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Finite exclusion process and independent random walks

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Abstract. We show that the total variational distance between a process of two particles interacting by exclusion and a process of two independent particles goes to 0 as time goes to infinity, when the underlying one particle system is a symmetric random walk on $\mathbb{Z}^d$ with finite second moments. Upper bounds for the speed of convergence are given.

1 Introduction

The exclusion process with infinitely many particles has been extensively studied [see Liggett (1985, 1999) and its references]. Less attention has been given to exclusion systems with finitely many particles. Nevertheless, these finite systems are a source of interesting problems and, in some cases, a better understanding of them allows us to prove results concerning infinite systems. In this paper we compare a system of two particles interacting by exclusion to a system composed by two independent random walks on $\mathbb{Z}^d$. This comparison is given in terms of an upper bound for the difference in total variation between the two systems.

Given a translation invariant transition matrix $p(x, y)$ on $\mathbb{Z}^d$, we define the two-particle exclusion process as follows: two particles are initially located in distinct points of $\mathbb{Z}^d$, each of these attempts to perform a continuous time random walk with exponential holding times of parameter one and jumps governed by $p(x, y)$. These random walks are independent except for the following rule: each time a particle chooses to jump to the point occupied by the other particle, the jump is suppressed and the particle waits another exponential time before attempting a new jump. We denote this process by $X(t)$ and we denote by $Y(t)$ the process composed by two independent random walks with exponential holding times of parameter one and jumps governed by $p(x, y)$. Since we do not distinguish the particles from each other, the state space of $Y(t)$ is

$$S = \frac{\mathbb{Z}^d \times \mathbb{Z}^d}{\sim},$$

where $\sim$ is the equivalence relation that identifies (x, y) to (y, x). Similarly, the state space of $X(t)$ is

$$\overline{S} = \frac{\mathbb{Z}^d \times \mathbb{Z}^d \setminus \{(x, x) : x \in \mathbb{Z}^d\}}{\sim}.$$

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In the sequel $\|X^{a,b}(t) - Y^{a,b}(t)\|$ denotes the difference in total variation between $X(t)$ and $Y(t)$, when both processes started from some $(a,b) \in \overline{S}$, that is,

$$\|X^{a,b}(t) - Y^{a,b}(t)\| = \sum_{(u,v)\in\overline{S}} \big|P^{a,b}\big(X(t) = (u,v)\big) - P^{a,b}\big(Y(t) = (u,v)\big)\big| + \sum_{u\in\mathbb{Z}^d} P^{a,b}\big(Y(t) = (u,u)\big).$$

The following two theorems give upper bounds for this expression.

Theorem 1.1. *If $p(x,y)$ is a symmetric, translation invariant transition matrix on $\mathbb{Z}^d$ and $\sum_{x\in\mathbb{Z}^d} \|x\|^2 p(0,x) < \infty$, then there exists a constant C such that $\forall (a,b) \in \overline{S}$ and $t \geq 2$,*

$$\|X^{a,b}(t) - Y^{a,b}(t)\| \leq \begin{cases} C\dfrac{\ln t}{\sqrt{t}} & \text{if } d = 1, \\ C\dfrac{\ln t}{t} & \text{if } d = 2, \\ \dfrac{C}{t} & \text{if } d \geq 3. \end{cases}$$

Theorem 1.2. *If $d = 1$ and $p(x,x+1) = p(x,x-1) = 1/2$, then there exists a constant C such that $\forall (a,b) \in \overline{S}$, $t > 0$,*

$$\|X^{a,b}(t) - Y^{a,b}(t)\| \leq \frac{C}{\sqrt{t}}.$$

The fact that $\lim_t \|X^{a,b}(t) - Y^{a,b}(t)\| = 0$ was already known in the one-dimensional cases treated by Theorem 1.1: De Masi and Presutti (1983) derived it from the main results of their paper where an upper bound of the order of $t^{-1/4+\varepsilon}$ can be obtained. For a different comparison between the two processes we refer the reader to Ferrari et al. (1991).

The proofs of Theorems 1.1 and 1.2 are quite different. They are given in Sections 2 and 3, respectively. Although the proof of Theorem 1.2 is quite technical, we include it because it shows that the bound of Theorem 1.1 is not always optimal and because the better bound is needed in Konno (1995). In Section 4 we make some remarks concerning the hypotheses of these theorems and state some problems we have been unable to solve.

2 Proof of Theorem 1.1

Let $p(x,y)$ be as in the statement of Theorem 1.1 and denote by $U(t)$ and $V(t)$ the semigroups associated to $Y(t)$ and $X(t)$, respectively. These semigroups act on the set of bounded real valued functions on S and $\overline{S}$, respectively, in the following way:

$$U(t)f(a,b) = E^{a,b}(f(Y(t)))$$

and

$$V(t)f(a,b) = E^{a,b}(f(X(t))),$$

where the superscripts on the expectation operator denote the initial position of the process. We identify in the obvious way $U(t)$ and $V(t)$ with semigroups acting on the sets of bounded symmetric functions on $\mathbb{Z}^d \times \mathbb{Z}^d$ and on $\mathbb{Z}^d \times \mathbb{Z}^d \setminus \{(x,x) : x \in \mathbb{Z}^d\}$, respectively. The new semigroups are also denoted by $U(t)$ and $V(t)$ and their generators are denoted by U and V, respectively. The transition matrix for the continuous time random walk associated to $p(x,y)$ is denoted by $p_s(x,y)$.

