $H \cdot N$ is a Q-local martingale. Let $V := B + U$ and $H^\alpha := H 1_{\{|H|\leq\alpha\}}$ for $\alpha \geq 0$. We have that $Q \in \mathcal{M}_{\sup}(S)$, N is a Q-local martingale and $V = S - N$. This implies that V^i is a Q-local supermartingale for $i > d$, and V^i is a Q-local martingale for $i \leq d$. We can further assume by localization that $N^i \in \mathcal{H}^1(Q)$ for all $i \leq N$ and that V has canonical decomposition $V = M - A$, where M^i in $\mathcal{H}^1(Q)$ and $A^i \geq 0$ is Q-integrable, predictable and nondecreasing for all $i \leq N$; see Theorem IV-51 in [32]. By Lemmas 3.3 and 3.4, these assumptions imply that for all $\alpha \geq 0$, $H^\alpha \cdot N$ and $H^\alpha \cdot M$ are Q-martingales and $H^\alpha \cdot V$ is a Q-supermartingale. In particular for all stopping times $\tau \leq T$, $E^Q[(H^\alpha \cdot N)_\tau] = 0$ and $E^Q[(H^\alpha \cdot V)_\tau] \leq 0$. This implies that for all stopping times $\tau \leq T$, $E^Q[|(H \cdot N)_\tau|] = 2E^Q[(H \cdot N)_\tau^-]$ and $E^Q[|(H \cdot V)_\tau|] \leq 2E^Q[(H \cdot V)_\tau^-]$. After these observations, by following the same argument as the one given in the proof of Proposition 3.3 in [1], we find a sequence of stopping times $(\tau_p)_{p\geq 0}$ increasing to ∞ such that $E^Q[|H \cdot V|_{\tau_p}] \leq 12p + 4E^Q[|\Theta|]$ and, for all $\alpha \geq 0$, $|(H^\alpha \cdot V)^{\tau_p}| \leq 4p + |H \cdot V|_{\tau_p}$. An application of the dominated convergence theorem yields that $(H \cdot V)^{\tau_p}$ is a Q-supermartingale for all $p \geq 0$. Since $H \cdot S = H \cdot N + H \cdot V$ and $(H \cdot N)$ is a Q-local martingale, we conclude that $(H \cdot S)$ is a Q-local supermartingale.

($\Rightarrow$) The Q-local supermartingale $H \cdot S$ is special. By Proposition 2 in [20], if $S = M - A$ is the canonical decomposition of S with respect to Q, where M^i is a Q-local martingale, $A_0 = 0$ and A^i is an nondecreasing, predictable and Q-locally integrable process for all $i \leq N$, then $H \cdot S = H \cdot M - H \cdot A$ is the canonical decomposition of $H \cdot S$, where $H \cdot M$ is a Q-local martingale and $H \cdot A$ is nondecreasing, predictable and Q-locally integrable. By Proposition 3.3 in [1] we can find a sequence of stopping times $(T_n)_{n\geq 0}$ that increases to ∞ and a sequence of nonpositive random variables $(\tilde{\Theta}_n)$ in $L^1(Q)$ such that

$$\Delta(H \cdot M)^{T_n} \geq \tilde{\Theta}_n.$$

We can further assume without loss of generality that $(H \cdot A)_{T_n} \in L^1(Q)$ for all n. By taking $\Theta_n = \tilde{\Theta}_n - (H \cdot A)_{T_n}$, we conclude that for all n

$$\Delta(H \cdot S)^{T_n} = \Delta(H \cdot M)^{T_n} - \Delta(H \cdot A)^{T_n} \geq \tilde{\Theta}_n - (H \cdot A)^{T_n} \geq \Theta_n. \qquad \square$$

LEMMA 3.6. *Let $Q \in \mathcal{M}_{\sup}(S)$ and $H \in \mathcal{A}$; see Definitions 2.1 and 3.1. Then $(H \cdot S)$ is a Q-supermartingale. In particular $(H \cdot S)_T \in L^1(Q)$ and $E^Q[(H \cdot S)_T] \leq 0$.*

PROOF. Assume that $(H \cdot S) \geq -\alpha$, with $\alpha \geq 0$. Let $q \geq 0$ be arbitrary. If we define $T_q = \inf\{t \geq 0 : (H \cdot S)_t \geq q - \alpha\}$, we have that $\Delta(H \cdot S)^{T_q} \geq -q$. By Proposition 3.5 we conclude that $(H \cdot S)$ is a Q-local supermartingale bounded from below. By Fatou's lemma we obtain that $(H \cdot S)$ is a Q-supermartingale as we wanted to prove. $\square$

REMARK 3.7. The statement of Lemma 3.6 corresponds to Lemma 2.2 and Proposition 3.1 in [25]. Here we have proved this result by methods similar to the ones appearing in the original proof of Ansel and Stricker in [1]. Additionally, we have given sufficient and necessary conditions for the σ-supermartingale property (see Definition 2.1 in [25]) to hold.

We are now ready to prove the main proposition of this section. The arguments below essentially correspond to those presented in [9, 24] and [26]. We include them here for completeness.

