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RANDOM G-EXPECTATIONS

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We construct a time-consistent sublinear expectation in the setting of volatility uncertainty. This mapping extends Peng's G-expectation by allowing the range of the volatility uncertainty to be stochastic. Our construction is purely probabilistic and based on an optimal control formulation with path-dependent control sets.

1. Introduction. The so-called G-expectation as introduced by Peng [13, 14] is a dynamic nonlinear expectation which advances the notions of g-expectations (Peng [10]) and backward SDEs (Pardoux and Peng [9]). Moreover, it yields a stochastic representation for a specific PDE and a risk measure for volatility uncertainty in financial mathematics (Avellaneda, Levy and Parás [1], Lyons [6]). The concept of volatility uncertainty also plays a key role in the existence theory for second order backward SDEs (Soner, Touzi and Zhang [18]) which were introduced as representations for a large class of fully nonlinear second order parabolic PDEs (Cheridito et al. [3]).

The G-expectation is a sublinear operator defined on a class of random variables on the canonical space Ω. Intuitively, it corresponds to the "worst-case" expectation in a model where the volatility of the canonical process B is seen as uncertain, but is postulated to take values in some bounded set D. The symbol G then stands for the support function of D. If $\mathcal{P}^G$ is the set of martingale laws on Ω under which the volatility of B behaves accordingly, the G-expectation at time $t = 0$ may be expressed as the upper expectation $\mathcal{E}_0^G(X) := \sup_{P\in\mathcal{P}^G} E^P[X]$. This description is due to Denis, Hu and Peng [4]. See also Denis and Martini [5] for a general study of related capacities.

For positive times t, the G-expectation is extended to a conditional expectation $\mathcal{E}_t^G(X)$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by B. When $X = f(B_T)$ for some sufficiently regular function f, then $\mathcal{E}_t^G(X)$ is defined via the solution of the nonlinear heat equation $\partial_t u - G(u_{xx}) = 0$ with boundary condition $u|_{t=0} = f$. The mapping $\mathcal{E}_t^G$ can be extended to random variables of the form $X = f(B_{t_1}, \dots, B_{t_n})$ by a stepwise evaluation of the PDE and finally to a suitable completion of the

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space of all such random variables. As a result, one obtains a family $(\mathcal{E}_t)_{t\geq 0}$ of conditional G-expectations satisfying the semigroup property $\mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_s$ for $s \leq t$, also called time-consistency property in this context. For an exhaustive overview of G-expectations and related literature we refer to Peng's recent ICM paper [16] and survey [15].

In this paper, we develop a formulation where the set D is allowed to be path-dependent, that is, we replace D by a set-valued process $\mathbf{D} = \{\mathbf{D}_t(\omega)\}$. Intuitively, this means that the function $G(\cdot)$ is replaced by a random function $G(t, \omega, \cdot)$ and that the a priori bounds on the volatility can be adjusted to the observed evolution of the system, which is highly desirable for applications. Our main result is the existence of a time-consistent family $(\mathcal{E}_t)_{t\geq 0}$ of sublinear operators corresponding to this formulation. When $\mathbf{D}$ depends on ω in a Markovian way, $\mathcal{E}_t$ can be seen as a stochastic representation for a class of state-dependent nonlinear heat equations $\partial_t u - G(x, u_{xx}) = 0$ which are not covered by [18].

At time $t = 0$, we again have a set $\mathcal{P}$ of probability measures and define $\mathcal{E}_0(X) := \sup_{P\in\mathcal{P}} E^P[X]$. For $t > 0$, we want to have

$$\mathcal{E}_t(X) \text{``=''} \sup_{P\in\mathcal{P}} E^P[X|\mathcal{F}_t] \qquad \text{in some sense.} \tag{1.1}$$

The main difficulty here is that the set $\mathcal{P}$ is not dominated by a finite measure. Moreover, as the resulting problem is non-Markovian in an essential way, the PDE approach outlined above seems unfeasible. We shall adopt the framework of regular conditional probability distributions and define, for each $\omega \in \Omega$, a quantity $\mathcal{E}_t(X)(\omega)$ by conditioning X and $\mathbf{D}$ (and hence, $\mathcal{P}$) on the path ω up to time t,

$$\mathcal{E}_t(X)(\omega) := \sup_{P\in\mathcal{P}(t,\omega)} E^P\big[X^{t,\omega}\big], \qquad \omega \in \Omega. \tag{1.2}$$

Then the right-hand side is well defined since it is simply a supremum of real numbers. This approach gives direct access to the underlying measures and allows for control theoretic methods. There is no direct reference to the function G, so that G is no longer required to be finite and we can work with an unbounded domain $\mathbf{D}$. The final result is the construction of a random variable $\mathcal{E}_t(X)$ which makes (1.1) rigorous in the form

$$\mathcal{E}_t(X) = \operatorname*{ess\,sup}_{P'\in\mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[X|\mathcal{F}_t], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P},$$

where $\mathcal{P}(t, P) = \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_t\}$ and $\operatorname{ess\,sup}^{(P,\mathcal{F}_t)}$ denotes the essential supremum with respect to the collection of $(P, \mathcal{F}_t)$-nullsets.

The approach via (1.2) is strongly inspired by the formulation of stochastic target problems in Soner, Touzi and Zhang [17]. There, the situation is more nonlinear in the sense that, instead of taking conditional expectations on the right-hand side, one solves under each P a backward SDE with terminal value X. On the other hand, those problems have (by assumption) a deterministic domain with respect to the volatility, which corresponds to a deterministic set $\mathbf{D}$ in our case, and therefore their control sets are not path-dependent.

The path-dependence of $\mathcal{P}(t,\omega)$ constitutes the main difficulty in the present paper, for example, it is not obvious under which conditions $\omega \mapsto \mathcal{E}_t(X)(\omega)$ in (1.2) is even measurable. The main problem turns out to be the following. In our formulation, the time-consistency of $(\mathcal{E}_t)_{t\geq 0}$ takes the form of a dynamic programming principle. The proof of such a result generally relies on a pasting operation performed on controls from the various conditional problems. However, we shall see that the resulting control in general violates the constraint given by $\mathbf{D}$, when $\mathbf{D}$ is stochastic. This feature, reminiscent of viability or state-constrained problems, renders our problem quite different from other known problems with path-dependence, such as the controlled SDE studied by Peng [11]. Our construction is based on a new notion of regularity which is tailored such that we can perform the necessary pastings at least on certain well-chosen controls.

One motivation for this work is to provide a model for superhedging in financial markets with a stochastic range of volatility uncertainty. Given a contingent claim X, this is the problem of finding the minimal capital x such that by trading in the stock market B, one can achieve a financial position greater or equal to X at time T. From a financial point of view, it is crucial that the trading strategy be universal, that is, it should not depend on the uncertain scenario P. It is worked out in Nutz and Soner [8] that the (right-continuous version of the) process $\mathcal{E}(X)$ yields the dynamic superhedging price; in particular, $\mathcal{E}_0(X)$ corresponds to the minimal capital x. Since the universal superhedging strategy is constructed from the quadratic covariation process of $\mathcal{E}(X)$ and B, it is crucial for their arguments that our model yields $\mathcal{E}(X)$ as a single, aggregated process. One then obtains an "optional decomposition" of the form

$$\mathcal{E}(X) = \mathcal{E}_0(X) + \int Z\,dB - K,$$

where K is an increasing process whose terminal value $K_T \geq 0$ indicates the difference between the financial position time T and the claim X.

The remainder of this paper is organized as follows. Section 2 introduces the basic set-up and notation. In Section 3 we formulate the control problem (1.2) for uniformly continuous random variables and introduce a regularity condition on $\mathbf{D}$. Section 4 contains the proof of the dynamic programming principle for this control problem. In Section 5 we extend $\mathcal{E}$ to a suitable completion.

2. Preliminaries. We fix a constant $T > 0$ and let $\Omega := \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths equipped with the uniform norm $\|\omega\|_T := \sup_{0\leq s\leq T}|\omega_s|$, where $|\cdot|$ is the Euclidean norm. We denote by B the canonical process $B_t(\omega) = \omega_t$, by P_0 the Wiener measure and by $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ the raw filtration generated by B. Unless otherwise stated, probabilistic notions requiring a filtration (such as adaptedness) refer to $\mathbb{F}$.

A probability measure P on Ω is called *local martingale measure* if B is a local martingale under P. We recall from Bichteler [2], Theorem 7.14, that, via the integration-by-parts formula, the quadratic variation process $\langle B\rangle(\omega)$ can be defined *pathwise* for all ω outside an exceptional set which is a P-nullset for every local martingale measure P. Taking componentwise limits, we can then define the $\mathbb{F}$-progressively measurable process

$$\hat{a}_t(\omega) := \limsup_{n\to\infty} n\big[\langle B\rangle_t(\omega) - \langle B\rangle_{t-1/n}(\omega)\big], \qquad 0 < t \le T,$$

taking values in the set of $d\times d$-matrices with entries in the extended real line. We also set $\hat{a}_0 = 0$.

Let $\bar{\mathcal{P}}_W$ be the set of all local martingale measures P such that $t \mapsto \langle B\rangle_t$ is absolutely continuous P-a.s. and $\hat{a}$ takes values in $\mathbb{S}_d^{>0}$ $dt \times P$-a.e., where $\mathbb{S}_d^{>0} \subset \mathbb{R}^{d\times d}$ denotes the set of strictly positive definite matrices. Note that $\hat{a}$ is then the quadratic variation density of B under any $P \in \bar{\mathcal{P}}_W$.

As in [4, 17, 18] we shall use the so-called strong formulation of volatility uncertainty in this paper, that is, we consider a subclass of $\bar{\mathcal{P}}_W$ consisting of the laws of stochastic integrals with respect to a fixed Brownian motion. The latter is taken to be the canonical process B under P_0: we define $\bar{\mathcal{P}}_S \subset \bar{\mathcal{P}}_W$ to be the set of laws

$$P^\alpha := P_0 \circ \big(X^\alpha\big)^{-1} \qquad \text{where } X_t^\alpha := {}^{(P_0)}\!\!\int_0^t \alpha_s^{1/2}\, dB_s, \qquad t\in[0,T]. \tag{2.1}$$

Here α ranges over all $\mathbb{F}$-progressively measurable processes with values in $\mathbb{S}_d^{>0}$ satisfying $\int_0^T |\alpha_t|\,dt < \infty$ P_0-a.s. The stochastic integral is the Itô integral under P_0, constructed as an $\mathbb{F}$-progressively measurable process with right-continuous and P_0-a.s. continuous paths, and, in particular, without passing to the augmentation of $\mathbb{F}$; cf. Stroock and Varadhan [19], page 97.