We start now the proof of Theorem 1.1 in the one-dimensional case and at the end of the section we say how to adapt it when $d \geq 2$. In the case $d = 1$, we can restate Theorem 1.1 as follows: $\exists C > 0$ such that $\forall t \geq 2$, we have

$$\sup_{f : \|f\|=1} \|U(t)f - V(t)f\| \leq C\frac{\ln t}{\sqrt{t}},$$

where $\|\cdot\|$ denotes, as in the rest of the section, the sup norm.

To prove this, we use the integration by parts formula:

$$[U(t) - V(t)]f = \int_0^t V(t-s)[U - V]U(s)f\,ds, \tag{2.1}$$

and give upper bounds to the right-hand side. Proving these bounds is the purpose of the next two lemmas.

Lemma 2.1. *$\forall s > 0$ and $x, y \in \mathbb{Z}$ $x \neq y$, we have*

$$|[U - V]U(s)f(x,y)| \leq p(x,y)\|f\|\Big(\sum_{v\in\mathbb{Z}} |p_s(x,v) - p_s(y,v)|\Big)^2.$$

Proof. First note that

$$(U - V)g(x,y) = p(x,y)[g(y,y) - g(x,y)] + p(y,x)[g(x,x) - g(x,y)],$$

and by the symmetry of p we get

$$(U - V)g(x,y) = p(x,y)[g(x,x) - 2g(x,y) + g(y,y)]. \tag{2.2}$$

Since $U(s)$ is the semigroup associated to two independent random walks, we have

$$U(s)f(x,y) = \sum_{u,v\in\mathbb{Z}} p_s(x,u)p_s(y,v)f(u,v).$$

Hence, applying (2.2) to $g = U(s)f$, we obtain

$$
\begin{aligned}
&(U - V)U(s)f(x,y)\\
&\quad = p(x,y) \sum_{u,v\in\mathbb{Z}} [p_s(x,u)p_s(x,v) - 2p_s(x,u)p_s(y,v)\\
&\qquad\qquad + p_s(y,u)p_s(y,v)]f(u,v)\\
&\quad = p(x,y)\Big[\sum_{u,v} p_s(x,u)\big(p_s(x,v) - p_s(y,v)\big)f(u,v)\\
&\qquad\qquad + \sum_{u,v} p_s(y,v)\big(p_s(y,u) - p_s(x,u)\big)f(u,v)\Big].
\end{aligned}
$$

Using the symmetry of f, this can be written as

$$p(x,y)\sum_{u,v}\big(p_s(x,u) - p_s(y,u)\big)\big(p_s(x,v) - p_s(y,v)\big)f(u,v).$$

Since the absolute value of this expression is bounded above by

$$p(x,y)\|f\|\Big(\sum_u |p_s(x,u) - p_s(y,u)|\Big)^2,$$

the lemma is proved. □

The following inequality is standard and we omit its proof: for some constant C_1 (depending on p) we have

$$\sum_{v\in\mathbb{Z}} |p_s(y,v) - p_s(x,v)| \le C_1\frac{|x-y|}{\sqrt{s}} \qquad \forall s > 0, x, y \in \mathbb{Z}.$$

Hence, by Lemma 2.1 we get

$$|(U - V)U(s)f(x,y)| \le C_1^2 p(x,y)\|f\|\frac{|x-y|^2}{s}. \tag{2.3}$$

Lemma 2.2. *There exists a constant C_2 such that*

$$\|V(t-s)(U-V)U(s)f\| \le \frac{C_2\|f\|}{s\sqrt{t-s}} \qquad \forall t > s > 0.$$

Proof. Since V is a positive operator, it follows from (2.3) that

$$|V(t-s)(U-V)U(s)f(x,y)| \le \frac{C_1^2\|f\|}{s}V(t-s)h(x,y), \tag{2.4}$$

where $h(x,y) = p(x,y)|x-y|^2$. To find an upper bound for $V(t-s)h$, first define

$$g(x,y) = \begin{cases} h(x,y) & \text{if } x \ne y,\\ \sum_z p(x,z)|z-x|^2 & \text{if } x = y.\end{cases}$$

Since $h \le g$ and V is positive, we have

$$V(t-s)h(x,y) \le V(t-s)g(x,y).$$

Then noting that g is a positive definite symmetric function, we may use Proposition VIII.1.7 in Liggett (1985) to conclude that

$$V(t-s)h(x,y) \le U(t-s)g(x,y). \tag{2.5}$$

For $r > 0$ we compute

$$\begin{aligned} U(r)g(x,y) &= \sum_{u,v\in\mathbb{Z}} p_r(x,u)p_r(y,v)g(u,v) \\ &= \sum_{\substack{u,v\in\mathbb{Z}\\ u\neq v}} p_r(x,u)p(y,v)g(u,v) + \sum_{u\in\mathbb{Z}} p_r(x,u)p_r(y,u)g(u,u). \end{aligned}$$

Noting that $g(u,u)$ is constant ($=\sum_z p(0,z)z^2$) and that $p_r(\cdot)$ is symmetric, we can write the second sum of the right-hand side above as

$$p_{2r}(x,y)g(0,0).$$

Since the first sum is equal to

$$\begin{aligned} &\sum_{\substack{u,v\in\mathbb{Z}\\ u\neq v}} p_r(x,u)p_r(y,v)p(u,v)|u-v|^2 \\ &\quad = \sum_{\substack{u,v\in\mathbb{Z}\\ u\neq v}} p_r(x,u)p_r(u,y-v+u)p(0,u-v)|u-v|^2, \end{aligned}$$

where the equality follows from the translation invariance and symmetry of p_r and p, we obtain

$$\begin{aligned} U(r)g(x,y) &= p_{2r}(x,y)g(0,0) + \sum_{u\in\mathbb{Z}}\sum_{w\neq 0} p_r(x,u)p_r(u,y+w)p(0,w)|w|^2 \\ &= p_{2r}(x,y)g(0,0) + \sum_{w\neq 0} p_{2r}(x,y+w)p(0,w)|w|^2. \end{aligned}$$

