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RANDOM INTERLACEMENTS AND AMENABILITY

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We consider the model of random interlacements on transient graphs, which was first introduced by Sznitman [*Ann. of Math.* (2) (2010) **171** 2039–2087] for the special case of $\mathbb{Z}^d$ (with $d \geq 3$). In Sznitman [*Ann. of Math.* (2) (2010) **171** 2039–2087], it was shown that on $\mathbb{Z}^d$: for any intensity $u > 0$, the interlacement set is almost surely connected. The main result of this paper says that for transient, transitive graphs, the above property holds if and only if the graph is amenable. In particular, we show that in nonamenable transitive graphs, for small values of the intensity u the interlacement set has infinitely many infinite clusters. We also provide examples of nonamenable transitive graphs, for which the interlacement set becomes connected for large values of u. Finally, we establish the monotonicity of the transition between the "disconnected" and the "connected" phases, providing the uniqueness of the critical value u_c where this transition occurs.

1. Introduction. In [21], Sznitman introduced the model of random interlacements on $\mathbb{Z}^d$, for $d \geq 3$. Intuitively speaking, this model describes the local picture left by the trace of a random walk on a discrete torus or a discrete cylinder; see [26] and [19]. Moreover, recent works have shown that random interlacements can be used to obtain a better understanding of the trace of a random walk on these graphs; see, for instance, [18, 20] and [24].

Later in [23] the construction of random interlacements was generalized to any transient weighted graph. This extension has already been useful to prove results concerning the random walk on random regular graphs; see [25]. There are also strong indications that a better understanding of the random interlacements model in more general classes of graphs could provide interesting results on the disconnection time of a discrete cylinder with general basis by a random walk; see [17, 22] and [27].

In this paper we study random interlacements on a given graph $G = (V, E)$, with special emphasis on the relations between the behavior of the interlacement set and geometric properties of G. First, let us give an intuitive description of the model which will be made precise in the next section. To define random interlacements, one first considers the space W^* of doubly infinite, transient trajectories

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in G modulo time shift; see (17). The random interlacements consist of a Poisson point process on the space W^*, constructed in a probability space $(\Omega, \mathcal{A}, \mathbb{P}_u)$. Usually one is interested in understanding the trace left by this soup of random walk trajectories, which is a random subset $\mathcal{I}$ of V. The parameter $u \in \mathbb{R}_+$ is a multiplicative factor of the intensity; therefore the bigger the value of u, the more trajectories enter the picture. Although we postpone the precise definition of random interlacements to the next section, we mention here that $\mathcal{I}$, under the law $\mathbb{P}_u$, is characterized as the unique random subset of V such that

$$\mathbb{P}_u[\mathcal{I} \cap K = \varnothing] = \exp\{-u \operatorname{cap}(K)\} \qquad \text{for every finite } K \subset V; \tag{1}$$

see (1.1) of [23]. In the equation above, $\operatorname{cap}(K)$ stands for the capacity of K defined in (21).

In the case of $\mathbb{Z}^d$, it was already shown that $\mathcal{I}$, regarded as a random subset of $\mathbb{Z}^d$, defines an ergodic percolation process; see, for instance, [21], Theorem 2.1. However, due to its long-range dependence, $\mathcal{I}$ behaves very differently from the usual independent site percolation on $\mathbb{Z}^d$. Let us stress here one such difference. Although the marginal density $\mathbb{P}_u[x \in \mathcal{I}]$ converges to zero as we drive the intensity parameter u to zero, the set $\mathcal{I}$ is $\mathbb{P}_u$-almost surely given by a single infinite connected subset of $\mathbb{Z}^d$; see [21], Corollary 2.3. In particular there is no phase transition for the connectivity of $\mathcal{I}$ as one varies u. For this reason, in the case of $\mathbb{Z}^d$, more attention has been given to the complement of $\mathcal{I}$, the so called vacant set denoted by $\mathcal{V}$, which exhibits a nontrivial phase transition for every $d \geq 3$; see, for instance, [21] and [16].

In this work, we address similar questions for more general graphs G. We first show that

(2) if G is a transient, vertex-transitive graph, then the interlacement set $\mathcal{I}$ under the law $\mathbb{P}_u$ is an ergodic subset of V;

see Proposition 2.4. In particular, any automorphism-invariant event on $\{0, 1\}^V$ has probability either 0 or 1.

Given this fact, it is natural to ask what special property of $\mathbb{Z}^d$ is responsible for the connectivity of the interlacement set at all intensities u. In this paper we show that the answer to this question is related to the amenability of $\mathbb{Z}^d$, as shown in the two results that we now describe. The main result of this paper is the following:

(3) if G is a vertex-transitive nonamenable graph, then for $u > 0$ small enough, the interlacement set $\mathcal{I}$ is $\mathbb{P}_u$-almost surely disconnected;

see Theorem 6.3. This shows a clear difference from the $\mathbb{Z}^d$ case.

This result motivates the definition of the critical parameter

$$u_c = u_c(G) = \inf\{u : \mathcal{I} \text{ is connected } \mathbb{P}_u\text{-a.s.}\}. \tag{4}$$

See also Corollary 7.5. A trivial consequence of (3) is that the critical threshold u_c is positive for vertex-transitive, nonamenable graphs. In fact, Theorem 6.3 gives a lower bound for u_c in terms of the spectral radius of G; see definition in (14).

The counterpart for Theorem 6.3 is given in Theorem 3.3, which gives the following generalization of Corollary 2.3 of [21]:

(5) if G is a vertex-transitive amenable transient graph, then for all $u > 0$, the set $\mathcal{I}$ is $\mathbb{P}_u$-almost surely connected.

In particular, (5) implies that for transitive amenable graphs G, $u_c(G) = 0$.

We also note that it is not always the case that u_c is finite. In fact according to Remark 7.1, in an infinite regular tree $\mathbb{T}^d$ for $d \geq 3$, the interlacement set is almost surely disconnected for every value of $u > 0$, so that $u_c(\mathbb{T}^d) = \infty$. However, in the same spirit as for Bernoulli percolation (see, e.g., [9]), we can prove that

(6) for random interlacements on $\mathbb{T}^d \times \mathbb{Z}^{d'}$, where $d \geq 3$ and $d' \geq 1$, we have $0 < u_c(\mathbb{T}^d \times \mathbb{Z}^{d'}) < \infty$;

see Proposition 7.2. Thus each of the three cases $u_c(G) = 0$, $u_c(G) \in (0, \infty)$ and $u_c(G) = \infty$ can happen. It is also of interest to study what happens regarding the connectivity of $\mathcal{I}$ above u_c (note that the connectivity of $\mathcal{I}$ is not a monotone property). In Theorem 7.4, we show that

(7) if G is a vertex-transitive and transient graph, then for any $u > u_c$, the set $\mathcal{I}$ is $\mathbb{P}_u$-almost surely connected.

In other words, there is monotonicity in the uniqueness transition.

In addition to the critical value u_c defined in (4), there is another critical value of interest, which was defined in [21], (0.13). Recall that $\mathcal{V}$ stands for the complement of $\mathcal{I}$. For $x \in V$, let

(8) $\eta_x(u) = \mathbb{P}_u[x \text{ belongs to an unbounded connected component of } \mathcal{V}]$,

(9) $u_*(G) = \inf\{u \geq 0, \eta_x(u) = 0\}$.

In Corollary 3.2 of [23], the critical intensity $u_*(G)$ was shown to be independent of the choice of the base point x.

The critical intensity of $\mathbb{Z}^d$ was first studied in [21], where it was shown that $u_*(\mathbb{Z}^d) < \infty$ for $d \geq 3$, and $u_*(\mathbb{Z}^d) > 0$ for $d \geq 7$. Later in [16], $u_*(\mathbb{Z}^d) > 0$ was established for any $d \geq 3$. In [23], the nondegeneracy of u_* was established for some families of graphs.

In Proposition 8.1 we join results from [3, 23] and (2) to show that

(10) $0 < u_*(G) < \infty$ whenever G is a nonamenable Cayley graph.

We now give a brief overview of the proof of the main result (3) of this paper. We rely on the following domination result:

(11) there is a coupling between $\mathbb{P}_u$ and the law of a certain branching random walk on G, such that the cluster of $\mathcal{I}$ containing a given point x is almost-surely contained in the trace left by the branching random walk from x.

Given this coupling, the proof of (3) is reduced to a calculation on the heat-kernel of the branching random walk.

The domination stated in (11) is given in two steps. First, in Proposition 4.1 we give the domination of random interlacements by the "frog model" or "$A + B \to 2A$ model," studied in [1] and [13]. Then Proposition 5.1 establishes the domination of the mentioned frog model by a specific branching random walk on G.

We now discuss the resemblance between our main results and a conjecture in Bernoulli percolation. As we explained above, our results characterize the amenability of transitive graphs in terms of the existence of infinitely many infinite clusters in the interlacements set, for some parameter $u > 0$. Our investigation was partially inspired by a similar, still open question for Bernoulli percolation. More precisely, Benjamini and Schramm conjectured that a transitive graph is nonamenable if and only if there is some $p \in [0, 1]$ for which Bernoulli site percolation with parameter p contains an infinite number of infinite clusters; see Conjecture 6 in [5]. This conjecture has been resolved in some cases: Benjamini and Schramm [6] solved it in the planar case and Pak and Smirnova-Nagnibeda [12] established the nonuniqueness phase for certain classes of nonamenable Cayley graphs. Theorem 4 of [5] gives a sufficient condition for having a nonuniqueness phase, and that theorem is the main inspiration for our result concerning disconnectedness of random interlacements. In [4], Benjamini et al. characterized the amenability property of Cayley graphs in terms of the connectedness of the union of the wired spanning forest and Bernoulli percolation.

The rest of the paper is organized as follows. The notation and the main properties of random interlacements we need are given in Section 2, where also the $\{0, 1\}$-law is provided. We then prove that the interlacement set is always connected on amenable graphs; see Section 3. The Propositions 4.1 and 5.1 concerning domination of random interlacements by the frog model and the branching random walk are given, respectively, in Sections 4 and 5. The proof of Theorem 6.3 concerning disconnectedness of the interlacement set for small u is given in Section 6. In the same section, we also obtain the required heat-kernel estimates on the mentioned branching random walk. In Section 7, we prove that u_c is finite for the graph $\mathbb{T}^d \times \mathbb{Z}^{d'}$ and deal with the monotonicity of the uniqueness transition. Finally, in Section 8, we obtain bounds on u_* for nonamenable Cayley graphs, establishing (10).

In this paper we use the following convention on constants, which are denoted by c and c'. These constants can change from line to line and solely depend on the given graph G which is fixed within proofs. Further dependence of constants are indicated by subscripts, that is, c_ε is a constant depending on ε and potentially on the graph G.

2. Notation and definitions. We let $G = (V, E)$ be an infinite connected graph, which we always assume to have finite geometry, that is, every vertex in V

belongs only to a finite number of edges. For $x, y \in V$, we write $x \leftrightarrow y$ if they are neighbors in G. For $x \in V$, let d_x be the degree of x. Let $\Delta = \Delta(G) = \sup_{x \in V} d_x$, which sometimes will be assumed to be finite. We recall the definitions of vertex-transitivity and amenability. A bijection $f : V \to V$ such that $\{f(x), f(y)\} \in E$ if and only if $\{x, y\} \in E$ is said to be a graph automorphism of G. We denote the set of automorphisms of G by $\mathrm{Aut}(G)$ and say that G is transitive if for any $x, y \in V$, there is a graph automorphism f such that $f(x) = y$. For $K \subset V$, we define the *outer vertex boundary* of K as $\partial_V K = \{y \notin K : \exists x \in K, d(x, y) = 1\}$, where d is the usual graph distance induced in G. We write $\overline{K}$ for the set $K \cup \partial_V K$. We also define the *outer edge boundary* of K as $\partial_E K = \{\{x, y\} \in E : x \in K, y \notin K\}$.

The *vertex isoperimetric constant* of the graph G is defined as

$$\kappa_V = \kappa_V(G) = \inf\left\{\frac{|\partial_V K|}{|K|} : |K| < \infty\right\}. \tag{12}$$

Similarly, the *edge isoperimetric constant* is defined to be

$$\kappa_E = \kappa_E(G) = \inf\left\{\frac{|\partial_E K|}{|K|} : |K| < \infty\right\}. \tag{13}$$

A graph G is said to be nonamenable if $\kappa_V(G) > 0$, otherwise it is said to be amenable. In Theorems 3.3 and 6.3, where we use the notion of amenability, we also assume the graph to have bounded degree; therefore we could have defined amenability in terms of κ_E instead.

We let P_x stand for the law of a simple random walk on G starting at x, while $(X_n)_{n \geq 0}$ stands for the canonical projections on V. For vertices $x, y \in V$, let $p^{(n)}(x, y)$ be the probability that a simple random walk started at x is at y after n steps, that is, $P_x[X_n = y]$. Let

$$\rho = \rho(G) := \limsup_{n \to \infty} \big(p^{(n)}(x, y)\big)^{1/n}. \tag{14}$$

The quantity ρ is called the *spectral radius* of G, and is independent of the choices of x and y. If G is nonamenable and has bounded degree, then $\rho < 1$; see [8], Theorem 2.3. We also define the Green's function of the simple random walk on G as $g(x, y) = \sum_{n \geq 0} p^{(n)}(x, y)$, for $x, y \in V$.