PROOF OF PROPOSITION 3.2. By Lemma 3.6

$$\mathcal{M}_{\sup}(S) \subset \Big\{Q \sim P : \sup_{f \in \mathcal{C}} E^Q[f] = 0\Big\}.$$

Now suppose that Q is a probability measure equivalent to P such that $E^Q[f] \leq 0$ for all $f \in \mathcal{C}$. Fix $1 \leq i \leq N$. Since S^i is locally bounded, there exists a sequence of stopping times (σ_n) increasing to ∞ such that $S^i_{\cdot \wedge \sigma_n}$ is bounded. Let $0 \leq s < t \leq T$, $A \in \mathcal{F}_s$ and $n \geq 0$ be arbitrary. Consider the process $H^i(r, \omega) = 1_A(\omega) 1_{(s\wedge\sigma_n, t\wedge\sigma_n]}(r)$. Let $H^j \equiv 0$ for $j \neq i$. We have that $H = (H_1, \dots, H_N) \in \mathcal{A}$, $(H \cdot S)_T \in \mathcal{C}$ and

$$0 \geq E^Q[(H \cdot S)_T] = E^Q[1_A(S^i_{t\wedge\sigma_n} - S^i_{s\wedge\sigma_n})].$$

This implies that $S^i_{\cdot\wedge\sigma_n}$ is a Q-supermartingale for all n and S^i is a Q-local supermartingale. Since S^i is nonnegative, by Fatou's lemma we conclude that S^i is a Q-supermartingale. For $1 \le i \le d$ we can apply the same argument to the process $H^i(r,\omega) = -1_A(\omega)1_{(s\wedge\sigma_n, t\wedge\sigma_n]}(r)$ to conclude that S^i is a Q-local martingale. Hence

$$\mathcal{M}_{\sup}(S) \supset \Big\{Q \sim P : \sup_{f\in\mathcal{C}} E^Q[f] = 0\Big\},$$

and the proposition follows. □

We have seen in the proof of this proposition that the following equality holds.

COROLLARY 3.8. *Let $\mathcal{M}_{\sup}(S)$ be as in Definition 3.1. Then*

$$\mathcal{M}_{\sup}(S) = \{Q \sim P\colon (H \cdot S) \text{ is a } Q\text{-supermartingale for all } H \in \mathcal{A}\}. \tag{3.1}$$

3.2. *The main theorem.*

THEOREM 3.9. (*NFLVR-S*) $\Leftrightarrow \mathcal{M}_{\sup}(S) \neq \varnothing$.

In order to prove this theorem we need the following lemma.

LEMMA 3.10. $\{(H \cdot S) : H \in \mathcal{A}, (H \cdot S) \ge -1\}$ *is a closed subset of the space of vector valued P-semimartingales on $[0,T]$ with the semimartingale topology given by the quasinorm*

$$D(X) = \sup\{E^P[1 \wedge |(H \cdot X)_T|]\colon H \text{ predictable and } |H| \le 1\}. \tag{3.2}$$

PROOF. Since $\{\overrightarrow{x} \in \mathbb{R}^N : x^i \ge 0 \text{ for } i > d\}$ is a closed convex polyhedral cone in $\mathbb{R}^N$, this result follows from the considerations made in [8]. □

REMARK 3.11. Notice that for the conclusion of Lemma 3.10 to hold, it is important to work with short sales constraints as explained in Definition 2.1. In order to consider general convex cone constraints an alternative approach is to consider constrained portfolios modulus those strategies with zero value. This is the approach taken in [26]. In our particular case, and as it is pointed out in [8], we have the advantage of considering portfolio constraints defined pointwise for (ω, t) in $\Omega \times [0, T]$. Given a particular strategy, it is easier to verify admissibility when pointwise restrictions are considered.

PROOF OF THEOREM 3.9. If K_1 and K_2 are nonnegative bounded predictable processes, $K_1K_2 = 0$, $H_1, H_2 \in \mathcal{A}$ are such that $(H_1 \cdot S), (H_2 \cdot S) \ge -1$, and $X := K_1 \cdot (H_1 \cdot S) + K_2 \cdot (H_2 \cdot S) \ge -1$, then associativity of the stochastic integral implies that $X \in \{(H \cdot S) : H \in \mathcal{A}, (H \cdot S) \ge -1\}$. This fact, Proposition 3.2, Lemma 3.10 and Theorem 1.2 in [24] imply that (NFLVR-S) is equivalent to existence of a measure $Q \in \mathcal{M}_{\sup}(S)$. □

REMARK 3.12. By using the results obtained by Kabanov in [24], Karatzas and Kardaras in [26] proved that the condition of (*NFLVR*), with predictable convex portfolio restrictions, is equivalent to the existence of a measure under which the value processes of admissible strategies are supermartingales. In their work the set of measures on the right-hand side of equation (3.1) is also referred to as the set of equivalent supermartingale measures. As mentioned in Remark 3.11, they considered convex portfolio constraints modulus strategies with zero value. We have shown that in the special case of short sales prohibition one can consider pointwise portfolio restrictions. More importantly, we have shown that in the case of short sales prohibition, the set of measures under which the values of admissible portfolios are supermartingales is precisely the set of measures under which the prices of the assets that cannot be sold short are supermartingales, and the prices of assets that can be admissibly sold short are local martingales; see Corollary 3.8. This provides a more precise characterization of the set of risk neutral measures under short sales prohibition. Given a particular model, this characterization simplifies the process of verifying that the model is consistent with the condition of (NFLVR-S).

This section demonstrates that the results obtained by Jouini and Kallal in [23], Schürger in [34], Frittelli in [18], Pham and Touzi in [30] and Napp in [29], can be extended to a more general class of models, similar to the ones used by Delbaen and Schachermayer in [9]. It is also clear from this characterization that the prices of the risky assets that cannot be sold short could be above their risk-neutral expectations at maturity time, because the condition of (NFLVR-S) only guarantees the existence of an equivalent supermartingale measure for those prices.