Using the fact that

$$\sup_{t>0}\Big[\sup_y p_t(x,y)\Big]\sqrt{t} < \infty, \tag{$*$}$$

[see Proposition 6 in page 72 of Spitzer (1976)], we get that for some constant K and all $t > 0$, we have

$$U(2r)g(x,y) \le \frac{K}{\sqrt{2r}} 2g(0,0). \tag{2.6}$$

Now, the lemma follows from (2.4), (2.5) and (2.6). □

To complete the proof of Theorem 1.1, we assume $t \geq 2$ and write the right-hand side of (2.1) as a sum of three integrals on the intervals $[0, 1]$, $[1, t-1]$ and $[t-1, t]$. Call these integrals I_1, I_2 and I_3, respectively. Then use (2.2) to conclude that

$$|(U - V)U(s)f(x, y)| \leq 4p(x, y)\|f\|.$$

Let h be as in the proof of Lemma 2.2, then

$$p(x, y) \leq h(x, y) \qquad \forall x \neq y,$$

hence,

$$V(t-s)p(x, y) \leq V(t-s)h(x, y).$$

Hence, by (2.5) and (2.6) we also have

$$V(t-s)p(x, y) \leq \frac{K}{\sqrt{2(t-s)}} 2g(0, 0).$$

Therefore,

$$|I_1| = \left| \int_0^1 V(t-s)(U - V)U(s)f(x, y)\, ds \right| \leq \frac{8K}{\sqrt{2}} \|f\| g(0, 0) \int_0^1 \frac{1}{\sqrt{t-s}}\, ds.$$

Hence, for some $L > 0$ and all $t \geq 2$, we have

$$|I_1| \leq \frac{L}{\sqrt{t}} \|f\|. \tag{2.7}$$

From (2.3) we get for $t \geq 2$

$$|I_3| \leq C_1^2 \|f\| \int_{t-1}^{t} V(t-s)h(x, y) \frac{1}{s}\, ds \leq \frac{C_1^2 \|f\|}{t-1} \|h\| \leq \frac{C_3 \|f\|}{t} \tag{2.8}$$

for some constant C_3.

And using Lemma 2.2, we get

$$\begin{aligned} |I_2| &\leq C_2 \|f\| \int_1^{t-1} \frac{1}{s\sqrt{t-s}}\, ds \leq C_2 \|f\| \sqrt{t} \int_1^{t-1} \frac{1}{s(t-s)}\, ds \\ &= C_2 \|f\| \sqrt{t} \left(2 \frac{\ln(t-1)}{t} \right) \leq C_4 \|f\| \frac{\ln t}{\sqrt{t}} \end{aligned} \tag{2.9}$$

for some constant C_4 and all $t \geq 2$.

The conclusion of Theorem 1.1 in the case $d = 1$ follows from (2.7), (2.8) and (2.9).

If $d \geq 2$, we can improve $(*)$ by replacing it by $\sup_{t>0}(\sup_y p_t(x, y)) t^{d/2} < \infty$. Proceeding then as in the one-dimensional case, we get

$$|I_1| \leq \frac{L}{t^{d/2}} \|f\|,$$

$$|I_3| \leq \frac{C_3}{t} \|f\|$$

and

$$|I_2| \le \begin{cases} C_4\|f\|\dfrac{\ln t}{t} & \text{if } d = 2, \\ \dfrac{C_4\|f\|}{t} & \text{if } d \ge 3, \end{cases}$$

thus completing the proof of Theorem 1.1.

3 Proof of Theorem 1.2

Throughout this section $d = 1$, and $p(x, x+1) = p(x, x-1) = 1/2$. To prove Theorem 1.2, we find it convenient to identify in the obvious way $\frac{\mathbb{Z}\times\mathbb{Z}\setminus\{(x,x):x\in\mathbb{Z}\}}{\sim}$ to

$$E = \{(x, y) \in \mathbb{Z}^2 : x > y\},$$

and consider $X(t)$ as a process on the state space E. It will also be convenient to consider $Y(t)$ as a process on $\mathbb{Z} \times \mathbb{Z}$. To prove Theorem 1.2, it now suffices to show that for some constant K the following two inequalities hold for all $a > b$ $a, b \in \mathbb{Z}$ and $t > 0$:

$$\sum_{u>v} \left| P\big(X^{a,b}(t) = (u, v)\big) - P\big(Y^{a,b}(t) = (u, v)\big) - P\big(Y^{a,b}(t) = (v, u)\big) \right| \le \frac{K}{\sqrt{t}}, \tag{3.1}$$

$$\sum_{u\in Z} P\big(Y^{a,b}(t) = (u, u)\big) \le \frac{K}{\sqrt{t}}.$$

Since the second of these inequalities follows from standard estimates, the rest of this section is dedicated to the proof of (3.1). To do this, we start introducing some notation: let $V_1, \dots, V_n, \dots$ and $W_1, \dots, W_n, \dots$ be independent random variables such that

$$P(V_i = -1) = P(V_i = 1) = \frac{1}{4}, \qquad P(V_i = 0) = \frac{1}{2}$$

and

$$P(W_i = 0) = P(W_i = 1) = \frac{1}{2} \qquad \forall i.$$

Let

$$S_n = \sum_{i=1}^{n} V_i \quad \text{and} \quad S'_n = \sum_{1=1}^{n} W_i.$$

Noting that $W_1 + W_2 - 1$ has the same distribution as V_1, we see that S_n has the same distribution as $S'_{2n} - n$. We start with an elementary lemma whose proof we include for the sake of completeness.

