

OPTIMAL MODEL SELECTION FOR DENSITY ESTIMATION OF STATIONARY DATA UNDER VARIOUS MIXING CONDITIONS

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We propose a block-resampling penalization method for marginal density estimation with nonnecessary independent observations. When the data are β or τ -mixing, the selected estimator satisfies oracle inequalities with leading constant asymptotically equal to 1.

We also prove in this setting the slope heuristic, which is a data-driven method to optimize the leading constant in the penalty.

1. Introduction. Model selection by penalization of an empirical loss is a general method that includes several famous procedures as cross-validation [Rudemo (1982)] or hard thresholding [Donoho et al. (1996)] as shown by Barron, Birgé and Massart (1999). The difficulty is to calibrate the penalty so that the selected estimator satisfies an oracle inequality. A good penalty has the shape of an ideal one [see definition (2.4)] and depends in general on a leading constant that should be chosen sufficiently large.

Resampling penalties provide a shape for the penalty term in a general statistical learning framework; see Arlot (2009). The resulting estimator satisfies sharp oracle inequalities in non-Gaussian heteroscedastic regression among histograms [Arlot (2009)] and in density estimation among more general collections of models [Lerasle (2011a)]. The validity of these theorems relies on the independence of the observations. In this paper, we study a generalization of these penalties, called block-resampling penalties and we prove that the resulting estimator satisfies sharp oracle inequalities when the data are only supposed to be β - or τ -mixing [the coefficient β has been defined by Volkonskiĭ and Rozanov (1959), the coefficient τ by Dedecker and Prieur (2005); see Section 2.4].

We use a coupling method to extend the results for independent data. It was introduced in Baraud, Comte and Viennet (2001) in a regression problem and used in Comte and Merlevède (2002) for density estimation with β -mixing observations. β is a well known “strong” mixing coefficient. We refer to the books of Doukhan (1994) and Bradley (2007) for examples of β -mixing processes. One of the most

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important is the following: a stationary, irreducible, aperiodic and positively recurrent Markov chain is β -mixing. “Strong” mixing coefficients cannot be used to study a lot of simple processes. For example, the stationary solution of the equation

$$(1.1) \quad X_n = \frac{1}{2}(X_{n-1} + \xi_n),$$

where $(\xi_n)_{n \in \mathbb{Z}}$ are i.i.d. Bernoulli random variables $\mathcal{B}(1/2)$ is not β -mixing [see Andrews (1984)]. This is why “weak” mixing coefficients such as τ have been introduced. They are easier to compute and allow us to cover more examples, as the process (1.1) [we refer to the papers of Dedecker and Prieur (2005), Comte, Dedecker and Taupin (2008) or the book of Dedecker et al. (2007) for examples of weakly-mixing processes]. In Lerasle (2009), we used a coupling result of Dedecker and Prieur (2005) to extend the coupling method to τ -mixing data.

In all these previous papers, the dimension of the models was used as a shape of the penalties. The leading constant was built with the mixing coefficients and could not in general be computed from the data. When it could, the theoretical upper bounds obtained are probably too pessimistic to be used by the statistician. We use in this paper block-resampling penalties as a shape of the penalty and a data-driven leading constant. The first main result of the paper is that the resulting estimator satisfies asymptotically optimal oracle inequalities. We propose also to optimize the choice of the leading constant in penalties using the slope algorithm. This procedure is based on the slope heuristic, introduced in Birgé and Massart (2007) and proved in Birgé and Massart (2007) for Gaussian regression, in Arlot and Massart (2009) for non-Gaussian heteroscedastic regression over histograms and in Lerasle (2011a) for density estimation. The second main result of this paper is a proof of the slope heuristic for the marginal density estimation problem with β - or τ -mixing data.

Block-resampling penalties and the slope heuristic can be defined in a more general statistical learning framework, including the problems of classification and regression [see Arlot (2009); Arlot and Massart (2009)]. Our results are contributions to the theoretical understanding of these generic methods. Up to our knowledge, they are the first ones obtained in a mixing framework.

The paper is organized as follows. Section 2 introduces the density estimation framework, the estimators, the penalties and the main assumptions. Sections 3 and 4 give the main results, respectively, for τ - and β -mixing processes. Section 5 gives the proofs of the main results. Some other proofs are available as Supplementary Material [Lerasle (2011b)].

2. Preliminaries.

2.1. *The density estimation framework.* We observe n real valued, identically distributed random variables X_1, \dots, X_n , defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common law P . We assume that P is absolutely continuous with respect to the Lebesgue measure μ on \mathbb{R} and we want to estimate the density s of P with

respect to μ . $L^2(\mu)$ denotes the Hilbert space of square integrable real valued functions and $\|\cdot\|$ the associated L^2 -norm. We assume that s belongs to $L^2(\mu)$. The risk of an estimator \hat{s} of s is measured with the L^2 -loss, that is $\|s - \hat{s}\|^2$, which is random when \hat{s} is.

Let p, q be two integers and assume that $n = 2pq$. For all $i = 0, \dots, p - 1$, let $I_i = (2iq + 1, \dots, (2i + 1)q)$, $A_i = (X_l)_{l \in I_i}$. For all functions t in $L^1(P)$, for all reals x_1, \dots, x_q , we define

$$L_q t(x_1, \dots, x_q) = \frac{1}{q} \sum_{i=1}^q t(x_i), \quad P t = \int_{\mathbb{R}} t(x) s(x) d\mu(x),$$

$$P_A t = \frac{1}{p} \sum_{i=0}^{p-1} L_q t(A_i).$$

Given a linear space S_m of measurable, real valued functions, and an orthonormal basis $(\psi_\lambda)_{\lambda \in \Lambda_m}$ of S_m , we define the projection estimator $\hat{s}_{A,m}$ of s onto S_m by

$$\hat{s}_{A,m} = \sum_{\lambda \in \Lambda_m} (P_A \psi_\lambda) \psi_\lambda \in \arg \min_{t \in S_m} \{\|t\|^2 - 2P_A t\}.$$

Given a finite collection $(S_m)_{m \in \mathcal{M}_n}$ of such linear spaces and a penalty function $\text{pen} : \mathcal{M}_n \rightarrow \mathbb{R}_+$, the Penalized Projection Estimator, hereafter PPE, is defined by

$$(2.1) \quad \tilde{s}_A = \hat{s}_{A,\hat{m}} \quad \text{where } \hat{m} \in \arg \min_{m \in \mathcal{M}_n} \{\|\hat{s}_{A,m}\|^2 - 2P_A \hat{s}_{A,m} + \text{pen}(m)\}.$$

We will say that the PPE satisfies an oracle inequality when one of the two following inequalities holds.

There exist constants $\kappa > 0, \gamma > 1$ and a positive sequence $(K_n)_{n \in \mathbb{N}^*}$ bounded away from 0 such that

$$(2.2) \quad \mathbb{P}\left(K_n \|s - \tilde{s}_A\|^2 \leq \inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2\right) \geq 1 - \frac{\kappa}{n^\gamma}.$$

There exists a positive sequence $(K_n)_{n \in \mathbb{N}^*}$ bounded away from 0 such that

$$(2.3) \quad K_n \mathbb{E}(\|s - \tilde{s}_A\|^2) \leq \mathbb{E}\left(\inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2\right).$$

The oracle inequality is said to be sharp when, moreover, the sequence $K_n \rightarrow 1$ when n grows to infinity. Inequalities (2.2) are usually preferred to (2.3) since they describe the typical behavior of the selected estimator and not only of its expectation.

It is worth mentioning that we only use $\text{Card}(\cup_{i=0}^{p-1} I_i) = pq = n/2$ data to build the estimator \tilde{s}_A . The consequences of this choice are discussed after Theorem 3.1 and in Section 4.3.

2.2. *Block-resampling penalties.* We introduce block-resampling penalties as natural generalizations of resampling penalties. The best estimator in the collection $(\widehat{s}_{A,m})_{m \in \mathcal{M}_n}$ minimizes among \mathcal{M}_n the ideal criterion

$$\|s - \widehat{s}_{A,m}\|^2 - \|s\|^2 = \|\widehat{s}_{A,m}\|^2 - 2P_A \widehat{s}_{A,m} + \text{pen}_{\text{id}}(m).$$

In this decomposition, the ideal penalty $\text{pen}_{\text{id}}(m)$ [Arlot (2009)] is equal to

$$(2.4) \quad \text{pen}_{\text{id}}(m) = 2(P_A - P)(\widehat{s}_{A,m}).$$

To adapt the approach of Arlot (2009) to a dependent setting, we replace the resampling step by a resampling procedure on the blocks $(A_i)_{i=0,\dots,p-1}$. Let (W_0, \dots, W_{p-1}) be a resampling scheme, that is, a vector of positive random variables, independent of $(X_i)_{i=1,\dots,n}$ and exchangeable, which means that, for all permutations ξ of $\{0, \dots, p-1\}$,

$$(W_{\xi(0)}, \dots, W_{\xi(p-1)}) \text{ has the same law as } (W_0, \dots, W_{p-1}).$$

Let $\overline{W} = p^{-1} \sum_{i=0}^{p-1} W_i$, for all t in $L^1(P)$, let P_A^W be the block-resampling empirical process defined by

$$P_A^W t = \frac{1}{P} \sum_{i=0}^{p-1} W_i L_q t(A_i).$$

For all integrable random variables $F(X_1, \dots, X_n, W_0, \dots, W_{p-1})$, let

$$\begin{aligned} &\mathbb{E}_W[F(X_1, \dots, X_n, W_0, \dots, W_{p-1})] \\ &= \mathbb{E}[F(X_1, \dots, X_n, W_0, \dots, W_{p-1}) | X_1, \dots, X_n]. \end{aligned}$$

Let $((\psi_\lambda)_{\lambda \in \Lambda_m})_{m \in \mathcal{M}_n}$ be orthonormal bases of $(S_m)_{m \in \mathcal{M}_n}$ and let $(\widehat{s}_{A,m}^W)_{m \in \mathcal{M}_n}$ be the collection of resampling projection estimators

$$\widehat{s}_{A,m}^W = \sum_{\lambda \in \Lambda_m} (P_A^W \psi_\lambda) \psi_\lambda.$$

The block-resampling penalties are defined as block-resampling estimators of the ideal penalty by

$$(2.5) \quad \text{pen}_W(m, C) = C \mathbb{E}_W(2(P_A^W - \overline{W}P_A)(\widehat{s}_{A,m}^W)).$$

The idea of resampling is to mimic the behavior of the empirical process P_A around P by the behavior of the resampling empirical process P_A^W around $\overline{W}P_A$. The resampling procedure is a plug-in method where the unknown functionals $F(P, P_n)$ are estimated by $F(\overline{W}P_n, P_n^W)$. Hence, $\widehat{s}_{A,m}$ in $\text{pen}_{\text{id}}(m)$ is replaced by $\widehat{s}_{A,m}^W$ in $\text{pen}_W(m, C)$ and, instead of applying the process $P_A - P$, we apply the process $P_A^W - \overline{W}P_A$. We take the expectation with respect to the distribution of the resampling scheme to stabilize the procedure. Finally, we let a normalizing constant C free for this general definition.

