

GLOBAL SELF-WEIGHTED AND LOCAL QUASI-MAXIMUM EXPONENTIAL LIKELIHOOD ESTIMATORS FOR ARMA–GARCH/IGARCH MODELS

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This paper investigates the asymptotic theory of the quasi-maximum exponential likelihood estimators (QMELE) for ARMA–GARCH models. Under only a fractional moment condition, the strong consistency and the asymptotic normality of the global self-weighted QMELE are obtained. Based on this self-weighted QMELE, the local QMELE is showed to be asymptotically normal for the ARMA model with GARCH (finite variance) and IGARCH errors. A formal comparison of two estimators is given for some cases. A simulation study is carried out to assess the performance of these estimators, and a real example on the world crude oil price is given.

1. Introduction. Assume that $\{y_t : t = 0, \pm 1, \pm 2, \dots\}$ is generated by the ARMA–GARCH model

$$(1.1) \quad y_t = \mu + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i} + \varepsilon_t,$$

$$(1.2) \quad \varepsilon_t = \eta_t \sqrt{h_t} \quad \text{and} \quad h_t = \alpha_0 + \sum_{i=1}^r \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^s \beta_i h_{t-i},$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ ($i = 1, \dots, r$), $\beta_j \geq 0$ ($j = 1, \dots, s$), and η_t is a sequence of i.i.d. random variables with $E\eta_t = 0$. As we all know, since Engle (1982) and Bollerslev (1986), model (1.1)–(1.2) has been widely used in economics and finance; see Bollerslev, Chou and Kroner (1992), Bera and Higgins (1993), Bollerslev, Engel and Nelson (1994) and Francq and Zakoïan (2010). The asymptotic theory of the quasi-maximum likelihood estimator (QMLE) was established by Ling and Li (1997) and by Francq and Zakoïan (2004) when $E\varepsilon_t^4 < \infty$. Under the strict stationarity condition, the consistency and the asymptotic normality of the QMLE were obtained by Lee and Hansen (1994) and Lumsdaine (1996) for the GARCH(1, 1) model, and by Berkes, Horváth and Kokoszka (2003) and Francq and Zakoïan (2004) for the GARCH(r, s) model. Hall and Yao (2003)

Received January 2011.

¹Supported in part by Hong Kong Research Grants Commission Grants HKUST601607 and HKUST602609.

MSC2010 subject classifications. 62F12, 62M10, 62P20.

Key words and phrases. ARMA–GARCH/IGARCH model, asymptotic normality, global self-weighted/local quasi-maximum exponential likelihood estimator, strong consistency.

established the asymptotic theory of the QMLE for the GARCH model when $E\varepsilon_t^2 < \infty$, including both cases in which $E\eta_t^4 = \infty$ and $E\eta_t^4 < \infty$. Under the geometric ergodicity condition, Lang, Rahbek and Jensen (2011) gave the asymptotic properties of the modified QMLE for the first order AR–ARCH model. Moreover, when $E|\varepsilon_t|^\iota < \infty$ for some $\iota > 0$, the asymptotic theory of the global self-weighted QMLE and the local QMLE was established by Ling (2007) for model (1.1)–(1.2).

It is well known that the asymptotic normality of the QMLE requires $E\eta_t^4 < \infty$ and this property is lost when $E\eta_t^4 = \infty$; see Hall and Yao (2003). Usually, the least absolute deviation (LAD) approach can be used to reduce the moment condition of η_t and provide a robust estimator. The local LAD estimator was studied by Peng and Yao (2003) and Li and Li (2005) for the pure GARCH model, Chan and Peng (2005) for the double AR(1) model, and Li and Li (2008) for the ARFIMA–GARCH model. The global LAD estimator was studied by Horváth and Liese (2004) for the pure ARCH model and by Berkes and Horváth (2004) for the pure GARCH model, and by Zhu and Ling (2011a) for the double AR(p) model. Except for the AR models studied by Davis, Knight and Liu (1992) and Ling (2005) [see also Knight (1987, 1998)], the nondifferentiable and nonconvex objective function appears when one studies the LAD estimator for the ARMA model with i.i.d. errors. By assuming the existence of a \sqrt{n} -consistent estimator, the asymptotic normality of the LAD estimator is established for the ARMA model with i.i.d. errors by Davis and Dunsmuir (1997) for the finite variance case and by Pan, Wang and Yao (2007) for the infinite variance case; see also Wu and Davis (2010) for the noncausal or noninvertible ARMA model. Recently, Zhu and Ling (2011b) proved the asymptotic normality of the global LAD estimator for the finite/infinite variance ARMA model with i.i.d. errors.

In this paper, we investigate the self-weighted quasi-maximum exponential likelihood estimator (QMELE) for model (1.1)–(1.2). Under only a fractional moment condition of ε_t with $E\eta_t^2 < \infty$, the strong consistency and the asymptotic normality of the global self-weighted QMELE are obtained by using the bracketing method in Pollard (1985). Based on this global self-weighted QMELE, the local QMELE is showed to be asymptotically normal for the ARMA–GARCH (finite variance) and –IGARCH models. A formal comparison of two estimators is given for some cases.

To motivate our estimation procedure, we revisit the GNP deflator example of Bollerslev (1986), in which the GARCH model was proposed for the first time. The model he specified is an AR(4)–GARCH(1, 1) model for the quarterly data from 1948.2 to 1983.4 with a total of 143 observations. We use this data set and his fitted model to obtain the residuals $\{\hat{\eta}_t\}$. The tail index of $\{\eta_t^2\}$ is estimated by Hill's estimator $\hat{\alpha}_\eta(k)$ with the largest k data of $\{\hat{\eta}_t^2\}$, that is,

$$\hat{\alpha}_\eta(k) = \frac{k}{\sum_{j=1}^k (\log \tilde{\eta}_{143-j} - \log \tilde{\eta}_{143-k})},$$

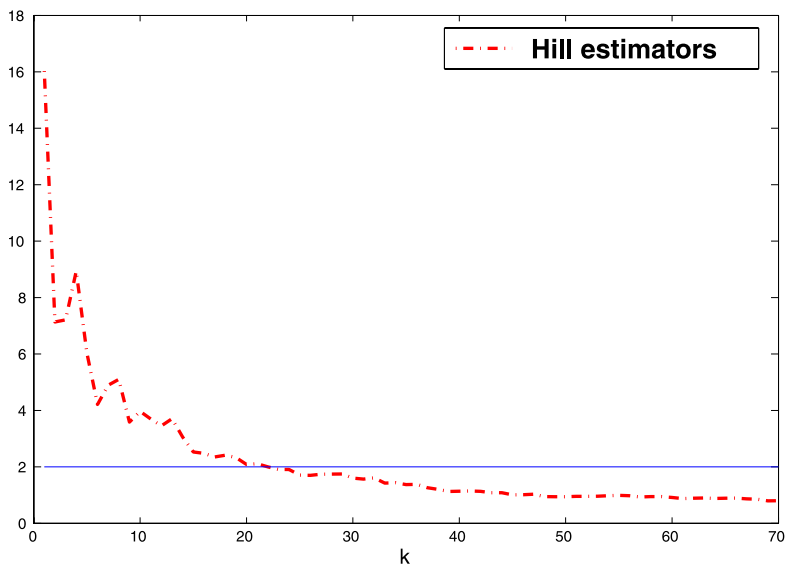


FIG. 1. The Hill estimators $\{\hat{\alpha}_\eta(k)\}$ for $\{\hat{\eta}_t^2\}$.

where $\tilde{\eta}_j$ is the j th order statistic of $\{\hat{\eta}_t^2\}$. The plot of $\{\hat{\alpha}_\eta(k)\}_{k=1}^{70}$ is given in Figure 1. From this figure, we can see that $\hat{\alpha}_\eta(k) > 2$ when $k \leq 20$, and $\hat{\alpha}_\eta(k) < 2$ when $k > 20$. Note that Hill’s estimator is not so reliable when k is too small. Thus, the tail of $\{\eta_t^2\}$ is most likely less than 2, that is, $E\eta_t^4 = \infty$. Thus, the setup that η_t has a finite fourth moment may not be suitable, and hence the standard QMLE procedure may not be reliable in this case. The estimation procedure in this paper only requires $E\eta_t^2 < \infty$. It may provide a more reliable alternative to practitioners. To further illustrate this advantage, a simulation study is carried out to compare the performance of our estimators and the self-weighted/local QMLE in Ling (2007), and a new real example on the world crude oil price is given in this paper.

This paper is organized as follows. Section 2 gives our results on the global self-weighted QMELE. Section 3 proposes a local QMELE estimator and gives its limiting distribution. The simulation results are reported in Section 4. A real example is given in Section 5. The proofs of two technical lemmas are provided in Section 6. Concluding remarks are offered in Section 7. The remaining proofs are given in the Appendix.

2. Global self-weighted QMELE. Let $\theta = (\gamma', \delta')'$ be the unknown parameter of model (1.1)–(1.2) and its true value be θ_0 , where $\gamma = (\mu, \phi_1, \dots, \phi_p, \psi_1, \dots, \psi_q)'$ and $\delta = (\alpha_0, \dots, \alpha_r, \beta_1, \dots, \beta_s)'$. Given the observations $\{y_n, \dots, y_1\}$ and the initial values $Y_0 \equiv \{y_0, y_{-1}, \dots\}$, we can rewrite the parametric model (1.1)–(1.2)

as

$$(2.1) \quad \varepsilon_t(\gamma) = y_t - \mu - \sum_{i=1}^p \phi_i y_{t-i} - \sum_{i=1}^q \psi_i \varepsilon_{t-i}(\gamma),$$

$$(2.2) \quad \eta_t(\theta) = \varepsilon_t(\gamma) / \sqrt{h_t(\theta)} \quad \text{and}$$

$$h_t(\theta) = \alpha_0 + \sum_{i=1}^r \alpha_i \varepsilon_{t-i}^2(\gamma) + \sum_{i=1}^s \beta_i h_{t-i}(\theta).$$

Here, $\eta_t(\theta_0) = \eta_t$, $\varepsilon_t(\gamma_0) = \varepsilon_t$ and $h_t(\theta_0) = h_t$. The parameter space is $\Theta = \Theta_\gamma \times \Theta_\delta$, where $\Theta_\gamma \subset R^{p+q+1}$, $\Theta_\delta \subset R_0^{r+s+1}$, $R = (-\infty, \infty)$ and $R_0 = [0, \infty)$. Assume that Θ_γ and Θ_δ are compact and θ_0 is an interior point in Θ . Denote $\alpha(z) = \sum_{i=1}^r \alpha_i z^i$, $\beta(z) = 1 - \sum_{i=1}^s \beta_i z^i$, $\phi(z) = 1 - \sum_{i=1}^p \phi_i z^i$ and $\psi(z) = 1 + \sum_{i=1}^q \psi_i z^i$. We introduce the following assumptions:

ASSUMPTION 2.1. For each $\theta \in \Theta$, $\phi(z) \neq 0$ and $\psi(z) \neq 0$ when $|z| \leq 1$, and $\phi(z)$ and $\psi(z)$ have no common root with $\phi_p \neq 0$ or $\psi_q \neq 0$.

ASSUMPTION 2.2. For each $\theta \in \Theta$, $\alpha(z)$ and $\beta(z)$ have no common root, $\alpha(1) \neq 1$, $\alpha_r + \beta_s \neq 0$ and $\sum_{i=1}^s \beta_i < 1$.

ASSUMPTION 2.3. η_t^2 has a nondegenerate distribution with $E\eta_t^2 < \infty$.

Assumption 2.1 implies the stationarity, invertibility and identifiability of model (1.1), and Assumption 2.2 is the identifiability condition for model (1.2). Assumption 2.3 is necessary to ensure that η_t^2 is not almost surely (a.s.) a constant. When η_t follows the standard double exponential distribution, the weighted log-likelihood function (ignoring a constant) can be written as follows:

$$(2.3) \quad L_{sn}(\theta) = \frac{1}{n} \sum_{t=1}^n w_t l_t(\theta) \quad \text{and} \quad l_t(\theta) = \log \sqrt{h_t(\theta)} + \frac{|\varepsilon_t(\gamma)|}{\sqrt{h_t(\theta)}},$$

where $w_t = w(y_{t-1}, y_{t-2}, \dots)$ and w is a measurable, positive and bounded function on R^{Z_0} with $Z_0 = \{0, 1, 2, \dots\}$. We look for the minimizer, $\hat{\theta}_{sn} = (\hat{\gamma}'_{sn}, \hat{\delta}'_{sn})'$, of $L_{sn}(\theta)$ on Θ , that is,

$$\hat{\theta}_{sn} = \underset{\Theta}{\text{arg min}} L_{sn}(\theta).$$

Since the weight w_t only depends on $\{y_t\}$ itself and we do not assume that η_t follows the standard double exponential distribution, $\hat{\theta}_{sn}$ is called the self-weighted quasi-maximum exponential likelihood estimator (QMELE) of θ_0 . When h_t is a constant, the self-weighted QMELE reduces to the weighted LAD estimator of the ARMA model in Pan, Wang and Yao (2007) and Zhu and Ling (2011b).

The weight w_t is to reduce the moment condition of ε_t [see more discussions in Ling (2007)], and it satisfies the following assumption:

ASSUMPTION 2.4. $E[(w_t + w_t^2)\xi_{\rho t-1}^3] < \infty$ for any $\rho \in (0, 1)$, where $\xi_{\rho t} = 1 + \sum_{i=0}^{\infty} \rho^i |y_{t-i}|$.

