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Differentiable approximation of diffusion equations driven by α-stable Lévy noise

H. Al-Talibi

Linnæus University

Abstract. Edward Nelson derived Brownian motion from the Ornstein–Uhlenbeck theory by a scaling limit. Previously we extended the scaling limit to an Ornstein–Uhlenbeck process driven by an α-stable Lévy process. In this paper we extend the scaling result to α-stable Lévy processes in the presence of a nonlinear drift, an external field of force in physical terms.

1 Introduction

In Nelson (1967) Brownian motion is constructed as a scaling limit of a one parameter family of Ornstein–Uhlenbeck position processes. In a further step he extended the scaling limit by adding a nonlinear drift to the evolution equation. The result goes back to a work by Chandrasekhar. In contrast to Einstein's model, the noise is introduced into a second order ordinary differential equation, a Newton equation in physical terms. In this way the approximating processes are differentiable almost surely. For this and further references see Nelson (1967). Processes of this type are solutions of stochastically perturbed Newton equations which where studied, for instance, in Albeverio et al. (1992, 1999) and Markus and Weerasinghe (1993). Stated in geometrical terms, the Ornstein–Uhlenbeck process is defined in the tangent bundle of the real line. The driving Brownian motion of the system is defined in the tangent space. The scaling procedure recovers the driving process in the limit and the drift term which physically represents the external field of force; see Nelson (1967). We point out that Nelson does not assign the notation B to standard Brownian motion.

In this paper we do not assume the drift to be of the form $K(x) = \beta \frac{\nabla\Psi}{\Psi}$ where the complex valued function Ψ solves a Schrödinger equation, which motivated E. Nelson to set up his beautiful kinetic theory of Brownian motion. It reveals a new physical interpretation of the stochastically perturbed Newton equations in terms of stochastic forward and backward velocities. A consistent probabilistic variational approach of dynamics is developed in Zambrini (1986). An extension to bridges of Lévy processes with jumps may be found in Privault and Zambrini (2004).

In our previous work (Al-Talibi et al., 2009) we have extended the result in Nelson (1967), Chapters 9 and 10, concerning the Ornstein–Uhlenbeck process

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to α-stable Lévy processes using time change. In this paper we treat the scaling limit for Newton equations perturbed by an α-stable Lévy process as in Al-Talibi et al. (2009) with an additional nonlinear drift term (βK), $\beta > 0$, satisfying a uniform Lipschitz condition. Applying integration by parts, we separate the terms to be found in the limit. The remaining terms are shown to converge to zero by using time change. In the course of this argument it becomes important to have a uniform estimate on the supremum of the position process. This can be achieved by introducing an adequate finite partition of the compact interval for which the limit holds.

In physical models $x(t)$ describes the position of a particle at time $t > 0$. It is assumed that the velocity $\frac{dx}{dt} = v$ exists and satisfies the so-called Langevin equation with an additional nonlinear drift. Mathematically, the two ordinary differential equations combine to the initial value problem:

$$\begin{aligned} dx_t &= v_t\, dt, \\ dv_t &= -\beta v_t\, dt + \beta K(x_t)\, dt + \beta\, dX_t, \end{aligned} \tag{1.1}$$

with initial value $(x_0, v_0) = (x(0), v(0))$, where $\beta > 0$, K is a nonlinear drift which satisfies sufficient conditions to guarantee existence and uniqueness of solutions [see, for example, Applebaum (2004) and Kolokoltsov et al. (2002)] and X_t is an α-stable Lévy process.

For simplicity reason we treat the case where K in (1.1) is independent of time.

2 The position process

We study the diffusion equation (1.1). Sufficient conditions for the existence of a unique solution may be found in Applebaum (2004), Ikeda and Watanabe (1989) and Kolokoltsov et al. (2002). In this case the solution of this stochastic differential equation can be represented as given in the proposition below.