We use a block-resampling scheme instead of a classical exchangeable resampling scheme in order to preserve the dependence of the data inside the blocks. This is a key point for the procedure to work. Examples of resampling schemes can be found in Arlot (2009). The classical block-bootstrap [Künsch (1989); Liu and Singh (1992)] is obtained when the distribution of (W_0, \dots, W_{p-1}) is the multinomial $\mathcal{M}(p, 1/p, \dots, 1/p)$.

2.3. *The slope algorithm.* The “slope heuristic” has been introduced by Birgé and Massart (2007) in order to calibrate the leading constant in a penalty term [e.g., the constant C in (2.5)]. It is based on the behavior of the complexity of the selected model [recall the definition (2.1)]. It states that there exist a family $(\Delta_m)_{m \in \mathcal{M}_n}$ and a constant K_{\min} satisfying the following properties:

- (SH1) When $\text{pen}(m) \leq K \Delta_m$, with $K < K_{\min}$, then $\Delta_{\hat{m}} \geq c_1 \max_{m \in \mathcal{M}_n} \Delta_m$.
- (SH2) When $\text{pen}(m) = K \Delta_m$, with $K > K_{\min}$, then $\Delta_{\hat{m}}$ is much smaller.
- (SH3) When $\text{pen}(m) = 2K_{\min} \Delta_m$, then \hat{s}_A satisfies a sharp oracle inequality.

Based on this heuristic, Birgé and Massart (2007) introduced the following slope algorithm. It can be used in practice when a family $(\Delta_m)_{m \in \mathcal{M}_n}$ satisfying the slope heuristic is known.

- For all $K > 0$, compute $\Delta_{\hat{m}(K)}$ where $\hat{m}(K)$ is defined as in (2.1) with $\text{pen}(m) = K \Delta_m$.
- Find \tilde{K} such that $\Delta_{\hat{m}(K)}$ is very large for $K < \tilde{K}$ and much smaller when $K > \tilde{K}$.
- Choose the final \hat{m} equal to $\hat{m}(2\tilde{K})$.

The idea is that $\tilde{K} \sim K_{\min}$ since we observe a jump of the complexity of the selected model around $K = \tilde{K}$ [thanks to (SH1), (SH2)] and thus that the final estimator, selected by the penalty $2\tilde{K} \Delta_m \sim 2K_{\min} \Delta_m$, satisfies an optimal oracle inequality [by (SH3)].

2.4. *Some measures of dependence.*

2.4.1. *β -mixing data.* Volkonskiĭ and Rozanov (1959) defined the coefficient β as follows. Let Y be a random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let \mathcal{M} be a σ -algebra in \mathcal{A} , let

$$\beta(\mathcal{M}, \sigma(Y)) = \mathbb{E} \left(\sup_{A \in \mathcal{B}} |\mathbb{P}_{Y|\mathcal{M}}(A) - \mathbb{P}_Y(A)| \right).$$

For all stationary sequences of random variables $(X_n)_{n \in \mathbb{Z}}$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$, let

$$\beta_k = \beta(\sigma(X_i, i \leq 0), \sigma(X_i, i \geq k)).$$

The process $(X_n)_{n \in \mathbb{Z}}$ is said to be β -mixing when $\beta_k \rightarrow 0$ as $k \rightarrow \infty$.

2.4.2. *τ-mixing data.* Dedecker and Prieur (2005) defined the coefficient τ as follows. For all l in \mathbb{N}^* , for all x, y in \mathbb{R}^l , let $d_l(x, y) = \sum_{i=1}^l |x_i - y_i|$. For all l in \mathbb{N}^* , for all functions t defined on \mathbb{R}^l , the Lipschitz semi-norm of t is defined by

$$\text{Lip}_l(t) = \sup_{x \neq y \in \mathbb{R}^l} \frac{|t(x) - t(y)|}{d_l(x, y)}.$$

For all functions t defined on \mathbb{R} , we will denote for short by $\text{Lip}(t) = \text{Lip}_1(t)$. Let λ_1 be the set of all functions $t : \mathbb{R}^l \rightarrow \mathbb{R}$ such that $\text{Lip}_l(t) \leq 1$. For all integrable, \mathbb{R}^l -valued, random variables Y defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and all σ -algebra \mathcal{M} in \mathcal{A} , let

$$\tau(\mathcal{M}, Y) = \mathbb{E} \left(\sup_{t \in \lambda_1} |\mathbb{P}_{Y|\mathcal{M}}(t) - \mathbb{P}_Y(t)| \right).$$

For all stationary sequences of integrable random variables $(X_n)_{n \in \mathbb{Z}}$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$, for all integers k, r , let

$$\tau_{k,r} = \max_{1 \leq l \leq r} \frac{1}{l} \sup_{k \leq i_1 < \dots < i_l} \{ \tau(\sigma(X_p, p \leq 0), (X_{i_1}, \dots, X_{i_l})) \}, \quad \tau_k = \sup_{r \in \mathbb{N}^*} \tau_{k,r}.$$

The process $(X_n)_{n \in \mathbb{Z}}$ is said to be τ -mixing when $\tau_k \rightarrow 0$ as $k \rightarrow \infty$.

2.5. *Main assumptions.*

2.5.1. *A specific collection for τ-mixing sequences.* Wavelet spaces have been widely used in density estimation since the oracle is adaptive over Besov spaces [see Birgé and Massart (1997)].

Dyadic wavelet spaces: Let r be a real number, $r \geq 1$. We work with an r -regular orthonormal multiresolution analysis of $L^2(\mu)$, associated with a compactly supported scaling function ϕ and a compactly supported mother wavelet ψ . Without loss of generality, we suppose that the support of the functions ϕ and ψ is included in an interval $[A_1, A_2]$ where A_1 and A_2 are integers such that $A_2 - A_1 = A \geq 1$. For all k in \mathbb{Z} and j in \mathbb{N}^* , let $\psi_{0,k} : x \rightarrow \sqrt{2}\phi(2x - k)$ and $\psi_{j,k} : x \rightarrow 2^{j/2}\psi(2^j x - k)$. The family $\{(\psi_{j,k})_{j \geq 0, k \in \mathbb{Z}}\}$ is an orthonormal basis of $L^2(\mu)$. The collection of dyadic wavelet spaces is described as follows.

[W] *dyadic wavelet generated spaces:* let $J_n = \lfloor \log_2(n) \rfloor$, for all $J_m = 1, \dots, J_n$, let

$$\Lambda_m = \{(j, k), 0 \leq j \leq J_m, k \in \mathbb{Z}\}$$

and let S_m be the linear span of $\{\psi_\lambda\}_{\lambda \in \Lambda_m}$.

2.5.2. *General framework.* We present in this section a set of assumptions sufficient to prove the theorems. None of them is used to build the penalties.

(H1) *There exists a constant κ_a such that, for all m, m' in \mathcal{M}_n , for all t in $S_m + S_{m'}$, with $\|t\| \leq 1$, there exist t_m in S_m and $t_{m'}$ in $S_{m'}$, with $\|t_m\| \vee \|t_{m'}\| \leq \kappa_a$ such that $t = t_m + t_{m'}$.*

(H1) is typically satisfied for nested collections as [W].

(H2) $N_n = \text{Card}(\mathcal{M}_n)$ is finite and there exist constants $c_{\mathcal{M}}, \alpha_{\mathcal{M}}$ such that $N_n \leq c_{\mathcal{M}}n^{\alpha_{\mathcal{M}}}$.

(H2) means that the collection is not too rich and thus that the model selection problem is not too hard. It is satisfied by the collection [W].

