Brazilian Journal of Probability and Statistics
2013, Vol. 27, No. 1, 54–69
DOI: 10.1214/11-BJPS150

On some fundamental aspects of polyominoes on random Voronoi tilings

Leandro P. R. Pimentel

Federal University of Rio de Janeiro

Abstract. Consider a Voronoi tiling of $\mathbb{R}^d$ based on a realization of an inhomogeneous Poisson random set. A Voronoi polyomino is a finite and connected union of Voronoi tiles. In this paper we provide tail bounds for the number of boxes that are intersected by a Voronoi polyomino, and vice-versa. These results will be crucial to analyze self-avoiding paths, greedy polyominoes and first-passage percolation models on Voronoi tilings and on the dual graph, named the Delaunay triangulation [Asymptotics for first-passage times on Delaunay triangulations (2011) Preprint, Greedy Polyominoes and first-passage times on random Voronoi tilings (2012) Preprint].

1 Introduction

To any locally finite subset $\mathcal{N}$ of $\mathbb{R}^d$ one can associate a partition of the plane as follows. To each point $\mathbf{v} \in \mathcal{N}$ corresponds a polygonal region $C_{\mathbf{v}}$, the *Voronoi tile* at $\mathbf{v}$, consisting of the set of points of $\mathbb{R}^d$ which are closer to $\mathbf{v}$ than to any other $\mathbf{v}' \in \mathcal{N}$. Closer is understood here in the euclidean sense, and the partition is not a real one, but the set of points which belong to more than one Voronoi tile has Lebesgue measure 0. From now on, $\mathcal{N}$ is understood to be distributed like a Poisson random set on $\mathbb{R}^d$ with intensity measure μ. We shall always assume that μ is comparable to Lebesgue's measure on $\mathbb{R}^d$, λ_d, in the sense that there exists a positive constant c_μ such that for every Lebesgue-measurable subset A of $\mathbb{R}^d$,

$$c_\mu^{-1}\lambda_d(A) \le \mu(A) \le c_\mu \lambda_d(A). \tag{1.1}$$

Notice that, with probability one, when two Voronoi tiles are connected, they share a $(d-1)$-dimensional face. The collection $\mathcal{V} = \mathcal{V}(\mathcal{N}) := \{C_{\mathbf{v}}\}_{\mathbf{v}\in\mathcal{N}}$ is called the *Voronoi tiling* (or tessellation) of the plane based on $\mathcal{N}$.

The study of Voronoi tilings has a very long history. The terminology is in honour of Voronoi (1908), who used these tilings to study quadratic forms. Our aim is to study some fundamental aspects of a finite and connected union of Voronoi tiles. These objects are called *Voronoi polyominoes* (Figure 1). Polyominoes were first introduced in periodic tilings of the plane. The Voronoi setting is rather different from the periodic one since we are now considering a random environment induced by the underlying Poisson random set.

Key words and phrases. Voronoi tilings, Delaunay triangulations, polyominoes, self-avoiding paths.

Received September 2010; accepted April 2011.

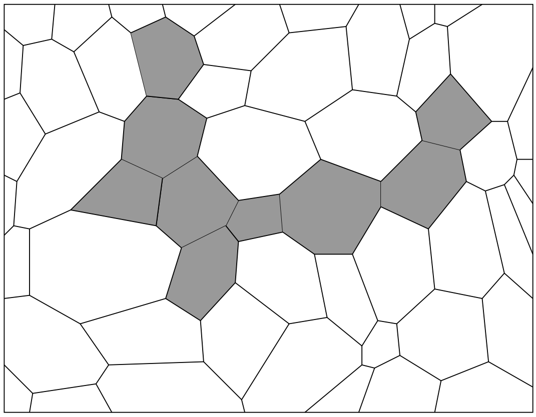

Figure 1 *A two-dimensional Voronoi tiling and a (gray colored) Voronoi polyomino of size* $n = 9$.

The main results of this article, that will be stated in Section 2, will provide tail bounds for the maximum and minimum number of square boxes intersected by a Voronoi polyomino that contains the origin $\mathbf{0}$ and has size r. They will be important tools to analyze self-avoiding paths, greedy polyominoes and first-passage percolation models on the Voronoi random setting (Pimentel, 2011, Pimentel and Rossignol, 2012). The idea to prove them is to combine block arguments with standard results for greedy lattice animal and site percolation models. The block argument is to consider a large box in $\mathbb{R}^d$ so that it contains with "high probability" some configuration of points which prevent a Voronoi tile to cross it completely. The "high probability" alluded to is some percolation probability: we need the "bad boxes" (those who can be crossed) to not percolate. In Section 3 we will prove some technical lemmas concerning greedy lattice animals and site percolation models that will be combined with the block argument to prove the theorems in Section 4. In Section 5 a slightly different random setup is introduced and we will draw an outline of how the results can be extended to this new setup.

2 Polyominoes on Voronoi tilings

Let $\#S$ denote the usual cardinality of a set S, and for each subset A of $\mathbb{R}^d$ let $\#_{\mathcal{N}} A := \#(A \cap \mathcal{N})$. For each natural number $r \geq 1$ a Voronoi *polyomino* $\mathcal{P}$ of size r is a connected union of r Voronoi tiles. Let $\Pi_{\geq r}$ denote the collection of all polyominoes $\mathcal{P}$ such that the origin $\mathbf{0} \in \mathcal{P}$ and $\#_{\mathcal{N}}\mathcal{P} \geq r$, and let $\Pi_{\leq r}$ be the collection of all polyominoes $\mathcal{P}$ such that $\mathbf{0} \in \mathcal{P}$ and $\#_{\mathcal{N}}\mathcal{P} \leq r$. Let $\mathbb{Z}^d$ denote the d-dimensional integer lattice. For each $\mathbf{z} \in \mathbb{Z}^d$ let

$$B_{\mathbf{z}} := \mathbf{z} + [-1/2, 1/2)^d,$$

and for each connected set $C \subseteq \mathbb{R}^d$ let

$$\mathbf{A}(C) := \{\mathbf{z} \in \mathbb{Z}^d : B_{\mathbf{z}} \cap C \neq \varnothing\}.$$

Theorem 1. *There exists a constant* $b_1 \in (0, \infty)$ *such that if* $r \geq b_1 s$, *then*

$$\mathbb{P}\Big(\min_{\mathcal{P}\in\Pi_{\geq r}} \#\mathbf{A}(\mathcal{P}) \leq s\Big) \leq e^{-r/2}. \tag{2.1}$$

Further, there exist constants $b_2, b_3 \in (0, \infty)$ *such that if* $s \geq b_2 r$, *then*

$$\mathbb{P}\Big(\max_{\mathcal{P}\in\Pi_{\leq r}} \#\mathbf{A}(\mathcal{P}) \geq s\Big) \leq e^{-b_3 s}. \tag{2.2}$$

In the same polyomino model one could consider max and min of $\#\mathbf{A}(\mathcal{P})$ over all polyominoes $\mathcal{P}$ of size r touching $B_\mathbf{0}$ ($\mathcal{P} \cap B_\mathbf{0} \neq \varnothing$). The same method to prove Theorem 1 can be extended to this situation, yielding to similar large deviations bounds (2.1) and (2.2).

