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QUENCHED LIMITS FOR THE FLUCTUATIONS OF TRANSIENT RANDOM WALKS IN RANDOM ENVIRONMENT ON $\mathbb{Z}$[1]

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We consider transient nearest-neighbor random walks in random environment on $\mathbb{Z}$. For a set of environments whose probability is converging to 1 as time goes to infinity, we describe the fluctuations of the hitting time of a level n, around its mean, in terms of an explicit function of the environment. Moreover, their limiting law is described using a Poisson point process whose intensity is computed. This result can be considered as the quenched analog of the classical result of Kesten, Kozlov and Spitzer [*Compositio Math.* **30** (1975) 145–168].

1. Introduction. Random walks in a one-dimensional random environment were first introduced in the late sixties as a toy model for DNA replication. The recent development of micromanipulation techniques such as DNA unzipping has raised a renewed interest in this model in genetics and biophysics; cf., for instance, [2] where it is involved in a DNA sequencing procedure. Its mathematical study was initiated by Solomon's 1975 article [20] characterizing the transient and recurrent regimes and proving a strong law of large numbers. A salient feature emerging from this work was the existence of an intermediary regime where the walk is transient with a zero asymptotic speed, in contrast with the case of simple random walks. Shortly after, Kesten, Kozlov and Spitzer [14] precised this result by giving limit laws in the transient regime. When suitably normalized, the (properly centered) hitting time of site n by the random walk was proved to converge toward a stable law as n tends to infinity, which implies a limit law for the random walk itself. In particular, this entailed that the ballistic case (i.e., with positive speed) further decomposes into a diffusive and a subdiffusive regime.

Note that these results, except when they deal with almost sure statements, concern only the annealed behavior. When dealing with applications, what we call the medium is usually fixed during the experiment (e.g., the DNA sequence), and we are naturally led to consider the quenched behavior of the walk. The first results in this direction by Peterson and Zeitouni [17] and Peterson [15] were unfortunately

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negative results, saying that, for almost all environment, the laws of the fluctuations of the walk along the time have several accumulation points. However, it was shown by three of the authors in [9], that, in the case of transient walks having 0 asymptotic speed, one can get some quenched localization result by slightly relaxing the point of view. Namely, for a set of media whose probability converges to 1 as time goes to infinity, the law of the (suitably normalized) position of the walk is getting close to a discrete probability measure whose weights and support are expressed in terms of the environment. In the same spirit, we focus in this work on the quenched fluctuations of hitting times in the case of a general transient subdiffusive random walk in random environment.

Adopting Sinai's now famous description of the medium by a potential [19], we introduce a notion of valley. We then prove that the fluctuations of the hitting time of x around its expectation mainly come from the times spent crossing a very small number of deep potential wells. Since these wells are well apart, their crossing times are almost independent. Moreover, it is shown that the laws of these crossing times are well approximated by exponential variables whose expectations are functions of the environment, functions which in turn happen to be closely related to the classical Kesten renewal series.

Thus, our main result states that the law of the difference of a hitting time with its expectation is close to the law of a sum of centered exponential variables which are weighted by heavy-tailed functions of the environment. This makes it possible to describe their law in terms of a Poisson point process whose intensity is explicitly computed.

To make the exposition clearer, we first present the main results and notation (Section 2) and defer to Section 3 the more precise description of the organization of the paper along with a sketch of the proof.

2. Notation and main results. Let $\omega := (\omega_x, x \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0, 1)$, which stands for the random environment. Let $\Omega := (0, 1)^{\mathbb{Z}}$ and denote by P the distribution of ω (on Ω) and by E the corresponding expectation. Conditioning on ω (i.e., choosing an environment), we define the random walk in random environment $X := (X_t, t \in \mathbb{N})$ starting from $x \in \mathbb{Z}$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: if we denote by $P_{x,\omega}$ the law of the Markov chain $(X_t, t \ge 0)$ defined by $P_{x,\omega}(X_0 = x) = 1$ and

$$P_{x,\omega}(X_{t+1} = z | X_t = y) := \begin{cases} \omega_y, & \text{if } z = y + 1, \\ 1 - \omega_y, & \text{if } z = y - 1, \\ 0, & \text{otherwise,} \end{cases}$$

then the joint law of (ω, X) is $\mathbb{P}_x(\mathrm{d}\omega, \mathrm{d}X) := P_{x,\omega}(\mathrm{d}X)P(\mathrm{d}\omega)$. For convenience, we let $\mathbb{P} := \mathbb{P}_0$. We refer to [21] for an overview of results on random walks in random environment. An important role is played by the sequence of variables

$$\rho_x := \frac{1 - \omega_x}{\omega_x}, \qquad x \in \mathbb{Z}. \tag{2.1}$$

We will make the following assumptions in the rest of this paper.

ASSUMPTIONS.

(a) There exists $0 < \kappa < 2$ for which $E[\rho_0^\kappa] = 1$ and $E[\rho_0^\kappa \log^+ \rho_0] < \infty$;
(b) The distribution of $\log \rho_0$ is nonlattice.

Let us recall here that, under assumptions (a) and (b), Kesten, Kozlov and Spitzer [14] proved a limit theorem toward a stable law of index κ, whose scaling parameter is obtained in [8] for the sub-ballistic case and in [6] for the ballisitic case.

We now introduce the hitting time $\tau(x)$ of site x for the random walk $(X_t, t \geq 0)$,

$$\tau(x) := \inf\{t \geq 0 : X_t = x\}, \qquad x \in \mathbb{Z},$$

and the inter-arrival time $\tau(x, y)$ between sites x and y by

$$\tau(x, y) := \inf\{t \geq 0 : X_{\tau(x)+t} = y\}, \qquad x, y \in \mathbb{Z}.$$

Following Sinai [19] (in the recurrent case), and more recently the study of the case $0 < \kappa < 1$ in [8], we define a notion of potential that enables us to visualize where the random walk spends most of its time.

The potential, denoted by $V = (V(x), x \in \mathbb{Z})$, is a function of the environment ω defined by $V(0) = 0$ and $\rho_x = \mathrm{e}^{V(x)-V(x-1)}$ for every $x \in \mathbb{Z}$, that is,

$$V(x) := \begin{cases} \sum_{1 \leq y \leq x} \log \rho_y, & \text{if } x \geq 1, \\ 0, & \text{if } x = 0, \\ -\sum_{x < y \leq 0}^{0} \log \rho_y, & \text{if } x \leq -1, \end{cases}$$

where the ρ_y's are defined in (2.1). Under hypothesis (a), Jensen's inequality gives $E[\log \rho_0^\kappa] \leq \log E[\rho_0^\kappa] = 0$, and hypothesis (b) excludes the equality case $\rho_0 = 1$ a.s., hence, $E[\log \rho_0] < 0$ and thus $V(x) \to \mp\infty$ a.s. when $x \to \pm\infty$.

The potential is subdivided into pieces, called "excursions," by its weak descending ladder epochs $(e_p)_{p \geq 0}$ defined by $e_0 := 0$ and

$$e_{p+1} := \inf\{x > e_p : V(x) \leq V(e_p)\}, \qquad p \geq 0. \tag{2.2}$$

The number of excursions before $x > 0$ is

$$n(x) := \max\{p : e_p \leq x\}. \tag{2.3}$$

Moreover, let us introduce the constant C_K describing the tail of Kesten's renewal series $R := \sum_{x \geq 0} \rho_0 \cdots \rho_x = \sum_{x \geq 0} \mathrm{e}^{V(x)}$ (see [13]) that plays a crucial role in this work:

$$P(R > t) \sim C_K t^{-\kappa}, \qquad t \to \infty.$$

Note that at least two probabilistic representations are available to compute C_K numerically, which are equally efficient. The first one was obtained by Goldie [10] and a second one was obtained in [7].

Finally, recall the definition of the Wasserstein metric W^1 between probability measures μ, ν on $\mathbb{R}$:

$$W^1(\mu,\nu) := \inf_{\substack{(X,Y):\\ X\sim\mu, Y\sim\nu}} E\big[|X-Y|\big],$$

where the infimum is taken over all couplings (X, Y) with marginals μ and ν. We will denote by $W^1_\omega(X, Y)$ the W^1 distance between the laws of random variables X and Y conditional to ω, that is, between the "quenched distributions" of X and Y.

Let us emphasize that the following results, which describe the quenched law of $\tau(x)$ in terms of the environment, can be stated in different ways, depending on the applications we have in mind, either practical or theoretical. We give two variants and mention that the following results hold for any $\kappa \in (0, 2)$ (so that the sub-ballistic regime is also included, even though a finer study was led for $\kappa \in (0, 1)$ in [9]).

THEOREM 1. *Under assumptions* (a) *and* (b) *we have*

$$W^1_\omega\left(\frac{\tau(x)-E_\omega[\tau(x)]}{x^{1/\kappa}}, \frac{1}{x^{1/\kappa}}\sum_{p=0}^{n(x)-1} E_\omega[\tau(e_p,e_{p+1})]\bar{\mathbf{e}}_p\right) \xrightarrow[x]{P\text{-probability}} 0,$$

with $\bar{\mathbf{e}}_p := \mathbf{e}_p - 1$, *where* $(\mathbf{e}_p)_p$ *are i.i.d. exponential random variables of parameter* 1 *independent of* ω; *the terms* $E_\omega[\tau(e_p, e_{p+1})]$ *can be made explicit* [*see* (4.4) *in the Preliminaries*], *and* $n(x)$ *may be replaced by* $\lfloor \frac{x}{E[e_1]} \rfloor$.

THEOREM 2. *Under assumptions* (a) *and* (b), *for every* $\delta > 0$ *and* $\varepsilon > 0$, *if* x *is large enough*, *we may enlarge the probability space so as to introduce i.i.d. random variables* $\widehat{Z} = (\widehat{Z}_p)_{p\geq 0}$ *such that*

$$P(\widehat{Z}_p > t) \sim 2^\kappa C_U t^{-\kappa}, \qquad t \to \infty, \tag{2.4}$$

where $C_U := E[\rho_0^\kappa \log \rho_0] E[e_1](C_K)^2$, *and*

$$P\left(W^1_{(\omega,\widehat{Z})}\left(\frac{\tau(x)-E_\omega[\tau(x)]}{x^{1/\kappa}}, \frac{1}{x^{1/\kappa}}\sum_{p=1}^{\lfloor x/E[e_1]\rfloor} \widehat{Z}_p\bar{\mathbf{e}}_p\right) > \delta\right) < \varepsilon,$$

with $\bar{\mathbf{e}}_p := \mathbf{e}_p - 1$, *where* $(\mathbf{e}_p)_p$ *are i.i.d. exponential random variables of parameter* 1 *independent of* $\widehat{Z}$, *and* $W^1_{(\omega,\widehat{Z})}(X, Y)$ *denotes the* W^1 *distance between the law of* X *given* ω *and the law of* Y *given* $\widehat{Z}$.

By a classical result (cf. [5], page 152, or [18], page 138, for a general statement), the set $\{n^{-1/\kappa}\widehat{Z}_p | 1 \le p \le n\}$ converges toward a Poisson point process of intensity $2^\kappa C_U \kappa x^{-(\kappa+1)}\,\mathrm{d}x$. It is therefore natural to expect the following corollary.

COROLLARY 1. *Under assumptions* (a) *and* (b) *we have*

$$\mathscr{L}\left(\frac{\tau(x) - E_\omega[\tau(x)]}{x^{1/\kappa}}\bigg|\omega\right) \xrightarrow[x]{W^1} \mathscr{L}\left(\sum_{p=1}^{\infty} \xi_p \bar{\mathbf{e}}_p \bigg| (\xi_p)_{p\ge1}\right) \qquad \textit{in law},$$

where the convergence is the convergence in law on the W^1 metric space of probability measures on $\mathbb{R}$ with finite first moment, and $(\xi_p)_{p\ge1}$ is a Poisson point process of intensity $\lambda\kappa u^{-(\kappa+1)}\,\mathrm{d}u$ where

$$\lambda := \frac{2^\kappa C_U}{E[e_1]} = 2^\kappa \kappa E\big[\rho_0^\kappa \log \rho_0\big] C_K^2,$$

$\bar{\mathbf{e}}_p := \mathbf{e}_p - 1$ where $(\mathbf{e}_p)_p$ are i.i.d. exponential random variables of parameter 1, and the two families are independent of each other. In the case $\kappa = 1$, $\lambda = \frac{2}{E[\rho_0 \log \rho_0]}$, and in the case where ω_0 has a distribution $\mathrm{Beta}(\alpha, \beta)$, *with $0 < \alpha - \beta < 2$,*

$$\lambda = 2^{\alpha-\beta} \frac{\Psi(\alpha) - \Psi(\beta)}{(\alpha-\beta) B(\alpha-\beta, \beta)^2},$$

where Ψ denotes the classical digamma function $\Psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and $B(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

REMARKS.

(1) Since the topology of convergence in W^1-distance is finer than the topology of weak convergence restricted to probability measures with finite first moment, we may replace W^1 by the topology of the convergence in law in the above limit.

(2) For every $\varepsilon > 0$, the mass of $(\varepsilon, +\infty)$ for the measure $\mu = \lambda\kappa \frac{\mathrm{d}u}{u^{\kappa+1}}$ is finite so that it makes sense to consider a decreasing ordering $(\xi^{(k)})_{k\ge1}$ of the Poisson process of intensity μ. A change of variable then shows that

$$\xi^{(p)} = \lambda^{1/\kappa}(\mathbf{f}_1 + \cdots + \mathbf{f}_p)^{-1/\kappa}, \qquad p \ge 1, \tag{2.5}$$

$(\mathbf{f}_p)_p$ being i.i.d. exponential random variables of parameter 1. In particular, by the law of large numbers,

$$\xi^{(p)} \sim \lambda^{1/\kappa} p^{-1/\kappa}, \qquad p \to \infty, \text{ a.s.}, \tag{2.6}$$

hence, $\sum_p (\xi_p)^2 = \sum_p (\xi^{(p)})^2 < \infty$ a.s. Thus, the random series $\sum_p \xi_p \bar{\mathbf{e}}_p$ converges a.s. Furthermore, since its characteristic function is also an absolutely convergent product, its law does not depend on the ordering of the points.

Corollary 1 can be easily deduced from the previous theorems. We give a short proof of this result in Section 9.

While finishing writing the present article, we learned about the article [16] by Peterson and Samorodnitsky giving a result close to Corollary 1. Another article [4] by Dolgopyat and Goldsheid was also submitted, that establishes a similar result (under the ellipticity condition). Our statement, however, gives the convergence in W^1 instead of the weak convergence and especially specifies the *value of the constant* λ that appears in the intensity of the limiting Poisson point process. Furthermore, the three proofs are rather different.

In the following, the constant C stands for a positive constant large enough, whose value can change from line to line.

3. Sketch of the proof. Along the sequence $(e_p)_{p\geq 0}$, hitting times decompose into crossing times of a linear number of excursions,

$$\tau(x) = \sum_{0\leq p<n(x)} \tau(e_p, e_{p+1}) + \tau(e_{n(x)}, x).$$

Although these terms are very correlated, the core of the proof consists of the fact that, as far as fluctuations are concerned, the main contribution only comes from a logarithmic subfamily of asymptotically i.i.d. terms which correspond to so-called "high excursions" (or "deep valleys"). This property (stemming from the fact that the random variables $E_\omega[\tau(e_p, e_{p+1})]$ are heavy-tailed) enables the proof to be divided into two parts detailed below.