4. The hedging problem and maximal claims. In this section we seek to understand the scope of the effects of short sales prohibition on the hedging problem of arbitrary contingent claims. We study, in financial markets with short sales prohibitions where prices are driven by nonnegative locally bounded semimartingales, the space of contingent claims that can be super-replicated and perfectly replicated. The duality type results presented in this section are robust because they characterize the claims that can be perfectly replicated or super-replicated in markets with prohibition on short-selling without relying on particular assumptions on the dynamics of the asset prices, other than the locally bounded semimartingale property. By using the results of Föllmer and Kramkov in [16] we extend the classic results of Ansel and Stricker in [1]. The results presented also extend those in Chapter 5 of [27] and Chapter 9 of [17] to more general semimartingale financial markets. Additionally, we establish, in our context, a connection to the concept of maximal claims as it was first introduced by Delbaen and Schachermayer in [9] and [10]. The (FTAP) (Theorem 3.9) can be generalized to the case of special convex cone portfolio constraints (see Theorem 4.4 in [26]), and some of the results presented in this section could be extended to this framework. In our study, we specialize to short sales prohibition because in this case the examples are simplified by the fact that the set of risk neutral measures is characterized by the behavior of the underlying price processes, rather than the behavior of the value processes of the trading strategies; see Remark 3.12. Additionally, in this case, the portfolio restrictions can be considered pointwise in $\Omega \times [0, T]$; see Remarks 3.11 and 3.12. A related study on the implications of short sales prohibitions on hedging strategies involving futures contracts can be found in [22]. We will use the same notation as described in Section 2. We will denote by $\mathcal{M}_{\text{loc}}(S)$ the set of measures equivalent to P under which the components of S are local martingales.

4.1. *The hedging problem.* This section shows how the results obtained by Föllmer and Kramkov in [16] extend the usual characterization of attainable claims and claims that can be super-replicated to markets with short sales prohibition. These results extend those presented in Chapter 5 of [27] and Chapter 9 of [17] to more general semimartingale financial models. We will assume that the condition of (NFLVR-S) (see Theorem 3.9) holds. Recent works (see, e.g., [31] and [33]) have shown that in order to find suitable trading strategies the condition of (NFLVR-S) can be weakened and the hedging problem can be studied in markets that admit certain types of arbitrage.

4.1.1. *Super-replication.* Regarding the super-replication of contingent claims in markets with short sales prohibition we have the following theorem.

THEOREM 4.1. *Suppose $\mathcal{M}_{\sup}(S) \neq \varnothing$. A nonnegative random variable f measurable with respect to $\mathcal{F}_T$ can be written as*

$$f = x + (H \cdot S)_T - C_T \tag{4.1}$$

with x constant, $H \in \mathcal{A}$ and $C \geq 0$ an adapted and nondecreasing càdlàg process with $C_0 = 0$ if and only if

$$\sup_{Q \in \mathcal{M}_{\sup}(S)} E^Q[f] < \infty.$$

In this case, $x = \sup_{Q \in \mathcal{M}_{\sup}(S)} E^Q[f]$ is the minimum amount of initial capital for which there exist $H \in \mathcal{A}$ and $C \geq 0$ an adapted and nondecreasing càdlàg process with $C_0 = 0$ such that (4.1) holds.

PROOF. This follows directly from Corollary 3.8 in this paper and Examples 2.2, 4.1 and Proposition 4.1 in [16]. □

Before we give an analogous result regarding perfect replication of contingent claims, we present an example of a contingent claim that cannot be super-replicated under short sales prohibition.

EXAMPLE 4.2. This example illustrates how, under certain market hypotheses, it is possible to explicitly exhibit a payoff that cannot be super-replicated without short selling. Suppose that S is of the form $S = \mathcal{E}(R)$. Suppose that R is a continuous P-martingale such that $R_0 = 0$ and there exist $\varepsilon, C > 0$ such that $P(\varepsilon \leq [R, R]_T \leq C) = 1$. Let $f = \exp(-R_T)$. We have, by Novikov's criterion (see Theorem III-45 in [32]) and by Girsanov's theorem (see Theorem III-40 in [32]), that for every $\alpha > 0$, $\frac{dQ^\alpha}{dP} = \mathcal{E}(-\alpha R)_T$ defines a measure $Q^\alpha \in \mathcal{M}_{\text{sup}}(S)$. Additionally,

$$
\begin{aligned}
E^{Q^\alpha}[f] &= E^P[\mathcal{E}(-\alpha R)_T f] \\
&= E^P\big[\mathcal{E}\big(-(1+\alpha)R\big)_T \exp\big((1/2+\alpha)[R,R]_T\big)\big] \\
&\geq E^P\big[\mathcal{E}\big(-(1+\alpha)R\big)_T\big]\exp\big((1/2+\alpha)\varepsilon\big) \\
&= \exp\big((1/2+\alpha)\varepsilon\big) \to \infty
\end{aligned}
$$

as α goes to infinity. Hence $\sup_{Q\in\mathcal{M}_{\text{sup}}(S)} E^Q[f] = \infty$, and Theorem 4.1 implies that f cannot be super-replicated without selling S short. However, if we assume that the market where S can be sold short is complete under P, that is, $\mathcal{M}_{\text{loc}}(S) = \{P\}$, then in the market where S can be sold short f can be replicated because it belongs to $L^1(P)$. Indeed, we have that

$$
0 \leq f \leq \exp\left(\frac{C}{2}\right)\mathcal{E}(-R)_T \in L^1(P).
$$

4.1.2. *Replication*. A question that remains open, however, is whether there exists a characterization of contingent claims that can be perfectly replicated. In this regard we have the following result analogous to the one proven by Ansel and Stricker in [1]; see also Theorems 5.8.1 and 5.8.4 in [27].