2.1 Self-avoiding paths on the Delaunay triangulation

An important graph for the study of a Voronoi tiling is its facial dual, the *Delaunay graph* based on $\mathcal{N}$. This graph, denoted by $\mathcal{D} = \mathcal{D}(\mathcal{N})$ is an unoriented graph embedded in $\mathbb{R}^d$ which has vertex set $\mathcal{N}$ and edges $\{\mathbf{u}, \mathbf{v}\}$ every time $C_\mathbf{u}$ and $C_\mathbf{v}$ share a $(d-1)$-dimensional face. We remark that, for our Poisson random set, a.s. no $d+1$ points are on the same hyperplane and no $d+2$ points are on the same hypersphere and that makes the Delaunay graph a well-defined triangulation (Figure 2). This triangulation divides $\mathbb{R}^d$ into bounded simplexes called *Delaunay cells*. For each Delaunay cell Δ no point in $\mathcal{N}$ is inside the circum-hypersphere of Δ. Polyominoes on the Voronoi tiling correspond to connected (in the graph topology) subsets of the Delaunay graph.

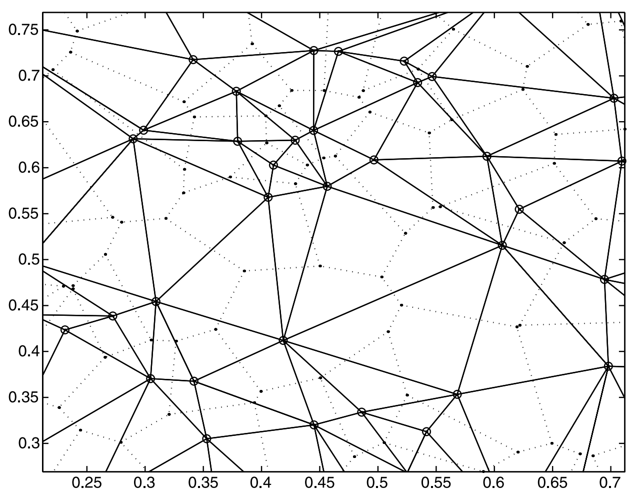

Figure 2 *The Voronoi tiling* $\mathcal{V}$ (*dotted lines*) *and the Delaunay triangulation* $\mathcal{D}$ (*solid lines*) *in the two-dimensional model.*

Let $\mathbf{v_x}$ be the nearest point $\mathbf{v} \in \mathcal{N}$ to $\mathbf{x} \in \mathbb{R}^d$, and let $\Gamma_{\geq r}$ (resp., $\Gamma_{\leq r}$) be the collection of all self-avoiding paths γ starting at $\mathbf{v_0}$ and of size (number of vertices) $\#\gamma \geq r$ (resp., $\#\gamma \leq r$). Recall that polyominoes on the Voronoi tiling correspond to connected (in the graph topology) subsets of the Delaunay graph. Therefore, for each $\gamma \in \Gamma_{\geq r}$ corresponds a unique polyomino $\mathcal{P}_\gamma \in \Pi_{\geq r}$ and, analogously, for each $\gamma \in \Gamma_{\leq r}$ corresponds a unique polyomino $\mathcal{P}_\gamma \in \Pi_{\leq r}$. Let $\mathbf{A}(\gamma) := \mathbf{A}(\mathcal{P}_\gamma)$. Thus, the following corollary is a straightforward consequence of Theorem 1.

Corollary 2. *There exists a constant $b_1 \in (0, \infty)$ such that if $r \geq b_1 s$, then*

$$\mathbb{P}\Big(\min_{\gamma \in \Gamma_{\geq r}} \#\mathbf{A}(\gamma) \leq s\Big) \leq e^{-r/2}. \tag{2.3}$$

Further, there exist constants $b_2, b_3 \in (0, \infty)$ such that if $s \geq b_2 r$, then

$$\mathbb{P}\Big(\max_{\gamma \in \Gamma_{\leq r}} \#\mathbf{A}(\gamma) \geq s\Big) \leq e^{-b_3 s}. \tag{2.4}$$

2.2 The inverse problem

Until now we have been concerned with the size of a lattice covering of a Voronoi polyomino of size r. It is also natural to consider the inverse problem, that is, the number of Voronoi tiles that one needs to cover a connected set composed by s lattice boxes. Precisely, for each connected set $\mathbf{A} \subseteq \mathbb{Z}^d$ (in the l_1-nearest-neighbor sense) let

$$B_{\mathbf{A}} := \bigcup_{\mathbf{z} \in \mathbf{A}} B_{\mathbf{z}}.$$

A connected subset of $\mathbb{Z}^d$ is also called a *lattice animal*. We denote $\Phi_{\leq s}$ the collection of all lattice animals such that $\mathbf{0} \in \mathbf{A}$ and $\#\mathbf{A} \leq s$. A Voronoi covering is defined by taking

$$\mathcal{P}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in C_{\mathbf{v}} \text{ and } C_{\mathbf{v}} \cap B_{\mathbf{A}} \neq \varnothing\}.$$

Theorem 3. *There exist constants $b_7, b_8 \in (0, \infty)$ such that if $r \geq b_7 s$, then*

$$\mathbb{P}\Big(\max_{\mathbf{A} \in \Phi_{\leq s}} \#_{\mathcal{N}} \mathcal{P}(\mathbf{A}) \geq r\Big) \leq 2e^{-b_8 r}. \tag{2.5}$$

One important consequence of Theorem 3 concerns the following construction: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and consider the Polyomino $\mathcal{P}([\mathbf{x}, \mathbf{y}])$ generated by all Voronoi tiles that intersect the line segment $[\mathbf{x}, \mathbf{y}]$. Then one can always find a self-avoiding path $\gamma(\mathbf{x}, \mathbf{y})$ with vertices in $\mathcal{P}([\mathbf{x}, \mathbf{y}])$ and that connects $\mathbf{v_x}$ to $\mathbf{v_y}$. Clearly,

$$\#\gamma(\mathbf{x}, \mathbf{y}) \leq \#_{\mathcal{N}} \mathcal{P}([\mathbf{x}, \mathbf{y}]),$$

and, hence, by Theorem 3, we have the following corollary:

Corollary 4. *There exist constants $b_7, b_8 \in (0,\infty)$ such that if $r \geq b_7 s$, then*

$$\mathbb{P}\Big(\max_{\mathbf{x}:\|\mathbf{x}\|_2\leq s} \#\gamma(\mathbf{0},\mathbf{x}) \geq r\Big) \leq 2e^{-b_8 r}, \tag{2.6}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

3 Technical lemmas

3.1 A greedy lattice animal model with Poisson weights

Let

$$N_{\mathbf{z}} = \#_{\mathcal{N}} B_{\mathbf{z}}.$$

Then $\{N_{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^d\}$ is a collection of independent Poisson random variables. By (1.1),

$$\sup_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}(e^{N_{\mathbf{z}}}) \leq e^{c_\mu(e-1)}. \tag{3.1}$$

It is a standard result in combinatorics[1] that

$$\#\Phi_{\leq s} \leq \alpha^s, \tag{3.2}$$

for a finite $\alpha = \alpha(d)$.