3.1. *Exit time from a deep valley* (*Section* 5). The crossing time of the excursion $[e_p, e_{p+1}]$ will mainly depend on its height

$$H_p := \max_{e_p\leq x<e_{p+1}} \big(V(x) - V(e_p)\big).$$

As p grows, the law of the potential V viewed from e_p converges to $P^{\geq 0} := P(\cdot|\forall x \leq 0, V(x) \geq 0)$, and therefore the time $\tau(e_p, e_{p+1})$ converges in law to $\tau(e_1)$ under $P^{\geq 0}$ which we have now to study. A classical Markov chain computation gives (cf. Section 4.2)

$$E_\omega\big[\tau(e_1)\big] = \sum_{0\leq y<e_1} \sum_{x\leq y} (2 - \mathbf{1}_{\{x=y\}}) \mathrm{e}^{V(y)-V(x)}.$$

When $H := H_0$ is large, factorizing by the largest term $2\mathrm{e}^H$ leads to

$$E_\omega\big[\tau(e_1)\big] \simeq 2\mathrm{e}^H \sum_x \mathrm{e}^{-V(x)} \sum_y \mathrm{e}^{-(H-V(y))},$$

where in the sums the significant terms are those indexed by values x close to 0 and values y close to T_H; cf. Figure 1. In particular, we have

$$E_\omega\big[\tau(e_1)\big] \simeq 2\mathrm{e}^H M_1 M_2,$$

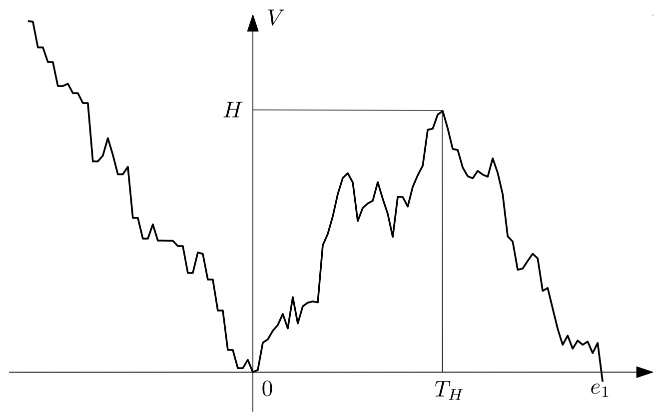

FIG. 1. *Height of an excursion.*

where M_1, M_2 are defined by (4.12). Due to the "locality" of M_1 and M_2, a key fact from [7] is that, when H is large, M_1, M_2 and H are asymptotically independent and M_1, M_2 have the same law. Now, Iglehart's tail estimate on e^H [see (4.10)] yields

$$P^{\geq 0}\big(E_\omega[\tau(e_1)] \geq t\big) \sim 2^\kappa C_I E\big[M^\kappa\big]^2 t^{-\kappa}, \qquad t \to \infty,$$

where C_I is given by (4.11). This is an important result of [7], rephrased in Lemma 2.

To complete the description of the law of the crossing time of a "high excursion," we furthermore prove in Section 5 that, for large H, the law of $\tau(e_1)$, given ω, is close to an exponential law with mean $E_\omega[\tau(e_1)]$. This follows from the fact that the number of returns to 0 before reaching e_1 follows a geometric law.

3.2. *Deep and shallow valleys* (*Sections* 6 *and* 7). As mentioned at the beginning of the section, we try to focus the study on the crossing times of high excursions. To this aim, we introduce a critical height h_n, adapted to the space scale n, defined by

$$h_n := \frac{1}{\kappa}\log n - \log\log n.$$

Then, let $(\sigma(i))_{i\geq 1}$ be the sequence of the indices of the successive excursions whose heights are greater than h_n. More precisely,

$$\sigma(1) := \inf\{p \geq 0 : H_p \geq h_n\},$$
$$\sigma(i+1) := \inf\big\{p > \sigma(i) : H_p \geq h_n\big\}, \qquad i \geq 1.$$

The *high excursions* (see Figure 2) are defined as the restriction of the potential to $[b_i, d_i]$, where

$$b_i := e_{\sigma(i)}, \qquad d_i := e_{\sigma(i)+1}.$$

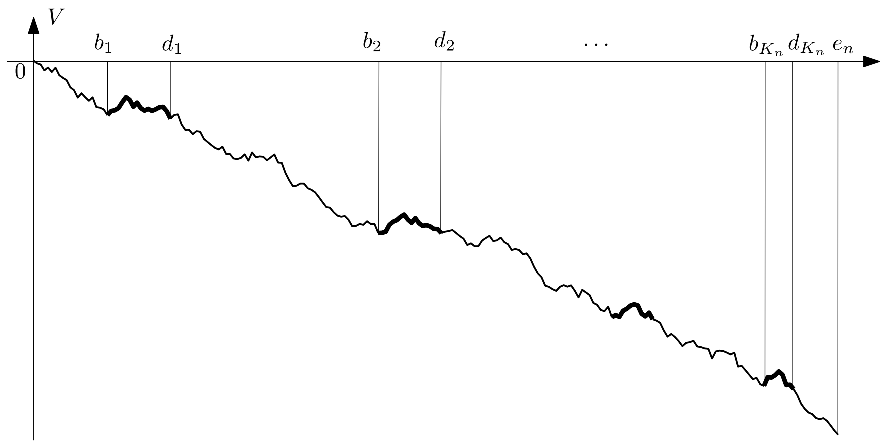

FIG. 2. *High excursions* (*in bold*) *among the n first excursions*.

Note that by Iglehart's estimate, the probability $P(H \geq h_n)$ is asymptotically equal to $C_I e^{-\kappa h_n}$, hence, the number of high excursions among the n first ones,

$$K_n := \#\{0 \leq i \leq n-1 : H_i \geq h_n\},$$

is of order $(\log n)^\kappa$.

It turns out that the crossing time $\tau(b_i, d_i)$ involves mainly the environment between a_i and d_i where a_i is defined as

$$a_i := e_{\sigma(i)-D_n}$$

and D_n is chosen in such a way that $V(a_i) - V(b_i)$ is slightly greater than h_n, that is,

$$D_n := \left\lceil \frac{1+\gamma}{A\kappa} \log n \right\rceil, \tag{3.1}$$

$\gamma > 0$ being arbitrary and A being equal to $E[-V(e_1)]$ if this expectation is finite, and otherwise being an arbitrary positive real number. The *deep valleys* are defined as the restriction of the potential to $[a_i, d_i]$.

We successively prove that:

(1) deep valleys are asymptotically disjoint and their exit times $\tau(b_i, d_i)$ are asymptotically i.i.d. (Section 6);

(2) the contribution to fluctuations of the crossing times of low excursions is negligible (Section 7).

This second point constitutes a novelty with respect to previous works in that the contribution of the crossing times of the numerous small excursions is not negligible with respect to $\tau(x)$ (for $1 \leq \kappa < 2$) but only their fluctuations are, and for this reason we have to control their covariances.

The behavior summarized above is emphasized in the following formulation which lies at the core of the proof: under assumptions (a) and (b),

$$W_\omega^1\left(\frac{\tau(x)-E_\omega[\tau(x)]}{x^{1/\kappa}}, \frac{1}{x^{1/\kappa}}\sum_{i=1}^{K(x)} E_\omega[\tau(b_i,d_i)]\bar{\mathbf{e}}_i\right) \xrightarrow[x]{P\text{-probability}} 0, \tag{3.2}$$

with $\bar{\mathbf{e}}_i := \mathbf{e}_i - 1$ where $(\mathbf{e}_i)_i$ are i.i.d. exponential random variables of parameter 1, independent of ω, and $K(x) := K_{n(x)}$ where $n(x)$ is defined by (2.3).

Note that the terms $E_\omega[\tau(b_i,d_i)]$ can be made explicit [see (4.4)]. Hence, this formula is well suited to derive practical information about $\tau(x)$ which, for instance, appears as an unzipping time in [2].

4. Preliminaries. This section is divided into three independent parts. The first part quickly recalls a stationarity property of the potential when suitably conditioned on $\mathbb{Z}_-$, which is used throughout the paper. The second one recalls usual formulas about random walks in a one-dimensional potential. Finally, the last part adapts the main results from [7] in the present context.

4.1. *Environment on the left of* 0. It will be convenient to extend the sequence $(e_p)_{p\ge 0}$ to negative indices by letting

$$e_{p-1} := \sup\{x < e_p : \forall y < x, V(y) \ge V(x)\}, \qquad p \le 0. \tag{4.1}$$

The structure of the sequence $(e_p)_{p\in\mathbb{Z}}$ will be better understood after Lemma 1.

We accordingly extend the sequence $(H_p)_{p\ge 0}$ of heights

$$H_p := \max_{e_p \le x \le e_{p+1}} \big(V(x) - V(e_p)\big), \qquad p \in \mathbb{Z}.$$

Note that the excursions $(V(e_p + x) - V(e_p))_{0\le x < e_{p+1}-e_p}$, $p \ge 0$, are i.i.d. Also, the intervals $(e_p, e_{p+1}]$, $p \in \mathbb{Z}$, stand for the excursions of the potential above its past minimum, provided $V(x) \ge 0$ when $x \le 0$.

By definition, the distribution of the environment is translation invariant. However, the distribution of the "environment seen from e_p," that is, of $(\omega_{e_p+x})_{x\in\mathbb{Z}}$, depends on $p \in \mathbb{Z}$. When suitably conditioning the environment on $\mathbb{Z}_-$, this problem vanishes.

Let us define the conditioned probabilities

$$P^{\ge 0} := P\big(\cdot|\forall x \le 0, V(x) \ge 0\big) \quad \text{and} \quad \mathbb{P}^{\ge 0} := P_\omega \times P^{\ge 0}(d\omega).$$

Then the definition of e_p for $p < 0$ classically implies the following useful property.

LEMMA 1. *Under $P^{\ge 0}$, the sequence $(V(e_p + x) - V(e_p))_{0\le x\le e_{p+1}-e_p}$, $p \in \mathbb{Z}$, of excursions is i.i.d. In particular, the sequence $(V(e_p + x) - V(e_p))_{x\in\mathbb{Z}}$ of potentials* [*and thus the sequence $(\omega_{e_p+x})_{x\in\mathbb{Z}}$, $p \in \mathbb{Z}$, of environments*] *is stationary under $P^{\ge 0}$.*

4.2. *Quenched formulas*. We recall here a few Markov chain formulas that are of repeated use in the paper.

Quenched exit probabilities. For any $a \le x \le b$ (see [21], formula (2.1.4))

$$P_{x,\omega}\big(\tau(b) < \tau(a)\big) = \frac{\sum_{a \le y < x} \mathrm{e}^{V(y)}}{\sum_{a \le y < b} \mathrm{e}^{V(y)}}. \tag{4.2}$$

In particular,

$$P_{a+1,\omega}\big(\tau(a) = \infty\big) = \Big(\sum_{y \ge a} \mathrm{e}^{V(y)-V(a)}\Big)^{-1}. \tag{4.3}$$

Thus, $P_{0,\omega}(\tau(1) = \infty) = (\sum_{x \le 0} \mathrm{e}^{V(x)})^{-1} = 0$, P-a.s. because $V(x) \to +\infty$ a.s. when $x \to -\infty$, and $P_{1,\omega}(\tau(0) = \infty) = (\sum_{x \ge 0} \mathrm{e}^{V(x)})^{-1} > 0$, P-a.s. by the root test (using $E[\log \rho_0] < 0$). This means that X is transient to $+\infty$, $\mathbb{P}$-a.s.

Quenched expectation. For any $a < b$, P-a.s. (cf. [21])

$$E_{a,\omega}\big[\tau(b)\big] = \sum_{a \le y < b} \sum_{x \le y} \alpha_{xy} \mathrm{e}^{V(y)-V(x)}, \tag{4.4}$$

where $\alpha_{xy} = 2$ if $x < y$, and $\alpha_{yy} = 1$. Thus, we have

$$E_{a,\omega}\big[\tau(b)\big] \le 2 \sum_{a \le y < b} \sum_{x \le y} \mathrm{e}^{V(y)-V(x)} \tag{4.5}$$

and, in particular,

$$E_{a,\omega}\big[\tau(a+1)\big] = 1 + 2 \sum_{x < a} \mathrm{e}^{V(a)-V(x)} \le 2 \sum_{x \le a} \mathrm{e}^{V(a)-V(x)}. \tag{4.6}$$

Quenched variance. For any $a < b$, P-a.s. (cf. [1] or [11])

$$\begin{aligned} \mathrm{Var}_{a,\omega}\big(\tau(b)\big) = 4 \sum_{a \le y < b} \sum_{x \le y} \mathrm{e}^{V(y)-V(x)}\big(1 + \mathrm{e}^{V(x-1)-V(x)}\big) \\ \times \Big(\sum_{z < x} \mathrm{e}^{V(x)-V(z)}\Big)^2, \end{aligned} \tag{4.7}$$

from where we get, after expansion, change of indices and addition of a few terms,

$$\mathrm{Var}_{a,\omega}\big(\tau(b)\big) \le 16 \sum_{a \le y < b} \sum_{z' \le z \le x \le y} \mathrm{e}^{V(y)+V(x)-V(z)-V(z')}. \tag{4.8}$$

4.3. *Renewal estimates*. In this section we recall and adapt results from [7], which are very useful to finely bound the expectations of exponential functionals of the potential.

Let us first observe that hypothesis (a) implies that e_1 is exponentially integrable. Indeed, for all $x \in \mathbb{N}$, for any $\lambda > 0$, $P(e_1 > x) \le P(V(x) > 0) =$

$P(\mathrm{e}^{\lambda V(x)} > 1) \le E[\mathrm{e}^{\lambda V(x)}] = E[\rho_0^\lambda]^x$, and $E[\rho_0^\lambda] < 1$ for any $0 < \lambda < \kappa$ by convexity of $s \mapsto E[\rho_0^s]$.

Let $R_- := \sum_{x \le 0} \mathrm{e}^{-V(x)}$. Then, Lemma 3.2 from [7] proves that

$$E^{\ge 0}[R_-] < \infty, \tag{4.9}$$

and that more generally all the moments of R_- are finite under $P^{\ge 0}$.

The study of "high excursions" involves the following key result of Iglehart [12] which gives the tail probability of H (recall $H := H_0$), namely,

$$P(H \ge h) \sim C_I \mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{4.10}$$

where

$$C_I := \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}. \tag{4.11}$$

Let us define

$$T_H := \min\{x \ge 0 : V(x) = H\}$$

and

$$M_1 := \sum_{x < T_H} \mathrm{e}^{-V(x)}, \qquad M_2 := \sum_{0 \le x < e_1} \mathrm{e}^{V(x) - H}. \tag{4.12}$$

Let $Z := M_1 M_2 \mathrm{e}^H$. Theorem 2.2 (together with Remark A.1) of [7] proves that

$$P^{\ge 0}(Z > t, H = S) \sim C_U t^{-\kappa}, \qquad t \to \infty, \tag{4.13}$$

where C_U was defined after (2.4); cf. also the sketch in Section 4.3 for heuristics. While the condition $\{H = S\}$ was natural in the context of [7], we will need to remark that we may actually drop it.

LEMMA 2. *We have*

$$P^{\ge 0}(Z > t) \sim C_U t^{-\kappa}, \qquad t \to \infty.$$

The proof of this lemma is postponed to Appendix A.1. We will often need moments involving

$$M_1' := \sum_{x < e_1} \mathrm{e}^{-V(x)},$$

instead of $M_1 (\le M_1')$. The next result is an adaptation of Lemma 4.1 from [7] to the present situation, together with (4.10), with a novelty coming from the difference between M_1' and M_1.

LEMMA 3. *For any $\alpha, \beta, \gamma \geq 0$, there is a constant C such that, for large $h > 0$,*

$$E^{\geq 0}\big[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} | H < h\big] \leq \begin{cases} C, & \text{if } \gamma < \kappa, \\ Ch, & \text{if } \gamma = \kappa, \\ C\mathrm{e}^{(\gamma-\kappa)h}, & \text{if } \gamma > \kappa, \end{cases} \tag{4.14}$$

and, if $\gamma < \kappa$,

$$E^{\geq 0}\big[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} | H \geq h\big] \leq C\mathrm{e}^{\gamma h}. \tag{4.15}$$

The proof of this lemma is technical and therefore postponed to Appendix A.1. Let us now give an important application of Lemma 3.

LEMMA 4. *We have, for all $h > 0$, if $0 < \kappa < 1$,*

$$E^{\geq 0}\big[E_\omega[\tau(e_1)]\mathbf{1}_{\{H<h\}}\big] \leq C\mathrm{e}^{(1-\kappa)h} \tag{4.16}$$

and, if $0 < \kappa < 2$,

$$E^{\geq 0}\big[E_\omega[\tau(e_1)]^2\mathbf{1}_{\{H<h\}}\big] \leq C\mathrm{e}^{(2-\kappa)h}. \tag{4.17}$$

PROOF. Since, by (4.5), we have $E_\omega[\tau(e_1)] \leq 2M_1' M_2 \mathrm{e}^H$, the result follows directly from Lemma 3. $\square$

5. Exit time from a deep valley. This section aims at proving that the quenched law of the crossing time

$$\tau := \tau(e_1)$$

of an excursion is close to that of $E_\omega[\tau]\mathbf{e}$, where $\mathbf{e}$ is an exponential random variable independent of ω, when the height H of the excursion is high. Let us give a precise statement. Define the critical height

$$\mathfrak{h}_t := \log t - \log\log t, \qquad t \geq \mathrm{e}^{\mathrm{e}}.$$

Heuristics suggest (and it would follow from later results) that when $H > \mathfrak{h}_t$, τ is on the order of $\mathrm{e}^H > \frac{t}{\log t}$. Proposition 1 shows that the distance between τ and $E_\omega[\tau]\mathbf{e}$ (for a suitable coupling) is no larger than $t^\beta \ll \frac{t}{\log t}$ in quenched average when $H > \mathfrak{h}_t$, in agreement with our aim.