When $w_t \equiv 1$, the $\hat{\theta}_{sn}$ is the global QMELE and it needs the moment condition $E|\varepsilon_t|^3 < \infty$ for its asymptotic normality, which is weaker than the moment condition $E\varepsilon_t^4 < \infty$ as for the QMLE of θ_0 in Francq and Zakoian (2004). It is well known that the higher is the moment condition of ε_t , the smaller is the parameter space. Figure 2 gives the strict stationarity region and regions for $E|\varepsilon_t|^{2\iota} < \infty$ of the GARCH(1, 1) model: $\varepsilon_t = \eta_t \sqrt{h_t}$ and $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$, where $\eta_t \sim \text{Laplace}(0, 1)$. From Figure 2, we can see that the region for $E|\varepsilon_t|^{0.1} < \infty$ is

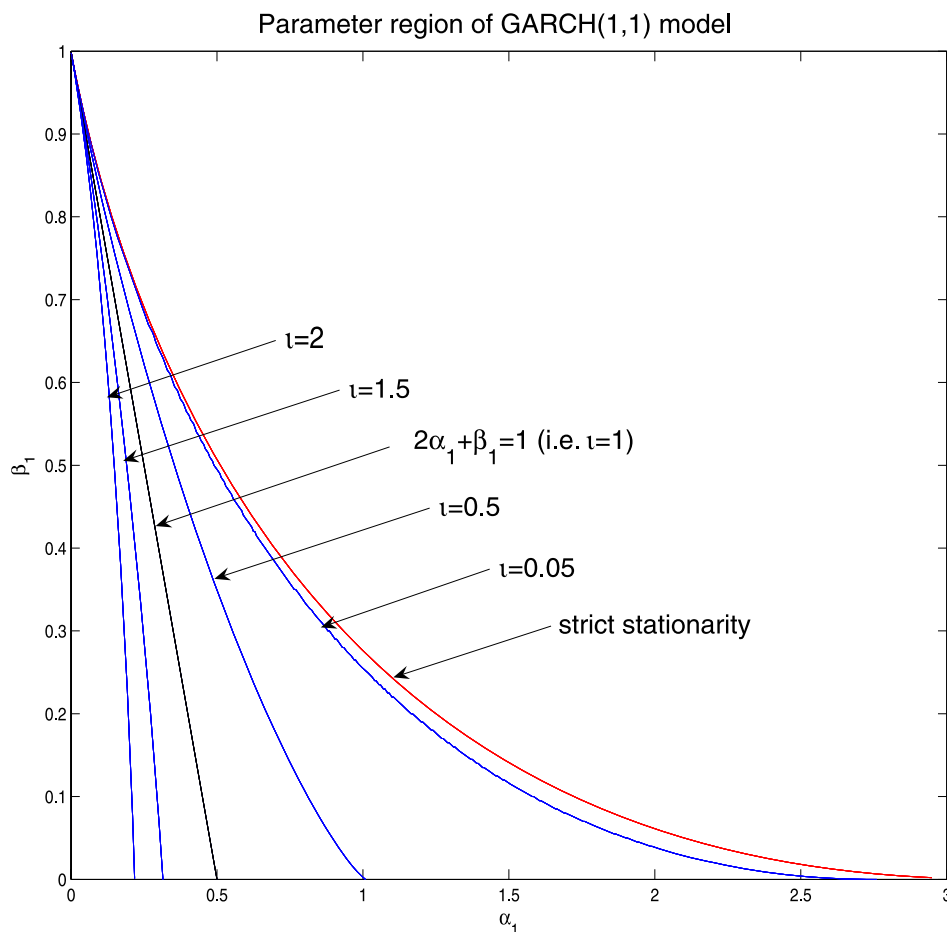


FIG. 2. The regions bounded by the indicated curves are for the strict stationarity and for $E|\varepsilon_t|^{2\iota} < \infty$ with $\iota = 0.05, 0.5, 1, 1.5$ and 2 , respectively.

very close to the region for strict stationarity of ε_t , and is much bigger than the region for $E\varepsilon_t^4 < \infty$.

Under Assumption 2.4, we only need a fractional moment condition for the asymptotic property of $\hat{\theta}_{sn}$ as follows:

ASSUMPTION 2.5. $E|\varepsilon_t|^{2\iota} < \infty$ for some $\iota > 0$.

The sufficient and necessary condition of Assumption 2.5 is given in Theorem 2.1 of Ling (2007). In practice, we can use Hill’s estimator to estimate the tail index of $\{y_t\}$ and its estimator may provide some useful guidelines for the choice of ι . For instance, the quantity 2ι can be any value less than the tail index $\{y_t\}$. However, so far we do not know how to choose the optimal ι . As in Ling (2007) and Pan, Wang and Yao (2007), we choose the weight function w_t according to ι . When $\iota = 1/2$ (i.e., $E|\varepsilon_t| < \infty$), we can choose the weight function as

$$(2.4) \quad w_t = \left(\max \left\{ 1, C^{-1} \sum_{k=1}^{\infty} \frac{1}{k^9} |y_{t-k}| I\{|y_{t-k}| > C\} \right\} \right)^{-4},$$

where $C > 0$ is a constant. In practice, it works well when we select C as the 90% quantile of data $\{y_1, \dots, y_n\}$. When $q = s = 0$ (AR-ARCH model), for any $\iota > 0$, the weight can be selected as

$$w_t = \left(\max \left\{ 1, C^{-1} \sum_{k=1}^{p+r} \frac{1}{k^9} |y_{t-k}| I\{|y_{t-k}| > C\} \right\} \right)^{-4}.$$

When $\iota \in (0, 1/2)$ and $q > 0$ or $s > 0$, the weight function need to be modified as follows:

$$w_t = \left(\max \left\{ 1, C^{-1} \sum_{k=1}^{\infty} \frac{1}{k^{1+8/\iota}} |y_{t-k}| I\{|y_{t-k}| > C\} \right\} \right)^{-4}.$$

Obviously, these weight functions satisfy Assumptions 2.4 and 2.7. For more choices of w_t , we refer to Ling (2005) and Pan, Wang and Yao (2007). We first state the strong convergence of $\hat{\theta}_{sn}$ in the following theorem and its proof is given in the Appendix.

THEOREM 2.1. *Suppose η_t has a median zero with $E|\eta_t| = 1$. If Assumptions 2.1–2.5 hold, then*

$$\hat{\theta}_{sn} \rightarrow \theta_0 \quad a.s., \text{ as } n \rightarrow \infty.$$

To study the rate of convergence of $\hat{\theta}_{sn}$, we reparameterize the weighted log-likelihood function (2.3) as follows:

$$L_n(u) \equiv nL_{sn}(\theta_0 + u) - nL_{sn}(\theta_0),$$

where $u \in \Lambda \equiv \{u = (u'_1, u'_2)' : u + \theta_0 \in \Theta\}$. Let $\hat{u}_n = \hat{\theta}_{sn} - \theta_0$. Then, \hat{u}_n is the minimizer of $L_n(u)$ on Λ . Furthermore, we have

$$(2.5) \quad L_n(u) = \sum_{t=1}^n w_t A_t(u) + \sum_{t=1}^n w_t B_t(u) + \sum_{t=1}^n w_t C_t(u),$$

where

$$A_t(u) = \frac{1}{\sqrt{h_t(\theta_0)}} [|\varepsilon_t(\gamma_0 + u_1)| - |\varepsilon_t(\gamma_0)|],$$

$$B_t(u) = \log \sqrt{h_t(\theta_0 + u)} - \log \sqrt{h_t(\theta_0)} + \frac{|\varepsilon_t(\gamma_0)|}{\sqrt{h_t(\theta_0 + u)}} - \frac{|\varepsilon_t(\gamma_0)|}{\sqrt{h_t(\theta_0)}},$$

$$C_t(u) = \left[\frac{1}{\sqrt{h_t(\theta_0 + u)}} - \frac{1}{\sqrt{h_t(\theta_0)}} \right] [|\varepsilon_t(\gamma_0 + u_1)| - |\varepsilon_t(\gamma_0)|].$$

Let $I(\cdot)$ be the indicator function. Using the identity

$$(2.6) \quad |x - y| - |x| = -y[I(x > 0) - I(x < 0)] + 2 \int_0^y [I(x \leq s) - I(x \leq 0)] ds$$

for $x \neq 0$, we can show that

$$(2.7) \quad A_t(u) = q_t(u)[I(\eta_t > 0) - I(\eta_t < 0)] + 2 \int_0^{-q_t(u)} X_t(s) ds,$$

where $X_t(s) = I(\eta_t \leq s) - I(\eta_t \leq 0)$, $q_t(u) = q_{1t}(u) + q_{2t}(u)$ with

$$q_{1t}(u) = \frac{u'}{\sqrt{h_t(\theta_0)}} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \quad \text{and} \quad q_{2t}(u) = \frac{u'}{2\sqrt{h_t(\theta_0)}} \frac{\partial^2 \varepsilon_t(\xi^*)}{\partial \theta \partial \theta'} u,$$

and ξ^* lies between γ_0 and $\gamma_0 + u_1$. Moreover, let $\mathcal{F}_t = \sigma\{\eta_k : k \leq t\}$ and

$$\xi_t(u) = 2w_t \int_0^{-q_{1t}(u)} X_t(s) ds.$$

Then, from (2.7), we have

$$(2.8) \quad \sum_{t=1}^n w_t A_t(u) = u' T_{1n} + \Pi_{1n}(u) + \Pi_{2n}(u) + \Pi_{3n}(u),$$

where

$$T_{1n} = \sum_{t=1}^n \frac{w_t}{\sqrt{h_t(\theta_0)}} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} [I(\eta_t > 0) - I(\eta_t < 0)],$$

$$\Pi_{1n}(u) = \sum_{t=1}^n \{\xi_t(u) - E[\xi_t(u) | \mathcal{F}_{t-1}]\},$$

$$\begin{aligned} \Pi_{2n}(u) &= \sum_{t=1}^n E[\xi_t(u)|\mathcal{F}_{t-1}], \\ \Pi_{3n}(u) &= \sum_{t=1}^n w_t q_{2t}(u)[I(\eta_t > 0) - I(\eta_t < 0)] \\ &\quad + 2 \sum_{t=1}^n w_t \int_{-q_{1t}(u)}^{-q_{2t}(u)} X_t(s) ds. \end{aligned}$$

By Taylor’s expansion, we can see that

$$(2.9) \quad \sum_{t=1}^n w_t B_t(u) = u' T_{2n} + \Pi_{4n}(u) + \Pi_{5n}(u),$$

where

$$\begin{aligned} T_{2n} &= \sum_{t=1}^n \frac{w_t}{2h_t(\theta_0)} \frac{\partial h_t(\theta_0)}{\partial \theta} (1 - |\eta_t|), \\ \Pi_{4n}(u) &= u' \sum_{t=1}^n w_t \left(\frac{3}{8} \left| \frac{\varepsilon_t(\gamma_0)}{\sqrt{h_t(\zeta^*)}} \right| - \frac{1}{4} \right) \frac{1}{h_t^2(\zeta^*)} \frac{\partial h_t(\zeta^*)}{\partial \theta} \frac{\partial h_t(\zeta^*)}{\partial \theta'} u, \\ \Pi_{5n}(u) &= u' \sum_{t=1}^n w_t \left(\frac{1}{4} - \frac{1}{4} \left| \frac{\varepsilon_t(\gamma_0)}{\sqrt{h_t(\zeta^*)}} \right| \right) \frac{1}{h_t(\zeta^*)} \frac{\partial^2 h_t(\zeta^*)}{\partial \theta \partial \theta'} u, \end{aligned}$$

and ζ^* lies between θ_0 and $\theta_0 + u$.

We further need one assumption and three lemmas. The first lemma is directly from the central limit theorem for a martingale difference sequence. The second- and third-lemmas give the expansions of $\Pi_{in}(u)$ for $i = 1, \dots, 5$ and $\sum_{t=1}^n C_t(u)$. The key technical argument is for the second lemma for which we use the bracketing method in Pollard (1985).

ASSUMPTION 2.6. η_t has zero median with $E|\eta_t| = 1$ and a continuous density function $g(x)$ satisfying $g(0) > 0$ and $\sup_{x \in R} g(x) < \infty$.

LEMMA 2.1. Let $T_n = T_{1n} + T_{2n}$. If Assumptions 2.1–2.6 hold, then

$$\frac{1}{\sqrt{n}} T_n \rightarrow_d N(0, \Omega_0) \quad \text{as } n \rightarrow \infty,$$

where \rightarrow_d denotes the convergence in distribution and

$$\Omega_0 = E \left(\frac{w_t^2}{h_t(\theta_0)} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta'} \right) + \frac{E \eta_t^2 - 1}{4} E \left(\frac{w_t^2}{h_t^2(\theta_0)} \frac{\partial h_t(\theta_0)}{\partial \theta} \frac{\partial h_t(\theta_0)}{\partial \theta'} \right).$$

LEMMA 2.2. *If Assumptions 2.1–2.6 hold, then for any sequence of random variables u_n such that $u_n = o_p(1)$, it follows that*

$$\Pi_{1n}(u_n) = o_p(\sqrt{n}\|u_n\| + n\|u_n\|^2),$$

where $o_p(\cdot) \rightarrow 0$ in probability as $n \rightarrow \infty$.