2.1. *Shifted paths and regular conditional distributions.* We now introduce the notation for the conditional problems of our dynamic programming. Since Ω is the canonical space, we can construct for any probability measure P on Ω and any $(t,\omega) \in [0,T]\times\Omega$ the corresponding regular conditional probability distribution P_t^ω; cf. [19], Theorem 1.3.4. We recall that P_t^ω is a probability kernel on $\mathcal{F}_t \times \mathcal{F}_T$, that is, it is a probability measure on $(\Omega,\mathcal{F}_T)$ for fixed ω and $\omega \mapsto P_t^\omega(A)$ is $\mathcal{F}_t$-measurable for each $A\in\mathcal{F}_T$. Moreover, the expectation under P_t^ω is the conditional expectation under P:

$$E^{P_t^\omega}[X] = E^P[X|\mathcal{F}_t](\omega), \qquad P\text{-a.s.}$$

whenever X is $\mathcal{F}_T$-measurable and bounded. Finally, P_t^ω is concentrated on the set of paths that coincide with ω up to t,

$$P_t^\omega\{\omega' \in \Omega : \omega' = \omega \text{ on } [0,t]\} = 1. \tag{2.2}$$

Next, we fix $0 \le s \le t \le T$ and define the following shifted objects. We denote by $\Omega^t := \{\omega \in C([t,T];\mathbb{R}^d) : \omega_t = 0\}$ the shifted canonical space, by B^t the canonical process on Ω^t, by P_0^t the Wiener measure on Ω^t and by $\mathbb{F}^t = \{\mathcal{F}_u^t\}_{t\le u\le T}$ the (raw) filtration generated by B^t. For $\omega \in \Omega^s$, the shifted path $\omega^t \in \Omega^t$ is defined by $\omega_u^t := \omega_u - \omega_t$ for $t \le u \le T$ and furthermore, if $\tilde\omega \in \Omega^t$, then the concatenation of ω and $\tilde\omega$ at t is the path

$$(\omega \otimes_t \tilde\omega)_u := \omega_u \mathbf{1}_{[s,t)}(u) + (\omega_t + \tilde\omega_u)\mathbf{1}_{[t,T]}(u), \qquad s \le u \le T.$$

If $\bar\omega \in \Omega$, we note the associativity $\bar\omega \otimes_s (\omega \otimes_t \tilde\omega) = (\bar\omega \otimes_s \omega) \otimes_t \tilde\omega$. Given an $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s and $\omega \in \Omega^s$, we define the shifted random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\tilde\omega) := \xi(\omega \otimes_t \tilde\omega), \qquad \tilde\omega \in \Omega^t.$$

Clearly $\tilde\omega \mapsto \xi^{t,\omega}(\tilde\omega)$ is $\mathcal{F}_T^t$-measurable and $\xi^{t,\omega}$ depends only on the restriction of ω to $[s,t]$. For a random variable ψ on Ω, the associativity of the concatenation yields

$$\left(\psi^{s,\bar\omega}\right)^{t,\omega} = \psi^{t,\bar\omega\otimes_s\omega}.$$

We note that for an $\mathbb{F}^s$-progressively measurable process $\{X_u, u \in [s,T]\}$, the shifted process $\{X_u^{t,\omega}, u \in [t,T]\}$ is $\mathbb{F}^t$-progressively measurable. If P is a probability on Ω^s, the measure $P^{t,\omega}$ on $\mathcal{F}_T^t$ defined by

$$P^{t,\omega}(A) := P_t^\omega(\omega \otimes_t A), \qquad A \in \mathcal{F}_T^t \qquad \text{where } \omega \otimes_t A := \{\omega \otimes_t \tilde\omega : \tilde\omega \in A\}$$

is again a probability by (2.2). We then have

$$E^{P^{t,\omega}}\left[\xi^{t,\omega}\right] = E^{P_t^\omega}[\xi] = E^P\left[\xi \mid \mathcal{F}_t^s\right](\omega), \qquad P\text{-a.s.}$$

In analogy to the above, we also introduce the set $\bar{\mathcal{P}}_W^t$ of martingale measures on Ω^t under which the quadratic variation density process $\hat a^t$ of B^t is well defined with values in $\mathbb{S}_d^{>0}$ and the subset $\bar{\mathcal{P}}_S^t \subseteq \bar{\mathcal{P}}_W^t$ induced by (P_0^t, B^t)-stochastic integrals of $\mathbb{F}^t$-progressively measurable integrands. (By convention, $\bar{\mathcal{P}}_S^T = \bar{\mathcal{P}}_W^T$ consists of the unique probability on $\Omega^T = \{0\}$.) Finally, we denote by $\Omega_t^s := \{\omega|_{[s,t]} : \omega \in \Omega^s\}$ the restriction of Ω^s to $[s,t]$ and note that Ω_t^s can be identified with $\{\omega \in \Omega^s : \omega_u = \omega_t \text{ for } u \in [t,T]\}$.

3. Formulation of the control problem. We start with a closed set-valued process $\mathbf{D} : \Omega \times [0,T] \to 2^{\mathbb{S}_d^+}$ taking values in the positive semidefinite matrices, that is, $\mathbf{D}_t(\omega)$ is a closed set of matrices for each $(t,\omega) \in [0,T] \times \Omega$. We assume that $\mathbf{D}$ is progressively measurable in the sense that for every compact $K \subset \mathbb{S}_d^+$, the lower inverse image $\{(t,\omega) : \mathbf{D}_t(\omega) \cap K \neq \varnothing\}$ is a progressively measurable subset of $[0,T] \times \Omega$. In particular, the value of $\mathbf{D}_t(\omega)$ depends only on the restriction of ω to $[0,t]$.

In view of our setting with a nondominated set of probabilities, we shall introduce topological regularity. As a first step to obtain some stability, we consider laws under which the quadratic variation density of B takes values in a uniform interior of $\mathbf{D}$. For a set $D \subseteq \mathbb{S}_d^+$ and $\delta > 0$, we define the δ-interior $\operatorname{Int}^\delta D := \{x \in D : B_\delta(x) \subseteq D\}$, where $B_\delta(x)$ denotes the open ball of radius δ.

DEFINITION 3.1. Given $(t,\omega)\in[0,T]\times\Omega$, we define $\mathcal{P}(t,\omega)$ to be the collection of all $P\in\bar{\mathcal{P}}_S^t$ for which there exists $\delta=\delta(t,\omega,P)>0$ such that

$$\hat{a}_s^t(\tilde{\omega})\in\operatorname{Int}^\delta\mathbf{D}_s^{t,\omega}(\tilde{\omega})\qquad\text{for } ds\times P\text{-a.e. } (s,\tilde{\omega})\in[t,T]\times\Omega^t.$$

Furthermore, if δ^* denotes the supremum of all such δ, we define the positive quantity $\deg(t,\omega,P):=(\delta^*/2)\wedge 1$. We note that $\mathcal{P}(0,\omega)$ does not depend on ω and denote this set by $\mathcal{P}$.

The formula $(\delta^*/2)\wedge 1$ ensures that $\deg(t,\omega,P)$ is finite and among the admissible δ. The following is the main regularity condition in this paper.

DEFINITION 3.2. We say that $\mathbf{D}$ is *uniformly continuous* if for all $\delta>0$ and $(t,\omega)\in[0,T]\times\Omega$ there exists $\varepsilon=\varepsilon(t,\omega,\delta)>0$ such that $\|\omega-\omega'\|_t\le\varepsilon$ implies

$$\operatorname{Int}^\delta\mathbf{D}_s^{t,\omega}(\tilde{\omega})\subseteq\operatorname{Int}^\varepsilon\mathbf{D}_s^{t,\omega'}(\tilde{\omega})\qquad\text{for all } (s,\tilde{\omega})\in[t,T]\times\Omega^t.$$

If the dimension is $d=1$ and $\mathbf{D}$ is a random interval, this property is related to the uniform continuity of the processes delimiting the interval (see also Example 3.8).

ASSUMPTION 3.3. We assume throughout that $\mathbf{D}$ is uniformly continuous and such that $\mathcal{P}(t,\omega)\neq\varnothing$ for all $(t,\omega)\in[0,T]\times\Omega$.

This assumption is in force for the entire paper. We now introduce the value function which will play the role of the sublinear (conditional) expectation. We denote by $\mathrm{UC}_b(\Omega)$ the space of bounded uniformly continuous functions on Ω.

DEFINITION 3.4. Given $\xi\in\mathrm{UC}_b(\Omega)$, we define for each $t\in[0,T]$ the value function

$$V_t(\omega):=V_t(\xi)(\omega):=\sup_{P\in\mathcal{P}(t,\omega)}E^P[\xi^{t,\omega}],\qquad\omega\in\Omega.$$

Until Section 5, *the function ξ is fixed* and often suppressed in the notation. The following result will guarantee enough separability for our proof of the dynamic programming principle; it is a direct consequence of the preceding definitions.

LEMMA 3.5. *Let $(t,\omega)\in[0,T]\times\Omega$ and $P\in\mathcal{P}(t,\omega)$. Then there exists $\varepsilon=\varepsilon(t,\omega,P)>0$ such that $P\in\mathcal{P}(t,\omega')$ and $\deg(t,\omega',P)\ge\varepsilon$ hold whenever $\|\omega-\omega'\|_t\le\varepsilon$.*

PROOF. Let $\delta:=\deg(t,\omega,P)$. Then, by definition,

$$\hat{a}_s^t(\tilde{\omega})\in\operatorname{Int}^\delta\mathbf{D}_s^{t,\omega}(\tilde{\omega})\qquad\text{for } ds\times P\text{-a.e. } (s,\tilde{\omega})\in[t,T]\times\Omega^t.$$

Let $\varepsilon = \varepsilon(t, \omega, \delta)$ be as in Definition 3.2 and ω' such that $\|\omega - \omega'\|_t \le \varepsilon$, then $\operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \subseteq \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega})$ by Assumption 3.3 and hence,

$$\hat{a}_s^t(\tilde{\omega}) \in \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \qquad \text{for } ds \times P\text{-a.e. } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

That is, $P \in \mathcal{P}(t, \omega')$ and $\deg(t, \omega', P) \ge \varepsilon(t, \omega, P) := (\varepsilon/2) \wedge 1$. $\square$

A first consequence of the preceding lemma is the measurability of V_t. We denote $\|\omega\|_t := \sup_{0 \le s \le t} |\omega_s|$.

