A CONTROL PROBLEM WITH FUEL CONSTRAINT AND DAWSON–WATANABE SUPERPROCESSES

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We solve a class of control problems with fuel constraint by means of the log-Laplace transforms of J-functionals of Dawson–Watanabe superprocesses. This solution is related to the superprocess solution of quasilinear parabolic PDEs with singular terminal condition. For the probabilistic verification proof, we develop sharp bounds on the blow-up behavior of log-Laplace functionals of J-functionals, which might be of independent interest.

1. Introduction. One of the most exciting aspects of Dawson–Watanabe superprocesses is their connection to quasilinear partial differential equations (PDEs) with singular boundary condition. This connection was pioneered by Dynkin (1991a, 1992); see also Dynkin (2004) for more recent developments and related literature. Similar quasilinear PDEs also appear in the Hamilton–Jacobi–Bellman (HJB) formulation of stochastic control problems with terminal state constraint, and so it is natural to ask whether these control problems possess solutions in terms of superprocesses. Establishing such a direct connection is the main goal of this paper.

The connection we find is direct insofar as it avoids the use of HJB equations and instead uses a probabilistic verification argument based solely on the log-Laplace equation for a certain J-functional of a superprocess. While the standard verification argument relies on the existence of smooth solutions to the HJB equation, whose existence is often very difficult to establish, the "mild solutions" provided by the log-Laplace functionals of superprocesses are ideally suited for carrying out the verification argument. They are also superior to the commonly used viscosity solutions, because the latter do not go well along with Itô calculus due to their possible lack of smoothness.

The problem we will consider here is the minimization of the functional

$$(1.1) \qquad E_{0,z}\left[\int_0^T |\dot{x}(t)|^p \eta(Z_t)\,dt + \int_{[0,T]} |x(t)|^p A(dt)\right]$$

over adapted and absolutely continuous strategies $x(t)$ satisfying the constraints $x(0) = x_0$ and $x(T) = 0$. We assume here that $p \in [2, \infty)$, η is a strictly positive function and A is a nonnegative additive functional of the (time-inhomogeneous) Markov process Z with $Z_0 = z$ $P_{0,z}$-a.s. This control problem is closely related to the monotone follower problems with fuel constraint that were introduced by Beneš, Shepp and Witsenhausen (1980) and further developed, for example, by Karatzas (1985). Also, as we will explain in Section 1.2, problems of this type have recently appeared in the context of mathematical finance. In the next section we will give some heuristic arguments that explain the connection of this problem with quasilinear PDEs with singular terminal condition that are related to super-processes.

1.1. *The connection between the control problem and superprocesses.* Let us assume that $A(du) = a(Z_u) \, du$ for some function $a \geq 0$ and define the value function of our problem as

$$V(t, z, x_0) := \inf_{x(\cdot)} E_{t,z}\left[\int_t^T |\dot{x}(u)|^p \eta(Z_u) \, du + \int_t^T |x(u)|^p a(Z_u) \, du\right],$$

where the infimum is taken over the class of all absolutely continuous and adapted strategies $x(\cdot)$ satisfying the constraints $x(t) = x_0$ and $x(T) = 0$. As usual $P_{t,z}$ denotes the probability measure under which the Markov process Z starts at z at time t. When L_t is the infinitesimal generator of Z, standard arguments from optimal control suggest that V should satisfy the following HJB equation:

$$V_t(t, z, x_0) + \inf_{\xi}\{\eta(z)|\xi|^p + V_{x_0}(t, z, x_0)\xi\}$$

(1.2)

$$+ a(z)|x_0|^p + L_t V(t, z, x_0) = 0$$

with singular terminal condition

(1.3)
$$V(T, z, x_0) = \begin{cases} 0, & \text{if } x_0 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that the singularity in this terminal condition is required by the fuel constraint $x(T) = 0$.

To see how this PDE is related to superprocesses, we consider the case $x_0 \geq 0$ and make the ansatz $V(t, z, x_0) = x_0^p v(t, z)$ for some function v. Plugging this ansatz into (1.2), minimizing over ξ, dividing by x_0^p and using (1.3) yields the equation

$$v_t - \frac{1}{\beta \eta^\beta} v^{1+\beta} + a + L_t v = 0,$$

(1.4)

$$v(T, z) = +\infty,$$

where $\beta = \frac{1}{p-1}$. This is just the type of PDE solved in Dynkin (1992) by means of superprocesses.

The minimizing ξ in (1.2) is given by $\xi = -x_0 v^\beta / \eta^\beta$, which suggests that the minimizing strategy x^* for the cost functional (1.1) is given in feedback form as the solution of the ordinary differential equation

$$\dot{x}(u) = -\frac{x(u)v(u, Z_u)^\beta}{\eta(Z_u)^\beta},$$

that is,

$$x^*(t) = x_0 \exp\left(-\int_0^t \frac{v(s, Z_s)^\beta}{\eta(Z_s)^\beta} \, ds\right).$$

As we will see later on, these heuristic computations will give the correct results. There are some difficulties, however, which need to be overcome to turn these heuristics into a full proof. For instance, we must make sure that the strategy x^* satisfies the fuel constraint $x^*(T) = 0$, which requires us to find a *lower* bound on $v(s, Z_s)$ when it approaches the singularity so as to ensure that $\int_0^t \frac{v(s,Z_s)^\beta}{\eta(Z_s)^\beta} \, ds$ diverges when $t \uparrow T$. On the other hand, we must also make sure that x^* has finite cost (1.1). To this end, we will need a sharp *upper* bound on $v(s, Z_s)$ for s close to T. These bounds are derived in Section 3 by extending existing bounds from Schied (1996) to J-functionals and to a generalized class of superprocesses with nonhomogeneous branching parameters. These bounds may be of independent interest. For instance, by means of the results in Dynkin (1992) they translate into sharp bounds for solutions $v(t, z)$ of the singular Cauchy problem (1.4) when t approaches the singularity T.

1.2. *Financial motivation of the control problem.* Let (S_t) be a square-integrable martingale, which will be interpreted as the price process of an asset. In the linear Almgren–Chriss market impact model, a strategy $x(\cdot)$ as described above is interpreted as the policy of an investor who wishes to liquidate x_0 shares of the asset throughout the time interval $[0, T]$. This liquidation creates price impact so that the investor trades at price $S_t^x = S_t + \gamma(x_t - x_0) + \eta_t \dot{x}_t$, where γ is a constant and the process η describes the intraday liquidity fluctuations; see Almgren and Chriss (2000), Almgren (2012) and the survey Gatheral and Schied (2013) for details. The liquidation costs arising from the strategy $x(\cdot)$ are then given by

$$C(x) = \frac{\gamma}{2}x_0^2 + \int_0^T \eta_t \dot{x}(t)^2 \, dt - \int_0^T x(t) \, dS_t.$$

The problem considered by practitioners is the minimization of the following mean-variance functional of the costs:

$$(1.5) \qquad\qquad E\big[C(x)\big] + \lambda \operatorname{var}\big(C(x)\big),$$

over all absolutely continuous policies satisfying the constraints $x(0) = x_0$ and $x(T) = 0$. This is a straightforward exercise when η is constant and strategies are

deterministic, but not so easy when strategies are adapted. The reason for this difficulty is the time inconsistency of the mean-variance functional, which precludes the use of control techniques; see, for example, Tse et al. (2011) and the references therein.

As a way out, one can use an infinitesimal re-optimization process as in Section 6.4 of Schöneborn (2008) or other, more generally available arguments such as those in Ekeland and Lazrak (2006) or Björk and Murgoci (2010) to replace the original, time-inconsistent problem by a time-consistent approximation. At least when η is deterministic, this process leads to the problem of minimizing the functional

$$(1.6) \qquad E\left[\int_0^T \eta_t \dot{x}(t)^2\, dt + \lambda \int_{[0,T]} x(t)^2\, d[S]_t\right];$$

see also Almgren (2012), Forsyth et al. (2012) and Tse et al. (2011) for other motivations of this problem and further studies with applications in mind. In particular, Almgren (2012) proposes to study the cost functional (1.6) also for nondeterministic η. When S_t and η_t are functions of an underlying Markov process Z_t, as it is the case for almost all probabilistic models of asset price processes, we see that this problem is precisely of the form (1.1). It is now also clear that using a general additive functional A and not just a functional of the form $A(dt) = a(Z_t)\, dt$ in the formulation of (1.1) is suggested by applications and not just an artificial generalization of our problem.

1.3. *Plan of the paper.* In Section 2.1 we introduce the basic setup of the paper. In Section 2.2 we first state the solution of problem (1.1) for $\eta \equiv 1$. Theorem 2.3, the corresponding result, is actually a special case of our main result, Theorem 2.7, but unlike Theorem 2.7 it only involves classical superprocesses, as constructed in Dynkin (1991b), while the case of nonconstant η requires an extended class of superprocesses. In Theorem 2.8 we consider a variant of problem (1.1) in which the fuel constraint $x(T) = 0$ is relaxed and replaced by a penalty term.

In Section 3 we collect some auxiliary results on superprocesses and their Laplace functionals, some of which may be of independent interest. In Section 3.1 we present a probabilistic version of the parabolic maximum principle for the log-Laplace equations associated with J-functionals. This result will be needed for comparing log-Laplace equations for superprocesses with inhomogeneous branching characteristics to the case with homogeneous branching. In Section 3.2 we derive estimates for the Laplace functionals of J-functionals of superprocesses with homogeneous branching characteristics by extending bounds obtained in Schied (1996) to the case of J-functionals. In Section 3.3, these bounds are then extended as well to a generalized class of superprocesses with nonhomogeneous branching parameters. This extension is needed for Theorem 2.7. The generalized class of superprocesses is constructed by means of an "h-transform" for superprocesses that was introduced independently by Engländer and Pinsky (1999) and Schied (1999).

The proofs of our main results are given in Section 4.