LEMMA 2.3. *If Assumptions 2.1–2.6 hold, then for any sequence of random variables u_n such that $u_n = o_p(1)$, it follows that:*

- (i) $\Pi_{2n}(u_n) = (\sqrt{n}u_n)' \Sigma_1(\sqrt{n}u_n) + o_p(n\|u_n\|^2),$
- (ii) $\Pi_{3n}(u_n) = o_p(n\|u_n\|^2),$
- (iii) $\Pi_{4n}(u_n) = (\sqrt{n}u_n)' \Sigma_2(\sqrt{n}u_n) + o_p(n\|u_n\|^2),$
- (iv) $\Pi_{5n}(u_n) = o_p(n\|u_n\|^2),$
- (v) $\sum_{t=1}^n C_t(u_n) = o_p(n\|u_n\|^2),$

where

$$\Sigma_1 = g(0)E\left(\frac{w_t}{h_t(\theta_0)} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta'}\right)$$

and

$$\Sigma_2 = \frac{1}{8}E\left(\frac{w_t}{h_t^2(\theta_0)} \frac{\partial h_t(\theta_0)}{\partial \theta} \frac{\partial h_t(\theta_0)}{\partial \theta'}\right).$$

The proofs of Lemmas 2.2 and 2.3 are given in Section 6. We now can state one main result as follows:

THEOREM 2.2. *If Assumptions 2.1–2.6 hold, then:*

- (i) $\sqrt{n}(\hat{\theta}_{sn} - \theta_0) = O_p(1),$
- (ii) $\sqrt{n}(\hat{\theta}_{sn} - \theta_0) \rightarrow_d N(0, \frac{1}{4}\Sigma_0^{-1}\Omega_0\Sigma_0^{-1}) \quad \text{as } n \rightarrow \infty,$

where $\Sigma_0 = \Sigma_1 + \Sigma_2$.

PROOF. (i) First, we have $\hat{u}_n = o_p(1)$ by Theorem 2.1. Furthermore, by (2.5), (2.8) and (2.9) and Lemmas 2.2 and 2.3, we have

$$(2.10) \quad L_n(\hat{u}_n) = \hat{u}_n' T_n + (\sqrt{n}\hat{u}_n)' \Sigma_0(\sqrt{n}\hat{u}_n) + o_p(\sqrt{n}\|\hat{u}_n\| + n\|\hat{u}_n\|^2).$$

Let $\lambda_{\min} > 0$ be the minimum eigenvalue of Σ_0 . Then

$$L_n(\hat{u}_n) \geq -\|\sqrt{n}\hat{u}_n\| \left[\left\| \frac{1}{\sqrt{n}} T_n \right\| + o_p(1) \right] + n\|\hat{u}_n\|^2[\lambda_{\min} + o_p(1)].$$

Note that $L_n(\hat{u}_n) \leq 0$. By the previous inequality, it follows that

$$(2.11) \quad \sqrt{n}\|\hat{u}_n\| \leq [\lambda_{\min} + o_p(1)]^{-1} \left[\left\| \frac{1}{\sqrt{n}} T_n \right\| + o_p(1) \right] = O_p(1),$$

where the last step holds by Lemma 2.1. Thus, (i) holds.

(ii) Let $u_n^* = -\Sigma_0^{-1} T_n/2n$. Then, by Lemma 2.1, we have

$$\sqrt{n}u_n^* \rightarrow_d N\left(0, \frac{1}{4}\Sigma_0^{-1}\Omega_0\Sigma_0^{-1}\right) \quad \text{as } n \rightarrow \infty.$$

Hence, it is sufficient to show that $\sqrt{n}\hat{u}_n - \sqrt{n}u_n^* = o_p(1)$. By (2.10) and (2.11), we have

$$\begin{aligned} L_n(\hat{u}_n) &= (\sqrt{n}\hat{u}_n)' \frac{1}{\sqrt{n}} T_n + (\sqrt{n}\hat{u}_n)' \Sigma_0(\sqrt{n}\hat{u}_n) + o_p(1) \\ &= (\sqrt{n}\hat{u}_n)' \Sigma_0(\sqrt{n}\hat{u}_n) - 2(\sqrt{n}\hat{u}_n)' \Sigma_0(\sqrt{n}u_n^*) + o_p(1). \end{aligned}$$

Note that (2.10) still holds when \hat{u}_n is replaced by u_n^* . Thus,

$$\begin{aligned} L_n(u_n^*) &= (\sqrt{n}u_n^*)' \frac{1}{\sqrt{n}} T_n + (\sqrt{n}u_n^*)' \Sigma_0(\sqrt{n}u_n^*) + o_p(1) \\ &= -(\sqrt{n}u_n^*)' \Sigma_0(\sqrt{n}u_n^*) + o_p(1). \end{aligned}$$

By the previous two equations, it follows that

$$(2.12) \quad \begin{aligned} L_n(\hat{u}_n) - L_n(u_n^*) &= (\sqrt{n}\hat{u}_n - \sqrt{n}u_n^*)' \Sigma_0(\sqrt{n}\hat{u}_n - \sqrt{n}u_n^*) + o_p(1) \\ &\geq \lambda_{\min} \|\sqrt{n}\hat{u}_n - \sqrt{n}u_n^*\|^2 + o_p(1). \end{aligned}$$

Since $L_n(\hat{u}_n) - L_n(u_n^*) = n[L_{sn}(\theta_0 + \hat{u}_n) - L_{sn}(\theta_0 + u_n^*)] \leq 0$ a.s., by (2.12), we have $\|\sqrt{n}\hat{u}_n - \sqrt{n}u_n^*\| = o_p(1)$. This completes the proof. \square

REMARK 2.1. When $w_t \equiv 1$, the limiting distribution in Theorem 2.2 is the same as that in Li and Li (2008). When $r = s = 0$ (ARMA model), it reduces to the case in Pan, Wang and Yao (2007) and Zhu and Ling (2011b). In general, it is not easy to compare the asymptotic efficiency of the self-weighted QMELE and the self-weight QMLE in Ling (2007). However, for the pure ARCH model, a formal comparison of these two estimators is given in Section 3. For the general ARMA–GARCH model, a comparison based on simulation is given in Section 4.

In practice, the initial values Y_0 are unknown, and have to be replaced by some constants. Let $\tilde{\varepsilon}_t(\theta)$, $\tilde{h}_t(\theta)$ and \tilde{w}_t be $\varepsilon_t(\theta)$, $h_t(\theta)$ and w_t , respectively, when Y_0 are constants not depending on parameters. Usually, Y_0 are taken to be zeros. The objective function (2.3) is modified as

$$\tilde{L}_{sn}(\theta) = \frac{1}{n} \sum_{t=1}^n \tilde{w}_t \tilde{l}_t(\theta) \quad \text{and} \quad \tilde{l}_t(\theta) = \log \sqrt{\tilde{h}_t(\theta)} + \frac{|\tilde{\varepsilon}_t(\gamma)|}{\sqrt{\tilde{h}_t(\theta)}}.$$

To make the initial values Y_0 ignorable, we need the following assumption.

ASSUMPTION 2.7. $E|w_t - \tilde{w}_t|^{t_0/4} = O(t^{-2})$, where $t_0 = \min\{t, 1\}$.

Let $\tilde{\theta}_{sn}$ be the minimizer of $\tilde{L}_{sn}(\theta)$, that is,

$$\tilde{\theta}_{sn} = \arg \min_{\Theta} \tilde{L}_{sn}(\theta).$$

Theorem 2.3 below shows that $\tilde{\theta}_{sn}$ and $\hat{\theta}_{sn}$ have the same limiting property. Its proof is straightforward and can be found in [Zhu \(2011\)](#).

THEOREM 2.3. *Suppose that Assumption 2.7 holds. Then, as $n \rightarrow \infty$,*

(i) *if the assumptions of Theorem 2.1 hold*

$$\tilde{\theta}_{sn} \rightarrow \theta_0 \quad \text{a.s.},$$

(ii) *if the assumptions of Theorem 2.2 hold*

$$\sqrt{n}(\tilde{\theta}_{sn} - \theta_0) \rightarrow_d N(0, \frac{1}{4} \Sigma_0^{-1} \Omega_0 \Sigma_0^{-1}).$$

3. Local QMELE. The self-weighted QMELE in Section 2 reduces the moment condition of ε_t , but it may not be efficient. In this section, we propose a local QMELE based on the self-weighted QMELE and derive its asymptotic property. For some special cases, a formal comparison of the local QMELE and the self-weighted QMELE is given.

Using $\hat{\theta}_{sn}$ in Theorem 2.2 as an initial estimator of θ_0 , we obtain the local QMELE $\hat{\theta}_n$ through the following one-step iteration:

$$(3.1) \quad \hat{\theta}_n = \hat{\theta}_{sn} - [2\Sigma_n^*(\hat{\theta}_{sn})]^{-1} T_n^*(\hat{\theta}_{sn}),$$

where

$$\Sigma_n^*(\theta) = \sum_{t=1}^n \left\{ \frac{g(0)}{h_t(\theta)} \frac{\partial \varepsilon_t(\gamma)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma)}{\partial \theta'} + \frac{1}{8h_t^2(\theta)} \frac{\partial h_t(\theta)}{\partial \theta} \frac{\partial h_t(\theta)}{\partial \theta'} \right\},$$

$$T_n^*(\theta) = \sum_{t=1}^n \left\{ \frac{1}{\sqrt{h_t(\theta)}} \frac{\partial \varepsilon_t(\gamma)}{\partial \theta} [I(\eta_t(\theta) > 0) - I(\eta_t(\theta) < 0)] \right. \\ \left. + \frac{1}{2h_t(\theta)} \frac{\partial h_t(\theta)}{\partial \theta} (1 - |\eta_t(\theta)|) \right\}.$$

In order to get the asymptotic normality of $\hat{\theta}_n$, we need one more assumption as follows:

ASSUMPTION 3.1. $E\eta_t^2 \sum_{i=1}^r \alpha_{0i} + \sum_{i=1}^s \beta_{0i} < 1$ or

$$E\eta_t^2 \sum_{i=1}^r \alpha_{0i} + \sum_{i=1}^s \beta_{0i} = 1$$

with η_t having a positive density on R such that $E|\eta_t|^\tau < \infty$ for all $\tau < \tau_0$ and $E|\eta_t|^{\tau_0} = \infty$ for some $\tau_0 \in (0, \infty]$.

Under Assumption 3.1, there exists a unique strictly stationary causal solution to GARCH model (1.2); see Bougerol and Picard (1992) and Basrak, Davis and Mikosch (2002). The condition $E\eta_t^2 \sum_{i=1}^r \alpha_{0i} + \sum_{i=1}^s \beta_{0i} < 1$ is necessary and sufficient for $E\varepsilon_t^2 < \infty$ under which model (1.2) has a finite variance. When $E\eta_t^2 \sum_{i=1}^r \alpha_{0i} + \sum_{i=1}^s \beta_{0i} = 1$, model (1.2) is called IGARCH model. The IGARCH model has an infinite variance, but $E|\varepsilon_t|^{2\iota} < \infty$ for all $\iota \in (0, 1)$ under Assumption 3.1; see Ling (2007). Assumption 3.1 is crucial for the ARMA–IGARCH model. From Figure 2 in Section 2, we can see that the parameter region specified in Assumption 3.1 is much bigger than that for $E|\varepsilon_t|^3 < \infty$ which is required for the asymptotic normality of the global QMELE. Now, we give one lemma as follows and its proof is straightforward and can be found in Zhu (2011).

LEMMA 3.1. *If Assumptions 2.1–2.3, 2.6 and 3.1 hold, then for any sequence of random variables θ_n such that $\sqrt{n}(\theta_n - \theta_0) = O_p(1)$, it follows that:*

- (i) $\frac{1}{n}[T_n^*(\theta_n) - T_n^*(\theta_0)] = [2\Sigma + o_p(1)](\theta_n - \theta_0) + o_p\left(\frac{1}{\sqrt{n}}\right)$,
- (ii) $\frac{1}{n}\Sigma_n^*(\theta_n) = \Sigma + o_p(1)$,
- (iii) $\frac{1}{\sqrt{n}}T_n^*(\theta_0) \rightarrow_d N(0, \Omega) \quad \text{as } n \rightarrow \infty$,

where

$$\Omega = E\left(\frac{1}{h_t(\theta_0)} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta'}\right) + \frac{E\eta_t^2 - 1}{4} E\left(\frac{1}{h_t^2(\theta_0)} \frac{\partial h_t(\theta_0)}{\partial \theta} \frac{\partial h_t(\theta_0)}{\partial \theta'}\right),$$

$$\Sigma = g(0)E\left(\frac{1}{h_t(\theta_0)} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta'}\right) + \frac{1}{8}E\left(\frac{1}{h_t^2(\theta_0)} \frac{\partial h_t(\theta_0)}{\partial \theta} \frac{\partial h_t(\theta_0)}{\partial \theta'}\right).$$

THEOREM 3.1. *If the conditions in Lemma 3.1 are satisfied, then*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N\left(0, \frac{1}{4}\Sigma^{-1}\Omega\Sigma^{-1}\right) \quad \text{as } n \rightarrow \infty.$$

PROOF. Note that $\sqrt{n}(\hat{\theta}_{sn} - \theta_0) = O_p(1)$. By (3.1) and Lemma 3.1, we have that

$$\begin{aligned} \hat{\theta}_n &= \hat{\theta}_{sn} - \left[\frac{2}{n}\Sigma_n^*(\hat{\theta}_{sn})\right]^{-1} \left[\frac{1}{n}T_n^*(\hat{\theta}_{sn})\right] \\ &= \hat{\theta}_{sn} - [2\Sigma + o_p(1)]^{-1} \left\{ \frac{1}{n}T_n^*(\theta_0) + [2\Sigma + o_p(1)](\hat{\theta}_{sn} - \theta_0) + o_p\left(\frac{1}{\sqrt{n}}\right) \right\} \\ &= \theta_0 + \frac{\Sigma^{-1}T_n^*(\theta_0)}{2n} + o_p\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

It follows that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{\Sigma^{-1}T_n^*(\theta_0)}{2\sqrt{n}} + o_p(1).$$

By Lemma 3.1(iii), we can see that the conclusion holds. This completes the proof. \square

REMARK 3.1. In practice, by using $\tilde{\theta}_{sn}$ in Theorem 2.3 as an initial estimator of θ_0 , the local QMELE has to be modified as follows:

$$\hat{\theta}_n = \tilde{\theta}_{sn} - [2\tilde{\Sigma}_n^*(\tilde{\theta}_{sn})]^{-1}\tilde{T}_n^*(\tilde{\theta}_{sn}),$$

where $\tilde{\Sigma}_n^*(\theta)$ and $\tilde{T}_n^*(\theta)$ are defined in the same way as $\Sigma_n^*(\theta)$ and $T_n^*(\theta)$, respectively, with $\varepsilon_t(\theta)$ and $h_t(\theta)$ being replaced by $\tilde{\varepsilon}_t(\theta)$ and $\tilde{h}_t(\theta)$. However, this does not affect the asymptotic property of $\hat{\theta}_n$; see Theorem 4.3.2 in Zhu (2011).