Let us introduce some notation. For all m in \mathcal{M}_n , for all orthonormal bases $(\psi_\lambda)_{\lambda \in \Lambda_m}$ of S_m , let

$$D_{A,m} = q \sum_{\lambda \in \Lambda_m} \text{Var}(L_q(\psi_\lambda)(A_0)), \quad R_{A,m} = n\|s - s_m\|^2 + 2D_{A,m},$$

$$B_m = \{t \in S_m, \|t\| \leq 1\}, \quad b_m = \sup_{t \in B_m} \|t\|_\infty.$$

$D_{A,m}$, and thus $R_{A,m}$, are well defined since we can check with Cauchy–Schwarz inequality that

$$D_{A,m} = q\mathbb{E}\left[\left(\sup_{t \in B_m} L_q t(A_0) - Pt\right)^2\right].$$

Two quantities will play a fundamental role to discuss the results. The first one is the risk of an oracle:

$$R_n = \inf_{m \in \mathcal{M}_n} R_{A,m}.$$

We are typically interested in non parametric problems where $R_n/n \sim n^{-\gamma}$ for some $0 < \gamma < 1$. This situation occurs, for example, when s is a regular function, in this case, we have $R_n/n = \kappa n^{-2\alpha/(2\alpha+1)}$, for some $\alpha > 0, \kappa > 0$. We will make the following assumption:

(H3) *There exists a constant $\kappa_R > 0$ such that $R_n \geq \kappa_R(\ln n)^8$.*

The constant 8 in $(\ln n)^8$ is technical, it yields the rate $\varepsilon_n = (\ln n)^{-1/2}$ in the oracle inequalities. Arlot (2009) replaced this assumption by a lower bound on the bias of the models. It implies that $R_n \geq \kappa n^\gamma$, for some constants $\kappa > 0, 1 > \gamma > 0$ and therefore assumption (H3).

(H4) *There exists a constant $c_D > 0$ such that*

$$\forall m \in \mathcal{M}_n \quad P\left(\sup_{t \in B_m} t^2\right) \geq c_D b_m^2.$$

It is shown in the [Appendix](#) that some classical examples of collections $(S_m, m \in \mathcal{M}_n)$ as regular histograms, Fourier spaces and [W] satisfy (H4).

The following assumptions will be used to prove the slope heuristic. We introduce a second quantity, that will play a fundamental role. Let

$$D_n^* = \max_{m \in \mathcal{M}_n} D_{A,m}.$$

In classical collection of models, like [W], $D_n^* \sim cn$. This is why we introduce the following assumption:

(H5) $D_n^*/R_n \rightarrow \infty$ when n grows to infinity.

We will prove that, when the data are mixing, $D_{A,m} \simeq n\mathbb{E}(\|s_m - \widehat{s}_{A,m}\|^2)$ represents the variance term of the risk. It is a natural measure of the complexity of the models. Hence, D_n^* represents the maximal complexity of the models. Moreover, R_n is the risk of the oracle. It balances the complexity and the bias term and has therefore the same order as the complexity of an oracle. Hence, assumption (H5) means that the largest complexity in the collection $(S_m)_{m \in \mathcal{M}_n}$ is much larger than the one of an oracle, which is a natural condition for the slope heuristic to hold. We need a final assumption.

(H6) For all m^* such that $D_{A,m^*} = D_n^*$, we have

$$\frac{n\|s - s_{m^*}\|^2}{D_n^*} \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$

When D_n^* is of order n , (H6) simply means that the distance between s and a complex model goes to 0. In general, it means that for these complex models, the bias part of the risk is negligible compared to the variance part. We conclude this section by the assumptions on the mixing coefficients. All mean that these coefficients are sufficiently small. Let $\gamma = \beta$ or τ .

[AR(θ)] *arithmetical γ -mixing with rate θ : there exists $C > 0$ such that, for all k in \mathbb{N} , $\gamma_k \leq C(1+k)^{-(1+\theta)}$.*

S(β) $\sum_{l \geq 1} (l+1)\beta_l \leq c_D/64$, where c_D is defined in (H4).

We prove in the [Appendix](#) that $c_D = 1$ for regular histograms and Fourier spaces.

S(τ, W) $\sum_{l \geq 1} (s^2\tau_l)^{1/3} \leq C(W)$, where $C(W)$ depends only on ϕ, ψ .

The value of the constant $C(W)$ is given in Lemma 5.2 of [Lerasle \(2011b\)](#).

3. Results for τ -mixing sequences.

3.1. *Resampling penalties.* The result of this section is that PPE selected by block-resampling penalties satisfy sharp oracle inequalities.

THEOREM 3.1. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection*

of regular wavelet spaces [W] satisfying (H3), (H4). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.

Let $\tilde{C}_W = \text{Var}(W_1 - \bar{W})^{-1}$, $C > \tilde{C}_W/2$ and let \tilde{s}_A be the PPE defined in (2.1) with the penalty $\text{pen}_W(m, C)$ defined in (2.5).

Assume that there exists $\theta > 5$ such that X_1, \dots, X_n are arithmetically [AR(θ)] τ -mixing and satisfy $S(\tau, W)$. Let $\varepsilon_n = (\ln n)^{-1/2}$, $\kappa(C) = 2(C\tilde{C}_W^{-1} - 1)$.

There exist constants κ_1, κ_2 such that we have

$$(3.1) \quad K_n \mathbb{E}(\|s - \tilde{s}_A\|^2) \leq \mathbb{E}\left(\inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2\right) + \frac{\kappa_2}{n},$$

with

$$K_n = \frac{(1 \wedge (1 + \kappa(C))) - \kappa_1 \varepsilon_n}{(1 \vee (1 + \kappa(C))) + \kappa_1 \varepsilon_n}.$$

Comments:

- The constant C has to be chosen asymptotically equal to \tilde{C}_W . If we choose $C > \tilde{C}_W$, we still get an oracle inequality, with a leading constant less sharp. On the other hand, if we choose $C < \tilde{C}_W$ we can have $K_n \leq 0$ in (3.1). This is a first reason why it is generally useful to over-penalize a little bit from a nonasymptotic point of view.

3.2. *Slope heuristic.* Theorem 3.1 gives a totally data driven penalty which satisfies a sharp oracle inequality, therefore, the heuristic is not necessary to obtain asymptotically optimal results. However, C can be optimized for small samples. Moreover, the slope algorithm is faster to compute than resampling penalties when a deterministic quantity can be used in the slope heuristic. Theorem 3.2 hereafter justifies property (SH1) of the heuristic. Δ_m is the variance term $D_{A,m}/n$ and $K_{\min} = 2$.

THEOREM 3.2. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of regular wavelet spaces [W] satisfying (H3)–(H6). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.*

Assume that there exists a constant $0 < \delta < 1$ such that, for all m in \mathcal{M}_n ,

$$(3.2) \quad 0 \leq \text{pen}(m) \leq (2 - \delta) \frac{D_{A,m}}{n},$$

and let \tilde{s}_A be the PPE defined in (2.1).

Assume that there exists $\theta > 5$ such that X_1, \dots, X_n are arithmetically [AR(θ)] τ -mixing and satisfy $S(\tau, W)$. There exist constants κ_1, κ_2 such that

$$(3.3) \quad \mathbb{E}(D_{A,\hat{m}}) \geq \frac{4\delta}{9} D_n^* - \kappa_1.$$

$$(3.4) \quad \mathbb{E}(\|s - \tilde{s}_A\|^2) \geq \frac{\delta}{5} \frac{D_n^*}{R_n} \left(\mathbb{E} \left(\inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2 \right) - \frac{\kappa_2}{n} \right).$$

Comments:

- Inequality (3.3) states that $D_{A,\hat{m}}$ is as large as possible when the penalty term is too small. This is exactly (SH1) with $\Delta_m = D_{A,m}$.
- Inequality (3.4) states that the model selected by a too small penalty is never an oracle. This is another reason why it is interesting to choose $C > \tilde{C}_W$ in Theorem 3.1.

The following theorem justifies properties (SH2), (SH3) of the slope heuristic.

THEOREM 3.3. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of regular wavelet spaces [W] satisfying (H3), (H4). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.*

Assume that there exist $\delta_+ \geq -\delta_- > -1, \varepsilon \geq 0$ and some constants κ_1, κ_2 satisfying, for all m in \mathcal{M}_n ,

$$(3.5) \quad \mathbb{E} \left[\sup_{m \in \mathcal{M}_n} \left((2 - \delta_-) \frac{2D_{A,m}}{n} - \text{pen}(m) - \varepsilon \frac{R_{A,m}}{n} \right)_+ \right] \leq \frac{\kappa_1}{n},$$

$$(3.6) \quad \mathbb{E} \left[\sup_{m \in \mathcal{M}_n} \left(\text{pen}(m) - (2 + \delta_+) \frac{D_{A,m}}{n} - \varepsilon \frac{R_{A,m}}{n} \right)_+ \right] \leq \frac{\kappa_2}{n}.$$

Let \tilde{s}_A be the PPE defined in (2.1) with $\text{pen}(m)$ and let $\varepsilon_n = (\ln n)^{-1/2}$.

Assume that there exists $\theta > 5$ such that X_1, \dots, X_n are arithmetically [AR(θ)] τ -mixing and satisfy S(τ, W). There exist constants $\kappa_1, \kappa_2, \kappa_3$, such that

$$(3.7) \quad K_n \mathbb{E}(\|\tilde{s}_A - s\|^2) \leq \mathbb{E} \left(\inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2 \right) + \frac{\kappa_2}{n},$$

with

$$K_n = \frac{(1 \wedge (1 - \delta_-)) - \kappa_1(\varepsilon_n + \varepsilon)}{(1 \vee (1 + \delta_+)) + \kappa_1(\varepsilon_n + \varepsilon)}.$$

Moreover, we have

$$(3.8) \quad K_n \mathbb{E}(D_{A,\hat{m}}) \leq R_n + \kappa_3.$$

Comments:

- When $\text{pen}(m)$ becomes larger than $2D_{A,m}/n$, $D_{A,\hat{m}}$ jumps from D_n^* (3.3) to R_n [(3.8) for δ_+ and $-\delta_-$ close to -1]. This justifies (SH2) since $R_n \ll D_n^*$.
- A model selected with a penalty $4D_{A,m}/n$ satisfies an oracle inequality (Theorem 3.3 for δ_+ and δ_- close to 0). This justifies (SH3).