COROLLARY 3.6. *Let $\xi \in \mathrm{UC}_b(\Omega)$. The value function $\omega \mapsto V_t(\xi)(\omega)$ is lower semicontinuous for $\|\cdot\|_t$ and, in particular, $\mathcal{F}_t$-measurable.*

PROOF. Fix $\omega \in \Omega$ and $P \in \mathcal{P}(t, \omega)$. Since ξ is uniformly continuous, there exists a modulus of continuity $\rho^{(\xi)}$,

$$\left|\xi(\omega) - \xi(\omega')\right| \le \rho^{(\xi)}\left(\|\omega - \omega'\|_T\right) \qquad \text{for all } \omega, \omega' \in \Omega.$$

It follows that for all $\tilde{\omega} \in \Omega^t$,

$$\begin{aligned} \left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| &= \left|\xi(\omega \otimes_t \tilde{\omega}) - \xi(\omega' \otimes_t \tilde{\omega})\right| \\ &\le \rho^{(\xi)}\left(\|\omega \otimes_t \tilde{\omega} - \omega' \otimes_t \tilde{\omega}\|_T\right) \\ &= \rho^{(\xi)}\left(\|\omega - \omega'\|_t\right). \end{aligned} \tag{3.1}$$

Consider a sequence (ω^n) such that $\|\omega - \omega^n\|_t \to 0$. The preceding lemma shows that $P \in \mathcal{P}(t, \omega^n)$ for all $n \ge n_0 = n_0(t, \omega, P)$ and thus

$$\begin{aligned} \liminf_{n\to\infty} V_t(\omega^n) &= \liminf_{n\to\infty} \sup_{P' \in \mathcal{P}(t,\omega^n)} E^{P'}[\xi^{t,\omega^n}] \\ &\ge \liminf_{n\to\infty} \Big[\sup_{P' \in \mathcal{P}(t,\omega^n)} E^{P'}[\xi^{t,\omega}] - \rho^{(\xi)}\left(\|\omega - \omega^n\|_t\right)\Big] \\ &= \liminf_{n\to\infty} \sup_{P' \in \mathcal{P}(t,\omega^n)} E^{P'}[\xi^{t,\omega}] \\ &\ge E^P[\xi^{t,\omega}]. \end{aligned}$$

As $P \in \mathcal{P}(t, \omega)$ was arbitrary, we conclude that $\liminf_n V_t(\omega^n) \ge V_t(\omega)$. $\square$

We note that the obtained regularity of V_t is significantly weaker than the uniform continuity of ξ; this is a consequence of the state-dependence in our problem. Indeed, the above proof shows that if $\mathcal{P}(t, \omega)$ is independent of ω, then V_t is again uniformly continuous with the same modulus of continuity as ξ (see also [17]). Similarly, in Peng's construction of the G-expectation, the preservation of Lipschitz-constants arises because the nonlinearity in the underlying PDE has no state-dependence.

REMARK 3.7. Since ξ is bounded and continuous, the value function $V_t(\xi)$ remains unchanged if $\mathcal{P}(t,\omega)$ is replaced by its weak closure (in the sense of weak convergence of probability measures). As an application, we show that we retrieve Peng's G-expectation under a nondegeneracy condition.

Given G, we recall from [4], Section 3, that there exists a compact and convex set $D \subset \mathbb{S}_d^+$ such that $2G$ is the support function of D and such that $\mathcal{E}_0^G(\psi) = \sup_{P \in \mathcal{P}^G} E^P[\psi]$ for sufficiently regular ψ, where

$$\mathcal{P}^G := \{P^\alpha \in \bar{\mathcal{P}}_S : \alpha_t(\omega) \in D \text{ for } dt \times P_0\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

We make the additional assumption that D has nonempty interior $\operatorname{Int} D$. In the scalar case $d = 1$, this precisely rules out the trivial case where $\mathcal{E}_0^G$ is an expectation in the usual sense.

We then choose $\mathbf{D} := D$. In this deterministic situation, our formulation boils down to

$$\mathcal{P} = \bigcup_{\delta > 0} \{P^\alpha \in \bar{\mathcal{P}}_S : \alpha_t(\omega) \in \operatorname{Int}^\delta D \text{ for } dt \times P_0\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

Clearly $\mathcal{P} \subset \mathcal{P}^G$, so it remains to show that $\mathcal{P}$ is dense. To this end, fix a point $\alpha^* \in \operatorname{Int} D$ and let $P^\alpha \in \mathcal{P}^G$, that is, α takes values in D. Then for $0 < \varepsilon < 1$, the process $\alpha^\varepsilon := \varepsilon\alpha^* + (1-\varepsilon)\alpha$ takes values in $\operatorname{Int}^\delta D$ for some $\delta > 0$, due to the fact that the disjoint sets ∂D and $\{\varepsilon\alpha^* + (1-\varepsilon)x : x \in D\}$ have positive distance by compactness. We have $P^{\alpha^\varepsilon} \in \mathcal{P}$ and it follows by dominated convergence for stochastic integrals that $P^{\alpha^\varepsilon} \to P^\alpha$ for $\varepsilon \to 0$.

While this shows that we can indeed recover the G-expectation, we should mention that if one wants to treat only deterministic sets $\mathbf{D}$, one can use a much simpler construction than in this paper, and, in particular, there is no need to use the sets $\operatorname{Int}^\delta \mathbf{D}$ at all.

Next, we give an example where our continuity assumption on $\mathbf{D}$ is satisfied.

EXAMPLE 3.8. We consider the case $d = 1$. Let $a, b : [0,T] \times \Omega \to \mathbb{R}$ be progressively measurable processes satisfying $0 \le a < b$. Assume that a is uniformly continuous in ω, uniformly in time, that is, that for all $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\|\omega - \omega'\|_T \le \varepsilon \quad \text{implies} \quad \sup_{0 \le s \le T} |a_s(\omega) - a_s(\omega')| \le \delta. \tag{3.2}$$

Assume that b is uniformly continuous in the same sense. Then the random interval

$$\mathbf{D}_t(\omega) := [a_t(\omega), b_t(\omega)]$$

is uniformly continuous. Indeed, given $\delta > 0$, there exists $\varepsilon' = \varepsilon'(\delta) > 0$ such that $|a_s(\omega) - a_s(\omega')| < \delta/2$ for all $0 \le s \le T$ whenever $\|\omega - \omega'\|_T \le \varepsilon'$, and the same for b. We set $\varepsilon := \varepsilon' \wedge \delta/2$. Then for ω, ω' such that $\|\omega - \omega'\|_t \le \varepsilon$, we have that $\|\omega \otimes_t \tilde{\omega} - \omega' \otimes_t \tilde{\omega}\|_T = \|\omega - \omega'\|_t \le \varepsilon$ and hence,

$$\begin{aligned}\operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) &= \big[a_s(\omega \otimes_t \tilde{\omega}) + \delta, b_s(\omega \otimes_t \tilde{\omega}) - \delta\big] \\ &\subseteq \big[a_s\big(\omega' \otimes_t \tilde{\omega}\big) + \varepsilon, b_s\big(\omega' \otimes_t \tilde{\omega}\big) - \varepsilon\big] \\ &= \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \qquad \text{for all } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.\end{aligned}$$

A multivariate version of the previous example runs as follows. Let $A : [0, T] \times \Omega \to \mathbb{S}_d^{>0}$ and $r : [0, T] \times \Omega \to [0, \infty)$ be two progressively measurable processes which are uniformly continuous in the sense of (3.2) and define the set-valued process

$$\mathbf{D}_t(\omega) := \big\{\Gamma \in \mathbb{S}_d^{>0} : \big|\Gamma - A_t(\omega)\big| \le r_t(\omega)\big\}.$$

Then $\mathbf{D}$ is uniformly continuous; the proof is a direct extension of the above.

We close this section by a remark relating the "random G"-expectations to a class of state-dependent nonlinear heat equations.

REMARK 3.9. We consider a Markovian case of Example 3.8, where the functions delimiting $\mathbf{D}$ depend only on the current state. Indeed, let $a, b : \mathbb{R} \to \mathbb{R}$ be bounded, uniformly continuous functions such that $0 \le a \le b$ and $b - a$ is bounded away from zero, and define

$$\mathbf{D}_t(\omega) := \big[a(\omega_t), b(\omega_t)\big].$$

(Of course, an additional time-dependence could also be included.) Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, uniformly continuous function and consider

$$\begin{aligned}&-\partial_t u - G(x, u_{xx}) = 0, \qquad u(T, \cdot) = f; \\ &G(x, q) := \sup_{a(x) \le p \le b(x)} pq/2.\end{aligned} \tag{3.3}$$

We claim that the (unique, continuous) viscosity solution u of (3.3) satisfies

$$u(0, x) = V_0(\xi) \qquad \text{for } \xi := f(x + B_T). \tag{3.4}$$

Indeed, by the standard Hamilton–Jacobi–Bellman theory, u is the value function of the control problem

$$u(0, x) = \sup_\alpha E^{P_0}\big[f\big(x + X_T^\alpha\big)\big], \qquad X_t^\alpha = \int_0^t \alpha_s^{1/2}\, dB_s,$$

where α varies over all positive, progressively measurable processes satisfying

$$\alpha_t \in \mathbf{D}\big(X_t^\alpha\big), \qquad dt \times P_0\text{-a.e.}$$

For each such α, let P^α be the law of X^α, then clearly

$$u(0,x) = \sup_\alpha E^{P^\alpha}\big[f(x+B_T)\big].$$

It follows from (the proof of) Lemma 4.2 below that the laws $\{P^\alpha\}$ are in one-to-one correspondence with $\mathcal{P}$, if Definition 3.1 is used with $\delta = 0$ (i.e., we use $\mathbf{D}$ instead of its interior). Let $G^\delta(x,q) := \sup_{a(x)+\delta \le p \le b(x)-\delta} pq/2$ be the nonlinearity corresponding to $\operatorname{Int}^\delta \mathbf{D}$ and let u^δ be the viscosity solution of the corresponding equation (3.3). Then the above yields

$$u(0,x) \ge V_0(\xi) \ge u^\delta(0,x)$$

for $\delta > 0$ small (so that $b - a \ge 2\delta$). It follows from the comparison principle and stability of viscosity solutions that $u^\delta(t,x)$ increases monotonically to $u(t,x)$ as $\delta \downarrow 0$; as a result, we have (3.4).