The space W_+ stands for the set of infinite trajectories that spend only a finite time in finite sets

$$\begin{aligned} W_+ = \big\{\gamma : \mathbb{N} \to V; \gamma(n) \leftrightarrow \gamma(n+1) \text{ for each } n \geq 0 \text{ and} \\ \big\{n; \gamma(n) = y\big\} \text{ is finite for all } y \in V\big\}. \end{aligned} \tag{15}$$

We endow W_+ with the σ-algebra $\mathcal{W}_+$ generated by the canonical coordinate maps X_n.

We further consider the space of doubly infinite trajectories that spend only a finite time in finite subsets of V

$$W = \{\gamma : \mathbb{Z} \to V; \gamma(n) \leftrightarrow \gamma(n+1) \text{ for each } n \in \mathbb{Z} \text{ and } \{n; \gamma(i) = y\} \text{ is finite for all } y \in V\}. \tag{16}$$

On the space W, for $k \in \mathbb{Z}$, we introduce the shift operator $\theta_k : W \to W$ which sends a trajectory w to w' such that $w'(\cdot) = w(\cdot - k)$. We also consider the space W^* of trajectories in W modulo time shift

$$W^* = W/\sim \qquad \text{where } w \sim w' \iff w = \theta_k(w') \qquad \text{for some } k \in \mathbb{Z} \tag{17}$$

and denote with π^* the canonical projection from W to W^*. The map π^* induces a σ-algebra in W^* given by $\mathcal{W}^* = \{A \subset W^*; (\pi^*)^{-1}(A) \in \mathcal{W}\}$, which is the largest σ-algebra on W^* for which $(W, \mathcal{W}) \xrightarrow{\pi^*} (W^*, \mathcal{W}^*)$ is measurable.

For any finite set $K \subset V$, we define for a trajectory $w \in W$ the entrance time of K as

$$H_K(w) = \inf\{k \in \mathbb{Z}; w(k) \in K\}. \tag{18}$$

For a trajectory $w \in W_+$, the hitting time of K is defined as

$$\tilde{H}_K(w) = \inf\{k \geq 1; w(k) \in K\}. \tag{19}$$

Considering still a finite $K \subset V$, we define the *equilibrium measure* e_K by

$$e_K(x) = 1_{\{x \in K\}} P_x[\tilde{H}_K = \infty] \cdot d_x \tag{20}$$

and the capacity

$$\operatorname{cap}(K) = \sum_{x \in K} e_K(x). \tag{21}$$

In addition, we introduce the measure

$$P_{e_K} = \sum_{x \in V} e_K(x) P_x. \tag{22}$$

Given a finite set $K \subset V$, write W_K for the space of trajectories in W that enter the set K, and denote with W_K^* the image of W_K under π^*.

The set of point measures on which one canonically defines random interlacements is given by

$$\Omega = \left\{\omega = \sum_{i \geq 1} \delta_{w_i^*}; w_i^* \in W^* \text{ and } \omega(W_K^*) < \infty, \text{ for every finite } K \subset V\right\}, \tag{23}$$

endowed with the σ-algebra $\mathcal{A}$ generated by the evaluation maps $\omega \mapsto \omega(D)$ for $D \in \mathcal{W}^* \otimes \mathcal{B}(\mathcal{R}_+)$.

The following theorem resembles Theorem 1.1 in [21]. It was established in Theorem 2.1 of [23], providing the existence of the intensity measure used to construct the random interlacements process.

THEOREM 2.1. *There exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ satisfying, for each finite set $K \subset V$,*

$$1_{W_K^*} \cdot \nu = \pi^* \circ Q_K, \tag{24}$$

where the finite measure Q_K on W_K is determined by the following. Given A and B in $\mathcal{W}_+$ and a point $x \in V$,

$$\begin{aligned} &Q_K\big[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B\big] \\ &\qquad = P_x[A | \tilde{H}_K = \infty] e_K(x) P_x[B]. \end{aligned} \tag{25}$$

We are now ready to define the random interlacements. Consider on Ω the law $\mathbb{P}_u$ of a Poisson point process with intensity measure given by $u\nu(dw^*)$; for a reference on the construction of Poisson point processes, see [14], Proposition 3.6. We define the *interlacement* and the *vacant set* at level u, respectively, as

$$\mathcal{I}(\omega) = \Big\{ \bigcup_{w^* \in \mathrm{supp}(\omega)} \mathrm{Range}(w^*) \Big\} \tag{26}$$

and

$$\mathcal{V}(\omega) = V \setminus \mathcal{I}(\omega) \tag{27}$$

for $\omega = \sum_{i\geq 0} \delta_{w_i^*}$ in Ω.

Observe that with (1), (20) and (21), we have

$$\mathbb{P}_u[x \in \mathcal{V}] = \exp\big\{-u d_x P_x[\tilde{H}_x = \infty]\big\}. \tag{28}$$

For finite $K \subset V$ and $\omega = \sum_{i\geq 0} \delta_{w_i^*} \in \Omega$ we define

$$\mu_K(\omega) = \sum_{i\geq 0} \delta_{(w_i^*)^{K,+}} 1\big\{w_i^* \in W_K^*\big\} \tag{29}$$

were for $w^* \in W_K^*$, $(w_i^*)^{K,+}$ is defined to be the trajectory in W_+ which follows w^* from the first time it hits K. Proposition 1.3, equation (1.45) in [21] says that

(30) under $\mathbb{P}_u$, μ_K is a Poisson point process on W_+ with intensity measure $u P_{e_K}(dw)$.

Actually, in [21], (30) is stated only for the $\mathbb{Z}^d$ case, but the proof does not use any special properties of $\mathbb{Z}^d$ and is thus also valid here.

REMARK 2.2. Recall the definition of H_K in (18) and note that for each $w^* \in W_K^*$, there is a unique $w \in W_K$ such that $\pi^*(w) = w^*$ and $H_K(w) = 0$, we call this $s_K(w^*)$. Therefore, since by (25) the measure Q_K is supported on $\{H_K = 0\}$, we conclude that for any $A \in \mathcal{W}$ contained in W_K,

$$
\begin{aligned}
&Q_K[(\pi^*)^{(-1)}(\pi^*(A))] \\
&\quad = Q_K[\{w \in W; \theta_l(w) \in A \text{ for some } l \in \mathbb{Z}\}] \\
&\quad = Q_K[\{w \in W; H_K(w) = 0 \text{ and } \theta_l(w) \in A \text{ for some } l \in \mathbb{Z}\}] \\
&\quad = Q_K[s_K(\pi^*(A))].
\end{aligned} \tag{31}
$$

For a measure γ on W^* and $g \in \mathrm{Aut}(G)$, let γ^g denote the image of γ under the mapping $w^* \to g(w^*)$. In Theorem 1.1 of [21] it was shown that on $\mathbb{Z}^d$, the measure ν is invariant under translations. We now show a similar statement for vertex-transitive graphs.

PROPOSITION 2.3. *For* $g \in \mathrm{Aut}(G)$,

$$
\nu^g = \nu. \tag{32}
$$

PROOF. We follow the proof of Theorem 1.1 in [21]. For $g \in \mathrm{Aut}(G)$ and $K \subset V$ finite, $w^* \to g(w^*)$ maps W_K^* one-to-one onto $W_{g(K)}^*$. Also note that $s_K(g(w^*)) = g(s_{g^{-1}(K)}(w^*))$ for every $w^* \in W_{g^{-1}(K)}^*$. Then, for $C \in \mathcal{W}$, we get that

$$
\begin{aligned}
s_K \circ (1_{W_K^*} \cdot \nu^g)(C) &= s_K \circ ((1_{W_{g^{-1}(K)}^*} \cdot \nu)^g)(C) \\
&= s_{g^{-1}(K)} \circ (1_{W_{g^{-1}(K)}^*} \cdot \nu)(g(w) \in C) \\
&= Q_{g^{-1}(K)}[g(w) \in C].
\end{aligned} \tag{33}
$$

Now fix $A, B \in \mathcal{W}_+$ and $y \in V$. Let C be the event $\{(X_{-n})_{n \ge 0} \in A, X_0 = y, (X_n)_{n\ge 0} \in B\}$. Using (25), (24) and (33), we obtain that

$$
\begin{aligned}
&s_K \circ (1_{W^*}\nu^g)(C) \\
&\quad = Q_{g^{-1}(K)}[(X_{-n})_{n\ge 0} \in g^{-1}(A), X_0 = g^{-1}(y), (X_n)_{n \ge 0} \in g^{-1}(B)] \\
&\quad = P_{g^{-1}(y)}^{g^{-1}(K)}[g^{-1}(A)] e_{g^{-1}(K)}(g^{-1}(y)) P_{g^{-1}(y)}[g^{-1}(B)] = Q_K[C] \\
&\quad = s_K \circ (1_{W_K^*}\nu)(C).
\end{aligned} \tag{34}
$$

This clearly implies that $\nu^g = \nu$. $\square$

For $g \in \mathrm{Aut}(G)$, we define $\tau_g : \Omega \to \Omega$, which maps $\omega = \sum_{i \ge 0} \delta_{w_i^*} \in \Omega$ to

$$
\tau_g \omega = \sum_{i \ge 0} \delta_{g(w_i^*)} \in \Omega. \tag{35}
$$

A consequence of (32) is that

$$\mathbb{P}_u \text{ is invariant under } (\tau_g)_{g\in \mathrm{Aut}(G)}. \tag{36}$$

More than that, we can prove a $\{0, 1\}$-law. First, we need to introduce some additional notation. A given $g \in \mathrm{Aut}(G)$ naturally induces a mapping from $\{0, 1\}^V$ to itself which we shall denote by t_g. Denote by Q_u the law on $\{0, 1\}^V$ of $(1\{x \in \mathcal{I}\})_{x\in V}$ under $\mathbb{P}_u$. We write $\mathcal{Y}$ for the σ-algebra generated by the coordinate maps Y_z, $z \in V$ on $\{0, 1\}^V$.

PROPOSITION 2.4. *If G is transient and transitive, then for any $u \geq 0$, $(t_g)_{g\in\mathrm{Aut}(G)}$ is a measure preserving flow on $(\{0, 1\}^V, \mathcal{Y}, Q_u)$ which is ergodic. In particular, for any $u \geq 0$ and any $A \in \mathcal{Y}$ which is invariant under* $\mathrm{Aut}(G)$,

$$Q_u[A] \in \{0, 1\}. \tag{37}$$

The proof of this proposition is very similar to the proof of Theorem 2.1 in [21], but for the reader's convenience we present the full proof with the necessary modifications in the Appendix.

3. The amenable case. In this section we prove that if G is an amenable transitive graph, then $\mathcal{I}$ is $\mathbb{P}_u$-almost surely connected for every $u > 0$. This is not a direct consequence of the results in [7] as we explain in Remark 3.1 below.

REMARK 3.1. In [7], Burton and Keane developed a quite general technique to prove uniqueness of the infinite components induced by a random subset of edges $\mathcal{U} \subset E$ of an amenable graph $G = (V, E)$; see also Theorem 12.2 in [10]. Their result applies under the rather general conditions that $\mathcal{U}$ is translation invariant and satisfies the so called *finite energy property*; that is, for every edge $e \in E$ and any $U \subset E \setminus \{e\}$,

$$0 < \mathcal{P}\big[e \in \mathcal{U} | \mathcal{U} \setminus \{e\} = U\big] < 1 \tag{38}$$

for some version of this conditional probability; see Definition 12.1 in [10]. We note here that this condition does not hold in general for the interlacement set $\mathcal{I}$ as observed in Remark 2.2(3) of [21].

Instead of considering $\mathcal{I}$, it will be convenient to consider the set of edges traversed by any of the trajectories in the Poisson point process ω. More precisely, consider the set

$$\begin{aligned} \hat{\mathcal{I}}(\omega) = \big\{&e \in E; e = \{w(k), w(k+1)\}, \\ &\text{for some } k \in \mathbb{Z} \text{ and some } w \text{ such that } \pi^*(w) \in \mathrm{supp}(\omega)\big\}. \end{aligned} \tag{39}$$

Observe, as in [21] below (2.18), that the connected components of V induced by the edges in $\hat{\mathcal{I}}$ are either isolated points or infinite sub-components of $\mathcal{I}$. We introduce some notation as in [21]. Denote by $\tilde{\psi}:\Omega\to\{0,1\}^E$ the map $\tilde{\psi}(\omega)=(1\{e\in\hat{\mathcal{I}}(\omega)\})_{e\in E}$. Let $\tilde{Q}_u$ be the probability measure on $(\{0,1\}^E,\tilde{\mathcal{Y}})$ defined as the image of $\mathbb{P}_u$ under $\tilde{\psi}$. Here $\tilde{\mathcal{Y}}$ stands for the canonical σ-algebra on $\{0,1\}^E$. Let $\tilde{t}_g$, $g\in\mathrm{Aut}(G)$ be the canonical shift on $\{0,1\}^E$. Then $\tilde{t}_g\circ\tilde{\psi}=\tilde{\psi}\circ\tau_g$, $g\in\mathrm{Aut}(G)$, and

(40) for $u\ge 0$, $(\tilde{t}_g)_{g\in\mathrm{Aut}(G)}$ is a measure preserving flow on $(\{0,1\}^E,\tilde{\mathcal{Y}},\tilde{Q}_u)$ which is ergodic.