PROPOSITION 1. *We may enlarge the probability space in order to introduce an exponential random variable $\mathbf{e}$ of parameter 1, independent of ω, such that, for some $\beta < 1$, as $t \to \infty$,*

$$P^{\geq 0}\big(E_\omega\big[|\tau - E_\omega[\tau]\mathbf{e}|\big] > t^\beta, H \geq \mathfrak{h}_t\big) = o\big(t^{-\kappa}\big). \tag{5.1}$$

This proposition can equivalently be phrased, using (4.10), as

$$P^{\geq 0}\big(W_\omega^1(\tau, E_\omega[\tau]\mathbf{e}) > t^\beta | H \geq \mathfrak{h}_t\big) = o\bigg(\frac{1}{\log t}\bigg),$$

where $\mathbf{e}$ is an exponential random variable of parameter 1 independent of ω.

5.1. *"Good" environments.* The proof relies on a precise control of the geometry of a typical valley, namely, that it is not too wide and smooth enough. Let us define the maximal "increments" of the potential in a window $[x, y]$ by

$$V^{\uparrow}(x, y) := \max_{x \le u \le v \le y} \big(V(v) - V(u)\big), \qquad x < y,$$

$$V^{\downarrow}(x, y) := \min_{x \le u \le v \le y} \big(V(v) - V(u)\big), \qquad x < y.$$

Then, we introduce the following events:

$$\Omega_t^{(1)} := \{e_1 \le C \log t\},$$

$$\Omega_t^{(2)} := \big\{\max\{-V^{\downarrow}(0, T_H), V^{\uparrow}(T_H, e_1)\} \le \alpha \log t\big\},$$

$$\Omega_t^{(3)} := \big\{R^- \le (\log t)^4 t^{\alpha}\big\},$$

where $\max\{0, 1-\kappa\} < \alpha < \min\{1, 2-\kappa\}$ is arbitrary, and R^- is defined by

$$R^- := \sum_{x=-\infty}^{-1} \left(1 + 2 \sum_{y=x+2}^{0} \mathrm{e}^{V(y)-V(x+1)}\right)\left(\mathrm{e}^{-V(x+1)} + 2 \sum_{y=-\infty}^{x-1} \mathrm{e}^{-V(y+1)}\right).$$

We define the set of "good" environments at time t by

$$\Omega_t := \Omega_t^{(1)} \cap \Omega_t^{(2)} \cap \Omega_t^{(3)}. \tag{5.2}$$

By the following result, "good" environments are asymptotically typical on $\{H \ge \mathfrak{h}_t\}$.

LEMMA 5. *The event Ω_t satisfies*

$$P^{\ge 0}\big(\Omega_t^c, H \ge \mathfrak{h}_t\big) = o\big(t^{-\kappa}\big), \qquad t \to \infty.$$

The proof of this result is easy but technical and therefore postponed to Appendix A.2.

5.2. *Preliminary results.* In order to finely estimate the time spent in a deep valley, we decompose the passage from 0 to e_1 into the sum of a random geometrically distributed number, denoted by N, of unsuccessful attempts to reach e_1 from 0 (i.e., excursions of the particle from 0 to 0 which do not hit e_1), followed by a successful attempt. More precisely, N is a geometrically distributed random variable with parameter $1 - p$ satisfying

$$1 - p = \frac{\omega_0}{\sum_{x=0}^{e_1-1} \mathrm{e}^{V(x)}} = \frac{\omega_0}{M_2 \mathrm{e}^{H}}, \tag{5.3}$$

and we can write $\tau(e_1) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the durations of the successive i.i.d. failures and G that of the first success. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two h-processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (this potential is denoted by $\widehat{V}$ in [8], page 2494) and the other to the conditioning on a success (denoted by $\bar{V}$ in [8], page 2497). Note that this approach was first introduced by three of the authors in [8] to estimate the quenched Laplace transform of the occupation time of a deep valley in the case $0 < \kappa < 1$. We refer to [8] for more details on these two h-processes. Moreover, using the properties of "good" environments introduced above, we can prove the following useful lemmata, whose proofs are postponed to Appendix A.2.

LEMMA 6. *For all $t \ge 1$, we have on Ω_t,*

$$\mathrm{Var}_\omega(F) \le C(\log t)^4 t^\alpha, \tag{5.4}$$

$$M_2 \le C \log t, \tag{5.5}$$

$$|\widehat{M}_1 - M_1| \le o(t^{-\delta}) M_1, \tag{5.6}$$

with $\delta \in (0, 1-\alpha)$ and where $\widehat{M}_1$ is defined by the relation $E_\omega[F] = 2\omega_0 \widehat{M}_1$.

LEMMA 7. *For all $t \ge 1$, we have on Ω_t,*

$$E_\omega[G] \le C(\log t)^4 t^\alpha.$$

5.3. *Definition of the coupling.* We recall here the coupling from [9] between the quenched distribution of the random walk before time τ and an exponential random variable $\mathbf{e}$ of parameter 1 independent of ω. Given ω and $\mathbf{e}$, let us define

$$N := \left\lfloor -\frac{1}{\log(1 - p(\omega))} \mathbf{e} \right\rfloor,$$

where $p(\omega) = P_{0,\omega}(\tau(0) < \tau(e_1))$; cf. (5.3). Note that, conditionally on ω, N is a geometric random variable of parameter $1 - p$, just like the number of returns to 0 before the walk reaches e_1.

Given ω and $\mathbf{e}$ (and hence, N), the random walk is sampled as usual as a Markov chain, except that the number of returns to 0 is conditioned on being equal to N, which amounts to saying that when the walk reaches 0 for the first N times, it is conditioned on coming back to 0 before reaching e_1 (this is still a Markov chain, namely, the h-process associated to $\widehat{V}$; see Section 5.2), while on the $(N + 1)$th visit of 0 it is conditioned on reaching e_1 first (this is the h-process associated to $\bar{V}$). Due to the definition of N, the distribution of the walk given ω only is $P_{0,\omega}$.

5.4. *Proof of Proposition* 1. We consider the same decomposition as in Section 5.2, that is, $\tau = F_1 + \cdots + F_N + G$. By Wald identity, $E_\omega[\tau] = E_\omega[N]E_\omega[F] + E_\omega[G]$. Thus, we have

$$\begin{aligned}\left|\tau - E_\omega[\tau]\mathbf{e}\right| \le \left|F_1 + \cdots + F_N - NE_\omega[F]\right| + E_\omega[F]\left|N - E_\omega[N]\mathbf{e}\right| \\ + G + E_\omega[G]\mathbf{e}.\end{aligned}$$

Let us consider each term, starting with the last two (with same P_ω-expectation). If we choose β such that $\alpha < \beta < 1$, then by Lemmas 7 and 5 we have, for large t,

$$P^{\ge 0}\left(E_\omega[G] \ge \frac{t^\beta}{4}, H \ge \mathfrak{h}_t\right) \le P^{\ge 0}\left((\Omega_t)^c, H \ge \mathfrak{h}_t\right) = o\left(t^{-\kappa}\right). \tag{5.7}$$

We turn to the first one. Conditioning first on N [which is independent of $(F_i)_i$] and then applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}E_\omega\left[\left|F_1 + \cdots + F_N - NE_\omega[F]\right|\right] &\le E_\omega\left[\mathrm{Var}_\omega(F_1 + \cdots + F_N|N)^{1/2}\right] \\ &= E_\omega\left[N^{1/2}\right]\mathrm{Var}_\omega(F)^{1/2}.\end{aligned}$$

Furthermore, $E_\omega[N^{1/2}] \le E_\omega[N]^{1/2} = ((1-p)^{-1} - 1)^{1/2} \le (M_2)^{1/2}\mathrm{e}^{H/2}\omega_0^{-1/2}$ and $\omega_0 \ge \frac{1}{2}$, $P^{\ge 0}$-almost surely. Thus, using Lemma 6 to bound $\mathrm{Var}_\omega(F)$, we get

$$\begin{aligned}&P^{\ge 0}\left(E_\omega\left[\left|F_1 + \cdots + F_N - NE_\omega[F]\right|\right] > \frac{t^\beta}{4}, H \ge \mathfrak{h}_t\right) \\ &\quad\le P^{\ge 0}\left((\Omega_t)^c, H \ge \mathfrak{h}_t\right) + P^{\ge 0}\left((M_2)^{1/2}\mathrm{e}^{H/2} \ge \frac{t^{\beta-\alpha/2}}{C(\log t)^2}, H \ge \mathfrak{h}_t\right).\end{aligned}$$

As before, the first term is $o(t^{-\kappa})$. And the second one is less than

$$\begin{aligned}&P\left(M_2 \ge (\log t)^2, H \ge \mathfrak{h}_t\right) + P\left(\mathrm{e}^{H/2} \ge \frac{t^{\beta-\alpha/2}}{C(\log t)^3}\right) \\ &\quad\le \frac{P(H \ge \mathfrak{h}_t)}{(\log t)^2}E[M_2|H \ge \mathfrak{h}_t] + P\left(\mathrm{e}^H \ge \frac{t^{2\beta-\alpha}}{C^2(\log t)^6}\right).\end{aligned}$$

Each term is $o(t^{-\kappa})$ if we additionally impose $\frac{1+\alpha}{2} < \beta < 1$, due to (4.15) and (4.10).

Finally, we have

$$\begin{aligned}\left|N - E_\omega[N]\mathbf{e}\right| &= \left|\left\lfloor \frac{1}{-\log(1-p)}\mathbf{e}\right\rfloor - \left(\frac{1}{p} - 1\right)\mathbf{e}\right| \\ &\le \left(1 + \left|-\frac{1}{\log(1-p)} - \frac{1}{p}\right|\right)\mathbf{e} + 1,\end{aligned}$$

and the function $x \mapsto -\frac{1}{\log(1-x)} - \frac{1}{x}$ extends continuously on $[0, 1]$ and is thus bounded by a constant C, hence,

$$\begin{aligned}
&P^{\geq 0}\Big(E_\omega[F]E_\omega\big[|N - E_\omega[N]\mathbf{e}|\big] \geq \frac{t^\beta}{4}, H \geq \mathbb{P}_t\Big) \\
&\quad \leq P^{\geq 0}\Big(E_\omega[F] \geq \frac{t^\beta}{4C},\ H \geq \mathfrak{h}_t\Big) \\
&\quad \leq P^{\geq 0}\big((\Omega_t)^c, H \geq \mathfrak{h}_t\big) + \frac{8CP(H \geq \mathfrak{h}_t)}{t^\beta} E^{\geq 0}[M_1 | H \geq \mathfrak{h}_t]
\end{aligned}$$

for large t, due to (5.6), recalling that $E_\omega[F] = 2\omega_0 \widehat{M}_1$ (see Lemma 6). We conclude as before that this is negligible compared to $t^{-\kappa}$.

Therefore, gathering all these estimates gives Proposition 1.

6. Independence of the deep valleys. The independence between deep valleys goes through imposing these valleys to be disjoint (i.e., $a_i > d_{i-1}$ for all i) and neglecting the time spent on the left of a valley while it is being crossed (i.e., the time spent on the left of a_i before d_i is reached).

NB. All the results and proofs from this section hold for any parameter $\kappa > 0$.

For any integers x, y, z, let us define

$$\widetilde{\tau}^{(z)}(x, y) := \#\big\{\tau(x) \leq k \leq \tau(y) : X_k \leq z\big\},$$

the time spent on the left of z between the first visit to x and the first visit to y, and

$$\widetilde{\tau}^{(z)} := \#\big\{k \geq \tau(z) : X_k \leq z\big\},$$

the total time spent on the left of z after the first visit to z.

Let us consider the event

$$\mathrm{NO}(n) := \{0 < a_1\} \cap \bigcap_{i=1}^{K_n - 1} \{d_i < a_{i+1}\},$$

which means that the large valleys before e_n lie entirely on $\mathbb{Z}_+$ and do not overlap. The following two propositions will enable us to reduce to i.i.d. deep valleys.

PROPOSITION 2. *We have*

$$P\big(\mathrm{NO}(n)\big) \xrightarrow[n]{} 1.$$

PROOF. Choose $\varepsilon > 0$ and define the event

$$A_K(n) := \big\{K_n \leq (1+\varepsilon)C_I(\log n)^\kappa\big\}.$$

Since K_n is a binomial random variable of mean $nq_n \sim_n C_I(\log n)^\kappa$, it follows from the law of large numbers that $P(A_K(n))$ converges to 1 as $n \to \infty$. On the other hand, if the event $\mathrm{NO}(n)^c$ occurs, then there exists $1 \le i \le K_n$ such that there is at least one high excursion among the first D_n excursions to the right of d_{i-1} (with $d_0 = 0$). Thus,

$$\begin{aligned} P\big(\mathrm{NO}(n)^c\big) &\le P\big(A_K(n)^c\big) + (1+\varepsilon)C_I(\log n)^\kappa\big(1-(1-q_n)^{D_n}\big) \\ &\le o(1) + (1+\varepsilon)C_I(\log n)^\kappa q_n D_n = o(1). \end{aligned}$$

Indeed, for any $0 < u < 1$ and $\alpha > 0$, we have $1-(1-u)^\alpha \le \alpha u$ by concavity of $u \mapsto 1-(1-u)^\alpha$. $\square$

PROPOSITION 3. *Under $P^{\ge 0}$,*

$$\frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n} E_\omega\big[\widetilde{\tau}^{(a_i)}(b_i,d_i)\big] = \frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1} E_\omega\big[\widetilde{\tau}^{(e_{p-D_n})}(e_p,e_{p+1})\big]\mathbf{1}_{\{H_p \ge h_n\}} \xrightarrow[n]{(p)} 0.$$

PROOF. The equality is trivial from the definitions. Note that the terms in the second expression have the same distribution under $P^{\ge 0}$ because of Lemma 1. As $E_\omega[\widetilde{\tau}^{(e_{-D_n})}(0,e_1)]\mathbf{1}_{\{H \ge h_n\}}$ is not integrable for $0 < \kappa \le 1$, we introduce the event

$$\begin{aligned} A_n &:= \big\{\text{for } i = 1,\dots,K_n, H_{\sigma(i)} \le V(a_i) - V(b_i)\big\} \\ &= \bigcap_{p=0}^{n-1}\{H_p < h_n\} \cup \big\{h_n \le H_p \le V(e_{p-D_n}) - V(e_p)\big\}. \end{aligned}$$

Let us prove that our choice of D_n ensures $P^{\ge 0}((A_n)^c) = o_n(1)$. By Lemma 1, we have $P^{\ge 0}((A_n)^c) \le nP^{\ge 0}(H \ge h_n, H > V(e_{-D_n}))$. Then, let us choose $0 < \gamma' < \gamma'' < \gamma$ [cf. (3.1)] and define $l_n := \frac{1+\gamma'}{\kappa}\log n$. We get

$$P^{\ge 0}\big((A_n)^c\big) \le n\big(P(H \ge l_n) + P(H \ge h_n)P^{\ge 0}\big(V(e_{-D_n}) < l_n\big)\big). \tag{6.1}$$

Equation (4.10) gives $P(H \ge l_n) \sim_n C_I \mathrm{e}^{-\kappa l_n} = C_I n^{-(1+\gamma')}$, and $P(H \ge h_n) \sim_n C_I n^{-1}(\log n)^\kappa$. Under $P^{\ge 0}$, $V(e_{-D_n})$ is the sum of D_n i.i.d. random variables distributed like $-V(e_1)$. Therefore, for any $\lambda > 0$,

$$P^{\ge 0}\big(V(e_{-D_n}) < l_n\big) \le \mathrm{e}^{\lambda l_n} E\big[\mathrm{e}^{-\lambda(-V(e_1))}\big]^{D_n}.$$

Since $\frac{1}{\lambda}\log E[\mathrm{e}^{-\lambda(-V(e_1))}] \to -E[-V(e_1)] \in [-\infty, 0)$ as $\lambda \to 0^+$, we can choose $\lambda > 0$ such that $\log E[\mathrm{e}^{-\lambda(-V(e_1))}] < -\lambda A\frac{1+\gamma''}{1+\gamma}$ [where A was defined after (3.1)], hence, $E[\mathrm{e}^{-\lambda(-V(e_1))}]^{D_n} \le n^{-\lambda(1+\gamma'')/\kappa}$. This gives the bound $P^{\ge 0}(V(e_{-D_n}) < l_n) \le n^{-\lambda(\gamma''-\gamma')/\kappa}$. Using these estimates in (6.1) concludes the proof that $P^{\ge 0}((A_n)^c) = o_n(1)$.

