The Annals of Applied Probability
2013, Vol. 23, No. 3, 1074–1085
DOI: 10.1214/12-AAP864

EXAMPLES OF NONPOLYGONAL LIMIT SHAPES IN I.I.D. FIRST-PASSAGE PERCOLATION AND INFINITE COEXISTENCE IN SPATIAL GROWTH MODELS

BY MICHAEL DAMRON[1] AND MICHAEL HOCHMAN[2]

Princeton University

We construct an edge-weight distribution for i.i.d. first-passage percolation on $\mathbb{Z}^2$ whose limit shape is not a polygon and whose extreme points are arbitrarily dense in the boundary. Consequently, the associated Richardson-type growth model can support coexistence of a countably infinite number of distinct species, and the graph of infection has infinitely many ends.

1. Introduction. Throughout this note μ denotes a Borel probability measure on $[0, \infty)$ with finite mean and such that $\mu(\{0\}) < p_c$, the critical probability for bond percolation in $\mathbb{Z}^2$, and $\mathcal{M}$ denotes the family of such measures. Let $\mathbb{E}$ denote the set of nearest-neighbor edges of the lattice $\mathbb{Z}^2$, and let $\{\tau_e : e \in \mathbb{E}\}$ be a family of i.i.d. random variables with marginal μ and joint distribution $\mathbb{P} = \mu^{\mathbb{E}}$. The *passage time* of a path $\gamma = (e_1, \dots, e_n) \in \mathbb{E}^n$ in the graph $(\mathbb{Z}^2, \mathbb{E})$ is $\tau(\gamma) = \sum_{i=1}^n \tau_{e_i}$, and for $x, y \in \mathbb{Z}^2$ the *passage time* from x to y is

$$\tau(x, y) = \min_{\gamma} \tau(\gamma),$$

where the minimum is over all paths γ joining x to y. A minimizing path is called a *geodesic* from x to y.

The theory of first passage percolation (FPP) is concerned with the large-scale geometry of the metric space $(\mathbb{Z}^2, \tau)$. The following fundamental result concerns the asymptotic geometry of balls. Write $B(t) = \{x \in \mathbb{Z}^2 : \tau(0, x) \le t\}$ for the ball of radius t at the origin, and for $S \subseteq \mathbb{R}^2$ and $a \ge 0$, write $aS = \{ax : x \in S\}$.

THEOREM 1.1 (Cox and Durrett [1]). *For every $\mu \in \mathcal{M}$ there exists a deterministic, compact,*[3] *convex set B_μ, with nonempty interior, such that for every*

Received September 2010; revised February 2011.
[1]Supported by an NSF postdoctoral fellowship.
[2]Supported by NSF Grant 0901534.
MSC2010 subject classifications. Primary 60K35; secondary 82B43.
Key words and phrases. First-passage percolation, limit shapes, extreme points, Richardson's growth model, graph of infection.

[3]This is the only place where we use the assumptions on μ. If $\mu(\{0\})$ exceeds the critical percolation probability, then $B_\mu = \mathbb{R}^2$ in an appropriate sense, and without finite mean we could have $B_\mu = \{0\}$.

$\varepsilon > 0$,

$$\mathbb{P}\left((1-\varepsilon)B_\mu \subseteq \frac{1}{t}B(t) \subseteq (1+\varepsilon)B_\mu \textit{ for all large } t\right) = 1.$$

Little is known about the geometry of B_μ, which is called the *limit shape*. It is conjectured to be strictly convex when μ is nonatomic, and nonpolygonal in all but the most degenerate cases, but, in fact, there are currently no known examples of μ for which these properties are verified (see [8]). For a compact, convex set $C \subseteq \mathbb{R}^2$ write $\mathrm{ext}(C)$ for the set of extreme points and $\mathrm{sides}(C) = |\mathrm{ext}(C)|$, so that C is a polygon if and only if $\mathrm{sides}(C) < \infty$. The best result to date, due to Marchand [13], is that under mild assumptions, $\mathrm{sides}(B_\mu) \geq 8$. Building on results of Marchand, our purpose of this note is to give the first examples of distributions for which the limit shape is not a polygon. If A and B are subsets of $\mathbb{R}^2$ (with, say, the ℓ^1-metric), we say that A is *ε-dense* in B if for each $x \in B$ there exists $y \in A$ such that $\|x - y\|_1 < \varepsilon$.

THEOREM 1.2. *For every $\varepsilon > 0$ there exists $\mu \in \mathcal{M}$ (with atoms) such that B_μ is not a polygon, that is,* $\mathrm{sides}(B_\mu) = \infty$, *and* $\mathrm{ext}(B_\mu)$ *is ε-dense in ∂B_μ. There exist nonatomic μ such that* $\mathrm{sides}(B_\mu) > 1/\varepsilon$ *and* $\mathrm{ext}(B_\mu)$ *is ε-dense in ∂B_μ.*

It is tempting to try to obtain a strictly convex limit shape by taking a limit of measures μ_n such that B_{μ_n} have progressively denser sets of extreme points, but unfortunately the limit one gets in our example is the unit ball of ℓ^1.