To prove (40), one proceeds in the same way as for Proposition 2.4; see the Appendix.

According to Remark 3.1, we need to adapt the technique of [7] in order to apply it to the sets $\mathcal{I}$ or $\hat{\mathcal{I}}$.

PROPOSITION 3.2. *Let G be a transient and transitive graph and fix $u>0$. The number of infinite connected components induced by the set $\hat{\mathcal{I}}$ as defined in* (39) *is $\mathbb{P}_u$-almost surely a constant, which is either* 1 *or* ∞.

PROOF. The proof of Corollary 2.3 of [21] would go through in the current setting with only minor modifications, but here we proceed in a slightly different manner. We denote by N the number of connected components induced by $\hat{\mathcal{I}}(\omega)$. The fact that N is almost surely a constant follows from ergodicity; see (40). Moreover, $\hat{\mathcal{I}}$ induces at least one unbounded connected component a.s. Thus it remains to show that for any $2\le k<\infty$, $\mathbb{P}_u[N=k]=0$. Assume for contradiction that for some $2\le k<\infty$ we have $\mathbb{P}_u[N=k]=1$. Fix $o\in V$, under the above assumption, we can pick $L<\infty$ so large that the event

$$A=\{\hat{\mathcal{I}}(\omega)\text{ has }k\text{ components, all intersecting }B(o,L)\}$$

has positive probability. Denote by W_1^* the set of trajectories in W^* that visit *every* vertex in $B(o,L)$. Decompose ω into $\omega^1=1_{W_1^*}\omega$ and $\omega^2=1_{W^*\setminus W_1^*}\cdot\omega$. Under $\mathbb{P}_u$, ω^1 and ω^2 are independent Poisson point processes with intensity measures $u1_{W_1^*}\,d\nu$ and $u1_{W^*\setminus W_1^*}\cdot d\nu$, respectively. Using only the transience and connectedness of G, it readily follows that the mass of $1_{W_1^*}\,d\nu$ is strictly positive. Since each trajectory in W_1^* intersects every trajectory of $W^*_{B(o,L)}$, it follows that if $\omega^1(W^*)>0$, only one connected component induced by $\hat{\mathcal{I}}(\omega)$ intersects $B(o,L)$. Thus we have the inclusion

$$A\subset\{\omega^1(W^*)=0\}. \tag{41}$$

From (41) it follows that $A\subset\tilde{A}$ where

$$\tilde{A}=\{\hat{\mathcal{I}}(\omega^2)\text{ has }k\text{ components, all intersecting }B(o,L)\} \tag{42}$$

and since A has positive probability the same holds for $\tilde{A}$. Since the events $\tilde{A}$ and $\{\omega^1(W^*) > 0\}$ are defined in terms of independent Poisson point processes, they are independent. Therefore,

$$\mathbb{P}_u[\tilde{A} \cap \{\omega^1(W^*) > 0\}] > 0. \tag{43}$$

But on $\tilde{A} \cap \{\omega^1(W^*) > 0\}$, we have $N = 1$. Thus (43) contradicts our assumption that $N = k$ a.s. for some $2 \le k < \infty$, completing the proof of the proposition. □

In the next theorem, we rule out the possibility of having infinitely many components in $\hat{\mathcal{I}}$ if the underlying graph is transitive and amenable.

THEOREM 3.3. *Let G be a transient, transitive and amenable graph. Then*

$$\mathbb{P}_u[\mathcal{I} \textit{ is connected}] = \mathbb{P}_u[\hat{\mathcal{I}} \textit{ is connected}] = 1 \qquad \textit{for every } u > 0. \tag{44}$$

PROOF. Throughout this proof, we use the convention that Range(w) is the set of edges that are traversed by the trajectory w where $w \in W$ or $w \in W^*$. Using Proposition 3.2, we conclude that for every $u > 0$, the number of infinite clusters in $\hat{\mathcal{I}}$ is $\mathbb{P}_u$-a.s. a constant which is either 1 or ∞. Since $\hat{\mathcal{I}}$ has no finite components, all we need to do is to rule out the case that it has infinitely many infinite clusters. For this let us suppose by contradiction the contrary. Under this assumption, we are going to construct, with positive probability, a trifurcation point for $\hat{\mathcal{I}}$ as defined below.

The definition of trifurcation point was first introduced in [7]. We say that a given point y is a trifurcation point for the configuration $\mathcal{E} \in \{0, 1\}^E$ if it belongs to an infinite connected component induced by $\{\mathcal{E} = 1\}$ but the removal of y would split its cluster into three distinct infinite connected components.

Using the assumption that $\mathbb{P}_u$-a.s. there are infinitely many infinite connected components in $\hat{\mathcal{I}}$, we conclude that there exist a finite, connected set $K \subseteq V$ which intersects three distinct infinite clusters of $\hat{\mathcal{I}}$ with positive probability. This implies that

$$\begin{aligned}\mathbb{P}_u\big[&\text{there are trajectories } w_1, w_2, w_3 \in \operatorname{supp}(\omega) \cap W_K^* \\ &\quad \text{whose ranges belong to distinct components of } \hat{\mathcal{I}}\big] > 0.\end{aligned} \tag{45}$$

It is useful now to decompose the point measure ω into $\omega_1 = \mathbf{1}_{W_K^*} \cdot \omega$ and $\omega_2 = \mathbf{1}_{W^* \setminus W_K^*} \cdot \omega$, which are independent under $\mathbb{P}_u$; see the notation below (18). We denote the laws of ω_1 and ω_2, respectively, by $\mathbb{P}_1$ and $\mathbb{P}_2$. We say that a given set $\mathcal{U} \subseteq E$ is *good* if

$$\begin{aligned}\mathbb{P}_1\big[&\text{there are } w_1, w_2, w_3 \text{ in } \operatorname{supp}(\omega_1) \\ &\quad \text{whose images are not connected through } \mathcal{U}\big] > 0.\end{aligned} \tag{46}$$

By (45) and Fubini's theorem, we conclude that with positive $\mathbb{P}_2$-probability, $\hat{\mathcal{I}}(\omega_2)$ is good.

From now on we fix a good set $\mathcal{U}$. Our aim is to show that with positive $\mathbb{P}_1$-probability $\hat{\mathcal{I}}(\omega_1) \cup \mathcal{U}$ has a trifurcation point in K. For this, we first observe from (46), (24) and (25) that for some triple $\{z_1, z_2, z_3\} \subset K$,

$$P_{z_1}^K \otimes P_{z_2}^K \otimes P_{z_3}^K\big[\text{Range}(X^i), i = 1, 2, 3 \text{ are not connected through } \mathcal{U}\big] > 0,$$

where X^i, $i = 1, 2, 3$ stand for the random walk trajectories, which are implicit in the above product space. The above can be rewritten as

$$\begin{aligned} P_{z_1}^K \otimes P_{z_2}^K \otimes P_{z_3}^K\big[&\text{Range}(X^1) \text{ and } \text{Range}(X^2) \text{ are not connected} \\ &\text{to each other through } \mathcal{U} \text{ and } \text{Range}(X^3) \text{ does} \\ &\text{not touch the clusters containing } z_1 \text{ and } z_2 \text{ in} \\ &\qquad \mathcal{U} \cup \text{Range}(X^1) \cup \text{Range}(X^2)\big] > 0. \end{aligned}$$

By conditioning in the range of the walks X^1 and X^2, we obtain that

$$\begin{aligned} P_{z_1}^K \otimes P_{z_2}^K \otimes P_{z_3}^K \otimes P_{z_3}^K\big[&\text{Range}(X^i), i = 1, 2 \text{ are not connected through} \\ &\mathcal{U} \text{ and } \text{Range}(X^3) \cup \text{Range}(X^4) \text{ do not touch} \\ &\text{the clusters containing } z_1 \text{ and } z_2 \\ &\qquad \text{in } \mathcal{U} \cup \text{Range}(X^1) \cup \text{Range}(X^2)\big] > 0, \end{aligned}$$

and repeating this argument twice, we get

$$\begin{aligned} (P_{z_1}^K)^{\otimes 2} \otimes (P_{z_2}^K)^{\otimes 2} \otimes (P_{z_3}^K)^{\otimes 2}\big[&(\text{Range}(X^1) \cup \text{Range}(X^2)), \\ &(\text{Range}(X^3) \cup \text{Range}(X^4)), \\ &(\text{Range}(X^5) \cup \text{Range}(X^6)) \\ &\text{are not connected through } \mathcal{U}\big] > 0. \end{aligned} \tag{47}$$

We can construct a deterministic subgraph G' of K such that:

- G' is connected,
- z_1, z_2 and z_3 belong to G' and
- the removal of some vertex y of G' separates the vertices z_1, z_2 and z_3 into disjoint components.

Indeed, to perform such construction, one could, for instance, take a shortest path γ from z_1 to z_2 in K and a shortest path γ' from z_3 to the range of γ. Then, the graph G' given by the union of the vertices and edges of γ and γ' would satisfy the above conditions with y being the intersection of γ and γ'. Note that the three above conditions imply that, for a subgraph $\mathcal{I}$ of G,

(48) if $\mathcal{I} \cap K = G'$, and z_1, z_2 and z_3 belong to distinct infinite connected components of $\mathcal{I} \setminus (K \setminus \{z_1, z_2, z_3\})$, then y is a trifurcation point of $\mathcal{I}$.

Using that G' is finite and connected, with positive P_{z_1}-probability, the random walk X^1 can cover G' and return to z_1 before escaping from K. This together with (48) and (47) gives that if $\mathcal{U}$ is good,

$$\prod_{i=1,2,3} (P_{z_i} \otimes P_{z_i}^K)\Bigg[y \text{ is a trifurcation point of } \bigcup_{i=1,\dots,6} \operatorname{Range}(X^i) \cup \mathcal{U} \Bigg] > 0.$$

Finally, we use (24) and (25) to conclude that if $\mathcal{U}$ is good,

$$\mathbb{P}_1[y \text{ is a trifurcation point of } \hat{\mathcal{I}}(\omega_1) \cup \mathcal{U}] > 0. \tag{49}$$

This, together with Fubini's theorem and the fact that $\mathcal{I}(\omega_2)$ is good with positive $\mathbb{P}_2$-probability, gives that y is a trifurcation point with positive $\mathbb{P}_u$-probability. The rest of the proof of the theorem follows the ideas of [7]; see also the proof of Theorem 2.4 in [10]. □

4. Domination by the frog model. Using the Poissonian character of random interlacements, we are going to show that the cluster of $\mathcal{I}(\omega)$ containing a given point $x \in V$ is dominated by the trace left by the particles in a certain particle system, the so-called frog model. In what follows we give an intuitive description of this process.

Place a random number of particles in each site of V with a prescribed distribution which will be given later. These particles should be understood as "sleeping" when the system starts. In the first iteration, only the particles sitting at the fixed site x are active, and they perform simple random walks. When an active particle reaches the neighborhood of a sleeping one, the latter becomes active, starting to perform simple random walk and so on. Note the two differences between this model and the one studied in [13]: we consider a random initial configuration instead of a deterministic one and the active particles can wake a sleeping particle by simply visiting its neighborhood, without the need to occupy the same site.

We now give a precise description of this process, while keeping a similar notation as that of previous sections. Consider the following measure on the space $(W, \mathcal{W})$ of doubly infinite trajectories:

$$\tilde{\nu} = \sum_{y \in V} d_y \tilde{P}_y, \tag{50}$$

where $\tilde{P}_y$ is supported in $\{X_0 = y\} \subset W$ and given by

$$\begin{aligned} &\tilde{P}_y[(X_{-i})_{i \geq 0} \in A, X_0 = y, (X_i)_{i \geq 0} \in B] \\ &\qquad = P_y[A] P_y[B] \qquad \text{for every } A, B \in \mathcal{W}_+. \end{aligned} \tag{51}$$

Intuitively speaking, the measure $\tilde{P}_y$ launches two independent random walks from y, one to the future and the other to the past.

Consider also the following space of point measures on W:

$$M = \Big\{\tilde{\omega} = \sum_{i\geq 1} \delta_{w_i}; w_i \in W \text{ and } \tilde{\omega}\big(\{X_0(w) \in K\}\big) < \infty, \tag{52}$$
$$\text{for all finite } K \subset V\Big\},$$

which is endowed with the sigma-algebra $\mathcal{M}$ generated by the evaluation maps $\tilde{\omega} \to \tilde{\omega}(D)$, for any $D \in \mathcal{W}$.