2. Setup, preliminaries, and main results.

2.1. *Assumptions and preliminaries.*

2.1.1. *The Markov process* Z. We will assume henceforth that the Markov process $Z = (Z_t, \mathcal{F}(I), P_{r,z})$ is a time-inhomogeneous right process with sample space $(\Omega, \mathcal{F})$ and state space $(S, \mathcal{B})$ in the sense of Section 2.2 in Dynkin (1994). Here, S is a metrizable Luzin space with Borel field $\mathcal{B}$. The σ-algebra $\mathcal{F}(I) \subset \mathcal{F}$ contains events observable during the time interval $I \subset [0, \infty)$. For every $r \geq 0$ and probability measure μ on $(S, \mathcal{B})$, we thus get a filtered probability space $(\Omega, (\mathcal{F}[r,t])_{t \geq r}, P_{r,\mu})$. Here, $P_{r,\mu}$ denotes as usual the probability measure under which Z starts at time $r \geq 0$ with initial distribution μ. The fact that Z is right essentially means that $t \mapsto Z_t(\omega)$ is càdlàg for each ω and that, for $r < t$, measurable $f : S \to \mathbb{R}_+$, and probability measures μ on $(S, \mathcal{B})$, the process $s \mapsto E_{s, Z_s}[f(Z_t)]$ is $P_{r,\mu}$-a.s. right-continuous on $[r, t]$. For all further details we refer to Sections 2.2.1 and 2.2.3 in Dynkin (1994).

2.1.2. *The* (Z, K, ψ)-*superprocess.* For $t \geq 0$, $z \in S$, $\xi \geq 0$, $\beta \in (0, 1]$ and bounded, measurable and positive $a : \mathbb{R}_+ \times S \to \mathbb{R}_+$ let

$$(2.1) \qquad \psi(t, z, \xi) = a(t, z)\xi^{1+\beta}.$$

Let moreover K be a continuous and nonnegative additive functional of Z such that

$$(2.2) \qquad \sup_\omega K[0, t](\omega) < \infty \qquad \text{for all } t \geq 0.$$

By Theorem 1.1 from Dynkin (1991b) one can then[2] construct the *superprocess with parameters* (Z, K, ψ). It is a time-inhomogeneous Markov process $X = (X_t, \mathcal{G}(I), \mathbb{P}_{r,\mu})$ with state space $\mathcal{M}$, the space of all nonnegative finite Borel measures on $(S, \mathcal{B})$. Its transition probabilities are determined as follows by the Laplace functionals of X. For any measurable function $f : S \to \mathbb{R}_+$ and $v \in \mathcal{M}$, we write $\langle f, v \rangle$ shorthand for the integral $\int f \, dv$. Then

$$\mathbb{E}_{r,\mu}[e^{-\langle f, X_T \rangle}] = e^{-\langle v(r, \cdot), \mu \rangle},$$

where v solves the integral equation

$$(2.3) \quad \begin{aligned} v(s, z) &= E_{s,z}[f(Z_T)] \\ &\quad - E_{s,z}\left[\int_s^T \psi(t, Z_t, v(t, Z_t)) K(dt)\right], \qquad r \leq s \leq T. \end{aligned}$$

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[2]Actually, a larger class of functions ψ is possible Dynkin (1991a, 1994), but here we will only need the class specified in (2.1).

When f is bounded, then v is the unique nonnegative solution of (2.3). For $T < r$ we make the convention that $X_T = 0$ $\mathbb{P}_{r,\mu}$-a.s.

Later on, we will need to allow for unbounded additive functionals K and thus have to extend this class of superprocesses; see Proposition 2.5.

It might be interesting to note that superprocesses provide an infinite-dimensional example of an affine process, a class of processes which has recently received considerable attention in mathematical finance; see Duffie, Filipović and Schachermayer (2003).

2.1.3. *A class of additive functionals of* Z. Following Dynkin (1991b), a nonnegative additive functional A of Z belongs to $\mathcal{A}_{(1)}$ if there exists a finite set $\{t_1, \ldots, t_n\} \subset \mathbb{R}_+$ with $t_1 < \cdots < t_n$ and bounded measurable functions $f_i \geq 0$, $i = 1, \ldots, n$, such that

$$(2.4) \qquad A[s, u] = \sum_{s \leq t_i \leq u} f_i(Z_{t_i}).$$

Next, $\mathcal{A}_{(2)}$ denotes the class of all nonnegative additive functionals A for which there exists a sequence (A_n) in $\mathcal{A}_{(1)}$ with the following three properties: $A_n[r, \infty) \to A[r, \infty)$ $P_{r,z}$-a.s. for all pairs (r, z); there exists $T > 0$ such that $A_n[T, \infty) = 0$ for all n; and $\sup_{\omega, n} A_n[0, T](\omega) < \infty$. Finally, $\mathcal{A}$ consists of all nonnegative additive functionals A for which there exists a sequence (A_n) in $\mathcal{A}_{(2)}$ such that $A_n(B) \nearrow A(B)$ for all measurable sets $B \subset \mathbb{R}_+$. For $q \geq 1$ and $T > 0$, we furthermore introduce the class

$$\mathcal{A}_T^q := \big\{ A \in \mathcal{A} \,|\, E_{r,z}\big[A[r, T]^q\big] < \infty \text{ and}$$
$$A(T, \infty) = 0 \ P_{r,z}\text{-a.s. for all } (r, z) \in [0, T] \times S \big\}.$$

REMARK 2.1 (Quadratic variation and path processes). Suppose that Y_t is a semimartingale of the form $Y_t = \phi(Z_t)$, where $\phi: S \to \mathbb{R}$ is a measurable function. Then the quadratic variation of Y gives rise to the nonnegative additive functional

$$(2.5) \qquad A(dt) := d[Y]_t$$

of Z. But in general it is not obvious whether A can be approximated by additive functionals of the form (2.4) unless, for example, Y is an Itô process of the form $Y_t = Y_0 + \int_0^t \sigma(s, Z_s)\,dW_s + \int_0^t b(s, Z_s)\,ds$ and so $d[Y]_t = \sigma(t, Z_t)^2\,dt$. Nevertheless, when we are interested primarily in Y and its quadratic variation as in the financial context of Section 1.2, then we may always assume that Z is the *path process* for Y [called *historical process* in Dawson and Perkins (1991) or Perkins (2002)]; that is, at each point t in time, Z_t is equal to the sample path of the entire history $(Y_{s \wedge t})_{s \geq 0}$. Then the dynamics of Z will automatically be Markovian, and Z will be a right process under mild assumptions. In this case, we can let

$$f_i(Z_{t_i}) := \big(Z_{t_i}(t_i) - Z_{t_i}(t_{i-1})\big)^2 \wedge K = (Y_{t_i} - Y_{t_{i-1}})^2 \wedge K$$

in (2.4), where $K > 0$ and $t_0 := 0$. The corresponding additive functional belongs to $\mathcal{A}_{(1)}$ and can be used to approximate the quadratic variation $[Y]$, so that under mild conditions (2.5) has a version in $\mathcal{A}$.

2.1.4. *J-functionals associated with an additive functional.* *J*-functionals are functionals of the (Z, K, ψ)-superprocess X associated with additive functionals $A \in \mathcal{A}$. They were introduced in Dynkin (1991b) as follows. Suppose that $A \in \mathcal{A}_{(1)}$ is given by (2.4). Then the corresponding *J*-functional is defined as

$$(2.6) \qquad J_A := \sum_{i=1}^{n} \langle f_{t_i}, X_{t_i} \rangle$$

(recall the convention that $X_t = 0$ $\mathbb{P}_{r,\mu}$-a.s. for $t < r$). For more general additive functionals A, the corresponding *J*-functionals can then be defined by a limiting procedure. By Theorem 1.2 in Dynkin (1991b), one has for $A \in \mathcal{A}_T^1$,

$$(2.7) \qquad \mathbb{E}_{r,\mu}\big[e^{-J_A}\big] = e^{-\langle v(r,\cdot),\mu \rangle},$$

where $v(r, z)$ solves

$$(2.8) \qquad v(r, z) = E_{r,z}\big[A[r, T]\big] - E_{r,z}\left[\int_r^T \psi\big(s, Z_s, v(s, Z_s)\big)K(ds)\right]$$

for $0 \le r \le T$. According to Dynkin [(1994), Theorem 3.4.2], solutions to (2.8) are unique when $E_{r,z}[A[r, T]]$ is uniformly bounded in z and $r \le T$, and hence in particular when $A \in \mathcal{A}_{(1)}$. Here we have the following result, which will be proved in Section 3.1.

PROPOSITION 2.2. *For $A \in \mathcal{A}_T^1$ and ψ as in (2.1), the function $v(r, z) := -\log \mathbb{E}_{r,\delta_z}[e^{-J_A}]$ is the unique finite and nonnegative solution of (2.8).*

2.2. *Statement of main results.* Let $T > 0$ be a fixed finite time horizon, $z \in S$ a fixed starting point for Z and $x_0 \in \mathbb{R}$ a given initial value. An *admissible strategy* will be a stochastic process $(x(t))_{0 \le t \le T}$ that is of the form $x(t) = x_0 + \int_0^t \dot{x}(s)\,ds$ for an integrable and $(\mathcal{F}[0, t])$-progressively measurable process $(\dot{x}(t))_{0 \le t \le T}$. We also assume that $x(\cdot)$ satisfies the fuel constraint

$$(2.9) \qquad x(T) = 0, \qquad P_{0,z}\text{-a.s.}$$

The *cost* of an admissible strategy will be

$$(2.10) \qquad E_{0,z}\left[\int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} |x(s)|^p A(ds)\right].$$

Here, $p \in [2, \infty)$ and $\eta: S \to (0, \infty)$ is a measurable function[3] that will be further specified below. Our goal is the minimization of this cost functional over all admissible strategies.

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[3]We can always assume that Z is of the form $Z_t = (t, \widetilde{Z}_t)$, and so there is no loss of generality in assuming the form $\eta(Z_t)$ rather than the form $\eta(t, Z_t)$.

We are now ready to state our first main result, pertaining to the case $\eta \equiv 1$. It is actually a corollary of the more general Theorem 2.7, but the latter needs additional assumptions and preparation. So we state this result here for the impatient reader.

THEOREM 2.3. *For $p \in [2, \infty)$ let q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and take $A \in \mathcal{A}_T^q$. Let J_A be the corresponding J-functional of the superprocess with parameters Z, $K(ds) = \frac{1}{\beta} ds$ and $\psi(t, z, \xi) = \xi^{1+\beta}$ for $\beta := \frac{1}{p-1}$, and define the function*

$$v_\infty(r, z) = -\log \mathbb{E}_{r, \delta_z}\big[e^{-J_A} \mathbb{1}_{\{X_T = 0\}}\big].$$

Then

$$x^*(t) := x_0 \exp\Big(-\int_0^t v_\infty(s, Z_s)^\beta \, ds\Big)$$

is the unique admissible strategy that minimizes the cost functional (2.10) *for the choice $\eta \equiv 1$. Moreover, the minimal costs are given by*

$$E_{0,z}\bigg[\int_0^T |\dot{x}^*(s)|^p \, ds + \int_{[0,T]} |x^*(s)|^p A(ds)\bigg] = |x_0|^p v_\infty(0, z).$$

REMARK 2.4. In the proof of Theorem 2.7 it will be shown that the optimal strategy is always below the linear strategy, that is,

$$(2.11) \qquad\qquad |x^*(t)| \leq |x_0| \frac{T - t}{T} \qquad \text{for } 0 \leq t \leq T;$$

see Remark 4.3.