4. Dynamic programming. The main goal of this section is to prove the dynamic programming principle for $V_t(\xi)$, which corresponds to the time-consistency property of our sublinear expectation. For the case where $\mathbf{D}$ is deterministic and $V_t(\xi) \in \mathrm{UC}_b(\Omega)$, the relevant arguments were previously given in [17].

4.1. *Shifting and pasting of measures.* As usual, one inequality in the dynamic programming principle will be the consequence of an invariance property of the control sets.

LEMMA 4.1 (Invariance). *Let $0 \le s \le t \le T$ and $\bar\omega \in \Omega$. If $P \in \mathcal{P}(s,\bar\omega)$, then*

$$P^{t,\omega} \in \mathcal{P}(t, \bar\omega \otimes_s \omega) \qquad \textit{for } P\textit{-a.e. } \omega \in \Omega^s.$$

PROOF. It is shown in [17], Lemma 4.1, that $P^{t,\omega} \in \bar{\mathcal{P}}^t_S$ and that under $P^{t,\omega}$, the quadratic variation density of B^t coincides with the shift of $\hat a^s$:

$$\hat a^t_u(\tilde\omega) = \big(\hat a^s_u\big)^{t,\omega}(\tilde\omega) \qquad \text{for } du \times P^{t,\omega}\text{-a.e. } (u,\tilde\omega) \in [t,T] \times \Omega^t \tag{4.1}$$

and P-a.e. $\omega \in \Omega^s$. Let $\delta := \deg(s,\bar\omega,P)$, then

$$\hat a^s_u(\omega') \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar\omega}_u(\omega') \qquad \text{for } du \times P\text{-a.e. } (u,\omega') \in [s,T] \times \Omega^s$$

and hence,

$$\hat a^s_u(\omega \otimes_t \tilde\omega) \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar\omega}_u(\omega \otimes_t \tilde\omega) \qquad \text{for } du \times P^{t,\omega}\text{-a.e. } (u,\tilde\omega) \in [t,T] \times \Omega^t.$$

Now (4.1) shows that for $du \times P^{t,\omega}$-a.e. $(u,\tilde\omega) \in [t,T] \times \Omega^t$ we have

$$\hat a^t_u(\tilde\omega) = \big(\hat a^s_u\big)^{t,\omega}(\tilde\omega) = \hat a^s_u(\omega \otimes_t \tilde\omega) \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar\omega}_u(\omega \otimes_t \tilde\omega) = \operatorname{Int}^\delta \mathbf{D}^{t,\bar\omega\otimes_s\omega}_u(\tilde\omega)$$

for P-a.e. $\omega \in \Omega^s$, that is, that $P^{t,\omega} \in \mathcal{P}(t, \bar\omega \otimes_s \omega)$. $\square$

The dynamic programming principle is intimately related to a stability property of the control sets under a pasting operation. More precisely, it is necessary to collect ε-optimizers from the conditional problems over $\mathcal{P}(t,\omega)$ and construct from them a control in $\mathcal{P}$ (if $s=0$). As a first step, we give a tractable criterion for the admissibility of a control. We recall the process X^α from (2.1) and note that since it has continuous paths P_0-a.s., X^α can be seen as a transformation of the canonical space under the Wiener measure.

LEMMA 4.2. *Let $(t,\omega)\in[0,T]\times\Omega$ and $P=P^\alpha\in\bar{\mathcal{P}}_S^t$. Then $P\in\mathcal{P}(t,\omega)$ if and only if there exists $\delta>0$ such that*

$$\alpha_s(\tilde\omega)\in \operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}\big(X^\alpha(\tilde\omega)\big) \qquad \text{for } ds\times P_0^t\text{-a.e. } (s,\tilde\omega)\in[t,T]\times\Omega^t.$$

PROOF. We first note that

$$\langle B^t\rangle=\int_t^\cdot \hat a_u^t(B^t)\,du,\qquad P^\alpha\text{-a.s.}\quad\text{and}\quad \langle X^\alpha\rangle=\int_t^\cdot \alpha_u(B^t)\,du,\qquad P_0^t\text{-a.s.}$$

Recalling that $P^\alpha=P_0^t\circ(X^\alpha)^{-1}$, we observe that the P^α-distribution of $(B^t,\int_t^\cdot \hat a^t(B^t)\,du)$ coincides with the P_0^t-distribution of $(X^\alpha,\int_t^\cdot \alpha(B^t)\,du)$. By definition, $P^\alpha\in\mathcal{P}(t,\omega)$ if and only if there exists $\delta>0$ such that

$$\hat a^t\big(B^t\big)\in \operatorname{Int}^\delta \mathbf{D}^{t,\omega}\big(B^t\big),\qquad ds\times P^\alpha\text{-a.e. on } [t,T]\times\Omega^t,$$

and by the above, this is further equivalent to

$$\alpha\big(B^t\big)\in \operatorname{Int}^\delta \mathbf{D}^{t,\omega}\big(X^\alpha\big),\qquad ds\times P_0^t\text{-a.e. on } [t,T]\times\Omega^t.$$

This was the claim. $\square$

To motivate the steps below, we first consider the admissibility of pastings in general. We can paste given measures $P=P^\alpha\in\bar{\mathcal{P}}_S$ and $\hat P=P^{\hat\alpha}\in\bar{\mathcal{P}}_S^t$ at time t to obtain a measure $\bar P$ on Ω and we shall see that $\bar P=P^{\bar\alpha}$ for

$$\bar\alpha_u(\omega)=\mathbf{1}_{[0,t)}(u)\alpha_u(\omega)+\mathbf{1}_{[t,T]}(u)\hat\alpha_u\big(X^\alpha(\omega)^t\big).$$

Now assume that $P\in\mathcal{P}$ and $\hat P\in\mathcal{P}(t,\hat\omega)$. By the previous lemma, these constraints may be formulated as $\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^\alpha)$ and $\hat\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^{\hat\alpha})^{t,\hat\omega}$, respectively. If $\mathbf{D}$ is deterministic, we immediately see that $\bar\alpha(\omega)\in\operatorname{Int}^\delta\mathbf{D}$ for all $\omega\in\Omega$ and therefore $\bar P\in\mathcal{P}$. However, in the stochastic case we merely obtain that the constraint on $\bar\alpha(\omega)$ is satisfied for ω such that $X^\alpha(\omega)^t=\hat\omega$. Therefore, we typically have $\bar P\notin\mathcal{P}$.

The idea to circumvent this difficulty is that, due to the formulation chosen in the previous section, there exists a neighborhood $B(\hat\omega)$ of $\hat\omega$ such that $\hat P\in\mathcal{P}(t,\omega')$ for all $\omega'\in B(\hat\omega)$. Therefore, the constraint $\bar\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^{\bar\alpha})$ is verified on the preimage of $B(\hat\omega)$ under X^α. In the next lemma, we exploit the separability of Ω to construct a sequence of $\hat P$'s such that the corresponding neighborhoods cover the space Ω, and in Proposition 4.4 below we shall see how to obtain an admissible pasting from this sequence. We denote $\|\omega\|_{[s,t]}:=\sup_{s\le u\le t}|\omega_u|$.

LEMMA 4.3 (Separability). *Let* $0 \le s \le t \le T$ *and* $\bar{\omega} \in \Omega$. *Given* $\varepsilon > 0$, *there exist a sequence* $(\hat{\omega}^i)_{i \ge 1}$ *in* Ω^s, *an* $\mathcal{F}^s_t$*-measurable partition* $(E^i)_{i \ge 1}$ *of* Ω^s *and a sequence* $(P^i)_{i \ge 1}$ *in* $\bar{\mathcal{P}}^t_S$ *such that*:

(i) $\|\omega - \hat{\omega}^i\|_{[s,t]} \le \varepsilon$ *for all* $\omega \in E^i$,
(ii) $P^i \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega)$ *for all* $\omega \in E^i$ *and* $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$,
(iii) $V_t(\bar{\omega} \otimes_s \hat{\omega}^i) \le E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}] + \varepsilon$.

PROOF. Fix $\varepsilon > 0$ and let $\hat{\omega} \in \Omega^s$. By definition of $V_t(\bar{\omega} \otimes_s \hat{\omega})$ there exists $P(\hat{\omega}) \in \mathcal{P}(t, \bar{\omega} \otimes_s \hat{\omega})$ such that

$$V_t(\hat{\omega}) \le E^{P(\hat{\omega})}\big[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}}\big] + \varepsilon.$$

Furthermore, by Lemma 3.5, there exists $\varepsilon(\hat{\omega}) = \varepsilon(t, \bar{\omega} \otimes_s \hat{\omega}, P(\hat{\omega})) > 0$ such that $P(\hat{\omega}) \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega')$ and $\deg(t, \bar{\omega} \otimes_s \omega', P(\hat{\omega})) \ge \varepsilon(\hat{\omega})$ for all $\omega' \in B(\varepsilon(\hat{\omega}), \hat{\omega}) \subseteq \Omega^s$. Here $B(\varepsilon, \hat{\omega}) := \{\omega' \in \Omega^s : \|\hat{\omega} - \hat{\omega}'\|_{[s,t]} < \varepsilon\}$ denotes the open $\|\cdot\|_{[s,t]}$-ball. By replacing $\varepsilon(\hat{\omega})$ with $\varepsilon(\hat{\omega}) \wedge \varepsilon$ we may assume that $\varepsilon(\hat{\omega}) \le \varepsilon$.

As the above holds for all $\hat{\omega} \in \Omega^s$, the collection $\{B(\varepsilon(\hat{\omega}), \hat{\omega}) : \hat{\omega} \in \Omega^s\}$ forms an open cover of Ω^s. Since the (quasi-)metric space $(\Omega^s, \|\cdot\|_{[s,t]})$ is separable and therefore Lindelöf, there exists a countable subcover $(B^i)_{i \ge 1}$, where $B^i := B(\varepsilon(\hat{\omega}^i), \hat{\omega}^i)$. As a $\|\cdot\|_{[s,t]}$-open set, each B^i is $\mathcal{F}^s_t$-measurable and

$$E^1 := B^1, \qquad E^{i+1} := B^{i+1} \setminus \big(E^1 \cup \cdots \cup E^i\big), \qquad i \ge 1,$$

defines a partition of Ω^s. It remains to set $P^i := P(\hat{\omega}^i)$ and note the fact that $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) \ge \varepsilon(\hat{\omega}^i) > 0$ for each $i \ge 1$. $\square$

For $A \in \mathcal{F}^s_T$ we denote $A^{t,\omega} = \{\tilde{\omega} \in \Omega^t : \omega \otimes_t \tilde{\omega} \in A\}$.