In analogy to (26), for any $\tilde{\omega} \in M$ we define

$$\tilde{\mathcal{I}}(\tilde{\omega}) = \Big\{ \bigcup_{w \in \operatorname{supp}(\tilde{\omega})} \operatorname{Range}(w)\Big\}, \tag{53}$$

which is the trace of the trajectories composing $\tilde{\omega}$.

We also introduce, in the probability space $(M, \mathcal{M}, \tilde{\mathbb{P}}_u)$, a Poisson point process $\tilde{\omega}$ in W with intensity measure given by $u\tilde{\nu}$. In Remark 4.2 below, we prove that, if G has bounded degree, then $\tilde{\mathcal{I}}(\tilde{\omega}) = V$, $\tilde{\mathbb{P}}_u$-a.s. In our pictorial description, this corresponds to the fact that if we wake up, all the particles at time zero all the sites will eventually be visited. In what follows we will instead consider $\tilde{\mathcal{I}}(\tilde{w}|.)$ where $\tilde{w}|.$ denotes the restriction of $\tilde{w}$ to some set $\cdot \in \mathcal{W}$.

Fix now $\tilde{\omega} \in M$. The following construction can be intuitively described as gradually "revealing" $\tilde{\omega}$. By this, we mean that in each step we observe $\tilde{\omega}$ restricted to larger and larger subsets of W. More precisely, recalling the definition of $\overline{K}$ in Section 2, define:

- $\tilde{A}_0(\tilde{\omega}) = \{x\}$ "particles sitting at x are activated at iteration 0," and supposing we have constructed $\overline{\tilde{A}_0, \dots, \tilde{A}_{k-1}}$,
- let $\tilde{A}_k(\tilde{\omega}) = \overline{\tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \tilde{A}_{k-1}(\tilde{\omega})\}})}$, "particles sitting at $\tilde{A}_{k-1}$ were activated and performed random walks, whose ranges will determine the next active set $\tilde{A}_k$."

Note that the above restriction of $\tilde{\omega}$ does not include all trajectories which hit $\tilde{A}_{k-1}$, solely the ones starting on it. Also, we sometimes write $\tilde{A}_k$ instead of $\tilde{A}_k(\tilde{\omega})$ in order to avoid an overly heavy notation.

Due to the Poissonian character of $\tilde{\mathbb{P}}_u$, since $\tilde{A}_k(\tilde{\omega})$ is determined by $\tilde{\omega}|_{\{X_0 \in \tilde{A}_{k-1}\}}$,

(54) conditioned on $\tilde{A}_0(\tilde{\omega}), \dots, \tilde{A}_k(\tilde{\omega})$, the point measure $\tilde{\omega}|_{\{X_0 \in \tilde{A}_k \setminus \tilde{A}_{k-1}\}}$, is distributed as a Poisson point process on $\{X_0 \in \tilde{A}_k \setminus \tilde{A}_{k-1}\}$, with intensity given by $u 1_{\{X_0 \in \tilde{A}_k \setminus \tilde{A}_{k-1}\}} \cdot \tilde{\nu}$;

see, for instance, [14], Proposition 3.6.

We can now state our domination result.

PROPOSITION 4.1. *Let $\mathcal{C}_x$ be the connected component of $\mathcal{I}(\omega)$ containing $x \in V$. We can find a coupling Q_1 between $\mathbb{P}_u$ and $\tilde{\mathbb{P}}_u$ such that*

$$\mathcal{C}_x \subset \tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \bigcup_k \tilde{A}_k\}}), \qquad Q_1\text{-a.s.} \tag{55}$$

PROOF. Recall the definition of W_K^* above (23), which we now extend to sets K which are possibly infinite.

For a given $\omega \in \Omega$, we are going to construct sets $A_0(\omega), A_1(\omega), \dots$ in a similar way as we did for $\tilde{A}_k(\tilde{\omega})$'s. For this:

- consider $A_0 = \{x\}$ "trajectories passing through x are activated at step 0," and suppose we have constructed $A_0, \dots, A_{k-1}$,
- let $A_k(\omega) = \overline{\mathcal{I}(\omega|_{W^*_{A_{k-1}}})}$ "trajectories meeting A_{k-1} were activated and their ranges will determine the next active set A_k."

Note that the Poissonian character of $\mathbb{P}_u$ gives us that for every $k \geq 0$,

$$\begin{aligned}&\text{conditioned on } A_0(\omega), \dots, A_k(\omega)\text{, the process } \omega|_{W^*_{A_k} \setminus W^*_{A_{k-1}}} \text{ is distributed}\\ &\text{as a Poisson point process on } W^*_{A_k} \text{ with intensity given by } u 1_{W^*_{A_k} \setminus W^*_{A_{k-1}}} \cdot \nu;\end{aligned} \tag{56}$$

compare with (54).

Another important remark is that, although $\mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$ could in principle be a proper subset of $\mathcal{I}(\omega)$, actually the connected component $\mathcal{C}_x$ of $\mathcal{I}(\omega)$ containing x is given by

$$\mathcal{C}_x = \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}}). \tag{57}$$

To see why this holds, note first that $\mathcal{C}_x$ is empty if and only if $\omega|_{W^*_{A_k}} = 0$ for every $k \geq 0$. Assume now that $\mathcal{C}_x$ is nonempty and take a point y in $\mathcal{C}_x$, which implies the existence of a path $x = x_0, x_1, \dots, x_n = y$ contained in $\mathcal{I}(\omega)$. Supposing by contradiction that $y \notin \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$, let j_o be the first $j \leq n$ such that $x_j \notin \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$. Since $\mathcal{C}_x$ is nonempty, we conclude that $j_o \geq 1$, and therefore x_{j_o} is within distance one from $x_{j_o-1} \in \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$. To obtain a contradiction, let k_o be such that $x_{j_o-1} \in \mathcal{I}(\omega|_{W^*_{\bigcup_{k \leq k_o} A_k}})$ and observe that x_{j_o} must belong to A_{k_o+1}, and since it belongs to $\mathcal{I}(\omega)$ it must also be in $\mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$, which is a contradiction. This proves that $\mathcal{C}_x \subseteq \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$. The other inclusion in obvious since $\mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}})$ is connected and contained in $\mathcal{I}(\omega)$. This establishes (57).

From (57), we see that the set $\mathcal{C}_x$ can be written in a way that resembles the set $\tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \bigcup_k \tilde{A}_k\}})$ appearing in (55). But to proceed with the proof of the proposition, we first need to find a coupling Q_1 between ω (under $\mathbb{P}_u$) and $\tilde{\omega}$ (under $\tilde{\mathbb{P}}_u$) such that

$$\omega|_{W^*_{A_k(\omega)}} \leq \pi^* \circ (\tilde{\omega}|_{\{X_0 \in A_k(\omega)\}}) \qquad \text{for every } k \geq 0,\ Q_1\text{-a.s.} \tag{58}$$

Note that the sets $\tilde{A}_k(\tilde{\omega})$ do not appear in the above equation. In order for Q_1 to satisfy the above, it is clearly enough that

$$\omega|_{W^*_{A_k} \setminus W^*_{A_{k-1}}} \leq \pi^* \circ (\tilde{\omega}|_{\{X_0 \in A_k \setminus A_{k-1}\}}) \qquad \text{for every } k \geq 0,\ Q_1\text{-a.s.,} \tag{59}$$

where we used the convention that $A_{-1} = \varnothing$. We now prove the existence Q_1 satisfying (59) and consequently (58).

Recall first that, conditioned on $A_1, A_2, \ldots, A_k$, the left-hand side of inequality (59) is independent of $\omega|_{W^*_{A_{k-1}}}$ while the right-hand side is independent of $\tilde{\omega}|_{\{X_0 \in A_{k-1}\}}$. Therefore, to be able to construct the coupling Q_1 satisfying (59), it suffices to show that for any $B \subseteq B'$,

$$\begin{array}{l} \text{we can couple } \omega|_{W^*_{B'} \setminus W^*_B} \text{ with } \tilde{\omega}|_{\{X_0 \in B' \setminus B\}} \text{ in a way that} \\ \omega|_{W^*_{B'} \setminus W^*_B} \leq \pi^* \circ (\tilde{\omega}|_{\{X_0 \in B' \setminus B\}}). \end{array} \tag{60}$$

In fact, once we have established the above, we can proceed by induction with $B' = A_k$ and $B = A_{k-1}$, for $k \geq 0$ and define $\omega|_{W^* \setminus \bigcup_k W^*_{A_k}}$ and $\tilde{\omega}|_{\{X_0 \notin \bigcup_k A_k\}}$ independently. This way, they will have the correct marginal distribution (see, e.g., Proposition 3.6 in [14]) and will satisfy (59).

As a further reduction, we claim that it is enough to establish (60) in the case where $B' = B \cup \{y\}$, with $y \notin B$. Indeed, if $B' = B \cup \{y_1, y_2, \ldots\}$, then we write $B_i = B \cup \{y_1, \ldots, y_i\}$ and use (60) repeatedly for the sets B_{i+1} and B_i, with $i \geq 0$ to obtain (60).

From now on, fix $B \subset V$ and $y \notin B$ and note, by (24), that

$$\text{the intensity measure of } \omega|_{W^*_{B \cup \{y\}} \setminus W^*_B} \text{ is given by } u\pi^* \circ (\mathbf{1}_{\{H_B = \infty\}} \cdot Q_{\{y\}}), \tag{61}$$

which we estimate as follows.

Fix $C, C' \in \mathcal{W}_+$ and consider the event $D \in \mathcal{W}$ given by $D = \{H_y = 0\} \cap \{(X_{-i})_{i \geq 0} \in C\} \cap \{(X_i)_{i \geq 0} \in C'\}$. Then

$$\begin{aligned} \mathbf{1}_{\{H_B = \infty\}} \cdot Q_{\{y\}}(D) &= Q_{\{y\}}[D, H_B = \infty] \\ &\overset{(25)}{=} P_y[C, H_B = \tilde{H}_y = \infty] d_y P_y\big[C', H_B = \infty\big] \\ &\leq P_y[C, \tilde{H}_y = \infty] d_y P_y\big[C'\big] \\ &\overset{(51)}{=} d_y \tilde{P}_y\big[(X_{-i})_{i \geq 0} \in C \cap \{\tilde{H}_y = \infty\}, X_0 = y, (X_i)_{i \geq 0} \in C'\big] \\ &= d_y \tilde{P}_y[D]. \end{aligned} \tag{62}$$

Since the above holds for every event D as above, we conclude that $\mathbf{1}_{\{H_B = \infty\}} \cdot Q_{\{y\}} \leq d_y \tilde{P}_y$. We can therefore construct a Poisson point process $\tilde{\omega}_-$ in W with intensity given by $u\mathbf{1}_{\{H_B = \infty\}} \cdot Q_{\{y\}}$ in a way that $\tilde{\omega}_- \leq \tilde{\omega}_{\{X_0 = y\}}$. Since $\pi^* \circ \tilde{\omega}_-$ has the same law as $\omega|_{W^*_{B \cup \{y\}} \setminus W^*_B}$, we conclude (60) from (61), for the case where $B' = B \cup \{y\}$. As we have discussed, this is enough to establish (60) in general and consequently (59) and (58).

Now that we have constructed Q_1, we are going to prove that it satisfies (55). First we claim that

$$A_k(\omega) \subseteq \tilde{A}_k(\tilde{\omega}) \qquad \text{for every } k \geq 0,\ Q_1\text{-a.s.} \tag{63}$$

To prove that, observe first that $A_0 = \tilde{A}_0 = \{x\}$ and suppose that we have established the above result for $k-1$. Then we use (58) to obtain that, Q_1-almost surely,

$$\mathcal{I}(\omega|_{W^*_{A_{k-1}}}) \subseteq \tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in A_{k-1}\}}) \tag{64}$$

and therefore

$$A_k(\omega) = \overline{\mathcal{I}(\omega|_{W^*_{A_{k-1}}})} \subseteq \overline{\tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in A_{k-1}\}})} \subseteq \overline{\tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \tilde{A}_{k-1}\}})} = \tilde{A}_k, \tag{65}$$

proving (63).

Finally, using (57), (58) and (63), we obtain

$$\mathcal{C}_x = \mathcal{I}(\omega|_{W^*_{\bigcup_k A_k}}) \subseteq \tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \bigcup_k A_k\}}) \subseteq \tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0 \in \bigcup_k \tilde{A}_k\}}), \qquad Q_1\text{-a.s.}, \tag{66}$$

proving (55). $\square$

REMARK 4.2. As promised below (53), let us show that if G has degree bounded by Δ, then $\tilde{\mathbb{P}}_u$-a.s. $\tilde{\mathcal{I}}(\tilde{\omega}) = V$. First, fix some $y \in V$ and consider the events $C_z = \{\text{some walker started at } z \text{ hits } y\}$, for $z \in V$. Clearly, under the measure $\tilde{\mathbb{P}}_u$ they are independent. We now estimate $\sum_{z \in V} \tilde{\mathbb{P}}_u[C_z]$, which is bounded from below by

$$\begin{aligned} &\sum_{z \in V} \tilde{\mathbb{P}}_u[\tilde{\omega}(X_0 \in z) > 0] P_z[H_y < \infty] \\ &\qquad \geq c_u \sum_{z \in V} P_z[H_y < \infty] \\ &\qquad = c_u \frac{1}{E_y[\text{number of visists to } y]} \sum_{z \in V} E_z[\text{number of visists to } y] \\ &\qquad \geq c_{u,y} \sum_{z \in V} \sum_{n \geq 0} p^{(n)}(z, y) \geq c_{u,y,\Delta} \sum_{n \geq 0} \sum_{z \in V} p^{(n)}(y, z) = \infty, \end{aligned} \tag{67}$$

where in the second line we used the strong Markov property, and in the third line we used the reversibility of the walk.