THEOREM 4.3. *Suppose $\mathcal{M}_{\text{sup}}(S) \neq \varnothing$. For a nonnegative random variable f measurable with respect to $\mathcal{F}_T$ the following statements are equivalent*:

(i) $f = x + (H\cdot S)_T$ *with x constant and $H \in \mathcal{A}$ such that $(H\cdot S)$ is an R^*-martingale for some $R^* \in \mathcal{M}_{\text{sup}}(S)$.*

(ii) *There exists $R^* \in \mathcal{M}_{\text{sup}}(S)$ such that*

$$
\sup_{Q\in\mathcal{M}_{\text{sup}}(S)} E^Q[f] = E^{R^*}[f] < \infty. \tag{4.2}
$$

PROOF. That (i) implies (ii) follows from the fact that $(H\cdot S)$ is a Q-supermartingale starting at 0 for all $Q \in \mathcal{M}_{\text{sup}}(S)$; see Corollary 3.8. To prove that (ii) implies (i) we define for all t in $[0, T]$

$$
V_t := \operatorname*{ess\,sup}_{Q\in\mathcal{M}_{\text{sup}}(S)} E^Q[f|\mathcal{F}_t]. \tag{4.3}
$$

By Lemma A.1 in [16] the process V is a supermartingale under any $Q \in \mathcal{M}_{\sup}(S)$. In particular V is an R^*-supermartingale. The fact that $V_T = f$ and (4.2) imply that $V_0 = E^{R^*}[V_T]$ and V is a martingale under R^*. On the other hand by Theorem 3.1 in [16], $V = V_0 + (H \cdot S) - C$ for some $H \in \mathcal{A}$ and $C \geq 0$ nondecreasing. Since $(H \cdot S)$ is an R^*-supermartingale (see Corollary 3.8) we conclude that

$$E^{R^*}[C_T] = V_0 + E^{R^*}[(H \cdot S)_T] - E^{R^*}[V_T] \leq 0.$$

Then, $C \equiv 0$ R^*-almost surely and $(H \cdot S)$ is an R^*-martingale. □

V_t in (4.3) is usually used to define the selling price of the claim f at time t. It represents the minimum cost of super-replication of the claim f at time t; see Proposition 4.1 in [16]. We now give an example of a payoff in markets with continuous price processes which cannot be attained with "martingale strategies."

EXAMPLE 4.4. Suppose that the market consists of a single risky asset with continuous price process S. Assume further that S is a P-martingale which is not constant P-almost surely. Then $f = 1_{\{S_T \leq S_0\}}$ does not belong to the space

$$\begin{aligned} \mathcal{G} := \{ & x + (H \cdot S)_T : x \in \mathbb{R}, H \in \mathcal{A}, \\ & (H \cdot S) \text{ is a } Q\text{-martingale for some } Q \in \mathcal{M}_{\sup}(S) \}. \end{aligned} \tag{4.4}$$

Indeed, for each $n \in \mathbb{N}$, let $(T_{n,m})_m$ be a localizing sequence for

$$\mathcal{E}(-n(S_t - S_0)).$$

Define $Q_{n,m} \in \mathcal{M}_{\sup}(S)$ by

$$\frac{dQ_{n,m}}{dP} = \mathcal{E}(-n(S_{T \wedge T_{n,m}} - S_0)).$$

We have that

$$\begin{aligned} E^{Q_{n,m}}[f] &= 1 - E^{Q_{n,m}}[1 - f] \\ &= 1 - E^P\left[1_{\{S_T > S_0\}} \exp\left(-n(S_{T \wedge T_{n,m}} - S_0) - \frac{n^2}{2}[S, S]_{T \wedge T_{n,m}}\right)\right]. \end{aligned}$$

Since the expression under the last expectation is dominated by $\exp(nS_0) \in \mathbb{R}$, the Dominated Convergence theorem implies that for fixed n

$$\lim_m E^{Q_{n,m}}[f] = 1 - E^P\left[1_{\{S_T > S_0\}} \exp\left(-n(S_T - S_0) - \frac{n^2}{2}[S, S]_T\right)\right].$$

Applying the dominated convergence theorem once again we obtain that

$$\lim_n \lim_m E^{Q_{n,m}}[f] = 1.$$

This allows us to conclude that

$$\sup_{Q \in \mathcal{M}_{\sup}(S)} E^Q[f] = 1.$$

However, since f is not P-almost surely constant, this supremum is never attained. By Theorem 4.3, f does not belong to the set $\mathcal{G}$ defined in (4.4).

REMARK 4.5. Example 4.4 illustrates that in nontrivial markets with continuous price processes, the minimum super-replicating cost of a digital option of the form $1_{\{S_T \le S_0\}}$ is 1; See Theorem 4.1. We will give other examples of claims that cannot be perfectly replicated with martingale strategies at the end of this section.

We now proceed to give an alternative characterization of the random variables in $\mathcal{G}$, with $\mathcal{G}$ as in (4.4), by extending the concept of maximal claims introduced by Delbaen and Schachermayer in [9] and [10].

4.2. *Maximal claims*. By using the extension of the (FTAP) proved in Section 3, this section generalizes the ideas presented in [10] to markets with short sales prohibition. For simplicity, we assume below that S, the price process of the underlying asset, is one-dimensional. The results can be easily extended to the multi-dimensional case. Recall the definitions of no arbitrage under short sales prohibition (NA-S) and no free lunch with vanishing risk under short sales prohibition (NFLVR-S) given in Section 2.

4.2.1. *The main theorem*.

DEFINITION 4.6. Let $\mathcal{J} \subset L^0(P)$. We say that an element f is maximal in $\mathcal{J}$ if:

(i) $f \in \mathcal{J}$ and
(ii) $f \le g$ P-almost surely and $g \in \mathcal{J}$ imply that $f = g$ P-almost surely.