PROPOSITION 4.4 (Pasting). *Let* $0 \le s \le t \le T$, $\bar{\omega} \in \Omega$ *and* $P \in \mathcal{P}(s, \bar{\omega})$. *Let* $(E^i)_{0 \le i \le N}$ *be a finite* $\mathcal{F}^s_t$*-measurable partition of* Ω^s. *For* $1 \le i \le N$, *suppose that* $P^i \in \bar{\mathcal{P}}^t_S$ *are such that* $P^i \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega)$ *for all* $\omega \in E^i$ *and such that* $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$. *Then*

$$\bar{P}(A) := P\big(A \cap E^0\big) + \sum_{i=1}^{N} E^P\big[P^i\big(A^{t,\omega}\big)\mathbf{1}_{E^i}(\omega)\big], \qquad A \in \mathcal{F}^s_T,$$

defines an element of $\mathcal{P}(s, \bar{\omega})$. *Furthermore*:

(i) $\bar{P} = P$ *on* $\mathcal{F}^s_t$,
(ii) $\bar{P}^{t,\omega} = P^{t,\omega}$ *for* P*-a.e.* $\omega \in E^0$,
(iii) $\bar{P}^{t,\omega} = P^i$ *for* P*-a.e.* $\omega \in E^i$ *and* $1 \le i \le N$.

PROOF. We first show that $\bar{P} \in \mathcal{P}(s, \bar{\omega})$. The proof that $\bar{P} \in \bar{\mathcal{P}}^s_S$ is the same as in [17], Appendix, Proof of Claim (4.19); the observation made there is that if α, α^i are the $\mathbb{F}^s$-, respectively, $\mathbb{F}^t$-progressively measurable processes such that $P = P^\alpha$ and $P^i = P^{\alpha^i}$, then $\bar{P} = P^{\bar{\alpha}}$ for $\bar{\alpha}$ defined by

$$\bar{\alpha}_u(\omega) := \mathbf{1}_{[s,t)}(u)\alpha_u(\omega) + \mathbf{1}_{[t,T]}(u)\left[\alpha_u(\omega)\mathbf{1}_{E^0}\big(X^\alpha(\omega)\big) + \sum_{i=1}^{N} \alpha_u^i(\omega^t)\mathbf{1}_{E^i}\big(X^\alpha(\omega)\big)\right]$$

for $(u, \omega) \in [s, T] \times \Omega^s$. To show that $\bar{P} \in \mathcal{P}(s, \bar{\omega})$, it remains to check that

$$\hat{a}_u^s(\omega) \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}(\omega) \qquad \text{for } du \times \bar{P}\text{-a.e. } (u, \omega) \in [s, T] \times \Omega^s$$

for some $\delta > 0$. Indeed, this is clear for $s \le u \le t$ since both sides are adapted and $\bar{P} = P$ on $\mathcal{F}_t^s$ by (i), which is proved below. In view of Lemma 4.2, it remains to show that

$$\bar{\alpha}_u(\omega) \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}\big(X^{\bar{\alpha}}(\omega)\big) \qquad \text{for } du \times P_0^s\text{-a.e. } (u, \omega) \in [t, T] \times \Omega^s. \tag{4.2}$$

Let $A^i := \{X^\alpha \in E^i\} \in \mathcal{F}_t^s$ for $0 \le i \le N$. Note that A^i is defined up to a P_0^s-nullset since X^α is defined as an Itô integral under P_0^s. Let $\omega \in A^0$, then $X^\alpha(\omega) \in E^0$ and thus $\bar{\alpha}_u(\omega) = \alpha_u(\omega)$ for $t \le u \le T$. With $\delta^0 := \deg(s, \bar{\omega}, P)$, Lemma 4.2 shows that

$$\bar{\alpha}_u(\omega) = \alpha_u(\omega) \in \operatorname{Int}^{\delta^0} \mathbf{D}_u^{s,\bar{\omega}}\big(X^\alpha(\omega)\big) = \operatorname{Int}^{\delta^0} \mathbf{D}_u^{s,\bar{\omega}}\big(X^{\bar{\alpha}}(\omega)\big)$$
$$\text{for } du \times P_0^s\text{-a.e. } (u, \omega) \in [t, T] \times A^0.$$

Next, consider $1 \le i \le N$ and $\omega^i \in E^i$. By assumption, $P^i \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega^i)$ and

$$\deg(t, \bar{\omega} \otimes_s \omega^i, P^i) \ge \delta^i := \inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0.$$

We set $\delta := \min\{\delta^0, \dots, \delta^N\} > 0$, then Lemma 4.2 yields

$$\alpha_u^i(\tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}_u^{t,\bar{\omega}\otimes_s\omega^i}\big(X^{\alpha^i}(\tilde{\omega})\big) = \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}\big(\omega^i \otimes_t X^{\alpha^i}(\tilde{\omega})\big)$$
$$\text{for } du \times P_0^t\text{-a.e. } (u, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

Now let $\omega \in A^i$ for some $1 \le i \le N$. Applying the previous observation with $\omega^i := X^\alpha(\omega) \in E^i$, we deduce that

$$\bar{\alpha}_u(\omega) = \alpha_u^i(\omega^t) \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}\big(X^\alpha(\omega) \otimes_t X^{\alpha^i}(\omega^t)\big) = \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}\big(X^{\bar{\alpha}}(\omega)\big)$$
$$\text{for } du \times P_0^s\text{-a.e. } (u, \omega) \in [t, T] \times A^i.$$

More precisely, here we have used the following two facts. First, to pass from $du \times P_0^t$-nullsets to $du \times P_0^s$-nullsets, we have used that if $G \subset \Omega^t$ is a P_0^t-nullset, then $P_0^s\{\omega \in \Omega^s : \omega^t \in G\} = P_0^t(G) = 0$ since the canonical process B^s has P_0^s-independent increments. Second, we have used that $\psi(\omega) := X^\alpha(\omega) \otimes_t X^{\alpha^i}(\omega^t) =$

$X^{\bar\alpha}(\omega)$ for $\omega\in A^i$. Indeed, for $s\le u<t$ we have $\psi_u(\omega)=X^\alpha_u(\omega)=X^{\bar\alpha}_u(\omega)$, while for $t\le u\le T$, $\psi_u(\omega)$ equals

$$\,^{(P_0^s)}\!\!\int_s^t \alpha^{1/2}\,dB(\omega)+\,^{(P_0^t)}\!\!\int_t^u (\alpha^i)^{1/2}\,dB^t(\omega^t)=\,^{(P_0^s)}\!\!\int_s^u (\bar\alpha)^{1/2}\,dB(\omega)=X^{\bar\alpha}_u(\omega).$$

As $P_0^s[\bigcup_{i=0}^N A^i]=1$, we have proved (4.2), therefore $\bar P\in\mathcal{P}(s,\bar\omega)$.

It remains to show (i)–(iii). These assertions are fairly standard; we include the proofs for completeness.

(i) Let $A\in\mathcal{F}_t^s$; we show that $\bar P(A)=P(A)$. Indeed, for $\omega\in\Omega$, the question whether $\omega\in A$ depends only on the restriction of ω to $[s,t]$. Therefore,

$$P^i(A^{t,\omega})=P^i\{\tilde\omega:\omega\otimes_t\tilde\omega\in A\}=\mathbf{1}_A(\omega),\qquad 1\le i\le N,$$

and thus $\bar P(A)=\sum_{i=0}^N E^P[\mathbf{1}_{A\cap E^i}]=P(A)$.

(ii), (iii) Let $F\in\mathcal{F}_T^t$; we show that

$$\bar P^{t,\omega}(F)=P^{t,\omega}(F)\mathbf{1}_{E^0}(\omega)+\sum_{i=1}^N P^i(F)\mathbf{1}_{E^i}(\omega),\qquad P\text{-a.s.}$$

Using the definition of conditional expectation and (i), this is equivalent to the following equality for all $\Lambda\in\mathcal{F}_t^s$:

$$\bar P\{\omega\in\Lambda:\omega^t\in F\}=P\{\omega\in\Lambda\cap E^0:\omega^t\in F\}+\sum_{i=1}^N P^i(F)P(\Lambda\cap E^i).$$

For $A:=\{\omega\in\Lambda:\omega^t\in F\}$ we have $A^{t,\omega}=\{\tilde\omega\in F:\omega\otimes_t\tilde\omega\in\Lambda\}$ and since $\Lambda\in\mathcal{F}_t^s$, $A^{t,\omega}$ equals F if $\omega\in\Lambda$ and is empty otherwise. Thus the definition of $\bar P$ yields $\bar P(A)=P(A\cap E^0)+\sum_{i=1}^N E^P[P^i(F)\mathbf{1}_\Lambda(\omega)\mathbf{1}_{E^i}(\omega)]=P(A\cap E^0)+\sum_{i=1}^N P^i(F)P(\Lambda\cap E^i)$ as desired. □

We remark that the above arguments apply also to a countably infinite partition $(E^i)_{i\ge1}$, provided that $\inf_{i\ge1}\inf_{\omega\in E^i}\deg(t,\omega,P^i)>0$. However, this condition is difficult to guarantee. A second observation is that the results of this subsection are based on the regularity property of $\omega\mapsto\mathcal{P}(t,\omega)$ stated in Lemma 3.5, but make no use of the continuity of ξ or the measurability of $V_t(\xi)$.

4.2. *Dynamic programming principle.* We can now prove the key result of this paper. We recall the value function $V_t=V_t(\xi)$ from Definition 3.4 and denote by $\operatorname{ess\,sup}^{(P,\mathcal{F}_s)}$ the essential supremum of a family of $\mathcal{F}_s$-measurable random variables with respect to the collection of $(P,\mathcal{F}_s)$-nullsets.