Lemma 5. *If $r \geq 2(\log\alpha + c_\mu(e-1))s$, then*

$$\mathbb{P}\Bigg(\max_{\mathbf{A}\in\Phi_{\leq s}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \geq r\Bigg) \leq e^{-r/2}.$$

Proof. Combining (3.1) and (3.2) together with Markov's inequality, one has that

$$\begin{aligned}
\mathbb{P}\Bigg(\max_{\mathbf{A}\in\Phi_{\leq s}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \geq r\Bigg) &\leq \sum_{\mathbf{A}\in\Phi_{\leq s}} \mathbb{P}\Bigg(\sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \geq r\Bigg)\\
&\leq \alpha^s\Big[\sup_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{E}(e^{N_{\mathbf{z}}})\Big]^s e^{-r}\\
&\leq \exp\{[\log\alpha + c_\mu(e-1)]s - r\} \leq e^{-r/2},
\end{aligned}$$

whenever $r \geq 2(\log\alpha + c_\mu(e-1))s$. □

---

[1] To see this, notice that for each lattice animal $\mathbf{A} \in \Phi_{\leq s}$ one can (injectively) associate an "exploration" nearest neighbor path $(\mathbf{0}, \mathbf{z}_1, \ldots, \mathbf{z}_l)$ such that $\mathbf{z}_i \in \mathbf{A}$ and $\#\{\mathbf{z}_i : \mathbf{z}_i = \mathbf{z}\} \leq 2d$ for each $\mathbf{z} \in \mathbf{A}$. Thus, $l \leq 2ds$ and $\alpha := (2d)^{2d}$ will do.

3.2 Site percolation schemes

Throughout this section $\mathcal{Y} := \{Y_{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^d\}$ will denote an i.i.d. site percolation scheme (or random field) with parameter $\rho \in (0, 1)$. If $Y_{\mathbf{z}} = 1$, we say that $\mathbf{z}$ is open. Otherwise, we say that it is closed.

Lemma 6. *If* $2\alpha\sqrt{1-\rho} < e^{-1}$ *and* $s \geq 2r$, *then*

$$\mathbb{P}\Big(\min_{\mathbf{A}\in\Phi_{\geq s}} \sum_{\mathbf{z}\in\mathbf{A}} Y_{\mathbf{z}} \leq r\Big) \leq e^{-s},$$

where $\Phi_{\geq s}$ *denotes the set of all connected sets* $\mathbf{A} \subseteq \mathbb{Z}^d$ *such that* $\#\mathbf{A} \geq s$ *and* $\mathbf{0} \in \mathbf{A}$.

Proof. Let $\binom{s}{r}$ denote the binomial coefficient. If $\mathbf{A} \in \Phi_{\geq s}$ and $\sum_{\mathbf{z}\in\mathbf{A}} Y_{\mathbf{z}} \leq r$, by taking a connected subset of $\mathbf{A}$ of size s we may assume that $\mathbf{A}$ has size exactly s. Then there exists some subset of exactly $s - r$ sites of $\mathbf{A}$ with $Y_{\mathbf{z}} = 0$. By (3.2), this shows that

$$\begin{aligned}\mathbb{P}\Big(\min_{\mathbf{A}\in\Phi_{\geq s}} \sum_{\mathbf{z}\in\mathbf{A}} Y_{\mathbf{z}} \leq r\Big) &\leq \alpha^s \binom{s}{r} (1-\rho)^{s-r} \\ &\leq \alpha^s 2^s (1-\rho)^{s-r} \\ &\leq \big(2\alpha\sqrt{1-\rho}\big)^s \leq e^{-s},\end{aligned} \tag{3.3}$$

whenever $2\alpha\sqrt{1-\rho} < e^{-1}$ and $s \geq 2r$. □

A closed cluster is a maximal connected set of closed vertices of $\mathbb{Z}^d$. Let $\mathbf{Cl}_{\mathbf{z}}$ denote the closed cluster that contains $\mathbf{z}$ (it is empty if $\mathbf{z}$ is open). For a finite subset $\mathbf{A}$ of $\mathbb{Z}^d$ let $\mathcal{C}_{\mathcal{Y}}(\mathbf{A})$ denote the collection of all closed clusters (with respect to $\mathcal{Y}$) intersecting $\mathbf{A}$.

Lemma 7. *If* f *is an increasing function from* $\mathbb{N}$ *to* $[1, +\infty[$, *then*

$$\mathbb{E}\Big(\prod_{\mathbf{Cl}\in\mathcal{C}_{\mathcal{Y}}(\mathbf{A})} f(\#\mathbf{Cl})\Big) \leq \{\mathbb{E} f(\#\mathbf{Cl}_{\mathbf{0}})\}^{\#\mathbf{A}}.$$

Proof. The proof of Lemma 7 is due to Raphaël Rossignol and the author is grateful for his help. This inequality is also a crucial tool for proving the results in Pimentel and Rossignol (2012). Let us recall Reimer's inequality (see Grimmett's book (Grimmett, 1999), page 39). Let n be a positive integer, let $\mathbf{B}(n) = \mathbb{Z}^d \cap [-n, n]^d$ and define $\Omega_n = \{0, 1\}^{\mathbf{B}(n)}$. For $\omega \in \Omega_n$ and $\mathbf{K} \subset \mathbf{B}(n)$, define the cylinder event $C(\omega, \mathbf{K})$ generated by ω on $\mathbf{K}$ by

$$C(\omega, \mathbf{K}) = \{\omega' \in \Omega_n \text{ s.t. } \omega'_i = \omega_i \ \forall i \in \mathbf{K}\}.$$

If A and B are two subsets of Ω_n, define their disjoint intersection $A \square B$ as follows:

$$A \square B = \{\omega \in \Omega_n : \exists \mathbf{K} \subset \mathbf{B}(n), C(\omega, \mathbf{K}) \subset A \text{ and } C(\omega, \mathbf{K}^c) \subset B\}.$$

Reimer's inequality states that

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A)\mathbb{P}(B).$$

Remark that $\square$ is a commutative and associative operation, and that, for any l subsets $A_1, \ldots, A_l$ of Ω_n,

$$A_1 \square \cdots \square A_l = \Big\{\omega \in \Omega_n : \exists \mathbf{K}_1, \ldots, \mathbf{K}_l \text{ disjoint subsets of } \mathbf{B}(n), \\ \bigcup_{i=1}^{l} \mathbf{K}_i = \mathbf{B}(n) \text{ and } C(\omega, \mathbf{K}_i) \subset A_i \ \forall i = 1, \ldots, l\Big\}.$$

Now take n large enough so that $\mathbf{A} \subset \mathbf{B}(n)$. Let $l = \#\mathbf{A}$, and order the elements of $\mathbf{A} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l\}$. For any $\mathbf{x}$ in $\mathbf{B}(n)$, and any $\omega \in \Omega_n$, let $\mathbf{Cl}_n(\mathbf{x}, \omega)$ be the closed cluster (for the configuration ω) in $\mathbf{B}(n)$ containing $\mathbf{x}$, which is empty if $\omega(\mathbf{x}) = 1$. Define

$$\forall i, \qquad \mathbf{Cl}_i(\omega) = \begin{cases} \mathbf{Cl}_n(\mathbf{x}_i, \omega) & \text{if } \mathbf{x}_i \notin \bigcup_{k=1}^{i-1} \mathbf{Cl}_n(\mathbf{x}_k, \omega), \\ \varnothing & \text{else.} \end{cases}$$