We also obtain examples of measures μ such that, at the points $v \in \mathrm{ext}(B_\mu)$ which lie on the boundary of the ℓ^1-unit ball, ∂B_μ is infinitely differentiable. This should be compared with the work of Zhang [16], where such behavior was ruled out for certain μ. Last, as we explain in Section 3, we can produce measures μ which are not purely atomic that have $\mathrm{sides}(B_\mu) = \infty$.

Theorem 1.2 has implications for the Richardson growth model, whose definition we recall next. Fix $x_1, \ldots, x_k \in \mathbb{Z}^2$ and imagine that at time 0 the site x_i is inhabited by a species of type i. Each species spreads at unit speed, taking time τ_e to cross an edge $e \in \mathbb{E}$. An uninhabited site is exclusively and permanently colonized by the first species that reaches it, that is, $y \in \mathbb{Z}^2$ is occupied at time t by the ith species if $\tau(y, x_i) \leq t$ and $\tau(y, x_i) < \tau(y, x_j)$ for all $j \neq i$. When there are *unique geodesics*, that is, $\mathbb{P}$-a.s. no two paths have the same passage time, as is the case when μ is continuous, each site is eventually occupied by a unique species. We shall also want to consider measures μ with atoms. The definition of the model in this case is formally the same, but note that there may be sites which are never colonized, that is, those sites y for which $\min_{1 \leq i \leq k} \tau(y, x_i)$ is achieved by multiple x_i's.

Given initial sites $x_1, \ldots, x_k$, consider the set colonized by the ith species,

$$C_i = \{y \in \mathbb{Z}^2 : y \text{ is eventually occupied by } i\}.$$

One says that μ *admits coexistence of k species* if for some choice of $x_1, \dots, x_k$,

$$\mathbb{P}(|C_i| = \infty \text{ for all } i = 1, \dots, k) > 0.$$

Coexistence of infinitely many species is defined similarly. Notice that if μ has atoms and a site x is colonized by species i, then the same will be true if we change the model by introducing an arbitrary tie-breaking rule governing the infections of sites which are reached simultaneously by more than one species. Thus, if coexistence of k species holds in our model, then the same is true under any tie-breaking rule.

In the past ten years, there have been many studies related to Richardson-type models, for instance, in questions related to the asymptotic shape of infected regions [7, 15] and to coexistence [6, 9–11]. Pertaining to the latter, it is not known, even in simple examples, how many species can coexist. When μ is the exponential distribution, Häggström and Pemantle [9] proved coexistence of 2 species (see [3] for a review of recent results on Richardson models, focused on exponential passage times). Shortly thereafter, Garet and Marchand [6] and Hoffman [10] independently extended these results to prove coexistence of 2 species for a broad class of translation-invariant measures on $(0, \infty)^{\mathbb{E}}$, including some non-i.i.d. ones. Later, Hoffman [11] demonstrated coexistence of 8 species for a similarly broad class of measures by establishing a relation with the number of sides of the limit shape in the associated FPP. Using the same relation we obtain the following theorem.

THEOREM 1.3. *There exists $\mu \in \mathcal{M}$ (with atoms) which admits coexistence of infinitely many species. For each k there exist nonatomic $\mu \in \mathcal{M}$ admitting coexistence of k species.*

When μ is nonatomic Theorem 1.3 follows from Theorem 1.2 and from Hoffman [11], Theorem 1.4. For the atomic case we provide the necessary modifications of Hoffman's arguments in Section 4.

Finally, the *graph of infection* $\Gamma(0) \subseteq \mathbb{E}$ is the union over $x \in \mathbb{Z}^d$ of the edges of geodesics from 0 to x. This terminology is consistent with the Richardson model when there are unique passage times, in which case it is a tree, but note that in general the graph of infection may also contain sites which were not infected, that is, those where a tie condition exists, and in this way we may obtain loops. A graph has *m ends* if, after removing a finite set of vertices, the induced graph contains at least m infinite connected components, and, if there are m ends for every $m \in \mathbb{N}$, we say there are infinitely many ends. Letting $K(\Gamma(0))$ be the number of ends in $\Gamma(0)$, Newman [14] has conjectured for a broad class of μ that $K(\Gamma(0)) = \infty$. Hoffman [11] showed for continuous distributions that in general $K(\Gamma(0)) \geq 4$ almost surely.

THEOREM 1.4. *There exist $\mu \in \mathcal{M}$ (with atoms) such that $\mathbb{P}$-a.s., $K(\Gamma(0)) = \infty$. For each k there exist nonatomic $\mu \in \mathcal{M}$ such that $\mathbb{P}$-a.s., $K(\Gamma(0)) \geq k$.*

When there are unique geodesics, Hoffman's results imply that $K(\Gamma(0))$ is at least $\text{sides}(B_\mu)/2$ ([11], Theorem 1.4), which proves Theorem 1.4 for nonatomic μ. In the case that μ has atoms, Theorem 1.4 follows directly from Theorem 1.2 (using the fact that the measure can be made to be not purely atomic) and the following result, which we prove in Section 5.

THEOREM 1.5. *If μ is not purely atomic and B_μ has at least $s \in \mathbb{N}$ sides, then $K(\Gamma(0))$ is $\mathbb{P}$-a.s. at least*

$$k = 4\left\lfloor \frac{s-4}{12} \right\rfloor. \tag{1.1}$$

See below Theorem 5.1 in Section 5 for an explanation of this bound.