- $D_{A,m}$ is unknown and cannot be used in the slope algorithm. We show [Lemma 5.2 in Lerasle (2011b)] that $D_{A,m}$ satisfies $\kappa_* 2^{J_m} \leq D_{A,m} \leq \kappa^* 2^{J_m}$. The slope heuristic might hold for $\Delta_m = 2^{J_m}/n$, but a complete proof requires moreover that $\kappa_* \simeq \kappa^*$. However, we obtain in the proof of Theorem 3.1 that $\text{pen}_W(m, \tilde{C}_W)$ satisfies (3.5) and (3.6) for $\delta_+ = \delta_- = 0$ and $\varepsilon = \kappa \varepsilon_n$. Since (3.2) can be modified to work with random penalties, we can apply the slope algorithm with $\text{pen}_W(m, 1)$ instead of $D_{A,m}/n$.

4. Results for β -mixing sequences. We show that block-resampling penalties select oracles and that the slope heuristic holds in this case.

4.1. *Resampling penalties.*

THEOREM 4.1. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of linear spaces satisfying (H1)–(H4). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.*

Let $\tilde{C}_W = \text{Var}(W_1 - \bar{W})^{-1}$, $C > \tilde{C}_W/2$ and let \tilde{s}_A be the PPE defined in (2.1) with the block-resampling penalty $\text{pen}_W(m, C)$ defined in (2.5).

Assume that there exists $\theta > 2$ such that X_1, \dots, X_n are arithmetically [AR(θ)] β -mixing and satisfy S(β). Let $\varepsilon_n = (\ln n)^{-1/2}$, $\kappa(C) = 2(C\tilde{C}_W^{-1} - 1)$.

There exist constants κ_1, κ_2 such that

$$(4.1) \quad P\left(K_n \|s - \tilde{s}_A\|^2 \leq \inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2\right) \geq 1 - \kappa_2 \left(\frac{1}{n^2} \vee \frac{(\ln n)^{4+2\theta}}{n^{\theta/2}}\right),$$

with

$$K_n = \frac{(1 \wedge (1 + \kappa(C))) - \kappa_1 \varepsilon_n}{(1 \vee (1 + \kappa(C))) + \kappa_1 \varepsilon_n}.$$

Comments:

- The coupling lemma of Berbee (1979) for β -mixing processes is much stronger than the one satisfied by τ -mixing data [Dedecker and Prieur (2005)]. This is why Theorem 4.1 covers more collections of models than Theorem 3.1 and why we prove oracle inequalities in probability.

4.2. *Slope heuristic.* The following theorems are adaptations to the β -mixing case of Theorems 3.2 and 3.3.

THEOREM 4.2. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of linear spaces satisfying (H1)–(H6). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.*

Let \tilde{s}_A be the PPE defined in (2.1) with a penalty $\text{pen}(m)$ satisfying, for all m in \mathcal{M}_n , condition (3.2) of Theorem 3.2.

Assume that there exists $\theta > 2$ such that X_1, \dots, X_n are arithmetically $[\text{AR}(\theta)]$ β -mixing and satisfy $S(\beta)$. There exists a constant κ and an event Ω_n such that

$$\mathbb{P}(\Omega_n) \geq 1 - \kappa \left(\frac{1}{n^2} \vee \frac{(\ln n)^{4+2\theta}}{n^{\theta/2}} \right),$$

and, on Ω_n ,

$$(4.2) \quad D_{A,\hat{m}} \geq \frac{4\delta}{9} D_n^*, \quad \|s - \tilde{s}_A\|^2 \geq \frac{\delta}{5} \frac{D_n^*}{R_n} \inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2.$$

THEOREM 4.3. Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of linear spaces satisfying (H1)–(H4). Let p, q be two integers such that $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.

Assume that there exist $\delta_+ \geq -\delta_- > -1$, $\varepsilon \geq 0$, $0 \leq \eta < 1$ and an event Ω_{pen} , with $\mathbb{P}(\Omega_{\text{pen}}) \geq 1 - \eta$ such that, on Ω_{pen} , for all m in \mathcal{M}_n ,

$$(4.3) \quad (2 - \delta_-) \frac{2D_{A,m}}{n} - \varepsilon \frac{R_{A,m}}{n} \leq \text{pen}(m) \leq (2 + \delta_+) \frac{2D_{A,m}}{n} + \varepsilon \frac{R_{A,m}}{n}.$$

Let \tilde{s}_A be the PPE defined in (2.1) with pen .

Assume that there exists $\theta > 2$ such that X_1, \dots, X_n are arithmetically $[\text{AR}(\theta)]$ β -mixing and satisfy $S(\beta)$. There exist constants κ_1, κ_2 and an event Ω_n^* such that

$$\mathbb{P}(\Omega_n^*) \geq 1 - \eta - \kappa_2 \left(\frac{1}{n^2} \vee \frac{(\ln n)^{4+2\theta}}{n^{\theta/2}} \right),$$

and, on Ω_n^* ,

$$(4.4) \quad K_n \|\tilde{s}_A - s\|^2 \leq \inf_{m \in \mathcal{M}_n} \|s - \hat{s}_{A,m}\|^2,$$

with

$$K_n = \frac{(1 \wedge (1 - \delta_-)) - \kappa_1(\varepsilon_n + \varepsilon)}{(1 \vee (1 + \delta_+)) + \kappa_1(\varepsilon_n + \varepsilon)}.$$

Moreover, $\Omega_n^*, 2K_n D_{A,\hat{m}} \leq 3R_n$.

Comments:

- We refer to the comments of Theorems 3.2 and 3.3 where we explain why Theorems 4.2 and 4.3 imply the slope heuristic with $\Delta_m = D_{A,m}/n$, $K_{\min} = 2$.
- As in Theorem 3.3, $D_{A,m}$ cannot be used to build a model selection procedure. A deterministic shape of $D_{A,m}$ is unknown, although we prove in the Supplementary Material that $D_{A,m}$ is bounded by b_m^* . However, $\text{pen}_W(m, 1)$ can be used instead of $D_{A,m}$.

4.3. *Discussion and perspectives.* Block-resampling penalties yield data driven procedures for the estimation of the marginal density in a mixing framework. The selected estimators satisfy sharp oracle inequalities without remainder term. This improves Theorems 3.1 and 4.1 in Lerasle (2009) and Theorem 3.1 in Comte and Merlevède (2002), where the leading constants was built with the mixing coefficients of the process. Moreover, our results hold for possibly infinite dimensional models.

Lacour (2008) gave also a model selection procedure to estimate the stationary density and the transition probability of a Markov Chain. She worked with a stationary chain, irreducible, aperiodic and positively recurrent, which is therefore β -mixing. Her density estimator is selected by a penalty equal to Kd_m/n with a constant K that “depends on the law of the chain” [see Remark 4 after Theorem 3 in Lacour (2008)]. She proposed to estimate K in the simulations by the slope algorithm. We prove the slope heuristic, justifying that the slope algorithm can be used to optimize the leading constant. It would be interesting to see if resampling penalties may be used in her context to estimate the transition probabilities.

Gannaz and Wintenberger (2009) worked with other weak mixing coefficients [namely λ and $\tilde{\phi}$; see Dedecker et al. (2007) for a definition] and studied a wavelet thresholded estimator. The main advantage is that the thresholded estimator is adaptive over a larger class of Besov spaces than the oracle over the collection $[W]$ [for details about this important issue see Barron, Birgé and Massart (1999)]. The main drawback is that their threshold is built with the mixing coefficients.

Block-resampling penalties can be extended to the statistical learning framework of Massart and Nédélec (2006), where the slope algorithm has already been defined [Arlot and Massart (2009)]. We believe that these procedures perform well in this context but the problem remains open.

The main drawback of our approach is that we use only $n/2$ data. Moreover, the deterministic choice of the number p of blocks is not optimized. For example, when the data are geometrically β -mixing, which means that, for some constants $\theta > 0$, $C > 0$, $\beta_k \leq Ce^{-\theta k}$, choosing p of order $n(\ln n)^{-2}$ would improve the rates of convergence of the leading constant. An interesting direction of research would be to provide data-driven choices of p and q to improve these rates, and a data-driven choice of blocks to use more data.

In practice, the computation time is also a very important issue. Actually, the conditional expectation is a bit long to evaluate and some efforts have to be done in this direction. Things can be improved if we obtain a deterministic shape of the ideal penalty, as in the independent case, since the slope heuristic is faster to compute with a deterministic Δ_m . We obtain upper and lower bounds on pen_{id} , but our inequalities are not sharp enough to justify completely the slope heuristic. We can also think of the V -fold cross validation penalties defined in Arlot (2008). These penalties are also faster to compute than the resampling penalties. They can be viewed as resampling penalties defined with nonexchangeable weights. These issues are far beyond the objectives of the present paper and will be addressed in forthcoming works.