DEFINITION 4.7. Given $H \in \mathcal{A}$, we define $\mathcal{B}(H)$ as the set of random variables of the form

$$((H^1, H^2) \cdot (S^1, S^2))_T,$$

where $S^1 = (H \cdot S)$, $S^2 = S$; $(H^1, H^2) \in L(S^1, S^2)$; $H^2 \ge 0$, $H_0^1 \equiv 1$, $H_0^2 \equiv 0$ and

$$(H^1 - 1, H^2) \cdot (S^1, S^2) \ge -\beta - \alpha S^1 \tag{4.5}$$

for some $\alpha, \beta > 0$.

The following is the main theorem of this section.

THEOREM 4.8. *Let $f \in L^0(P)$ be a random variable bounded from below. The following statements are equivalent*:

(i) $f = (H \cdot S)_T$ *for some* $H \in \mathcal{A}$ *such that*:
  (a) *the market where* $S^1 = (H \cdot S)$ *and* $S^2 = S$ *trade with short selling prohibition on* S^2 *satisfies* (*NFLVR-S*) *and*
  (b) f *is maximal in* $\mathcal{B}(H)$ (*see Definition* 4.7).

(ii) *There exists* $R^* \in \mathcal{M}_{\sup}(S)$ *such that*

$$\sup_{Q \in \mathcal{M}_{\sup}(S)} E^Q[f] = E^{R^*}[f] = 0.$$

(iii) *There exists* $H \in \mathcal{A}$ *such that* $f = (H \cdot S)_T$ *and* $(H \cdot S)$ *is an* R^**-martingale for some* R^* *in* $\mathcal{M}_{\sup}(S)$.

If we further assume that f *is bounded and* $\mathcal{M}_{\mathrm{loc}}(S) \neq \varnothing$, *the above statements are equivalent to*:

(iv) *There exists* $H \in \mathcal{A}$ *such that* $f = (H \cdot S)_T$ *for some* $H \in \mathcal{A}$ *and* $(H \cdot S)$ *is an* R*-martingale for all* R *in* $\mathcal{M}_{\mathrm{loc}}(S)$.

REMARK 4.9. It is important to point out that we can take the same measure R^* in (ii) and (iii), and the same strategy H in (i), (iii) and (iv).

Before establishing some lemmas necessary to prove this theorem we make some additional remarks.

REMARK 4.10. A related result for diffusion price processes can be found in Theorem 5.8.4 in [27]. This theorem uses the alternative assumption that

$$\{(H \cdot S)_\rho \colon \rho \text{ is a stopping time in } [0, T]\}$$

is Q-uniformly integrable for all $Q \in \mathcal{M}_{\sup}(S)$. This hypothesis also implies that $(H \cdot S)$ is a Q-martingale for all $Q \in \mathcal{M}_{\mathrm{loc}}(S)$.

REMARK 4.11. Condition (4.5) resembles the definition of workable contingent claims studied in [11].

REMARK 4.12. If $f = (H \cdot S)_T$, $(H \cdot S)$ is an R^*-martingale for some $R^* \in \mathcal{M}_{\sup}(S)$ and $1_{\{H=0\}} \cdot S$ is indistinguishable from 0, then $R^* \in \mathcal{M}_{\mathrm{loc}}(S) \neq \varnothing$. Indeed, observe that if we call $M = (H \cdot S)$, then, by Corollary 3.5 in [1], $(\frac{1}{H} 1_{\{H \neq 0\}}) \cdot M = 1_{\{H \neq 0\}} \cdot S = S - S_0$ is an R^*-local martingale. Theorem 11.4.4 in [14] implies that the claim f is also maximal in

$$\tilde{\mathcal{K}} = \{(H \cdot S)_T : H \in \tilde{A}\}, \tag{4.6}$$

where $\tilde{A}$ is the set of strategies that satisfy (i), (ii) and (iii) in Definition 2.1. Additionally, also by Theorem 11.4.4 in [14], Theorem 4.8 shows that when $\mathcal{M}_{\mathrm{loc}}(S) \neq \varnothing$, all bounded maximal claims in $\mathcal{B}(H)$ (see Definition 4.7) of the form $(H \cdot S)_T$ for some $H \in \mathcal{A}$ are maximal in $\tilde{\mathcal{K}}$ as defined in (4.6).

The proof of Theorem 4.8 that we present below mimics the argument presented in [10]. In this generalization, the (FTAP) under short sales prohibition (Theorem 3.9) and the results presented by Kabanov in [24] are fundamental.

4.2.2. *Some lemmas*. We first recall the following definition.

DEFINITION 4.13. A subset $\mathcal{N}$ of $L^0(P)$ is bounded in $L^0(P)$ if for all $\varepsilon > 0$ there exists $M > 0$ such that $P(|Y| > M) < \varepsilon$ for all $Y \in \mathcal{N}$.

The following lemmas will be used.

LEMMA 4.14. *The condition of* (*NFLVR-S*) *holds if and only if* (*NA-S*) *holds, and the set*

$$\mathcal{K}_1 = \{(H \cdot S)_T \colon H \in \mathcal{K} \text{ and } (H \cdot S) \geq -1\}$$

is bounded in $L^0(P)$.

PROOF. This corresponds to Lemma 2.2 in [24]. As already noticed before in the proof of Theorem 3.9, the results in [24] can be applied to our case because the convex portfolio constraints satisfy the desired hypotheses. □

LEMMA 4.15. *The condition of* (*NFLVR-S*) *holds if and only if* (*NA-S*) *holds and there exists a strictly positive* P-*local martingale* $L = (L_t)_{0 \leq t \leq T}$ *such that* $L_0 = 1$ *and* $P \in \mathcal{M}_{\mathrm{sup}}(LS)$.