2. Background on the limit shape. For any $x \in \mathbb{Z}^2$ let $m_\mu(x) = \lim_{n\to\infty} \tau(0, nx)/n$. This limit exists by Theorem 1.1 and by a theorem of Cox and Kesten, the map $\mu \mapsto m_\mu((1,0))$ is continuous [2]. By [12], Remark 6.18, this continuity can be extended to other unit vectors x and is actually uniform over all of them. To describe this, endow $\mathcal{M}$ with the topology of weak convergence and for convenience fix a compatible metric $d(\cdot,\cdot)$ on $\mathcal{M}$. Next, fix the ℓ^1-metric on $\mathbb{R}^2$, and write $A^{(\varepsilon)}$ for the ε-neighborhood of $A \subseteq \mathbb{R}$. Let $\mathcal{C}$ denote the space of nonempty, closed, convex subsets of $\mathbb{R}^2$ endowed with the Hausdorff metric d_H,

$$d_H(A,B) = \inf\{\varepsilon : A \subseteq B^{(\varepsilon)} \text{ and } B \subseteq A^{(\varepsilon)}\}.$$

THEOREM 2.1 (Kesten). *The map $\mu \mapsto B_\mu$ from $\mathcal{M}$ to $\mathcal{C}$ is continuous.*

We shall use the following elementary semicontinuity property of the map $A \mapsto \text{ext}(A)$ for $A \in \mathcal{C}$.

LEMMA 2.1. *Let $A \in \mathcal{C}$ and $x \in \text{ext}\,A$. For every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $A' \in \mathcal{C}$ and $d_H(A,A') < \delta$, then there exists $x' \in \text{ext}\,A'$ with $\|x - x'\|_1 < \varepsilon$.*

PROOF. Choose a linear functional $T : \mathbb{R}^2 \to \mathbb{R}$ and $\beta > 0$ such that $T(x) > \beta$ and the set $B = \{y \in A : T(y) \ge \beta\}$ has diameter less than $\varepsilon/2$. Note that $x \in B$. Since T is continuous, for small enough δ, if $d_H(A,A') < \delta$ then the set $B' = \{y \in A' : T(y) \ge \beta\}$ is nonempty and satisfies $d_H(B,B') < \varepsilon/2$. Since T is linear, its maximum on the convex set A' is attained at some extreme point $x' \in \text{ext}\,A'$, and by definition $x' \in B'$. Now if y is the closest point in B to x' then $\|x' - y\|_1 < \varepsilon/2$ and we also have $\|x - y\|_1 < \varepsilon/2$, so $\|x - x'\|_1 < \varepsilon$. □

Combining this lemma with Theorem 2.1, we have the following corollary.

COROLLARY 2.1. *Let $\mu \in \mathcal{M}$. For every $x_1, \dots, x_k \in \text{ext}(B_\mu)$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, if $\nu \in \mathcal{M}$ and $d(\nu, \mu) < \delta$ then there are $y_1, \dots, y_k \in \text{ext}(B_\nu)$ such that $\|x_i - y_i\| < \varepsilon$ for $i = 1, \dots, k$.*

Next we recall some results about limit shapes for a special class of measures. Given $0 < p < 1$, let $\mathcal{M}_p \subseteq \mathcal{M}$ denote the set of measures $\mu \in \mathcal{M}$ with an atom of mass p located at $x = 1$, that is, $\mu(\{1\}) = p$, and no mass to the left of 1, that is, $\mu((-\infty, 1)) = 0$. Limit shapes for μ of this form were first studied by Durrett and Liggett [5]. Writing $\vec{p}_c$ for the critical parameter of oriented percolation on $\mathbb{Z}^2$ (see Durrett [4] for background), it was shown that when $p > \vec{p}_c$ and $\mu \in \mathcal{M}_p$, the limit shape B_μ contains a "flat edge," or more precisely, ∂B_μ has sides which lie on the boundary of the ℓ^1-unit ball. The nature of this edge was fully characterized by Marchand in [13]. For $p \geq \vec{p}_c$, let α_p be the asymptotic speed of super-critical oriented percolation on $\mathbb{Z}^2$ with parameter p (see [4]). Define points $w_p, w'_p \in \mathbb{R}^2$ by

$$w_p = (1/2 + \alpha_p/\sqrt{2}, 1/2 - \alpha_p/\sqrt{2}),$$
$$w'_p = (1/2 - \alpha_p/\sqrt{2}, 1/2 + \alpha_p/\sqrt{2}).$$

Let $[w_p, w'_p] \subseteq \mathbb{R}^2$ denote the line segment with endpoints w_p and w'_p. It will be important to note that α_p is strictly increasing in $p > \vec{p}_c$, so the same is true of $[w_p, w'_p]$.