Let $k_1, \ldots, k_l$ be non-negative integers and

$$A_i = \{\omega : \#\mathbf{Cl}_i(\omega) \geq k_i\} \qquad \forall 1 \leq i \leq l,$$
$$\widetilde{A}_i = \{\omega : \#\mathbf{Cl}_n(\mathbf{x}_i, \omega) \geq k_i\} \qquad \forall 1 \leq i \leq l.$$

Then, we claim that

$$\bigcap_{i=1}^{l} A_i \subset \widetilde{A}_1 \square \cdots \square \widetilde{A}_l.$$

Indeed, let $\omega \in \bigcap_{i=1}^{l} A_i$. Then, for every $i = 1, \ldots, l-1$, $C(\omega, \mathbf{Cl}_i(\omega)) \subset A_i$. Furthermore, $C(\omega, \mathbf{B}(n) \setminus \bigcup_{i=1}^{l-1} \mathbf{Cl}_i(\omega)) \subset A_l$. This shows that $\omega \in \widetilde{A}_1 \square \cdots \square \widetilde{A}_l$, and proves the claim above. Therefore, using Reimer's inequality,

$$\mathbb{P}\left(\bigcap_{i=1}^{l} A_i\right) \leq \prod_{i=1}^{l} \mathbb{P}(\widetilde{A}_i). \tag{3.4}$$

Now, let f be an increasing function from $\mathbb{N}$ to $[1, \infty)$. Define f_1 as follows:

$$f_1(0) = 1 \quad \text{and} \quad \forall k \geq 1, \qquad f_1(k) = f(k).$$

Denote by $\{\beta_0 = 1 < \beta_1 < \cdots < \beta_j < \cdots\}$ the range of f_1. Define

$$k_j = \inf\{k : f(k) \geq \beta_j\} \qquad \forall j \in \mathbb{N},$$

$$A_{i,j} = \{\omega : \#\mathbf{Cl}_i(\omega) \geq k_j\} = \{\omega : f_1(\#\mathbf{Cl}_i(\omega)) \geq \beta_j\},$$

and

$$\widetilde{A}_{i,j} = \{\omega : \#\mathbf{Cl}_n(\mathbf{x}_i, \omega) \geq k_j\} = \{\omega : f_1(\#\mathbf{Cl}_n(\mathbf{x}_i, \omega)) \geq \beta_j\}.$$

By convention, set $\beta_{-1} = 0$ and define $a_j = \beta_j - \beta_{j-1}$. We can write

$$f_1(\#\mathbf{Cl}_i(\omega)) = \sum_{j \in \mathbb{N}} (\beta_j - \beta_{j-1}) \mathbb{1}_{A_{i,j}} = \sum_{j \in \mathbb{N}} a_j \mathbb{1}_{A_{i,j}}.$$

Define $\mathcal{C}_n(\mathbf{A})$ as the set of nonempty components $\mathbf{Cl}_n(\mathbf{x}_i, \omega)$, for $i \in \{1, \ldots, l\}$. Since $f_1(0) = 1$, we can write

$$\begin{aligned}
\mathbb{E}\Bigg(\prod_{\mathbf{Cl} \in \mathcal{C}_n(\mathbf{A})} f(\#\mathbf{Cl})\Bigg) &= \mathbb{E}\Bigg(\prod_{\mathbf{Cl} \in \mathcal{C}_n(\mathbf{A})} f_1(\#\mathbf{Cl})\Bigg) \\
&= \mathbb{E}\Bigg(\prod_{i=1}^{l} f_1(\#\mathbf{Cl}_i(\omega))\Bigg) \\
&= \mathbb{E}\Bigg(\prod_{i=1}^{l} \sum_{j \in \mathbb{N}} a_j \mathbb{1}_{A_{i,j}}\Bigg) \\
&= \sum_{j_1, \ldots, j_l} a_{j_1} \cdots a_{j_l} \mathbb{E}\Bigg(\prod_{i=1}^{l} \mathbb{1}_{A_{i,j_i}}\Bigg) \\
&\leq \sum_{j_1, \ldots, j_l} a_{j_1} \cdots a_{j_l} \prod_{i=1}^{l} \mathbb{P}(\widetilde{A}_{i,j_i}) \\
&= \prod_{i=1}^{l} \sum_{j \in \mathbb{N}} a_j \mathbb{P}(\widetilde{A}_{i,j}) \\
&= \prod_{i=1}^{l} \mathbb{E}(f_1(\#\mathbf{Cl}_n(\mathbf{x}_i, \omega))) \leq \prod_{i=1}^{l} \mathbb{E}(f(\#\mathbf{Cl}_n(\mathbf{x}_i, \omega))),
\end{aligned}$$

where the first inequality follows from (3.4). Finally, we may let n tend to infinity, and then use the Lebesgue's monotone convergence theorem for the right-hand side and Fatou's lemma for the left-hand side. □

Let $\partial_\infty \mathbf{A}$ denote the lattice boundary of $\mathbf{A}$ with respect to the l_∞-norm, and define

$$\bar{\mathbf{A}} := \mathbf{A} \cup \partial_\infty \mathbf{A} \quad \text{and} \quad \mathbf{Cl}_{\mathcal{Y}}(\mathbf{A}) := \bar{\mathbf{A}} \cup \Bigg\{\bigcup_{\mathbf{Cl} \in \mathcal{C}_{\mathcal{Y}}(\mathbf{A})} \bar{\mathbf{Cl}}\Bigg\}.$$

Lemma 8. *If* $\xi = \xi(\rho) := \mathbb{E}e^{\#\mathbf{Cl}_\mathbf{0}} < \infty$ *and* $r \geq 2(\log\alpha + 3^d + \log\xi)s$, *then*

$$\mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\leq s}} \#\mathbf{Cl}_\mathcal{Y}(\mathbf{A}) \geq r\Big) \leq e^{-r/2}.$$

Proof. Notice that

$$\#\mathbf{Cl}_\mathcal{Y}(\mathbf{A}) \leq \#\bar{\mathbf{A}} + \sum_{\mathbf{Cl}\in\mathcal{C}_\mathcal{Y}(\mathbf{A})} \#\mathbf{Cl} \leq 3^d\#\mathbf{A} + \sum_{\mathbf{Cl}\in\mathcal{C}_\mathcal{Y}(\mathbf{A})} \#\mathbf{Cl}.$$

By Markov's inequality and Lemma 7, if $\mathbf{A} \in \Phi_{\leq s}$, then

$$\mathbb{P}\Big(\sum_{\mathbf{Cl}\in\mathcal{C}_\mathcal{Y}(\mathbf{A})} \#\mathbf{Cl} \geq y\Big) \leq e^{-y}\mathbb{E}\big(e^{\sum_{\mathbf{Cl}\in\mathcal{C}_\mathcal{Y}(\mathbf{A})}\#\mathbf{Cl}}\big) \leq e^{-y}(\mathbb{E}e^{\#\mathbf{Cl}_\mathbf{0}})^s = e^{-y}\xi^s.$$