We now compare the asymptotic efficiency of the local QMELE and the self-weighted QMELE. First, we consider the pure ARMA model, that is, model (1.1)–(1.2) with h_t being a constant. In this case,

$$\begin{aligned} \Omega_0 &= E(w_t^2 X_{1t} X'_{1t}), & \Sigma_0 &= g(0)E(w_t X_{1t} X'_{1t}), \\ \Omega &= E(X_{1t} X'_{1t}) & \text{and } \Sigma &= g(0)\Omega, \end{aligned}$$

where $X_{1t} = h_t^{-1/2}\partial\varepsilon_t(\gamma_0)/\partial\theta$. Let b and c be two any m -dimensional constant vectors. Then,

$$\begin{aligned} c'\Sigma_0 b b'\Sigma_0 c &= \{E[(c'\sqrt{g(0)}w_t X_{1t})(\sqrt{g(0)}X'_{1t}b)]\}^2 \\ &\leq E(c'\sqrt{g(0)}w_t X_{1t})^2 E(\sqrt{g(0)}X'_{1t}b)^2 \\ &= [c'g(0)\Omega_0 c][b'\Sigma b] = c'[g(0)\Omega_0 b'\Sigma b]c. \end{aligned}$$

Thus, $g(0)\Omega_0 b'\Sigma b' - \Sigma_0 b b'\Sigma_0 \geq 0$ (a positive semi-definite matrix) and hence $b'\Sigma_0 \Omega_0^{-1} \Sigma_0 b = \text{tr}(\Omega_0^{-1/2} \Sigma_0 b b' \Sigma_0 \Omega_0^{-1/2}) \leq \text{tr}(g(0)b'\Sigma b) = g(0)b'\Sigma b$. It follows that $\Sigma_0^{-1} \Omega_0 \Sigma_0^{-1} \geq [g(0)\Sigma]^{-1} = \Sigma^{-1} \Omega \Sigma^{-1}$. Thus, the local QMELE is more efficient than the self-weighted QMELE. Similarly, we can show that the local QMELE is more efficient than the self-weighted QMELE for the pure GARCH model.

For the general model (1.1)–(1.2), it is not easy to compare the asymptotic efficiency of the self-weighted QMELE and the local QMELE. However, when $\eta_t \sim \text{Laplace}(0, 1)$, we have

$$\begin{aligned} \Sigma_0 &= E\left(\frac{w_t}{2} X_{1t} X'_{1t} + \frac{w_t}{8} X_{2t} X'_{2t}\right), \\ \Omega_0 &= E\left(w_t^2 X_{1t} X'_{1t} + \frac{w_t^2}{4} X_{2t} X'_{2t}\right), \\ \Sigma &= E\left(\frac{1}{2} X_{1t} X'_{1t} + \frac{1}{8} X_{2t} X'_{2t}\right) \quad \text{and} \quad \Omega = 2\Sigma, \end{aligned}$$

where $X_{2t} = h_t^{-1} \partial h_t(\theta_0) / \partial \theta$. Then, it is easy to see that

$$\begin{aligned} & c' \Sigma_0 b b' \Sigma_0 c \\ &= \{E[(c' 2^{-1/4} w_t X_{1t})(2^{-3/4} X'_{1t} b) + (c' 2^{-5/4} w_t X_{2t})(2^{-7/4} X'_{2t} b)]\}^2 \\ &\leq \left\{ \sqrt{E(c' 2^{-1/4} w_t X_{1t})^2 E(2^{-3/4} X'_{1t} b)^2} \right. \\ &\quad \left. + \sqrt{E(c' 2^{-5/4} w_t X_{2t})^2 E(2^{-7/4} X'_{2t} b)^2} \right\}^2 \\ &\leq [E(c' 2^{-1/4} w_t X_{1t})^2 + E(c' 2^{-5/4} w_t X_{2t})^2] \\ &\quad \times [E(2^{-3/4} X'_{1t} b)^2 + E(2^{-7/4} X'_{2t} b)^2] \\ &= [c' 2^{-1/2} \Omega_0 c][b' 2^{-1/2} \Sigma b] = c'[2^{-1} \Omega_0 b' \Sigma b]c. \end{aligned}$$

Thus, $2^{-1} \Omega_0 b' \Sigma b' - \Sigma_0 b b' \Sigma_0 \geq 0$ and hence $b' \Sigma_0 \Omega_0^{-1} \Sigma_0 b = \text{tr}(\Omega_0^{-1/2} \Sigma_0 b b' \times \Sigma_0 \Omega_0^{-1/2}) \leq \text{tr}(2^{-1} b' \Sigma b) = 2^{-1} b' \Sigma b$. It follows that $\Sigma_0^{-1} \Omega_0 \Sigma_0^{-1} \geq 2 \Sigma^{-1} = \Sigma^{-1} \Omega \Sigma^{-1}$. Thus, the local QMELE is more efficient than the global self-weighted QMELE.

In the end, we compare the asymptotic efficiency of the self-weighted QMELE and the self-weighted QMLE in Ling (2007) for the pure ARCH model, when $E \eta_t^4 < \infty$. We reparametrize model (1.2) when $s = 0$ as follows:

$$(3.2) \quad y_t = \eta_t^* \sqrt{h_t^*} \quad \text{and} \quad h_t^* = \alpha_0^* + \sum_{i=1}^r \alpha_i^* y_{t-i}^2,$$

where $\eta_t^* = \eta_t / \sqrt{E \eta_t^2}$, $h_t^* = (E \eta_t^2) h_t$ and $\theta^* = (\alpha_0^*, \alpha_1^*, \dots, \alpha_r^*)' = (E \eta_t^2) \theta$. Let $\tilde{\theta}_{sn}^*$ be the self-weighted QMLE of the true parameter, θ_0^* , in model (3.2). Then, $\tilde{\theta}_{sn} = \tilde{\theta}_{sn}^* / E \eta_t^2$ is the self-weighted QMLE of θ_0 , and its asymptotic covariance is

$$\Gamma_1 = \kappa_1 [E(w_t X_{2t} X'_{2t})]^{-1} E(w_t^2 X_{2t} X'_{2t}) [E(w_t X_{2t} X'_{2t})]^{-1},$$

where $\kappa_1 = E \eta_t^4 / (E \eta_t^2)^2 - 1$. By Theorem 2.2, the asymptotic variance of the self-weighted QMELE is

$$\Gamma_2 = \kappa_2 [E(w_t X_{2t} X'_{2t})]^{-1} E(w_t^2 X_{2t} X'_{2t}) [E(w_t X_{2t} X'_{2t})]^{-1},$$

where $\kappa_2 = 4(E \eta_t^2 - 1)$. When $\eta_t \sim \text{Laplace}(0, 1)$, $\kappa_1 = 5$ and $\kappa_2 = 4$. Thus, $\Gamma_1 > \Gamma_2$, meaning that the self-weighted QMELE is more efficient than the self-weighted QMLE. When $\eta_t = \tilde{\eta}_t / E|\tilde{\eta}_t|$, with $\tilde{\eta}_t$ having the following mixing normal density:

$$f(x) = (1 - \varepsilon) \phi(x) + \frac{\varepsilon}{\tau} \phi\left(\frac{x}{\tau}\right),$$

we have $E|\eta_t| = 1$,

$$E \eta_t^2 = \frac{\pi(1 - \varepsilon + \varepsilon \tau^2)}{2(1 - \varepsilon + \varepsilon \tau)}$$

and

$$E\eta_t^4 = \frac{3\pi(1 - \varepsilon + \varepsilon\tau^4)}{2(1 - \varepsilon + \varepsilon\tau)^2(1 - \varepsilon + \varepsilon\tau^2)},$$

where $\phi(x)$ is the pdf of standard normal, $0 \leq \varepsilon \leq 1$ and $\tau > 0$. The asymptotic efficiencies of the self-weighted QMELE and the self-weighted QMLE depend on ε and τ . For example, when $\varepsilon = 1$ and $\tau = \sqrt{\pi/2}$, we have $\kappa_1 = (6 - \pi)/\pi$ and $\kappa_2 = 2\pi - 4$, and hence the self-weighted QMLE is more efficient than the self-weighted QMELE since $\Gamma_1 < \Gamma_2$. When $\varepsilon = 0.99$ and $\tau = 0.1$, we have $\kappa_1 = 28.1$ and $\kappa_2 = 6.5$, and hence the self-weighted QMELE is more efficient than the self-weighted QMLE since $\Gamma_1 > \Gamma_2$.

4. Simulation. In this section, we compare the performance of the global self-weighted QMELE ($\hat{\theta}_{sn}$), the global self-weighted QMLE ($\bar{\theta}_{sn}$), the local QMELE ($\hat{\theta}_n$) and the local QMLE ($\bar{\theta}_n$). The following AR(1)–GARCH(1, 1) model is used to generate data samples:

$$(4.1) \quad \begin{aligned} y_t &= \mu + \phi_1 y_{t-1} + \varepsilon_t, \\ \varepsilon_t &= \eta_t \sqrt{h_t} \quad \text{and} \quad h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}. \end{aligned}$$

We set the sample size $n = 1,000$ and use 1,000 replications, and study the cases when η_t has Laplace(0, 1), $N(0, 1)$ and t_3 distribution. For the case with $E\varepsilon_t^2 < \infty$ (i.e., $E\eta_t^2\alpha_{01} + \beta_{01} < 1$), we take $\theta_0 = (0.0, 0.5, 0.1, 0.18, 0.4)$. For the IGARCH case (i.e., $E\eta_t^2\alpha_{01} + \beta_{01} = 1$), we take $\theta_0 = (0.0, 0.5, 0.1, 0.3, 0.4)$ when $\eta_t \sim \text{Laplace}(0, 1)$, $\theta_0 = (0.0, 0.5, 0.1, 0.6, 0.4)$ when $\eta_t \sim N(0, 1)$ and $\theta_0 = (0.0, 0.5, 0.1, 0.2, 0.4)$ when $\eta_t \sim t_3$. We standardize the distribution of η_t to ensure that $E|\eta_t| = 1$ for the QMELE. Tables 1–3 list the sample biases, the sample standard deviations (SD) and the asymptotic standard deviations (AD) of $\hat{\theta}_{sn}, \bar{\theta}_{sn}, \hat{\theta}_n$ and $\bar{\theta}_n$. We choose w_t as in (2.4) with C being 90% quantile of $\{y_1, \dots, y_n\}$ and $y_i \equiv 0$ for $i \leq 0$. The ADs in Theorems 2.2 and 3.1 are estimated by $\hat{\chi}_{sn} = 1/4 \hat{\Sigma}_{sn}^{-1} \hat{\Omega}_{sn} \hat{\Sigma}_{sn}^{-1}$ and $\hat{\chi}_n = 1/4 \hat{\Sigma}_n^{-1} \hat{\Omega}_n \hat{\Sigma}_n^{-1}$, respectively, where

$$\begin{aligned} \hat{\Sigma}_{sn} &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{g(0)w_t}{h_t(\hat{\theta}_{sn})} \frac{\partial \varepsilon_t(\hat{\gamma}_{sn})}{\partial \theta} \frac{\partial \varepsilon_t(\hat{\gamma}_{sn})}{\partial \theta'} + \frac{w_t}{8h_t^2(\hat{\theta}_{sn})} \frac{\partial h_t(\hat{\theta}_{sn})}{\partial \theta} \frac{\partial h_t(\hat{\theta}_{sn})}{\partial \theta'} \right\}, \\ \hat{\Omega}_{sn} &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{w_t^2}{h_t(\hat{\theta}_{sn})} \frac{\partial \varepsilon_t(\hat{\gamma}_{sn})}{\partial \theta} \frac{\partial \varepsilon_t(\hat{\gamma}_{sn})}{\partial \theta'} + \frac{E\eta_t^2 - 1}{4} \frac{w_t^2}{h_t^2(\hat{\theta}_{sn})} \frac{\partial h_t(\hat{\theta}_{sn})}{\partial \theta} \frac{\partial h_t(\hat{\theta}_{sn})}{\partial \theta'} \right\}, \\ \hat{\Sigma}_n &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{g(0)}{h_t(\hat{\theta}_n)} \frac{\partial \varepsilon_t(\hat{\gamma}_n)}{\partial \theta} \frac{\partial \varepsilon_t(\hat{\gamma}_n)}{\partial \theta'} + \frac{1}{8h_t^2(\hat{\theta}_n)} \frac{\partial h_t(\hat{\theta}_n)}{\partial \theta} \frac{\partial h_t(\hat{\theta}_n)}{\partial \theta'} \right\}, \\ \hat{\Omega}_n &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{1}{h_t(\hat{\theta}_n)} \frac{\partial \varepsilon_t(\hat{\gamma}_n)}{\partial \theta} \frac{\partial \varepsilon_t(\hat{\gamma}_n)}{\partial \theta'} + \frac{E\eta_t^2 - 1}{4} \frac{1}{h_t^2(\hat{\theta}_n)} \frac{\partial h_t(\hat{\theta}_n)}{\partial \theta} \frac{\partial h_t(\hat{\theta}_n)}{\partial \theta'} \right\}. \end{aligned}$$