THEOREM 2.2 (Marchand [13]). *Let $\mu \in \mathcal{M}_p$. Then*:

(1) $B_\mu \subseteq \{x \in \mathbb{R}^2 : \|x\|_1 \leq 1\}$.
(2) *If $p < \vec{p}_c$, then $B_\mu \subseteq \{x \in \mathbb{R}^2 : \|x\|_1 < 1\}$.*
(3) *If $p > \vec{p}_c$, then $\partial B_\mu \cap \{(x, y) \in \mathbb{R}^2 : x + y = 1\} = [w_p, w'_p]$.*
(4) *If $p = \vec{p}_c$, then $\partial B_\mu \cap \{(x, y) \in \mathbb{R}^2 : x + y = 1\} = \{(1/2, 1/2)\}$.*

As noted by Marchand, this implies $\text{sides}(B_\mu) \geq 8$ for $\mu \in \mathcal{M}_p$ and $\vec{p}_c < p < 1$, since w_p, w'_p and their reflections about the axes are extreme points.

3. Proof of Theorem 1.2. Our aim is to construct a $\mu \in \mathcal{M}$ with $\text{sides}(B_\mu) = \infty$. Fix any $p_0 > \vec{p}_c$, $\mu_0 \in \mathcal{M}_{p_0}$ and a real parameter $\eta_0 > 0$. We will inductively define a sequence $p_1 > p_2 > \cdots > \vec{p}_c$, measures $\mu_1 \in \mathcal{M}_{p_1}, \mu_2 \in \mathcal{M}_{p_2}, \dots,$ and $\eta_1, \eta_2, \dots > 0$ such that for every $n \geq 0$ and all $k \leq n$,

(1) If $\nu \in \mathcal{M}$ and $d(\nu, \mu_k) < \eta_k$ then $\text{sides}(B_\nu) \geq k$, and
(2) $d(\mu_k, \mu_n) < \frac{1}{2}\eta_k$.

Note that, in particular, $\text{sides}(B_{\mu_k}) \geq k$ by (1). Assuming p_k, μ_k and η_k are defined for $k \leq n$, we define them for $n + 1$. Fix $p_{n+1} \in (\vec{p}_c, p_n)$ and set

$r = p_n - p_{n+1} > 0$. For $y > 1$ construct μ_{n+1}^y from μ_n by moving an amount r of mass from the atom at 1 to y, that is,

$$\mu_{n+1}^y = \mu_n - r\delta_1 + r\delta_y.$$

We claim that for small enough $y > 1$ and any sufficiently small choice of $\eta_{n+1} > 0$ (depending on the previous parameters), the measure $\mu_{n+1} = \mu_{n+1}^y$ has the desired properties. First, $\mu_{n+1}^y \to \mu_n$ weakly as $y \downarrow 1$, so, since μ_n satisfies (2), so does μ_{n+1}^y for all sufficiently small y.

Second, we claim that $\text{sides}(B_{\mu_{n+1}^y}) \geq n + 1$ for y close enough to 1. Indeed, since $r > 0$ we have $w_{p_{n+1}} \neq w_{p_n}$. Using (1), choose n extreme points $x_1, \dots, x_n \in \text{ext}(B_{\mu_n})$ and let

$$a = \min\{\|x_i - x_j\|_1, \|x_i - w_{p_{n+1}}\|_1 : i \neq j\}.$$

Note that $a > 0$ by Marchand's theorem. By Corollary 2.1, for y close enough to 1, for each $i = 1, \dots, n$ we can choose an extreme point x_i' of $B_{\mu_{n+1}^y}$ with $\|x_i' - x_i\| < a/2$. By Marchand's theorem, $B_{\mu_{n+1}^y}$ also has an extreme point at $w_{p_{n+1}}$. By definition of a, these extreme points are distinct, giving $\text{sides}(B_{\mu_{n+1}^y}) \geq n + 1$.

Finally, by Corollary 2.1, μ_{n+1}^y satisfies (1) for any sufficiently small choice of η_{n+1}.

Let μ be a weak limit of μ_n. Then $d(\mu, \mu_n) \leq \frac{1}{2}\eta_n$ for all n, so by (2), $\text{sides}(B_\mu) = \infty$. The proof is complete.

One can modify the construction in a number of ways in order to control the resulting measure μ. First, at each step, rather than creating a new atom at y, one can instead add, for example, Lebesgue measure on a small interval around y. In this way one can make the atom at 1 be the only atom of μ.

Regarding the degree of denseness of the extreme points in the boundary, note that at each stage, if y is small enough and p_{n+1} is close enough to p_n, the new extreme point we introduce can be made arbitrarily close to w_{p_n} (here we use that α_p is continuous in $p > \vec{p}_c$, from [4]), and in the limit we can ensure an extreme point close to it. Thus, if we begin from $\mu_0 = \delta_1$ and choose p_n so that $\lim p_n = \vec{p}_c$, and using Marchand's result that the flat edge in B_{μ_n} then shrinks to a point (and symmetry of the limit shape about the axes), we can ensure ε-density of the extreme points of B_μ.

For the second part of Theorem 1.2, choose a sequence $\nu_n \in \mathcal{M}$ of continuous measures converging weakly to μ. By Corollary 2.1, $\text{sides}(B_{\nu_n}) \to \infty$ and if $\text{ext}(B_\mu)$ is ε-dense in ∂B_μ, then the same holds for B_{ν_n} for sufficiently large n.