5. Proofs.

5.1. *Notation.* Recall that p and q are integers such that $2pq = n$, and that $\sqrt{n}(\ln n)^2/2 \leq p \leq \sqrt{n}(\ln n)^2$. For all $k = 0, \dots, p - 1$, let $I_k = (2kq + 1, \dots, (2k + 1)q)$, $A_k = (X_i)_{i \in I_k}$ and $I = \bigcup_{k=0}^{p-1} I_k$. For all t in $L^2(\mu)$ and all x_1, \dots, x_q in \mathbb{R} ,

$$L_q(t)(x_1, \dots, x_q) = \frac{1}{q} \sum_{i=1}^q t(x_i), \quad P_A t = \frac{1}{p} \sum_{k=0}^{p-1} L_q(t)(A_k) = \frac{2}{n} \sum_{i \in I} t(X_i),$$

$$v_A(t) = (P_A - P)(t).$$

For all m in \mathcal{M}_n , we denote by $(\psi_\lambda)_{\lambda \in \Lambda_m}$ an orthonormal basis of S_m . The estimator $\widehat{s}_{A,m}$ associated to the model S_m , is defined as

$$\widehat{s}_{A,m} = \sum_{\lambda \in \Lambda_m} (P_A \psi_\lambda) \psi_\lambda.$$

Classical computations show that, if s_m denotes the orthogonal projection of s onto S_m ,

$$s_m = \sum_{\lambda \in \Lambda_m} (P \psi_\lambda) \psi_\lambda, \quad \text{hence } \|\widehat{s}_{A,m} - s_m\|^2 = \sum_{\lambda \in \Lambda_m} (v_A \psi_\lambda)^2.$$

The ideal penalty, $2v_A(\widehat{s}_{A,m})$ satisfies

$$v_A(\widehat{s}_{A,m} - s_m) + v_A(s_m) = \sum_{\lambda \in \Lambda_m} (v_A \psi_\lambda)^2 + v_A(s_m) = \|\widehat{s}_{A,m} - s_m\|^2 + v_A(s_m).$$

For all m, m' in \mathcal{M}_n , let

$$p(m) = \|s_m - \widehat{s}_{A,m}\|^2 = \sup_{t \in B_m} (v_A(t))^2 = \sum_{\lambda \in \Lambda_m} (v_A(\psi_\lambda))^2,$$

$$\delta(m, m') = 2v_A(s_m - s_{m'}).$$

Hereafter W_0, \dots, W_{p-1} denotes a resampling scheme, $\overline{W} = p^{-1} \sum_{i=0}^{p-1} W_i$, P_A^W denotes the resampling empirical process, defined for all measurable functions t by

$$P_A^W t = \frac{1}{p} \sum_{i=0}^{p-1} W_i L_q t(A_i).$$

We introduce also $v_A^W = P_A^W - \overline{W} P_A$ and $\tilde{C}_W = (\text{Var}(W_1 - \overline{W}))^{-1}$. For any orthonormal basis $(\psi_\lambda)_{\lambda \in \Lambda_m}$ of S_m , let

$$p_W(m) = \tilde{C}_W \sum_{\lambda \in \Lambda_m} \mathbb{E}_W((v_A^W(\psi_\lambda))^2).$$

$p_W(m)$ is well defined since, from the Cauchy–Schwarz inequality,

$$p_W(m) = \tilde{C}_W \mathbb{E}_W \left(\sup_{t \in B_m} (v_A^W t)^2 \right).$$

Let $\varepsilon_n = (\ln n)^{-1/2}$ and let $\kappa > 0$. Let \mathcal{M} denote one of the set \mathcal{M}_n or \mathcal{M}_n^2 . When $\mathcal{M} = \mathcal{M}_n$, for all \bar{m} in \mathcal{M} let $R_{A,\bar{m}} = R_{A,m}$ and when $\mathcal{M} = \mathcal{M}_n^2$, for all $\bar{m} = (m, m')$ in \mathcal{M} , let $R_{A,\bar{m}} = R_{A,m} \vee R_{A,m'}$. For all \bar{m} in \mathcal{M} , let

$$(5.1) \quad f_1(\bar{m}, \kappa) = p(m) - \frac{2D_{A,m}}{n} - \kappa \varepsilon_n \frac{R_{A,m}}{n},$$

$$(5.2) \quad f_2(\bar{m}, \kappa) = \frac{2D_{A,m}}{n} - p(m) - \kappa \varepsilon_n \frac{R_{A,m}}{n},$$

$$(5.3) \quad f_3(\bar{m}, \kappa) = p(m) - p_W(m) - \kappa \varepsilon_n \frac{R_{A,m}}{n},$$

$$(5.4) \quad f_4(\bar{m}, \kappa) = p_W(m) - p(m) - \kappa \varepsilon_n \frac{R_{A,m}}{n},$$

$$(5.5) \quad f_5(\bar{m}, \kappa) = \delta(m, m') - \kappa \varepsilon_n \frac{R_{A,m} \vee R_{A,m'}}{n}.$$

We will use the following fact.

FACT 0. *The resampling penalty $\text{pen}_W(m, C)$ defined in (2.5) satisfies*

$$\text{pen}_W(m, C) = 2C \tilde{C}_W^{-1} p_W(m).$$

PROOF. Let $(\psi_\lambda)_{\lambda \in \Lambda_m}$ be an orthonormal basis of S_m . Recall that $\hat{s}_{A,m}^W = \sum_{\lambda \in \Lambda_m} (P_A^W \psi_\lambda) \psi_\lambda$, so that

$$\hat{s}_{A,m}^W - \bar{W} \hat{s}_{A,m} = \sum_{\lambda \in \Lambda_m} (v_A^W \psi_\lambda) \psi_\lambda.$$

Hence, $v_A^W (\hat{s}_{A,m}^W - \bar{W} \hat{s}_{A,m}) = \sum_{\lambda \in \Lambda_m} (v_A^W \psi_\lambda)^2$.

We conclude the proof showing that $\mathbb{E}_W(v_A^W (\bar{W} \hat{s}_{A,m})) = 0$, hence

$$\frac{p_W(m)}{\tilde{C}_W} = \mathbb{E}_W(v_A^W (\hat{s}_{A,m}^W - \bar{W} \hat{s}_{A,m})) = \mathbb{E}_W(v_A^W (\hat{s}_{A,m}^W)) = \frac{\text{pen}_W(m, C)}{2C}.$$

Since W_0, \dots, W_{p-1} are independent of X_1, \dots, X_n ,

$$\mathbb{E}_W(v_A^W (\bar{W} \hat{s}_{A,m})) = \frac{1}{p^2} \sum_{i,j=0}^{p-1} L_q(\psi_\lambda)(A_i) L_q(\psi_\lambda)(A_j) \mathbb{E}_W(W_i \bar{W} - (\bar{W})^2).$$

Then, by exchangeability of the weights,

$$\begin{aligned} \mathbb{E}_W(W_i \bar{W} - (\bar{W})^2) &= \frac{1}{p} \left(\mathbb{E}(W_i^2) + \sum_{j \neq i} \mathbb{E}(W_i W_j) \right) \\ &\quad - \frac{1}{p^2} \left(\sum_i \mathbb{E}(W_i^2) + \sum_{i \neq j} \mathbb{E}(W_i W_j) \right) = 0. \end{aligned} \quad \square$$

5.2. *Proof of Theorem 3.1.* The proof is based on the following lemma, whose proof is given in Lerasle (2011b).

LEMMA 5.1. *Let X_1, \dots, X_n be a strictly stationary sequence of real valued random variables with common density s and let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of regular wavelet spaces $[W]$ satisfying assumptions (H3), (H4). Let p, q be two integers satisfying $2pq = n$ and $\frac{1}{2}\sqrt{n}(\ln n)^2 \leq p \leq \sqrt{n}(\ln n)^2$.*

Assume that there exists $\theta > 5$ such that X_1, \dots, X_n are arithmetically $[\text{AR}(\theta)]$ τ -mixing and satisfy $S(\tau, W)$. There exist constants κ_1, κ_2 , such that, for all $i = 1, \dots, 5$, for all \bar{m} in \mathcal{M} ,

$$(5.6) \quad \mathbb{E} \left(\sup_{\bar{m} \in \mathcal{M}} (f_i(\bar{m}, \kappa_1))_+ \right) \leq \frac{\kappa_2}{n}.$$

It comes from Fact 0 and the equality $2C\tilde{C}_W = \kappa(C) + 2$ that, for all m in \mathcal{M}_n ,

$$(5.7) \quad \text{pen}_W(m, C) - (2 + \kappa(C))p(m) = 2C\tilde{C}_W^{-1}(p_W(m) - p(m)).$$

Hence, from (5.6) with $i = 3, 4$, $\text{pen}_W(m, C)$ satisfies conditions (3.6) and (3.5) of Theorem 3.3 with $\delta_+ = -\delta_- = \kappa(C)$ and $\varepsilon = 2\kappa_1 C\tilde{C}_W^{-1} \varepsilon_n$. Theorem 3.1 follows from (3.7).

5.3. *Proof of Theorem 4.1.* The proof is based on the following lemma whose proof is given in additional material.

LEMMA 5.2. *Let $\theta > 1$ and let $(X_n)_{n \in \mathbb{Z}}$ be an arithmetically $[\text{AR}(\theta)]$ β -mixing process satisfying $S(\beta)$. Let $(S_m)_{m \in \mathcal{M}_n}$ be a collection of linear spaces satisfying assumptions (H1)–(H4). Let p, q such that $2pq = n$, $\sqrt{n}(\ln n)^2/2 \leq p \leq \sqrt{n}(\ln n)^2$. There exist constants κ_1, κ_2 and an event Ω_n satisfying*

$$\mathbb{P}(\Omega_n) \geq 1 - \kappa_2 \left(\frac{(\ln n)^{2(1+\theta)}}{n^{\theta/2}} \vee \frac{1}{n^2} \right),$$

such that, on Ω_n ,

$$(5.8) \quad \forall \bar{m} \in \mathcal{M}, \forall i = 1, \dots, 5 \quad f_i(\bar{m}) \leq 0.$$

Hence, from (5.6) with $i = 3, 4$, $\text{pen}_W(m, C)$ satisfies condition (4.3) of Theorem 4.3 with $\delta_+ = -\delta_- = \kappa(C)$ and $\varepsilon = 2\kappa_1 C\tilde{C}_W^{-1} \varepsilon_n$. Theorem 4.1 follows from (4.4).