TABLE 1
Estimators for model (4.1) when $\eta_t \sim \text{Laplace}(0, 1)$

	$\theta_0 = (0.0, 0.5, 0.1, 0.18, 0.4)$					$\theta_0 = (0.0, 0.5, 0.1, 0.3, 0.4)$				
	Self-weighted QMELE ($\hat{\theta}_{sn}$)					Self-weighted QMELE ($\hat{\theta}_{sn}$)				
	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$
Bias	0.0004	-0.0023	0.0034	0.0078	-0.0154	0.0003	-0.0049	0.0031	0.0054	-0.0068
SD	0.0172	0.0317	0.0274	0.0548	0.1125	0.0195	0.0318	0.0219	0.0640	0.0673
AD	0.0166	0.0304	0.0255	0.0540	0.1061	0.0192	0.0311	0.0218	0.0624	0.0664
	Local QMELE ($\hat{\theta}_n$)					Local QMELE ($\hat{\theta}_n$)				
	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$
Bias	0.0008	-0.0019	0.0027	0.0002	-0.0094	0.0010	-0.0044	0.0024	-0.0008	-0.0025
SD	0.0170	0.0253	0.0249	0.0400	0.0989	0.0192	0.0261	0.0203	0.0502	0.0591
AD	0.0162	0.0245	0.0234	0.0407	0.0920	0.0190	0.0258	0.0206	0.0499	0.0591
	Self-weighted QMLE ($\bar{\theta}_{sn}$)					Self-weighted QMLE ($\bar{\theta}_{sn}$)				
	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$
Bias	-0.0003	-0.0016	0.0041	0.0114	-0.0227	0.0005	-0.0039	0.0031	0.0104	-0.0127
SD	0.0243	0.0451	0.0301	0.0624	0.1237	0.0283	0.0458	0.0242	0.0750	0.0755
AD	0.0240	0.0443	0.0285	0.0607	0.1184	0.0283	0.0461	0.0243	0.0704	0.0741
	Local QMLE ($\bar{\theta}_n$)					Local QMLE ($\bar{\theta}_n$)				
	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$
Bias	0.0007	-0.0034	0.0026	0.0037	-0.0144	0.0022	-0.0045	0.0020	0.0044	-0.0081
SD	0.0243	0.0368	0.0279	0.0461	0.1115	0.0282	0.0377	0.0227	0.0579	0.0674
AD	0.0236	0.0361	0.0261	0.0459	0.1026	0.0281	0.0384	0.0230	0.0564	0.0659

From Table 1, when $\eta_t \sim \text{Laplace}(0, 1)$, we can see that the self-weighted QMELE has smaller AD and SD than those of both the self-weighted QMLE and the local QMLE. When $\eta_t \sim N(0, 1)$, in Table 2, we can see that the self-weighted QMLE has smaller AD and SD than those of both the self-weighted QMELE and the local QMELE. From Table 3, we note that the SD and AD of both the self-weighted QMLE and the local QMLE are not close to each other since their asymptotic variances are infinite, while the SD and AD of the self-weighted QMELE and the local QMELE are very close to each other. Except $\bar{\theta}_n$ in Table 3, we can see that all four estimators in Tables 1–3 have very small biases, and the local QMELE and local QMLE always have the smaller SD and AD than those of the self-weighted QMELE and self-weighted QMLE, respectively. This conclusion holds no matter with GARCH errors (finite variance) or IGARCH er-

TABLE 2
Estimators for model (4.1) when $\eta_t \sim N(0, 1)$

	$\theta_0 = (0.0, 0.5, 0.1, 0.18, 0.4)$					$\theta_0 = (0.0, 0.5, 0.1, 0.6, 0.4)$				
	Self-weighted QMELE ($\hat{\theta}_{sn}$)					Self-weighted QMELE ($\hat{\theta}_{sn}$)				
	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$
Bias	0.0003	-0.0042	0.0075	0.0065	-0.0372	-0.0008	-0.0034	0.0029	-0.0019	-0.0028
SD	0.0192	0.0457	0.0366	0.0600	0.1738	0.0255	0.0437	0.0204	0.0815	0.0512
AD	0.0189	0.0443	0.0379	0.0604	0.1812	0.0257	0.0424	0.0202	0.0809	0.0491
	Local QMELE ($\hat{\theta}_n$)					Local QMELE ($\hat{\theta}_n$)				
	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$
Bias	0.0006	-0.0051	0.0061	0.0019	-0.0268	0.0000	-0.0040	0.0029	-0.0048	-0.0015
SD	0.0184	0.0372	0.0357	0.0487	0.1674	0.0252	0.0364	0.0197	0.0671	0.0472
AD	0.0183	0.0370	0.0350	0.0488	0.1652	0.0252	0.0359	0.0194	0.0685	0.0453
	Self-weighted QMLE ($\bar{\theta}_{sn}$)					Self-weighted QMLE ($\bar{\theta}_{sn}$)				
	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$
Bias	-0.0001	-0.0039	0.0069	0.0089	-0.0361	-0.0006	-0.0016	0.0024	0.0027	-0.0045
SD	0.0151	0.0366	0.0333	0.0566	0.1599	0.0196	0.0337	0.0189	0.0770	0.0481
AD	0.0150	0.0352	0.0345	0.0568	0.1658	0.0200	0.0329	0.0188	0.0757	0.0459
	Local QMLE ($\bar{\theta}_n$)					Local QMLE ($\bar{\theta}_n$)				
	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$
Bias	0.0009	-0.0048	0.0055	0.0038	-0.0252	0.0004	-0.0031	0.0024	-0.0019	-0.0027
SD	0.0145	0.0300	0.0322	0.0454	0.1535	0.0195	0.0287	0.0183	0.0633	0.0442
AD	0.0145	0.0294	0.0320	0.0460	0.1517	0.0197	0.0279	0.0181	0.0644	0.0424

rors. This coincides with what we expected. Thus, if the tail index of the data is greater than 2 but $E\eta_t^4 = \infty$, we suggest to use the local QMELE in practice; see also Ling (2007) for a discussion.

Overall, the simulation results show that the self-weighted QMELE and the local QMELE have a good performance in the finite sample, especially for the heavy-tailed innovations.

5. A real example. In this section, we study the weekly world crude oil price (dollars per barrel) from January 3, 1997 to August 6, 2010, which has in total 710 observations; see Figure 3(a). Its 100 times log-return, denoted by $\{y_t\}_{t=1}^{709}$, is plotted in Figure 3(b). The classic method based on the Akaike’s information

TABLE 3
Estimators for model (4.1) when $\eta_t \sim t_3$

	$\theta_0 = (0.0, 0.5, 0.1, 0.18, 0.4)$					$\theta_0 = (0.0, 0.5, 0.1, 0.2, 0.4)$				
	Self-weighted QMELE ($\hat{\theta}_{sn}$)					Self-weighted QMELE ($\hat{\theta}_{sn}$)				
	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$	$\hat{\mu}_{sn}$	$\hat{\phi}_{1sn}$	$\hat{\alpha}_{0sn}$	$\hat{\alpha}_{1sn}$	$\hat{\beta}_{1sn}$
Bias	0.0004	-0.0037	0.0059	0.0081	-0.0202	-0.0005	-0.0026	0.0032	0.0088	-0.0158
SD	0.0231	0.0416	0.0289	0.0600	0.1084	0.0221	0.0404	0.0252	0.0619	0.0968
AD	0.0233	0.0393	0.0282	0.0620	0.1101	0.0238	0.0393	0.0266	0.0637	0.1001
	Local QMELE ($\hat{\theta}_n$)					Local QMELE ($\hat{\theta}_n$)				
	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$	$\hat{\mu}_n$	$\hat{\phi}_{1n}$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\beta}_{1n}$
Bias	0.0011	-0.0039	0.0041	0.0011	-0.0115	0.0001	-0.0028	0.0019	0.0029	-0.0092
SD	0.0229	0.0328	0.0256	0.0429	0.0955	0.0218	0.0325	0.0226	0.0450	0.0842
AD	0.0228	0.0314	0.0252	0.0461	0.0918	0.0233	0.0317	0.0243	0.0483	0.0851
	Self-weighted QMLE ($\bar{\theta}_{sn}$)					Self-weighted QMLE ($\bar{\theta}_{sn}$)				
	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$	$\bar{\mu}_{sn}$	$\bar{\phi}_{1sn}$	$\bar{\alpha}_{0sn}$	$\bar{\alpha}_{1sn}$	$\bar{\beta}_{1sn}$
Bias	-0.0056	-0.0151	0.0029	0.0503	-0.0594	0.0036	-0.0141	0.0115	0.0442	-0.0543
SD	0.9657	0.1045	0.0868	0.2521	0.1740	0.1827	0.1065	0.3871	0.2164	0.1605
AD	0.0536	0.0907	33.031	0.1795	34.498	0.0517	0.0876	138.38	0.1875	11.302
	Local QMLE ($\bar{\theta}_n$)					Local QMLE ($\bar{\theta}_n$)				
	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$	$\bar{\mu}_n$	$\bar{\phi}_{1n}$	$\bar{\alpha}_{0n}$	$\bar{\alpha}_{1n}$	$\bar{\beta}_{1n}$
Bias	-0.0048	-0.0216	-2.1342	0.0185	3.7712	-0.0010	-0.0203	1.3241	0.0253	-0.1333
SD	0.0517	0.1080	38.535	0.3596	83.704	0.0521	0.1318	42.250	0.2524	3.4539
AD	0.0508	0.0661	55.717	0.1447	45.055	0.0520	0.0707	13.761	0.1535	1.1343

criterion (AIC) leads to the following model:

$$(5.1) \quad y_t = 0.2876\varepsilon_{t-1} + 0.1524\varepsilon_{t-3} + \varepsilon_t, \quad (0.0357) \quad (0.0357)$$

where the standard errors are in parentheses, and the estimated value of σ_ε^2 is 16.83. Model (5.1) is stationary, and none of the first ten autocorrelations or partial autocorrelations of the residuals $\{\hat{\varepsilon}_t\}$ are significant at the 5% level. However, looking at the autocorrelations of $\{\hat{\varepsilon}_t^2\}$, it turns out that the 1st, 2nd and 8th all exceed two asymptotic standard errors; see Figure 4(a). Similar results hold for the partial autocorrelations of $\{\hat{\varepsilon}_t^2\}$ in Figure 4(b). This shows that $\{\varepsilon_t^2\}$ may be highly correlated, and hence there may exist ARCH effects.

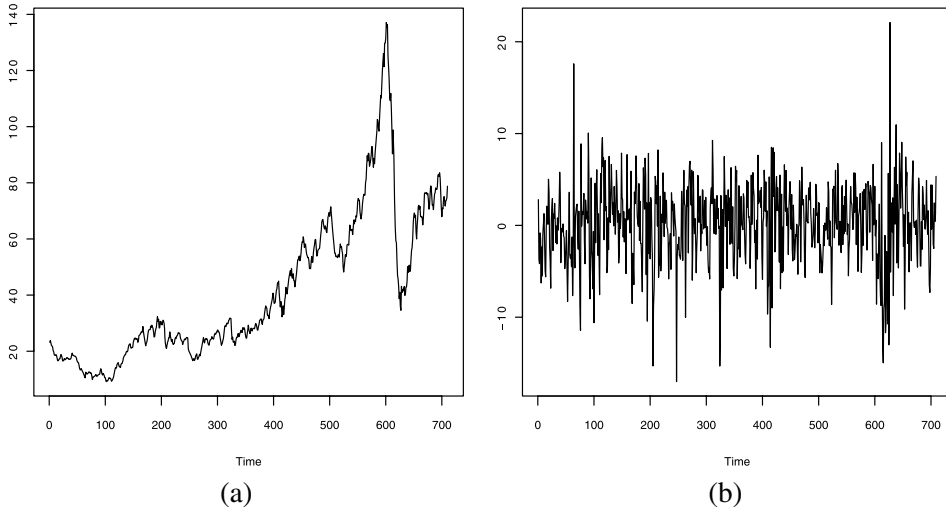


FIG. 3. (a) The weekly world crude oil prices (dollars per barrel) from January 3, 1997 to August 6, 2010 and (b) its 100 times log return.

Thus, we try to use a MA(3)–GARCH(1, 1) model to fit the data set $\{y_t\}$. To begin with our estimation, we first estimate the tail index of $\{y_t^2\}$ by using Hill’s estimator $\{\hat{\alpha}_y(k)\}$ with $k = 1, \dots, 180$, based on $\{y_t^2\}_{t=1}^{709}$. The plot of $\{\hat{\alpha}_y(k)\}_{k=1}^{180}$ is given in Figure 5, from which we can see that the tail index of $\{y_t^2\}$ is between 1 and 2, that is, $E y_t^4 = \infty$. So, the standard QMLE procedure is not suitable. Therefore, we first use the self-weighted QMELE to estimate the MA(3)–GARCH(1, 1) model, and then use the one-step iteration as in Section 3 to obtain

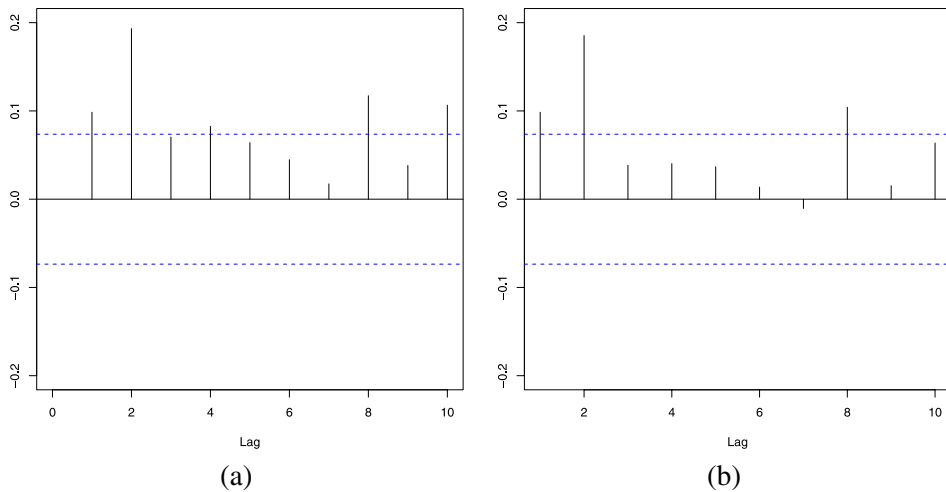


FIG. 4. (a) The autocorrelations for $\{\hat{\varepsilon}_t^2\}$ and (b) the partial autocorrelations for $\{\hat{\varepsilon}_t^2\}$.