Hence,

$$\begin{aligned}\mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\leq s}} \#\mathbf{Cl}_\mathcal{Y}(\mathbf{A}) \geq r\Big) &\leq \mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\leq s}} \sum_{\mathbf{Cl}\in\mathcal{C}_\mathcal{Y}(\mathbf{A})} \#\mathbf{Cl} \geq (r - 3^d s)\Big)\\ &\leq \alpha^s e^{-(r-3^d s)}\xi^s\\ &= \exp\{-r + s(\log\alpha + 3^d + \log\xi)\}\\ &\leq e^{-r/2},\end{aligned}$$

whenever $r \geq 2(\log\alpha + 3^d + \log\xi)s$. □

3.3 Domination by product measures

Let $\mathcal{X} = \{X_\mathbf{z} : \mathbf{z} \in \mathbb{Z}^d\}$ be a collection of random variables that take values 0 and 1 and which satisfy the following conditions: (i) for each pair $\mathbf{A}, \mathbf{B} \in \mathbb{Z}^d$ such that all sites in $\mathbf{A}$ are at distance greater than k from all sites in $\mathbf{B}$ (in the sup-norm sense), the collections of random variables $\{X_\mathbf{z} : \mathbf{z} \in \mathbf{A}\}$ and $\{X_\mathbf{z} : \mathbf{z} \in \mathbf{B}\}$ are independent; (ii) $\inf_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}(X_\mathbf{z} = 1) \geq p$. In this case we say that $\mathcal{X}$ is a k-dependent random field whose marginals are at least p, and denote $\mathcal{C}(d, k, p)$ the class of all such fields.

Let $\mathcal{Y}$ and $\mathcal{X}$ be two random fields. We say that $\mathcal{Y}$ dominates $\mathcal{X}$ from below if there is a coupling (joint realization) between $\mathcal{Y}$ and $\mathcal{X}$ such that $Y_\mathbf{z} \leq X_\mathbf{z}$ for all $\mathbf{z} \in \mathbb{Z}^d$. We refer to Liggett's book (Liggett, 1985) for more details in stochastic domination and couplings. Theorem 0.0 of Liggett, Schonmann and Stacey (1997) states that when p is close to 1, the random fields in $\mathcal{C}(d, k, p)$ are dominated from below by an i.i.d. random field $\mathcal{Y}$ with density $\rho = \rho(d, k, p)$. Further, one can make ρ arbitrarily close to 1 by taking p close enough to 1.

Lemma 9. *Let* $\mathcal{X} \in \mathcal{C}(d, k, p)$. *There exists* $\bar{p} \in (0, 1)$ *such that for all* $p \in [\bar{p}, 1]$ *if* $s \geq 2r$, *then*

$$\mathbb{P}\Big(\min_{\mathbf{A}\in\Phi_{\geq s}} \sum_{\mathbf{z}\in\mathbf{A}} X_\mathbf{z} \leq r\Big) \leq e^{-s}.$$

Further, there exists a constant $c_0 > 0$ *such that if* $r \geq c_0 s$, *then*

$$\mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\leq s}} \#\mathbf{Cl}_{\mathcal{X}}(\mathbf{A}) \geq r\Big) \leq e^{-r/2}.$$

Proof. We note that if $\mathcal{Y}$ dominates $\mathcal{X}$ from below, then

$$\sum_{\mathbf{z}\in\mathbf{A}} Y_{\mathbf{z}} \leq \sum_{\mathbf{z}\in\mathbf{A}} X_{\mathbf{z}} \quad \text{and} \quad \#\mathbf{Cl}_{\mathcal{X}}(\mathbf{A}) \leq \#\mathbf{Cl}_{\mathcal{Y}}(\mathbf{A})$$

for any lattice animal $\mathbf{A}$. Hence, Lemma 9 follows by combining Theorem 0.0 in Ligget, Schonmann and Stacey (1997) together with Lemma 6 and Lemma 8. □

The proof of the first part of Lemma 9 (in the k-dependent setup) could be done directly without using Ligget, Schonmann and Stacey (1997). One needs to notice that given any set of $s - r$ boxes, we can pick a subset of independent boxes of size $(s - r)/k^d$, and then use the same argument as before. However, the proof of the second part is more delicate and it is not clear (for the author) that it could be easily adapted to the k-dependent situation.

3.4 The block argument

For each $\mathbf{z} \in \mathbb{Z}^d$, $L > 0$ and $s \in \{j/2 : j \in \mathbb{N}\}$ let

$$B_{\mathbf{z}}^{s,L} := L\mathbf{z} + [-sL, sL]^d.$$

Given a locally finite set $\mathcal{N} \subseteq \mathbb{R}^d$, we say that a (square) box B is a $\mathcal{N}$*-full* box if cutting it regularly into $(4\lceil\sqrt{d}\rceil + 1)^d$ sub-boxes, each one of these boxes contains at least one point of the set $\mathcal{N}$. Let

$$B(\mathbf{A}) := \bigcup_{\mathbf{z}\in\mathbf{A}} B_{\mathbf{z}}^{1/2,L} \quad \text{and}$$

$$\tilde{B}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^d : \exists \mathbf{y} \in B(\mathbf{A}) \text{ s.t. } \|\mathbf{x} - \mathbf{y}\|_2 \leq L/2\}.$$

Lemma 10. *Let* $\mathbf{A}$ *be a finite and connected subset of* $\mathbb{Z}^d$ *and assume that* $B_{\mathbf{z}}^{1/2,L}$ *is a* $\mathcal{N}$*-full box for all* $\mathbf{z} \in \partial_\infty \mathbf{A}$. *If* $C_{\mathbf{v}} \in \mathcal{V}$ *and* $C_{\mathbf{v}} \cap B(\mathbf{A}) \neq \varnothing$, *then* $C_{\mathbf{v}} \subseteq \tilde{B}(\mathbf{A})$.

Proof. Assume that $C_{\mathbf{v}} \cap B(\mathbf{A}) \neq \varnothing$ but $C_{\mathbf{v}} \not\subseteq \tilde{B}(\mathbf{A})$. Then there will exist $\mathbf{x}_1, \mathbf{x}_2 \in C_{\mathbf{v}}$ such that

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_2 \geq L/2.$$

On the other hand, $B_{\mathbf{z}}^{1/2,L}$ is a full box for all $\mathbf{z} \in \partial_\infty \mathbf{A}$. By picking $\mathbf{x}_1$ and $\mathbf{x}_2$ in the (euclidean) boundary of $B(\mathbf{A})$ and $\tilde{B}(\mathbf{A})$, respectively, this implies that there exist $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{N}$ such that

$$\|\mathbf{v}_1 - \mathbf{x}_1\|_2 \leq \frac{\sqrt{d}}{4\lceil\sqrt{d}\rceil + 1} L \quad \text{and} \quad \|\mathbf{v}_2 - \mathbf{x}_2\|_2 \leq \frac{\sqrt{d}}{4\lceil\sqrt{d}\rceil + 1} L$$

(the right-hand side of the inequality is the length of the diagonal of a subsquare). However,

$$\|\mathbf{v}-\mathbf{x}_1\|_2 \le \|\mathbf{v}_1-\mathbf{x}_1\|_2 \quad \text{and} \quad \|\mathbf{v}-\mathbf{x}_2\|_2 \le \|\mathbf{v}_2-\mathbf{x}_2\|_2,$$

and, hence,

$$\begin{aligned}\frac{1}{2}L &\le \|\mathbf{x}_1-\mathbf{x}_2\|_2\\ &\le \|\mathbf{x}_1-\mathbf{v}\|_2+\|\mathbf{x}_2-\mathbf{v}\|_2\\ &\le \|\mathbf{x}_1-\mathbf{v}_1\|_2+\|\mathbf{x}_2-\mathbf{v}_2\|_2 \le \frac{2\sqrt{d}}{4\lceil\sqrt{d}\rceil+1}L,\end{aligned}$$

which yields to a contradiction since $4\lceil\sqrt{d}\rceil+1>4\sqrt{d}$. □

Lemma 11. *Under* (1.1),

$$\sup_{\mathbf{z}\in\mathbb{Z}^d}\mathbb{P}(B_{\mathbf{z}}^{1/2,L} \textit{ is not a } \mathcal{N}\textit{-full box}) \le (4\lceil\sqrt{d}\rceil+1)^d \exp\left\{-c_\mu^{-1}\left(\frac{L}{4\lceil\sqrt{d}\rceil+1}\right)^d\right\}.$$