PROOF. The same proof of Theorem 11.2.9 in [14] can be applied to our context. □

We now state from our framework a result that is analogous to Theorem 11.4.2 in [14]. This theorem gives necessary and sufficient conditions under which the condition of (NA-S) holds after a change of numéraire. We will need the following lemma, that proves that the self-financing condition [see (2.1)] is independent of the choice of numéraire; see also [21].

LEMMA 4.16. *Let* V *be a positive* P-*semimartingale*, $M = (\frac{S}{V}, \frac{1}{V}, 1)$ *and* $N = (S, 1, V)$. *For a* (*three-dimensional*) *predictable process* H *the following statements are equivalent*:

(i) $H \in L(M)$ *and*

$$H \cdot M = HM - H_0 M_0 = H^1 \frac{S}{V} + H^2 \frac{1}{V} + H^3 - H_0^1 \frac{S_0}{V_0} - H_0^2 \frac{1}{V_0} - H_0^3;$$

(ii) $H \in L(N)$ *and*

$$H \cdot N = HN - H_0 N_0 = H^1 S + H^2 + H^3 V - H_0^1 S_0 - H_0^2 - H_0^3 V_0.$$

PROOF. ($\Rightarrow$) Let $W = H \cdot M$. By (i), $\Delta W = H \Delta M = HM - HM_-$ and $W_- = W - \Delta W = HM_- - H_0 M_0$. The integration by parts formula implies that

$$\begin{aligned} d(VW) &= W_- \, dV + V_- \, dW + d[W, V] \\ &= (HM_- - H_0 M_0) \, dV + V_- H \, dM + d[W, V]. \end{aligned}$$

Since $d[W, V] = H \, d[M, V]$ regrouping terms and using integration by parts once more we obtain that

$$\begin{aligned} d(VW) &= H\big(M_- \, dV + V_- \, dM + d[M, V]\big) - H_0 M_0 \, dV \\ &= H \, d(VM) - H_0 M_0 \, dV. \end{aligned}$$

We have that $VM = N$, and hence $d(VW) = H \, dN - H_0 M_0 \, dV$. By (i), $VW = HN - V H_0 M_0$ and

$$\begin{aligned} H \, dN &= d(VW) + H_0 M_0 \, dV \\ &= \big(d(HN) - H_0 M_0 \, dV\big) + H_0 M_0 \, dV \\ &= d(HN) \end{aligned}$$

as we wanted to show.

($\Leftarrow$) The proof of this direction is analogous to the one just presented since M is obtained after multiplying N by the nonnegative semimartingale $\frac{1}{V}$. $\square$

LEMMA 4.17. *Suppose that V is a strictly positive P-semimartingale. The market with multi-dimensional price process $(\frac{1}{V}, \frac{S}{V})$, where short selling prohibition is imposed on $\frac{S}{V}$, satisfies the condition of* (NA-S) *if and only if $V_T - V_0$ is maximal in $\mathcal{D}$, where $\mathcal{D}$ is the set of random variables of the form $(H \cdot (S, V))_T$ where $H^1 \geq 0$, $H_0^1 \equiv 0$, $H_0^2 \equiv 1$ and*

$$\big(H^1, H^2 - 1\big) \cdot (S, V) \geq -\alpha V \qquad \textit{for some } \alpha > 0.$$

PROOF. ($\Leftarrow$) Let $M = (\frac{1}{V}, \frac{S}{V})$ and $N = (S, V)$. Suppose that $H = (H^1, H^2)$ is an arbitrage in the market with multi-dimensional price process $(\frac{1}{V}, \frac{S}{V})$. In other words, assume that $H^2 \geq 0$, $H_0 \equiv 0$, $(H \cdot M)_T \geq 0$, $P((H \cdot M)_T > 0) > 0$ and $H \cdot M \geq -\alpha$ for some $\alpha > 0$. If we define

$$\begin{aligned} H^3 &= 1 + H \cdot M - HM, \\ \tilde{M} &= \left(\frac{1}{V}, \frac{S}{V}, 1\right), \\ \tilde{N} &= (1, S, V) \end{aligned}$$

and

$$\tilde{H} = \big(H^1, H^2, H^3\big),$$

we have that $\tilde{H} \cdot \tilde{M} = \tilde{H}\tilde{M} - 1$. By Lemma 4.16 we have that

$$\tilde{H} \cdot \tilde{N} = \tilde{H}\tilde{N} - V_0.$$

But observe that

$$\tilde{H}\tilde{N} = VHM + (1 + H \cdot M - HM)V = (1 + H \cdot M)V$$

and

$$\tilde{H} \cdot \tilde{N} = K \cdot N,$$

where $K = (H^2, H^3)$. Hence $(K \cdot N)_T$ is an element of $\mathcal{D}$ such that $(K \cdot N)_T \geq V_T - V_0$ P-almost surely and $P((K \cdot N)_T > V_T - V_0) > 0$, whence $V_T - V_0$ is not maximal in $\mathcal{D}$.