We now turn to extending the results from Theorem 2.3 to the minimization of the cost functional (2.10) with nonconstant η. Unless η is bounded away from zero, this problem cannot be solved by the standard class of superprocesses considered in Dynkin (1991b, 1994); we need the extended class of superprocesses constructed in Schied (1999) by means of an "h-transform." An analytical version of this transform was found independently by Engländer and Pinsky (1999). Here we will use the probabilistic version. To make it work, we need the following assumption, which we will impose from now on: for given $T > 0$ there exists a constant $c_T > 0$ such that

$$(2.12) \qquad \frac{1}{c_T} \eta(z) \leq E_{r,z}\big[\eta(Z_t)\big] \leq c_T \eta(z) \qquad \text{for } 0 \leq r \leq t \leq T \text{ and } z \in S.$$

We assume moreover that

$$(2.13) \qquad\qquad E_{t,z}\big[\eta(Z_T)\big] \to \eta(z) \qquad \text{uniformly in } z \text{ as } t \uparrow T.$$

By $\mathcal{M}^\eta$ we denote the class of all nonnegative measures μ on $(S, \mathcal{B})$ for which $\int \eta \, d\mu < \infty$ and by $\mathcal{B}_\eta^+$ the class of all bounded $\mathcal{B}$-measurable functions $f : S \to \mathbb{R}_+$ for which there is a constant c such that $f \leq c\eta$. The proof of the following result will be based on Schied [(1999), Theorem 2] and given in Section 3.3.

PROPOSITION 2.5. *For $\beta \in (0, 1]$ and under assumption* (2.12), *there exists an $\mathcal{M}^\eta$-valued Markov process $X = ((X_t)_{t \leq T}, \mathcal{G}(I), \mathbb{P}_{r,\mu})$, the superprocess with parameters Z, $K(ds) = \frac{1}{\beta} \eta(Z_s) \, ds$ and $\psi(\xi) = (\frac{\xi}{\eta(z)})^{1+\beta}$, whose Laplace functionals are given by*

$$(2.14) \qquad \mathbb{E}_{r,\mu}\big[e^{-\langle f, X_t \rangle}\big] = e^{-\langle u(r,\cdot), \mu \rangle}, \qquad f \in B_\eta^+, \mu \in \mathcal{M}^\eta, t \leq T,$$

where u is the unique solution in B_η^+ of the integral equation

$$(2.15) \qquad u(r, z) = E_{r,z}\bigg[f(Z_t) - \int_r^t u(s, Z_s)^{1+\beta} \frac{1}{\beta \eta(Z_s)^\beta} \, ds \bigg].$$

Moreover, to each $A \in \mathcal{A}_T^1$ there exists a corresponding J-functional, J_A, satisfying

$$(2.16) \qquad \mathbb{E}_{r,\mu}\big[e^{-J_A}\big] = e^{-\langle v(r,\cdot), \mu \rangle},$$

where v solves

$$(2.17) \qquad v(r, z) = E_{r,z}\bigg[A[r, T] - \int_r^t v(s, Z_s)^{1+\beta} \frac{1}{\beta \eta(Z_s)^\beta} \, ds \bigg].$$

Furthermore, v is the unique finite and nonnegative solution of (2.17).

EXAMPLE 2.6 (Hyperbolic superprocesses). Let Z be a one-dimensional Brownian motion stopped when first hitting zero. Then one can take $\eta(z) = |z|^{1/\beta + \lambda}$ for some $\lambda \in [0, 1]$; see Schied (1999), Example 1(i). For $\beta = \lambda = 1$, the corresponding superprocess was constructed in Fleischmann and Mueller (1997).

We are now ready to state our main result.

THEOREM 2.7. *Suppose that η is as above. For $p \in [2, \infty)$ let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, and take $A \in \mathcal{A}_T^1$ such that*

$$(2.18) \qquad \int_0^T E_{0,z}\big[\eta(Z_t)^{1-q} A[t, T]^q\big] \, dt < \infty.$$

For $\beta := \frac{1}{p-1}$ let moreover $X = (X_t, \mathcal{G}(I), \mathbb{P}_{r,\mu})$ be the superprocess constructed in Proposition 2.5, *and define*

$$(2.19) \qquad v_\infty(r, y) := -\log \mathbb{E}_{r,\delta_y}\big[e^{-J_A} \mathbb{1}_{\{X_T = 0\}}\big]$$

and

$$(2.20) \qquad x^*(t) := x_0 \exp\bigg(-\int_0^t \frac{v_\infty(s, Z_s)^\beta}{\eta(Z_s)^\beta} \, ds\bigg).$$

Then x^* is an admissible strategy and the unique minimizer of the cost functional (2.10). *Moreover, the minimal costs are given by*

$$E_{0,z}\bigg[\int_0^T |\dot{x}^*(s)|^p \eta(Z_s) \, ds + \int_{[0,T]} |x^*(s)|^p A(ds) \bigg] = |x_0|^p v_\infty(0, z).$$

Note that the function $v_\infty(t, y)$ blows up as $t \uparrow T$. This fact will create considerable difficulties. In fact, the most difficult part in the proof of Theorem 2.7 will be to show that the strategy x^* defined through (2.20) is an admissible strategy with finite cost. To prove this, we need sharp upper and lower bound for the behavior of $v_\infty(t, y)$ as $t \uparrow T$. These estimates are of independent interest and will be developed in Section 3.

The blow-up of the function v_∞ is of course linked to the fuel constraint (2.9) required from admissible strategies. A common question one therefore encounters in relation to finite-fuel control problems is whether it may not be reasonable to replace the sharp fuel constraint by a suitable penalization term and thus avoid the singularity of the value function. It turns out that in our context such a penalization approach can be carried out without much additional effort. To describe it, we define a *relaxed strategy* as a stochastic process $(x(t))_{0 \le t \le T}$ that is of the form $x(t) = x_0 + \int_0^t \dot{x}(s)\,ds$ for an integrable and $(\mathcal{F}[0, t])$-progressively measurable process $(\dot{x}(t))_{0 \le t \le T}$. The *cost* of a relaxed strategy $x(\cdot)$ will be

$$(2.21) \quad E_{0,z}\left[\int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} |x(s)|^p A(ds) + \varrho(Z_T)|x(T)|^p\right],$$

where p, A and η are as above, and $\varrho : S \to \mathbb{R}_+$ is a measurable penalty function that satisfies the following assumptions for given T:

$$(2.22) \quad \begin{aligned} \varrho(y) &\le c_\varrho \eta(y) \qquad \text{for a constant } c_\varrho \text{ and all } y; \\ E_{t,Z_t}\big[\varrho(Z_T)\big] &\longrightarrow \varrho(Z_T) \qquad P_{0,z}\text{-a.s. as } t \uparrow T. \end{aligned}$$

By martingale convergence, the second condition is satisfied as soon as Z_T is $P_{0,z}$-a.s. measurable with respect to $\sigma(\bigcup_{t<T} \mathcal{F}[0, t])$, which in turn holds when Z is a Hunt process. Moreover, it follows from (2.13) that both conditions in (2.22) are satisfied when $\varrho = c\eta$ for some constant $c \ge 0$.

THEOREM 2.8. *Suppose that η, p, β and X are as in Theorem 2.7 and that the measurable penalty function $\varrho : S \to \mathbb{R}_+$ satisfies (2.22). Let $A \in \mathcal{A}_T^1$ be such that $A\{T\} = 0$ $P_{0,z}$-a.s. For*

$$(2.23) \quad v_\varrho(r, y) = -\log \mathbb{E}_{r,\delta_y}\big[e^{-J_A - \langle \varrho, X_T \rangle}\big],$$

the relaxed strategy

$$(2.24) \quad x_\varrho(t) := x_0 \exp\left(-\int_0^t \frac{v_\varrho(s, Z_s)^\beta}{\eta(Z_s)^\beta}\,ds\right)$$

is the unique minimizer of the cost functional (2.21) in the class of relaxed strategies. Moreover, the minimal costs are

$$\begin{aligned} E_{0,z}&\left[\int_0^T |\dot{x}_\varrho(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} |x_\varrho(s)|^p A(ds) + \varrho(Z_T)|x_\varrho(T)|^p\right] \\ &= |x_0|^p v_\varrho(0, z). \end{aligned}$$

3. Auxiliary results on superprocesses and their Laplace functionals.

3.1. *A probabilistic version of the parabolic maximum principle.* Our first result is the following proposition, which can be regarded as a probabilistic version of a parabolic maximum principle for equations of the form (2.8).

PROPOSITION 3.1. *Suppose that $A, \widetilde{A} \in \mathcal{A}_T^1$. Let furthermore L and $\widetilde{L}$ be nonnegative continuous additive functionals of Z. Suppose that $A[r, t] \leq \widetilde{A}[r, t]$ and $L[r, t] \geq \widetilde{L}[r, t]$ for $0 \leq r \leq t \leq T$. For some $q \geq 1$ let v and $\widetilde{v}$ be finite and nonnegative solutions of the integral equations*

$$v(r, z) = E_{r,z}\left[A[r, T] - \int_r^T v(t, Z_t)^q L(dt)\right],$$

$$\widetilde{v}(r, z) = E_{r,z}\left[\widetilde{A}[r, T] - \int_r^T \widetilde{v}(t, Z_t)^q \widetilde{L}(dt)\right].$$

Then we have $v \leq \widetilde{v}$.

For this and other proofs we will need the following special version of the general Feynman–Kac formula from Dynkin (1994), Theorem 4.1.1.