Regarding the remark after Theorem 1.2, one may verify that if at each stage y is chosen close enough to 1 and $p = \lim p_n$, then w_p is a C^∞-point of ∂B_μ.

4. Proof of Theorem 1.3. Let us recall Hoffman's argument relating coexistence to the geometry of the limit shape for continuous μ ([11], Theorem 1.6).

Extend τ to $\mathbb{R}^2 \times \mathbb{R}^2$ by $\tau(x, y) = \tau(x', y')$ where x' is the unique lattice point in $x + [-1/2, 1/2)^2$. Similarly, a geodesic between x, y is a geodesic between x', y'. For $S \subseteq \mathbb{R}^2$, the Busemann function $B_S : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$B_S(x, y) = \inf_{z \in S} \tau(x, z) - \inf_{w \in S} \tau(y, w).$$

For $v \in \mathbb{R}^2$, write $S + v = \{s + v : s \in S\}$. If $v \in \partial B_\mu$ is a point of differentiability and w is a tangent vector at v, let π_v denote the linear functional $av + bw \mapsto a$. Define the lower density of a set $A \subseteq \mathbb{N}$ by $\underline{d}(A) = \liminf \frac{1}{N} |A \cap \{1, \dots, N\}|$. The following is a rephrasing of [11], Lemma 4.6.

THEOREM 4.1 (Hoffman). *Let $\mu \in \mathcal{M}$ and let $v \in B_\mu$ be a point of differentiability of ∂B_μ with tangent line $L \subseteq \mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists an $M = M(v, \varepsilon) > 0$ such that, if $x, y \in \mathbb{R}^2$ satisfy $\pi_v(x - y) > M$, then the set of n such that*

$$\mathbb{P}\big(B_{L+nv}(y, x) > (1 - \varepsilon)\pi_v(x - y)\big) > 1 - \varepsilon$$

has lower density at least $1 - \varepsilon$.

Hoffman's proof of this result does not use unique passage times.

Theorem 4.1 is related to coexistence as follows. Suppose $\text{sides}(B_\mu) \geq k$. We can then find k points of differentiability $v_1, \dots, v_k \in \partial B_\mu$ with distinct tangent lines L_i, and in particular $\pi_{v_i}(v_i - v_j) > 0$ for all $j \neq i$. Fix $\varepsilon > 0$ and choose $R > 0$ large enough so that the points $x_i = Rv_i$ satisfy $\pi_{v_i}(x_i - x_j) > M(v_i, \varepsilon/k^2)$. Using the elementary relation $\underline{d}(\bigcap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n (1 - \underline{d}(A_i))$, for each i we see that the set of n such that

$$\mathbb{P}\big(B_{L+nv_i}(x_j, x_i) > 0 \text{ for all } j \neq i\big) > 1 - \frac{\varepsilon}{k}$$

has lower density at least $1 - \varepsilon/k$. Hence, with positive probability (which can be made arbitrarily close to 1 by decreasing ε), for each i there are infinitely many n such that $B_{L_i+nv_i}(x_j, x_i) > 0$ for all $j \neq i$. For such an n, take $y_{i,n} \in L_i + nv_i$ to be the closest point (in the sense of passage times) to x_i; by definition $y_{i,n}$ is reached first by species i. The points $y_{i,n}$ are in C_i, so $|C_i| = \infty$ for $i = 1, \dots, k$, that is, coexistence occurs. Note that this argument does not use unique passage times.

When $\text{sides}(B_\mu) = \infty$, one proves coexistence of infinitely many types using the above arguments. Choose a sequence $\{v_i\}_{i=1}^\infty \subseteq \partial B_\mu$ of points of differentiability of the boundary, ordered clockwise, say. Given $\varepsilon > 0$, define the points x_i inductively by $x_{i+1} = x_i + R_i(v_{i+1} - v_i)$ for a sufficiently large $R_i > 0$ so as to ensure that for $i \neq j$, $\pi_{v_i}(x_i - x_j) > M(v_i, \varepsilon_{i,j})$, where $\sum_{i,j} \varepsilon_{i,j} < \varepsilon$. Using the argument above, we see that with probability $> 1 - \varepsilon$, any finite subcollection of the x_i's coexist. We now need to show that this implies that all species coexist with probability $> 1 - \varepsilon$. Fix a configuration in which every finite set of species coexist, and suppose that they do not all coexist. Let C_i denote the set of sites colonized by the ith species when all species compete simultaneously. Since we are assuming that coexistence does not occur, we have $|C_{i_0}| < \infty$ for some i_0. Let ∂C_{i_0} denote the set of sites in $\mathbb{Z}^2 \setminus C_{i_0}$ which are adjacent to C_{i_0}, so that this is a finite set. Every $u \in \partial C_{i_0}$ is colonized by some species $j = j(u)$ at or before the time the species i_0 reaches u. It follows that when the species i_0 competes against the finite set of species $\{j(u) : u \in \partial C_{i_0}\}$, it still only colonizes the set C_{i_0}. This contradiction completes the proof.