Using the independence of the events $\{C_z\}_{z \in V}$ and the Borel–Cantelli lemma, we conclude that $\tilde{\mathbb{P}}_u$-a.s. the point y is visited by infinitely many walks. Since this holds true for any of the countable y's in V, we conclude that $\tilde{\mathbb{P}}_u$-a.s. $\tilde{\mathcal{I}}(\tilde{\omega}) = V$, as desired.

5. Domination by branching random walk. In this section we show that the so-called frog model defined in Section 4 can be dominated by a certain multitype branching random walk which we introduce below. Although the main result of this section is very intuitive, its proof is somewhat technical. This is mainly due to the infection of neighbors present in our version of the frog model (see comment in the second paragraph of Section 4) and the notation we chose for the frog model, resembling random interlacements' formalism.

Consider two types of individuals, namely $\circ$ and $\bullet$, and four probability distributions on $\mathbb{Z}_+$: $q_{\circ\to\circ}$, $q_{\circ\to\bullet}$, $q_{\bullet\to\circ}$ and $q_{\bullet\to\bullet}$. These are the offspring distributions which should be understood as follows: $q_{\circ\to\bullet}$ is the distribution of the number of $\bullet$-descendants of a $\circ$-individual, and analogously for $q_{\circ\to\circ}$, $q_{\bullet\to\circ}$ and $q_{\bullet\to\bullet}$. If one supposes that all individuals have an independent number of $\circ$ and $\bullet$ descendants, then these four distributions completely characterize the branching mechanism.

We now make a specific choice for the q's, namely:

- $q_{\circ\to\bullet} = \delta_2$, "$\circ$-individuals always have two descendants of type $\bullet$,"
- $q_{\bullet\to\bullet} = \delta_1$, "$\bullet$-individuals always have one descendants of type $\bullet$" and
- $q_{\circ\to\circ} = q_{\bullet\to\circ} = \text{Poisson}(u\Delta^2)$.

Finally we need to introduce the initial distribution:

(68) at the first generation, all individuals are of type $\circ$ and their number is $\text{Poisson}(u\Delta^2)$ distributed.

We now construct in some abstract probability space $(S, \mathcal{S}, \mathbf{P}_u)$ the tree $\mathcal{T} = (T, \mathcal{E})$ originated by the above 2-type Galton–Watson process, which is completely characterized by the above properties. Rigorously, $\mathcal{T}$ should be called a forest, since it has as many connected components as the number of individuals at generation one. However, we keep the notation "tree" for simplicity.

Here, the set of vertices T is given by the disjoint union of T° and $T^\bullet$, corresponding to the type of each individual; see Figure 1. Supposing that the degree of G is bounded by Δ, we construct in the same probability space $(S, \mathcal{S}, \mathbf{P}_u)$ a

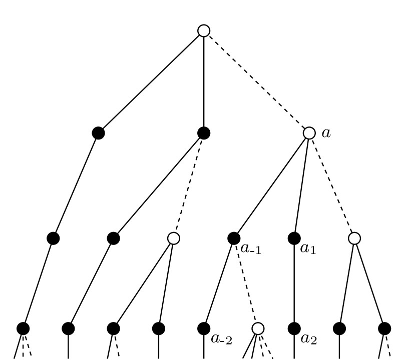

FIG. 1. *The multitype Galton–Watson tree $\mathcal{T} = (T, \mathcal{E})$, where $T = T^\circ \cup T^\bullet$.*

collection $(\ell_e)_{e\in\mathcal{E}}$ of i.i.d. random variables with uniform distribution over the set $\{1,2,\dots,\Delta!\}$. These variables ℓ_e should be regarded as random labels assigned to the edges of $\mathcal{T}$.

We now use the above construction to define a special branching random walk on the original graph $G=(V,E)$. For this, for every vertex $y\in V$, we fix a bijection $\phi_y:(\mathbb{Z}/d_y\mathbb{Z})\to\mathcal{N}(y)$. Recall that d_y is the degree of y, and $\mathcal{N}(y)$ stands for the set of its neighbors.

Fix now $x\in V$, a tree $\mathcal{T}$, the labels $(\ell_e)_{e\in\mathcal{E}}$ and $(\phi_y)_{y\in V}$ as above. We define the coordinate processes $(Z_a^x)_{a\in T}$ of the branching random walk by:

(69) $Z_a^x=x$ for every a in the first generation of $\mathcal{T}$. If $Z_a^x=y$ for some $a\in T$, then for any descendant b of a, we define Z_b^x as $\phi_y(\ell_{\{a,b\}} \bmod d_y)$.

Recall that ℓ_e was uniformly chosen in $\{1,2,\dots,\Delta!\}$, and note that d_y divides $\Delta!$. This implies that, conditioned on $Z_a^x=y$, Z_b^x is a uniformly chosen neighbor of y, for every b descendant of a in $\mathcal{T}$. Using this fact inductively, we conclude that given $\mathcal{T}$, for any $b\in T$,

(70) Z_b^x under the probability distribution $\mathbf{P}_u$ has the same law as the random walk X_n under P_x, where n is the generation of b in $\mathcal{T}$.

From our specific choice of the offspring distribution, note that $\mathbf{P}_u$-a.s.

(71) for every point $a\in T^\circ$, there exist exactly two infinite lines of $\bullet$-descendants going down from a.

We denote the union of these two lines by $L_a\subset\mathcal{T}$, which are drawn with continuous segments in Figure 1. Denote by $E(L_a)\subset\mathcal{E}$ the set of edges in L_a and by $V(L_a)\subset T$ the set of individuals in L_a.

Having constructed the branching random walk $(Z_a^x)_{a\in T}$, we can use the above observation to associate, for every $a\in T^\circ$ a doubly infinite trajectory $w_a\in W$ given by the image of L_a under the branching random walk Z. And using (70), we conclude that

(72) the law of w_a as an element of W is given by $\tilde{P}_{Z_a}$ as in (51).

The above allows us to define

$$\mathring{\omega}=\sum_{a\in T^\circ}\delta_{w_a}\in M.$$

We can now prove the domination result, which gives a way to control the particle system defined in the previous section with the above branching random walk.

PROPOSITION 5.1. *If the degree of G is bounded by Δ, there is a coupling Q_2 between the laws $\tilde{\mathbb{P}}_u$ and $\mathbf{P}_u$, such that Q_2-a.s. we have*

$$\tilde{\mathcal{I}}(\tilde{\omega}|_{\{X_0\in\bigcup_k\tilde{A}_k\}})\subseteq\tilde{\mathcal{I}}(\mathring{\omega})=\{Z_a^x;a\in T\}. \tag{73}$$

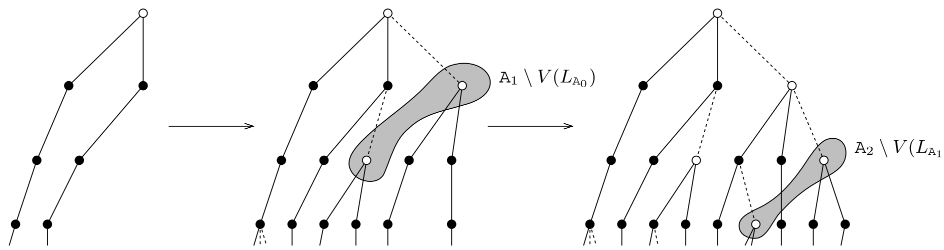

FIG. 2. *The sets $L_{\mathbb{A}_0}$, $L_{\mathbb{A}_1}$ and $L_{\mathbb{A}_2}$ corresponding to $\mathcal{T}$ in Figure 1.*

PROOF. We recall the definition of the sets $\tilde{A}_k(\tilde{\omega})$ constructed above Proposition 4.1. We will need an analogous way to gradually reveal the measure $\mathring{\omega}$ which can be informally described as "slowly revealing the lines L_a" composing $\mathcal{T}$. More precisely:

- consider $\mathbb{A}_0 = T_0^\circ$, "all individuals in the first generation of $\mathcal{T}$" and supposing we have constructed $\mathbb{A}_{k-1}$,
- let $L_{\mathbb{A}_{k-1}}$ be the union of all L_a where $a \in \mathbb{A}_{k-1}$ and all edges in $\mathcal{T}$ that connect any such a to its parent. Then define $\mathbb{A}_k$ to be the union of $V(L_{\mathbb{A}_{k-1}})$ with its descendants.

In Figure 2, we show the sets $L_{\mathbb{A}_0}$, $L_{\mathbb{A}_1}$ and $L_{\mathbb{A}_2}$.

Observe also the following two consequences of our particular choice for the offspring distribution of $\mathcal{T}$:

- $\mathbb{A}_k \setminus V(L_{\mathbb{A}_{k-1}})$ consists solely of $\circ$-type individuals,
- conditioned on $L_{\mathbb{A}_{k-1}}$, the set $\mathbb{A}_k \setminus V(L_{\mathbb{A}_{k-1}})$ can be obtained by independently attaching a Poisson$(u\Delta^2)$ number of $\circ$-offsprings to each site of $L_{\mathbb{A}_{k-1}} \setminus L_{\mathbb{A}_{k-2}}$, for any $k \geq 1$,

where we use the convention that $L_{\mathbb{A}_{-1}} = \varnothing$.

In order to compare $\tilde{\omega}$ and $\mathring{\omega}$, we claim that

(74) given $L_{\mathbb{A}_{k-1}}$ and $(\ell_e)_{e \in E(L_{\mathbb{A}_{k-1}})}$, let $\mathcal{Z}_k = (Z_a^x)_{a \in V(L_{\mathbb{A}_{k-1}})}$. Then $\sum_{a \in \mathbb{A}_k \setminus V(L_{\mathbb{A}_{k-1}})} \delta_{w_a}$ is a Poisson point process with intensity measure larger or equal than $\sum_{x \in \overline{\mathcal{Z}}_k \setminus \mathcal{Z}_k} u d_x \tilde{P}_x$.

To prove the above statement, observe first that $\mathcal{Z}_k$ can be written in terms of $L_{\mathbb{A}_{k-1}}$ and $(\ell_e)_{e \in E(L_{\mathbb{A}_{k-1}})}$. By using the comment above (74), we conclude that $(Z_a^x)_{a \in \mathbb{A}_k \setminus V(L_{\mathbb{A}_{k-1}})}$ is a Poisson process in $\overline{\mathcal{Z}}_k$ with intensity given by

$$\sum_{a \in V(L_{\mathbb{A}_{k-1}}) \setminus V(L_{\mathbb{A}_{k-2}})} \frac{u\Delta^2}{\deg(Z_a^x)} \sum_{y \in \mathcal{N}(Z_a^x)} \delta_y. \tag{75}$$

One can now conclude (74) from (75) and (72) and $u\Delta^2/\deg(Z_a^x) \geq u d_x$.

We now prove that

(76) there exist a coupling Q_2 between $\tilde{\mathbb{P}}_u$ and $\mathbf{P}_u$, such that $\tilde{\omega}|_{\{X_0 \in \tilde{A}_k\}} \leq \sum_{a \in \mathbb{A}_k} \delta_{w_a}$, Q_2-a.s. for every $k \geq 0$.

For this, fix a point measure $\tilde{\omega}$ sampled according to $\tilde{\mathbb{P}}_u$. This gives rise to a sequence of sets $\tilde{A}_0, \tilde{A}_1, \ldots$. Note that $\tilde{\omega}(\{X_0 \in \tilde{A}_0\})$ is Poisson distributed, with a parameter not bigger than $u\Delta$. We can therefore couple this random variable with the initial number $|T_0|$ of individuals in the Galton–Watson tree, in a way that almost surely

$$\tilde{\omega}(\{X_0 \in \tilde{A}_0\}) \leq |T_0|.$$

And on the above event, one can construct the coupling in (76) for $k = 0$, using (72).

We now proceed by induction. Suppose that we have obtained the coupling between $\tilde{\omega}|_{\{X_0 \in \tilde{A}_{k-1}\}}$ under $\tilde{\mathbb{P}}_u$ and $L_{\mathbb{A}_{k-1}}$ under $\mathbf{P}_u$ [together with its labels $(\ell_e)_{e \in E(L_{\mathbb{A}_{k-1}})}$] in a way that $\tilde{\omega}|_{\{X_0 \in \tilde{A}_{k-1}\}} \leq \sum_{a \in \mathbb{A}_{k-1}} \delta_{w_a}$ for a given k. Then $\tilde{A}_k \subseteq \overline{\mathcal{Z}}_k$. We can now obtain the statement for k using (74) and comparing the intensity measure of the referred Poisson point process with that of (50).