Proposition 2.1. *Let a be a constant. Furthermore, let X be a Lévy process on $\mathbb{R}$. Let $f:[0,\infty) \to \mathbb{R}$ be a continuous function such that $\int_0^t e^{a(t-s)} f(s)\, ds$ is finite for almost all t. Then the global solution of the stochastic differential equation*

$$d\zeta_t = a\zeta_t\, dt + f(t)\, dt + dX_t, \qquad t \geq 0,$$

where initial value $\zeta(0) = \zeta_0$ exists and is of the form

$$\zeta_t = e^{at}\zeta_0 + \int_0^t e^{a(t-s)} f(s)\, ds + \int_0^t e^{a(t-s)}\, dX_s.$$

Proof. Due to the assumption on f,

$$\zeta_t := e^{at}\left[\zeta_0 + \int_0^t e^{-as} f(s)\, ds + \int_0^t e^{-as}\, dX_s\right]$$

exists almost everywhere for arbitrary constant ζ_0. Equivalently,

$$\int_0^t e^{-as} f(s)\, ds + \int_0^t e^{-as}\, dX_s = e^{-at}\zeta_t - \zeta_0.$$

Moreover, calculating the total derivative of ζ_t and reinserting ζ_t reveals the given stochastic differential equations, that is, ζ_t solves the given differential equations. □

Let us focus on the position process $\{x_t\}_{t\geq 0}$ in (1.1). Due to Proposition 2.1, it has the form

$$\begin{aligned} x_t = x_0 &+ \int_0^t e^{-\beta s} v_0\, ds + \beta \int_0^t \int_0^s e^{-\beta(s-u)} K(x_u)\, du\, ds \\ &+ \int_0^t \int_0^s \beta e^{-\beta(s-u)}\, dX_u\, ds. \end{aligned} \tag{2.1}$$

There is a natural extension of these results to $\mathbb{R}^d$, $d > 1$. We observe that the third term in (2.1), a double integral, includes a stochastic integral with respect to a Lévy process.

Our notation coincides with the one in Applebaum (2004) from where we also recall that for arbitrary Lévy processes Y the characteristic function is of the form $\phi_{Y_t}(u) = e^{t\eta(u)}$ for each $u \in \mathbb{R}, t \geq 0$, where η is the Lévy-symbol of $Y(1)$. For centered α-stable Lévy processes, the Lévy-symbol at $t = 1$ for $\alpha \neq 1$ is given by

$$\eta(u) = -\sigma^\alpha |u|^\alpha, \tag{2.2a}$$

and for $\alpha = 1$ is given by

$$\eta_1(u) = -\sigma|u|. \tag{2.2b}$$

Proposition 2.2. *Assume that Y is an α-stable Lévy process*, $0 < \alpha < 2$, *and g is a continuous function on the interval $[s,t] \subset T \subsetneq \mathbb{R}$. Let η be the Lévy symbol of Y_1 and ξ_t be the Lévy symbol of $\psi(t) = \int_s^t g(r)\, dY_r$. Then we have*

$$\xi_t(u) = \int_s^t \eta(ug(r))\, dr.$$

The proof is a direct consequence of Theorem 1 in Lukacs (1969).

For $g(\ell) = e^{\beta(\ell - t)}$, $\ell \geq 0$, and the α-stable process X in (2.1) the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is

$$\xi(u) = \begin{cases} \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta(u) & \text{for } 0 < \alpha < 2, \alpha \neq 1, \\ \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta_1(u) & \text{for } \alpha = 1 \end{cases}$$

with η, η_1 as in (2.2a) and (2.2b), respectively, and $0 \le s \le t$. We are thus led to introduce the time change $\tau^{-1}(t)$ where

$$\tau(t) = \int_0^t e^{-\alpha\beta t} e^{\alpha\beta u} \, du = \frac{1}{\alpha\beta}(1 - e^{-\alpha\beta t}), \tag{2.3}$$

which is actually deterministic. This means that X_t and $Z_{\tau^{-1}(t)}$ have the same distribution.

3 The main result

Let us now formulate the main result of this paper. Let

$$dy_t = K(y_t)\, dt + dX_t, \tag{3.1}$$

with $y(0) = x_0$ and assume that $K : \mathbb{R} \to \mathbb{R}$ satisfies a global Lipschitz condition.

Theorem 3.1. *Let $t_1 < t_2$, $t_1, t_2 \in T$, T a compact subset of $[0, \infty)$, and $\beta > 0$. Then the limit*

$$\lim_{\beta \to \infty} x_t = y_t$$

exists almost surely for any $t \in T$, where $\{x_t\}_{t \ge 0}$ is the position process (2.1) and $\{y_t\}_{t \ge 0}$ is the solution of (3.1) with $\{X_t\}_{t \ge 0}$ as its driving α-stable Lévy noise.