Let us now prove the proposition itself. By the Markov inequality, for all $\delta > 0$,

$$P^{\geq 0}\left(\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1} E_\omega[\widetilde{\tau}^{(e_{p-D_n})}(e_p, e_{p+1})]\mathbf{1}_{\{H_p\geq h_n\}} > \delta\right)$$

$$\begin{aligned}(6.2)\qquad &\leq P^{\geq 0}\big((A_n)^c\big) + \frac{1}{\delta n^{1/\kappa}} E^{\geq 0}\left[\sum_{p=0}^{n-1} E_\omega\big[\widetilde{\tau}^{(e_{p-D_n})}(e_p, e_{p+1})\big]\mathbf{1}_{\{H_p\geq h_n\}}\mathbf{1}_{A_n}\right]\\ &\leq o_n(1) + \frac{n}{\delta n^{1/\kappa}} E^{\geq 0}\big[E_\omega\big[\widetilde{\tau}^{(e_{-D_n})}(0, e_1)\big]\mathbf{1}_{\{H\geq h_n, H<V(e_{-D_n})\}}\big].\end{aligned}$$

Note that we have $E_\omega[\widetilde{\tau}^{(e_{-D_n})}(0, e_1)] = E_\omega[N]E_\omega[T_1]$, where N is the number of crossings from $e_{-D_n}+1$ to e_{-D_n} before the first visit at e_1, and T_1 is the time for the random walk to go from e_{-D_n} to $e_{-D_n}+1$ (for the first time, e.g.); furthermore, these two terms are independent under $P^{\geq 0}$. Using (4.2), we have

$$\begin{aligned}E_\omega[N] &= \frac{P_{0,\omega}(\tau(e_{-D_n}) < \tau(e_1))}{P_{e_{-D_n}+1,\omega}(\tau(e_1) < \tau(e_{-D_n}))} = \sum_{0\leq x<e_1} \mathrm{e}^{V(x)-V(e_{-D_n})}\\ &= M_2 \mathrm{e}^{H-V(e_{-D_n})},\end{aligned}$$

hence, on the event $\{H < V(e_{-D_n})\}$, $E_\omega[N] \leq M_2$.

The length of an excursion to the left of e_{-D_n} is computed as follows, due to (4.6):

$$E_\omega[T_1] = E_{e_{-D_n},\omega}\big[\tau(e_{-D_n}+1)\big] \leq 2\sum_{x\leq e_{-D_n}} \mathrm{e}^{-(V(x)-V(e_{-D_n}))}.$$

The law of $(V(x) - V(e_{-D_n}))_{x\leq e_{-D_n}}$ under $P^{\geq 0}$ is $P^{\geq 0}$ because of Lemma 1. Therefore,

$$E^{\geq 0}\big[E_\omega[T_1]\big] \leq 2E^{\geq 0}\left[\sum_{x\leq 0}\mathrm{e}^{-V(x)}\right] = 2E^{\geq 0}[R_-] < \infty,$$

with (4.9). Then, we conclude that the right-hand side of (6.2) is less than $o_n(1) + 2\delta^{-1}n^{1-1/\kappa}E^{\geq 0}[R_-]E[M_2\mathbf{1}_{\{H\geq h_n\}}]$. Since Lemma 3 gives the bound $E[M_2\mathbf{1}_{\{H\geq h_n\}}] \leq CP(H \geq h_n) \sim_n C'\mathrm{e}^{-\kappa h_n} = C'n^{-1}(\log n)^\kappa$, this whole expression converges to 0, which concludes the proof of the proposition. □

7. Fluctuation of interarrival times. For any $x \leq y$, recall that the interarrival time $\tau(x, y)$ between sites x and y is defined by $\tau(x, y) := \inf\{n \geq 0 : X_{\tau(x)+n} = y\}$. Then, let

$$\tau_{\mathrm{IA}} := \sum_{i=0}^{K_n} \tau(d_i, b_{i+1}\wedge e_n) = \sum_{p=0}^{n-1} \tau(e_p, e_{p+1})\mathbf{1}_{\{H_p<h_n\}}$$

(with $d_0 = 0$) be the time spent at crossing small excursions before $\tau(e_n)$. The aim of this section is the following bound on the fluctuations of τ_{IA}.

PROPOSITION 4. *For any* $0 < \kappa < 2$, *under* $P^{\geq 0}$,

$$\frac{1}{n^{1/\kappa}} E_\omega\big[\big|\tau_{\mathrm{IA}} - E_\omega[\tau_{\mathrm{IA}}]\big|\big] \xrightarrow[n]{(p)} 0. \tag{7.1}$$

This proposition holds for $0 < \kappa < 1$ in a simple way: we have, in this case, using Lemmas 1 and 4,

$$\begin{aligned} E^{\geq 0}\big[E_\omega[\tau_{\mathrm{IA}}]\big] &= nE^{\geq 0}\big[E_\omega\big[\tau(e_1)\big]\mathbf{1}_{\{H<h_n\}}\big] \leq nE^{\geq 0}\big[2M_1' M_2 \mathrm{e}^H \mathbf{1}_{\{H<h_n\}}\big] \\ &\leq Cn\mathrm{e}^{(1-\kappa)h_n} = o\big(n^{1/\kappa}\big), \end{aligned}$$

hence, $n^{-1/\kappa} E_\omega[\tau_{\mathrm{IA}}]$ itself converges to 0 in $L^1(P^{\geq 0})$-norm and thus in probability.

We now consider the case $1 \leq \kappa < 2$. The proposition will directly follow from the fact that, under $P^{\geq 0}$,

$$\frac{1}{n^{2/\kappa}} \mathrm{Var}_\omega(\tau_{\mathrm{IA}}) \xrightarrow[n]{(p)} 0,$$

which in turn will come from Lemma 9 proving $E^{\geq 0}[\mathrm{Var}_\omega(\tau_{\mathrm{IA}})] = o(n^{2/\kappa})$. However, a specific caution is necessary in the case $\kappa = 1$; indeed, $\mathrm{Var}_\omega(\tau_{\mathrm{IA}})$ is not integrable in this case, because of the rare but significant fluctuations originating from the time spent by the walk when it backtracks into deep valleys. Our proof in this case consists of first proving that we may neglect in probability (using a first-moment method) the time spent backtracking into these deep valleys; and then that this brings us to the computation of the variance of τ_{IA} in an environment where small excursions have been substituted for the high ones (thus removing the nonintegrability problem).

Section 7.1 is dedicated to this reduction to an integrable setting, which is only involved in the case $\kappa = 1$ of Proposition 4 and of the theorems (but holds in greater generality), while Section 7.2 states and proves the bounds on the variance, implying Proposition 4.

7.1. *Reduction to small excursions* (*required for the case* $\kappa = 1$). Let $h > 0$. Let us denote by d_- the right end of the first excursion on the left of 0 that is higher than h:

$$d_- := \max\{e_p : p \leq 0, H_{p-1} \geq h\}.$$

Remember $\widetilde{\tau}^{(d_-)}(0, e_1)$ is the time spent on the left of d_- before the walk reaches e_1.

LEMMA 8. *There exists* $C > 0$, *independent of* h, *such that*

$$\mathbb{E}^{\geq 0}\big[\widetilde{\tau}^{(d_-)}(0, e_1)\mathbf{1}_{\{H<h\}}\big] \leq C \begin{cases} \mathrm{e}^{-(2\kappa-1)h}, & \text{if } \kappa < 1, \\ h\mathrm{e}^{-h}, & \text{if } \kappa = 1, \\ \mathrm{e}^{-\kappa h}, & \text{if } \kappa > 1. \end{cases} \tag{7.2}$$

PROOF. Let us decompose $\widetilde{\tau}^{(d_-)}(0, e_1)$ into the successive excursions to the left of d_-:

$$\widetilde{\tau}^{(d_-)}(0, e_1) = \sum_{m=1}^{N} T_m,$$

where N is the number of crossings from $d_- + 1$ to d_- before $\tau(e_1)$, and T_m is the time for the walk to go from d_- to $d_- + 1$ on the mth time. Under P_ω, the times T_m, $m \geq 1$, are i.i.d. and independent of N [i.e., more properly, the sequence $(T_m)_{1 \leq m \leq N}$ can be prolonged to an infinite sequence with these properties]. We have, using the Markov property and then (4.2),

$$E_\omega[N] = \frac{P_{0,\omega}(\tau(d_-) < \tau(e_1))}{P_{d_-+1,\omega}(\tau(e_1) < \tau(d_-))} = \sum_{0 \leq x < e_1} \mathrm{e}^{V(x) - V(d_-)}$$

and, from (4.6), $E_\omega[T_1] = E_{d_-,\omega}[\tau(d_- + 1)] \leq 2 \sum_{x \leq d_-} \mathrm{e}^{-(V(x) - V(d_-))}$. Therefore, by Wald identity and Lemma 1,

$$\begin{aligned} &\mathbb{E}^{\geq 0}\big[\widetilde{\tau}^{(d_-)}(0, e_1)\mathbf{1}_{\{H<h\}}\big] \\ &\quad = E^{\geq 0}\big[E_\omega[N]E_\omega[T_1]\mathbf{1}_{\{H<h\}}\big] \qquad (7.3)\\ &\quad \leq 2E\bigg[\sum_{0 \leq x < e_1} \mathrm{e}^{V(x)}\mathbf{1}_{\{H<h\}}\bigg] E^{\geq 0}\big[\mathrm{e}^{-V(d_-)}\big] E\bigg[\sum_{x \leq 0} \mathrm{e}^{-V(x)} \Big| \Lambda(h)\bigg], \end{aligned}$$

where $\Lambda(h) := \{\forall x \leq 0, V(x) \geq 0\} \cap \{H_{-1} \geq h\}$. The first expectation can be written as $E[M_2 \mathrm{e}^H \mathbf{1}_{\{H<h\}}]$. For the second one, note that $d_- = e_{-W}$, where W is a geometric random variable of parameter $q := P(H \geq h)$; and, conditional on $\{W = n\}$, the distribution of $(V(x))_{e_{-W} \leq x \leq 0}$ under $P^{\geq 0}$ is the same as that of $(V(x))_{e_{-n} \leq x \leq 0}$ under $P^{\geq 0}(\cdot | \text{for } p = 0, \ldots, n-1, H_{-p} < h)$. Therefore,

$$E^{\geq 0}\big[\mathrm{e}^{-V(d_-)}\big] = E^{\geq 0}\big[E\big[\mathrm{e}^{-V(e_1)} | H < h\big]^W\big] = \frac{q}{1 - (1-q)E[\mathrm{e}^{V(e_1)} | H < h]},$$

and $(1-q)E[\mathrm{e}^{V(e_1)} | H < h]$ converges to $E[\mathrm{e}^{V(e_1)}] < 1$ when $h \to \infty$ [the inequality comes from assumption (b)], hence, this quantity is uniformly bounded from above by $c < 1$ for large h. In addition, (4.10) gives $q \sim C_I \mathrm{e}^{-\kappa h}$ when $h \to \infty$, hence, $E^{\geq 0}[\mathrm{e}^{-V(d_-)}] \leq C\mathrm{e}^{-\kappa h}$, where C is independent of h.

Finally, let us consider the last term of (7.3). We have

$$\begin{aligned} &E\bigg[\sum_{x \leq 0} \mathrm{e}^{-V(x)} \Big| \Lambda(h)\bigg] \\ &\quad = E\bigg[\sum_{e_{-1} < x \leq 0} \mathrm{e}^{-V(x)} \Big| H_{-1} \geq h\bigg] + E^{\geq 0}\bigg[\sum_{x \leq e_{-1}} \mathrm{e}^{-(V(x) - V(e_{-1}))}\bigg] E\big[\mathrm{e}^{-V(e_{-1})}\big] \\ &\quad \leq E\big[M_1' | H \geq h\big] + E^{\geq 0}[R_-]E\big[\mathrm{e}^{V(e_1)}\big], \end{aligned}$$

hence, using Lemma 3, (4.9) and $V(e_1) \le 0$, this term is bounded by a constant. The statement of the lemma then follows from the application of Lemma 3 to the expectation $E[M_2 \mathrm{e}^H \mathbf{1}_{\{H<h\}}]$. $\square$

The part of the interarrival time τ_{IA} spent at backtracking in high excursions can be written as follows:

$$\begin{aligned}\widetilde{\tau}_{\mathrm{IA}} &:= \widetilde{\tau}^{(d_-)}(0, b_1 \wedge e_n) + \sum_{i=1}^{K_n} \widetilde{\tau}^{(d_i)}(d_i, b_{i+1} \wedge e_n)\\ &= \sum_{p=0}^{n-1} \widetilde{\tau}^{(d(e_p))}(e_p, e_{p+1}) \mathbf{1}_{\{H_p<h_n\}},\end{aligned}$$

where, for $x \in \mathbb{Z}$, $d(x) := \max\{e_p : p \in \mathbb{Z}, e_p \le x, H_{p-1} \ge h_n\}$. In particular, $d(0) = d_-$ in the previous notation with $h = h_n$.

Note that under $\mathbb{P}^{\ge 0}$, because of Lemma 1, the terms of the above sum have the same distribution as $\widetilde{\tau}^{(d(0))}(0, e_1)\mathbf{1}_{\{H<h_n\}}$, hence,

$$\mathbb{E}^{\ge 0}[\widetilde{\tau}_{\mathrm{IA}}] = n\mathbb{E}^{\ge 0}\bigl[\widetilde{\tau}^{(d(0))}(0, e_1)\mathbf{1}_{\{H<h_n\}}\bigr].$$

Thus, for $\mathbb{E}^{\ge 0}[\widetilde{\tau}_{\mathrm{IA}}]$ to be negligible with respect to $n^{1/\kappa}$, it suffices that the expectation on the right-hand side be negligible with respect to $n^{1/\kappa-1}$. In particular, for $\kappa = 1$, it suffices that it converges to 0, which is readily seen from (7.2). Thus, for $\kappa = 1$,

$$n^{-1/\kappa}\mathbb{E}^{\ge 0}[\widetilde{\tau}_{\mathrm{IA}}] \underset{n}{\longrightarrow} 0, \tag{7.4}$$

hence, in particular, $n^{-1/\kappa}E_\omega[\widetilde{\tau}_{\mathrm{IA}}] \to 0$ in probability under $\mathbb{P}^{\ge 0}$. Note that (7.4) actually holds for any $\kappa \ge 1$.

Let us introduce the modified environment, where independent small excursions are substituted for the high excursions. In order to avoid obfuscating the redaction, we will only introduce little notation regarding this new environment.

Let us enlarge the probability space in order to accommodate a new family of independent excursions indexed by $\mathbb{N}^* \times \mathbb{Z}$ such that for all n, k the excursion with index (n, k) has the same distribution as $(V(x))_{0\le x\le e_1}$ under $P(\cdot|H < h_n)$. Thus we are given, for every $n \in \mathbb{N}^*$, a countable family of independent excursions lower than h_n. For every fixed n, we define the *modified environment of height less than* h_n by replacing all the excursions of V that are higher than h_n by new independent ones that are lower than h_n. Because of Lemma 1, this construction is especially natural under $P^{\ge 0}$, where it has stationarity properties.

In the following, we will denote by P' the law of the modified environment relative to the height h_n given in the context [hence, also a definition of $(P^{\ge 0})'$, e.g.].