PROPOSITION 3.2 [Dynkin (1994)]. *Suppose that B is a signed additive functional of Z whose total variation belongs to $\mathcal{A}_T^1$. Let furthermore C be a continuous and nonnegative additive functional of Z. Let*

$$g(r, z) := E_{r,z}\left[\int_{[r,T]} e^{-C[r,s]} B(ds)\right].$$

When $E_{r,z}[\int_r^t g(s, Z_s) C(ds)]$ is well defined and finite for all (r, z), then g is the unique solution of the linear integral equation

$$(3.1) \qquad g(r, z) = E_{r,z}[B[r, T]] - E_{r,z}\left[\int_r^t g(s, Z_s) C(ds)\right].$$

PROOF OF PROPOSITION 3.1. Via the outer regularity of finite Borel measures on $[0, T]$, our condition $\widetilde{L}[r, t] \leq L[r, t]$ implies that $\widetilde{L}(dt) = \varphi_t L(dt)$ for some $[0, 1]$-valued function φ_t, which can be chosen to be progressively measurable. One sees that $u := \widetilde{v} - v$ satisfies

$$u(r, z) = E_{r,z}\left[\widetilde{A}[r, T] - A[r, T] + \int_r^T v(t, Z_t)^q (1 - \varphi_t) L(dt)\right.$$

$$\left. - \int_r^T u(t, Z_t) w(t, Z_t) \widetilde{L}(dt)\right],$$

where

$$w(t, z) := \begin{cases} \dfrac{\widetilde{v}(t, z)^q - v(t, z)^q}{\widetilde{v}(t, z) - v(t, z)}, & \text{if } \widetilde{v}(t, z) \neq v(t, z), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $w(t, z)$ is nonnegative. We define an additive functional B of Z via

$$B[r, t] := \widetilde{A}[r, t] - A[r, t] + \int_r^t v(s, Z_s)^q (1 - \varphi_s) L(ds).$$

It is nonnegative and belongs to $\mathcal{A}_T^1$ because

$$E_{r,z}\left[\int_r^T v(s, Z_s)^q (1 - \varphi_s) L(ds)\right] \leq E_{r,z}\left[\int_r^T v(s, Z_s)^q L(ds)\right],$$

and the expectation on the right is finite by assumption. Moreover, $C(dt) := w(t, Z_t)\widetilde{L}(dt)$ is a continuous and nonnegative additive functional. We have

$$E_{r,z}\left[\int_r^T |u(s, Z_s)| C(ds)\right]$$

$$\leq E_{r,z}\left[\int_r^T \widetilde{v}(t, Z_t)^q \mathbb{1}_{\{w(t, Z_t) \neq 0\}} \widetilde{L}(dt)\right]$$

$$+ E_{r,z}\left[\int_r^T v(t, Z_t)^q \mathbb{1}_{\{w(t, Z_t) \neq 0\}} \widetilde{L}(dt)\right].$$

Due to our assumptions, both expectations on the right are finite, and so $E_{r,z}[\int_r^T u(s, Z_s) C(ds)]$ is well defined and finite for all (r, z). Therefore, Proposition 3.2 yields that

$$u(r, z) = E_{r,z}\left[\int_{[r,T]} e^{-C[r,s]} B(ds)\right],$$

which is nonnegative. Hence, $\widetilde{v} \geq v$. $\square$

PROOF OF PROPOSITION 2.2.   By Dynkin [(1991b), Theorem 1.2], we have $\mathbb{E}_{r,\delta_z}[J_A] \leq E_{r,z}[A[r, T]]$, which is finite for all (r, z) since $A \in \mathcal{A}_T^1$. Hence $J_A < \infty$ $\mathbb{P}_{r,\delta_z}$-a.s. and so $v(r, z)$ is finite for all (r, z). Note that

$$(3.2) \qquad \begin{aligned} &E_{r,z}\left[\int_r^T v(s, Z_s)^{1+\beta} a(s, Z_s) K(ds)\right] \\ &= E_{r,z}\left[\int_r^T \psi\big(s, Z_s, v(s, Z_s)\big) K(ds)\right] < \infty \end{aligned}$$

for all (r, z). When $\widetilde{v}$ is another finite and nonnegative solution of (2.8), then (3.2) holds also for $\widetilde{v}$. Therefore we may apply Proposition 3.1 with $L(ds) = \widetilde{L}(ds) = a(s, Z_s) K(ds)$ and $q = 1 + \beta$ to get $v \leq \widetilde{v}$. Interchanging the roles of v and $\widetilde{v}$ yields the uniqueness of solutions. $\square$

EXAMPLE 3.3 (Laplace functionals for the total mass process). Consider the superprocess with parameters Z, $K(ds) = \gamma\, ds$ for a constant $\gamma > 0$, and $\psi(s, z, \xi) = \xi^{1+\beta}$ for $\beta \in (0, 1]$. Let ν be a finite and nonnegative Borel measure on $[0, T]$. Clearly, ν can be regarded as an element of $\mathcal{A}_T^1$. The corresponding J-functional is given by

$$(3.3) \qquad J_\nu = \int \langle 1, X_t \rangle \nu(dt)$$

as can easily be seen from (2.6). Its log-Laplace functional

$$v(r, z) := -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_\nu}\big] = -\log \mathbb{E}_{r,\delta_z}\big[e^{-\int_{[r,T]} \langle 1, X_t \rangle \nu(dt)}\big]$$

is in fact independent of z. Indeed, when $\nu = \lambda \delta_t$ for some $\lambda \geq 0$ and $t \in [r, T]$, then $v(r, z) = 0$ for $r > t$ and

$$(3.4) \qquad v(r, z) = -\log \mathbb{E}\big[e^{-\lambda \langle 1, X_t \rangle}\big] = \frac{\lambda}{\big(1 + \gamma \beta (t - r) \lambda^\beta\big)^{1/\beta}}$$

for $r \leq t$ and $\lambda \geq 0$, as can be shown by a straightforward computation based on the integral equation (2.3). When ν is a positive linear combination of Dirac measures, then we can use the Markov property of X to conclude that $v(r, z)$ is independent of z. For general ν we use an approximation argument. Alternatively, one can use the fact that, for superprocesses with homogeneous branching, the total mass process, $\langle 1, X_t \rangle$, is itself a one-dimensional Markov process. It follows that the function v is the unique nonnegative solution of the integral equation

$$v(r) = \nu[r, T] - \gamma \int_r^T v(s)^{1+\beta}\, ds.$$

3.2. *Estimates for the Laplace functionals of J-functionals.* Throughout this section, let $X = (X_t, \mathcal{G}(I), \mathbb{P}_{r,\mu})$ be the superprocess with one-particle motion Z, $K(ds) = \gamma\, ds$ for a constant $\gamma > 0$ and $\psi(s, z, \xi) = \xi^{1+\beta}$ for $\beta \in (0, 1]$. A key ingredient in the proof of Theorem 2.7 will be inequality (3.7) in the following theorem. This inequality gives a bound on the Laplace transform of a J-functional and is also of independent interest. It extends the upper bound of the following estimate for the Laplace functionals of X from Section 5 of Schied (1996):

$$(3.5) \qquad E_{r,z}\big[V_{T-r} f(Z_T)\big] \leq -\log \mathbb{E}_{r,\delta_z}\big[e^{-\langle f, X_T \rangle}\big] \leq V_{T-r} E_{r,z}\big[f(Z_T)\big]$$

for $r < T$ and $f \geq 0$, where $V_t : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the nonlinear semigroup

$$V_t y = \frac{y}{(1 + \gamma \beta t y^\beta)^{1/\beta}}, \qquad y \geq 0, t \geq 0.$$

For $\beta = 1$, estimate (3.5) can actually be extended to functions f with arbitrary sign. The lower bound in (3.5) will be needed in the proof of Theorem 2.8. Note that, by means of Example 3.3, the following inequality (3.7) coincides with the upper bound in (3.5) for the additive functional $A(dt) = f(Z_t)\delta_T(dt)$.

THEOREM 3.4. *For the superprocess with homogeneous branching rate γ, let J_A be the J-functional associated with a given $A \in \mathcal{A}_T^1$. For $r \leq T$ and $z \in S$ fixed, define moreover the finite and nonnegative Borel measure $\alpha_{r,z}(ds)$ on $[r, T]$ by*

$$\int f(s)\alpha_{r,z}(ds) = E_{r,z}\left[\int_{[r,T]} f(s)A(ds)\right]$$

for bounded measurable $f : [r, T] \to \mathbb{R}$. Then the conditional expectation of J_A given the evolution of the total mass process, $(\langle 1, X_t \rangle)_{r \leq t \leq T}$, is given by

$$(3.6) \qquad \mathbb{E}_{r,\delta_z}[J_A | \langle 1, X_t \rangle, r \leq t \leq T] = \int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt).$$

Moreover,

$$(3.7) \qquad \mathbb{E}_{r,\delta_z}[e^{-J_A}] \geq \mathbb{E}_{r,\delta_z}[e^{-\int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt)}].$$

PROOF. To prove (3.6), we can assume without loss of generality that A is bounded. For $\lambda \geq 0$ and a bounded nonnegative Borel measure μ on $[0, T]$, let

$$v_\lambda(s, y) := -\log \mathbb{E}_{s,\delta_y}[e^{-\lambda J_A - \int \langle 1, X_t \rangle \mu(dt)}].$$

Then v_λ is the unique nonnegative solution of

$$(3.8) \qquad v_\lambda(s, y) = E_{s,y}[\lambda A[s, T]] + \mu[s, T] - \gamma \int_s^T E_{s,y}[v_\lambda(t, Z_t)^{1+\beta}]dt.$$

It follows that

$$(3.9) \qquad \mathbb{E}_{r,\delta_z}[J_A e^{-\int \langle 1, X_t \rangle \mu(dt)}] = -\frac{d}{d\lambda}\bigg|_{\lambda=0} e^{-v_\lambda(r,z)} = e^{-v_0(r,z)} \frac{\partial v_\lambda(r, z)}{\partial \lambda}\bigg|_{\lambda=0}.$$

For $\lambda = 0$, we have

$$v_0(s, y) = -\log \mathbb{E}_{s,\delta_y}[e^{-\int \langle 1, X_t \rangle \mu(dt)}],$$

which is in fact independent of z due to the assumed homogeneity of the branching mechanism; see Example 3.3. Hence, $w := \partial v_\lambda / \partial \lambda|_{\lambda=0}$ solves

$$w(s, y) = E_{s,y}[A[s, T]] - \gamma(1 + \beta) \int_s^T E_{s,y}[w(t, Z_t)]v_0(t)^\beta dt.$$