Proof. The statement of the theorem means that the position process x_t in (2.1) converges in almost sure sense to y_t on any compact subset of the time axis $[0, \infty)$, as β tends to infinity. The increment of the process (2.1), according to Proposition 2.1, is given by

$$\begin{aligned} x_{t_2} - x_{t_1} = \int_{t_1}^{t_2} e^{-\beta s} v_0 \, ds + \beta \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} K(x_u) \, du \, ds \\ + \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} \beta \, dX_u \, ds. \end{aligned} \tag{3.2}$$

From now on let us denote $\Delta t = t_2 - t_1$. The first integral of (3.2) tends to zero as β tends to infinity; see Al-Talibi et al. (2009). Indeed, we demonstrate the technique which uses time change on the third part of (3.2). But first we split the double integral into two integrals. We have

$$\beta \left[\int_{t_1}^{t_2} \int_{t_1}^{s} e^{-\beta s} e^{\beta u} \, dX_u \, ds + \int_{t_1}^{t_2} \int_0^{t_1} e^{-\beta s} e^{\beta u} \, dX_u \, ds \right]. \tag{3.3}$$

Let us look to the first part of (3.3) which reveals the increment of the driving Lévy process. We use partial integration to have

$$\beta \int_{t_1}^{t_2} \int_{t_1}^{s} e^{-\beta s} e^{\beta u} \, dX_u \, ds = -e^{-\beta t_2} \int_{t_1}^{t_2} e^{\beta u} \, dX_u + (X_{t_2} - X_{t_1}). \tag{3.4}$$

By introducing a time change in analogy to (2.3) on the right-hand side of (3.4), we obtain

$$-e^{-\beta t_2}\int_{t_1}^{t_2} e^{\beta u}\,dX_u = Z_{(1/\alpha\beta)(1-e^{-\alpha\beta\Delta t})} = \frac{1}{\sqrt[\alpha]{\beta}} Z_{(1/\alpha)(1-e^{-\alpha\beta\Delta t})},$$

where we used the scaling property of α-stable Lévy processes, that is, $Z_{\gamma\tau} \triangleq \gamma^{\alpha} Z_{\tau}$ with $\gamma > 0$.

We see that $e^{-\alpha\beta\Delta t}$ tends to zero when β tends to infinity and simultaneously $Z_{(1/\alpha)(1-e^{-\alpha\beta\Delta t})}$ converges to $Z_{(1/\alpha)}$.

In analogy to the argument above, the product $\frac{1}{\sqrt[\alpha]{\beta}} Z_{(1/\alpha)(1-e^{-\alpha\beta\Delta t})}$ tends to zero almost surely for β tending to infinity. The double integral of the second part of (3.3) tends to zero as β tends to infinity. For more details we refer to Al-Talibi et al. (2009). We now turn to the term in (3.2) which has not been dealt with in Al-Talibi et al. (2009); the second term is given by

$$I_2 := \int_{t_1}^{t_2} \beta e^{-\beta s} \int_0^s e^{\beta u} K(x_u)\,du\,ds.$$

Using integration by parts, we obtain

$$\begin{aligned} I_2 &= \left[-e^{-\beta s}\int_0^s e^{\beta u} K(x_u)\,du\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} K(x_s)\,ds \\ &= -e^{-\beta t_2}\int_0^{t_2} e^{\beta u} K(x_u)\,du + e^{-\beta t_1}\int_0^{t_1} e^{\beta u} K(x_u)\,du + \int_{t_1}^{t_2} K(x_s)\,ds. \end{aligned} \tag{3.5}$$

The last term will appear in the limit where β tends to infinity. In the sequel we show that other two terms converge to zero in this limit. The first integral of (3.5) can be estimated by

$$\begin{aligned} \left|\int_0^{t_2} e^{-\beta(t_2-u)} K(x_u)\,du\right| &\le \int_0^{t_2} e^{-\beta(t_2-u)} |K(x_u) - K(x_0)|\,du \\ &\quad + K(x_0)\int_0^{t_2} e^{-\beta(t_2-u)}\,du. \end{aligned} \tag{3.6}$$

The last integral of (3.6) is $K(x_0)(-\frac{1}{\beta} + \frac{1}{\beta}e^{-\beta t_2})$, which tends to zero as β tends to infinity. Let κ be the Lipschitz constant of K, that is, $|K(x_1) - K(x_2)| \le \kappa |x_1 - x_2|$ for $x_1, x_2 \in \mathbb{R}$. Looking at the first integral in (3.6), we see that it is bounded by

$$\begin{aligned} \int_0^{t_2} e^{-\beta(t_2-u)} |K(x_u) - K(x_0)|\,du &\le \kappa \sup_{0\le u\le t_2} |x_u - x_0| \int_0^{t_2} e^{-\beta(t_2-u)}\,du \\ &= \frac{\kappa}{\beta}(-1 + e^{-\beta t_2}) \sup_{0\le u\le t_2} |x_u - x_0|. \end{aligned} \tag{3.7}$$

As mentioned before, $\frac{1}{\beta}(-1 + e^{-\beta t_2})$ converges to zero as β tends to infinity, hence, it suffices that the supremum is uniformly bounded, which is done below.