($\Rightarrow$) Conversely, suppose that $V_T - V_0$ is not maximal in $\mathcal{D}$. With the notation used above, let $K = (K^1, K^2)$ be a strategy such that $(K \cdot N)_T \geq V_T - V_0$ P-almost surely and $P((K \cdot N)_T > V_T - V_0) > 0$, with $K^1 \geq 0$, $K_0^1 \equiv 0$, $K_0^2 \equiv 1$ and $(K^1, K^2 - 1) \cdot N \geq -\alpha V$ for some $\alpha > 0$. Define $H^2 = K^1$, $H^3 = K^2 - 1$, $H^1 = (H^2, H^3) \cdot N - (H^2, H^3)N$ and $H = (H^1, H^2, H^3)$. We have that $H \cdot \tilde{N} = H\tilde{N} - H_0\tilde{N}_0$. By Lemma 4.16 we have that

$$H \cdot \tilde{M} = H\tilde{M} - H_0\tilde{M}_0 = H\tilde{M}.$$

Hence,

$$\big(H^1, H^2\big) \cdot M = H\tilde{M}.$$

We have that

$$H\tilde{M} = \frac{1}{V}H\tilde{N} = \frac{1}{V}\big((H^2, H^3) \cdot N\big) = \frac{1}{V}\big(K \cdot N - (V - V_0)\big) \geq -\alpha.$$

Therefore,

$$\big((H^1, H^2) \cdot M\big)_T = \frac{1}{V_T}\big((K \cdot N)_T - (V_T - V_0)\big),$$

$((H^1, H^2) \cdot M)_T \in L_+^0(P)$ and $P(((H^1, H^2) \cdot M)_T > 0) > 0$. Since $H_0^1 = H_0^2 = 0$, (H^1, H^2) is an arbitrage strategy in the market with multi-dimensional price process $(\frac{1}{V}, \frac{S}{V})$. $\square$

REMARK 4.18. It is important to observe that the no arbitrage condition (NA-S) over $(\frac{1}{V}, \frac{S}{V})$ holds for strategies that are nonnegative on the second component, but can be negative in an admissible way [see condition (iii) in Definition 2.1] over the first component. Lemma 4.17 gives a necessary and sufficient condition under which the introduction of V as a numéraire does not introduce arbitrage in a market with short sales prohibition. A related discussion on numéraires over convex sets of random variables can be found in [28].

These lemmas allow us to prove Theorem 4.8.

4.2.3. *Proof of the main theorem.*

PROOF OF THEOREM 4.8. Since f is bounded from below there exists a constant x such that $\tilde{f} := f + x$ is nonnegative. Theorem 4.3, applied to $\tilde{f}$, proves the equivalence between (ii) and (iii). We will prove now that (iii) implies (i). The (FTAP) (Theorem 3.9) shows that (NFLVR-S) holds for the market consisting of S and $(H \cdot S)$ with short selling prohibition on S. Now assume that $f \le ((H^1, H^2) \cdot (S^1, S^2))_T$ with $((H^1, H^2) \cdot (S^1, S^2))_T \in \mathcal{B}(H)$. Then

$$(H^1 - 1, H^2) \cdot (S^1, S^2) \ge -\beta - \alpha S^1$$

for some $\alpha, \beta > 0$ and $((H^1 - 1, H^2) \cdot (S^1, S^2))_T \ge 0$. Since

$$(H^1 - 1 + \alpha, H^2) \cdot (S^1, S^2) \ge -\beta$$

by Lemma 3.6 (extended to the case when the integrand is not identically 0 at time 0) we conclude that

$$(H^1 - 1 + \alpha, H^2) \cdot (S^1, S^2)$$

is an R^*-supermartingale, which in turn implies that $((H^1 - 1, H^2) \cdot (S^1, S^2))$ is an R^*-supermartingale starting at 0. Since $((H^1 - 1, H^2) \cdot (S^1, S^2))_T \ge 0$, we conclude that $((H^1 - 1, H^2) \cdot (S^1, S^2))_T = 0$ P-almost surely. This shows that f is maximal in $\mathcal{B}(H)$.

Let us prove now that (i) implies (iii). By the (FTAP) we know that there exists $\tilde{P} \in \mathcal{M}_{\text{sup}}(S)$ such that $(H \cdot S)$ is a $\tilde{P}$-local martingale. Let a be such that $V := a + (H \cdot S)$ is positive and bounded away from 0. Since f is maximal in $\mathcal{B}(H)$, $V_T - V_0$ is maximal in $\mathcal{D}$, where $\mathcal{D}$ is as in Lemma 4.17. By Lemma 4.17 (NA-S) holds in the market where $\frac{S}{V}$ and $\frac{1}{V}$ trade with short selling prohibition on $\frac{S}{V}$. By Lemma 4.15 we conclude that (NFLVR-S) holds in this market with respect to the measure $\tilde{P}$. Hence, by the (FTAP) there exists $\tilde{Q} \sim \tilde{P}$ (and hence $\tilde{Q} \sim P$) such that $\frac{S}{V}$ is a $\tilde{Q}$-supermartingale, and $\frac{1}{V}$ is a bounded $\tilde{Q}$-local martingale and therefore a $\tilde{Q}$-martingale. By defining R^* by $V_T \, dR^* = (E^{\tilde{Q}}[\frac{1}{V_T}])^{-1} \, d\tilde{Q}$, we observe that $R^* \in \mathcal{M}_{\text{sup}}(S)$ and V is an R^*-martingale. This implies that $(H \cdot S)$ is an R^*-martingale as well.

Finally to prove that (iii) implies (iv) we observe that if $R \in \mathcal{M}_{\text{loc}}(S)$ and (τ_n) is an R-localizing sequence for $(H \cdot S)$ then $(H \cdot S)_{\tau_n \wedge T} = E^{R^*}[f | \mathcal{F}_{\tau_n \wedge T}]$ is a dominated sequence of random variables with zero R-expectation. By the dominated convergence theorem we conclude that $E^R[f] = 0$, and $(H \cdot S)$ is an R-martingale (it is an R-supermartingale with constant expectation). □

4.3. *Final remarks.*

REMARK 4.19. Condition (i) in Theorem 4.8 can be interpreted as follows. The market where S^1 and S^2 trade with short sales prohibition on S^2 satisfies the no arbitrage paradigm of (NFLVR-S). In this market the strategy of buying and holding S^1 cannot be dominated by any strategy with initial holdings of one share of S^1 and none of S^2 that does not sell S^2 short.