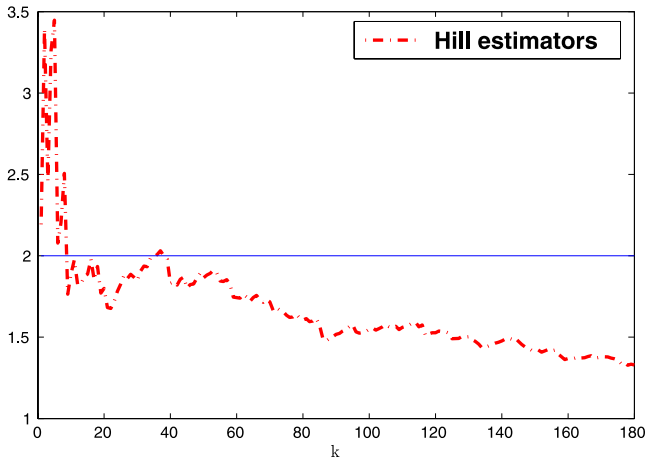


FIG. 5. Hill estimators $\{\hat{\alpha}_y(k)\}$ for $\{y_t^2\}$.

its local QMELE. The fitted model is as follows:

$$\begin{aligned}
 y_t &= 0.3276\varepsilon_{t-1} + 0.1217\varepsilon_{t-3} + \varepsilon_t, \\
 &\quad (0.0454) \quad (0.0449) \\
 (5.2) \quad h_t &= 0.5147 + 0.0435\varepsilon_{t-1}^2 + 0.8756h_{t-1}, \\
 &\quad (0.3248) \quad (0.0159) \quad (0.0530)
 \end{aligned}$$

where the standard errors are in parentheses. Again model (5.2) is stationary, and none of first ten autocorrelations or partial autocorrelations of the residuals $\hat{\eta}_t \triangleq \hat{\varepsilon}_t \hat{h}_t^{-1/2}$ are significant at the 5% level. Moreover, the first ten autocorrelations and partial autocorrelations of $\{\hat{\eta}_t^2\}$ are also within two asymptotic standard errors; see Figure 6(a) and (b). All these results suggest that model (5.2) is adequate for the data set $\{y_t\}$.

Finally, we estimate the tail index of η_t^2 in model (5.2) by using Hill’s estimator $\hat{\alpha}_\eta(k)$ with $k = 1, \dots, 180$, base on $\{\hat{\eta}_t^2\}$. The plot of $\{\hat{\alpha}_\eta(k)\}_{k=1}^{180}$ is given in Figure 7, from which we can see that $E\eta_t^2$ is most likely finite, but $E\eta_t^4$ is infinite. Furthermore, the estimator of $E\eta_t^2$ is $\sum_{t=1}^n \hat{\eta}_t^2/n = 1.6994$, and it turns out that $\hat{\alpha}_{1n}(\sum_{t=1}^n \hat{\eta}_t^2/n) + \hat{\beta}_{1n} = 0.9495$. This means that $E\varepsilon_t^2 < \infty$. Therefore, all the assumptions of Theorem 3.1 are most likely satisfied. In particular, the estimated tail indices of $\{y_t^2\}$ and $\{\hat{\eta}_t^2\}$ show the evidence that the self-weighted/local QMELE is necessary in modeling the crude oil price.

6. Proofs of Lemmas 2.2 and 2.3. In this section, we give the proofs of Lemmas 2.2 and 2.3. In the rest of this paper, we denote C as a universal constant, and $G(x)$ be the distribution function of η_t .

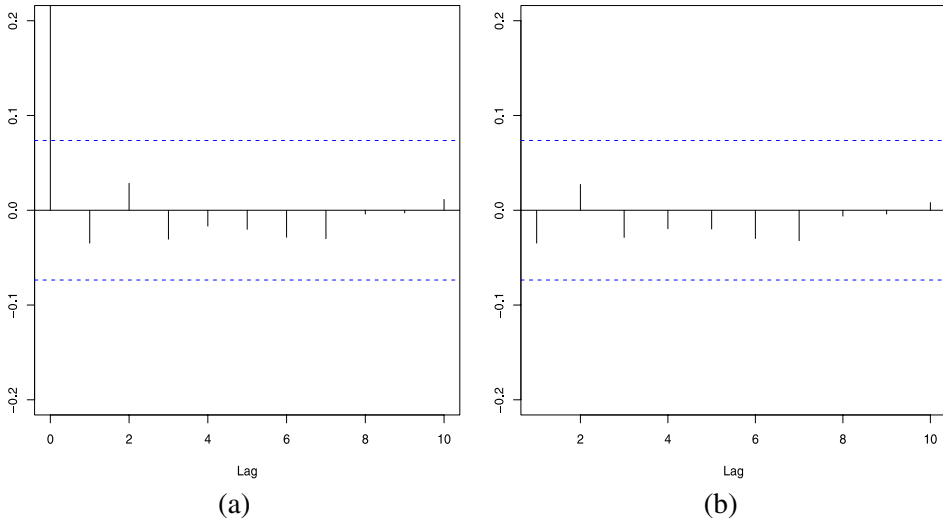


FIG. 6. (a) The autocorrelations for $\{\hat{\eta}_T^2\}$ and (b) the partial autocorrelations for $\{\hat{\eta}_T^2\}$.

PROOF OF LEMMA 2.2. A direct calculation gives

$$\xi_t(u) = -u' \frac{2w_t}{\sqrt{h_t}} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} M_t(u),$$

where $M_t(u) = \int_0^1 X_t(-q_{1t}(u)s) ds$. Thus, we have

$$|\Pi_{1n}(u)| \leq 2\|u\| \sum_{j=1}^m \left| \frac{w_t}{\sqrt{h_t}} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta_j} \sum_{t=1}^n \{M_t(u) - E[M_t(u)|\mathcal{F}_{t-1}]\} \right|.$$

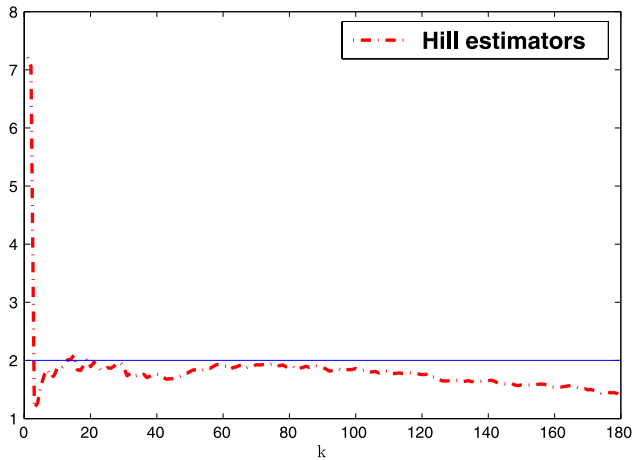


FIG. 7. The Hill estimators $\{\hat{\alpha}_\eta(k)\}$ for $\{\hat{\eta}_T^2\}$ of model (5.2).

It is sufficient to show that

$$(6.1) \quad \left| \frac{w_t}{\sqrt{h_t}} \frac{\partial \varepsilon_t(\theta_0)}{\partial \theta_j} \sum_{t=1}^n \{M_t(u_n) - E[M_t(u_n)|\mathcal{F}_{t-1}]\} \right| = o_p(\sqrt{n} + n\|u_n\|),$$

for each $1 \leq j \leq m$. Let $m_t = w_t h_t^{-1/2} \partial \varepsilon_t(\theta_0) / \partial \theta_j$, $f_t(u) = m_t M_t(u)$ and

$$D_n(u) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \{f_t(u) - E[f_t(u)|\mathcal{F}_{t-1}]\}.$$

Then, in order to prove (6.1), we only need to show that for any $\eta > 0$,

$$(6.2) \quad \sup_{\|u\| \leq \eta} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} = o_p(1).$$

Note that $m_t = \max\{m_t, 0\} - \max\{-m_t, 0\}$. To make it simple, we only prove the case when $m_t \geq 0$.

We adopt the method in Lemma 4 of Pollard (1985). Let $\mathfrak{F} = \{f_t(u) : \|u\| \leq \eta\}$ be a collection of functions indexed by u . We first verify that \mathfrak{F} satisfies the bracketing condition in Pollard (1985), page 304. Denote $B_r(\zeta)$ be an open neighborhood of ζ with radius $r > 0$. For any fix $\varepsilon > 0$ and $0 < \delta \leq \eta$, there is a sequence of small cubes $\{B_{\varepsilon\delta/C_1}(u_i)\}_{i=1}^{K_\varepsilon}$ to cover $B_\delta(0)$, where K_ε is an integer less than $c_0\varepsilon^{-m}$ and c_0 is a constant not depending on ε and δ ; see Huber (1967), page 227. Here, C_1 is a constant to be selected later. Moreover, we can choose $U_i(\delta) \subseteq B_{\varepsilon\delta/C_1}(u_i)$ such that $\{U_i(\delta)\}_{i=1}^{K_\varepsilon}$ be a partition of $B_\delta(0)$. For each $u \in U_i(\delta)$, we define the bracketing functions as follows:

$$f_t^\pm(u) = m_t \int_0^1 X_t \left(-q_{1t}(u)s \pm \frac{\varepsilon\delta}{C_1\sqrt{h_t}} \left\| \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \right\| \right) ds.$$

Since the indicator function is nondecreasing and $m_t \geq 0$, we can see that, for any $u \in U_i(\delta)$,

$$f_t^-(u_i) \leq f_t(u) \leq f_t^+(u_i).$$

Note that $\sup_{x \in R} g(x) < \infty$. It is straightforward to see that

$$(6.3) \quad E[f_t^+(u_i) - f_t^-(u_i)|\mathcal{F}_{t-1}] \leq \frac{2\varepsilon\delta}{C_1} \sup_{x \in R} g(x) \frac{w_t}{h_t} \left\| \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \right\|^2 \equiv \frac{\varepsilon\delta\Delta_t}{C_1}.$$

Setting $C_1 = E(\Delta_t)$, we have

$$E[f_t^+(u_i) - f_t^-(u_i)] = E\{E[f_t^+(u_i) - f_t^-(u_i)|\mathcal{F}_{t-1}]\} \leq \varepsilon\delta.$$

Thus, the family \mathfrak{F} satisfies the bracketing condition.

Put $\delta_k = 2^{-k}\eta$. Define $B(k) \equiv B_{\delta_k}(0)$, and $A(k)$ to be the annulus $B(k)/B(k+1)$. Fix $\varepsilon > 0$, for each $1 \leq i \leq K_\varepsilon$, by the bracketing condition, there exists a partition $\{U_i(\delta_k)\}_{i=1}^{K_\varepsilon}$ of $B(k)$.

We first consider the upper tail. For $u \in U_i(\delta_k)$, by (6.3) with $\delta = \delta_k$, we have

$$\begin{aligned} D_n(u) &\leq \frac{1}{\sqrt{n}} \sum_{t=1}^n \{f_t^+(u_i) - E[f_t^-(u_i)|\mathcal{F}_{t-1}]\} \\ &= D_n^+(u_i) + \frac{1}{\sqrt{n}} \sum_{t=1}^n E[f_t^+(u_i) - f_t^-(u_i)|\mathcal{F}_{t-1}] \\ &\leq D_n^+(u_i) + \sqrt{n}\varepsilon\delta_k \left[\frac{1}{nC_1} \sum_{t=1}^n \Delta_t \right], \end{aligned}$$

where

$$D_n^+(u_i) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \{f_t^+(u_i) - E[f_t^+(u_i)|\mathcal{F}_{t-1}]\}.$$

Denote the event

$$E_n = \left\{ \omega : \frac{1}{nC_1} \sum_{t=1}^n \Delta_t(\omega) < 2 \right\}.$$

On E_n with $u \in U_i(\delta_k)$, it follows that

$$(6.4) \quad D_n(u) \leq D_n^+(u_i) + 2\sqrt{n}\varepsilon\delta_k.$$

On $A(k)$, the divisor $1 + \sqrt{n}\|u\| > \sqrt{n}\delta_{k+1} = \sqrt{n}\delta_k/2$. Thus, by (6.4) and Cheby-
shev’s inequality, it follows that

$$\begin{aligned} &P\left(\sup_{u \in A(k)} \frac{D_n(u)}{1 + \sqrt{n}\|u\|} > 6\varepsilon, E_n \right) \\ &\leq P\left(\sup_{u \in A(k)} D_n(u) > 3\sqrt{n}\varepsilon\delta_k, E_n \right) \\ &\leq P\left(\max_{1 \leq i \leq K_\varepsilon} \sup_{u \in U_i(\delta_k) \cap A(k)} D_n(u) > 3\sqrt{n}\varepsilon\delta_k, E_n \right) \\ (6.5) \quad &\leq P\left(\max_{1 \leq i \leq K_\varepsilon} D_n^+(u_i) > \sqrt{n}\varepsilon\delta_k, E_n \right) \\ &\leq K_\varepsilon \max_{1 \leq i \leq K_\varepsilon} P(D_n^+(u_i) > \sqrt{n}\varepsilon\delta_k) \\ &\leq K_\varepsilon \max_{1 \leq i \leq K_\varepsilon} \frac{E[(D_n^+(u_i))^2]}{n\varepsilon^2\delta_k^2}. \end{aligned}$$