THEOREM 4.5. *Let $0\le s\le t\le T$. Then*

$$V_s(\omega)=\sup_{P\in\mathcal{P}(s,\omega)}E^P[(V_t)^{s,\omega}]\qquad \textit{for all } \omega\in\Omega. \tag{4.3}$$

With $\mathcal{P}(s,P) := \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_s\}$, *we also have*

$$V_s = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[V_t|\mathcal{F}_s], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P} \tag{4.4}$$

and, *in particular*,

$$V_s = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\xi|\mathcal{F}_s], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P}. \tag{4.5}$$

PROOF. (i) We first show the inequality "$\leq$" in (4.3). Fix $\bar{\omega} \in \Omega$ as well as $P \in \mathcal{P}(s,\bar{\omega})$. Lemma 4.1 shows that $P^{t,\omega} \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega)$ for P-a.e. $\omega \in \Omega^s$, yielding the inequality in

$$\begin{aligned} E^{P^{t,\omega}}\big[(\xi^{s,\bar{\omega}})^{t,\omega}\big] &= E^{P^{t,\omega}}\big[\xi^{t,\bar{\omega}\otimes_s\omega}\big] \\ &\leq \sup_{P' \in \mathcal{P}(t,\bar{\omega}\otimes_s\omega)} E^{P'}\big[\xi^{t,\bar{\omega}\otimes_s\omega}\big] \\ &= V_t(\bar{\omega} \otimes_s \omega) \\ &= V_t^{s,\bar{\omega}}(\omega) \qquad \text{for } P\text{-a.e. } \omega \in \Omega^s, \end{aligned}$$

where $V_t^{s,\bar{\omega}} := (V_t)^{s,\bar{\omega}}$. Since V_t is measurable by Corollary 3.6, we can take $P(d\omega)$-expectations on both sides to obtain that

$$E^P\big[\xi^{s,\bar{\omega}}\big] = E^P\big[E^{P^{t,\omega}}\big[(\xi^{s,\bar{\omega}})^{t,\omega}\big]\big] \leq E^P\big[V_t^{s,\bar{\omega}}\big].$$

Thus taking supremum over $P \in \mathcal{P}(s,\bar{\omega})$ yields the claim.

(ii) We now show the inequality "$\geq$" in (4.3). Fix $\bar{\omega} \in \Omega$ and $P \in \mathcal{P}(s,\bar{\omega})$ and let $\delta > 0$. We start with a preparatory step.

(ii.a) We claim that there exists a $\|\cdot\|_{[s,t]}$-compact set $E \in \mathcal{F}_t^s$ satisfying $P(E) > 1-\delta$ such that the restriction

$$V_t^{s,\bar{\omega}}(\cdot)|_E \qquad \text{is uniformly continuous for } \|\cdot\|_{[s,t]}.$$

In particular, there exists a modulus of continuity $\rho^{(V_t^{s,\bar{\omega}}|E)}$ such that

$$\big|V_t^{s,\bar{\omega}}(\omega) - V_t^{s,\bar{\omega}}(\omega')\big| \leq \rho^{(V_t^{s,\bar{\omega}}|E)}\big(\|\omega-\omega'\|_{[s,t]}\big) \qquad \text{for all } \omega,\omega' \in E.$$

Indeed, since P is a Borel measure on the Polish space Ω_t^s, there exists a compact set $K = K(P,\delta) \subset \Omega_t^s$ such that $P(K) > 1-\delta/2$. As $V_t^{s,\bar{\omega}}$ is $\mathcal{F}_t^s$-measurable (and thus Borel-measurable as a function on Ω_t^s), there exists, by Lusin's theorem, a closed set $\Lambda = \Lambda(P,\delta) \subseteq \Omega_t^s$ such that $P(\Lambda) > 1-\delta/2$ and such that $V_t^{s,\bar{\omega}}|_\Lambda$ is $\|\cdot\|_{[s,t]}$-continuous. Then $E' := K \cap \Lambda \subset \Omega_t^s$ is compact and hence, the restriction of $V_t^{s,\bar{\omega}}$ to E' is even uniformly continuous. It remains to set $E := \{\omega \in \Omega^s : \omega|_{[s,t]} \in E'\}$.

(ii.b) Let $\varepsilon > 0$. We apply Lemma 4.3 to E (instead of Ω^s) and obtain a sequence $(\hat{\omega}^i)$ in E, an $\mathcal{F}_t^s$-measurable partition (E^i) of E and a sequence (P^i) in $\bar{\mathcal{P}}_S^t$ such that:

(a) $\|\omega - \hat{\omega}^i\|_{[s,t]} \leq \varepsilon$ for all $\omega \in E^i$,
(b) $P^i \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega)$ for all $\omega \in E^i$ and $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$,
(c) $V_t(\bar{\omega} \otimes_s \hat{\omega}^i) \leq E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}] + \varepsilon$.

Let $A_N := E^1 \cup \cdots \cup E^N$ for $N \geq 1$. In view of (a)–(c), we can apply Proposition 4.4 to the finite partition $\{A_N^c, E^1, \ldots, E^N\}$ of Ω^s and obtain a measure $\bar{P} = \bar{P}_N \in \mathcal{P}(s, \bar{\omega})$ such that

$$\bar{P} = P \qquad \text{on } \mathcal{F}_t^s \quad \text{and} \quad \bar{P}^{t,\omega} = \begin{cases} P^{t,\omega}, & \text{for } \omega \in A_N^c, \\ P^i, & \text{for } \omega \in E^i, 1 \leq i \leq N. \end{cases}$$

Since ξ is uniformly continuous, we obtain, similar to (3.1), that there exists a modulus of continuity $\rho^{(\xi)}$ such that

$$\left|\xi^{t, \bar{\omega} \otimes_s \omega} - \xi^{t, \bar{\omega} \otimes_s \omega'}\right| \leq \rho^{(\xi)}\left(\|\omega - \omega'\|_{[s,t]}\right).$$

Let $\omega \in E^i \subset \Omega^s$ for some $1 \leq i \leq N$. Then using (a) and (c),

$$\begin{aligned} V_t^{s,\bar{\omega}}(\omega) &\leq V_t^{s,\bar{\omega}}\left(\hat{\omega}^i\right) + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &\leq E^{P^i}\left[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}\right] + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &\leq E^{P^i}\left[\xi^{t, \bar{\omega} \otimes_s \omega}\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &= E^{\bar{P}^{t,\omega}}\left[\xi^{t, \bar{\omega} \otimes_s \omega}\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &= E^{\bar{P}^{t,\omega}}\left[\left(\xi^{s,\bar{\omega}}\right)^{t,\omega}\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &= E^{\bar{P}}\left[\xi^{s,\bar{\omega}} | \mathcal{F}_t^s\right](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \end{aligned}$$

for $\bar{P}$-a.e. (and thus P-a.e.) $\omega \in E^i$. This holds for all $1 \leq i \leq N$. As $P = \bar{P}$ on $\mathcal{F}_t^s$, taking P-expectations yields

$$E^P\left[V_t^{s,\bar{\omega}} \mathbf{1}_{A_N}\right] \leq E^{\bar{P}}\left[\xi^{s,\bar{\omega}} \mathbf{1}_{A_N}\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon).$$

Recall that $\bar{P} = \bar{P}_N$. Using dominated convergence on the left-hand side, and on the right-hand side that $\bar{P}_N(E \setminus A_N) = P(E \setminus A_N) \to 0$ as $N \to \infty$ and that

$$\begin{aligned} E^{\bar{P}_N}\left[\xi^{s,\bar{\omega}} \mathbf{1}_{A_N}\right] &= E^{\bar{P}_N}\left[\xi^{s,\bar{\omega}} \mathbf{1}_E\right] - E^{\bar{P}_N}\left[\xi^{s,\bar{\omega}} \mathbf{1}_{E \setminus A_N}\right] \\ &\leq E^{\bar{P}_N}\left[\xi^{s,\bar{\omega}} \mathbf{1}_E\right] + \|\xi\|_\infty P_N(E \setminus A_N), \end{aligned} \tag{4.6}$$

we conclude that

$$\begin{aligned} E^P\left[V_t^{s,\bar{\omega}} \mathbf{1}_E\right] &\leq \limsup_{N \to \infty} E^{\bar{P}_N}\left[\xi^{s,\bar{\omega}} \mathbf{1}_E\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon) \\ &\leq \sup_{P' \in \mathcal{P}(s, \bar{\omega}, t, P)} E^{P'}\left[\xi^{s,\bar{\omega}} \mathbf{1}_E\right] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar{\omega}}|E)}(\varepsilon), \end{aligned}$$

where $\mathcal{P}(s,\bar{\omega},t,P) := \{P' \in \mathcal{P}(s,\bar{\omega}) : P' = P \text{ on } \mathcal{F}_t^s\}$. As $\varepsilon > 0$ was arbitrary, this shows that

$$E^P\bigl[V_t^{s,\bar{\omega}}\mathbf{1}_E\bigr] \le \sup_{P' \in \mathcal{P}(s,\bar{\omega},t,P)} E^{P'}\bigl[\xi^{s,\bar{\omega}}\mathbf{1}_E\bigr].$$

Finally, since $P'(E) = P(E) > 1-\delta$ for all $P' \in \mathcal{P}(s,\bar{\omega},t,P)$ and $\delta > 0$ was arbitrary, we obtain by an argument similar to (4.6) that

$$E^P\bigl[V_t^{s,\bar{\omega}}\bigr] \le \sup_{P' \in \mathcal{P}(s,\bar{\omega},t,P)} E^{P'}\bigl[\xi^{s,\bar{\omega}}\bigr] \le \sup_{P' \in \mathcal{P}(s,\bar{\omega})} E^{P'}\bigl[\xi^{s,\bar{\omega}}\bigr] = V_s(\bar{\omega}).$$

The claim follows as $P \in \mathcal{P}(s,\bar{\omega})$ was arbitrary. The proof of (4.3) is complete.

(iii) The next step is to prove that

$$V_t \le \mathop{\mathrm{ess\,sup}}_{P' \in \mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[\xi|\mathcal{F}_t], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P}. \tag{4.7}$$

Fix $P \in \mathcal{P}$. We use the previous step (ii) for the special case $s = 0$ and obtain that given $\varepsilon > 0$ there exists for each $N \ge 1$ a measure $\bar{P}_N \in \mathcal{P}(t,P)$ such that

$$V_t(\omega) \le E^{\bar{P}_N}[\xi|\mathcal{F}_t](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t|E)}(\varepsilon) \qquad \text{for } P\text{-a.s. } \omega \in E^1 \cup \cdots \cup E^N.$$

Therefore, since $E = \bigcup_{i\ge1} E^i$, we have

$$V_t(\omega) \le \sup_{N\ge1} E^{\bar{P}_N}[\xi|\mathcal{F}_t](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t|E)}(\varepsilon) \qquad \text{for } P\text{-a.s. } \omega \in E.$$

We recall that the set E depends on δ, but not on ε. Thus, letting $\varepsilon \to 0$ yields

$$\begin{aligned} V_t\mathbf{1}_E &\le \mathop{\mathrm{ess\,sup}}_{P' \in \mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} \bigl(E^{P'}[\xi|\mathcal{F}_t]\mathbf{1}_E\bigr) \\ &= \Bigl(\mathop{\mathrm{ess\,sup}}_{P' \in \mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[\xi|\mathcal{F}_t]\Bigr)\mathbf{1}_E, \qquad P\text{-a.s.}, \end{aligned}$$

where we have used that $E \in \mathcal{F}_t$. In view of $P(E) > 1-\delta$, the claim follows by taking the limit $\delta \to 0$.