The statement in Proposition 5.1 clearly follows from (76). $\square$

6. The disconnected phase. The aim of this section is to show that on any nonamenable graph of bounded degree, $u_c > 0$; that is, for some $u > 0$ small enough the interlacement set is $\mathbb{P}_u$-a.s. disconnected; see Theorem 6.3. First, in Section 6.1, we obtain estimates for the branching random walk introduced in Section 5, and then those estimates are used in Section 6.2 to establish the existence of such a disconnectedness phase.

6.1. *Results for the branching random walk.* The goal of this subsection is to prove a heat-kernel estimate for the branching random walk; see Proposition 6.2 below. In Section 6.2 this will be used as a key ingredient for showing Theorem 6.3. Recall the definitions of $\mathcal{T} = (T, \mathcal{E})$, T° and $T^\bullet$ from Section 5. We introduce some additional notation. For $n \in \mathbb{N}$, let

$$T_n = \{a \in T; a \text{ is in generation } n\}, \tag{77}$$

$$T_n^\circ = \{a \in T^\circ; a \text{ is in generation } n\} \tag{78}$$

and

$$T_n^\bullet = \{a \in T^\bullet; a \text{ is in generation } n\}. \tag{79}$$

Then clearly $|T_n| = |T_n^\circ| + |T_n^\bullet|$.

We begin by bounding the expected number of members of T_n:

LEMMA 6.1. *For every $n \geq 0$ and $u \geq 0$,*

$$\mathbf{E}_u\big[|T_n|\big] \leq c_{u,\lambda}(1 + 2u\Delta^2)^n. \tag{80}$$

PROOF. To show (80), we proceed by fairly standard recursion arguments. First, using the fact that all members in generation 0 are of type $\circ$, we observe that

$$\mathbf{E}_u\big[|T_0|\big] = \mathbf{E}_u\big[|T_0^\circ|\big] = u\Delta^2 \tag{81}$$

and

$$\mathbf{E}_u\big[|T_1|\big] = (2 + u\Delta^2)\mathbf{E}_u\big[|T_0^\circ|\big] = (2 + u\Delta^2)u\Delta^2. \tag{82}$$

In addition, we have

$$\begin{aligned} \mathbf{E}_u\big[|T_n^\circ|\big] &= u\Delta^2\mathbf{E}_u\big[|T_{n-1}^\circ|\big] + u\Delta^2\mathbf{E}_u\big[|T_{n-1}^\bullet|\big] \\ &= u\Delta^2\mathbf{E}_u\big[|T_{n-1}|\big] \qquad \text{for every } n \geq 1. \end{aligned} \tag{83}$$

Hence, for every $n \geq 2$,

$$\begin{aligned} \mathbf{E}_u\big[|T_n|\big] &= (1 + u\Delta^2)\mathbf{E}_u\big[|T_{n-1}^\bullet|\big] + (2 + u\Delta^2)\mathbf{E}_u\big[|T_{n-1}^\circ|\big] \\ &= (1 + u\Delta^2)\mathbf{E}_u\big[|T_{n-1}|\big] + \mathbf{E}_u\big[|T_{n-1}^\circ|\big] \\ &\overset{(83)}{=} (1 + u\Delta^2)\mathbf{E}_u\big[|T_{n-1}|\big] + u\Delta^2\mathbf{E}_u\big[|T_{n-2}|\big] \\ &\leq (1 + u\Delta^2)\mathbf{E}_u\big[|T_{n-1}|\big] + u\Delta^2\mathbf{E}_u\big[|T_{n-1}|\big] \\ &= (1 + 2u\Delta^2)\mathbf{E}\big[|T_{n-1}|\big]. \end{aligned} \tag{84}$$

Now (80) follows from (84), (82) and induction. $\square$

The proposition below can be seen as a heat-kernel estimate for the above constructed branching random walk.

PROPOSITION 6.2. *Suppose $G = (V, E)$ is nonamenable, with degree bounded by $\Delta < \infty$. Recall the definition of the branching random walk starting at a $x \in V$, introduced in Section 5. There is $u_0 > 0$ such that, for every $x \in V$ fixed,*

$$\lim_{y;d(x,y)\to\infty} \mathbf{P}_u\big[\exists a \in T : Z_a^x = y\big] = 0 \qquad \textit{for every } u < u_0. \tag{85}$$

PROOF. By [28], Lemma (8.1), page 84, we have

$$p^{(n)}(x, y) \leq c\rho^n \qquad \text{for every } x, y \in V. \tag{86}$$

Denote the event appearing in the probability on the left-hand side of (85) with H_y (where the index x is omitted since it is kept fixed during the proof).

We observe the following inclusion:

$$H_y \subset \bigcup_{n=d(x,y)}^{\infty} \{|\{a \in T_n : Z_a^x = y\}| \geq 1\} \tag{87}$$

and suppose that

$$u < \frac{1}{2\Delta^2}\left(\frac{1}{\rho} - 1\right). \tag{88}$$

We now obtain

$$\begin{aligned}
\mathbf{P}_u[H_y] &\overset{(87)}{\leq} \sum_{n=d(x,y)}^{\infty} \mathbf{P}_u\big[|\{a \in T_n : Z_a^x = y\}| \geq 1\big] \\
&\leq \sum_{n=d(x,y)}^{\infty} \mathbf{E}_u\big[|\{a \in T_n : Z_a^x = y\}|\big] \\
&= \sum_{n=d(x,y)}^{\infty} \sum_{m \geq 1} \mathbf{E}_u\big[|\{a \in T_n : Z_a^x = y\}|, |T_n| = m\big] \\
&= \sum_{n=d(x,y)}^{\infty} \sum_{m \geq 1} \mathbf{E}_u\Big[\sum_{a \in T_n} \mathbf{E}_u[Z_a^x = y | \mathcal{T}], |T_n| = m\Big].
\end{aligned}$$

By using the independence of $\mathcal{T}$ and the labels $(\ell_e)_{e \in \mathcal{E}}$, we deduce from (80) and (70) that the above is bounded by

$$\begin{aligned}
&\leq \sum_{n=d(x,y)}^{\infty} c_{u,\lambda}(1 + 2u\Delta^2)^n P_x[X_n = y] \\
&\overset{(86)}{\leq} \sum_{n=d(x,y)}^{\infty} c_{u,\lambda}(\rho(1 + 2u\Delta^2))^n,
\end{aligned}$$

which converges to zero as $d(x, y)$ goes to infinity, due to (88).

Therefore, we see that (85) holds with

$$u_0 = \frac{1}{2\Delta^2}\left(\frac{1}{\rho} - 1\right) > 0, \tag{89}$$

completing the proof of the theorem. $\square$

6.2. *Disconnectedness when u is small.* We now collect the results of previous sections in order to obtain the almost sure disconnectedness of random interlacements at low levels for a large class of nonamenable graphs of bounded degree. Before, recall the definitions of the spectral radius ρ in (14) and the capacity in (21).

THEOREM 6.3. *Consider a nonamenable graph G with degree bounded by $\Delta < \infty$. Then*

$$u_c \geq \frac{1}{2\Delta^2}\left(\frac{1}{\rho} - 1\right). \tag{90}$$

PROOF. We first observe that the nonamenability, together with the assumption that the degrees are uniformly bounded imply that

$$\inf_{x \in V} P_x[\tilde{H}_x = \infty] > 0 \qquad \text{see, for instance, (4.4) of [23].} \tag{91}$$

For $u, v \in V$, write $u \overset{\mathcal{I}(\omega)}{\longleftrightarrow} v$ if u and v belong to the same connected component of the subgraph induced by $\mathcal{I}(\omega)$. If $u > 0$ is such that $\mathcal{I}(\omega)$ is connected $\mathbb{P}_u$-a.s., then for $o \in V$,

$$\begin{aligned}
\inf_{y \in V} \mathbb{P}_u\big[o \overset{\mathcal{I}(\omega)}{\longleftrightarrow} y\big] &= \inf_{y \in V} \mathbb{P}_u\big[o \in \mathcal{I}(\omega), y \in \mathcal{I}(\omega)\big] \\
&\geq \inf_{y \in V} \mathbb{P}_u\big[o \in \mathcal{I}(\omega)\big]\mathbb{P}_u\big[y \in \mathcal{I}(\omega)\big] \\
&\geq \inf_{y \in V} \big(1 - \exp\{-u d_y P_y[\tilde{H}_y = \infty]\}\big)^2 \\
&\geq \Big(1 - \exp\Big\{-u \inf_{x \in V} P_x[\tilde{H}_x = \infty]\Big\}\Big)^2 \\
&> 0,
\end{aligned} \tag{92}$$

where the FKG-inequality (see [23], Theorem 3.1) was used in the first inequality, and in the third inequality we used the fact that $d_x \geq 1$. Consequently, to show that with positive probability $\mathcal{I}(\omega)$ is disconnected for small u, it is enough to show that

$$\lim_{d(o,y) \to \infty} \mathbb{P}_u\big[o \overset{\mathcal{I}(\omega)}{\longleftrightarrow} y\big] = 0,$$

when u is sufficiently small. From Propositions 4.1 and 5.1 we know that

$$\mathbb{P}_u\big[o \overset{\mathcal{I}(\omega)}{\longleftrightarrow} y\big] \leq \mathbf{P}_u[H_y], \tag{93}$$

where we recall the definition of the event H_y from the proof of Proposition 6.2. Proposition 6.2 says that

$$\lim_{d(x,y) \to \infty} \mathbf{P}_u[H_y] = 0 \qquad \text{whenever } u < \frac{1}{2\Delta^2}\left(\frac{1}{\rho} - 1\right), \tag{94}$$

and therefore, we obtain (90). □

7. The connected phase. The aim of this section is to provide results regarding the phase $u > u_c$. In Section 7.1 we give an example of a graph for which this phase is nonempty. In Section 7.2 we show that for any $u > u_c$, the interlacement set $\mathcal{I}$ is connected.

7.1. *Finiteness of u_c for $\mathbb{T}^d \times \mathbb{Z}^{d'}$.* In Proposition 7.2 below we prove that the value u_c is finite for the classical example of the product between a d-regular tree and the d'-dimensional Euclidean lattice: $\mathbb{T}^d \times \mathbb{Z}^{d'}$, where $d \geq 3$ and $d' \geq 1$. A similar result was proved in [9] for Bernoulli percolation. It is worth mentioning that the finiteness of u_c does not hold true for every vertex-transitive nonamenable graph G, as we note in the following:

REMARK 7.1. (1) For the infinite d regular tree $\mathbb{T}^d$ ($d \geq 3$), we have

$$u_c(\mathbb{T}^d) = \infty. \tag{95}$$

Indeed, using Theorem 5.1, (5.7) and (5.9) of [23], we conclude that for any $u > 0$, with $\mathbb{P}_u$-positive probability, the root $\varnothing \in \mathbb{T}^d$ is an isolated component of $\mathbb{T}^d \setminus \mathcal{I}$. In this event, any two neighbors y and y' of $\varnothing$ are contained in $\mathcal{I}$ ruling out the almost sure connectedness of $\mathcal{I}$ and yielding (95).

(2) Considering again the above mentioned event, since the set $\mathcal{I}$ has $\mathbb{P}_u$-a.s. no finite components [see (17) and [23] (2.26)] we conclude that with positive probability there are at least two distinct infinite clusters in $\mathcal{I}$. This together with Proposition 3.2 gives that for every $u > 0$ the interlacement set $\mathcal{I}$ has infinitely many connected components $\mathbb{P}_u$-a.s.

In what follows we use the same convention as in (4.1) of [23] for the product of two graphs. More precisely, if $G = (V, E)$ and $G' = (V', E')$ are graphs, the product $G \times G'$ has vertex set $V \times V'$, and there is an edge between (x, x') and (y, y') if and only if $d(x, y) + d'(x', y') = 1$, where $d(\cdot, \cdot)$ and $d'(\cdot, \cdot)$ denote, respectively, the distances in G and G'.

PROPOSITION 7.2. *For any $d \geq 3$ and $d' \geq 1$ we have that*

$$0 < u_c(\mathbb{T}^d \times \mathbb{Z}^{d'}) < \infty. \tag{96}$$

PROOF. The fact that u_c is positive follows directly from Theorem 6.3 and the nonamenability of the graph under consideration; see [28], 4.10, page 44. Therefore we focus on establishing that $u_c < \infty$.

We first obtain a characterization of the random interlacements law on $\mathbb{T}^d \times \mathbb{Z}^{d'}$, which resembles Theorem 5.1 in [23]. For this, let us first note that the set of trajectories

$$W^*_{\mathrm{bad}(n)} = \{w \in W^*; w \text{ intersects } B(\varnothing, n) \times \mathbb{Z}^{d'} \text{ infinitely many times}\}$$

has ν-measure zero for all $n \geq 0$, where $\varnothing$ denotes the origin of the tree. To see why this is true, observe that the first projection of a random walk on $\mathbb{T}^d \times \mathbb{Z}^{d'}$ is transient. This, together with equations (25) and (24), shows that $W^*_{\mathrm{bad}(n)} \cap W^*_{B(\varnothing,n)\times B(0,m)}$ has ν-measure zero for every $m \geq 0$, yielding the claim.