Proof. Cut $B_{\mathbf{z}}^{1/2,L}$ regularly into $(4\lceil\sqrt{d}\rceil+1)^d$ sub-boxes, so that

$$B_{\mathbf{z}}^{1/2,L}=\bigcup_{i=1}^{(4\lceil\sqrt{d}\rceil+1)^d} B_{\mathbf{z},i} \quad \text{and} \quad \mathbb{P}(\#_{\mathcal{N}}B_{\mathbf{z},i}=0)\le \exp\left\{-c_\mu^{-1}\left(\frac{L}{4\lceil\sqrt{d}\rceil+1}\right)^d\right\}.$$

Hence,

$$\begin{aligned}\mathbb{P}(B_{\mathbf{z}}^{1/2,L} \text{ is not a } \mathcal{N}\text{-full box}) &\le \sum_{i=1}^{(4\lceil\sqrt{d}\rceil+1)^d}\mathbb{P}(\#_{\mathcal{N}}B_{\mathbf{z},i}=0)\\ &\le (4\lceil\sqrt{d}\rceil+1)^d\exp\left\{-c_\mu^{-1}\left(\frac{L}{4\lceil\sqrt{d}\rceil+1}\right)^d\right\}.\end{aligned}$$ □

4 Proof of the theorems

Proof of (2.1). For any connected set $C\subseteq\mathbb{R}^d$,

$$C\subseteq\bigcup_{\mathbf{z}\in\mathbf{A}(C)}B_{\mathbf{z}},$$

and, thus,

$$r\le\#_{\mathcal{N}}\mathcal{P}\le\sum_{\mathbf{z}\in\mathbf{A}(\mathcal{P})}\#_{\mathcal{N}}B_{\mathbf{z}}=\sum_{\mathbf{z}\in\mathbf{A}(\mathcal{P})}N_{\mathbf{z}},$$

if $\mathcal{P} \in \Pi_{\geq r}$. Therefore, by Lemma 5,

$$\mathbb{P}\Big(\min_{\mathcal{P}\in\Pi_{\geq r}} \#\mathbf{A}(\mathcal{P}) \leq s\Big) \leq \mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\leq s}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \geq r\Big) \leq e^{-r/2}, \tag{4.1}$$

whenever $r \geq b_1 s$ and $b_1 := 2(\log\alpha + c_\mu(e-1))$. □

Proof of (2.2). For each $L \geq 1$ let

$$\mathbf{A}_L(\mathcal{P}) := \{\mathbf{z} \in \mathbb{Z}^d : B_{\mathbf{z}}^{1/2,L} \cap \mathcal{P} \neq \varnothing\},$$

and recall that $\mathbf{A}(\mathcal{P}) = \mathbf{A}_1(C)$. Then, for any $L \geq 1$,

$$\#\mathbf{A}_L(\mathcal{P}) \leq \#\mathbf{A}_1(\mathcal{P}) \leq L^d \#\mathbf{A}_L(\mathcal{P}). \tag{4.2}$$

Consider the nonhomogeneous 3-dependent percolation scheme $\mathcal{X}^L$ in $\mathbb{Z}^d$ defined by

$$X_{\mathbf{z}}^L := \mathbf{1}\{B_{\mathbf{z}'}^{1/2,L} \text{ is a full box } \forall \mathbf{z}' \text{ s.t. } \|\mathbf{z}'-\mathbf{z}\|_\infty \leq 1\}.$$

($\mathbf{1}$ denotes the indicator function of an event.) If $X_{\mathbf{z}}^L = 1$, we say that $\mathbf{B}_{\mathbf{z}}^{1/2,L}$ is a good box. By Lemma 11, $\mathcal{X}^L \in \mathcal{C}(d, 3, p_L)$, where

$$1 - p_L := \sup_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}(X_{\mathbf{z}}^L = 0) \leq 3^d (4\lceil\sqrt{d}\rceil + 1)^d \exp\Big\{-c_\mu^{-1}\Big(\frac{L}{4\lceil\sqrt{d}\rceil+1}\Big)^d\Big\}.$$

By Lemma 9, if we pick L_0 such that

$$3^d (4\lceil\sqrt{d}\rceil + 1)^d \exp\Big\{-c_\mu^{-1}\Big(\frac{L_0}{4\lceil\sqrt{d}\rceil+1}\Big)^d\Big\} \leq 1 - \bar{p},$$

then $p_{L_0} \in [\bar{p}, 1]$ and

$$\mathbb{P}\Big(\min_{\mathbf{A}\in\Phi_{\geq s}} \sum_{\mathbf{z}\in\mathbf{A}} X_{\mathbf{z}}^{L_0} \leq r\Big) \leq e^{-s}, \tag{4.3}$$

whenever $s \geq 2r$. Now, let

$$\mathcal{S}_{\mathcal{X}^L}(\mathcal{P}) := \{\mathbf{z} \in \mathbf{A}_L(\mathcal{P}) : X_{\mathbf{z}}^L = 1\}.$$

Notice that there exists at least one set $\mathcal{S}'_{\mathcal{X}} \subseteq \mathcal{S}_{\mathcal{X}}$ such that $|\mathbf{z}-\mathbf{z}'|_\infty \geq 2$ for all $\mathbf{z}, \mathbf{z}' \in \mathcal{S}'_{\mathcal{X}}$ and $k = \#\mathcal{S}'_{\mathcal{X}} \geq \#\mathcal{S}_{\mathcal{X}}/3^d$. Now, write $\mathcal{S}'_{\mathcal{X}} = \{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$. By Lemma 10, if $\mathbf{z}_i, \mathbf{z}_j \in \mathcal{S}'$ and $C_{\mathbf{v}_i} \cap B_{\mathbf{z}_i}^{1/2,L_0} \neq \varnothing$ and $C_{\mathbf{v}_j} \cap B_{\mathbf{z}_j}^{1/2,L_0} \neq \varnothing$, then $\mathbf{v}_i \neq \mathbf{v}_j$, and, thus,

$$\#_{\mathcal{N}}\mathcal{P} \geq k \geq \frac{\#\mathcal{S}_{\mathcal{X}}}{3^d} \geq \frac{\sum_{\mathbf{z}\in\mathbf{A}(\mathcal{P})} X_{\mathbf{z}}^L}{3^d}.$$