5.4. *Proof of Theorems 3.2 and 4.2.* It is sufficient to prove the results for sufficiently large n since we can increase the constant κ_2 if necessary. Let m_o be a model such that $R_{A,m_o} = R_n$. Now, by definition, \widehat{m} minimizes among \mathcal{M}_n the following criterion:

$$\text{Crit}(m) = \|\widehat{s}_{A,m}\|^2 - 2P_A\widehat{s}_{A,m} + \text{pen}(m) + \|s\|^2 + 2v_A(s_{m_o}).$$

FACT 1. *For all m in \mathcal{M}_n ,*

$$\text{Crit}(m) = \|s_m - s\|^2 + \text{pen}(m) - p(m) + 2v_A(s_{m_o} - s_m).$$

PROOF. Recalling that $\|s - \widehat{s}_{A,m}\|^2 = \|\widehat{s}_{A,m}\|^2 - 2P\widehat{s}_{A,m} + \|s\|^2$ and that $(P_A - P)(\widehat{s}_{A,m} - s_m) = \|\widehat{s}_{A,m} - s_m\|^2 = p(m)$, we have,

$$\begin{aligned} \text{Crit}(m) &= \|s - \widehat{s}_{A,m}\|^2 - 2v_A(\widehat{s}_{A,m} - s_m) + 2v_A(s_{m_o} - s_m) + \text{pen}(m) \\ &= (\|s - \widehat{s}_{A,m}\|^2 - \|\widehat{s}_{A,m} - s_m\|^2) - p(m) + \text{pen}(m) + 2v_A(s_{m_o} - s_m). \end{aligned}$$

We conclude the proof with the Pythagoras equality. \square

FACT 2. *For all m in \mathcal{M}_n , for all constants κ_1 ,*

$$\begin{aligned} (1 + 2\kappa_1\varepsilon_n)\frac{2D_{A,m}}{n} &\geq -\text{Crit}(m) + (1 - 2\kappa_1\varepsilon_n)\|s - s_m\|^2 \\ &\quad - \sup_{m \in \mathcal{M}_n} (f_1(m, \kappa_1)) - \sup_{(m,m') \in \mathcal{M}_n^2} (f_5((m, m'), \kappa_1)). \end{aligned}$$

PROOF. From Fact 1, for all m in \mathcal{M}_n , for all κ_1 , since $\text{pen}(m) \geq 0$,

$$\text{Crit}(m) \geq \|s_m - s\|^2 - f_1(m, \kappa_1) - \frac{2D_{A,m}}{n} - 2\kappa_1\varepsilon_n \frac{R_{A,m}}{n} - f_5((m_o, m), \kappa_1).$$

We conclude the proof using that $R_{A,m} = n\|s - s_m\|^2 + 2D_{A,m}$. \square

FACT 3. *For all m in \mathcal{M}_n , for all constants κ_1 ,*

$$\begin{aligned} (\delta - 4\kappa_1\varepsilon_n)\frac{D_{A,m}}{n} &\leq -\text{Crit}(m) + (1 + 2\kappa_1\varepsilon_n)\|s - s_m\|^2 \\ &\quad + \sup_{m \in \mathcal{M}_n} (f_2(m, \kappa_1)) + \sup_{(m,m') \in \mathcal{M}_n^2} (f_5((m, m'), \kappa_1)). \end{aligned}$$

PROOF. From Fact 1, for all m in \mathcal{M}_n , for all κ_1 , since $\text{pen}(m) \leq (2 - \delta)D_{A,m}/n$,

$$\text{Crit}(m) \leq \|s_m - s\|^2 + f_2(m, \kappa_1) - \delta \frac{D_{A,m}}{n} + 2\kappa_1\varepsilon_n \frac{R_{A,m}}{n} + f_5((m, m_o), \kappa_1).$$

We conclude the proof using that $R_{A,m} = n\|s - s_m\|^2 + 2D_{A,m}$. \square

From Fact 2, we have, for all κ_1 ,

$$(1 + 2\kappa_1\varepsilon_n)\frac{2D_{A,\widehat{m}}}{n} \geq -\text{Crit}(\widehat{m}) + (1 - 2\kappa_1\varepsilon_n)\|s - s_{\widehat{m}}\|^2 - \sup_{m \in \mathcal{M}_n} (f_1(m, \kappa_1)) - \sup_{(m,m') \in \mathcal{M}_n^2} (f_5((m, m'), \kappa_1)).$$

Let us now consider a model m^* such that $D_{A,m^*} = D_n^*$. By definition of \widehat{m} , we have $\text{Crit}(\widehat{m}) \leq \text{Crit}(m^*)$. Hence, from Fact 3, we deduce that

$$(5.9) \quad (1 + 2\kappa_1\varepsilon_n)\frac{2D_{A,\widehat{m}}}{n} \geq -\text{Crit}(m^*) + (1 - 2\kappa_1\varepsilon_n)\|s - s_{\widehat{m}}\|^2 - \sup_{m \in \mathcal{M}_n} (f_1(m, \kappa_1)) - \sup_{(m,m') \in \mathcal{M}_n^2} (f_5((m, m'), \kappa_1)).$$

$$\geq \left(\delta - 4\kappa_1\varepsilon_n - (1 + 2\kappa_1\varepsilon_n)\frac{\|s - s_{m^*}\|^2}{D_n^*} \right) \frac{D_{A,m^*}}{n} + (1 - 2\kappa_1\varepsilon_n)\|s - s_{\widehat{m}}\|^2 - 4 \sup_{i \in \{1,2,5\}, \overline{m} \in \mathcal{M}} (f_i(\overline{m}, \kappa_1)).$$

From Lemma 5.1, there exist κ_1 and κ_2 such that

$$\mathbb{E}\left(4 \sup_{i \in \{1,2,5\}, \overline{m} \in \mathcal{M}} (f_i(\overline{m}, \kappa_1))_+\right) \leq \frac{\kappa_2}{n}.$$

From Lemma 5.2, there exists κ_1 such that, on Ω_n ,

$$4 \sup_{i \in \{1,2,5\}, \overline{m} \in \mathcal{M}} (f_i(\overline{m}, \kappa_1)) \leq 0.$$

Now, assume that n is sufficiently large to ensure that

$$4\kappa_1\varepsilon_n \leq \frac{\delta}{4} \leq \frac{1}{4}, \quad \frac{n\|s - s_{m^*}\|^2}{D_n^*} \leq \frac{2\delta}{9}.$$

Then, taking the expectation in (5.9), we obtain that

$$\frac{9\mathbb{E}(D_{A,\widehat{m}})}{8n} \geq \frac{\delta}{2} \frac{D_n^*}{n} - \frac{\kappa_4}{n}.$$

Hence, (3.3) is proved for n sufficiently large.

Moreover, on Ω_n , we have

$$\frac{9D_{A,\widehat{m}}}{8n} \geq \frac{\delta}{2} \frac{D_n^*}{n}.$$

Hence, the first inequality of (4.2) is proved for n sufficiently large. (3.4) and the second inequality of (4.2) follow from the inequality

$$\|s - \tilde{s}_A\|^2 \geq (1 - \kappa_1\varepsilon_n)\frac{R_{A,\widehat{m}}}{n} - f_2(\widehat{m}, \kappa_1).$$

From Lemma 5.1, there exist constants κ_1, κ_2 , such that $\mathbb{E}(f_2(\widehat{m}, \kappa_1)) \leq \kappa_2/n$. We choose n sufficiently large to ensure that $\kappa_1 \varepsilon_n \leq 1/2$, we use (3.3) and we obtain that there exists a constant κ such that

$$\mathbb{E}(\|s - \tilde{s}_A\|^2) \geq \frac{2\delta}{9} \frac{D_n^* - \kappa}{n}.$$

We conclude the proof of (3.4) with the following fact.

FACT 4.

$$\frac{R_n}{n} \geq \frac{16}{17} \mathbb{E}\left(\inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2\right) - \frac{\kappa}{n},$$

thus

$$\frac{D_n^*}{n} \geq \frac{16D_n^*}{17R_n} \left(\mathbb{E}\left(\inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2\right) - \frac{\kappa}{n}\right).$$

PROOF. Let κ_1 be the constant previously defined,

$$\inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2 \leq (1 + \kappa_1 \varepsilon_n) \inf_{m \in \mathcal{M}_n} \left\{ \frac{R_{A,m}}{n} \right\} + \sup_{m \in \mathcal{M}_n} f_1(m, \kappa_1).$$

We conclude the proof with Lemma 5.1. \square

We use the first inequality of (4.2) and we obtain that, on Ω_n ,

$$\|s - \tilde{s}_A\|^2 \geq \frac{2\delta}{9} \frac{D_n^*}{n}.$$

We conclude the proof of Theorem 4.2, saying that, on Ω_n , we have

$$\begin{aligned} \frac{R_n}{n} &= \inf_{m \in \mathcal{M}_n} \left\{ \|s - s_m\|^2 + \frac{2D_{A,m}}{n} \right\} \geq (1 - \kappa_1 \varepsilon_n) \inf_{m \in \mathcal{M}_n} \{ \|s - s_m\|^2 + p(m) \} \\ &= (1 - \kappa_1 \varepsilon_n) \inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2 \geq \frac{15}{16} \inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2. \end{aligned}$$

Thus,

$$\|\tilde{s}_A - s\|^2 \geq \frac{2\delta}{9} \frac{D_n^*}{R_n} \frac{R_n}{n} \geq \frac{\delta}{9} \frac{D_n^*}{R_n} \inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2.$$

5.5. *Proofs of Theorems 3.3 and 4.3.* As in the previous proof, it is sufficient to obtain the results for sufficiently large n . Let us first prove the oracle inequalities. Let κ_1 be a constant to be chosen later. Let Ω_n be the set defined on Lemma 5.2. The key point to prove oracle inequalities is the following fact.