Lemma 3.1. *There exists a constant C such that*

$$\sum_{a\in\mathbb{Z}} |P(S_n = a) - P(S_{n+k} = a)| \le C\frac{k}{n} \qquad \forall n, k \in \mathbb{N}.$$

Proof. The result follows by induction once it is proved for $k = 1$. First observe that

$$\sum_{a\in\mathbb{Z}} |P(S_n = a) - P(S_{n+1} = a)|$$
$$= \sum_{a\in\mathbb{Z}} \left| P(S_n = a) - \left(\frac{1}{2}P(S_n = a) + \frac{1}{4}P(S_n = a-1) + \frac{1}{4}P(S_n = a+1)\right)\right|$$
$$= \sum_{a\in\mathbb{Z}} \frac{1}{2}\left| P(S_n = a) - \frac{1}{2}\big(P(S_n = a+1) + P(S_n = a-1)\big)\right|.$$

In view of our remark on the distribution of S_n and S'_{2n}, this is equal to

$$\frac{1}{2}\sum_{a\in\mathbb{Z}} \left| P(S'_{2n} = a+n) - \frac{1}{2}\big(P(S'_{2n} = a+1+n) + P(S'_{2n} = a-1+n)\big)\right|$$
$$= \frac{1}{2}\sum_{a\in\mathbb{Z}} \left| P(S'_{2n} = a) - \frac{1}{2}\big(P(S'_{2n} = a+1) + P(S'_{2n} = a-1)\big)\right|.$$

A straightforward calculation shows that

$$\begin{aligned} &P(S'_{2n} = a) - \frac{1}{2}\big(P(S'_{2n} = a+1) + P(S'_{2n} = a-1)\big) \\ &\quad = \frac{1}{2^{2n}}\binom{2n}{a}\left[\frac{-2(a-n)^2 + n + 1}{(a+1)(2n-a+1)}\right], \qquad a = 0, \dots, 2n, \end{aligned} \tag{3.2}$$

hence, these terms are positive if and only if $|a - n| < \sqrt{\frac{n+1}{2}}$.

Since the sum over all $a \in \mathbb{Z}$ of the left-hand side of (3.2) is 0, we have

$$\sum_{a\in\mathbb{Z}} \left| P(S'_{2n} = a) - \frac{1}{2}\big(P(S'_{2n} = a+1) + P(S'_{2n} = a-1)\big)\right|$$
$$= 2 \sum_{a:|a-n|>\sqrt{(n+1)/2}} \left| P(S'_{2n} = a) - \frac{1}{2}\big(P(S'_{2n} = a+1) + P(S'_{2n} = a-1)\big)\right|$$

and, since the terms we add are symmetric around $a = n$, we obtain

$$\begin{aligned} &\sum_{a\in\mathbb{Z}} |P(S_n = a) - P(S_{n+1} = a)| \\ &\quad = 2 \sum_{a>n+\sqrt{(n+1)/2}} -P(S'_{2n} = a) \end{aligned} \tag{3.3}$$

$$+\frac{1}{2}\big(P(S'_{2n}=a+1)+P(S'_{2n}=a-1)\big)$$
$$=P(S'_{2n}=k_0-1)-P(S'_{2n}=k_0),$$

where $k_0=\inf\{a\in\mathbb{N}:a>n+\sqrt{\frac{n+1}{2}}\}$.

The right-hand side of (3.3) is equal to

$$\frac{1}{2^{2n}}\left[\binom{2n}{k_0-1}-\binom{2n}{k_0}\right]=\frac{1}{2^{2n}}\binom{2n}{k_0}\left(\frac{k_0}{2n-k_0+1}-1\right),$$

which is bounded above by

$$\frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{2(k_0-n)-1}{2n-k_0+1}\right)\le\frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{2(\sqrt{(n+1)/2}+1)-1}{n-\sqrt{(n+1)/2}+1}\right).$$

Since $\frac{1}{2^{2n}}\binom{2n}{n}$ is asymptotic to some constant over $\sqrt{n}$, the lemma follows. □

Our next step is to introduce a Markov process on a subspace of $E\times\mathbb{Z}^2\times\mathbb{Z}$. This process constitutes a coupling of $X(t)$ and $Y(t)$. Let $T:\mathbb{Z}^2\to\mathbb{Z}^2$ be defined by

$$T(y_1,y_2)=(y_2,y_1)$$

and let

$$A=\{(\mathbf{x},\mathbf{y},u)\in E\times\mathbb{Z}^2\times\mathbb{Z}:\mathbf{x}=\mathbf{y}-(u,u)\text{ or }\mathbf{x}=T(\mathbf{y})-(u-1,u)\}.$$

We now consider the continuous time Markov process $(X(t),Y(t),U(t))$ on A whose rates are defined as follows:

If $k\ge 2$, and $x,u\in\mathbb{Z}$, then from $(x+k,x,x+u+k,x+u,u)$ the process jumps at rate $\frac{1}{2}$ to each of the following 4 elements:

$$(x+k+1,x,x+u+k+1,x+u,u),$$
$$(x+k-1,x,x+u+k-1,x+u,u),$$
$$(x+k,x+1,x+u+k,x+u+1,u),$$
$$(x+k,x-1,x+u+k,x+u-1,u),$$

and from $(x+k,x,x+u,x+k+u-1,u)$ it jumps at rate $\frac{1}{2}$ to each of the following 4 elements:

$$(x+k+1,x,x+u,x+k+u,u),$$
$$(x+k-1,x,x+u,x+k+u-2,u),$$
$$(x+k,x+1,x+u+1,x+k+u-1,u),$$
$$(x+k,x-1,x+u-1,x+k+u-1,u).$$