By (4.2), this shows that

$$\mathbb{P}\Big(\max_{\mathcal{P}\in\Pi_{\le r}} \#\mathbf{A}(\mathcal{P}) \ge s\Big) \le \mathbb{P}\Big(\max_{\mathcal{P}\in\Pi_{\le r}} \#\mathbf{A}_{L_0}(\mathcal{P}) \ge s/L_0\Big)$$

$$\le \mathbb{P}\Bigg(\min_{\mathbf{A}\in\Phi_{\ge s/L_0}} \sum_{\mathbf{z}\in\mathbf{A}} X_{\mathbf{z}}^{L_0} \le 3^d r\Bigg) \le e^{-s/L_0},$$

whenever $s \ge 2(L_0 3^d)r$. □

Proof of (2.5). By Lemma 10 (recall the definition of **Cl**. in Lemma 8),

$$\mathcal{P}(\mathbf{A}) \subseteq B(\mathbf{Cl}_{\mathcal{X}^L}(\mathbf{A}))$$

and

$$\#_{\mathcal{N}}\mathcal{P}(\mathbf{A}) \le \#_{\mathcal{N}} B(\mathbf{Cl}_{\mathcal{X}^L}(\mathbf{A})) = \sum_{\mathbf{z}\in\mathbf{Cl}_{\mathcal{X}^L}(\mathbf{A})} N_{\mathbf{z}},$$

which yields to

$$\mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\le s}} \#_{\mathcal{N}}\mathcal{P}(\mathbf{A}) \ge r\Big) \le \mathbb{P}\Bigg(\max_{\mathbf{A}\in\Phi_{\le cr}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \ge r\Bigg) + \mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\le s}} \#\mathbf{Cl}_{\mathcal{X}^L}(\mathbf{A}) \ge cr\Big)$$

for any $c > 0$. By Lemma 5 and Lemma 8,

$$\mathbb{P}\Bigg(\max_{\mathbf{A}\in\Phi^{cr}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \ge r\Bigg) \le e^{-r/2} \quad \text{and} \quad \mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\le s}} \#\mathbf{Cl}_{\mathcal{X}^L}(\mathbf{A}) \ge cr\Big) \le e^{-cr/2}$$

for $c = [2(\log\alpha + c_\mu(e-1))]^{-1}$ and $r \ge (c_0/c)s$ [notice that $r \ge (\log\alpha + c_\mu(e-1))cr = r/2$], which shows that

$$\mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\le s}} \#_{\mathcal{N}}\mathcal{P}(\mathbf{A}) \ge r\Big) \le e^{-r/2} + e^{-cr/2},$$

and finishes the proof of (2.5). □

5 A two-dimensional modified Poisson model

In Pimentel (2011) a random set is constructed from a realization of a two-dimensional homogeneous Poisson random set $\mathcal{N}$ as follows. Order the points of $\mathbb{Z}^2$ in some arbitrary fashion, say, $\mathbb{Z}^2 := \{\mathbf{u}_1, \mathbf{u}_2, \dots\}$. Fix $\delta \in (0, 1)$ and $n \ge 1$. For each $k \ge 1$ let

$$B_k^n := B_{\mathbf{u}_k}^{1/2, n^\delta}.$$

Divide B_k^n into 36 sub-boxes (as before) of the same length $n^\delta/6$, say, $B_{k,1}^n, \dots, B_{k,36}^n$. Now we construct the modified random set $\mathcal{N}(n) := \mathcal{N}(n, \mathcal{N})$ (whose distribution will also depend on n and δ) by changing the original Poisson random set $\mathcal{N}$ inside each $B_{k,j}^n$, as follows:

(1) If $1 \le \#_{\mathcal{N}} B^n_{k,j} \le n^{2\delta}$, then set $B^n_{k,j} \cap \mathcal{N}(n) := B^n_{k,j} \cap \mathcal{N}$.
(2) If $\#_{\mathcal{N}} B^n_{k,j} > n^{2\delta}$, then set $\mathcal{N}(n)$ by uniformly selecting $n^{2\delta}$ points from $B^n_{k,j} \cap \mathcal{N}$.
(3) If $\#_{\mathcal{N}} B^n_{k,j} = 0$, then set $B^n_k \cap \mathcal{N}(n)$ by adding an extra point uniformly distributed on $B^n_{k,j}$.

In a few words, we tile the plane into boxes B^n_k of size n^δ and we insist that each tile is a full box, and that no tile contains more than $36n^{2\delta}$ Poisson points. We make the convention $\mathcal{N}(\infty) = \mathcal{N}$ and denote by $\mathcal{D}(n)$ the Delaunay triangulation based on $\mathcal{N}(n)$. It is clear that method can be applied in this setup for each fixed n. However, we want to emphasize that it allows us to do so simultaneously for all n sufficiently large.

Theorem 12. *Theorems 1 and 3, as well as Corollaries 2 and 4, hold in the modified model $\mathcal{N}(n)$, where the constant b_j does not depend on $n \ge 1$.*

Proof (outline). The proof of (2.1), in the $\mathcal{N}(n)$ context, uses that

$$\#_{\mathcal{N}(n)} B_{\mathbf{z}} \le \#_{\mathcal{N}} B_{\mathbf{z}} + 2^d,$$

(the small box could intersect at most 2^d boxes $B^n_{k,j}$) and, hence [see (4.1)]

$$\mathbb{P}\Big(\min_{\mathcal{P}\in\Pi_{\ge r}} \#\mathbf{A}(\mathcal{P}) \le s\Big) \le \mathbb{P}\Big(\max_{\mathbf{A}\in\Phi_{\le s}} \sum_{\mathbf{z}\in\mathbf{A}} N_{\mathbf{z}} \ge r - 2^d s\Big),$$

which implies (2.1) (by Lemma 5).

The proof of (2.2) and (2.5) were essentially based on the fact that we can compare the original problem to a site percolation scheme with minimal marginal density $p_L \to 1$ as $L \to \infty$. The same comparison method works here as soon as we prove that

$$\inf_n p_L(n) \to 1 \qquad \text{as } L \to \infty,$$

or, equivalently,

$$\lim_{L\to\infty} \sup_{n\ge 1} \mathbb{P}\big(B^{1/2,L}_{\mathbf{z}} \text{ is not a } \mathcal{N}(n)\text{-full box}\big) = 0.$$

Notice that, if $L \ge 2n^\delta$, then a box B of size $L/6$ must contain at least one box $B^n_{k,j}$ of length $n^\delta/6$ (for some k and j) which means that B is a $\mathcal{N}(n)$-full box. On the other hand, if $L \le 2n^\delta$, then the probability of the event that $B^{1/2,L}_{\mathbf{z}}$ is not a $\mathcal{N}(n)$-full box is bounded by the probability of the event that $B^{1/2,L}_{\mathbf{z}}$ is not a $\mathcal{N}$-full box plus the probability of the event that $\mathcal{N} \cap B^{1/2,L}_{\mathbf{z}} \ne \mathcal{N}(n) \cap B^{1/2,L}_{\mathbf{z}}$, which decays to 0 as $\max\{e^{-L^2}, e^{-n^{2\delta}}\} \le e^{-L^2}$. The rest of the proof follows mutatis-mutandis the method applied to prove (2.2) and (2.5). □

The techniques developed in this paper also fit to study the minimal density of open edges among all self-avoiding paths $\gamma \in \Gamma_{\geq r}$, in the two-dimensional bond percolation model in the Delaunay triangulation. This model is constructed by attaching i.i.d. Bernoulli random variables $\tau_{\mathbf{e}}$ with parameter $p \in [0, 1]$ to edges $\mathbf{e} \in \mathcal{D}$. Let $p_c^* \in (0, 1)$ be the critical probability for the bond percolation model in the $\mathcal{N}$-Voronoi tessellation (Pimentel, 2006). We denote by $(\mathcal{N}(n), \tau)$ the the bond percolation model associated to the modified random set $\mathcal{N}(n)$.