The following observation is important. It shows that the elements $f \in L^0(P)$ that satisfy any of the conditions of Theorem 4.8 are maximal in $\mathcal{K}$. A related result was discussed in Remark 4.12, where it was shown that, under stronger assumptions on the replicating strategy for f, a stronger form of maximality holds, namely maximality in $\tilde{\mathcal{K}}$; see (4.6).

PROPOSITION 4.20. *If* (i), (ii) *or* (iii) *in Theorem* 4.8 *holds*, *then* f *is maximal in* $\mathcal{K}$.

PROOF. Assume that $E^{R^*}[f] = 0$ for some $R^* \in \mathcal{M}_{\text{sup}}(S)$. If $f \leq (K \cdot S)_T$ with $K \in \mathcal{A}$, by Lemma 3.6, we conclude that $E^{R^*}[(K \cdot S)_T] = 0$. This implies that $f = (K \cdot S)_T$ P-almost surely and f is maximal in $\mathcal{K}$. □

As shown in [13], without the assumption that f is bounded, (iv) of Theorem 4.8 is not a necessary condition. Theorem 4.8 is useful to argue why certain types of contingent claims in certain financial models cannot be replicated by using a strategy that is maximal in the sense of (i) of Theorem 4.8 above.

EXAMPLE 4.21. Let $K \in (0, \infty)$ be fixed. Assume that S is a continuous P-martingale, $[S, S]_T$ is deterministic and $P(S_T < K, \tau < T) > 0$ where

$$\tau = \inf\{t \leq T : S_t \geq K + \tfrac{1}{2}([S, S]_T - [S, S]_t)\} \wedge T.$$

By Novikov's criterion (Theorem III-45 in [32]) and by Girsanov's theorem (Theorem III-40 in [32]) we know that

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^T 1_{[\tau,T]}(s)\, dS_s\right)$$

defines a probability measure $Q \in \mathcal{M}_{\text{sup}}(S)$. If $g : [0, \infty) \to [0, \infty)$ is a function that vanishes on $[K, \infty)$ and is strictly positive on $[0, K)$, then

$$\begin{aligned} E^Q[g(S_T)] &= E^Q[g(S_T)1_{\{S_T<K\}}] \\ &\geq E^P[1_{\{\tau=T\}}g(S_T)1_{\{S_T<K\}}] \\ &\quad + E^P[1_{\{S_T<K,\tau<T\}}g(S_T)\exp(-(S_T-K))] \\ &> E^P[g(S_T)]. \end{aligned} \tag{4.7}$$

If we further assume that g is bounded, then by Theorem 4.8 [condition (iv)] we conclude that $g(S_T)$ does not belong to $\mathcal{G}$ as in (4.4). Indeed, if $g(S_T) = x + (H \cdot S)_T$, with $x \in \mathbb{R}$, $H \in \mathcal{A}$ and $(H \cdot S)$ an R^*-martingale for some

$R^* \in \mathcal{M}_{\sup}(S)$, then by Theorem 4.8, $(H \cdot S)$ would be an R-martingale for all $R \in \mathcal{M}_{\mathrm{loc}}(S)$. In particular, we would have that $E^P[g(S_T)] = x = E^Q[g(S_T)]$, which contradicts (4.7). The function $g(x) = (K - x)_+$ satisfies the above mentioned conditions. Hence under these assumptions, the put option's payoff does not belong to $\mathcal{G}$. This example is similar to Example 7.2 of [6].

REMARK 4.22. In Example 5.7.4 in [27] and Section 8.1 in [7], it is proven that for diffusion models with constant coefficients and stochastic volatility models with additional properties, respectively, the minimum super-replication price of an European put option, $\sup_{Q\in\mathcal{M}_{\sup}(S)} E^Q[(K - S_T)_+]$, is equal to K. In particular if $P(S_T \neq 0) > 0$, then this supremum is never attained and $(K - S_T)_+$ is not in $\mathcal{G}$ as defined by (4.4).

In this section we have studied the space of contingent claims that can be super-replicated and perfectly replicated with martingale strategies in a market with short sales prohibition. We extended results found in [1, 27] and [17] to the short sales prohibition case. We additionally have extended the results in [10] to our framework and modified the concept of maximality accordingly (see Theorem 4.8). Additionally, we presented explicit payoffs in general markets that cannot be replicated without selling the spot price process short.

5. Open questions. It is still unclear whether (NFLVR) for a market without short sales prohibition, implies that all claims that are maximal in the sense of (i) in Theorem 4.8 are maximal in $\tilde{\mathcal{K}}$; see (4.6). Equivalently, it is unclear whether for a claim f that is bounded from below, the conditions $\mathcal{M}_{\mathrm{loc}}(S) \neq \varnothing$ and

$$\sup_{Q\in\mathcal{M}_{\sup}(S)} E^Q[f] = E^{R^*}[f]$$

for some $R^* \in \mathcal{M}_{\sup}(S)$, imply that there exists $P^* \in \mathcal{M}_{\mathrm{loc}}(S)$ such that $E^{P^*}[f] = E^{R^*}[f]$. Also, it would be interesting to obtain a characterization of the set of claims that are maximal in $\mathcal{K}$ [as in (2.2)] and explore whether maximality in $\mathcal{K}$ implies maximality in $\tilde{\mathcal{K}}$; see (4.6).

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