REMARK. Repeating the proof done under $P^{\geq 0}$ for $(P^{\geq 0})'$, we see that R_- still has all finite moments in the modified environment, and that these moments are bounded uniformly in n. In particular, the bound for the quantity $E^{\geq 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} \mathbf{1}_{\{H<h_n\}}]$ given in Lemma 3 is unchanged for $(E^{\geq 0})'$ [writing $M_1' = R_- + \sum_{0\leq x<e_1} \mathrm{e}^{-V(x)}$ and using $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$]. On the other hand,

$$E'[R] = \sum_{p=0}^{\infty} E'[\mathrm{e}^{V(e_p)}]E'\Bigg[\sum_{e_p\leq x<e_{p+1}} \mathrm{e}^{V(x)-V(e_p)}\Bigg]$$
$$= \sum_{p=0}^{\infty} E[\mathrm{e}^{V(e_1)}|H<h_n]^p E[M_2 \mathrm{e}^H|H<h_n],$$

and $E[\mathrm{e}^{V(e_1)}|H<h_n] \leq c$ for some $c<1$ independent of n because this expectation is smaller than 1 for all n and it converges toward $E[\mathrm{e}^{V(e_1)}]<1$ as $n\to\infty$. Hence, by Lemma 3,

$$\text{if } \kappa = 1 \qquad E'[R] \leq Ch_n. \tag{7.5}$$

This is the only difference that will appear in the following computations.

Assuming that $d(0)$ keeps being defined with respect to the usual heights, (7.2) (with $h=h_n$) is still true for the walk in the modified environment. Indeed, the change only affects the environment on the left of $d(0)$, hence, the only difference in the proof involves the times T_m; in (7.3), one should substitute $(E^{\geq 0})'$ for $E[\cdot|\Lambda(h)]$, and this factor is uniformly bounded in both cases because of the above remark about R_-.

We deduce that the time $\widetilde{\tau}'_{\mathrm{IA}}$, defined as similar to $\widetilde{\tau}_{\mathrm{IA}}$ except that the excursions on the left of the points $d(e_i)$ (i.e., the times similar to T_m in the previous proof) are performed in the modified environment, still satisfies, for $\kappa=1$,

$$n^{-1/\kappa}\mathbb{E}^{\geq 0}[\widetilde{\tau}'_{\mathrm{IA}}] \xrightarrow[n]{} 0. \tag{7.6}$$

Now note that

$$\tau'_{\mathrm{IA}} := \tau_{\mathrm{IA}} - \widetilde{\tau}_{\mathrm{IA}} + \widetilde{\tau}'_{\mathrm{IA}} \tag{7.7}$$

is the time spent at crossing the (original) small excursions, in the environment where the high excursions have been replaced by new independent small excursions. Indeed, the high excursions are only involved in τ_{IA} during the backtracking of the walk to the left of $d(e_i)$ for some $0\leq i<n$. Assembling (7.4) and (7.4), it is equivalent (for $\kappa=1$) to prove (7.1) or

$$n^{-1/\kappa}E_\omega[|\tau'_{\mathrm{IA}} - E_\omega[\tau'_{\mathrm{IA}}]|] \xrightarrow[n]{(p)} 0,$$

and it is thus sufficient to prove $E^{\geq 0}[\mathrm{Var}_\omega(\tau'_{\mathrm{IA}})] = o_n(n^{2/\kappa})$.

7.2. *Bounding the variance of* τ_{IA}. Because of the previous subsection, Proposition 4 will follow from the next lemma.

LEMMA 9. *We have, for* $1 < \kappa < 2$,

$$E^{\geq 0}[\mathrm{Var}_\omega(\tau_{\mathrm{IA}})] = o_n(n^{2/\kappa}), \tag{7.8}$$

and, for $1 \leq \kappa < 2$,

$$E^{\geq 0}[\mathrm{Var}_\omega(\tau'_{\mathrm{IA}})] = o_n(n^{2/\kappa}).$$

We recall that the second bound is only introduced to settle the case $\kappa = 1$; it would suffice for $1 < \kappa < 2$ as well, but introduces unnecessary complication. The computations being very close for τ_{IA} and τ'_{IA}, we will write below the proof for τ_{IA} and indicate line by line where changes happen for τ'_{IA}. Let us stress that, when dealing with τ'_{IA}, all the indicator functions $\mathbf{1}_{\{H_\cdot < h_n\}}$ (which define the small valleys) would refer to the original heights, while all the potentials $V(\cdot)$ appearing along the computation (which come from quenched expectations of times spent by the walk) would refer to the modified environment.

PROOF OF LEMMA 9. We have

$$\tau_{\mathrm{IA}} = \sum_{p=0}^{n-1} \tau(e_p, e_{p+1}) \mathbf{1}_{\{H_p < h_n\}}, \tag{7.9}$$

and by the Markov property, the above times are independent under $P_{o,\omega}$. Hence,

$$\mathrm{Var}_\omega(\tau_{\mathrm{IA}}) = \sum_{p=0}^{n-1} \mathrm{Var}_\omega(\tau(e_p, e_{p+1})) \mathbf{1}_{\{H_p < h_n\}}.$$

Under $P^{\geq 0}$, the distribution of the environment seen from e_p does not depend on p, hence,

$$E^{\geq 0}[\mathrm{Var}_\omega(\tau_{\mathrm{IA}})] = n E^{\geq 0}[\mathrm{Var}_\omega(\tau(e_1)) \mathbf{1}_{\{H < h_n\}}]. \tag{7.10}$$

We use formula (4.8):

$$\mathrm{Var}_\omega(\tau(e_1)) \mathbf{1}_{\{H < h_n\}} \leq 16 \sum_{z' \leq z \leq x \leq y \leq e_1, 0 \leq y} \mathrm{e}^{V(y)+V(x)-V(z)-V(z')} \mathbf{1}_{\{H < h_n\}}. \tag{7.11}$$

Let us first consider the part of the sum where $x \geq 0$. By noting that the indices satisfy $z' \leq x$ and $z \leq y$, this part is seen to be less than $(M'_1 M_2 \mathrm{e}^H)^2 \mathbf{1}_{\{H < h_n\}}$. Lemma 3 shows that its expectation is smaller than $C\mathrm{e}^{(2-\kappa)h_n}$. *For* τ'_{IA}: *The same holds*, *because of the remark on page* 1169.

It remains to deal with the indices $x < 0$. This part rewrites as

$$\sum_{z', z \leq x < 0} \mathrm{e}^{V(x)-V(z)-V(z')} \cdot \sum_{0 \leq y < e_1} \mathrm{e}^{V(y)} \mathbf{1}_{\{H < h_n\}}. \tag{7.12}$$

Since $V_{|\mathbb{Z}_+}$ and $V_{|\mathbb{Z}_-}$ are independent under P, so are the two above factors. The second one equals $\mathrm{e}^H M_2 \mathbf{1}_{\{H<h_n\}}$. Let us split the first one according to the excursion $[e_{p-1}, e_p)$ containing x; it becomes

$$\sum_{p\le 0} \mathrm{e}^{-V(e_{p-1})} \sum_{e_{p-1}\le x<e_p} \mathrm{e}^{V(x)-V(e_{p-1})}\Bigl(\sum_{z\le x} \mathrm{e}^{-(V(z)-V(e_{p-1}))}\Bigr)^2. \tag{7.13}$$

We have by definition $V(e_{p-1}) \ge V(e_p)$ and, under $P^{\ge 0}$, $V(e_p)$ is independent of $(V(e_p+x)-V(e_p))_{x\le 0}$ and thus of $(V(e_{p-1}+x)-V(e_{p-1}))_{x\le e_p-e_{p-1}}$, which has same distribution as $(V(x))_{x\le e_1}$. Therefore, the expectation of (7.13) with respect to $P^{\ge 0}$ is less than

$$\begin{aligned}\sum_{p\le 0} E^{\ge 0}[\mathrm{e}^{-V(e_p)}] E^{\ge 0}\Biggl[\sum_{0\le x<e_1} \mathrm{e}^{V(x)}\Bigl(\sum_{z\le x}\mathrm{e}^{-V(z)}\Bigr)^2\Biggr]\\ \le \bigl(1-E[\mathrm{e}^{V(e_1)}]\bigr)^{-1} E^{\ge 0}[\mathrm{e}^H (M_1')^2 M_2].\end{aligned}$$

Thus the expectation of (7.12) with respect to $P^{\ge 0}$ is bounded by

$$\bigl(1-E[\mathrm{e}^{V(e_1)}]\bigr)^{-1} E^{\ge 0}[\mathrm{e}^H (M_1')^2 M_2] E^{\ge 0}[\mathrm{e}^H M_2 \mathbf{1}_{\{H<h_n\}}].$$

From Lemma 3, we conclude that this term is less than a constant if $\kappa > 1$. The part corresponding to $x \ge 0$ therefore dominates; this finishes the proof of (7.8). *For τ_{IA}': The first factor is $(1-E[\mathrm{e}^{V(e_1)}|H<h_n])^{-1}$, which is uniformly bounded because it converges to $(1-E[\mathrm{e}^{V(e_1)}])^{-1} < \infty$ and, using Lemma 3, the two other factors are each bounded by a constant if $\kappa > 1$ and by Ch_n if $\kappa = 1$; cf. again the remark page 1169. Thus, the part corresponding to $x \ge 0$ still dominates in this case.*

We have proved $E^{\ge 0}[\mathrm{Var}_\omega(\tau_{\mathrm{IA}})] \le Cn\mathrm{e}^{(2-\kappa)h_n}$. Since $n\mathrm{e}^{(2-\kappa)h_n} = \frac{n^{2/\kappa}}{(\log n)^{2-\kappa}}$, this concludes the proof of (7.8). □

7.3. *A subsequent lemma.* The proof of (7.8) entails the following bound for the crossing time of one low excursion.

LEMMA 10. *For all $h > 0$ we have, if $1 < \kappa < 2$,*

$$E^{\ge 0}\bigl[E_\omega[\tau(e_1)^2]\mathbf{1}_{\{H<h\}}\bigr] \le C\mathrm{e}^{(2-\kappa)h},$$

and similarly for $(E^{\ge 0})'$ if $1 \le \kappa < 2$.

PROOF. We have $E_\omega[\tau(e_1)^2] = \mathrm{Var}_\omega(\tau(e_1)) + E_\omega[\tau(e_1)]^2$. Equation (7.10) and the remainder of the proof of (7.8) give

$$E^{\ge 0}\bigl[\mathrm{Var}_\omega(\tau(e_1))\mathbf{1}_{\{H<h\}}\bigr] \le C\mathrm{e}^{(2-\kappa)h}.$$

Together with Lemma 4, this concludes the proof. □

8. Proof of Theorems 1 and 2. Note that we first prove the results under $P^{\geq 0}$. We will also prove (3.2) as a tool.

8.1. *Joint coupling*. Extending what we did in Section 5.3, we introduce an i.i.d. family $(\mathbf{e}_i)_{i\geq 1}$ of exponential random variables of parameter 1 and define, for $i \geq 1$,

$$N_i := \left\lfloor -\frac{1}{\log(1-p_i(\omega))}\mathbf{e}_i \right\rfloor,$$

where $p_i(\omega) = P_{b_i,\omega}(\tau(b_i) < \tau(d_i))$. Since, by the Markov property, the numbers of returns to b_i before the walk reaches d_i are independent given ω, conditioning these numbers to be equal to N_i realizes a coupling, as in Section 5.3.

8.2. *Reduction to one valley.* The above coupling enables us to give the following bound:

$$\begin{aligned}
&W_\omega^1\left(\tau(e_n) - E_\omega[\tau(e_n)], \sum_{i=1}^{K_n} E_\omega[\tau(b_i,d_i)]\bar{\mathbf{e}}_i\right)\\
&\quad\leq E_\omega\left[\left|\tau(e_n) - E_\omega[\tau(e_n)] - \sum_{i=1}^{K_n} E_\omega[\tau(b_i,d_i)]\bar{\mathbf{e}}_i\right|\right]\\
&\quad\leq E_\omega\big[|\tau_{\mathrm{IA}} - E_\omega[\tau_{\mathrm{IA}}]|\big] + \sum_{i=1}^{K_n} E_\omega\big[|\tau(b_i,d_i) - E_\omega[\tau(b_i,d_i)]\mathbf{e}_i|\big],
\end{aligned}$$

where τ_{IA} is defined in Section 7 (note that for the K_n high excursions the centerings simplify). We deduce, for all $\delta > 0$,

$$\begin{aligned}
&P^{\geq 0}\left(W_\omega^1\left(\tau(e_n) - E_\omega[\tau(e_n)], \sum_{i=1}^{K_n} E_\omega[\tau(b_i,d_i)]\bar{\mathbf{e}}_i\right) > \delta n^{1/\kappa}\right)\\
&\quad\leq P^{\geq 0}\left(E_\omega\big[|\tau_{\mathrm{IA}} - E_\omega[\tau_{\mathrm{IA}}]|\big] > \frac{\delta}{2}n^{1/\kappa}\right)\\
&\qquad + P^{\geq 0}\left(\bigcup_{p=0}^{n-1}\left\{E_\omega\big[|\tau(e_p,e_{p+1}) - E_\omega[\tau(e_p,e_{p+1})]\mathbf{e}_p|\big]\mathbf{1}_{\{H_p\geq h_n\}}\right.\right.\\
&\qquad\qquad\left.\left.\geq \frac{\delta}{2K_n}n^{1/\kappa}\right\}\right).
\end{aligned}$$

By Proposition 4, the first term is known to converge to 0 as $n \to \infty$ (using for $\kappa = 1$ the same reduction as in Section 7). By Lemma 1, the last term is bounded by

$$P\big(K_n \geq 2(\log n)^\kappa\big) + nP^{\geq 0}\left(E_\omega\big[|\tau - E_\omega[\tau]\mathbf{e}|\big] \geq \frac{\delta}{4(\log n)^\kappa}n^{1/\kappa}, H \geq h_n\right),$$

where τ and $\mathbf{e}$ stand for $\tau(e_1)$ $(=\tau(b_1,d_1)$ on $\{H\ge h_n\})$ and $\mathbf{e}_1$. By the proof of Proposition 2, the first probability goes to 0, when n tends to infinity. As for the other probability, it follows from Proposition 1 with $t=n^{1/\kappa}$ that it is $o(n^{-1})$.

This yields, under $P^{\ge 0}$,

$$W_\omega^1\left(\frac{\tau(e_n)-E_\omega[\tau(e_n)]}{n^{1/\kappa}},\frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n}E_\omega[\tau(b_i,d_i)]\bar{\mathbf{e}}_i\right)\xrightarrow[n]{(p)}0, \tag{8.1}$$

which is the statement of (3.2) along the random subsequence $x=e_n$, and under $P^{\ge 0}$ instead of P. Before proceeding to the interpolation from e_n to any x, let us show how the statements of Theorems 1 and 2 can be quickly deduced from (8.1), modulo the same restriction.

8.3. *Addition of small excursions and independence of the high ones.* More specifically, if (with a convenient abuse of notation) we extend the i.i.d. sequence $(\bar{\mathbf{e}}_i)_{i\ge 1}$ to an i.i.d. sequence $(\bar{\mathbf{e}}_p)_{p\ge 0}$ such that $\bar{\mathbf{e}}_i=\bar{\mathbf{e}}_p$ for $p=\sigma(i)$, the only addition in Theorem 1 is the following term which we shall prove is negligible:

$$\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1}Z_p\mathbf{1}_{\{H_p<h_n\}}\bar{\mathbf{e}}_p, \tag{8.2}$$

where we define

$$Z_i:=E_\omega[\tau(e_i,e_{i+1})],\qquad i\ge 0.$$

Note that $(Z_i)_{i\ge 0}$ is a stationary sequence under $P^{\ge 0}$; cf. Lemma 1.

For $0<\kappa<1$, it suffices to note that the $L^1(P_\omega)$-norm of this term is bounded by $n^{-1/\kappa}E_\omega[\tau_{\mathrm{IA}}]$ (since $E_\omega[|\bar{\mathbf{e}}_p|]=2/\mathrm{e}<1$), which converges to 0 in $L^1(P)$ and thus in probability in this case; cf. after Proposition 4.

For $1<\kappa<2$, let us write that the $L^1(P_\omega)$-norm of (8.2) is bounded, using the Cauchy–Schwarz inequality, by

$$\frac{1}{n^{1/\kappa}}\operatorname{Var}_\omega\left(\sum_{p=0}^{n-1}Z_p\mathbf{1}_{\{H_p<h_n\}}\bar{\mathbf{e}}_p\right)^{1/2}=\frac{1}{n^{1/\kappa}}\left(\sum_{p=0}^{n-1}Z_p^2\mathbf{1}_{\{H_p<h_n\}}\right)^{1/2},$$

hence,

$$P^{\ge 0}\left(E_\omega\left[\left|\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1}Z_p\mathbf{1}_{\{H_p<h_n\}}\bar{\mathbf{e}}_p\right|\right]\ge\delta\right)\le\frac{1}{\delta^2n^{2/\kappa}}nE^{\ge 0}[E_\omega[\tau]^2\mathbf{1}_{\{H<h_n\}}].$$

Lemma 4 shows that the last expectation is less than $Cn^{\frac{2}{\kappa}-1}(\log n)^{-(2-\kappa)}$ so that the right-hand side converges to 0.