Here, interchanging differentiation and integration is justified due to the uniform boundedness of A and, hence, of v_λ and w [the boundedness of the latter being implied by (3.9)]. Due to the general Feynman–Kac formula of Proposition 3.2, w is given by

$$w(r, z) = E_{r,z}\left[\int_{[r,T]} e^{-\gamma(1+\beta)\int_r^t v_0(u)^\beta du} A(dt)\right]$$

$$= \int_{[r,T]} e^{-\gamma(1+\beta)\int_r^t v_0(u)^\beta du} \alpha_{r,z}(dt).$$

Therefore

$$(3.10) \quad \mathbb{E}_{r,\delta_z}\big[J_A e^{-\int \langle 1, X_t \rangle \mu(dt)}\big] = e^{-v_0(r)} \int_{[r,T]} e^{-\gamma(1+\beta)\int_r^t v_0(u)^\beta \, du} \alpha_{r,z}(dt).$$

Now we take $A(dt) = \nu(dt)$ for a nonnegative finite Borel measure ν on $[0, T]$ and recall from (3.3) that in this case $J_A = J_\nu = \int \langle 1, X_t \rangle \nu(dt)$. We then get

$$(3.11) \quad \mathbb{E}_{r,\delta_z}\big[J_\nu e^{-\int \langle 1, X_t \rangle \mu(dt)}\big] = e^{-v_0(r)} \int_{[r,T]} e^{-\gamma(1+\beta)\int_r^t v_0(u)^\beta \, du} \nu(dt).$$

Comparing (3.10) with (3.11) and recalling (3.3) yields

$$\mathbb{E}_{r,\delta_z}\big[J_A e^{-\int \langle 1, X_t \rangle \mu(dt)}\big] = \mathbb{E}_{r,\delta_z}\big[J_{\alpha_{r,z}} e^{-\int \langle 1, X_t \rangle \mu(dt)}\big]$$

$$= \mathbb{E}_{r,\delta_z}\left[\int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt) e^{-\int \langle 1, X_t \rangle \mu(dt)}\right].$$

Varying μ and applying a monotone class argument yields (3.6).

Now we prove (3.7). We have

$$\mathbb{E}_{r,\delta_z}\big[e^{-J_A}\big] = \mathbb{E}_{r,\delta_z}\big[\mathbb{E}_{r,\delta_z}\big[e^{-J_A} | \langle 1, X_t \rangle, r \le t \le T\big]\big]$$

$$\ge \mathbb{E}_{r,\delta_z}\big[e^{-\mathbb{E}_{r,\delta_z}[J_A | \langle 1, X_t \rangle, r \le t \le T]}\big]$$

$$= \mathbb{E}_{r,\delta_z}\big[e^{-\int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt)}\big],$$

where we have used Jensen's inequality for conditional expectations in the second step and (3.6) in the third. $\square$

Recall that $K(ds) = \gamma \, ds$ and $\psi(s, z, \xi) = \xi^{1+\beta}$. Let us also mention that the following estimates will be extended to the case of nonhomogeneous branching in Proposition 3.7.

PROPOSITION 3.5.   *Let J_A be the J-functional associated with $A \in \mathcal{A}_T^1$. Then, for $k \ge 0$ and $r < T$,*

$$(3.12) \quad -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A - k\langle 1, X_T \rangle}\big] \le E_{r,z}[A[r, T]] + \frac{k}{(1 + \gamma\beta(T - r)k^\beta)^{1/\beta}}$$

and

$$(3.13) \quad -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A} \mathbb{1}_{\{X_T = 0\}}\big] \le E_{r,z}[A[r, T]] + \frac{1}{(\gamma\beta(T - r))^{1/\beta}}.$$

PROOF.   We can assume without loss of generality that A is bounded. From (3.7) we get

$$(3.14) \quad -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A - k\langle 1, X_T \rangle}\big] \le -\log \mathbb{E}_{r,\delta_z}\big[e^{-\int_{[r,T]} \langle 1, X_t \rangle \mu(dt) - k\langle 1, X_T \rangle}\big],$$

where $\mu([s, t]) = E_{r,z}[A[s, t]]$ for $r \le s \le t \le T$. As noted in Example 3.3, the right-hand side of (3.14) is independent of z and equal to $v(r)$, where v solves the integral equation

$$v(t) = k + \mu([t, T]) - \gamma \int_t^T v(s)^{1+\beta} \, ds, \qquad 0 \le t \le T.$$

Assertion (3.12) now follows from an application of Lemma 3.6, which is stated below. Inequality (3.13) is obtained by sending k to infinity in (3.12). $\quad \square$

LEMMA 3.6. *Suppose that $a : [0, T] \to \mathbb{R}_+$ is a measurable function, $k \ge 0$ is a constant and $v : [0, T] \to \mathbb{R}_+$ solves the integral equation*

$$v(r) = k + a(r) - \gamma \int_r^T v(s)^{1+\beta} \, ds, \qquad 0 \le r \le T.$$

Then

$$(3.15) \qquad v(t) \le a(t) + \frac{k}{(1 + \gamma \beta (T - t) k^\beta)^{1/\beta}}, \qquad 0 \le t \le T.$$

PROOF. The function

$$u(t) := \frac{k}{(1 + \gamma \beta (T - t) k^\beta)^{1/\beta}}$$

satisfies $u(T) = k$ and solves

$$u(r) = k - \gamma \int_r^T u(s)^{1+\beta} \, ds.$$

Let $\tilde{v}(t) := v(T - t)$, and define $\tilde{u}$ and $\tilde{a}$ accordingly. The function $w(t) := \tilde{v}(t) - \tilde{u}(t) - \tilde{a}(t)$ is absolutely continuous and satisfies for a.e. t

$$w'(t) = -\gamma (\tilde{v}(t)^{1+\beta} - \tilde{u}(t)^{1+\beta}) = -\gamma (\tilde{v}(t) - \tilde{u}(t)) f(t),$$

where

$$f(t) = \begin{cases} \dfrac{\tilde{v}(t)^{1+\beta} - \tilde{u}(t)^{1+\beta}}{\tilde{v}(t) - \tilde{u}(t)}, & \text{for } \tilde{v}(t) \ne \tilde{u}(t), \\ 0, & \text{otherwise.} \end{cases}$$

Since $f \ge 0$ and $a \ge 0$, it follows that $w'(t) \le -\gamma w(t) f(t)$ for a.e. $t \in [0, T]$. When letting $w_0(t) := e^{-\gamma \int_0^t f(s) \, ds}$, we have

$$\left(\frac{w(t)}{w_0(t)} \right)' = \frac{w'(t) w_0(t) - w(t) w_0'(t)}{w_0(t)^2} \le 0$$

and so

$$\frac{w(t)}{w_0(t)} \le \frac{w(0)}{w_0(0)} = v(T) - u(T) - a(T) = 0.$$

It follows that $w(t) \le 0$ and in turn that $v \le a + u$. $\quad \square$

3.3. *An "h-transform" for superprocesses.*   In this section, we prove Proposition 2.5 and extend the estimates from Proposition 3.5 to certain superprocesses with inhomogeneous branching characteristics. Our approach is based on the "*h*-transform" for superprocesses that was introduced independently by Engländer and Pinsky (1999) and Schied (1999). Whereas the first approach is primarily analytical, the latter approach is probabilistic, and it is the one we are going to use here. It is based on the following space–time harmonic function of Z:

$$(3.16) \qquad\qquad h(r, z) := E_{r,z}[\eta(Z_T)], \qquad 0 \le r \le T, z \in S.$$

We define the function $\psi(z, \xi) = (\frac{\xi}{\eta(z)})^{1+\beta}$, $\xi \ge 0$, $z \in S$ and a continuous nonnegative additive functional K of Z by $K(dt) = \frac{1}{\beta}\eta(Z_t) \, dt$. Then we have $\widetilde{\psi}(z, \xi) := \psi(z, \eta(z)\xi) = \xi^{1+\beta}$. Moreover, by (2.12),

$$E_{r,z}[K[r,t]] = \frac{1}{\beta} \int_r^t E_{r,z}[\eta(Z_t)] \, dt \le \frac{1}{\beta} c_T (t - r)\eta(z)$$

for $0 \le r \le t \le T$ and $z \in S$. Therefore, both ψ and K satisfy the conditions of Theorem 2 from Schied (1999), which hence implies the existence of a (Z, K, ψ)-superprocess X, defined here up to time T, for which the function u from (2.14) uniquely solves

$$u(r, z) = E_{r,z}\bigg[f(Z_t) - \int_r^t \psi(Z_s, u(s, Z_s)) K(ds) \bigg]$$

$$= E_{r,z}\bigg[f(Z_t) - \int_r^t u(s, Z_s)^{1+\beta} \frac{1}{\beta\eta(Z_s)^\beta} \, ds \bigg].$$

This implies the first part in the assertion of Proposition 2.5.

To prove the remaining part of Proposition 2.5 and to prepare for the proof of Theorem 2.7, we need to recall the construction of X given in Schied (1999). One first introduces Doob's h-transform of the process Z, that is, the Markov process $Z^h = (Z_t, \mathcal{F}(I), P_{r,z}^h)$ (defined up to the time horizon T) where

$$P_{r,z}^h[A] = \frac{1}{h(r, z)} E_{r,z}[\eta(Z_T)\mathbb{1}_A], \qquad A \in \mathcal{F}[r, T]$$

[note that $h > 0$ by (2.12)]. For any additive functional B of Z one then defines an additive functional B_h of Z^h by

$$B_h(ds) = \frac{1}{h(s, Z_s)} B(ds).$$

By the right property of Z, the process $h(s, Z_s)$ is a right continuous $P_{r,z}$-martingale. Hence, $h(s, Z_s)$ is equal to the optional projection of the constant process $t \mapsto \eta(Z_T)$. Therefore, for $0 \le r \le t \le T$,

$$E_{r,z}[B[r,t]] = E_{r,z}\bigg[\int_{[r,t]} h(s, Z_s) B_h(ds) \bigg] = E_{r,z}[B_h[r,t]\eta(Z_T)]$$

$$(3.17) \qquad\qquad\qquad = h(r, z) E_{r,z}^h[B_h[r,t]],$$

where we have used Theorem 57 in Chapter VI of Dellacherie and Meyer (1982) in the second step.