For arbitrary $t_1, t_2 \in T$ it might be necessary to introduce a partition of the interval $[0, t_2]$. Therefore, we reconsider (3.2) for increments $x_t - x_{t_1}$, $0 \le t_1 \le t$. The absolute value of this difference may be estimated by using monotonicity of Lebesgue integrals, triangle inequality, partial integrations and by neglecting negative terms as follows:

$$\begin{aligned}
|x_t - x_{t_1}| &\le \int_{t_1}^{t} e^{-\beta s}|v_0|\,ds + \beta \int_{t_1}^{t}\int_0^s e^{-\beta(s-u)}|K(x_u)|\,du\,ds \\
&\quad + \beta\left|\int_{t_1}^{t}\int_0^s e^{-\beta s}e^{\beta u}\,dX_u\,ds\right| \\
&= \int_{t_1}^{t} e^{-\beta s}|v_0|\,ds - e^{-\beta t}\int_{t_1}^{t} e^{\beta u}|K(x_u)|\,du + e^{-\beta t_1}\int_0^{t_1} e^{-\beta u}|K(x_u)|\,du \\
&\quad + \int_{t_1}^{t}|K(x_s)|\,ds + \left|-e^{-\beta t}\int_0^t e^{\beta u}\,dX_u\right| \\
&\quad + \left|e^{-\beta t_1}\int_0^{t_1} e^{-\beta u}\,dX_u\right| + \left|\int_{t_1}^{t} dX_s\,ds\right| \\
&\le \int_{t_1}^{t} e^{-\beta s}|v_0|\,ds + e^{-\beta t_1}\int_0^{t_1} e^{\beta u}|K(x_u)|\,du + \int_{t_1}^{t}|K(x_s)|\,ds \\
&\quad + \left|-e^{-\beta t}\int_0^t e^{\beta u}\,dX_u\right| + \left|e^{-\beta t_1}\int_0^{t_1} e^{\beta u}\,dX_u\right| + |(X_t - X_{t_1})|.
\end{aligned}$$

Due to the Lipschitz continuity of K with constant κ, taking suprema on both sides of the inequality and observing that $\int_0^s e^{-\beta(s-u)}\,du \le 1$ reveals

$$\begin{aligned}
\sup_{t_1\le t\le t_2}|x_t - x_{t_1}| \le |v_0| &+ e^{-\beta t_1}\kappa \sup_{t_1\le u\le t_2}|x_u - x_{t_1}|\int_0^{t_1} e^{\beta u}\,du \\
&+ e^{-\beta t_1}\int_0^{t_1} e^{\beta u}|K(x_{t_1})|\,du + (t - t_1)\kappa \sup_{t_1\le s\le t_2}|x_s - x_{t_1}| \\
&+ (t - t_1)|K(x_{t_1})| + \sup_{t_1\le t\le t_2}\left|e^{-\beta t}\int_0^t e^{\beta u}\,dX_u\right| \\
&+ \left|e^{-\beta t_1}\int_0^{t_1} e^{\beta u}\,dX_u\right| + \sup_{t_1\le u\le t_2}|(X_u - X_{t_1})|.
\end{aligned}$$

Letting $(t - t_1)\kappa \le \frac{1}{2}$ and observing that the second and the third integrals vanish as β tends to infinity, then algebraic calculation yields

$$\begin{aligned}
\sup_{t_1\le t\le t_2}|x_t - x_{t_1}| \le |v_0| + \frac{1}{2}\sup_{t_1\le s\le t_2}|x_s - x_{t_1}| + \frac{1}{2\kappa}|K(x_{t_1})| \\
+ \sup_{t_1\le t\le t_2}\left|e^{-\beta t}\int_0^t e^{\beta u}\,dX_u\right|
\end{aligned}$$

$$+\left|e^{-\beta t_1}\int_0^{t_1} e^{\beta u}\,dX_u\right| + \sup_{t_1\le u\le t_2}|(X_u - X_{t_1})|,$$

$$\frac{1}{2}\sup_{t_1\le t\le t_2}|x_t - x_{t_1}| \le |v_0| + \frac{1}{2\kappa}|K(x_{t_1})| + \sup_{t_1\le t\le t_2}\left|e^{-\beta t}\int_0^{t} e^{\beta u}\,dX_u\right|$$
$$+\left|e^{-\beta t_1}\int_0^{t_1} e^{\beta u}\,dX_u\right| + \sup_{t_1\le u\le t_2}|(X_u - X_{t_1})|.$$