For $\kappa=1$, we do the same as for $\kappa>1$, by means of the reduction to the modified environment (cf. Section 7.1): the decomposition $\tau_{\mathrm{IA}}=\tau'_{\mathrm{IA}}-\widetilde{\tau}_{\mathrm{IA}}+\widetilde{\tau}'_{\mathrm{IA}}$ of (7.7) induces a decomposition similar to (8.2) (with the only addition of quenched expectations and weights). The terms corresponding to $\widetilde{\tau}_{\mathrm{IA}}$ and $\widetilde{\tau}'_{\mathrm{IA}}$ are neglected using their first moment by the results (7.4) and (7.6) in Section 7.1, thus reducing the problem to the modified environment, where Lemma 10 applies. This would conclude the proof of Theorem 1, up to the previous restrictions.

To deduce Theorem 2 (along the random subsequence $x = e_n$ and under $P^{\geq 0}$) from (8.1), we have to replace $Z_{\sigma(i)} = E_\omega[\tau(b_i, d_i)]$, $i = 1, \ldots, K_n$, by independent terms having the same distribution, and to add new terms corresponding to small excursions, just like above but independent of each other. Note that the new independent terms $\widehat{Z}_p$ will depend on n, even though their distribution does not, which explains the wording of Theorem 2.

To this aim, let us enlarge the probability space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{B}, \mathbb{P}^{\geq 0})$ in order to introduce a sequence $(\omega^{(p)}, (X_t^{(p)})_{t\in\mathbb{N}})_{p\geq 0}$ of environments and random walks coupled with ω in the following way, for $p \geq 0$:

(1) if $H_p < h_n$, then $\omega^{(p)}$ is an independent environment sampled according to the distribution $P^{\geq 0}(\cdot | H < h_n)$;

(2) if $H_p \geq h_n$, that is, $p = \sigma(i)$ for some $i \geq 1$, then $\omega^{(p)}$ is built from the piece of ω from $d_{i-1} + 1$ to d_i, translated so that b_i is now at 0, and bordered by independent environments with law P on the right and law $P^{\geq 0}(\cdot | H_{-1} \geq h_n, V_{|\mathbb{Z}_-} \geq -A_i)$ on the left where $A_i := V(d_{i-1}) - V(b_i)$ (function of ω);

(3) for all $p \geq 0$, conditionally on $\omega^{(p)}$, $(X_t^{(p)})_{t\in\mathbb{N}}$ has law $P_{\omega^{(p)}}$.

Due to the independence between the excursions of ω under $P^{\geq 0}$, the sequence $(\omega^{(p)})_{p\geq 0}$ is seen to be independent. Furthermore, for every $p \geq 0$, the construction ensures that $\omega^{(p)}$ follows the law $P^{\geq 0}$. We will denote with a superscript ${}^{(p)}$ the quantities relative to $\omega^{(p)}$ instead of ω.

We may thus introduce

$$\widehat{Z}_p := E_\omega[\tau^{(p)}(e_1^{(p)})], \qquad p \geq 0,$$

which is defined as $Z_1 (:= E_\omega[\tau(e_1)])$ but relative to $(\omega^{(p)}, X^{(p)})$ instead of (ω, X). By the previous claims, $(\widehat{Z}_p)_{p\geq 0}$ is a sequence of i.i.d. random variables distributed as Z_1 under $P^{\geq 0}$.

For $i \geq 1$, to compare $Z_{\sigma(i)}$ with $\widehat{Z}_{\sigma(i)}$, we further decompose

$$Z_{\sigma(i)} =: \widetilde{Z}_{\sigma(i)} + Z^*_{\sigma(i)} \quad \text{and} \quad \widehat{Z}_{\sigma(i)} =: \widehat{\widetilde{Z}}_{\sigma(i)} + \widehat{Z}^*_{\sigma(i)},$$

where we let $\widetilde{Z}_{\sigma(i)} := E_\omega[\widetilde{\tau}^{(a_i)}(b_i, d_i)]$ and similarly, $\widehat{\widetilde{Z}}_{\sigma(i)}$ is defined as $E_\omega[\widetilde{\tau}^{(e_{-D_n})}(0, e_1)]$ with respect to $\omega^{(\sigma(i))}$ instead of ω, so that $Z^*_{\sigma(i)}$ is the quenched expectation of the time to go from b_i to d_i for a random walk reflected at a_i and thus only depends on the environment between a_i and d_i. Using this last remark, it is important to note that, on the event $\mathrm{NO}(n)$ (cf. Proposition 2), $Z^*_{\sigma(i)}$

and $\widehat{Z}^*_{\sigma(i)}$ are equal for $i = 1, \dots, K_n$. Indeed, since $P(\mathrm{NO}(n)) \to_n 1$, this gives us directly

$$W^1_\omega\left(\frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n} Z^*_{\sigma(i)}\bar{\mathbf{e}}_{\sigma(i)}, \frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n} \widehat{Z}^*_{\sigma(i)}\bar{\mathbf{e}}_{\sigma(i)}\right) \xrightarrow[n]{(p)} 0. \tag{8.3}$$

In addition, Proposition 3 and the triangular inequality give

$$W^1_\omega\left(\frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n} \widetilde{Z}_{\sigma(i)}\bar{\mathbf{e}}_{\sigma(i)}, 0\right) \xrightarrow[n]{(p)} 0. \tag{8.4}$$

It remains to prove that the same holds for $\widehat{\widehat{Z}}_{\sigma(i)}$ in order to get (8.1) with $\widehat{Z}^{(\sigma(i))}$ in place of $Z^{(\sigma(i))}$. And finally, Theorem 2 will be proved (under the above-mentioned restrictions) if, furthermore, the small independent excursions may be harmlessly introduced, that is, if

$$W^1_\omega\left(\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1} \widehat{Z}_p\bar{\mathbf{e}}_p\mathbf{1}_{\{H^{(p)}<h_n\}}, 0\right) \xrightarrow[n]{(p)} 0. \tag{8.5}$$

These two facts are given by the following lemma.

LEMMA 11. *We have, under $P^{\geq 0}$,*

$$\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1} \widehat{\widehat{Z}}_p\mathbf{1}_{\{H^{(p)}\geq h_n\}} \xrightarrow[n]{(p)} 0$$

and

$$\frac{1}{n^{1/\kappa}}\sum_{p=0}^{n-1} \widehat{Z}_p\mathbf{1}_{\{H^{(p)}<h_n\}}\bar{\mathbf{e}}_p \xrightarrow[n]{(p)} 0. \tag{8.6}$$

PROOF. These results follow, respectively, from the proofs of Proposition 3 and (8.2), made easier by the independence of the random variables $\widehat{Z}_0, \dots, \widehat{Z}_{n-1}$. More precisely, the proof of Proposition 3 holds in this i.i.d. context almost without a change, while the above derivation of (8.2) did not involve the correlation between $Z_0, \dots, Z_{n-1}$ in any way, hence, the proof may as well be conducted for independent copies. $\square$

8.4. *Interpolation from $\tau(e_n)$ to $\tau(x)$.* We now replace the subsequence $\tau(e_n)$ by the whole sequence $\tau(x)$. We write the proof in the setting of Theorem 1, from which the other cases follow, up to very minor modifications.

Choose $\frac{1}{2} < \alpha < \min\{1, \frac{1}{\kappa}\}$. For $x \in \mathbb{N}$, we define the following event about the environment:

$$A_x := \{e_{\lfloor (x-x^\alpha)/(E[e_1])\rfloor} < x < e_{\lfloor (x+x^\alpha)/(E[e_1])\rfloor}\}. \tag{8.7}$$

Since $\alpha > \frac{1}{2}$, it follows from the central limit theorem, applied to the i.i.d. sequence $(e_{n+1} - e_n)_n$, that

$$P(A_x) \to 1, \qquad x \to \infty. \tag{8.8}$$

Starting from the version of Theorem 1 we have obtained so far, that is, for every $\delta > 0$,

$$P^{\geq 0}\left(\left|\tau(e_n) - E_\omega[\tau(e_n)] - \sum_{p=0}^{n-1} E_\omega[\tau(e_p, e_{p+1})]\bar{\mathbf{e}}_p\right| > \delta n^{1/\kappa}\right) \xrightarrow[n]{} 0,$$

the limit still holds along the *deterministic* subsequences

$$n_x^- := \left\lfloor \frac{x - x^\alpha}{E[e_1]} \right\rfloor \quad \text{and} \quad n_x^+ := \left\lfloor \frac{x + x^\alpha}{E[e_1]} \right\rfloor,$$

and according to (8.8) it is legitimate to restrict to the event A_x in the above probability for $n = n_x^\pm$. From that remark and $n_x^\pm \sim_x \frac{x}{E[e_1]}$, we conclude that the result of Theorem 1 will follow from (under $P^{\geq 0}$)

$$\frac{1}{x^{1/\kappa}} E_\omega\big[|\tau(x) - \tau(e_{n_x^+})|\big] \xrightarrow[x]{(p)} 0, \qquad \frac{1}{x^{1/\kappa}} \big|E_\omega[\tau(x)] - E_\omega[\tau(e_{n_x^+})]\big| \xrightarrow[x]{(p)} 0,$$

the corresponding limits for n_x^- and

$$\frac{1}{x^{1/\kappa}} \sum_{n_x^- \leq p \leq n_x^+} E_\omega[\tau(e_p, e_{p+1})]\bar{\mathbf{e}}_p \xrightarrow[x]{(p)} 0.$$

Of course, the second limit will follow from the first one. Furthermore, on A_x we have

$$E_\omega\big[|\tau(x) - \tau(e_{n_x^\pm})|\big] \leq E_\omega\big[\tau(e_{n_x^+}) - \tau(e_{n_x^-})\big] = \sum_{n_x^- \leq p < n_x^+} E_\omega[\tau(e_p, e_{p+1})],$$

so that the three limits will come as a consequence of the following application of the Markov inequality:

$$\begin{aligned}
&P^{\geq 0}\left(\sum_{n_x^- \leq p \leq n_x^+} E_\omega[\tau(e_p, e_{p+1})] > \delta x^{1/\kappa}\right) \\
&\qquad \leq P(\exists n_x^- \leq p \leq n_x^+, H_p \geq h_x) + \frac{n_x^+ - n_x^- + 1}{\delta x^{1/\kappa}} E^{\geq 0}[E_\omega[\tau(e_1)], H < h_x] \\
&\qquad \leq \frac{2x^\alpha}{E[e_1]} P(H \geq h_x) + \frac{2x^\alpha + 1}{\delta x^{1/\kappa}} E^{\geq 0}[2M_1' M_2 \mathrm{e}^H, H < h_x].
\end{aligned}$$

By (4.10) and $\alpha < 1$, the first term goes to 0. By Lemma 3 and since $\alpha < \frac{1}{\kappa}$, the second term goes to 0 as well. This proves Theorem 1, under $P^{\geq 0}$.

8.5. *Conclusion.* Let us finally discuss the change of probability from $P^{\geq 0}$ to P. In fact, it suffices to note that the quenched expectation of the time spent on $\mathbb{Z}_-$ is finite a.s. under P and $P^{\geq 0}$, which follows from (4.3) and (4.6) (and $E[\log\rho]<0$) since this expectation is seen to be equal to $E_{0,\omega}[\tau(1)]P_{1,\omega}(\tau(0)=\infty)^{-1}$. This ends the proof of (3.2) and Theorems 1 and 2.

Note that the tail estimate (2.4) of $\widehat{Z}_i$ (i.e., of $E_\omega[\tau(e_1)]$ under $P^{\geq 0}$) given in Theorem 2, while not being exactly a consequence of Lemma 2, follows simply from it. Indeed, the expression $E_\omega[\tau(e_1)]=E_\omega[N]E_\omega[F]+E_\omega[G]=2\mathrm{e}^H\widehat{M}_1M_2+E_\omega[G]$, together with (5.6) and Lemma 7, gives the following lower and upper bounds, for some $\alpha<1$ and $\delta>0$:

$$
\begin{aligned}
P^{\geq 0}\biggl(2Z\geq\frac{t}{1+o(t^{-\delta})}\biggr)&\leq P^{\geq 0}\bigl(E_\omega\bigl[\tau(e_1)\bigr]\geq t\bigr)\\
&\leq P^{\geq 0}\bigl(\Omega_t^c\bigr)+P^{\geq 0}\biggl(2Z\geq\frac{t-C(\log t)^4t^\alpha}{1+o(t^{-\delta})}\biggr),
\end{aligned}
$$

and $P^{\geq 0}(\Omega_t^c)=o(t^{-\kappa})$ by Lemma 5, hence, with Lemma 2,

$$
P^{\geq 0}\bigl(E_\omega\bigl[\tau(e_1)\bigr]\geq t\bigr)\sim 2^\kappa C_Ut^{-\kappa},\qquad t\to\infty. \tag{8.9}
$$

9. Proof of Corollary 1. We show here how Corollary 1 follows from Theorem 2. With the notation of this theorem, it suffices to prove

$$
\mathscr{L}\Biggl(\frac{1}{x^{1/\kappa}}\sum_{p=1}^{x}\widehat{Z}_p\bar{\mathbf{e}}_p\Bigg|(\widehat{Z}_p)_{p\geq 1}\Biggr)\xrightarrow[x]{\mathrm{W}^1}\mathscr{L}\Biggl(\sum_{p=1}^{\infty}\xi_p\bar{\mathbf{e}}_p\Bigg|(\xi_p)_{p\geq 1}\Biggr)\qquad\text{in law,}
$$

where $(\widehat{Z}_p)_{p\geq 1}$ are i.i.d., independent of $(\bar{\mathbf{e}}_p)_{p\geq 1}$, such that $P(\widehat{Z}_1>t)\sim 2^\kappa C_Ut^{-\kappa}$, and $(\xi_p)_{p\geq 1}$ is a Poisson point process of intensity $2^\kappa C_U\kappa u^{-(\kappa+1)}\,\mathrm{d}u$, independent of $(\bar{\mathbf{e}}_p)_{p\geq 1}$. This reduction comes from the following easy property.

LEMMA 12. *If random variables $(X_n)_n$, $(Y_n)_n$ and Y take values in a metric space (E,d), $d(X_n,Y_n)\to_n 0$ in probability and $Y_n\to_n Y$ in law imply $X_n\to_n Y$ in law.*

Let us recall a simple result about order statistics of heavy-tailed random variables.

PROPOSITION 5. *Let $(Z_i)_{i\geq 1}$ be i.i.d. copies of a random variable $Z\geq 0$ such that*

$$
P(Z>t)\sim C_Zt^{-\kappa},\qquad t\to\infty, \tag{9.1}
$$

for some constant $C_Z>0$. For all $n\geq 1$, denote by $Z_n^{(1)}\geq\cdots\geq Z_n^{(n)}$ an ordering of the finite subsequence $(Z_1,\dots,Z_n)$. Then we have, for every $k\geq 1$,

$$
\frac{1}{n^{1/\kappa}}\bigl(Z_n^{(1)},\dots,Z_n^{(k)}\bigr)\xrightarrow[n]{\text{law}}\bigl(\xi^{(1)},\dots,\xi^{(k)}\bigr),
$$

where $\xi^{(k)} = C_Z^{1/\kappa}(\mathbf{f}_1 + \cdots + \mathbf{f}_k)^{-1/\kappa}$ *for* $k \ge 1$, $(\mathbf{f}_k)_k$ *being i.i.d. exponential random variables of parameter* 1; *cf.* (2.5).