If $x, u \in \mathbb{Z}$, then from $(x+1, x, x+1+u, x+u, u)$ the process jumps at rate $\frac{1}{2}$ to each of the following 4 elements:

$$(x+1, x-1, x+1+u, x-1+u, u),$$
$$(x+2, x, x+2+u, x+u, u),$$
$$(x+1, x, x+u, x+u, u),$$
$$(x+1, x, x+1+u, x+1+u, u+1),$$

and from $(x+1, x, x+u, x+u, u)$ the process jumps at rate $\frac{1}{2}$ to each of the following four elements:

$$(x+2, x, x+u, x+u+1, u),$$
$$(x+1, x-1, x-1+u, x+u, u),$$
$$(x+1, x, x+u+1, x+u, u),$$
$$(x+1, x, x+u, x+u-1, u-1).$$

A tedious but straightforward verification shows that all these jumps land in points of A when their departure point is in A, and that the Markov process obtained satisfies the following three properties:

1. Its projection on the first two coordinates, denoted by $X(t)$, evolves as the two-particle exclusion process,
2. Its projection on the third and fourth coordinates, denoted by $Y(t)$, evolves as the process given by two independent random walks,
3. Its projection on the last three coordinates, denoted by $(Y(t), U(t))$, is Markovian.

In the sequel for $a > b$ $a, b \in \mathbb{Z}$, we denote by $P^{a,b}$ the probability associated to the process $(X(t), Y(t), U(t))$ starting from $(a, b, a, b, 0)$. Since the process evolves on A, we have that $\forall (x_1, x_2) \in E$

$$\begin{aligned} P^{a,b}&\big(X(t) = (x_1, x_2)\big) \\ &= \sum_k P^{a,b}\big(Y(t) = (x_1+k, x_2+k),\ U(t) = k\big) \\ &\quad + \sum_k P^{a,b}\big(Y(t) = (x_2+k, x_1-1+k), U(t) = k\big). \end{aligned}$$

Hence, Theorem 1.2 is a consequence of the following proposition:

Proposition 3.2. *There exists a constant C such that*

$$\sum_{y_2\in\mathbb{Z}} \sum_{y_1>y_2} \Big| \sum_{k\in\mathbb{Z}} P^{a,b}\big(Y(t) = (y_1+k, y_2+k), U(t) = k\big) - P^{a,b}\big(Y(t) = (y_1, y_2)\big) \Big| < \frac{C}{\sqrt{t}} \tag{3.4}$$

and

$$\sum_{y_2\in\mathbb{Z}}\sum_{y_1>y_2}\bigg|\sum_{k\in\mathbb{Z}} P^{a,b}\big(Y(t)=(y_2+k,y_1-1+k),U(t)=k\big) - P^{a,b}\big(Y(t)=(y_2,y_1)\big)\bigg| < \frac{C}{\sqrt{t}} \tag{3.5}$$

for all $a>b$, $a,b\in\mathbb{Z}$ *and all* $t>0$.

Remark. The conclusion also holds when $a\le b$. This case can be treated similarly, but we only need the result when $a<b$.

Notation. In the proof we write P instead of $P^{a,b}$ and denote by $Y_1(t)$ and $Y_2(t)$ the coordinates of $Y(t)$. We also adopt the following conventions: $\sum_{0\le\ell\le n}$ and $\bigcup_{0\le\ell\le n}$ will mean $\sum_{0\le n<\infty}\sum_{0\le\ell\le n}$ and $\bigcup_{0\le n<\infty}\bigcup_{0\le\ell\le n}$, respectively.

Proof of Proposition 3.2. We start proving (3.4). Let $\tau_1<\tau_2<\cdots$ be the instants at which the successive jumps of $Y(t)$ occur and let $N(t)$ be the number of such jumps up to time t. Define

$$\begin{aligned}A(t)&=\#\{i:\tau_i\le t, Y_1(\tau_i-)-Y_2(\tau_i-)=1 \text{ and } Y_1(\tau_i)-Y_2(\tau_i)=0\}\\&\quad+\#\{i:\tau_i\le t, Y_1(\tau_i-)-Y_2(\tau_i-)=0 \text{ and } Y_1(\tau_i)-Y_2(\tau_i)=1\},\end{aligned}$$

that is, $A(t)$ counts the number of jumps from $\{(x,y):x-y=1\}$ to $\{(x,y):x-y=0\}$ and back. For $\ell,n,m\in\mathbb{Z}_+$ and $t\ge 0$ let

$$B_{n,\ell,m}(t)=\{N(t)=2n+|m-(a-b)|, A(t)=2\ell, Y_1(t)-Y_2(t)=m\}.$$

Consider for $y_1>y_2$,

$$P\big(Y(t)=(y_1+k,y_2+k),U(t)=k|A(t),N(t),Y(t)\big),$$

since this conditional probability is zero off the set $\bigcup_{0\le\ell\le n}B_{n,\ell,y_1-y_2}(t)$, we have

$$\begin{aligned}P\big(Y(t)&=(y_1+k,y_2+k),U(t)=k\big)\\&=\sum_{0\le\ell\le n}P\big(Y(t)=(y_1+k,y_2+k),U(t)=k|B_{n,\ell,y_1-y_2}(t)\big)\\&\qquad\times P(B_{n,\ell,y_1-y_2}(t))\\&=\sum_{0\le\ell\le n}P\big(Y_2(t)=y_2+k,U(t)=k|B_{n,\ell,y_1-y_2}(t)\big)P(B_{n,\ell,y_1-y_2}(t)).\end{aligned}\tag{3.6}$$

To deal with this expression, we need the following:

Lemma 3.3. *Conditioned on* $B_{n,\ell,y_1-y_2}(t)$, *the random variables* $Y_2(t)-U(t)-b$ *and* $U(t)$ *are independent and are distributed as* $S_{n-\ell}+\operatorname{sign}[a-b-(y_1-y_2)]S'_{|y_1-y_2-(a-b)|}$ *and* S_ℓ, *respectively.*

Proof. Let $Z(t) = Y_2(t) - Y_1(t)$ and denote by $Z_1, Z_2, \dots$ and $Y_{2,1}, Y_{2,2}, \dots$ the successive increments of the processes $Z(t)$ and $Y_2(t)$, respectively. We start looking at the conditional joint distribution of $Y_2(t) - U(t) - b$ and $U(t)$ given events of the form

$$A(n, a_1, \dots, a_{2n+|m-(a-b)|}) = \{N(t) = 2n + |m - (a - b)|, Z_i = a_i, 0 < i \le 2n + |m - (a - b)|\},$$

where $a_1, \dots, a_{2n+|m-(a-b)|}$ ranges over all sequences such that:

(1) $a_i \in \{-1, 1\}$ $i = 1, \dots, 2n + |m - (a - b)|$ and $\sum_{i=1}^{2n+|m-(a-b)|} a_i = m - (a - b)$

(2) the equalities $a - b + \sum_{i=1}^{k-1} a_i = 1$ and $a_k = -1$ hold for exactly ℓ values of $k \in \{1, \dots, 2n + |m - (a - b)|\}$ (by convention $\sum_{i=1}^{0} a_i = 0$)

(3) the equalities $a - b + \sum_{i=1}^{k-1} a_i = 0$ and $a_k = 1$ hold for exactly ℓ values of $k \in \{1, \dots, 2n + |m - (a - b)|\}$ (by convention $\sum_{i=1}^{0} a_i = 0$).

In the sequel $i_1 < i_2 < \cdots < i_\ell$ will denote the values of k satisfying the equalities in (2) and $j_1 < j_2 < \cdots < j_\ell$ will denote the values of k satisfying the equalities in (3). We will prove that the conditional joint distribution of $Y_2(t) - U(t) - b$ and $U(t)$ given any of the events $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$ is the same as in the conclusion of the lemma. Since the event $B_{n,\ell,m}(t)$ is a disjoint union of the events $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$, the lemma will follow. Note that $U(t)$ represents the increment in the time interval $[0, t]$ of $Y_2(t)$ due to jumps of $Y(\cdot)$ from $D_1 = \{(x, y) : x - y = 1\}$ to $D_0 = \{(x, y) : x - y = 0\}$ and back, while $Y_2(t) - U(t) - b$ represents the increment due to all the other jumps. Therefore, on the event $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$ $U(t) = \sum_{r=1}^{\ell}(Y_{2,i_r} + Y_{2,j_r})$ and $Y_2(t) - U(t) - b = \sum_{i\in I} Y_{2,i}$ where $I = \{1, \dots, 2n + |m - (a - b)|\} \setminus \{i_1, \dots, i_\ell, j_1, \dots, j_\ell\}$. Note also that, for any $n \in \mathbb{N}$, after conditioning on $N(t), Z_1, \dots, Z_n$ the random variables $Y_{2,1}, \dots, Y_{2,n}$ are independent and their marginal distributions are given by

$$P(Y_{2,i} = 0 | Z_i = 1) = P(Y_{2,i} = -1 | Z_i = 1) = \frac{1}{2}$$

and

$$P(Y_{2,i} = 0 | Z_i = -1) = P(Y_{2,i} = 1 | Z_i = -1) = \frac{1}{2}.$$

Hence, the random variables $U(t)$ and $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$ are conditionally independent given $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$. It also follows from the above considerations that the conditional distributions of $U(t)$ given $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$ is the same as the distribution of $X_1 + \cdots + X_\ell + Y_1 + \cdots + Y_\ell$ where all the r.v.'s involved are independent, the X's are distributed as W_1 and the Y's as $-W_1$. Hence, given $A(n, a_1, \dots, a_{2n+|m-(a-b)|})$, $U(t)$ is distributed as S'_ℓ. Similarly, the conditional distribution of $Y_2(t) - U(t) - b$ given

$A(n, a_1, \ldots, a_{2n+|m-(a-b)|})$ is equal to the distribution of $S_{n-\ell} + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|}$. □

From this lemma we immediately get the following:

Corollary 3.4. *Conditioned on $B_{n,\ell,y_1-y_2}(t)$, the random variable $Y_2(t) - b$ is distributed as $S_n + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|}$.*

We now return to the proof of Proposition 3.2. By (3.6) and Lemma 3.3 we have

$$\begin{aligned}
&P\big(Y(t) = (y_1 + k, y_2 + k), U(t) = k\big) \\
&\quad = \sum_{0\le\ell\le n} P(S_\ell = k) \\
&\qquad\quad \times P\big(S_{n-\ell} + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|} = y_2 - b\big) \\
&\qquad\quad \times P(B_{n,\ell,y_1-y_2}(t)).
\end{aligned}$$

Summing on k, we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} P\big(Y(t) = (y_1 + k, y_2 + k), U(t) = k\big) \\
&\quad = \sum_{0\le\ell\le n} P\big(S_{n-\ell} + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|} = y_2 - b\big) \\
&\qquad\quad \times P(B_{n,\ell,y_1-y_2}(t)).
\end{aligned}$$