Having a continuous function as an integrand and for β tending to infinity, $\sup_{t_1\le t\le t_2}|e^{-\beta t}\int_0^t e^{\beta u}\,dX_u|$ and $|e^{-\beta t_1}\int_0^{t_1} e^{\beta u}\,dX_u|$ vanish. Hence, we neglect these terms in the sequel and find that

$$\sup_{t_1\le t\le t_2}|x_t - x_{t_1}| \le 2|v_0| + c|K(x_{t_1})| + 2\sup_{t_1\le u\le t_2}|X_u - X_{t_1}|, \tag{3.8}$$

where $c = \frac{1}{\kappa} > 0$. The term $K(x_{t_1})$ can be written by introducing a partition of the time interval $[0, t_1]$ as

$$K(x_{t_1}) = |K(x_{t_1}) - K(x_{t'_n})| + K(x_{t'_n}) - \cdots + K(x_0),$$

which is finite with $(t'_n - t'_{n-1})\kappa \le \frac{1}{2}$. We see that the right-hand side of the inequality (3.8) is bounded almost surely. If $(t_2 - t_1)\kappa > \frac{1}{2}$, we introduce a finite partition $t_1 = \tau_1 < \cdots < \tau_n < \tau_{n+1} = t_2$ of the time interval $[t_1, t_2]$ such that $(\tau_{p+1} - \tau_p)\kappa \le \frac{1}{2}$, $1 \le p \le n$, and iterate the above. Let

$$\sup_{t_1\le t\le \tau_n}|x_t - x_{\tau_n}|.$$

For $n = 1$ we have seen that (3.8) holds. We assume that $\sup_{t_1\le t\le\tau_n}|x_t - x_{\tau_n}|$ is bounded, then we use the supremum property to show that the supremum of the increment is bounded for $t_1 \le t \le \tau_{p+1}$, $\tau_{p+1} \le t_2$, namely,

$$\sup_{t_1\le t\le \tau_{p+1}}|x_t - x_{t_1}| \le \sup_{t_1\le t\le \tau_p}|x_t - x_{t_1}| + \sup_{\tau_p\le t\le \tau_{p+1}}|x_t - x_{\tau_p}|,$$

where the first term of the right-hand side is bounded by assumption and the second term of the right-hand side is bounded by an analogous argument to the one given earlier. Inserting (3.8) into (3.7), we obtain

$$\int_{t_1}^{t_2} e^{-\beta(t_2-u)}|K(x_u) - K(x_{t_1})|\,du$$
$$\le \kappa\Big[2|v_0| + c|K(x_{t_1})| + 2\sup_{t_1\le u\le t_2}|X_u - X_{t_1}|\Big]\Big[\frac{1}{\beta}(1 - e^{-\beta t_2})\Big].$$

Then, the integral $\int_0^{t_2} e^{-\beta(t_2-u)}|K(x_u) - K(x_0)|\,du$ vanishes when β tends to infinity. Moreover, the second integral in I_2 can be estimated in the same manner as above. This means that the increments related to the position process, that is, the terms independent of the drift K, are the sum of the increments of the originally driving α-stable Lévy process

$$X_{t_2} - X_{t_1},$$

and three terms which are uniformly bounded by $e^{-\Delta t \beta}$ for all $t_1, t_2 \in T$, T a compact subset of $[0, \infty)$, and which converge to zero as β tends to infinity. Finally, the remaining, nonvanishing part of (3.5) is the integral $\int_{t_1}^{t_2} K(x_s)\, ds$ as proposed in the limit of the theorem above. □

Interesting applications of the Nelson-type scaling limit for α-stable Lévy processes are to study Lévy processes on manifolds. A generalization of Nelson's result on Brownian motion to Banach spaces and Riemannian manifolds is proven in Dowell (1980).

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Linnæus University
School of Computer Science, Physics and Mathematics
SE-351 95 Växjö
Sweden
E-mail: haidar.al-talibi@lnu.se