PROOF. Let $Y_n^{(i)} := nC_Z(Z_n^{(i)})^{-\kappa}$, and $Y_n = nC_Z(Z_1)^{-\kappa}$. From (9.1) we deduce $nP(Y_n \in [a,b]) \to_n b - a$ for all $0 < a < b$. Then, for all $t_1, \dots, t_k > 0$,

$$\begin{aligned}
P\big(t_1 < Y_n^{(1)} < t_2 < Y_n^{(2)} < \cdots < t_k < Y_n^{(k)}\big)
&= n(n-1)\cdots\big(n-(k-1)+1\big)P\big(Y_n \in [t_1,t_2]\big)\cdots P\big(Y_n \in [t_{k-1},t_k]\big)\\
&\quad \times P\big(Y_n \notin [0,t_k]\big)^{n-k}\\
&\to_n (t_2 - t_1)\cdots(t_k - t_{k-1})\mathrm{e}^{-t_k}\\
&= P(t_1 < \mathbf{f}_1 < t_2 < \mathbf{f}_1 + \mathbf{f}_2 < \cdots < t_k < \mathbf{f}_1 + \cdots + \mathbf{f}_k),
\end{aligned}$$

by a simple computation, from where the proposition follows. $\square$

Thanks to the previous lemma and Skorohod's representation theorem, there exists a copy $(\widetilde{\xi}^{(p)})_{p\ge1}$ of $(\xi^{(p)})_{p\ge1}$ and, for all $k \ge 1$, there exist random variables $(\widetilde{Z}_{k,n}^{(1)}, \dots, \widetilde{Z}_{k,n}^{(k)})_{n\ge k}$ such that (borrowing notation from the lemma) for every $n \ge k$ $(\widetilde{Z}_{k,n}^{(1)}, \dots, \widetilde{Z}_{k,n}^{(k)})$ is a copy of $(\widehat{Z}_n^{(1)}, \dots, \widehat{Z}_n^{(k)})$ and

$$\frac{1}{n^{1/\kappa}}\big(\widetilde{Z}_{k,n}^{(1)}, \dots, \widetilde{Z}_{k,n}^{(k)}\big) \xrightarrow[n]{(p)} \big(\widetilde{\xi}^{(1)}, \dots, \widetilde{\xi}^{(k)}\big).$$

We chose $(\widetilde{\xi}^{(p)})_{p\ge1}$ not depending on k to ease notation but this is unessential since we only need to understand the convergences in probability $X_n \xrightarrow[n]{(p)} X$ as properties of the law of (X_n, X) for every n, no matter on which space Ω_n this couple is defined.

We may also introduce additional random variables $(\widetilde{Z}_{k,n}^{(k+1)}, \dots, \widetilde{Z}_{k,n}^{(n)})_{n\ge1}$ such that for every n $(\widetilde{Z}_{k,n}^{(1)}, \dots, \widetilde{Z}_{k,n}^{(n)})$ is a copy of $(\widehat{Z}_n^{(1)}, \dots, \widehat{Z}_n^{(n)})$.

Then, by a diagonal argument, we can define $(\widetilde{Z}_n^{(p)})_{1\le p\le n}$ such that, for every n, $(\widetilde{Z}_n^{(p)})_{1\le p\le n}$ is a copy of $(Z_n^{(1)}, \dots, Z_n^{(n)})$ and, *for every* k,

$$\frac{1}{n^{1/\kappa}}\big(\widetilde{Z}_n^{(1)}, \dots, \widetilde{Z}_n^{(k)}\big) \xrightarrow[n]{(p)} \big(\widetilde{\xi}^{(1)}, \dots, \widetilde{\xi}^{(k)}\big). \tag{9.2}$$

Indeed, there is an increasing sequence $(N(k))_k$ such that for all $k \ge 1$, for $n \ge N(k)$,

$$P\left(\left\|\frac{1}{n^{1/\kappa}}\big(\widetilde{Z}_{k,n}^{(1)}, \dots, \widetilde{Z}_{k,n}^{(k)}\big) - \big(\widetilde{\xi}^{(1)}, \dots, \widetilde{\xi}^{(k)}\big)\right\|_1 > \frac{1}{k}\right) < \frac{1}{k}$$

(hence, the same bound also holds for the first $k' \le k$ components) and then we define, for $n \ge N(1)$ and $1 \le p \le n$, $\widetilde{Z}_n^{(p)} = \widetilde{Z}_{k,n}^{(p)}$, where k is given by $N(k) \le n <$

$N(k+1)$; and, for instance, $\widetilde{Z}_n^{(p)} = \widetilde{Z}_{1,n}^{(p)}$ when $1 \le p \le n < N(1)$. This is easily seen to satisfy (9.2).

We have, for all $n \ge k$,

$$
\begin{aligned}
W^1_{\widetilde{Z},\widetilde{\xi}}&\left(\sum_{p=1}^{n} \frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\bar{\mathbf{e}}_p, \sum_{p=1}^{\infty} \widetilde{\xi}^{(p)}\bar{\mathbf{e}}_p\right) \\
&\le E_{\widetilde{Z},\widetilde{\xi}}\left[\left|\sum_{p=1}^{n} \frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\bar{\mathbf{e}}_p - \sum_{p=1}^{\infty} \widetilde{\xi}^{(p)}\bar{\mathbf{e}}_p\right|\right] \\
&\le E_{\widetilde{Z}}\left[\left|\sum_{p=k+1}^{n} \frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\bar{\mathbf{e}}_p\right|\right] + E_{\widetilde{Z},\widetilde{\xi}}\left[\left|\sum_{p=1}^{k}\left(\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}} - \widetilde{\xi}^{(p)}\right)\bar{\mathbf{e}}_p\right|\right] \\
&\quad + E_{\widetilde{\xi}}\left[\left|\sum_{p=k+1}^{\infty} \widetilde{\xi}^{(p)}\bar{\mathbf{e}}_p\right|\right] \qquad (9.3)\\
&\le \sqrt{\sum_{p=k+1}^{n}\left(\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\right)^2} + \sum_{p=1}^{k}\left|\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}} - \widetilde{\xi}^{(p)}\right| + \sqrt{\sum_{p=k+1}^{\infty} (\widetilde{\xi}^{(p)})^2},
\end{aligned}
$$

using $E[|\bar{\mathbf{e}}_p|] = 2/\mathrm{e} \le 1$ and the inequality $E[|W|]^2 \le E[W^2] = \mathrm{Var}(W)$ for any centered random variable W. Let $\varepsilon_k > 0$ be such that $k^{-1/\kappa} \ll \varepsilon_k \ll 1$, when $k \to \infty$. Since $\widehat{Z}_n^{(k)} \ge \widehat{Z}_n^{(p)}$ for $p \ge k$,

$$
\begin{aligned}
P&\left(\sqrt{\sum_{p=k+1}^{n}\left(\frac{\widehat{Z}_n^{(p)}}{n^{1/\kappa}}\right)^2} \ge \frac{\delta}{3}\right) \\
&\le P\left(\frac{\widehat{Z}_n^{(k)}}{n^{1/\kappa}} \ge \varepsilon_k\right) + P\left(\sum_{p=1}^{n}\left(\frac{\widehat{Z}_p}{n^{1/\kappa}}\right)^2 \mathbf{1}_{\{\widehat{Z}_p/n^{1/\kappa} < \varepsilon_k\}} \ge \left(\frac{\delta}{3}\right)^2\right) \\
&\le P\left(\frac{\widehat{Z}_n^{(k)}}{n^{1/\kappa}} \ge \varepsilon_k\right) + \frac{9}{\delta^2} n E\left[\left(\frac{\widehat{Z}_1}{n^{1/\kappa}}\right)^2 \mathbf{1}_{\{\widehat{Z}_1/n^{1/\kappa} < \varepsilon_k\}}\right],
\end{aligned}
$$

hence, using (9.2) and (2.4), for all $\delta > 0$,

$$
\limsup_n P\left(\sqrt{\sum_{p=k+1}^{n}\left(\frac{\widehat{Z}_n^{(p)}}{n^{1/\kappa}}\right)^2} \ge \frac{\delta}{3}\right) \le P(\xi^{(k)} \ge \varepsilon_k) + \frac{9}{\delta^2}\frac{2C}{2-\kappa}\varepsilon_k^{1-\kappa/2} =: \varphi_\delta(k),
$$

where $C > C_Z := 2^\kappa C_U$ is arbitrary. Note that $\varphi_\delta(k) \to_k 0$ due to the choice of ε_k and to (2.6). We also have, respectively, because of (9.2) and of $\sum_p (\xi^{(p)})^2 < \infty$

a.s. [cf. (2.6)],

$$P\left(\sum_{p=1}^{k}\left|\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}-\widetilde{\xi}^{(p)}\right|\geq\frac{\delta}{3}\right)\underset{n}{\longrightarrow}0\quad\text{and}\quad P\left(\sqrt{\sum_{p=k+1}^{\infty}(\xi^{(p)})^2}\geq\frac{\delta}{3}\right)\underset{k}{\longrightarrow}0.$$

Denote by $\psi_\delta(k)$ the latter probability. Thus, from (9.3), for all $\delta>0$,

$$\limsup_n P\left(W^1_{\widetilde{Z},\widetilde{\xi}}\left(\sum_{p=1}^{n}\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\bar{\mathbf{e}}_p,\sum_{p=1}^{\infty}\widetilde{\xi}^{(p)}\bar{\mathbf{e}}_p\right)\geq\delta\right)\leq\varphi_\delta(k)+\psi_\delta(k)\to_k 0.$$

Thanks to our diagonal argument, the left-hand side does not depend on k. Thus,

$$\mathscr{L}\left(\sum_{p=1}^{n}\frac{\widetilde{Z}_n^{(p)}}{n^{1/\kappa}}\bar{\mathbf{e}}_p\Big|(\widetilde{Z}_n^{(p)})_{1\leq p\leq n}\right)\xrightarrow[n]{W^1}\mathscr{L}\left(\sum_{p=1}^{\infty}\widetilde{\xi}^{(p)}\bar{\mathbf{e}}_p\Big|\widetilde{\xi}\right)\qquad\text{in probability,}$$

and therefore in law. Since the convergence in law only deals with the laws of $\widetilde{Z}_n$ for $n\geq 1$ and of $\widetilde{\xi}$ (and not on their coupling), this concludes the proof of Corollary 1.

Finally, we mention that the expression of the parameter λ obtained for Dirichlet environments [i.e., when ω_0 follows a distribution Beta(α,β) with $0<\alpha-\beta<2$] can be easily deduced from a computation of C_K by Chamayou and Letac [3] (see [8] for more details).

APPENDIX

A.1. Proofs of Lemmas 2 and 3.

PROOF OF LEMMA 2. Compared to (4.13), it appears sufficient to prove that $P^{\geq 0}(Z>t,S>H)=o(t^{-\kappa})$, which is understood as follows: when Z is large, the height H of the first excursion tends to be large as well, while the other excursions are independent of Z, hence, H is likely to be the maximum S of V over all of $\mathbb{Z}_+$. More precisely: first, for $\ell_t>0$,

$$P^{\geq 0}(Z>t,H<\ell_t)\leq P^{\geq 0}\big(M_1M_2>t\mathrm{e}^{-\ell_t}\big)\leq\frac{E^{\geq 0}[(M_1M_2)^2]}{(t\mathrm{e}^{-\ell_t})^2},$$

and all moments of M_1M_2 are finite under $P^{\geq 0}$ [indeed we have $M_2\leq e_1$, $M_1\leq e_1+R_-$ and the random variables e_1 and R_- have all moments finite under $P^{\geq 0}$; cf. Section 4.3 and (4.9)]. Thus, if (recalling that $\kappa<2$) we choose ℓ_t such that $\ell_t\to\infty$ and $t^\kappa=o(t^2\mathrm{e}^{-2\ell_t})$ as $t\to\infty$, we have $P^{\geq 0}(Z>t,H<\ell_t)=o(t^{-\kappa})$. On the other hand, Z is independent of $S':=\sup_{x\geq e_1}(V(x)-V(e_1))$ which is larger than S on the event $\{H<S\}$, hence,

$$\begin{aligned}P^{\geq 0}(Z>t,H<S)&=P^{\geq 0}(Z>t,H\geq\ell_t,H<S)+o\big(t^{-\kappa}\big)\\&\leq P^{\geq 0}(Z>t)P^{\geq 0}\big(S'>\ell_t\big)+o\big(t^{-\kappa}\big)\\&=P^{\geq 0}(Z>t)o(1)+o\big(t^{-\kappa}\big),\end{aligned}$$

as $t \to \infty$, such that, using (4.13),

$$\begin{aligned} P^{\geq 0}(Z > t) &= P^{\geq 0}(Z > t, H = S) + P^{\geq 0}(Z > t, H < S) \\ &= C_U t^{-\kappa} + o(t^{-\kappa}) + P^{\geq 0}(Z > t) o(1) + o(t^{-\kappa}), \end{aligned}$$

which implies the lemma. □

PROOF OF LEMMA 3. The very first bound results simply, by monotone convergence, from $E^{\geq 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H}] < \infty$ when $\gamma < \kappa$, which is a consequence, via Hölder inequality, of the fact that all the moments of M_1' and M_2 are finite under $P^{\geq 0}$ (because $M_1' \leq R_- + e_1$ and $M_2 \leq e_1$), and of the fact that, due to (4.10), e^H has moments up to order κ (not included). Let us turn to the other bounds.

Note that, if M_1' and M_2 were positive constants, then the bounds would follow by an elementary computation from the tail estimate (4.10) and the classical formulas

$$E[\mathrm{e}^{\gamma H} \mathbf{1}_{\{H \geq h\}}] = \mathrm{e}^{\gamma h} P(H \geq h) + \int_h^\infty \gamma \mathrm{e}^{\gamma u} P(H \geq u) \,\mathrm{d}u$$

and $E[\mathrm{e}^{\gamma H} \mathbf{1}_{\{H < h\}}] = 1 - \mathrm{e}^{\gamma h} P(H \geq h) + \int_0^h \gamma \mathrm{e}^{\gamma u} P(H \geq u) \,\mathrm{d}u$.

As recalled in Section 3, it was proved in [7] that indeed M_1 and M_2 depend little on H, in that (Lemma 4.1 of [7]) for any integer $r > 0$ there is a constant C such that

$$E^{\geq 0}[(M_1)^r | \lfloor H \rfloor, H = S] \leq C, \tag{A.1}$$

and similarly for M_2 (due to a symmetry property under $P^{\geq 0}(\cdot | H = S)$; see Lemma 3.4 in [7]). Admitting that furthermore,

$$E^{\geq 0}[(M_1')^r | \lfloor H \rfloor, H = S] \leq C, \tag{A.2}$$

we would first get by the Cauchy–Schwarz inequality that, with $M := (M_1')^\alpha (M_2)^\beta$,

$$\begin{aligned} &E^{\geq 0}[M | \lfloor H \rfloor, H = S] \\ &\quad \leq E^{\geq 0}[(M_1')^{2\alpha} | \lfloor H \rfloor, H = S]^{1/2} E^{\geq 0}[(M_2)^{2\beta} | \lfloor H \rfloor, H = S]^{1/2} \\ &\quad \leq C, \end{aligned}$$

and, using conditioning on $\lfloor H \rfloor$, conclude that

$$E^{\geq 0}[M \mathrm{e}^{\gamma H} \mathbf{1}_{\{H < h\}} | H = S] \leq C' E[\mathrm{e}^{\gamma(\lfloor H \rfloor + 1)} \mathbf{1}_{\{\lfloor H \rfloor < h\}}] \leq C'' E[\mathrm{e}^{\gamma H} \mathbf{1}_{\{H < h+1\}}],$$

and similarly $E^{\geq 0}[M \mathrm{e}^{\gamma H} \mathbf{1}_{\{H \geq h\}} | H = S] \leq C'' E[\mathrm{e}^{\gamma H} \mathbf{1}_{\{H \geq h-1\}}]$ which brings us back to the situation where M_1' and M_2 would be constants. Thus, it remains to prove (A.2) and, first, justify why introducing the convenient condition $\{H = S\}$ is harmless.

As in Lemma 2, the condition $\{H = S\}$ is typically satisfied when H is large; thus it suffices to note that the contribution to the expectations of small values of H is not too significant. Let $\ell = \ell(h) := \frac{1}{\gamma} \log h$. We have

$$E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}\big] \leq E^{\geq 0}[M]h + E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h, H>\ell\}}\big]. \tag{A.3}$$

Since M and H are independent of $S' := \sup_{x\geq e_1} V(x) - V(e_1)$, and also $\{S > H > \ell\} \subset \{S' > \ell\}$, we have on the other hand

$$E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h, H>\ell\}}\mathbf{1}_{\{S>H\}}\big] \leq E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}\big]P\big(S' > \ell\big)$$

and $P(S' > \ell) = o(1)$ when $h \to \infty$, hence, substracting this quantity to (A.3) gives

$$E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}\big]\big(1 + o(1)\big) \leq E^{\geq 0}[M]h + E^{\geq 0}\big[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}\mathbf{1}_{\{H=S\}}\big].$$

Given that $P(H = S) > 0$, and $h \leq \mathrm{e}^{(\gamma-\kappa)h}$ for large h when $\gamma > \kappa$, it thus suffices to prove the last two bounds of (4.14) with the left-hand side replaced by $E^{\geq 0}[M\mathrm{e}^{\gamma H}|H < h, H = S]$. As for (4.15), the introduction of ℓ is useless to similarly prove [skipping (A.3)] that we may condition by $\{H = S\}$.