Theorem 13. *If $\mathbb{P}(\tau_{\mathbf{e}} = 0) < 1 - p_c^*$, then there exists $n_0 \geq 0$ and $b_9, b_{10} > 0$ such that, for all $n \geq n_0$, if $r \geq b_9 s$, then*

$$\mathbb{P}\left(\min_{\gamma \in \Gamma_{\geq r}} \sum_{\mathbf{e} \in \gamma} \tau_{\mathbf{e}} \leq s\right) \leq 3e^{-b_{10} r}. \tag{5.1}$$

Proof (outline). Let $\Gamma_{\mathbf{z}}^L$ be the collection of all self-avoiding paths $\gamma = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_l)$ in $\mathcal{D}$ such that $C_{\mathbf{v}_0} \cap \partial B_{\mathbf{z}}^{1/2,L} \neq \varnothing$, $C_{\mathbf{v}_l} \cap \partial B_{\mathbf{z}}^{3/2,L} \neq \varnothing$ and $\mathbf{v}_j \in B_{\mathbf{z}}^{3/2,L} \setminus B_{\mathbf{z}}^{1/2,L}$ for $j = 1, \ldots, l-1$. We say that $B_{\mathbf{z}}^{1/2,L}$ is a $(\mathcal{N}, \tau)$-good box if the following holds:

(1) $B_{\mathbf{z}'}^{1/2,L}$ is a $\mathcal{N}$-full box for all $\mathbf{z}' \in \mathbb{Z}^2$ s.t. $|\mathbf{z}' - \mathbf{z}|_\infty \leq 2$;
(2) $\sum_{\mathbf{e} \in \gamma} \tau_{\mathbf{e}} \geq 1$ for all $\gamma \in \Gamma_{\mathbf{z}}^L$.

We have seen that the probability of the event (1) is uniformly bounded for all $n \geq 1$. For $n = \infty$ (we are in classical Poisson model), Lemma 1 in Pimentel (2006) implies that the probability of the event (2) goes to 0 as $L \to \infty$, if $\mathbb{P}(\tau_{\mathbf{e}} = 0) < 1 - p_c^*$. For a finite $n \geq 1$ the probability that $\mathcal{N} \neq \mathcal{N}(n)$ inside $B_{\mathbf{z}}^{1/2,L}$ is of order $L^2 e^{-2n^\delta}$. Thus,

$$\begin{aligned}\mathbb{P}((2)|(1)) &\leq \mathbb{P}\big((2) \text{ for } n = \infty | (1)\big) + cL^2 e^{-2c' n^\delta} \\ &\leq \mathbb{P}\big((2) \text{ for } n = \infty | (1)\big) + cL^2 e^{-2c' L^\delta}\end{aligned}$$

for $L \leq n$ (c and c' are constants). This means that, given any $p \in (0, 1)$ there exists $L_0 > 0$ such that for all $n, L \geq L_0$,

$$\mathbb{P}\big(B_{\mathbf{z}}^{1/2,L} \text{ is a } (\mathcal{N}(n), \tau)\text{-good box}\big) \geq p. \tag{5.2}$$

Consider the homogeneous percolation scheme $\mathcal{Z}^L$ defined by

$$Z_{\mathbf{z}}^L := \mathbf{1}\{B_{\mathbf{z}}^{1/2,L} \text{ is a } (\mathcal{N}(n), \tau)\text{-good box}\}.$$

By Lemma 3 in Pimentel (2006), it is a 5-dependent percolation scheme. Together with (5.2), this implies that $\mathcal{Z}^{L_0} \in \mathcal{C}(d, 5, p)$ for all $n \geq L_0$, if $\mathbb{P}(\tau_{\mathbf{e}} = 0) < 1 - p_c^*$. Now, let

$$\mathcal{S}_{\mathcal{Z}^L}(\gamma) := \{\mathbf{z} \in \mathbf{A}_L(\gamma) : Z_{\mathbf{z}}^L = 1\}.$$

Notice that there exists at least one set $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathbf{z}-\mathbf{z}'|_\infty \geq 4$ for all $\mathbf{z}, \mathbf{z}' \in \mathcal{S}'$ and $k = |\mathcal{S}'| \geq |\mathcal{S}|/4^d$. Now, write $\mathcal{S}' = \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$. By Lemma 10, one can find disjoint pieces of γ, say, $\gamma_1, \dots, \gamma_k$, such that, for $i = 1, \dots, k$, $\sum_{\mathbf{e}\in\gamma_i} \tau_\mathbf{e} \geq 1$. Hence,

$$\sum_{\mathbf{e}\in\gamma} \tau_\mathbf{e} \geq \sum_{i=1}^{k} \Big(\sum_{\mathbf{e}\in\gamma_i} \tau_\mathbf{e}\Big) \geq k = |\mathcal{S}'| \geq \frac{|\mathcal{S}|}{4^d} = \frac{\sum_{\mathbf{z}\in\mathbf{A}(\gamma)} Z_\mathbf{z}^L}{4^d},$$

which shows that

$$\mathbb{P}\Big(\min_{\gamma\in\Gamma_{\geq r}} \sum_{\mathbf{e}\in\gamma} \tau_\mathbf{e} \leq s\Big) \leq \mathbb{P}\Big(\min_{\gamma\in\Gamma_{\geq r}} |\mathbf{A}_{L_0}(\gamma)| \leq \frac{r}{b_1}\Big) + \mathbb{P}\Big(\min_{\mathbf{A}\in\Phi_{\geq r/b_1}} \sum_{\mathbf{z}\in\mathbf{A}} Z_\mathbf{z}^{L_0} \leq 4^d s\Big)$$
$$\leq e^{-r/2} + 2e^{-r/b_1},$$

whenever $r \geq 2b_1(4^d)s$. In the last line, we have used (2.1) (in the modified context) and Lemma 9 (by choosing L_0 large enough). □

References

Grimmett, G. (1999). *Percolation*, 2nd ed. Berlin: Springer. MR1707339

Liggett, T. M. (1985). *Interacting Particles Systems*. New York: Springer. MR0776231

Ligget, T. M., Schonmann, R. H. and Stacey, A. M. (1997). Domination by product measures. *The Annals of Probability* **25**, 71–95. MR1428500

Pimentel, L. P. R. (2006). The time constant and critical probabilities in percolation models. *Electronic Communications in Probability* **11**, 160–167. MR2240709

Pimentel, L. P. R. (2011). Asymptotics for first-passage times on Delaunay triangulations. *Combinatorics, Probabability and Computing* **20**, 435–453. MR2784636

Pimentel, L. P. R. and Rossignol, R. (2012). Greedy Polyominoes and first-passage times on random Voronoi tilings. *Electronic Journal of Probability* **17**, Art. 12.

Voronoi, G. (1908). Nouvelles applications des paramètres continus à la thèorie des formes quadratiques. *Journal für die reine und angewandte Mathematik* **133**, 97–178.

Institute of Mathematics
Federal University of Rio de Janeiro
Caixa Postal 68530
CEP 21941-909
Rio de Janeiro, RJ
Brazil
E-mail: leandro@im.ufrj.br