With this notation,

$$K_h(ds) = \frac{1}{h(s, Z_s)} K(ds) = \frac{\eta(Z_s)}{\beta h(s, Z_s)} \, ds$$

is a bounded and continuous additive functional of Z^h, and the function

$$\psi_h(t, z, \xi) = \psi(z, h(t, z)\xi)$$

is of the form (2.1). Therefore, up to the time horizon T, we can define the (Z^h, K_h, ψ_h)-superprocess $X^h = (X_t^h, \mathcal{G}(I), \mathbb{P}_{r,\mu}^h)$, for example, via Theorem 1.1 in Dynkin (1991b). The (Z, K, ψ)-superprocess X under $\mathbb{P}_{r,\mu}$ is then defined as the law of

$$(3.18) \qquad\qquad X_t(dz) := \frac{1}{h(t, z)} X_t^h(dz)$$

under $\mathbb{P}_{r, h.\mu}^h$, where $h.\mu$ denotes the measure $h(r, z)\mu(dz)$. Using once again Theorem 57 in Chapter VI of Dellacherie and Meyer (1982), one checks that the log-Laplace functionals of X are indeed given by (2.14), (2.15).

PROOF OF THE SECOND PART OF PROPOSITION 2.5. For $A \in \mathcal{A}_{(1)} \cap \mathcal{A}_T^1$ it is clear from (2.6) and (3.18) that J_A must be defined as the J-functional J_{A_h} for X^h, and this identification carries over to all $A \in \mathcal{A}_T^1$ by approximation. As above, one then checks that

$$v(r, z) := -\log \mathbb{E}_{r, \delta_z}\big[e^{-J_A}\big] = -\log \mathbb{E}_{r, h(r,z)\delta_z}^h\big[e^{-J_{A_h}}\big]$$

$$= -h(r, z) \log \mathbb{E}_{r, \delta_z}^h\big[e^{-J_{A_h}}\big]$$

solves (2.17).

Conversely, when $A \in \mathcal{A}_T^1$ and $\widetilde{v}$ is a nonnegative solution of (2.17), then $\widetilde{v}^h(r, z) := \widetilde{v}(r, z)/h(r, z)$ solves

$$(3.19) \qquad \widetilde{v}^h(r, z) = E_{r,z}^h[A_h[r, T]] - E_{r,z}^h\left[\int_r^T \psi_h(s, Z_s, \widetilde{v}^h(s, Z_s)) K_h(ds)\right].$$

By (3.17), A_h belongs to the class $\mathcal{A}_T^1$ for Z^h, and so Proposition 2.2 implies that $\widetilde{v}^h$ is the unique finite and nonnegative solution of (3.19). But this equation is also solved by $v^h(r, z) = h(r, z)v(r, z)$, which gives the uniqueness of solutions to the equation (2.17). $\square$

Now we turn toward generalizing the results from Section 3.2 to superprocesses with inhomogeneous, state-dependent branching mechanism as constructed in Proposition 2.5.

PROPOSITION 3.7. *Let X be the superprocess constructed in Proposition* 2.5, *and let J_A be the J-functional associated with $A \in \mathcal{A}_T^1$. Then*

$$(3.20) \quad -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A - k\langle \eta, X_T \rangle}\big] \leq E_{r,z}[A[r,T]] + \frac{h(r,z)k}{(1 + c_T^{-\beta}(T-r)k^\beta)^{1/\beta}}$$

and

$$(3.21) \quad -\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A} \mathbb{1}_{\{X_T = 0\}}\big] \leq E_{r,z}[A[r,T]] + \frac{c_T h(r,z)}{(T-r)^{1/\beta}},$$

where c_T is the constant from (2.12).

PROOF. With the notation introduced above, we have

$$-\log \mathbb{E}_{r,\delta_z}\big[e^{-J_A - k\langle \eta, X_T \rangle}\big] = -\log \mathbb{E}_{r,h(r,z)\delta_z}^h\big[e^{-J_{A_h} - k\langle 1, X_T^h \rangle}\big] = h(r,z)v^h(r,z),$$

where $v^h(r,z)$ solves

$$(3.22) \quad \begin{aligned} v^h(r,z) &= k + E_{r,z}^h[A_h[r,T]] - E_{r,z}^h\left[\int_r^T \psi_h(t, Z_t, v^h(t, Z_t)) K_h(dt)\right] \\ &= k + E_{r,z}^h[A_h[r,T]] - E_{r,z}^h\left[\int_r^T \frac{1}{\beta}\left(\frac{h(t, Z_t)}{\eta(Z_t)}\right)^\beta v^h(t, Z_t)^{1+\beta} dt\right]. \end{aligned}$$

By (2.12), $\frac{h(t,Z_t)}{\eta(Z_t)} \geq \frac{1}{c_T}$. Applying Proposition 3.1 with $L(dt) = \frac{1}{\beta}\left(\frac{h(t,Z_t)}{\eta(Z_t)}\right)^\beta$ and $\widetilde{L}(dt) := \beta^{-1} c_T^{-\beta} dt$ yields that

$$v^h(r,z) \leq \widetilde{v}^h(r,z) := -\log \widetilde{\mathbb{E}}_{r,\delta_z}\big[e^{-J_{A_h} - k\langle 1, \widetilde{X}_T \rangle}\big],$$

where $\widetilde{X} = (\widetilde{X}_t, \mathcal{G}(I), \widetilde{\mathbb{P}}_{r,\mu})$ is the superprocess with one-particle motion Z^h, branching function $\widetilde{\psi}(\xi) = \xi^{1+\beta}$ and branching functional $\widetilde{L}$. Proposition 3.5 yields that

$$\begin{aligned} \widetilde{v}^h(r,z) &\leq E_{r,z}^h[A_h[r,T]] + \frac{k}{(1 + \beta^{-1} c_T^{-\beta}\beta(T-r)k^\beta)^{1/\beta}} \\ &= \frac{1}{h(r,z)} E_{r,z}[A[r,T]] + \frac{k}{(1 + c_T^{-\beta}(T-r)k^\beta)^{1/\beta}}. \end{aligned}$$

This proves (3.20). Sending k to infinity gives (3.21). □

We also need a lower bound in case $A = 0$. To this end, we define

$$(3.23) \quad c_{r,T} := \sup_{t \in [r,T]} \sup_z \frac{h(t,z)}{\eta(z)}.$$

It follows from (2.12) that $c_{r,T}$ is finite for all $r \in [0, T]$ and from (2.13) that $c_{r,T} \searrow 1$ as $r \uparrow T$.

LEMMA 3.8. *Let X be the superprocess constructed in Proposition 2.5. Then*

$$-\log \mathbb{P}_{r,\delta_z}[X_T = 0] \geq \frac{h(r,z)}{c_{r,T}(T-r)^{1/\beta}}.$$

PROOF. We have

$$-\log \mathbb{P}_{r,\delta_z}[X_T = 0] = -\lim_{k\uparrow\infty} \log \mathbb{E}_{r,\delta_z}\big[e^{-k\langle \eta, X_T\rangle}\big]$$

$$= -\lim_{k\uparrow\infty} \log \mathbb{E}^h_{r,h(r,z)\delta_z}\big[e^{-k\langle 1, X_T^h\rangle}\big]$$

$$= h(r,z) \lim_{k\uparrow\infty} v_k^h(r,z),$$

where v_k^h solves (3.22) for $A_h = 0$. Using $\frac{h(t,Z_t)}{\eta(Z_t)} \leq c_{r,T}$ for $r \leq t \leq T$ and applying Proposition 3.1 with $L(dt) = \beta^{-1} c_{r,T}^\beta\, dt$ and $\widetilde{L}(dt) = \frac{1}{\beta}(\frac{h(t,Z_t)}{\eta(Z_t)})^\beta$ hence yields that

$$v_k^h(s,z) \geq -\log \widehat{\mathbb{E}}_{s,\delta_z}\big[e^{-k\langle 1, \widehat{X}_T\rangle}\big], \qquad r \leq s \leq T,$$

where $\widehat{X} = (\widehat{X}_t, \mathcal{G}(I), \widehat{\mathbb{P}}_{s,\mu})$ is the superprocess with one-particle motion Z^h, branching function $\widehat{\psi}(\xi) = \xi^{1+\beta}$ and branching functional L. By (3.4),

$$-\log \widehat{\mathbb{E}}_{s,\delta_z}\big[e^{-k\langle 1, \widehat{X}_T\rangle}\big] = \frac{k}{(1 + c_{r,T}^\beta(T-s)k^\beta)^{1/\beta}}.$$

Taking $s = r$ and sending k to infinity yields the assertion. $\quad\square$

4. Proofs of the main results.

4.1. *Proof of Theorem 2.7.* We start by making the following simple observation.

LEMMA 4.1. *For any admissible strategy that is not monotone, there exists another admissible strategy that is monotone and has strictly lower cost.*

PROOF. We may suppose without loss of generality that $x_0 > 0$. Let $x(\cdot)$ be an admissible strategy that is not monotone. Define $y(t) := x_0 + \int_0^t \dot{x}(t)\mathbb{1}_{\{\dot{x}(t)<0\}}\, dt$ and $\tau := \inf\{t \geq 0 | y(t) = 0\}$. Then y is monotone, $\tau \leq T$ and $\widetilde{x}(t) := y(t \wedge \tau)$ is an admissible strategy with strictly lower cost than $x(\cdot)$. $\quad\square$

Let X be the superprocess constructed in Proposition 2.5, and recall the definitions (2.19) and (2.20) for v_∞ and x^*. In addition to (3.23) we define

$$(4.1) \qquad \overline{c}_{r,T} := \inf_{t\in[r,T]} \inf_z \frac{h(t,z)}{\eta(z)} \quad \text{and} \quad C_{r,T} := \frac{\overline{c}_{r,T}}{c_{r,T}}.$$

It follows from (2.12) that $C_{r,T}$ is strictly positive for all $r \in [0, T]$ and from (2.13) that $C_{r,T} \to 1$ as $r \uparrow T$. Proposition 3.7, and hence Theorem 3.4, play a crucial role in proving the following key lemma.

LEMMA 4.2. *For $A \in \mathcal{A}_T^1$ satisfying (2.18), the process*

$$x^*(t) = x_0 \exp\left(-\int_0^t \left(\frac{v_\infty(s, Z_s)}{\eta(Z_s)}\right)^\beta ds\right), \qquad 0 \le t < T,$$

is an admissible strategy with finite cost. That is, $x^*(t) \to 0$ as $t \uparrow T$ and $E_{0,z}[\int_0^T |\dot{x}^*(t)|^p \eta(Z_t) \, dt] < \infty$.