(iv) We now prove the inequality "$\le$" in (4.4); we shall reduce this claim to its special case (4.7). Fix $P \in \mathcal{P}$. For any $P' \in \mathcal{P}(s,P)$ we have that $(P')^{t,\omega} \in \mathcal{P}(t,\omega)$ for P'-a.s. $\omega \in \Omega$ by Lemma 4.1. Thus we can infer from (4.3), applied with $s := t$ and $t := T$, that

$$V_t(\omega) \ge E^{(P')^{t,\omega}}\bigl[\xi^{t,\omega}\bigr] = E^{P'}[\xi|\mathcal{F}_t](\omega), \qquad P'\text{-a.s.}$$

and, in particular, that $E^{P'}[V_t|\mathcal{F}_s] \ge E^{P'}[\xi|\mathcal{F}_s]$ P'-a.s. on $\mathcal{F}_s$, hence, also P-a.s. This shows that

$$\mathop{\mathrm{ess\,sup}}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[V_t|\mathcal{F}_s] \ge \mathop{\mathrm{ess\,sup}}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\xi|\mathcal{F}_s], \qquad P\text{-a.s.}$$

But (4.7), applied with s instead of t, yields that the right-hand side P-a.s. dominates V_s. This proves the claim.

(v) It remains to show the inequality "$\geq$" in (4.4). Fix $P \in \mathcal{P}$ and $P' \in \mathcal{P}(s, P)$. Since $(P')^{s,\omega} \in \mathcal{P}(s, \omega)$ for P'-a.s. $\omega \in \Omega$ by Lemma 4.1, (4.3) yields

$$V_s(\omega) \geq E^{(P')^{s,\omega}}\big[V_t^{s,\omega}\big] = E^{P'}[V_t|\mathcal{F}_s](\omega)$$

P'-a.s. on $\mathcal{F}_s$ and hence, also P-a.s. The claim follows as $P' \in \mathcal{P}(s, P)$ was arbitrary. $\square$

5. Extension to the completion. So far, we have studied the value function $V_t = V_t(\xi)$ for $\xi \in \mathrm{UC}_b(\Omega)$. We now set $\mathcal{E}_t(\xi) := V_t$ and extend this operator to a completion of $\mathrm{UC}_b(\Omega)$ by the usual procedure (e.g., Peng [12]). The main result in this section is that the dynamic programming principle carries over to the extension.

Given $p \in [1, \infty)$ and $t \in [0, T]$, we define $L^p_{\mathcal{P}}(\mathcal{F}_t)$ to be the space of $\mathcal{F}_t$-measurable random variables X satisfying

$$\|X\|_{L^p_{\mathcal{P}}} := \sup_{P \in \mathcal{P}} \|X\|_{L^p(P)} < \infty,$$

where $\|X\|^p_{L^p(P)} := E^P[|X|^p]$. More precisely, we take equivalences classes with respect to $\mathcal{P}$-quasi-sure equality so that $L^p_{\mathcal{P}}(\mathcal{F}_t)$ becomes a Banach space. [Two functions are equal $\mathcal{P}$-quasi-surely ($\mathcal{P}$-q.s. for short) if they are equal P-a.s. for all $P \in \mathcal{P}$.] Furthermore,

$$\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t) \text{ is defined as the } \|\cdot\|_{L^p_{\mathcal{P}}}\text{-closure of } \mathrm{UC}_b(\Omega_t) \subseteq L^p_{\mathcal{P}}(\mathcal{F}_t).$$

For brevity, we shall sometimes write $\mathbb{L}^p_{\mathcal{P}}$ for $\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_T)$ and $L^p_{\mathcal{P}}$ for $L^p_{\mathcal{P}}(\mathcal{F}_T)$.

LEMMA 5.1. *Let $p \in [1, \infty)$. The mapping $\mathcal{E}_t$ on $\mathrm{UC}_b(\Omega)$ is 1-Lipschitz for the norm $\|\cdot\|_{L^p_{\mathcal{P}}}$,*

$$\big\|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)\big\|_{L^p_{\mathcal{P}}} \leq \|\xi - \psi\|_{L^p_{\mathcal{P}}} \qquad \text{for all } \xi, \psi \in \mathrm{UC}_b(\Omega).$$

As a consequence, $\mathcal{E}_t$ uniquely extends to a Lipschitz-continuous mapping

$$\mathcal{E}_t : \mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_T) \to L^p_{\mathcal{P}}(\mathcal{F}_t).$$

PROOF. Note that $|\xi - \psi|^p$ is again in $\mathrm{UC}_b(\Omega)$. The definition of $\mathcal{E}_t$ and Jensen's inequality imply that $|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)|^p \leq \mathcal{E}_t(|\xi - \psi|)^p \leq \mathcal{E}_t(|\xi - \psi|^p)$. Therefore,

$$\big\|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)\big\|_{L^p_{\mathcal{P}}} \leq \sup_{P \in \mathcal{P}} E^P\big[\mathcal{E}_t\big(|\xi - \psi|^p\big)\big]^{1/p} = \sup_{P \in \mathcal{P}} E^P\big[|\xi - \psi|^p\big]^{1/p},$$

where the equality is due to (4.3). $\square$

Since we shall use $\mathbb{L}^p_{\mathcal{P}}$ as the domain of $\mathcal{E}_t$, we also give an explicit description of this space. We say that (an equivalence class) $X \in L^1_{\mathcal{P}}$ is *$\mathcal{P}$-quasi uniformly continuous* if X has a representative X' with the property that for all $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $P(G) < \varepsilon$ for all $P \in \mathcal{P}$ and such that the restriction $X'|_{\Omega \setminus G}$ is uniformly continuous. We define $\mathcal{P}$-quasi continuity in an analogous way and denote by $C_b(\Omega)$ the space of bounded continuous functions on Ω. The following is very similar to the results in [4].

PROPOSITION 5.2. *Let $p \in [1, \infty)$. The space $\mathbb{L}^p_{\mathcal{P}}$ consists of all $X \in L^p_{\mathcal{P}}$ such that X is $\mathcal{P}$-quasi uniformly continuous and* $\lim_{n\to\infty} \|X\mathbf{1}_{\{|X|\geq n\}}\|_{L^p_{\mathcal{P}}} = 0$.

If $\mathbf{D}$ is uniformly bounded, then $\mathbb{L}^p_{\mathcal{P}}$ coincides with the $\|\cdot\|_{L^p_{\mathcal{P}}}$-closure of $C_b(\Omega) \subset L^p_{\mathcal{P}}$ and "uniformly continuous" can be replaced by "continuous."

PROOF. For the first part, it suffices to go through the proof of Theorem 25 of [4] and replace continuity by uniform continuity everywhere. The only difference is that one has to use a refined version of Tietze's extension theorem which yields uniformly continuous extensions; cf. Mandelkern [7].

If $\mathbf{D}$ is uniformly bounded, $\mathcal{P}$ is a set of laws of continuous martingales with uniformly bounded quadratic variation density and therefore $\mathcal{P}$ is tight. Together with the aforementioned extension theorem we derive that $C_b(\Omega)$ is contained in $\mathbb{L}^p_{\mathcal{P}}$ and now the result follows from [4], Theorem 25. $\square$

Before extending the dynamic programming principle, we prove the following auxiliary result which shows, in particular, that the essential suprema in Theorem 4.5 can be represented as increasing limits. This is a consequence of a standard pasting argument which involves only controls with the same "history" and hence, there are no problems of admissibility as in Section 4.

LEMMA 5.3. *Let τ be an $\mathbb{F}$-stopping time and $X \in L^1_{\mathcal{P}}(\mathcal{F}_T)$. For each $P \in \mathcal{P}$ there exists a sequence $P_n \in \mathcal{P}(\tau, P)$ such that*

$$\operatorname*{ess\,sup}_{P' \in \mathcal{P}(\tau, P)}{}^{(P,\mathcal{F}_\tau)} E^{P'}[X|\mathcal{F}_\tau] = \lim_{n\to\infty} E^{P_n}[X|\mathcal{F}_\tau], \qquad P\text{-a.s.},$$

where the limit is increasing and $\mathcal{P}(\tau, P) := \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_\tau\}$.

PROOF. It suffices to show that the set $\{E^{P'}[X|\mathcal{F}_\tau] : P' \in \mathcal{P}(\tau, P)\}$ is P-a.s. upward filtering. Indeed, we prove that for $\Lambda \in \mathcal{F}_\tau$ and $P_1, P_2 \in \mathcal{P}(\tau, P)$ there exists $\bar{P} \in \mathcal{P}(\tau, P)$ such that

$$E^{\bar{P}}[X|\mathcal{F}_\tau] = E^{P_1}[X|\mathcal{F}_\tau]\mathbf{1}_\Lambda + E^{P_2}[X|\mathcal{F}_\tau]\mathbf{1}_{\Lambda^c}, \qquad P\text{-a.s.},$$

then the claim follows by letting $\Lambda := \{E^{P_1}[X|\mathcal{F}_\tau] > E^{P_2}[X|\mathcal{F}_\tau]\}$. Similarly as in Proposition 4.4, we define

$$\bar{P}(A) := E^P[P^1(A|\mathcal{F}_\tau)\mathbf{1}_\Lambda + P^2(A|\mathcal{F}_\tau)\mathbf{1}_{\Lambda^c}], \qquad A \in \mathcal{F}_T. \tag{5.1}$$

Let $\alpha, \alpha^1, \alpha^2$ be such that $P^\alpha = P$, $P^{\alpha^1} = P_1$ and $P^{\alpha^2} = P_2$. The fact that $P = P^1 = P^2$ on $\mathcal{F}_\tau$ translates to $\alpha = \alpha^1 = \alpha^2$ $du \times P_0$-a.e. on $[\![0, \tau(X^\alpha)[\![$ and with this observation we have as in Proposition 4.4 that $\bar{P} = P^{\bar{\alpha}} \in \bar{\mathcal{P}}_S$ for the $\mathbb{F}$-progressively measurable process

$$\begin{aligned}\bar{\alpha}_u(\omega) := {} & \mathbf{1}_{[\![0,\tau(X^\alpha)[\![}(u)\alpha_u(\omega) \\ & + \mathbf{1}_{[\![\tau(X^\alpha),T]\!]}(u)\big[\alpha^1_u(\omega)\mathbf{1}_\Lambda\big(X^\alpha(\omega)\big) + \alpha^2_u(\omega)\mathbf{1}_{\Lambda^c}\big(X^\alpha(\omega)\big)\big].\end{aligned}$$

Since $P, P^1, P^2 \in \mathcal{P}$, Lemma 4.2 yields that $\bar{P} \in \mathcal{P}$. Moreover, we have $\bar{P} = P$ on $\mathcal{F}_\tau$ and $\bar{P}^{\tau(\omega),\omega} = P_1^{\tau(\omega),\omega}$ for $\omega \in \Lambda$ and $\bar{P}^{\tau(\omega),\omega} = P_2^{\tau(\omega),\omega}$ for $\omega \in \Lambda^c$. Thus $\bar{P}$ has the required properties. $\square$

We now show that the extension $\mathcal{E}_t$ from Lemma 5.1 again satisfies the dynamic programming principle.