FACT 5. For all m in \mathcal{M}_n , for all real numbers δ_-, δ_+ and for all nonnegative reals x, y ,

$$(5.10) \quad \begin{aligned} & [(1 \wedge (1 - \delta_-)) - x - y] \|s - \tilde{s}_A\|^2 \\ & \leq [(1 \vee (1 + \delta_+)) + x + y] \|s - \hat{s}_{A,m}\|^2 \\ & \quad + \sup_{m \in \mathcal{M}_n} \{ \text{pen}(m) - (2 + \delta_+) \|\hat{s}_{A,m} - s_m\|^2 - x \|s - \hat{s}_{A,m}\|^2 \}_+ \end{aligned}$$

$$(5.11) \quad + \sup_{m \in \mathcal{M}_n} \{ (2 - \delta_-) \|\hat{s}_{A,m} - s_m\|^2 - \text{pen}(m) - x \|s - \hat{s}_{A,m}\|^2 \}_+$$

$$(5.12) \quad + 2 \sup_{(m,m') \in \mathcal{M}_n^2} \{ v_A(s_{m'} - s_m) - y (\|s - \hat{s}_{A,m}\|^2 + \|s - \hat{s}_{A,m'}\|^2) \}_+.$$

PROOF. By definition of \tilde{s}_A , for all m in \mathcal{M}_n , we have

$$\|\tilde{s}_A\|^2 - 2P_A \tilde{s}_A + \text{pen}(\hat{m}) + \|s\|^2 \leq \|\hat{s}_{A,m}\|^2 - 2P_A \hat{s}_{A,m} + \text{pen}(m) + \|s\|^2.$$

Now, for all m in \mathcal{M}_n , since $\|\hat{s}_{A,m} - s\|^2 = \|\hat{s}_{A,m}\|^2 - 2P \hat{s}_{A,m} + \|s\|^2$,

$$\|\hat{s}_{A,m}\|^2 - 2P_A \hat{s}_{A,m} + \|s\|^2 = \|\hat{s}_{A,m} - s\|^2 - 2(P_A - P) \hat{s}_{A,m}.$$

Thus, for all m in \mathcal{M}_n ,

$$\|\tilde{s}_A - s\|^2 - 2(P_A - P) \tilde{s}_A + \text{pen}(\hat{m}) \leq \|\hat{s}_{A,m} - s\|^2 - 2(P_A - P) \hat{s}_{A,m} + \text{pen}(m).$$

For all m in \mathcal{M}_n , since $(P_A - P)(\hat{s}_{A,m} - s_m) = \|\hat{s}_{A,m} - s\|^2$,

$$2(P_A - P) \hat{s}_{A,m} = 2\|s_m - \hat{s}_{A,m}\|^2 + 2(P_A - P) s_m.$$

This yields

$$\begin{aligned} \|s - \tilde{s}_A\|^2 & \leq \|s - \hat{s}_{A,m}\|^2 + \text{pen}(m) - 2\|\hat{s}_{A,m} - s_m\|^2 \\ & \quad + 2\|\hat{s}_{A,\hat{m}} - s_{\hat{m}}\|^2 - \text{pen}(\hat{m}) + 2v_A(s_{\hat{m}} - s_m). \end{aligned}$$

We add $-[(\delta_- \vee 0) + (x + y)] \|\tilde{s}_A - s\|^2$ to the left-hand side of the previous inequality and $-\delta_- \|\tilde{s}_A - s_{\hat{m}}\|^2 - (x + y) \|\tilde{s}_A - s\|^2 + [(\delta_+ \vee 0) + x + y] \|s - \hat{s}_{A,m}\|^2 - \delta_+ \|\hat{s}_{A,m} - s_m\|^2 - (x + y) \|s - \hat{s}_{A,m}\|^2$ to the right-hand side. This is valid because, for all m in \mathcal{M}_n , for all reals δ ,

$$[(\delta \vee 0) + x + y] \|\hat{s}_{A,m} - s\|^2 \geq \delta \|\hat{s}_{A,m} - s_m\|^2 + (x + y) \|\hat{s}_{A,m} - s\|^2.$$

We obtain

$$\begin{aligned} & [(1 \wedge (1 - \delta_-)) - x - y] \|s - \tilde{s}_A\|^2 \\ & \leq [(1 \vee (1 + \delta_+)) + x + y] \|s - \hat{s}_{A,m}\|^2 \\ & \quad + \text{pen}(m) - (2 + \delta_+) \|\hat{s}_{A,m} - s_m\|^2 - x \|\hat{s}_{A,m} - s\|^2 \\ & \quad + (2 - \delta_-) \|\hat{s}_{A,\hat{m}} - s_{\hat{m}}\|^2 - \text{pen}(\hat{m}) - x \|\hat{s}_{A,\hat{m}} - s\|^2 \\ & \quad + 2v_A(s_{\hat{m}} - s_m) - y \|\hat{s}_{A,m} - s\|^2 - x \|\hat{s}_{A,\hat{m}} - s\|^2. \quad \square \end{aligned}$$

We will also use the following fact.

FACT 6. For all reals κ such that $\kappa\varepsilon_n \leq 1/2$,

$$\frac{R_{A,m}}{n} \leq 2\|s - \widehat{s}_{A,m}\|^2 + 2\{f_2(m, \kappa)\}_+.$$

PROOF. We write

$$\frac{R_{A,m}}{n} = \frac{1}{1 - \kappa\varepsilon_n} \left(\frac{R_{A,m}}{n} - \|s - \widehat{s}_{A,m}\|^2 - \kappa\varepsilon_n \frac{R_{A,m}}{n} \right) + \frac{1}{1 - \kappa\varepsilon_n} \|s - \widehat{s}_{A,m}\|^2.$$

We use that $\kappa\varepsilon_n \leq 1/2$ and that $R_{A,m} = 2D_{A,m} + ns - s_m^2$ to conclude the proof. □

Control of (5.10). Assume that n is sufficiently large to ensure that $\kappa_1\varepsilon_n \leq 1/2$. We have, from Fact 6,

$$\begin{aligned} & \text{pen}(m) - (2 + \delta_+)p(m) - 2\varepsilon\|\widehat{s}_{A,m} - s\|^2 \\ & \leq \text{pen}(m) - (2 + \delta_+)p(m) - \varepsilon\frac{R_{A,m}}{n} + 2\varepsilon\{f_2(m, \kappa_1)\}_+. \end{aligned}$$

Applying Lemma 5.1, we obtain constants κ_1 and κ_2 such that

$$\mathbb{E} \left(\sup_{m \in \mathcal{M}_n} \{f_2(m, \kappa_1)\}_+ \right) \leq \frac{\kappa_2}{n}.$$

Applying Lemma 5.2, we obtain a constant κ_1 such that, on Ω_n ,

$$\sup_{m \in \mathcal{M}_n} \{f_2(m, \kappa_1)\}_+ \leq 0.$$

Moreover, (3.6) ensures that

$$\mathbb{E} \left(\sup_{m \in \mathcal{M}_n} \left\{ \text{pen}(m) - (2 + \delta_+)p(m) - \varepsilon\frac{R_{A,m}}{n} \right\}_+ \right) \leq \frac{\kappa}{n}.$$

On Ω_{pen} , we have

$$\sup_{m \in \mathcal{M}_n} \left\{ \text{pen}(m) - (2 + \delta_+)p(m) - \varepsilon\frac{R_{A,m}}{n} \right\}_+ \leq 0.$$

We choose $x = 2\varepsilon$. We obtain that, for Theorem 3.1, the expectation of (5.10) is upper bounded by κn^{-1} and for Theorem 4.1, the term (5.10) is equal to 0 on $\Omega_n \cap \Omega_{\text{pen}}$.

Control of (5.11). Assume that n is sufficiently large to ensure that $\kappa_1\varepsilon_n < 1/2$, we deduce from Fact 6 that

$$\begin{aligned} & (2 - \delta_-)p(m) - \text{pen}(m) - 2\varepsilon\|\widehat{s}_{A,m} - s\|^2 \\ & \leq (2 - \delta_-)p(m) - \text{pen}(m) - \varepsilon\frac{R_{A,m}}{n} + 2\varepsilon\{f_2(m, \kappa_1)\}_+. \end{aligned}$$

Applying Lemma 5.1, we obtain constants κ_1 and κ_2 such that

$$\mathbb{E}\left(\sup_{m \in \mathcal{M}_n} \{f_2(m, \kappa_1)\}_+\right) \leq \frac{\kappa_2}{n}.$$

Applying Lemma 5.2, we obtain a constant κ_1 such that, on Ω_n ,

$$\sup_{m \in \mathcal{M}_n} \{f_2(m, \kappa_1)\}_+ \leq 0.$$

Moreover, (3.5) ensures that

$$\mathbb{E}\left(\sup_{m \in \mathcal{M}_n} \left\{ (2 - \delta_-)p(m) - \text{pen}(m) - \varepsilon \frac{R_{A,m}}{n} \right\}_+\right) \leq \frac{\kappa}{n}.$$

On Ω_{pen} , we have

$$\sup_{m \in \mathcal{M}_n} \left\{ (2 - \delta_-)p(m) - \text{pen}(m) - \varepsilon \frac{R_{A,m}}{n} \right\}_+ \leq 0.$$

We choose $x = 2\varepsilon$. We obtain that, for Theorem 3.1, the expectation of (5.11) is upper bounded by κn^{-1} and for Theorem 4.1, the term (5.11) is equal to 0 on $\Omega_n \cap \Omega_{\text{pen}}$.