PROOF. By Lemma 3.8 and (4.1),

$$(4.2) \qquad v_\infty(r, z) \ge -\log \mathbb{P}_{r,\delta_z}[X_T = 0] \ge \frac{h(r, z)}{c_{r,T}(T - r)^{1/\beta}} \ge \frac{C_{r,T} \cdot \eta(z)}{(T - r)^{1/\beta}}.$$

We thus get the upper bound

$$(4.3) \quad x^*(t) \le x_0 \exp\left(-C_{r,T}^\beta \int_r^t \frac{1}{T - s} ds\right) = x_0\left(\frac{T - t}{T - r}\right)^{C_{r,T}^\beta}, \qquad r \le t < T.$$

In particular, we have $x^*(t) \to 0$ as $t \uparrow T$.

Next, recalling the identity $p = \frac{1+\beta}{\beta}$, estimate (4.3) implies that for a.e. $t \ge r$,

$$|\dot{x}^*(t)|^p = \left(\frac{v_\infty(t, Z_t)}{\eta(Z_t)}\right)^{1+\beta} x^*(t)^{(\beta+1)/\beta} \le c_1\left(\frac{v_\infty(t, Z_t)}{h(t, Z_t)}(T - t)^{C_{r,T}^\beta/\beta}\right)^{1+\beta},$$

where here and in the sequel c_i, $i \in \mathbb{N}$, denote constants depending on r, T, β, c_T, z and x_0. Using (3.21) and identity (3.17), we obtain that

$$\left(\frac{v_\infty(t, Z_t)}{h(t, Z_t)}(T - t)^{C_{r,T}^\beta/\beta}\right)^{1+\beta}$$

$$\le \left(\frac{E_{t,Z_t}[A[t, T]]}{h(t, Z_t)}(T - t)^{C_{r,T}^\beta/\beta} + c_T(T - t)^{(C_{r,T}^\beta - 1)/\beta}\right)^{1+\beta}$$

$$= (E_{t,Z_t}^h[A_h[t, T]](T - t)^{1/\beta} + c_T)^{1+\beta}(T - t)^{((1+\beta)(C_{r,T}^\beta - 1))/\beta}$$

$$\le c_2(1 + E_{t,Z_t}^h[A_h[t, T]]^{1+\beta}(T - t)^{(1+\beta)/\beta})(T - t)^{((1+\beta)(C_{r,T}^\beta - 1))/\beta}$$

$$= c_2(T - t)^{-\delta} + c_2 E_{t,Z_t}^h[A_h[t, T]]^{1+\beta}(T - t)^{(1+\beta)/\beta - \delta},$$

where

$$\delta := -\frac{(1 + \beta)(C_{r,T}^\beta - 1)}{\beta} \ge 0.$$

Now we fix r so that $\delta < 1$, which is possible since $C_{r,T} \to 1$ as $r \uparrow T$ by (2.13). Then

(4.4)
$$|\dot{x}^*(t)|^p \leq c_1 \left(\frac{v_\infty(t, Z_t)}{h(t, Z_t)} (T-t)^{C_{r,t}^\beta/\beta} \right)^{1+\beta}$$
$$\leq c_1 c_2 (T-t)^{-\delta} + c_1 c_3 E_{t,Z_t}^h [A_h[t, T]]^{1+\beta}.$$

We also need to estimate $|\dot{x}^*(t)|$ for $0 \leq t \leq r$. To this end, we will use the trivial bound $x^*(t) \leq x_0$ to get as above that

$$|\dot{x}^*(t)|^p \leq c_4 \left(\frac{v_\infty(t, Z_t)}{h(t, Z_t)} \right)^{1+\beta} \leq c_5 \big(1 + E_{t,Z_t}^h [A_h[t, T]]^{1+\beta}\big)$$

for $0 \leq t \leq r$.

Putting everything together and using $\int_r^T (T-t)^{-\delta}\, dt < \infty$, the fact that $q = 1 + \beta$, (2.12), (3.17), Jensen's inequality, the Markov property of Z and once again (2.12) yields

$$E_{0,z}\left[\int_0^T |\dot{x}^*(t)|^p \eta(Z_t)\, dt \right]$$

$$\leq c_6 c_T T \eta(z) + c_5 E_{0,z}\left[\int_0^r \eta(Z_t) E_{t,Z_t}^h [A_h[t, T]]^q\, dt \right]$$

$$+ c_1 c_3 E_{0,z}\left[\int_r^T \eta(Z_t) E_{t,Z_t}^h [A_h[t, T]]^q\, dt \right]$$

$$\leq c_7 + c_8 \int_0^T E_{0,z}\big[\eta(Z_t) E_{t,Z_t}^h [A_h[t, T]]^q\big]\, dt$$

$$\leq c_7 + c_8 \int_0^T E_{0,z}\big[\eta(Z_t) h(t, Z_t)^{-q} E_{t,Z_t}[A[t, T]^q]\big]\, dt$$

$$\leq c_7 + c_9 \int_0^T E_{0,z}\big[\eta(Z_t)^{1-q} A[t, T]^q\big]\, dt,$$

which is finite due to our assumption (2.18). $\square$

REMARK 4.3. When $\eta \equiv 1$ we will also have $C_{r,T} = 1$. Hence (4.3) implies the bound (2.11).

For $p \geq 2$, let us introduce the function

(4.5)
$$\phi_p(\xi, \zeta) := \xi^p - p\zeta^{p-1}\xi + (p-1)\zeta^p, \qquad \xi, \zeta \geq 0.$$

Note that $\phi_p(\xi, \zeta) \geq 0$ with equality if and only if $\xi = \zeta$, because Young's inequality gives

(4.6)
$$\xi\zeta^{p-1} \leq \frac{1}{p}\xi^p + \frac{p-1}{p}\zeta^p \qquad \text{for } \xi, \zeta \geq 0.$$

PROPOSITION 4.4. *For $A \in \mathcal{A}_T^1$ satisfying* (2.18), *the following inequality holds for any monotone admissible strategy x with finite cost*:

$$E_{0,z}\left[\int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} |x(s)|^p A(ds)\right]$$

(4.7)
$$\geq |x_0|^p v_\infty(0, z)$$

$$+ E_{0,z}\left[\int_0^T \eta(Z_t)\phi_p\left(|\dot{x}(t)|, |x(t)|\left(\frac{v_\infty(t, Z_t)}{\eta(Z_t)}\right)^{1/(p-1)}\right)dt\right].$$

Moreover, there is equality in (4.7) *when $x = x^*$.*

Before proving Proposition 4.4, let us show how it implies Theorem 2.7.

PROOF OF THEOREM 2.7. Since $\phi_p(\xi, \zeta) \geq 0$, Proposition 4.4 yields that for any monotone admissible strategy $x(\cdot)$ with finite cost,

(4.8)
$$E_{0,z}\left[\int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} |x(s)|^p A(ds)\right] \geq |x_0|^p v_\infty(0, z).$$

Since moreover $\phi_p(\xi, \zeta) = 0$ if and only if $\xi = \zeta$, equality holds in (4.8) if and only if there is equality in (4.7) and also

(4.9)
$$|\dot{x}(t)| = |x(t)|\left(\frac{v_\infty(t, Z_t)}{\eta(Z_t)}\right)^{1/(p-1)}$$

$$= |x(t)|\left(\frac{v_\infty(t, Z_t)}{\eta(Z_t)}\right)^\beta \qquad \text{for a.e. } t, P_{0,z}\text{-a.s.}$$

By the monotonicity of x the latter condition holds if and only if $x = x^*$. Moreover, x^* is an admissible strategy with finite cost by Lemma 4.2 and satisfies equality in (4.7). □

PROOF OF PROPOSITION 4.4. For $k \in \mathbb{N}$, we introduce the functions

(4.10)
$$v_k(r, z) = -\log \mathbb{E}_{r,\delta_z}\left[e^{-J_A - k\langle 1, X_T\rangle}\right].$$

Then $v_k \nearrow v_\infty$ and v_k uniquely solves

$$v_k(r, z) = k + E_{r,z}[A[r, T]] - E_{r,z}\left[\int_r^T v_k(s, Z_s)^{1+\beta} \frac{1}{\beta\eta(Z_s)^\beta}\,ds\right].$$

We now fix a monotone admissible strategy $x(\cdot)$ with finite cost. We may assume without loss of generality that $x_0 \geq 0$ and hence also $x(t) \geq 0$ for all t. We define

(4.11)
$$C_t^k := \int_0^t |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,t)} x(s)^p A(ds) + x(t)^p v_k(t, Z_t).$$

The first two terms on the right represent the cost accumulated by the strategy x over the time interval $[0, t)$. The rightmost term approximates our guess for the minimal cost incurred over the time interval $[t, T]$ when starting at time t with the remainder $x(t)$.

We next define M^k as a rightcontinuous version of the martingale

$$(4.12) \qquad M_t^k := k + E_{0,z}\left[A[0, T] - \int_0^T v_k(s, Z_s)^{1+\beta} \frac{1}{\beta\eta(Z_s)^\beta} ds \Big| \mathcal{F}_t\right].$$

Such a version exists when we replace the filtration $(\mathcal{F}[0, t])_{t \geq 0}$ by its $P_{0,z}$-augmentation $(\overline{\mathcal{F}}[r, t])_{t \geq 0}$ so that the resulting filtered probability space $(\Omega, (\overline{\mathcal{F}}[0, t]), P_{0,z})$ satisfies the usual conditions; see Section A.1.1 in Dynkin (1994).

By the Markov property of Z, we have

$$(4.13) \qquad v_k(t, Z_t) = M_t^k - A[0, t) + \int_0^t v_k(s, Z_s)^{1+\beta} \frac{1}{\beta\eta(Z_s)^\beta} ds.$$

In particular,

$$(4.14) \qquad\qquad\qquad V_t := v_k(t, Z_t) - A\{t\}$$

is right continuous, and

$$(4.15) \qquad C_t^k = \int_0^t |\dot{x}(s)|^p \eta(Z_s) ds + \int_{[0,t]} x(s)^p A(ds) + x(t)^p V_t$$

is rightcontinuous as well.