Similarly, from Corollary 3.4 we get

$$\begin{aligned}
&P\big(Y(t) = (y_1, y_2)\big) \\
&\quad = \sum_{0\le\ell\le n} P\big(Y(t) = (y_1, y_2)|B_{n,\ell,y_1-y_2}(t)\big)P(B_{n,\ell,y_1-y_2}(t)) \\
&\quad = \sum_{0\le\ell\le n} P\big(S_n + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|} = y_2 - b\big) \\
&\qquad\quad \times P(B_{n,\ell,y_1-y_2}(t)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{y_2\in\mathbb{Z}} \sum_{y_1>y_2} \Big|\sum_{k\in\mathbb{Z}} \big[P\big(Y(t) = (y_1 + k, y_2 + k), U(t) = k\big)\big] - P\big(Y(t) = (y_1, y_2)\big)\Big| \\
&\quad \le \sum_{y_2\in\mathbb{Z}} \sum_{y_1>y_2} \sum_{0\le\ell\le n} \big|P\big(S_{n-\ell} + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|} \\
&\qquad\qquad\qquad = y_2 - b\big) \\
&\qquad\qquad - P\big(S_n + \text{sign}[a - b - (y_1 - y_2)]S'_{|y_1-y_2-(a-b)|}
\end{aligned}$$

$$= y_2 - b)\big|P(B_{n,\ell,y_1-y_2}(t))$$

$$= \sum_{m\in\mathbb{N}} \sum_{0\le\ell\le n} P(B_{n,\ell,m}(t)) \sum_{k\in\mathbb{Z}} \big|P\big(S_{n-\ell} + \text{sign}[a-b-m]S'_{|m-(a-b)|} = k\big) - P\big(S_n + \text{sign}[a-b-m]S'_{|m-(a-b)|} = k\big)\big|,$$

where the equality is obtained by changing variables as follows:

$$m = y_1 - y_2, \qquad k = y_2 - b.$$

Using the fact that $S_{n-\ell}$ and S_n are independent of $S'_{|m-(a-b)|}$, we get that for fixed n, ℓ and m the above sum on k is bounded by

$$\sum_{k\in\mathbb{Z}} |P(S_{n-\ell} = k) - P(S_n = k)|.$$

Hence, using Lemma 3.1 and adopting the convention $\frac{\ell}{0} \wedge 1 = 1$, we get that for some constant L

$$\sum_{y_2\in\mathbb{Z}} \sum_{y_1>y_2} \Big|\sum_{k\in\mathbb{Z}} [P\big(Y(t) = (y_1+k, y_2+k), U(t) = k\big)] - P\big(Y(t) = (y_1, y_2)\big)\Big| \le L \sum_{m\in\mathbb{N}} \sum_{0\le\ell\le n} \left[\left(\frac{\ell}{n-\ell}\right) \wedge 1\right] P(B_{n,\ell,m}(t)). \tag{3.7}$$

Standard results on large deviations and on the number of visits of random walks to a given point yield the existence of strictly positive constants C_1, C_2, C_3, α_1 and α_2 such that the following inequalities hold $\forall t > 0$:

$$P\left(N(t) \le \frac{t}{2}\right) \le C_1 e^{-\alpha_1 t}, \tag{3.8}$$

$$P\left(|Y_1(t) - Y_2(t) - (a-b)| \ge \frac{N(t)}{4}\Big|N(t)\right) \le C_2 e^{-\alpha_2 N(t)} \tag{3.9}$$

and

$$E(A(t)|N(t)) \le C_3\sqrt{N(t)}. \tag{3.10}$$

This last inequality implies that

$$P\left(A(t) \ge \frac{N(t)}{4}\Big|N(t)\right) \le \frac{4C_3}{\sqrt{N(t)}}. \tag{3.11}$$

Since

$$P\left(\left\{|Y_1(t) - Y_2(t) - (a-b)| \ge \frac{N(t)}{4}\right\} \cup \left\{A(t) \ge \frac{N(t)}{4}\right\} \cup \left\{N(t) \le \frac{t}{2}\right\}\right)$$

$$\le P\left(|Y_1(t) - Y_2(t) - (a-b)| \ge \frac{N(t)}{4}\Big|N(t) > \frac{t}{2}\right)$$

$$+ P\left(A(t) \ge \frac{N(t)}{4}\Big|N(t) > \frac{t}{2}\right) + P\left(N(t) \le \frac{t}{2}\right),$$

we conclude from (3.8), (3.9) and (3.10) that

$$P\Big(\Big\{|Y_1(t)-Y_2(t)-(a-b)|\geq\frac{N(t)}{4}\Big\}\cup\Big\{A(t)\geq\frac{N(t)}{4}\Big\}\cup\Big\{N(t)\leq\frac{t}{2}\Big\}\Big)$$
$$\leq C_1e^{-\alpha_1 t}+\frac{4\sqrt{2}C_2}{\sqrt{t}}+C_3e^{-\alpha_2 t/2},$$

which is bounded above by $\frac{C_4}{\sqrt{t}}$ for some constant C_4. Since the sets $B_{n,\ell,m}(t)$ are disjoint, this implies that the right-hand side of (3.7) is bounded above by

$$\begin{aligned}L\Big[&\frac{C_4}{\sqrt{t}}\\ &+\sum_{m\in\mathbb{N}}\sum_{0\leq\ell\leq n}\Big[\Big(\frac{\ell}{n-\ell}\Big)\wedge 1\Big]\\ &\qquad\times P\Big(B_{n,\ell,m}(t)\cap\Big\{|Y_1(t)-Y_2(t)-(a-b)|<\frac{N(t)}{4}\Big\}\\ &\qquad\cap\Big\{A(t)<\frac{N(t)}{4}\Big\}\cap\Big\{N(t)>\frac{t}{2}\Big\}\Big)\Big].\end{aligned}\tag{3.12}$$

On the intersection of the four sets which appear in the above expression, we have

$$\frac{\ell}{n-\ell}=\frac{A(t)}{N(t)-|Y_1(t)-Y_2(t)-(a-b)|-A(t)}\leq\frac{2A(t)}{N(t)}\leq\frac{4A(t)}{t}.$$

Hence, (3.12) is bounded above by

$$K\Big[\frac{C_4}{\sqrt{t}}+\frac{4E(A(t))}{t}\Big]$$

for some constant K.