5. Proof of Theorem 1.5. Recall that $\Gamma(0)$ denotes the infection graph and that $K(\Gamma(0))$ is the number of ends of $\Gamma(0)$. In the following discussion we fix a measure $\mu \in \mathcal{M}$ and assume that μ is not purely atomic. Consequently, there exists a Borel set $Q \subset (0, \infty)$ such that $\mu(Q) > 0$ and $\mu(\{q\}) = 0$ for all $q \in Q$. Note that B_μ has nonempty interior, is symmetric with respect to reflections through the axes and is convex. Therefore, the origin is in its interior.

PROPOSITION 5.1. *Suppose that $\mathbb{P}$-a.s. there exist k infinite geodesics $\gamma_1, \dots, \gamma_k$ starting at* 0, *edges $e_1, \dots, e_k$ and a finite set $V \subseteq \mathbb{Z}^2$, such that*: (a) *e_i lies on γ_i but not on γ_j for $i \neq j$*, (b) *the endpoints of e_i are in V*, (c) *$\tau_{e_i} \in Q$ and* (d) *each pair of geodesics is disjoint outside of V. Then $\mathbb{P}$-a.s.*, $K(\Gamma(0)) \geq k$.

PROOF. Under our assumptions on μ, with probability 1 every pair of edges with passage times in Q has distinct passage times. Consequently, geodesics between $x, y \in \mathbb{Z}^2$ can differ only in edges e with $\tau_e \notin Q$, and must share edges with $\tau_e \in Q$. We assume we are in this probability-1 event.

We claim that no two of the given geodesics are connected in $\Gamma(0) \setminus V$. Suppose, for instance, that γ_1, γ_2 were connected in $\Gamma(0) \setminus V$ by a path σ which we may assume is simple (not self-intersecting) and with endpoints $y_1 \in \gamma_1$ and $y_2 \in \gamma_2$. Denote the sequence of vertices in σ by $y_1 = v_1, v_2, \dots, v_m = y_2$. Write $e = e_1$ and let $J \subseteq \{1, \dots, m\}$ denote the set of j such that there exists a geodesic σ_j from 0 to v_j which contains e. We claim that $m \in J$. This leads to a contradiction because then σ_1 and γ_2 are both geodesics connecting 0 and y_2, but only one of them, σ_m, contains e.

Clearly $1 \in J$. Suppose now that $j \in J$ with corresponding geodesic σ_j. Write f for the edge between v_j and v_{j+1}, and note that $f \neq e$ because the endpoints of e are in V while those of f are not. If $\tau(f) = 0$, then we can adjoin f to σ_j to form σ_{j+1}. Suppose, therefore, that $\tau(f) > 0$, so that $\tau(0, v_j) \neq \tau(0, v_{j+1})$. If $\tau(0, v_{j+1}) > \tau(0, v_j)$ then we adjoin f to σ_j and obtain a geodesic σ_{j+1} with the desired properties. If $\tau(0, v_{j+1}) < \tau(0, v_j)$ and v_{j+1} lies on σ_j, we remove f from σ_j to obtain σ_{j+1}. On the other hand, if v_{j+1} does not lie on σ_j, but $\tau(0, v_{j+1}) < \tau(0, v_j)$, then there is a geodesic σ'_{j+1} from 0 to v_j whose last edge is f. Because σ'_{j+1} must reach v_j in the same time as σ_j does, it must pass through each of those edges of σ_j which have passage times in Q, and in particular through e. We remove f from σ'_{j+1} to obtain σ_{j+1}. □

Our goal is to establish the hypotheses of the proposition for k as in (1.1). It is enough to show that there exist random variables $m < M$ such that with probability one:

(1) There exist k infinite geodesics $\gamma_1, \ldots, \gamma_k$ which are disjoint outside of mB_μ.
(2) There are edges e_i in γ_i, with endpoints in $MB_\mu \setminus mB_\mu$, such that $\tau_{e_i} \in Q$.

This suffices because we can then set $V = MB_\mu$ in the proposition. To show that such m, M exist, it is enough to show that for every $\varepsilon > 0$ there exist deterministic integers $m < M$ such that each of the conditions above holds on an event of probability $> 1 - \varepsilon$.

Given $u, v, w \in \partial B_\mu$ which are points of differentiability of ∂B_μ, let $C(u, v, w)$ denote the open arc in ∂B_μ from u to w containing v. We rely on the following result, whose proof does not require unique geodesics. It is a rephrasing of [11], Lemma 4.7.

THEOREM 5.1 (Hoffman). *Let $u, v, w \in \partial B_\mu$ be points of differentiability of ∂B_μ, let L be the tangent line at v and write $C = C(u, v, w)$. Then for every $\varepsilon > 0$, there is an $M_0 = M_0(\varepsilon)$ such that for every $M > M_0$, the set of n such that*

$$\mathbb{P}(\gamma \cap M\partial B_\mu \subseteq MC \text{ for all geodesics } \gamma \text{ from } 0 \text{ to } L + nv) > 1 - \varepsilon$$

has lower density at least $1 - \varepsilon$.