For $x \in V$, we write $n(x) = \inf\{n; x \in B(\varnothing, n) \times \mathbb{Z}^{d'}\}$ and define a map $\phi : W^* \to \mathbb{T}^d \times \mathbb{Z}^{d'}$ in the following way. If w belongs to some $\{W^*_{\mathrm{bad}(n)}\}_{n\geq 0}$, then $\phi(w)$ can be defined arbitrarily, otherwise $\phi(w)$ returns the unique point $x \in \mathbb{T}^d \times \mathbb{Z}^{d'}$ such that:

- w does not enter $B(\varnothing, n(x) - 1) \times \mathbb{Z}^{d'}$ [trivially true if $n(x) = 1$] and
- x is the first point visited in $B(\varnothing, n(x)) \times \mathbb{Z}^{d'}$.

Note that this is well defined out of $\bigcup_{n\geq 0} W^*_{\mathrm{bad}(n)}$.

A second observation is that the sets

$$W^*_x = \{w \in W^*; \phi(w) = x\} \tag{97}$$

form a disjoint partition of W^*. This decomposition provides us with an alternative way to construct the set $\mathcal{I}$. More precisely, in some auxiliar probability space $(\Omega', \mathcal{A}', P')$:

- we consider for every $x \in V$, independent random variables $(J_x)_{x\in V}$, distributed as Poisson$(u \cdot \nu(W^*_x))$;
- for $x \in V$ and $i \leq J_x$ we let X^i_x and Y^i_x be random walks with distributions $P_x[\cdot|\tilde{H}_{B(\varnothing,n(x))\times\mathbb{Z}^{d'}} = \infty]$ and $P_x[\cdot|H_{B(\varnothing,n(x)-1)\times\mathbb{Z}^{d'}} = \infty]$.

Using (25), (24) and Proposition 3.6 of [14], we conclude that under $\mathbb{P}_u$,

$$\text{the set } \mathcal{I} \text{ is distributed as } \bigcup_{x\in V}\bigcup_{i\leq J_x} \mathrm{Range}(X^i_x) \cup \mathrm{Range}(Y^i_x). \tag{98}$$

The main advantage of this representation is that the variables J_x are independent, providing us with an (inhomogeneous) Bernoulli percolation $Z_x = \mathbf{1}_{\{J_x>0\}}$, for $x \in V$, such that $P'[Z_x = 0] = P'[J_x = 0] = \exp\{-u\nu(W^*_x)\}$. It is also important to note by (25) and (24) that

$$\begin{aligned}\nu(W^*_x) &= P_x[\tilde{H}_{B(\varnothing,n(x))\times\mathbb{Z}^{d'}} = \infty, \tilde{H}_{\{x\}} = \infty] d_x P_x[\tilde{H}_{B(\varnothing,n(x)-1)\times\mathbb{Z}^{d'}} = \infty] \\ &\geq \beta > 0,\end{aligned} \tag{99}$$

uniformly on $x \in V$. This implies that $(Z_x)_{x\in V}$ stochastically dominates a Bernoulli$(1 - \exp\{-u\beta\})$ i.i.d. percolation.

We now state a known result on the uniqueness of the infinite cluster of Bernoulli percolation, first proved in [9]. More precisely, Proposition 8.1 of [10] states that for $d \geq 3$, $d' \geq 1$ and p close enough to 1,

$$\text{there is a.s. a unique infinite cluster } \mathcal{C}_\infty \text{ in Bernoulli}(p) \text{ site percolation on } \mathbb{T}^d \times \mathbb{Z}^{d'}. \tag{100}$$

A careful reader would observe that the result in [10] is stated for bond percolation instead. However, the second part of the proof of Lemma 4.1 applies without modifications to site percolation, if one uses the fact that Lemma 4.3 of [10] was proved in [15] for site percolation as well. Moreover, using equation (8.9) of [10] we obtain that this unique infinite cluster satisfies

$$P_x[H_{\mathcal{C}_\infty} < \infty] = 1 \qquad \text{for every } x \in V. \tag{101}$$

We now take u_o large enough so that for every $u \geq u_o$, $(Z_x)_{x\in V}$ dominates a Bernoulli percolation satisfying the two above claims. Then, for these values of u, the $\mathbb{P}_u$-almost sure connectivity of $\mathcal{I}$ will follow from the characterization in (98), proving that $u_c \leq u_o < \infty$. □

REMARK 7.3. A natural question that the above proof raises is: Is it true that for any nonamenable graph, random interlacements dominate Bernoulli site percolation, as we obtained in the proof of Proposition 7.2? This question was posed to the authors by Itai Benjamini.

Note also that random interlacements on $\mathbb{Z}^d$ do not dominate or become dominated by any Bernoulli site percolation; see Remark 1.1 in [16].

7.2. *Monotonicity of the uniqueness transition.* Observe that the event $\{\mathcal{I}$ is connected$\}$ is not monotone with respect to the set $\mathcal{I}$. This immediately raises the question of whether there could be some $u > u_c$ for which $\mathcal{I}^u$ is $\mathcal{P}_u$-a.s. disconnected. The next result rules out this possibility. Note that a similar question was considered in [11] in the case of Bernoulli percolation on unimodular transitive graphs and in [15] in the general quasi-transitive case.

Fix intensities $u' > u \geq 0$. For the statement of Theorem 7.4 below, we need to couple the random interlacement at levels u and u'. Actually, this is done in [23] for all values of $u \geq 0$ simultaneously, but for the purpose of this paper, it is enough to consider the following. Put $\mathbb{P} = \mathbb{P}_u \otimes \mathbb{P}_{u'-u}$. For $(\omega_1, \omega_2) \in \Omega \times \Omega$, put $\mathcal{I}^u = \mathcal{I}(\omega_1)$ and $\mathcal{I}^{u'-u} = \mathcal{I}(\omega_2)$. Finally, we define $\mathcal{I}^{u'}$ as $\mathcal{I}^u \cup \mathcal{I}^{u'-u}$. Then clearly $\mathcal{I}^u \subset \mathcal{I}^{u'}$, and under $\mathbb{P}$, the random sets $\mathcal{I}^u$ and $\mathcal{I}^{u'-u}$ are independent. Moreover, due to the Poissonian character of random interlacements, we have that under $\mathbb{P}$, the sets $\mathcal{I}^u$, $\mathcal{I}^{u'}$, $\mathcal{I}^{u'-u}$ have laws $\mathbb{P}_u$, $\mathbb{P}_{u'}$, $\mathbb{P}_{u'-u}$, respectively.

THEOREM 7.4. *Suppose that G is such that $P_x[\tilde{H}_x = \infty]$ is uniformly bounded from below by $\gamma > 0$. Then, for any $u' > u \geq 0$, we have that*

$$\textit{all components of } \mathcal{I}^{u'} \textit{ contain an infinite component of } \mathcal{I}^u,\ \mathbb{P}\textit{-a.s.} \tag{102}$$

Observe that transience and transitivity imply the above hypothesis.

PROOF OF THEOREM 7.4. To obtain the statement in (102), it is enough to prove that, for every $x \in V$, the following event has $\mathbb{P}$-probability one:

$$\begin{aligned}\{&\text{the connected component of } \mathcal{I}^{u'} \text{ containing } x\\ &\text{is either empty or contains an infinite component of } \mathcal{I}^u\}.\end{aligned}$$

Indeed, if this is the case, we can intersect the countable collection of such events (as x runs in V), and we obtain (102).

Now fix $x \in V$ and recall from (17) and (26) that the set $\mathcal{I}^u$ has no finite components. Then, since $\mathcal{I}^u \subseteq \mathcal{I}^{u'}$, the above mentioned event equals

$$\begin{aligned}(103)\qquad &\big\{\text{the connected component of } \mathcal{I}^{u'} \text{ containing } x\\ &\qquad \text{is either empty or contains a point in } \mathcal{I}^u\big\}.\end{aligned}$$

Now let us see that the above event has $\mathbb{P}$-probability one. For this, recall that if $x \in \mathcal{I}^{u'} \setminus \mathcal{I}^u$, then $x \in \mathcal{I}^{u'-u}$, which is equivalent to $\omega_2(W^*_{\{x\}}) > 0$.

Given the Poissonian character of ω_2, we can construct its restriction to the set $W^*_{\{x\}}$ in the following way. First simulate the random variable $J = \omega_2(W^*_{\{x\}})$ which has Poisson distribution [with parameter $(u' - u)\nu(W^*_{\{x\}})$]. Then for every $i = 1, \dots, J$ we throw an independent trajectory $w \in W^*_{\{x\}}$ with distribution given by ν restricted to $W^*_{\{x\}}$ and normalized to become a probability measure. According to (24), this probability distribution is given by $\pi^* \circ Q_{\{x\}}/e_{\{x\}}(x)$.

Conditioned on the event $\omega_2(W^*_{\{x\}}) > 0$, we know by the above construction and (25) that $\mathcal{I}^{u'-u}$ contains a random walk trajectory with law P_x. Thus, all we have to do in order to prove that (103) has probability one is to show that $\mathbb{P}$-a.s. the set $\mathcal{I}^u$ is such that

$$(104)\qquad P_x[H_{\mathcal{I}^u} < \infty] = 1,$$

which, by Fubini's theorem, is equivalent to proving that P_x-a.s. the range of a random walk is a sequence $(x_i)_{i\geq 0}$ such that

$$(105)\qquad \mathbb{P}\big[\mathcal{I}^u \cap \{x_i\}_{i\geq 0} \neq \varnothing\big] = 1.$$

In fact we will exhibit a subsequence $(x_{i_j})_{j\geq 0}$ of the random walk trace, such that $\mathcal{I}^u$ almost surely intersects $\{x_{i_j}\}_{j\geq 0}$. For this, let us show that P_x-a.s. there exists a subsequence $(x_{i_j})_{j\geq 0}$ of the random walk trace such that

$$(106)\qquad P_{x_{i_j}}[H_{\{x_{i_0},\dots,x_{i_{j-1}}\}} < \infty] < \tfrac{1}{2} \qquad \text{for all } j \geq 1.$$

To see why this is true, we first claim that for every finite $K \subset V$, the set

$$(107)\qquad A_K = \big\{z \in V; P_z[H_K < \infty] \geq \tfrac{1}{2}\big\} \text{ is } P_x\text{-a.s. visited finitely many times.}$$

Indeed, if the random walk had a positive probability of visiting the set A_K infinitely many times, we could use Borel–Cantelli's lemma to prove that the set K would also be visited infinitely often, which is a contradiction, proving (107). Note that for some graphs the set A_K could be infinite.

We can now define the subsequence $(i_j)_{j\geq 0}$ that we mentioned in (106) above. First fix a sequence $(x_i)_{i\geq 0}$ that visits $A_{B(x,n)}$ finitely often for every $n \geq 0$ (note that $A_{B(x,n)}$, $n \geq 0$ is a countable family). According to (107) these sequences have P_x-probability one. Now, let $i_0 = 0$ and supposing we have defined i_j for $j \leq j_o$, take i_{j_o+1} to be such that $x_{i_{j_o+1}}$ is outside of $A_{B(x,n)}$ with $K_{j-1} := \{x_{i_0}, \dots, x_{i_{j-1}}\} \subset B(x,n)$. This shows (106).

We are now in position to prove (105) by considering the following disjoint subsets of W^*:

$$W_j^* = \{w \in W^*; \text{Range}(w) \text{ intersects } x_{i_j} \text{ but not } K_{j-1}\} \qquad \text{for } j \geq 1. \tag{108}$$

They have ν-measure bounded away from 0, as the following calculation shows:

$$\begin{aligned} \nu(W_j^*) &= \nu(W^*_{\{x_{i_j}\}} \setminus W^*_{K_{j-1}}) \overset{(24)}{=} Q_{\{x_{i_j}\}}[(X_n)_{n\in\mathbb{Z}} \cap K_{j-1} = \varnothing] \\ &\overset{(25)}{=} P_{x_{i_j}}[H_{K_{j-1}} = \infty | \tilde{H}_{x_{i_j}} = \infty] e_{\{x_{i_j}\}}(x_{i_j}) P_{x_{i_j}}[H_{K_{j-1}} = \infty], \end{aligned} \tag{109}$$

which by (106) and the hypothesis of the theorem, is bounded from below by $\gamma/4$.

Finally, we estimate

$$\begin{aligned} \mathbb{P}[\mathcal{I}^u \cap \{x_i\}_{i\geq 0} \neq \varnothing] &\geq \mathbb{P}[\mathcal{I}^u \cap \{x_{i_j}\}_{j\geq 0} \neq \varnothing] \\ &\geq \mathbb{P}[\omega_1(W_j^*) > 0, \text{ for some } j \geq 1], \end{aligned} \tag{110}$$

which has probability one, since the above random variables are independent Poisson random variables with parameter bounded away from zero. This proves (105), therefore establishing that (103) has probability one, completing the proof of Theorem 7.4. $\square$

The following corollary is an immediate consequence of Theorem 7.4.