Control of (5.12). Let m, m' in \mathcal{M}_n and let m_s be the index such that $R_{A,m_s} = R_{A,m} \vee R_{A,m'}$ and let κ_1 be a constant to be chosen later. Assume that n is sufficiently large to ensure that $\kappa_1 \varepsilon_n \leq 1/2$. It comes from Fact 6 that

$$\begin{aligned} \delta(m, m') &= f_5((m, m'), \kappa_1) + \kappa_1 \varepsilon_n \frac{R_{A,m_s}}{n} \\ &\leq f_5((m, m'), \kappa_1) + 2\kappa_1 \varepsilon_n \|\widehat{s}_{A,m_s} - s\|^2 + 2\kappa_1 \varepsilon_n \{f_2(m_s, \kappa_1)\}_+. \end{aligned}$$

We deduce from Lemma 5.1 that there exist κ_1 and κ_2 such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{(m,m') \in \mathcal{M}_n^2} \{\delta(m, m') - 2\kappa_1 \varepsilon_n (\|\widehat{s}_{A,m} - s\|^2 + \|\widehat{s}_{A,m'} - s\|^2)\}\right) \\ &\leq \mathbb{E}\left(\sup_{(m,m') \in \mathcal{M}_n^2} \{f_5((m, m'), \kappa_1) + 2\kappa_1 \varepsilon_n f_2(m_s, \kappa_1)\}_+\right) \leq \frac{\kappa_2}{n}. \end{aligned}$$

Applying Lemma 5.2, we obtain a constant κ_1 such that, on Ω_n ,

$$\sup_{(m,m') \in \mathcal{M}_n^2} \{\delta(m, m') - 2\kappa_1 \varepsilon_n (\|\widehat{s}_{A,m} - s\|^2 + \|\widehat{s}_{A,m'} - s\|^2)\} \leq 0.$$

Conclusion of the proofs. We use Fact 5 with $x = 2\varepsilon$ and $y = 2\kappa_1 \varepsilon_n$. We take the expectation for the proof of Theorem 3.1, we have obtained that the expectation of the remainder terms (5.10)–(5.12) are upper bounded by κn^{-1} for a sufficiently large n . For the proof of Theorem 4.1, we have obtained that the remainder terms (5.10)–(5.12) with $x = 2\varepsilon$ and $y = 2\kappa_1 \varepsilon_n$ are equal to 0 on $\Omega_n \cap \Omega_{\text{pen}}$ when n

is sufficiently large. As explained in the beginning of the proof, this is sufficient to conclude the proof of (3.7) and (4.4).

Let us prove (3.8). Let $\kappa_1 < 1/(2\varepsilon_n)$, from Fact 6 and (3.7), we have

$$\begin{aligned} \frac{K_n}{n} \mathbb{E}(2D_{A,\widehat{m}}) &\leq K_n \left(\mathbb{E}(p(\widehat{m})) + \mathbb{E}(f_2(\widehat{m}, \kappa_1)) + \kappa_1 \varepsilon_n \mathbb{E}\left(\frac{R_{A,\widehat{m}}}{n}\right) \right) \\ &\leq (1 + 2\kappa_1 \varepsilon_n) \mathbb{E}((f_2(\widehat{m}, \kappa_1))_+) + (1 + 2\kappa_1 \varepsilon_n) K_n \mathbb{E}(\|s - \tilde{s}_A\|^2) \\ &\leq 2\mathbb{E}((f_2(\widehat{m}, \kappa_1))_+) + 2K_n \mathbb{E}(\|s - \tilde{s}_A\|^2) \\ &\leq 2 \left(\mathbb{E}((f_2(\widehat{m}, \kappa_1))_+) + \frac{\kappa}{n} \right) + 2R_n. \end{aligned}$$

We used that, by definition $K_n \leq 1$. We conclude the proof with Lemma 5.1.

In order to get the bound on $D_{A,\widehat{m}}$ in Theorem 4.3, we use that, on $\Omega_n \cap \Omega_{\text{pen}}$, (4.4) holds and there exists a constant κ_1 such that, $\kappa_1 \varepsilon_n < 1/2$ satisfying

$$\begin{aligned} K_n \frac{2D_{A,\widehat{m}}}{n} &\leq \frac{K_n}{1 - \kappa_1 \varepsilon_n} (\|s - s_{\widehat{m}}\|^2 + p^*(\widehat{m})) = \frac{K_n}{1 - \kappa_1 \varepsilon_n} \|s - \tilde{s}_A\|^2 \\ &\leq \frac{1}{1 - \kappa_1 \varepsilon_n} \inf_{m \in \mathcal{M}_n} \|s - \widehat{s}_{A,m}\|^2 \leq \frac{1 + \kappa_1 \varepsilon_n}{1 - \kappa_1 \varepsilon_n} \frac{R_n}{n} \leq 3 \frac{R_n}{n}. \end{aligned}$$

APPENDIX

We present in this section some classical collections of models and prove that they satisfy (H4).

Regular histograms: Let d be an integer and let S_d be the space of functions t constant on all the intervals $([k/d, (k + 1)/d))_{k \in \mathbb{Z}}$. S_d is called the space of regular histograms with size $1/d$. The family $(\psi_k)_{k \in \mathbb{Z}}$, where, for all k in \mathbb{Z} , $\psi_k = \sqrt{d} \mathbf{1}_{[k/d, (k+1)/d)}$ is an orthonormal basis of S_d . Let $B_d = \{t \in S_d, t^2 \leq 1\}$. From the Cauchy–Schwarz inequality, we have

$$\sup_{t \in B_d} t^2 = \sum_{k \in \mathbb{Z}} \psi_k^2 = d \mathbf{1}_{\mathbb{R}}.$$

Hence,

$$b_m^2 = \left\| \sup_{t \in B_d} t^2 \right\|_{\infty} = d, \quad P\left(\sup_{t \in B_d} t^2\right) = dP(\mathbf{1}_{\mathbb{R}}) = d.$$

(H4) holds on all the spaces S_d with $c_D = 1$, therefore, it holds on the collection $(S_d)_{d=1, \dots, n}$ called the regular histograms collection.

Fourier spaces: Let $k \geq 1$ be an integer and let, for all x in $[0, 1]$,

$$\psi_{1,k}(x) = \sqrt{2} \cos(2\pi kx), \quad \psi_{2,k}(x) = \sqrt{2} \sin(2\pi kx), \quad \psi_0 = \mathbf{1}_{[0,1]}.$$

Let $\mathcal{M}_n = \{1, \dots, n\}$ and $\forall m \in \mathcal{M}_n$, let $\Lambda_m = \{0, (1, k), (2, k), k = 1, \dots, m\}$. The space S_m , spanned by the family $(\psi_{\lambda})_{\lambda \in \Lambda_m}$ is called the Fourier space with

harmonic smaller than m and the collection $(S_m, m \in \mathcal{M}_n)$ is called the collection of Fourier spaces. Let $B_m = \{t \in S_m, t^2 \leq 1\}$. From the Cauchy–Schwarz inequality, for all x in $[0, 1]$,

$$\sup_{t \in B_m} t^2(x) = \sum_{\lambda \in \Lambda_m} \psi_\lambda^2(x) = 1 + 2 \sum_{k=1}^m (\cos^2(2\pi kx) + \sin^2(2\pi kx)) = 1 + 2m.$$

Hence, if P is supported in $[0, 1]$,

$$b_m^2 = \left\| \sup_{t \in B_m} t^2 \right\|_\infty = 1 + 2m, P\left(\sup_{t \in B_m} t^2\right) = 1 + 2m.$$

(H4) holds with $c_D = 1$ on the collection of Fourier spaces when P is supported on $[0, 1]$.

Wavelet spaces: Assume that $(S_m, m \in \mathcal{M}_n)$ is a collection of wavelet spaces [W]. Assume moreover that the scaling function ϕ and the mother wavelet ψ satisfy the following relation. There exists a constant $K_o > 0$ such that, for all x in \mathbb{R} ,

$$\frac{1}{K_o} \leq \sum_{k \in \mathbb{Z}} \phi^2(x - k) \leq K_o, \quad \frac{1}{K_o} \leq \sum_{k \in \mathbb{Z}} \psi^2(x - k) \leq K_o.$$

This condition is satisfied by the Haar basis, where $\phi = \mathbf{1}_{[0,1]}$, $\psi = \mathbf{1}_{[0,1/2]} - \mathbf{1}_{[1/2,1]}$, with $K_o = 1$. Then, for all $j \geq 0$, we have

$$\frac{1}{K_o} \leq \sum_{k \in \mathbb{Z}} \phi^2(2^j x - k) \leq K_o, \quad \frac{1}{K_o} \leq \sum_{k \in \mathbb{Z}} \psi^2(2^j x - k) \leq K_o.$$

Let $B_m = \{t \in S_m, t^2 \leq 1\}$. From the Cauchy–Schwarz inequality, we have

$$\Psi_m(x) = \sup_{t \in B_m} t^2(x) = \sum_{\lambda \in \Lambda_m} \psi_\lambda^2(x) = \sum_{k \in \mathbb{Z}} 2\phi^2(2x - k) + \sum_{j=1}^{J_m} 2^j \sum_{k \in \mathbb{Z}} \psi^2(2^j x - k).$$

We deduce that

$$\frac{2^{J_m}}{K_o} \leq \frac{1}{K_o} \left(2 + \sum_{j=1}^{J_m} 2^j \right) \leq \Psi_m(x) \leq K_o \left(2 + \sum_{j=1}^{J_m} 2^j \right) \leq 2K_o 2^{J_m}.$$

Hence, $b_m^2 = \Psi_{m\infty} \leq 2K_o 2^{J_m}$, $P(\Psi_m) \geq 2^{J_m} / K_o$.

(H4) holds with $c_D = 1/(2K_o^2)$ on the collection [W].

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SUPPLEMENTARY MATERIAL

Proofs of Lemmas 5.1 and 5.2 (DOI: 10.1214/11-AOS888SUPP; .pdf). In the Supplementary Material, we give complete proofs of the concentrations Lemmas 5.1 and 5.2. We use coupling results, respectively, of Berbee (1979) and Dedecker and Prieur (2005), to build sequences of independent random variables $(A_0^*, \dots, A_{p-1}^*)$ approximating the sequence of blocks (A_0, \dots, A_{p-1}) , respectively in the β and τ mixing case. We prove concentration lemmas equivalent to Lemmas 5.1 and 5.2 for these approximating random variables. The main tools here are the concentration inequalities of Bousquet (2002) and Klein and Rio (2005) for the maximum of the empirical process. We prove finally some covariance inequalities to evaluate the expectation of $p(m)$ and deduce the rates $\varepsilon_n = (\ln n)^{-1/2}$.

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