Henceforth, fix k as in (1.1) and $\varepsilon > 0$ and for $i = 1, \ldots, k$ choose points $u_i, v_i, w_i \in \partial B_\mu$ and lines L_i as in the theorem, and such that the closed sets $C_i = \overline{C(u_i, v_i, w_i)}$ are pairwise disjoint and do not intersect the boundary of the ℓ^1 unit ball; write $C = \bigcup_{i=1}^k C_i$. Note that k was picked so that such a choice is possible: $\frac{1}{4}(\text{sides}(B_\mu) - 4)$ is the number of distinct sides on each of the four curves in ∂B_μ which constitute the complement of the ℓ^1 unit ball; dividing this number by 3 gives an upper bound on the number of triples we can choose in each of these curves. Taking integer part and multiplying by 4 gives k.

CLAIM 5.1. *There exists M_0 and $\rho > 0$ such that with probability at least $1 - \varepsilon$, for all $M > M_0$, every $x \in MC$ and every geodesic γ from 0 to x, at least ρM edges of γ have passage times in Q.*

PROOF. Define edge weights $\{\tau'_e : e \in \mathbb{E}\}$ by the rule that if $\tau_e \in Q$ then $\tau'_e = \tau_e + 1$ and $\tau'_e = \tau_e$ otherwise. Let μ' denote the marginal distribution of τ'_e.

Choose $0 < \eta < 1$ so that $(1 - \eta)C \cap B_{\mu'} = \varnothing$ (we can do so by a theorem of Marchand [13], Theorem 1.5, and the fact that C is disjoint from the ℓ^1 unit ball).

For a path σ let $N_Q(\sigma)$ be the number of edges of σ with passage time in Q. By Theorem 1.1, there is an event A with $\mathbb{P}(A) > 1-\varepsilon$ and an M_0 such that for all $M > M_0$ and $y \in M\partial B_\mu$, the τ-geodesic γ from 0 to y satisfies $(1-\eta^2)M < \tau(\gamma) < (1+\eta^2)M$, and similarly for $y' \in M\partial B_{\mu'}$ and the τ'-length of τ'-geodesics from 0 to y'. We claim that A is the desired event. Indeed, let $M > M_0$ and let γ be a τ-geodesic from 0 to some $x \in MC$. We have

$$\tau'(\gamma) = \tau(\gamma) + N_Q(\gamma) \le (1+\eta^2)M + N_Q(\gamma).$$

On the other hand, $x = \frac{M}{s}y$ for some $y \in \partial B_{\mu'}$ and $s < 1-\eta$, so

$$\tau'(\gamma) \ge (1-\eta^2)\frac{M}{1-\eta}.$$

Combining these we find that $N_Q(\gamma) \ge (\eta - \eta^2)M$. We take this to be ρM. □

Let $\alpha > 0$ denote the quantity

$$\alpha = \tfrac{1}{2}\min\{\pi_{v_i}(x_i - x_j) : x_i \in C_i, x_j \in C_j, i \ne j\}. \tag{5.1}$$

Choose finite sets $D_i \subseteq C_i \cap \partial B_\mu$ with the property that for every $i = 1, \dots, k$,

$$C_i \subseteq \bigcup_{x \in D_i}\left(x + \frac{\alpha}{10}B_\mu\right).$$

This property can be satisfied by compactness of $\bigcup_i C_i$ and the fact that B_μ contains a neighborhood of the origin. Write $D = \bigcup_{i=1}^k D_i$.

We can choose large integers m and $M \gg m$ and a set $I \subseteq \mathbb{N}$ of density $> 1-\varepsilon$ such that, for $n \in I$, the following statements hold with probability $> 1-\varepsilon$.

(A) If $i \ne j$, then $B_{L_i+nv_i}(x_j, x_i) \ge m\alpha$ for all $x_i \in mD_i$ and $x_j \in mD_j$.

(B) Every geodesic $\gamma_{i,n}$ from 0 to $L_i + nv_i$ intersects $m\partial B_\mu$ in mC_i and intersects $M\partial B_\mu$ in MC_i.

(C) $|\tau(0,x) - m| < \frac{m\alpha}{10}$ for all $x \in mD$.

(D) If $x \in mD$ and $y \in x + \frac{m\alpha}{10}B_\mu$, then $\tau(y,x) < \frac{m\alpha}{5}$.

(E) At least one edge on $\gamma_{i,n} \cap (MB_\mu \setminus mB_\mu)$ has passage time in Q.

Indeed, for m, M large enough the first two properties follow from Theorems 5.1 and 4.1, and the third and fourth from Theorem 1.1 [for (D) we apply Theorem 1.1 to each of the finitely many points in mD and intersect the events; note that the probabilities do not depend on the point in question, only on m]. Last, (E) follows from the previous claim, since when M is large the number of edges on any geodesic from 0 to $m\partial B_\mu$ is smaller than ρM.

Call A_n the intersection of the above five events. Since $\mathbb{P}(A_n) > 1-\varepsilon$ for all $n \in I$, the event A that A_n occurs for infinitely many n has $\mathbb{P}(A) > 1-\varepsilon$. We now consider only configurations in A. For each n and i, fix $\gamma_{i,n}$ as in (B). We may choose a (random) infinite set $I' \subseteq I$ such that for all $n \in I'$, A_n occurs and $J \subseteq I'$

such that $\lim_{n\in J}\gamma_{i,n}\to\gamma_i$ for some infinite geodesics γ_i originating at 0, that is, for every $r>0$ we have $\gamma_i\cap[-r,r]^2=\gamma_{i,n}\cap[-r,r]^2$ for all large enough $n\in J$. Henceforth, we only consider such n.