Note that $|q_{1t}(u_i)| \leq C\delta_k\xi_{\rho t-1}$ and $m_t^2 \leq Cw_t^2\xi_{\rho t-1}^2$ for some $\rho \in (0, 1)$ by Lem-
ma A.1(i), and $\sup_{x \in R} g(x) < \infty$ by Assumption 2.6. By Taylor’s expansion, we

have

$$\begin{aligned}
 E[(f_t^+(u_i))^2] &= E\{E[(f_t^+(u_i))^2|\mathcal{F}_{t-1}]\} \\
 &\leq E\left[m_t^2 \int_0^1 E\left[X_t\left(-q_{1t}(u_i)s + \frac{\varepsilon\delta_k}{C_1\sqrt{h_t}}\left\|\frac{\partial\varepsilon_t(\gamma_0)}{\partial\theta}\right\|\right)\right]|\mathcal{F}_{t-1}\right] ds \Big] \\
 &\leq CE\left[\sup_{|x|\leq\delta_k C\xi_{\rho t-1}} |G(x) - G(0)|w_t^2\xi_{\rho t-1}^2\right] \\
 &\leq \delta_k CE(w_t^2\xi_{\rho t-1}^3).
 \end{aligned}$$

Since $f_t^+(u_i) - E[f_t^+(u_i)|\mathcal{F}_{t-1}]$ is a martingale difference sequence, by the previous inequality, it follows that

$$\begin{aligned}
 E[(D_n^+(u_i))^2] &= \frac{1}{n} \sum_{t=1}^n E\{f_t^+(u_i) - E[f_t^+(u_i)|\mathcal{F}_{t-1}]\}^2 \\
 &\leq \frac{1}{n} \sum_{t=1}^n E[(f_t^+(u_i))^2] \\
 (6.6) \quad &\leq \frac{\delta_k}{n} \sum_{t=1}^n CE(w_t^2\xi_{\rho t-1}^3) \\
 &\equiv \pi_n(\delta_k).
 \end{aligned}$$

Thus, by (6.5) and (6.6), we have

$$P\left(\sup_{u\in A(k)} \frac{D_n(u)}{1 + \sqrt{n}\|u\|} > 6\varepsilon, E_n\right) \leq K_\varepsilon \frac{\pi_n(\delta_k)}{n\varepsilon^2\delta_k^2}.$$

By a similar argument, we can get the same bound for the lower tail. Thus, we can show that

$$(6.7) \quad P\left(\sup_{u\in A(k)} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} > 6\varepsilon, E_n\right) \leq 2K_\varepsilon \frac{\pi_n(\delta_k)}{n\varepsilon^2\delta_k^2}.$$

Since $\pi_n(\delta_k) \rightarrow 0$ as $k \rightarrow \infty$, we can choose k_ε so that

$$2\pi_n(\delta_k)K_\varepsilon/(\varepsilon\eta)^2 < \varepsilon$$

for $k \geq k_\varepsilon$. Let k_n be an integer so that $n^{-1/2} \leq 2^{-k_n} < 2n^{-1/2}$. Split $\{u : \|u\| \leq \eta\}$ into two sets $B(k_n + 1)$ and $B(k_n + 1)^c = \bigcup_{k=0}^{k_n} A(k)$. By (6.7), since $\pi_n(\delta_k)$ is bounded, we have

$$\begin{aligned}
 &P\left(\sup_{u\in B(k_n+1)^c} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} > 6\varepsilon\right) \\
 (6.8) \quad &\leq \sum_{k=0}^{k_n} P\left(\sup_{u\in A(k)} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} > 6\varepsilon, E_n\right) + P(E_n^c)
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{n} \sum_{k=0}^{k_\varepsilon-1} \frac{CK_\varepsilon}{\varepsilon^2\eta^2} 2^{2k} + \frac{\varepsilon}{n} \sum_{k=k_\varepsilon}^{k_n} 2^{2k} + P(E_n^c) \\ &\leq O\left(\frac{1}{n}\right) + 4\varepsilon \frac{2^{2k_n}}{n} + P(E_n^c) \\ &\leq O\left(\frac{1}{n}\right) + 4\varepsilon + P(E_n^c). \end{aligned}$$

Since $1 + \sqrt{n}\|u\| > 1$ and $\sqrt{n}\delta_{k_n+1} < 1$, using a similar argument as for (6.5) together with (6.6), we have

$$\begin{aligned} P\left(\sup_{u \in B(k_n+1)} \frac{D_n(u)}{1 + \sqrt{n}\|u\|} > 3\varepsilon, E_n\right) &\leq P\left(\max_{1 \leq i \leq K_\varepsilon} D_n^+(u_i) > \varepsilon, E_n\right) \\ &\leq \frac{K_\varepsilon \pi_n(\delta_{k_n+1})}{\varepsilon^2}. \end{aligned}$$

We can get the same bound for the lower tail. Thus, we have

$$\begin{aligned} &P\left(\sup_{u \in B(k_n+1)} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} > 3\varepsilon\right) \\ (6.9) \quad &= P\left(\sup_{u \in B(k_n+1)} \frac{|D_n(u)|}{1 + \sqrt{n}\|u\|} > 3\varepsilon, E_n\right) + P(E_n^c) \\ &\leq \frac{2K_\varepsilon \pi_n(\delta_{k_n+1})}{\varepsilon^2} + P(E_n^c). \end{aligned}$$

Note that $\pi_n(\delta_{k_n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $P(E_n) \rightarrow 1$ by the ergodic theorem. Hence,

$$P(E_n^c) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, (6.2) follows by (6.8) and (6.9). This completes the proof. \square

PROOF OF LEMMA 2.3. (i). By a direct calculation, we have

$$\begin{aligned} \Pi_{2n}(u) &= 2 \sum_{t=1}^n w_t \int_0^{-q_{1t}(u)} G(s) - G(0) ds \\ (6.10) \quad &= 2 \sum_{t=1}^n w_t \int_0^{-q_{1t}(u)} sg(\zeta^*) ds \\ &= (\sqrt{nu})' [K_{1n} + K_{2n}(u)] (\sqrt{nu}), \end{aligned}$$

where ζ^* lies between 0 and s , and

$$K_{1n} = \frac{g(0)}{n} \sum_{t=1}^n \frac{w_t}{h_t(\theta_0)} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta} \frac{\partial \varepsilon_t(\gamma_0)}{\partial \theta'}$$

$$K_{2n}(u) = \frac{2}{n\|u\|^2} \sum_{t=1}^n w_t \int_0^{-q_{1t}(u)} s[g(\zeta^*) - g(0)] ds.$$

By the ergodic theorem, it is easy to see that

$$(6.11) \quad K_{1n} = \Sigma_1 + o_p(1).$$

Furthermore, since $|q_{1t}(u)| \leq C\|u\|\xi_{\rho t-1}$ for some $\rho \in (0, 1)$ by Lemma A.1(i), it is straightforward to see that for any $\eta > 0$,

$$\begin{aligned} \sup_{\|u\| \leq \eta} |K_{2n}(u)| &\leq \sup_{\|u\| \leq \eta} \frac{2}{n\|u\|^2} \sum_{t=1}^n w_t \int_{-|q_{1t}(u)|}^{|q_{1t}(u)|} s|g(\zeta^*) - g(0)| ds \\ &\leq \frac{1}{n} \sum_{t=1}^n \left[\sup_{|s| \leq C\eta\xi_{\rho t-1}} |g(s) - g(0)| w_t \xi_{\rho t-1}^2 \right]. \end{aligned}$$

By Assumptions 2.4 and 2.6, $E(w_t \xi_{\rho t-1}^2) < \infty$ and $\sup_{x \in R} g(x) < \infty$. Then, by the dominated convergence theorem, we have

$$\lim_{\eta \rightarrow 0} E \left[\sup_{|s| \leq C\eta\xi_{\rho t-1}} |g(s) - g(0)| w_t \xi_{\rho t-1}^2 \right] = 0.$$

Thus, by the stationarity of $\{y_t\}$ and Markov's theorem, for $\forall \varepsilon, \delta > 0, \exists \eta_0(\varepsilon) > 0$, such that

$$(6.12) \quad P \left(\sup_{\|u\| \leq \eta_0} |K_{2n}(u)| > \delta \right) < \frac{\varepsilon}{2}$$

for all $n \geq 1$. On the other hand, since $u_n = o_p(1)$, it follows that

$$(6.13) \quad P(\|u_n\| > \eta_0) < \frac{\varepsilon}{2}$$

as n is large enough. By (6.12) and (6.13), for $\forall \varepsilon, \delta > 0$, we have

$$\begin{aligned} P(|K_{2n}(u_n)| > \delta) &\leq P(|K_{2n}(u_n)| > \delta, \|u_n\| \leq \eta_0) + P(\|u_n\| > \eta_0) \\ &< P \left(\sup_{\|u\| \leq \eta_0} |K_{2n}(u)| > \delta \right) + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

as n is large enough, that is, $K_{2n}(u_n) = o_p(1)$. Furthermore, combining (6.10) and (6.11), we can see that (i) holds.

(ii) Let $\Pi_{3n}(u) = (\sqrt{n}u)' K_{3n}(\xi^*)(\sqrt{n}u) + K_{4n}(u)$, where

$$K_{3n}(\xi^*) = \frac{1}{n} \sum_{t=1}^n \frac{w_t}{\sqrt{h_t}} \frac{\partial^2 \varepsilon_t(\xi^*)}{\partial \theta \partial \theta'} [I(\eta_t > 0) - I(\eta_t < 0)],$$

$$K_{4n}(u) = 2 \sum_{t=1}^n w_t \int_{-q_{1t}(u)}^{-q_{1t}(u)} X_t(s) ds.$$

By Assumption 2.4 and Lemma A.1(i), there exists a constant $\rho \in (0, 1)$ such that

$$E\left(\sup_{\xi^* \in \Lambda} \frac{w_t}{\sqrt{h_t}} \left| \frac{\partial^2 \varepsilon_t(\xi^*)}{\partial \theta \partial \theta'} [I(\eta_t > 0) - I(\eta_t < 0)] \right| \right) \leq CE(w_t \xi_{\rho t-1}) < \infty.$$

Since η_t has median 0, the conditional expectation property gives

$$E\left(\frac{w_t}{\sqrt{h_t}} \frac{\partial^2 \varepsilon_t(\xi^*)}{\partial \theta \partial \theta'} [I(\eta_t > 0) - I(\eta_t < 0)]\right) = 0.$$

Then, by Theorem 3.1 in Ling and McAleer (2003), we have

$$\sup_{\xi^* \in \Lambda} |K_{3n}(\xi^*)| = o_p(1).$$

On the other hand,

$$\begin{aligned} \frac{K_{4n}(u)}{n \|u\|^2} &= \frac{2}{n} \sum_{t=1}^n w_t \int_0^{-q_{2t}(u)/\|u\|^2} X_t(\|u\|^2 s - q_{1t}(u)) ds \\ &\equiv \frac{2}{n} \sum_{t=1}^n J_{1t}(u). \end{aligned}$$

By Lemma A.1, we have $\|u\|^{-2} q_{2t}(u) \leq C \xi_{\rho t-1}$ and $|q_{1t}(u)| \leq C \|u\| \xi_{\rho t-1}$ for some $\rho \in (0, 1)$. Then, for any $\eta > 0$, we have

$$\begin{aligned} \sup_{\|u\| \leq \eta} |J_{1t}(u)| &\leq w_t \int_{-C \xi_{\rho t-1}}^{C \xi_{\rho t-1}} \{X_t(C \eta^2 \xi_{\rho t-1} + C \eta \xi_{\rho t-1}) \\ &\quad - X_t(-C \eta^2 \xi_{\rho t-1} - C \eta \xi_{\rho t-1})\} ds \\ &\leq 2C w_t \xi_{\rho t-1} \{X_t(C \eta^2 \xi_{\rho t-1} + C \eta \xi_{\rho t-1}) \\ &\quad - X_t(-C \eta^2 \xi_{\rho t-1} - C \eta \xi_{\rho t-1})\}. \end{aligned}$$

By Assumptions 2.4 and 2.6 and the double expectation property, it follows that

$$\begin{aligned} E\left[\sup_{\|u\| \leq \eta} |J_{1t}(u)|\right] &\leq 2CE[w_t \xi_{\rho t-1} \{G(C \eta^2 \xi_{\rho t-1} + C \eta \xi_{\rho t-1}) \\ &\quad - G(-C \eta^2 \xi_{\rho t-1} - C \eta \xi_{\rho t-1})\}] \\ &\leq C(\eta^2 + \eta) \sup_x g(x) E(w_t \xi_{\rho t-1}^2) \rightarrow 0 \end{aligned}$$

as $\eta \rightarrow 0$. Thus, as for (6.12) and (6.13), we can show that $K_{4n}(u_n) = o_p(n \|u_n\|^2)$. This completes the proof of (ii).