COROLLARY 7.5. *For any graph G satisfying the hypothesis of Theorem 7.4 and for any $u > u_c$, the set $\mathcal{I}$ is $\mathbb{P}_u$-a.s. connected. In particular, for these graphs we could have alternatively used the definition*

$$u_c = \sup\{u \geq 0; \mathcal{I} \textit{ is not } \mathbb{P}_u\textit{-a.s. connected}\}. \tag{111}$$

Furthermore, the connectedness transition for $\mathcal{I}$ is unique for these graphs.

REMARK 7.6. (1) Note that we do not necessarily suppose that the underlying graph is transitive (or quasi-transitive) which is a standard assumption in the case of Bernoulli percolation to obtain the monotonicity of the uniqueness transition.

(2) In the above result we do not rule out the existence of exceptional intensities $u > u_c$ for which $\mathcal{I}$ is disconnected. According to Corollary 7.5 and Fubini's theorem, the set of such exceptional intensities must have zero Lebesgue measure.

8. Bounds for u_*. Recall that in Theorem 4.1 of [23], it was shown that for any nonamenable graph of bounded degree, the critical value u_* is finite. In this section, we provide a lower bound for u_* which implies the positivity of u_* on any nonamenable Cayley graph. First we recall the definition of a Cayley graph: given a finitely generated group H with symmetric generating set S, the (right) Cayley graph $G = G(H, S)$ is the graph with vertex set H and such that $\{u, v\}$ is an edge if and only if $u = vs$ for some $s \in S$. Recall that the critical value u_c can be degenerated on nonamenable Cayley graphs; see Remark 7.1. In Proposition 8.1 below, we show that in contrast, u_* is always nondegenerate on such graphs.

PROPOSITION 8.1. *Let G be a Cayley graph of degree d, and fix $o \in V$. Then*

$$-\frac{1}{d P_o[\tilde{H}_o = \infty]} \log\left(\frac{d}{d + \kappa_V}\right) \leq u_* \leq 2d^2\kappa_E^{-2} \leq 2d^2\kappa_V^{-2}. \tag{112}$$

PROOF. We begin with the lower bound. Since G is a Cayley graph, and the law of $\mathcal{V}$ is invariant under $\mathrm{Aut}(G)$, Theorem 2.1 of [3] implies that if

$$\mathbb{P}_u[o \in \mathcal{V}] \geq \frac{d}{d + \kappa_V}, \tag{113}$$

then $\mathcal{V}$ contains unbounded connected components with positive probability. We recall that [see (28)]

$$\mathbb{P}_u[o \in \mathcal{V}] = \exp\{-u d P_o[\tilde{H}_o = \infty]\}. \tag{114}$$

We now conclude the lower bond in (112) from (113) and (114).

We now proceed with the upper bound. The proof of Theorem 4.1 in [23] shows that $u_* \leq \overline{\kappa}$ where $\overline{\kappa}$ is the constant appearing in statement (b) of Theorem 10.3 in [28]. However, the proof of Theorem 10.3 in [28] shows that $\overline{\kappa}$ can be chosen to equal $2\kappa^2$ where κ is the constant appearing in the statement of Proposition 4.3 of [28]. The constant κ is the same as appears in Definition 4.1 of [28], and comparing that definition with (13), one sees that $\kappa = d\kappa_E^{-1}$. Thus, $u_* \leq 2d^2\kappa_E^{-2}$. Since $|\partial_E K| \geq |\partial_V K|$ we obtain $\kappa_E \geq \kappa_V$, and consequently $2d^2\kappa_E^{-2} \leq 2d^2\kappa_V^{-2}$, completing the proof of the proposition. □

APPENDIX

In this section we present the proof of Proposition 2.4. We follow here the same arguments as those of Theorem 2.1 of [21], but they are included in detail for the reader's convenience. Recall the definitions of t_g, $\mathcal{Y}$ and Q_u from Section 2.

Before proving Proposition 2.4, we formulate and prove Lemma A.1. The proof of this lemma is similar to that of Lemma 2.1 of [2], where the analogous result for random interlacements on $\mathbb{Z}^d$ was proved; see also [22], Remark 1.5.

LEMMA A.1. *Let $u \geq 0$ and K_1 and K_2 be finite disjoint subsets of V. Let F_1 and F_2 be* $[0, 1]$*-valued measurable functions on the set of finite point-measures* *on W_+ (endowed with its canonical σ-field). Then*

$$\begin{aligned}&\big|\mathbb{E}_u[F_1(\mu_{K_1})F_2(\mu_{K_2})] - \mathbb{E}_u[F_1(\mu_{K_1})]\mathbb{E}_u[F_2(\mu_{K_2})]\big|\\ &\qquad \le c_u \operatorname{cap}(K_1)\operatorname{cap}(K_2) \sup_{\substack{x\in K_1,\\ y\in K_2}} g(x,y).\end{aligned}$$

PROOF. We decompose the Poisson point process $\mu_{K_1\cup K_2}$ into four independent Poisson point processes as follows:

$$\mu_{K_1\cup K_2} = \mu_{1,1} + \mu_{1,2} + \mu_{2,1} + \mu_{2,2}, \tag{115}$$

where

$$\mu_{1,1} = \sum_{i\ge 0} \delta_{w_i} 1\{X_0 \in K_1, H_{K_2} = \infty\}, \qquad \mu_{1,2} = \sum_{i\ge 0} \delta_{w_i} 1\{X_0 \in K_1, H_{K_2} < \infty\},$$

$$\mu_{2,1} = \sum_{i\ge 0} \delta_{w_i} 1\{X_0 \in K_2, H_{K_1} < \infty\}, \qquad \mu_{2,2} = \sum_{i\ge 0} \delta_{w_i} 1\{X_0 \in K_2, H_{K_1} = \infty\}.$$

From (30) we conclude that the $\mu_{i,j}$'s are independent Poisson point processes on W_+, and their corresponding intensity measures are given by

$$u1\{X_0 \in K_1, H_{K_2} = \infty\}P_{e_{K_1\cup K_2}}, \qquad u1\{X_0 \in K_1, H_{K_2} < \infty\}P_{e_{K_1\cup K_2}},$$
$$u1\{X_0 \in K_2, H_{K_1} < \infty\}P_{e_{K_1\cup K_2}}, \qquad u1\{X_0 \in K_2, H_{K_1} = \infty\}P_{e_{K_1\cup K_2}}.$$

We observe that $\mu_{K_1} - \mu_{1,1} - \mu_{1,2}$ is determined by $\mu_{2,1}$ and therefore independent of $\mu_{1,1}, \mu_{2,2}$ and $\mu_{1,2}$. In the same way, $\mu_{K_2} - \mu_{2,2} - \mu_{2,1}$ is independent of $\mu_{2,2}, \mu_{2,1}$ and $\mu_{1,1}$. We can therefore introduce the auxiliary Poisson processes $\mu'_{2,1}$ and $\mu'_{1,2}$ such that they have the same law as $\mu_{K_1} - \mu_{1,1} - \mu_{1,2}$ and $\mu_{K_2} - \mu_{2,2} - \mu_{2,1}$, respectively, and $\mu'_{2,1}, \mu'_{1,2}, \mu_{i,j}$, $1 \le i, j \le 2$ are independent. Then

$$\begin{aligned}\mathbb{E}_u[F_1(\mu_{K_1})] &= \mathbb{E}_u[F_1((\mu_{K_1} - \mu_{1,1} - \mu_{1,2}) + \mu_{1,1} + \mu_{1,2})]\\ &= \mathbb{E}_u[F_1(\mu'_{2,1} + \mu_{1,1} + \mu_{1,2})]\end{aligned} \tag{116}$$

and in the same way,

$$\mathbb{E}_u[F_2(\mu_{K_2})] = \mathbb{E}_u[F_2(\mu'_{1,2} + \mu_{2,2} + \mu_{2,1})]. \tag{117}$$

Using (116), (117) and the independence of the Poisson processes $\mu'_{2,1} + \mu_{1,1} + \mu_{1,2}$ and $\mu'_{1,2} + \mu_{2,2} + \mu_{2,1}$, we get

$$\begin{aligned}&\mathbb{E}_u[F_1(\mu_{K_1})]\mathbb{E}_u[F_2(\mu_{K_2})]\\ &\qquad = \mathbb{E}_u[F_1(\mu'_{2,1} + \mu_{1,1} + \mu_{1,2})F_2(\mu'_{1,2} + \mu_{2,2} + \mu_{2,1})].\end{aligned} \tag{118}$$

From (118) we see that

$$
\begin{aligned}
&\bigl|\mathbb{E}_u\bigl[F_1(\mu_{K_1})F_2(\mu_{K_2})\bigr] - \mathbb{E}_u\bigl[F_1(\mu_{K_1})\bigr]\mathbb{E}_u\bigl[F_2(\mu_{K_2})\bigr]\bigr| \\
&\quad \le \mathbb{P}_u\bigl[\mu'_{2,1} \neq 0 \text{ or } \mu'_{1,2} \neq 0 \text{ or } \mu_{2,1} \neq 0 \text{ or } \mu_{1,2} \neq 0\bigr] \\
&\quad \le 2\bigl(\mathbb{P}_u[\mu_{2,1} \neq 0] + \mathbb{P}_u[\mu_{1,2} \neq 0]\bigr) \\
&\quad \le 2u\bigl(P_{e_{K_1 \cup K_2}}[X_0 \in K_1, H_{K_2} < \infty] + P_{e_{K_1 \cup K_2}}[X_0 \in K_2, H_{K_1} < \infty]\bigr).
\end{aligned} \tag{119}
$$

We now bound the two last terms in the above equation,

$$
\begin{aligned}
P_{e_{K_1 \cup K_2}}[X_0 \in K_1, H_{K_2} < \infty] &\le \sum_{x \in K_1} e_{K_1}(x) P_x[H_{K_2} < \infty] \\
&= \sum_{x \in K_1, y \in K_2} e_{K_1}(x) g(x, y) e_{K_2}(y) \\
&\le \operatorname{cap}(K_1)\operatorname{cap}(K_2) \sup_{x \in K_1, y \in K_2} g(x, y).
\end{aligned} \tag{120}
$$

A similar estimate holds for $P_{e_{K_1 \cup K_2}}[X_0 \in K_2, H_{K_1} < \infty]$, and the result follows. □

PROOF OF PROPOSITION 2.4. We follow the proof of Theorem 2.1 in [21], which goes through with only minor modifications. We define the map $\psi : \Omega \to \{0, 1\}^V$ given by $\psi(\omega) = (1\{x \in \mathcal{V}(\omega)\})_{x \in V}$. Then $Q_u = \psi \circ \mathbb{P}_u$ and moreover

$$
t_g \circ \psi = \psi \circ \tau_g, \qquad g \in \operatorname{Aut}(G). \tag{121}
$$

Choose a sequence of vertices $v, v_1, v_2, \ldots \in V$ such that $d(v, v_i) \to \infty$ as $i \to \infty$. For each $i \ge 1$, let $g_i \in \operatorname{Aut}(G)$ be such that $g_i(v) = v_i$. These choices are possible due to our assumption that G is an infinite transitive graph. To prove the ergodicity statement, it suffices to establish that for any finite $K \subset V$ and any $[0, 1]$-valued $\sigma(Y_z, z \in K)$-measurable function f on $\{0, 1\}^{\mathbb{Z}^d}$, the following limit holds:

$$
\lim_{i \to \infty} E^{Q_u}[f f \circ \tau_{g_i}] = E^{Q_u}[f]^2. \tag{122}
$$

Bound (122) gives the mixing of the interlacements set with respect to the automorphisms of G. The $\{0, 1\}$-law (37) can be classically deduced as follows: if $A \in \mathcal{Y}$ is invariant under $\operatorname{Aut}(G)$, then one can do $L^1(Q_u)$-approximation of its indicator function by functions f as above. With (122), one obtains in a standard way that $Q_u(A) = Q_u(A)^2$, so that $Q_u(A) \in \{0, 1\}$.

Using (121), equation (122) will follow if we show that for any finite $K \subset V$ and any $[0, 1]$-valued function F on the set of finite point-measures on W_+,

$$
\lim_{i \to \infty} \mathbb{E}_u\bigl[F(\mu_K) F(\mu_K) \circ \tau_{g_i}\bigr] = \mathbb{E}_u\bigl[F(\mu_K)\bigr]^2. \tag{123}
$$

However, $\mathbb{E}_u[F(\mu_K)F(\mu_K) \circ \tau_{g_i}] = \mathbb{E}_u[F(\mu_K)H(\mu_{g_i(K)})]$ for some H in the same class of functions as F. Since $d(K, g_i(K)) \to \infty$ as $i \to \infty$ and G is transient, we conclude that $\sup_{x\in K, y\in g_i(K)} g(x, y) \to 0$ as $i \to \infty$. An appeal to Lemma A.1 now gives (123), and the theorem follows. $\square$

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