Let us finally prove (A.2). Let $r > 0$. We have $M_1' = M_1 + \sum_{T_H<x<e_1} \mathrm{e}^{-V(x)}$. It results from Lemma 3.4 of [7] that $(H, \sum_{T_H\leq x<e_1} \mathrm{e}^{-V(x)})$ has the same distribution under $P^{\geq 0}(\cdot|H = S)$ as $(H, \sum_{T_H^-<x\leq 0} \mathrm{e}^{V(x)-H})$ where

$$T_H^- := \sup\{x \leq 0 \colon V(x) > H\},$$

and we claim that there is $C_r' > 0$ such that, for all $N \in \mathbb{N}$,

$$E\bigg[\bigg(\sum_{T_N^-<x\leq 0} \mathrm{e}^{V(x)}\bigg)^r\bigg] \leq C_r'\mathrm{e}^{rN}. \tag{A.4}$$

Before we prove this inequality, let us use it to conclude that

$$\begin{aligned} &E^{\geq 0}\big[(M_1')^r\,|\lfloor H\rfloor, H = S\big] \\ &\quad\leq 2^r\big(E^{\geq 0}\big[(M_1)^r\,|\lfloor H\rfloor, H = S\big] + \mathrm{e}^{-r\lfloor H\rfloor}C\mathrm{e}^{r(\lfloor H\rfloor+1)}\big) \leq C'. \end{aligned} \tag{A.5}$$

For readability reasons, we write the proof of (A.4) when $r = 2$, the case of higher integer values being exactly similar and implying the general case (if $0 < r < s$, $E[X^r] \leq E[X^s]^{r/s}$ for any positive X). We have

$$\begin{aligned} &E\bigg[\bigg(\sum_{T_N^-<x\leq 0} \mathrm{e}^{V(x)}\bigg)^2\bigg] \\ &\quad\leq \sum_{0\leq m,n<N} \mathrm{e}^{n+1}\mathrm{e}^{m+1}E\big[\nu\big([n, n+1)\big)\nu\big([m, m+1)\big)\big], \end{aligned} \tag{A.6}$$

where $\nu(A) := \#\{x \leq 0 \colon V(x) \in A\}$ for all $A \subset \mathbb{R}$. For any $n \in \mathbb{N}$, applying the Markov property at time $\sup\{x \leq 0 \colon V(x) \in [n, n+1)\}$ gives us that $E[\nu([n, n+1))^2] \leq E[\nu([-1, 1))^2]$. This latter expectation is finite because $V(1)$ has a negative mean and is exponentially integrable; more precisely, $\nu([-1, 1))$ is exponentially integrable as well: for $\lambda > 0$, for all $x \ge 0$, $P(V(-x) < 1) \le \mathrm{e}^{\lambda} E[\mathrm{e}^{\lambda V(1)}]^x = \mathrm{e}^{\lambda} E[\rho^{\lambda}]^x$ hence, choosing $\lambda > 0$ small enough so that $E[\rho^{\lambda}] < 1$ [cf. Assumption (a)], we have, for all $p \ge 0$,

$$\begin{aligned} P\big(\nu([-1,1))>p\big) &\le P\big(\exists x \ge p \text{ s.t. } V(-x)<1\big) \\ &\le \sum_{x\ge p} P\big(V(-x)<1\big) \le \mathrm{e}^{\lambda}\big(1-E[\rho^{\lambda}]\big)^{-1} E[\rho^{\lambda}]^p. \end{aligned}$$

Thus, using the Cauchy–Schwarz inequality to bound the expectations uniformly, the right-hand side of (A.6) is less than $C\mathrm{e}^{2N}$ for some constant C. This proves (A.4) and therefore concludes the proof of Lemma 3. □

A.2. Proofs of Lemmas 5, 6 and 7.

PROOF OF LEMMA 5. By the union bound the proof of Lemma 5 boils down to showing that for $i = 1, 2, 3$,

$$P\big((\Omega_t^{(i)})^c, H \ge \mathfrak{h}_t\big) = o\big(t^{-\kappa}\big), \qquad t \to \infty.$$

The case $i = 1$ is trivial. Indeed, the fact that e_1 has some finite exponential moments (see Section 4.3) implies that $P((\Omega_t^{(1)})^c) = o(t^{-\kappa})$ when t tends to infinity (for C large enough). The case $i = 2$ can be proved by a minor adaptation of the proof of Lemma 5.5 in [8].

Let us consider the last case $i = 3$. Since R^- depends only on the variables $V(x), x \le 0$, and $P(H > \mathfrak{h}_t) \sim C_I t^{-\kappa}(\log t)^{\kappa}$ when $t \to \infty$, it suffices to prove $P^{\ge 0}(R^- > (\log t)^4 t^{\alpha}) = o((\log t)^{-\kappa})$. This would follow (for any $\alpha > 0$) from the Markov property if $E^{\ge 0}[R^-] < \infty$. We have (changing indices and incorporating the single terms into the sums)

$$\begin{aligned} R^- &= \sum_{x\le 0}\Big(1 + 2\sum_{x<y\le 0} \mathrm{e}^{V(y)-V(x)}\Big)\Big(\mathrm{e}^{-V(x)} + 2\sum_{z\le x-1}\mathrm{e}^{-V(z)}\Big) \\ &\le 4\sum_{z\le x\le y\le 0} \mathrm{e}^{V(y)-V(x)-V(z)}, \end{aligned} \tag{A.7}$$

and this latter quantity was already seen to be integrable under $P^{\ge 0}$, after (7.12), when $1 < \kappa < 2$. In order to deal with the case $0 < \kappa \le 1$, let us introduce the event

$$A_t = \bigcap_{p=1}^{\infty}\Big\{H_{-p} < \frac{1}{\kappa}\log p^2 + \log t + \log\log t\Big\}.$$

On one hand, by (4.10), $P((A_t)^c) \le \sum_{p=1}^{\infty} \frac{C}{p^2(t\log t)^{\kappa}} = (\sum_{p=1}^{\infty}\frac{C}{p^2})\frac{t^{-\kappa}}{(\log t)^{\kappa}} = o(t^{-\kappa})$. On the other hand, proceeding as after (7.12),

$$\begin{aligned} &E^{\ge 0}\big[R^- \mathbf{1}_{A_t}\big] \\ &\quad \le 4\sum_{p\le 0} E^{\ge 0}\big[\mathrm{e}^{-V(e_p)}\big]E^{\ge 0}\big[(M_1')^2 M_2 \mathrm{e}^{H}\mathbf{1}_{\{H<(1/\kappa)\log p^2 + \log t + \log\log t\}}\big] \end{aligned}$$

and $E^{\geq 0}[\mathrm{e}^{-V(e_p)}] = E[\mathrm{e}^{V(e_1)}]^p$ hence, using Lemma 3, when $0 < \kappa < 1$,

$$E^{\geq 0}[R^-\mathbf{1}_{A_t}] \leq 4\biggl(\sum_{p\leq 0} E[\mathrm{e}^{V(e_1)}]^p \frac{1}{(p^2)^{(1-\kappa)/\kappa}}\biggr)(t\log t)^{1-\kappa} \leq C(t\log t)^{1-\kappa},$$

and when $\kappa = 1$,

$$E^{\geq 0}[R^-\mathbf{1}_{A_t}] \leq 4\sum_{p\leq 0} E[\mathrm{e}^{V(e_1)}]^p \biggl(\frac{1}{\kappa}\log p^2 + \log t + \log\log t\biggr) \leq C\log t.$$

Finally, by the Markov inequality,

$$P^{\geq 0}(R^- > t^\alpha(\log t)^4) \leq P^{\geq 0}((A_t)^c) + \frac{1}{t^\alpha(\log t)^4}E^{\geq 0}[R^-\mathbf{1}_{A_t}]$$

is negligible with respect to $(\log t)^{-\kappa}$ for any $\alpha \geq 1-\kappa$ when $0 < \kappa < 1$, and for any $\alpha > 0$ when $\kappa = 1$. □

PROOF OF LEMMA 6. Since $\mathrm{Var}_\omega(F) \leq E_\omega[F^2]$, the proof of (5.4) is a consequence of (5.10) in [8] together with a minor adaptation of equation (5.26) in [8] and the definition of Ω_t. The proof of (5.5) is a direct consequence of the definitions of M_2 [see equation (4.12)] and Ω_t [see equation (5.2)]. Finally, the proof of (5.6) is straightforward by looking at the expression of $E_\omega[F] = 2\omega_0\widehat{M}_1$ in terms of the modified potential $\widehat{V}$ (see Lemma 5.2 in [8]) together with the properties of good environments ω in Ω_t. □

PROOF OF LEMMA 7. The proof of Lemma 7 can be deduced from Lemma 5.4 in [8] (which gives an upper bound for $E_\omega[G]$ in terms of the modified potential $\bar{V}$), the definition of the modified potential $\bar{V}$ (see equation (5.15) in [8]) and the definition of good environments ω in Ω_t. □

A.3. An annealed result. The techniques of this paper enable us to prove the following annealed counterpart to (8.9) which has its own interest.

PROPOSITION 6. *The tail distribution of the hitting time of the first negative record e_1 satisfies*

$$t^\kappa \mathbb{P}^{\geq 0}(\tau(e_1) \geq t) \longrightarrow C_T, \qquad t \to \infty, \tag{A.8}$$

where the constant C_T is given by

$$C_T := 2^\kappa\Gamma(\kappa+1)C_U. \tag{A.9}$$

Let us write τ for $\tau(e_1)$ in this section. The idea of the proof is the following. We first show that, on the event $\{\tau \geq t\}$, the height of the first excursion is typically larger than the function $\mathfrak{h}_t$ [of order $\log t$, defined in (5)]. We may then invoke Proposition 1 to reduce the tail of τ to that of $E_\omega[\tau]\mathbf{e}$ and conclude.

LEMMA 13. *We have*

$$\mathbb{P}^{\geq 0}(\tau(e_1) \geq t, H < \mathfrak{h}_t) = o(t^{-\kappa}), \qquad t \to \infty.$$

PROOF. Let us first assume that $0 < \kappa < 1$. Then, by the Markov inequality, we get

$$\mathbb{P}^{\geq 0}(\tau \geq t, H < \mathfrak{h}_t) = E^{\geq 0}[P_\omega(\tau \geq t)\mathbf{1}_{\{H<\mathfrak{h}_t\}}] \leq \frac{1}{t} E^{\geq 0}[E_\omega[\tau]\mathbf{1}_{\{H<\mathfrak{h}_t\}}]$$
$$\leq \frac{1}{t} E^{\geq 0}[2M_1' M_2 \mathrm{e}^H \mathbf{1}_{\{H<\mathfrak{h}_t\}}] \leq \frac{1}{t} C\mathrm{e}^{(1-\kappa)\mathfrak{h}_t},$$

where the last inequality follows from Lemma 3. Since we have $t^{-1}\mathrm{e}^{(1-\kappa)\mathfrak{h}_t} = t^{-\kappa}(\log t)^{-(1-\kappa)}$, this settles this case.

Let us now assume $1 < \kappa < 2$. By the Markov inequality, we get

$$\mathbb{P}^{\geq 0}(\tau \geq t, H < \mathfrak{h}_t) \leq \frac{1}{t^2} E^{\geq 0}[E_\omega[\tau^2]\mathbf{1}_{\{H<\mathfrak{h}_t\}}].$$

Applying Lemma 10 yields $\mathbb{P}^{\geq 0}(\tau \geq t, H < \mathfrak{h}_t) \leq Ct^{-2}\mathrm{e}^{(2-\kappa)\mathfrak{h}_t}$, which concludes the proof of Lemma 13 when $\kappa \neq 1$.

For $\kappa = 1$, neither of the above techniques works; the first one is too rough, and $\mathrm{Var}_\omega(\tau)$ is not integrable hence, the second does not make sense as is. We shall modify τ so as to make $\mathrm{Var}_\omega(\tau)$ integrable. To this end, let us refer to Section 7.1 and denote by d_- the right end of the first excursion on the left of 0 that is higher than h_t, and by $\widetilde{\tau} := \widetilde{\tau}^{(d_-)}(0, e_1)$ the time spent on the left of d_- before reaching e_1. By Lemma 8 we have $\mathbb{E}^{\geq 0}[\widetilde{\tau}\mathbf{1}_{\{H<\mathfrak{h}_t\}}] \leq C\mathfrak{h}_t \mathrm{e}^{-\mathfrak{h}_t} \leq C(\log t)^2 t^{-1}$. Let us also introduce $\widetilde{\tau}'$, which is defined like $\widetilde{\tau}$ but in the modified environment, that is, by replacing the high excursions (on the left of d_-) by small ones; cf. after Lemma 8. Then we have

$$\begin{aligned}
&\mathbb{P}^{\geq 0}(\tau \geq t, H < \mathfrak{h}_t)\\
&\quad \leq \mathbb{P}^{\geq 0}(\widetilde{\tau} \geq (\log t)^3, H < \mathfrak{h}_t) + \mathbb{P}^{\geq 0}(\tau - \widetilde{\tau} \geq t - (\log t)^3, H < \mathfrak{h}_t)\\
&\quad \leq \frac{1}{(\log t)^3}\mathbb{E}^{\geq 0}[\widetilde{\tau}\mathbf{1}_{\{H<\mathfrak{h}_t\}}] + \mathbb{P}^{\geq 0}(\tau - \widetilde{\tau} + \widetilde{\tau}' \geq t - (\log t)^3, H < \mathfrak{h}_t)\\
&\quad = o(t^{-1}) + (\mathbb{P}^{\geq 0})'(\tau \geq t - (\log t)^3, H < \mathfrak{h}_t)\\
&\quad \leq o(t^{-1}) + \frac{1}{(t - (\log t)^3)^2}(E^{\geq 0})'[E_\omega[\tau^2]\mathbf{1}_{\{H<\mathfrak{h}_t\}}],
\end{aligned}$$

and Lemma 10 allows us to conclude just like in the case $1 < \kappa < 2$. □

PROOF OF PROPOSITION 6. From the tail of $E_\omega[\tau]$ [cf. (8.9)], a simple computation gives

$$P^{\geq 0}(E_\omega[\tau]\mathbf{e} \geq t) \sim C_T t^{-\kappa}, \qquad t \to \infty. \tag{A.10}$$

Let us prove that this is also the tail of τ.

For any function $t \mapsto u_t$ we have, using, respectively, the previous lemma for the first bound and Proposition 1 and the Markov inequality (with respect to P_ω)

for the second,

$$\begin{aligned}\mathbb{P}^{\geq 0}\big(\tau - E_\omega[\tau]\mathbf{e} \geq u_t, \tau > t\big) &\leq \mathbb{P}^{\geq 0}\big(\tau - E_\omega[\tau]\mathbf{e} \geq u_t, H \geq \mathfrak{h}_t\big) + o\big(t^{-\kappa}\big) \\ &\leq \frac{t^\beta}{u_t}\mathbb{P}(H \geq \mathfrak{h}_t) + o\big(t^{-\kappa}\big) \\ &= t^{-\kappa}\bigg(\frac{t^\beta \log t}{u_t}\big(1+o(1)\big) + o(1)\bigg).\end{aligned}$$

If we choose u_t such that $t^\beta(\log t)^\kappa \ll u_t \ll t$ then we get, assembling this with (A.10),

$$\begin{aligned}\mathbb{P}^{\geq 0}(\tau > t) &= \mathbb{P}^{\geq 0}\big(\tau - E_\omega[\tau]\mathbf{e} \geq u_t, \tau > t\big) + \mathbb{P}^{\geq 0}\big(\tau - E_\omega[\tau]\mathbf{e} < u_t, \tau > t\big) \\ &\leq o\big(t^{-\kappa}\big) + \mathbb{P}^{\geq 0}\big(E_\omega[\tau]\mathbf{e} \geq t - u_t\big) \sim C_T t^{-\kappa}.\end{aligned}$$

The lower bound is identical, starting with

$$\mathbb{P}^{\geq 0}(\tau > t) \geq \mathbb{P}^{\geq 0}\big(E_\omega[\tau]\mathbf{e} \geq t + u_t\big) - \mathbb{P}^{\geq 0}\big(\tau - E_\omega[\tau]\mathbf{e} \leq -u_t, \tau > t\big).$$

This concludes the proof of Proposition 6. $\square$

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