Our next goal is to investigate the limit of C_t^k as $t \uparrow T$. To this end, we define N as a rightcontinuous version of the martingale

$$N_t = k + E_{0,z}[A[0, T] | \mathcal{F}_t].$$

We have $V_t \leq k + E_{t,Z_t}[A[t, T]]$ for each t $P_{0,z}$-a.s. and hence, by right continuity, $V_t \leq N_t$ for all t $P_{0,z}$-a.s. The martingale convergence theorem implies that $\sup_{t \leq T} N_t < \infty$ $P_{0,z}$-a.s. Hence also $\sup_{t \leq T} V_t < \infty$ $P_{0,z}$-a.s. It thus follows from $x(t) \to 0$ that $x(t)^p V_t \to 0$ $P_{0,z}$-a.s. as $t \uparrow T$. Moreover, we have $\int_{[0,T)} x(s) A(ds) = \int_{[0,T]} x(s) A(ds)$ because $x(T) = 0$. Therefore,

$$(4.16) \qquad C_t^k \longrightarrow C_T^k := \int_0^T |\dot{x}(s)|^p \eta(Z_s) ds + \int_{[0,T]} x(s)^p A(ds),$$

$P_{0,z}$-a.s. as $t \uparrow T$.

Next, applying Itô's formula to (4.15) and using (4.13) and (4.14) yields

$$dC_t^k = |\dot{x}(t)|^p \eta(Z_t) dt + x(t)^p A(dt) + px(t)^{p-1}\dot{x}(t) V_t dt + x(t)^p dV_t$$

$$= \left(|\dot{x}(t)|^p \eta(Z_t) + px(t)^{p-1}\dot{x}(t) V_t + x(t)^p V_t^{1+\beta} \frac{1}{\beta\eta(Z_t)^\beta}\right) dt$$

$$+ x(t)^p dM_t^k$$

$$= \eta(Z_t)\phi_p\left(|\dot{x}(t)|, x(t)\left(\frac{v_k(t, Z_t)}{\eta(Z_t)}\right)^{1/(p-1)}\right) dt + x(t)^p dM_t^k,$$

where, in the last step, we have used the relation $1 + \beta = p/(p-1)$ and the fact that $V_t = v_k(t, Z_t)$ for a.e. t.

Using (4.16), we obtain in the limit $t \uparrow T$ that $P_{0,z}$-a.s.

$$\int_0^T |\dot{x}(s)|^p \, ds + \int_{[0,T]} x(s)^p A(ds) - x_0^p v_k(0, z)$$

$$= C_T^k - C_0^k$$

$$= \int_0^T \eta(Z_t) \phi_p \left(|\dot{x}(t)|, x(t) \left(\frac{v_k(t, Z_t)}{\eta(Z_t)} \right)^{1/(p-1)} \right) dt + \int_0^T x(t)^p \, dM_t^k.$$

Taking expectations yields

$$E_{0,z} \left[\int_0^T |\dot{x}(s)|^p \eta(Z_s) \, ds + \int_{[0,T]} x(s)^p A(ds) \right]$$

(4.17) $$= x_0^p v_k(0, z)$$

$$+ E_{0,z} \left[\int_0^T \eta(Z_t) \phi_p \left(|\dot{x}(t)|, x(t) \left(\frac{v_k(t, Z_t)}{\eta(Z_t)} \right)^{1/(p-1)} \right) dt \right].$$

Fatou's lemma and the facts that $\phi_p \geq 0$ and $v_k \uparrow v_\infty$ thus yield the first part of the assertion when passing to the limit $k \uparrow \infty$ on the right-hand side of this identity.

Now we show that we may retain equality in (4.17) when passing to the limit $k \uparrow \infty$ and taking $x = x^*$. By (4.9), (4.6) and dominated convergence, this will hold when

$$E_{0,z} \left[\int_0^T \eta(Z_t) \left(x^*(t) \left(\frac{v_\infty(t, Z_t)}{\eta(Z_t)} \right)^{1/(p-1)} \right)^p dt \right] = E_{0,z} \left[\int_0^T |\dot{x}^*(t)|^p \eta(Z_t) \, dt \right]$$

is finite. But this is true by Lemma 4.2, and so the second part of the assertion is proved. □

4.2. *Proof of Theorem* 2.8.

LEMMA 4.5.   *If ϱ is as in Theorem* 2.8, *then $v_\varrho(t, Z_t) \to \varrho(Z_T)$ in $P_{0,z}$-probability as $t \uparrow T$.*

PROOF.   First, using the Markov property of Z,

(4.18) $$v_\varrho(t, Z_t) \leq E_{t,Z_t} [A[t, T] + \varrho(Z_T)]$$

$$= E_{0,z} [A[t, T] | \mathcal{F}[0, t]] + E_{t,Z_t} [\varrho(Z_T)].$$

By (2.22), the second term on the right converges $P_{0,z}$-a.s. to $\varrho(Z_T)$. As for the first term on the right, we first note that $A[t, T] \to 0$ in $L^1(P_{0,z})$ due to our assumption $A\{T\} = 0$ $P_{0,z}$-a.s., the fact that $E_{0,z}[A[0, T]] < \infty$, and dominated convergence.

Therefore, $E_{0,z}[A[t,T]|\mathcal{F}[0,t]] \to 0$ in $P_{0,z}$-probability, and so the entire right-hand side of (4.18) converges to $\varrho(Z_T)$ in $P_{0,z}$-probability.

We also need a lower bound on $v_\varrho(t, Z_t)$. Using the results and notation from Section 3.3, we have

$$
\begin{aligned}
(4.19) \quad v_\varrho(t, Z_t) &= -\log \mathbb{E}_{t,\delta_{Z_t}}\big[e^{-J_A-\langle \varrho, X_T\rangle}\big] \geq -\log \mathbb{E}_{t,\delta_{Z_t}}\big[e^{-\langle \varrho, X_T\rangle}\big] \\
&= -\log \mathbb{E}^h_{t,h(t,Z_t)\delta_{Z_t}}\big[e^{-\langle \varrho/\eta, X_T^h\rangle}\big] = h(t, Z_t)v^h(t, Z_t),
\end{aligned}
$$

where v^h solves

$$
v^h(r, y) = E^h_{r,y}\bigg[\frac{\varrho(Z_T)}{\eta(Z_T)}\bigg] - E^h_{r,y}\bigg[\int_r^T \frac{1}{\beta}\bigg(\frac{h(t, Z_t)}{\eta(Z_t)}\bigg)^\beta v^h(t, Z_t)^{1+\beta}\, dt\bigg].
$$

By (2.12), we have $h(t, Z_t)/\eta(Z_t) \leq c_T$. Let $\widetilde{X} = (\widetilde{X}_t, \mathcal{G}(I), \widetilde{\mathbb{P}}_{r,\mu})$ be the super-process with one-particle motion Z^h, $K(ds) = \frac{c_T^\beta}{\beta}\, ds$, and $\psi(\xi) = \xi^{1+\beta}$. Due to the fact that $E^h_{r,y}[\frac{\varrho(Z_T)}{\eta(Z_T)}] = \frac{1}{h(r,y)}E_{r,y}[\varrho(Z_T)] < \infty$ for all (r, y) by (2.12) and the first condition in (2.22), we may apply Proposition 3.1, which then yields that

$$
v^h(r, y) \geq -\log \widetilde{\mathbb{E}}_{r,\delta_y}\big[e^{-\langle \varrho/\eta, \widetilde{X}_T\rangle}\big].
$$

Since $\widetilde{X}$ has a homogeneous branching mechanism, we may apply the lower bound from (3.5) to get

$$
\begin{aligned}
-\log \widetilde{\mathbb{E}}_{r,\delta_y}\big[e^{-\langle \varrho/\eta, \widetilde{X}_T\rangle}\big] &\geq E^h_{r,y}\bigg[\frac{\varrho(Z_T)/\eta(Z_T)}{(1 + c_T^\beta(T-r)(\varrho(Z_T)/\eta(Z_T))^\beta)^{1/\beta}}\bigg] \\
&= \frac{1}{h(r, y)}E_{r,y}\bigg[\frac{\varrho(Z_T)}{(1 + c_T^\beta(T-r)(\varrho(Z_T)/\eta(Z_T))^\beta)^{1/\beta}}\bigg] \\
&\geq \frac{E_{r,y}[\varrho(Z_T)]}{h(r, y)(1 + c_T^\beta c_\varrho^\beta(T-r))^{1/\beta}},
\end{aligned}
$$

where we have used the first condition from (2.22) in the third step. Combining the preceding inequality with (4.19) yields

$$
\liminf_{t\uparrow T} v_\varrho(t, Z_t) \geq \liminf_{t\uparrow T} \frac{E_{t,Z_t}[\varrho(Z_T)]}{(1 + c_T^\beta c_\varrho^\beta(T-t))^{1/\beta}} = \varrho(Z_T), \qquad P_{0,z}\text{-a.s.}
$$

and hence the assertion. $\square$

PROOF OF THEOREM 2.8. Just as in Lemma 4.1 one first notes that we can restrict our attention to monotone relaxed strategies $x(\cdot)$ with $x(T) \geq 0$ for $x_0 \geq 0$ and $x(T) \leq 0$ for $x_0 \leq 0$.

We may assume $x_0 \geq 0$ without loss of generality. For a given monotone relaxed strategy $x(\cdot)$ with $x(T) \geq 0$ we define

$$
C_t := \int_0^t |\dot{x}(s)|^p \eta(Z_s)\, ds + \int_{[0,t)} x(s)^p A(ds) + x(t)^p v_\varrho(t, Z_t).
$$

It follows from our assumptions $A\{T\} = 0$ and Lemma 4.5 that in $P_{0,z}$-probability

$$C_t \longrightarrow C_T = \int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} x(s)^p A(ds) + x(t)^p \varrho(Z_T).$$

The function v_ϱ solves

$$v_\varrho(r, z) = E_{r,z}\big[A[r, T] + \varrho(Z_T)\big] - E_{r,z}\bigg[\int_r^T v_\varrho(s, Z_s)^{1+\beta} \frac{1}{\beta\eta(Z_s)^\beta}\,ds\bigg].$$

Thus, arguing as in the proof of Proposition 4.4, we find that

$$E_{0,z}\bigg[\int_0^T |\dot{x}(s)|^p \eta(Z_s)\,ds + \int_{[0,T]} x(s)^p A(ds) + \varrho(Z_T)x(T)^p\bigg]$$

$$= x_0^p v_\varrho(0, z) + E_{0,z}\bigg[\int_0^T \eta(Z_t)\phi_p\bigg(|\dot{x}(t)|, x(t)\bigg(\frac{v_\varrho(t, Z_t)}{\eta(Z_t)}\bigg)^{1/(p-1)}\bigg)\,dt\bigg].$$

In contrast to the proof of Proposition 4.4, note that here is no need for a limiting procedure since, unlike v_∞, the function v_ϱ has no singularity.

Now we can proceed as in the proof of Theorem 2.7 to get the assertion. □

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