Using (3.10) and Jensen's inequality, we obtain a bound of the form $\frac{C_5}{\sqrt{t}}$. This proves the first inequality of the proposition.

The proof of (3.5) is similar. For this reason we only write a sketch of it. As before, assume that $y_1>y_2$ and for $m\leq 0$ define

$$B'_{n,\ell,m}(t)=\{N(t)=2n+|m-(a-b)|,\,A(t)=2\ell+1,\,Y_1(t)-Y_2(t)=m\}.$$

Then

$$\begin{aligned}&P\big(Y(t)=(y_2+k,y_1-1+k),U(t)=k\big)\\ &\quad=\sum_{0\leq\ell\leq n}P\big(Y(t)=(y_2+k,y_1-1+k),U(t)=k|B'_{n,\ell,y_2-y_1+1}(t)\big)\\ &\qquad\qquad\times P(B'_{n,\ell,y_2-y_1+1}(t)).\end{aligned}$$

Conditioned on $B'_{n,\ell,y_2-y_1+1}(t)$, the random variables $Y_2(t)-U(t)-b$ and $U(t)$ are independent and distributed as $S_{n-\ell}+S'_{(a-b-(y_2-y_1+2))}$ and $S_\ell+S'_1$, respectively. Then, proceeding as in the proof of (3.4), we get

$$\sum_{k\in\mathbb{Z}} P\big(Y(t)=(y_2+k, y_1-1+k), U(t)=k\big)$$
$$=\sum_{0\le\ell\le n} P\big(S_{n-\ell}+S'_{(a-b-(y_2-y_1+2))}=y_1-1-b\big)P(B'_{n,\ell,y_2-y_1+1}(t))$$

and

$$P\big(Y(t)=(y_2, y_1-1)\big)$$
$$=\sum_{0\le\ell\le n} P\big(S_n+S'_{(a-b-(y_2-y_1+1))}=y_1-1-b\big)P(B'_{n,\ell,y_2-y_1+1}(t)).$$

Then write

$$\Big|\sum_{k\in\mathbb{Z}}\big[P\big(Y(t)=(y_2+k, y_1-1+k), U(t)=k\big)\big]-P\big(Y(t)=(y_2, y_1)\big)\Big|$$
$$\le\Big|\sum_{k\in\mathbb{Z}} P\big(Y(t)=(y_2+k, y_1-1+k), U(t)=k\big)-P\big(Y(t)=(y_2, y_1-1)\big)\Big|$$
$$+\big|P\big(Y(t)=(y_2, y_1-1)\big)-P\big(Y(t)=(y_2, y_1)\big)\big|.$$

The first term in the right-hand side above is treated as before and the second satisfies

$$\sum_{y_1,y_2\in\mathbb{Z}}\big|P\big(Y(t)=(y_2, y_1-1)\big)-P\big(Y(t)=(y_2, y_1)\big)\big|<\frac{C_6}{\sqrt{t}}$$

for some constant C_6. This proves (3.5). □

4 Related results and open problems

1. Since random walks on $\mathbb{Z}^d$ are either transient or null recurrent, the interaction mechanism in the two-particle exclusion system does not intervene frequently. However, there are cases for which $\|X(t)-Y(t)\|$ does not converge to zero. The simplest example is as follows: take $d=1$ and $p(x, x+1)=1$ and start the processes $X(t)$ and $Y(t)$ from $(0, 1)$. Then letting $N(t)$ and $N'(t)$ be two independent Poisson processes of parameter one, the position of the rightmost particle of $X(t)$ is distributed as $1+N(t)$, while the position of the rightmost particle of $Y(t)$ is distributed as $\max\{1+N(t), N'(t)\}$. Since the total variation distance of these two distributions remains bounded away from 0 as $t\to\infty$, the same happens to $\|X(t)-Y(t)\|$. In this example the underlying random walk is nearest neighbor

and totally asymmetric, but similar results can be proved with some extra work for one-dimensional random walks which are irreducible and admit jumps of size bigger than one.

2. Since the interaction between particles occurs more often when $p(x,y)$ is "more recurrent," it seems that the conclusion of Theorem 1.1 (or even of Theorem 1.2) should hold for any symmetric $p(x,y)$. Unfortunately, when $\sum x^2 p(0,x) = \infty$ our methods are not as powerful.

3. When $d = 1$ and $0 < \sum x^2 p(0,x) < \infty$,

$$\lim_t \sqrt{t} \sum_{u \in \mathbb{Z}} P\big(Y(t) = (u,u)\big) = c \in (0,\infty).$$

Therefore,

$$\liminf_t \sqrt{t} \|X(t) - Y(t)\| > 0.$$

Since under the hypothesis of Theorem 1.2 we proved that

$$\limsup_t \sqrt{t} \|X(t) - Y(t)\| < \infty,$$

it is natural to conjecture that

$$\lim \sqrt{t} \|X(t) - Y(t)\|$$

exists, and hence belongs to $(0,\infty)$. We also expect this to hold in any symmetric one-dimensional case for which

$$0 < \sum x^2 p(0,x) < \infty.$$

4. As shown by N. Konno (private communication), the proof of Theorem 1.1 can be generalized to a system with n particles. Does the conclusion of Theorem 1.2 also hold for n-particle systems?

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