Let $y_{i,n}$ be the first intersection point of $\gamma_{i,n}$ with mC_i, and choose $x_{i,n}\in mD_i$ such that $y_{i,n}\in x_{i,n}+\frac{m\alpha}{10}B_\mu$. Then by (D) we have $\tau(x_{i,n},y_{i,n})\le\frac{m\alpha}{5}$, so by (A),

$$|B_{L_i+nv_i}(y_{j,n},y_{i,n})-B_{L_i+nv_i}(x_{j,n},x_{i,n})|<\frac{2m\alpha}{5}\qquad\text{for } i\neq j. \tag{5.2}$$

CLAIM 5.2. *The γ_i's are disjoint outside of mB_μ.*

PROOF. Suppose, for example, that γ_1,γ_2 intersect at some point z outside of mB_μ. Then for large enough $n\in J$ the same is true of $\gamma_{1,n}$ and $\gamma_{2,n}$. Then

$$\tau(0,y_{1,n})+\tau(y_{1,n},z)=\tau(0,y_{2,n})+\tau(y_{2,n},z).$$

By (C) we have $|\tau(0,y_{1,n})-\tau(0,y_{2,n})|<\frac{2m\alpha}{10}$, so

$$|\tau(y_{1,n},z)-\tau(y_{2,n},z)|<\frac{2m\alpha}{10}.$$

Write σ_1 for the part of $\gamma_{1,n}$ from $y_{1,n}$ to L_1+nv_1. Let σ_2 be path which starts at $y_{2,n}$, follows $\gamma_{2,n}$ until z and then follows $\gamma_{1,n}$ until L_1+nv_1. We find that $|\tau(\sigma_1)-\tau(\sigma_2)|<\frac{2m\alpha}{10}$. But $\gamma_{1,n}$ is a shortest path from 0 to L_1+nv_1, so σ_1 is a shortest path from $y_{1,n}$ to L_1+nv_1. Hence, $B_{L_1+nv_1}(y_{2,n},y_{1,n})\le\frac{2m\alpha}{10}$. Combined with (5.2), this contradicts (A). ☐

Finally, combining the last claim with (E) establishes the two claims stated after Proposition 5.1. This completes the proof of Theorem 1.4.

REFERENCES

[1] COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603. MR0624685

[2] COX, J. T. and KESTEN, H. (1981). On the continuity of the time constant of first-passage percolation. *J. Appl. Probab.* **18** 809–819. MR0633228

[3] DEIJFEN, M. and HÄGGSTRÖM, O. (2008). The pleasures and pains of studying the two-type Richardson model. In *Analysis and Stochastics of Growth Processes and Interface Models* 39–54. Oxford Univ. Press, Oxford. MR2603218

[4] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

[5] DURRETT, R. and LIGGETT, T. M. (1981). The shape of the limit set in Richardson's growth model. *Ann. Probab.* **9** 186–193. MR0606981

[6] GARET, O. and MARCHAND, R. (2005). Coexistence in two-type first-passage percolation models. *Ann. Appl. Probab.* **15** 298–330. MR2115045

[7] GOUÉRÉ, J.-B. (2007). Shape of territories in some competing growth models. *Ann. Appl. Probab.* **17** 1273–1305. MR2344307

[8] HÄGGSTRÖM, O. and MEESTER, R. (1995). Asymptotic shapes for stationary first passage percolation. *Ann. Probab.* **23** 1511–1522. MR1379157

[9] HÄGGSTRÖM, O. and PEMANTLE, R. (1998). First passage percolation and a model for competing spatial growth. *J. Appl. Probab.* **35** 683–692. MR1659548

[10] HOFFMAN, C. (2005). Coexistence for Richardson type competing spatial growth models. *Ann. Appl. Probab.* **15** 739–747. MR2114988

[11] HOFFMAN, C. (2008). Geodesics in first passage percolation. *Ann. Appl. Probab.* **18** 1944–1969. MR2462555

[12] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV*—1984. *Lecture Notes in Math.* **1180** 125–264. Springer, Berlin. MR0876084

[13] MARCHAND, R. (2002). Strict inequalities for the time constant in first passage percolation. *Ann. Appl. Probab.* **12** 1001–1038. MR1925450

[14] NEWMAN, C. M. (1995). A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol.* 1, 2 (*Zürich*, 1994) 1017–1023. Birkhäuser, Basel. MR1404001

[15] PIMENTEL, L. P. R. (2007). Multitype shape theorems for first passage percolation models. *Adv. in Appl. Probab.* **39** 53–76. MR2307871

[16] ZHANG, Y. (2007). Shape curvatures and transversal fluctuations in the first passage percolation model. Preprint. Available at http://arxiv.org/abs/math.PR/0701689.

MATHEMATICS DEPARTMENT
PRINCETON UNIVERSITY
FINE HALL, WASHINGTON RD.
PRINCETON, NEW JERSEY 08544
USA
E-MAIL: mdamron@math.princeton.edu
hochman@math.princeton.edu