THEOREM 5.4. *Let $0 \le s \le t \le T$ and $X \in \mathbb{L}^1_{\mathcal{P}}$. Then*

$$\mathcal{E}_s(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}\big[\mathcal{E}_t(X)|\mathcal{F}_s\big], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P} \tag{5.2}$$

and, in particular,

$$\mathcal{E}_s(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s], \qquad P\text{-a.s.} \qquad \text{for all } P \in \mathcal{P}. \tag{5.3}$$

PROOF. Fix $P \in \mathcal{P}$. Given $\varepsilon > 0$, there exists $\psi \in \mathrm{UC}_b(\Omega)$ such that

$$\big\|\mathcal{E}_s(X) - \mathcal{E}_s(\psi)\big\|_{L^1_{\mathcal{P}}} \le \|X - \psi\|_{L^1_{\mathcal{P}}} \le \varepsilon.$$

For any $P' \in \mathcal{P}(s, P)$, we also note the trivial identity

$$\begin{aligned} & E^{P'}[X|\mathcal{F}_s] - \mathcal{E}_s(X) \\ & \quad = E^{P'}[X - \psi|\mathcal{F}_s] + \big(E^{P'}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big) \\ & \qquad + \big(\mathcal{E}_s(\psi) - \mathcal{E}_s(X)\big), \qquad P\text{-a.s.}\end{aligned} \tag{5.4}$$

(i) We first prove the inequality "$\le$" in (5.3). By (4.5) and Lemma 5.3 there exists a sequence (P_n) in $\mathcal{P}(s, P)$ such that

$$\mathcal{E}_s(\psi) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\psi|\mathcal{F}_s] = \lim_{n\to\infty} E^{P_n}[\psi|\mathcal{F}_s], \qquad P\text{-a.s.} \tag{5.5}$$

Using (5.4) with $P' := P_n$ and taking $L^1(P)$-norms we find that

$$\begin{aligned} & \big\|E^{P_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\big\|_{L^1(P)} \\ & \quad \le \|X - \psi\|_{L^1(P_n)} + \big\|E^{P_n}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big\|_{L^1(P)} + \big\|\mathcal{E}_s(\psi) - \mathcal{E}_s(X)\big\|_{L^1(P)} \\ & \quad \le \big\|E^{P_n}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big\|_{L^1(P)} + 2\varepsilon.\end{aligned}$$

Now, bounded convergence and (5.5) yield that

$$\limsup_{n\to\infty}\big\|E^{P_n}[X|\mathcal{F}_s]-\mathcal{E}_s(X)\big\|_{L^1(P)}\le 2\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this implies that there is a sequence $\tilde{P}_n\in\mathcal{P}(s,P)$ such that $E^{\tilde{P}_n}[X|\mathcal{F}_s]\to\mathcal{E}_s(X)$ P-a.s. In particular, we have proved the claimed inequality.

(ii) We now complete the proof of (5.3). By Lemma 5.3 we can choose a sequence (P'_n) in $\mathcal{P}(s,P)$ such that

$$\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)}E^{P'}[X|\mathcal{F}_s]=\lim_{n\to\infty}E^{P'_n}[X|\mathcal{F}_s],\qquad P\text{-a.s.}$$

with an increasing limit. Let $A_n:=\{E^{P'_n}[X|\mathcal{F}_s]\ge\mathcal{E}_s(X)\}$. As a result of Step (i), the sets A_n increase to Ω P-a.s. Moreover,

$$0\le\big(E^{P'_n}[X|\mathcal{F}_s]-\mathcal{E}_s(X)\big)\mathbf{1}_{A_n}\nearrow\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)}E^{P'}[X|\mathcal{F}_s]-\mathcal{E}_s(X),\qquad P\text{-a.s.}$$

By (5.4) with $P':=P'_n$ and by the first equality in (5.5), we also have that

$$E^{P'_n}[X|\mathcal{F}_s]-\mathcal{E}_s(X)\le E^{P'_n}[X-\psi|\mathcal{F}_s]+\mathcal{E}_s(\psi)-\mathcal{E}_s(X),\qquad P\text{-a.s.}$$

Taking $L^1(P)$-norms and using monotone convergence, we deduce that

$$\begin{aligned}
&\Big\|\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)}E^{P'}[X|\mathcal{F}_s]-\mathcal{E}_s(X)\Big\|_{L^1(P)}\\
&\qquad=\lim_{n\to\infty}\big\|\big(E^{P'_n}[X|\mathcal{F}_s]-\mathcal{E}_s(X)\big)\mathbf{1}_{A_n}\big\|_{L^1(P)}\\
&\qquad\le\limsup_{n\to\infty}\|X-\psi\|_{L^1(P'_n)}+\big\|\mathcal{E}_s(\psi)-\mathcal{E}_s(X)\big\|_{L^1(P)}\\
&\qquad\le 2\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, this proves (5.3).

(iii) It remains to show (5.2) for a given $P\in\mathcal{P}$. In view of (5.3), it suffices to prove that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)}E^{P'}[X|\mathcal{F}_s]\\
&\quad=\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)}E^{P'}\Big[\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)}E^{P''}[X|\mathcal{F}_t]\Big|\mathcal{F}_s\Big],\qquad P\text{-a.s.}
\end{aligned}$$

The inequality "$\le$" is obtained by considering $P'':=P'\in\mathcal{P}(t,P')$ on the right-hand side. To see the converse inequality, fix $P'\in\mathcal{P}(s,P)$ and choose by Lemma 5.3 a sequence (P''_n) in $\mathcal{P}(t,P')$ such that

$$\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)}E^{P''}[X|\mathcal{F}_t]=\lim_{n\to\infty}E^{P''_n}[X|\mathcal{F}_t],\qquad P'\text{-a.s.}$$

with an increasing limit. Then monotone convergence and the observation that $\mathcal{P}(t, P') \subseteq \mathcal{P}(s, P)$ yield

$$\begin{aligned} E^{P'}\Big[\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)} E^{P''}[X|\mathcal{F}_t]\big|\mathcal{F}_s\Big] &= \lim_{n\to\infty} E^{P''_n}[X|\mathcal{F}_s] \\ &\le \operatorname*{ess\,sup}_{P'''\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'''}[X|\mathcal{F}_s], \qquad P\text{-a.s.} \end{aligned}$$

As $P' \in \mathcal{P}(s, P)$ was arbitrary, this proves the claim. $\square$

We note that (5.3) determines $\mathcal{E}_s(X)$ $\mathcal{P}$-q.s. and can therefore be used as an alternative definition. For most purposes, it is not necessary to go back to the construction. Relation (5.2) expresses the time-consistency property of $\mathcal{E}_t$. With a mild abuse of notation, it can also be stated as

$$\mathcal{E}_s\big(\mathcal{E}_t(X)\big) = \mathcal{E}_s(X), \qquad 0 \le s \le t \le T, X \in \mathbb{L}^1_{\mathcal{P}};$$

indeed, the domain of $\mathcal{E}_s$ has to be slightly enlarged for this statement as in general we do not know whether $\mathcal{E}_t(X) \in \mathbb{L}^1_{\mathcal{P}}$.

We close by summarizing some of the basic properties of $\mathcal{E}_t$.

PROPOSITION 5.5. *Let $X, X' \in \mathbb{L}^p_{\mathcal{P}}$ for some $p \in [1,\infty)$ and let $t \in [0, T]$. Then the following relations hold $\mathcal{P}$-q.s.*:

(i) $\mathcal{E}_t(X) \ge \mathcal{E}_t(X')$ *if* $X \ge X'$,
(ii) $\mathcal{E}_t(X + X') = \mathcal{E}_t(X) + X'$ *if X' is $\mathcal{F}_t$-measurable*,
(iii) $\mathcal{E}_t(\eta X) = \eta^+\mathcal{E}_t(X) + \eta^-\mathcal{E}_t(-X)$ *if η is $\mathcal{F}_t$-measurable and* $\eta X \in \mathbb{L}^1_{\mathcal{P}}$,
(iv) $\mathcal{E}_t(X) - \mathcal{E}_t(X') \le \mathcal{E}_t(X - X')$,
(v) $\mathcal{E}_t(X + X') = \mathcal{E}_t(X) + \mathcal{E}_t(X')$ *if* $\mathcal{E}_t(-X') = -\mathcal{E}_t(X')$,
(vi) $\|\mathcal{E}_t(X) - \mathcal{E}_t(X')\|_{L^p_{\mathcal{P}}} \le \|X - X'\|_{L^p_{\mathcal{P}}}$.

PROOF. Statements (i)–(iv) follow directly from (5.2). The argument for (v) is as in [15], Proposition III.2.8: we have $\mathcal{E}_t(X + X') - \mathcal{E}_t(X') \le \mathcal{E}_t(X)$ by (iv) while $\mathcal{E}(X + X') \ge \mathcal{E}_t(X) - \mathcal{E}_t(-X') = \mathcal{E}_t(X) + \mathcal{E}_t(X')$ by (iv) and the assumption on X'. Of course, (vi) is contained in Lemma 5.1. $\square$

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