(iii) Let $\Pi_{4n}(u) = (\sqrt{n}u)' [n^{-1} \sum_{t=1}^n J_{2t}(\zeta^*)](\sqrt{n}u)$, where

$$J_{2t}(\zeta^*) = w_t \left(\frac{3}{8} \left| \frac{\varepsilon_t(\gamma_0)}{\sqrt{h_t(\zeta^*)}} \right| - \frac{1}{4} \right) \frac{1}{h_t^2(\zeta^*)} \frac{\partial h_t(\zeta^*)}{\partial \theta} \frac{\partial h_t(\zeta^*)}{\partial \theta'}.$$

By Assumption 2.4 and Lemma A.1(ii)–(iv), there exists a constant $\rho \in (0, 1)$ and a neighborhood Θ_0 of θ_0 such that

$$E \left[\sup_{\zeta^* \in \Theta_0} |J_{2t}(\zeta^*)| \right] \leq CE[w_t \xi_{\rho t-1}^2 (|\eta_t| \xi_{\rho t-1} + 1)] < \infty.$$

Then, by Theorem 3.1 of Ling and McAleer (2003), we have

$$\sup_{\zeta^* \in \Theta_0} \left| \frac{1}{n} \sum_{t=1}^n J_{2t}(\zeta^*) - E[J_{2t}(\zeta^*)] \right| = o_p(1).$$

Moreover, since $\zeta_n^* \rightarrow \theta_0$ a.s., by the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} E[J_{2t}(\zeta_n^*)] = E[J_{2t}(\theta_0)] = \Sigma_2.$$

Thus, (iii) follows from the previous two equations. This completes the proof of (iii).

(iv) Since $E|\eta_t| = 1$, a similar argument as for part (iii) shows that (iv) holds.

(v) By Taylor’s expansion, we have

$$\frac{1}{\sqrt{h_t(\theta_0 + u)}} - \frac{1}{\sqrt{h_t(\theta_0)}} = \frac{-u'}{2(h_t(\zeta^*))^{3/2}} \frac{\partial h_t(\zeta^*)}{\partial \theta},$$

where ζ^* lies between θ_0 and $\theta_0 + u$. By identity (2.6), it is easy to see that

$$\begin{aligned} |\varepsilon_t(\gamma_0 + u_1)| - |\varepsilon_t(\gamma_0)| &= u' \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta} [I(\eta_t > 0) - I(\eta_t < 0)] \\ &\quad + 2u' \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta} \int_0^1 X_t \left(-\frac{u'}{\sqrt{h_t}} \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta} s \right) ds, \end{aligned}$$

where ξ^* lies between γ_0 and $\gamma_0 + u_1$. By the previous two equations, it follows that

$$\sum_{t=1}^n w_t C_t(u) = (\sqrt{nu})' [K_{5n}(u) + K_{6n}(u)] (\sqrt{nu}),$$

where

$$\begin{aligned} K_{5n}(u) &= \frac{1}{n} \sum_{t=1}^n \frac{w_t}{2h_t^{3/2}(\zeta^*)} \frac{\partial h_t(\zeta^*)}{\partial \theta} \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta'} [I(\eta_t < 0) - I(\eta_t > 0)], \\ K_{6n}(u) &= -\frac{1}{n} \sum_{t=1}^n \frac{w_t}{h_t^{3/2}(\zeta^*)} \frac{\partial h_t(\zeta^*)}{\partial \theta} \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta'} \int_0^1 X_t \left(-\frac{u'}{\sqrt{h_t}} \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta} s \right) ds. \end{aligned}$$

By Lemma A.1(i), (iii), (iv) and a similar argument as for part (ii), it is easy to see that $K_{5n}(u_n) = o_p(1)$ and $K_{6n}(u_n) = o_p(1)$. Thus, it follows that (v) holds. This completes all of the proofs. \square

7. Concluding remarks. In this paper, we first propose a self-weighted QMELE for the ARMA–GARCH model. The strong consistency and asymptotic normality of the global self-weighted QMELE are established under a fractional moment condition of ε_t with $E\eta_t^2 < \infty$. Based on this estimator, the local QMELE is showed to be asymptotically normal for the ARMA–GARCH (finite variance) and –IGARCH models. The empirical study shows that the self-weighted/local QMELE has a better performance than the self-weighted/local QMLE when η_t has a heavy-tailed distribution, while the local QMELE is more efficient than the self-weighted QMELE for the cases with a finite variance and –IGARCH errors. We also give a real example to illustrate that our new estimation procedure is necessary. According to our limit experience, the estimated tail index of most of data sets lies in $[2, 4)$ in economics and finance. Thus, the local QMELE may be the most suitable in practice if there is a further evidence to show that $E\eta_t^4 = \infty$.

APPENDIX

The Lemma A.1 below is from Ling (2007).

LEMMA A.1. *Let $\xi_{\rho t}$ be defined as in Assumption 2.4. If Assumptions 2.1 and 2.2 hold, then there exists a constant $\rho \in (0, 1)$ and a neighborhood Θ_0 of θ_0 such that:*

- (i)
$$\sup_{\Theta} |\varepsilon_{t-1}(\gamma)| \leq C\xi_{\rho t-1},$$

$$\sup_{\Theta} \left\| \frac{\partial \varepsilon_t(\gamma)}{\partial \gamma} \right\| \leq C\xi_{\rho t-1} \quad \text{and}$$
- (ii)
$$\sup_{\Theta} \left\| \frac{\partial^2 \varepsilon_t(\gamma)}{\partial \gamma \partial \gamma'} \right\| \leq C\xi_{\rho t-1},$$

$$\sup_{\Theta} h_t(\theta) \leq C\xi_{\rho t-1}^2,$$
- (iii)
$$\sup_{\Theta_0} \left\| \frac{1}{h_t(\theta)} \frac{\partial h_t(\theta)}{\partial \delta} \right\| \leq C\xi_{\rho t-1}^{\iota_1} \quad \text{for any } \iota_1 \in (0, 1),$$
- (iv)
$$\sup_{\Theta} \left\| \frac{1}{\sqrt{h_t(\theta)}} \frac{\partial h_t(\theta)}{\partial \gamma} \right\| \leq C\xi_{\rho t-1}.$$

LEMMA A.2. *For any $\theta^* \in \Theta$, let $B_\eta(\theta^*) = \{\theta \in \Theta : \|\theta - \theta^*\| < \eta\}$ be an open neighborhood of θ^* with radius $\eta > 0$. If Assumptions 2.1–2.5 hold, then:*

- (i) $E \left[\sup_{\theta \in \Theta} w_t l_t(\theta) \right] < \infty,$
- (ii) $E[w_t l_t(\theta)]$ has a unique minimum at $\theta_0,$
- (iii) $E \left[\sup_{\theta \in B_\eta(\theta^*)} w_t |l_t(\theta) - l_t(\theta^*)| \right] \rightarrow 0 \quad \text{as } \eta \rightarrow 0.$

PROOF. First, by (A.13) and (A.14) in Ling (2007) and Assumptions 2.4 and 2.5, it follows that

$$E \left[\sup_{\theta \in \Theta} \frac{w_t |\varepsilon_t(\gamma)|}{\sqrt{h_t(\theta)}} \right] \leq C E[w_t \xi_{\rho t-1} (1 + |\eta_t|)] < \infty$$

for some $\rho \in (0, 1)$, and

$$E \left[\sup_{\theta \in \Theta} w_t \log \sqrt{h_t(\theta)} \right] < \infty;$$

see Ling (2007), page 864. Thus, (i) holds.

Next, by a direct calculation, we have

$$\begin{aligned} E[w_t l_t(\theta)] &= E \left[w_t \log \sqrt{h_t(\theta)} + \frac{w_t |\varepsilon_t(\gamma_0) + (\gamma - \gamma_0)' (\partial \varepsilon_t(\xi^*) / \partial \theta)|}{\sqrt{h_t(\theta)}} \right] \\ &= E \left[w_t \log \sqrt{h_t(\theta)} + \frac{w_t}{\sqrt{h_t(\theta)}} E \left\{ \left| \varepsilon_t(\gamma_0) + (\gamma - \gamma_0)' \frac{\partial \varepsilon_t(\xi^*)}{\partial \theta} \right| \middle| \mathcal{F}_{t-1} \right\} \right] \\ &\geq E \left[w_t \log \sqrt{h_t(\theta)} + \frac{w_t}{\sqrt{h_t(\theta)}} E(|\varepsilon_t| | \mathcal{F}_{t-1}) \right] \\ &= E \left[w_t \left(\log \sqrt{\frac{h_t(\theta)}{h_t(\theta_0)}} + \sqrt{\frac{h_t(\theta_0)}{h_t(\theta)}} \right) \right] + E[w_t \log \sqrt{h_t(\theta_0)}], \end{aligned}$$

where the last inequality holds since η_t has a unique median 0, and obtains the minimum if and only if $\gamma = \gamma_0$ a.s.; see Ling (2007). Here, ξ^* lies between γ and γ_0 . Considering the function $f(x) = \log x + a/x$ when $a \geq 0$, it reaches the minimum at $x = a$. Thus, $E[w_t l_t(\theta)]$ reaches the minimum if and only if $\sqrt{h_t(\theta)} = \sqrt{h_t(\theta_0)}$ a.s., and hence $\theta = \theta_0$; see Ling (2007). Thus, we can claim that $E[w_t l_t(\theta)]$ is uniformly minimized at θ_0 , that is, (ii) holds.

Third, let $\theta^* = (\gamma^{*'}, \delta^{*'})' \in \Theta$. For any $\theta \in B_\eta(\theta^*)$, using Taylor's expansion, we can see that

$$\log \sqrt{h_t(\theta)} - \log \sqrt{h_t(\theta^*)} = \frac{(\theta - \theta^*)'}{2h_t(\theta^{**})} \frac{\partial h_t(\theta^{**})}{\partial \theta},$$

where θ^{**} lies between θ and θ^* . By Lemma A.1(iii)–(iv) and Assumption 2.4, for some $\rho \in (0, 1)$, we have

$$E \left[\sup_{\theta \in B_\eta(\theta^*)} w_t |\log \sqrt{h_t(\theta)} - \log \sqrt{h_t(\theta^*)}| \right] \leq C \eta E(w_t \xi_{\rho t-1}) \rightarrow 0$$

as $\eta \rightarrow 0$. Similarly,

$$E \left[\sup_{\theta \in B_\eta(\theta^*)} \frac{w_t}{\sqrt{h_t(\theta)}} \left| |\varepsilon_t(\gamma)| - |\varepsilon_t(\gamma^*)| \right| \right] \rightarrow 0 \quad \text{as } \eta \rightarrow 0,$$

$$E \left[\sup_{\theta \in B_\eta(\theta^*)} w_t |\varepsilon_t(\gamma^*)| \left| \frac{1}{\sqrt{h_t(\theta)}} - \frac{1}{\sqrt{h_t(\theta^*)}} \right| \right] \rightarrow 0 \quad \text{as } \eta \rightarrow 0.$$

Then, it follows that (iii) holds. This completes all of the proofs of Lemma A.2. \square

PROOF OF THEOREM 2.1. We use the method in Huber (1967). Let V be any open neighborhood of $\theta_0 \in \Theta$. By Lemma A.2(iii), for any $\theta^* \in V^c = \Theta/V$ and $\varepsilon > 0$, there exists an $\eta_0 > 0$ such that

$$(A.1) \quad E \left[\inf_{\theta \in B_{\eta_0}(\theta^*)} w_t l_t(\theta) \right] \geq E[w_t l_t(\theta^*)] - \varepsilon.$$

From Lemma A.2(i), by the ergodic theorem, it follows that

$$(A.2) \quad \frac{1}{n} \sum_{t=1}^n \inf_{\theta \in B_{\eta_0}(\theta^*)} w_t l_t(\theta) \geq E \left[\inf_{\theta \in B_{\eta_0}(\theta^*)} w_t l_t(\theta) \right] - \varepsilon$$

as n is large enough. Since V^c is compact, we can choose $\{B_{\eta_0}(\theta_i) : \theta_i \in V^c, i = 1, 2, \dots, k\}$ to be a finite covering of V^c . Thus, from (A.1) and (A.2), we have

$$(A.3) \quad \begin{aligned} \inf_{\theta \in V^c} L_{sn}(\theta) &= \min_{1 \leq i \leq k} \inf_{\theta \in B_{\eta_0}(\theta_i)} L_{sn}(\theta) \\ &\geq \min_{1 \leq i \leq k} \frac{1}{n} \sum_{t=1}^n \inf_{\theta \in B_{\eta_0}(\theta_i)} w_t l_t(\theta) \\ &\geq \min_{1 \leq i \leq k} E \left[\inf_{\theta \in B_{\eta_0}(\theta_i)} w_t l_t(\theta) \right] - \varepsilon \end{aligned}$$

as n is large enough. Note that the infimum on the compact set V^c is attained. For each $\theta_i \in V^c$, from Lemma A.2(ii), there exists an $\varepsilon_0 > 0$ such that

$$(A.4) \quad E \left[\inf_{\theta \in B_{\eta_0}(\theta_i)} w_t l_t(\theta) \right] \geq E[w_t l_t(\theta_0)] + 3\varepsilon_0.$$

Thus, from (A.3) and (A.4), taking $\varepsilon = \varepsilon_0$, it follows that

$$(A.5) \quad \inf_{\theta \in V^c} L_{sn}(\theta) \geq E[w_t l_t(\theta_0)] + 2\varepsilon_0.$$

On the other hand, by the ergodic theorem, it follows that

$$(A.6) \quad \inf_{\theta \in V} L_{sn}(\theta) \leq L_{sn}(\theta_0) = \frac{1}{n} \sum_{t=1}^n w_t l_t(\theta_0) \leq E[w_t l_t(\theta_0)] + \varepsilon_0.$$

Hence, combining (A.5) and (A.6), it gives us

$$\inf_{\theta \in V^c} L_{sn}(\theta) \geq E[w_t l_t(\theta_0)] + 2\varepsilon_0 > E[w_t l_t(\theta_0)] + \varepsilon_0 \geq \inf_{\theta \in V} L_{sn}(\theta),$$

which implies that

$$\hat{\theta}_{sn} \in V \quad \text{a.s. for } \forall V, \text{ as } n \text{ is large enough.}$$

By the arbitrariness of V , it yields $\hat{\theta}_{sn} \rightarrow \theta_0$ a.s. This completes the proof. \square

Acknowledgments. The authors greatly appreciate the very helpful comments of two anonymous referees, the Associate Editor and the Editor T. Tony, Cai.

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