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CONVERGENCE OF CLOCK PROCESSES IN RANDOM ENVIRONMENTS AND AGEING IN THE *p*-SPIN SK MODEL

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We derive a general criterion for the convergence of clock processes in random dynamics in random environments that is applicable in cases when correlations are not negligible, extending recent results by Gayrard [(2010), (2011), forthcoming], based on general criterion for convergence of sums of dependent random variables due to Durrett and Resnick [*Ann. Probab.* **6** (1978) 829–846]. We demonstrate the power of this criterion by applying it to the case of random hopping time dynamics of the p-spin SK model. We prove that on a wide range of time scales, the clock process converges to a stable subordinator *almost surely* with respect to the environment. We also show that a time-time correlation function converges to the arcsine law for this subordinator, almost surely. This improves recent results of Ben Arous, Bovier and Černý [*Comm. Math. Phys.* **282** (2008) 663–695] that obtained similar convergence results in law, with respect to the random environment.

1. Introduction and main results. Over the last decades, random motion in random environments have been one of the main foci of research in applied probability theory and mathematical physics. This is due to the wide range of real life systems that can be modeled in this way, but also to the exciting, unforeseen and often counter-intuitive effects they exhibit. In fact, the early works of Solomon [25] and Sinai [24] on random walks in one-dimensional random environment were already striking examples of this feature.

While the most straightforward model class, the random walk in random environments on the lattice $\mathbb{Z}^d$, received the bulk of attention in the probability community, over the last decade, the study of the dynamics of spin glass models has attracted considerable attention in connection with the concept of aging. See, for example, [6] for a review. The dynamics of these models is expected to show very slow convergence to equilibrium, measurable in the anomalous behavior of certain time-time correlation functions.

Interesting models of the dynamics of spin glasses are Glauber dynamics on state spaces $\Sigma_n = \{-1, 1\}^n$, reversible with respect to Gibbs measures associated to random Hamiltonians, given by correlated Gaussian processes indexed by the

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hypercube Σ_n. Even on the nonrigorous level, predictions on their behavior were mostly based on the basis of drastically simplified *trap models* [10–12, 21, 22], based in turn on the ideas of Goldstein [19] to describe dynamics on long times scales in terms of thermally activated barrier crossings.

A rigorous analysis of many variants of such models was carried out over the last years [5, 7–9]. A striking feature that emerged in these works was the universal recurrence of the α-stable Lévy subordinators as basic random mechanisms in the description of the asymptotic properties of their dynamics. Another line of research tried to give a rigorous justification of the connection between spin glass dynamics and trap models. This was successful for the *Random Energy Model (REM)* of Derrida under a particular variant of the Glauber dynamics (the random hopping time dynamics, see below), first on times scales close to equilibrium [2–4] and later also on shorter time scales [8]. These results were partially extended to spin glasses with nontrivial correlations, the so-called p-spin SK models, by Ben Arous, Bovier and Černý [1]. Their results cover a limited range of times scales (in fact one expects a change of behavior at longer scales), and only in law with respect to the random environment, which in this case appears unnatural.

The recurrent appearance of stable subordinators in such a large variety of model systems asks for a simple and robust explanation. Such an explanation was given in a limited context of trap models by Ben Arous and Černý [8].

A more direct and general view on this problem was presented in a recent paper by one of us [15] and applied to more complicated situations in [16] and [17]. It emerges that the entire problem links up directly to a classical and well-studied field of probability theory, the convergence of sums of random variables to Lévy processes. The case of independent random variables has been well known since the work of Gnedenko and Kolmogorov [18], but a lot of work was done for the case of dependent random variables as well. In particular, there is a very amenable and useful criterion due to Durrett and Resnick [13] that we will rely on here.

Before entering in more detail, let us briefly describe the general setting of *Markov jump processes* in random environments that we consider here. Our arena is a sequence of loop-free graphs, $G_n(\mathcal{V}_n, \mathcal{L}_n)$ with set of vertices, $\mathcal{V}_n$, and set of edges, $\mathcal{L}_n$.

A *random environment* is a family of positive random variables, $\tau_n(x), x \in \mathcal{V}_n$, defined on some abstract probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. Note that we do not assume independence.

Next we define discrete time Markov processes, J_n, with state space $\mathcal{V}_n$ and nonzero transition probabilities along the edges, $\mathcal{L}_n$. We denote by μ_n its initial distribution and by $p_n(x, y)$ the elements of its transition matrix. Note that the p_n may be random variables on the space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the process J_n is reversible and admits a unique invariant measure π_n.

We construct our process of interest, X_n, as a time change of J_n. To this end we set

$$\lambda_n(x) \equiv C\pi_n(x)/\tau_n(x), \tag{1.1}$$

for some (model dependent) constant $C > 0$, and define the *clock process*

$$\widetilde{S}_n(k) = \sum_{i=0}^{k-1} \lambda_n^{-1}(J_n(i))e_{n,i}, \qquad k \in \mathbb{N}, \tag{1.2}$$

where $(e_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$ is a family of independent mean one exponential[2] random variables, independent of J_n.

We now define our continuous time process of interest, X_n, as

$$X_n(t) = J_n(i), \qquad \text{if } \widetilde{S}_n(i) \le t < \widetilde{S}_n(i+1) \text{ for some } i. \tag{1.3}$$

One can readily verify that X_n is a continuous time Markov process with infinitesimal generator λ_n, whose elements are

$$\lambda_n(x, y) = \lambda_n(x) p_n(x, y), \tag{1.4}$$

and whose unique invariant measure is given by

$$C\pi_n(x)\lambda_n^{-1}(x) = \tau_n(x). \tag{1.5}$$

Note that the numbers $\lambda_n^{-1}(x)$ play the role of the mean holding time of the process X_n in a site x.

For future reference, we refer to the σ-algebra generated by the variables J_n and X_n as $\mathcal{F}^J$ and $\mathcal{F}^X$, respectively. We write P_{μ_n} for the law of the process J_n, conditional on the σ-algebra $\mathcal{F}$, that is, for fixed realizations of the random environment. Likewise we call $\mathcal{P}_{\mu_n}$ the law of X_n conditional on $\mathcal{F}$.

This construction brings out the crucial role played by the clock process. If the chain J_n is rather fast mixing, convergence to equilibrium can only be slowed through an erratic behavior of the clock process. This process, on the other hand, is a sum of positive random variables, albeit in general dependent ones. The approach of [15] (and already [1]) is to abstract from all other issues and to focus on the analysis of the asymptotic behavior of the clock process. From that point onward, it is not surprising that stable subordinators will emerge as a standard class of limit processes; the universality appearing here is simply linked to the universal appearance of stable processes in the theory of sums of random variables.

In this paper we are mainly concerned with establishing criteria for the convergence of processes like (1.2) under suitable scaling; that is, we will ask when there are constants, a_n, c_n, such that the process

$$S_n(t) \equiv c_n^{-1}\widetilde{S}_n(\lfloor a_n t\rfloor) = c_n^{-1} \sum_{i=0}^{\lfloor a_n t\rfloor -1} \lambda_n^{-1}(J_n(i))e_{n,i}, \qquad t > 0, \tag{1.6}$$

converges in some sense to a limit process. Note that in physical terms, the constants c_n correspond to the time scale on which we observe our continuous time

[2]One can consider more general situations when $e_{n,i}$ have different distributions as well, leaving the setting of Markov processes.

Markov process X_n, while a_n corresponds to the number of steps the underlying process J_n makes during that time.

Due to the doubly stochastic nature of our processes, convergence can be considered in various modes, that is, under various laws. The physically most desirable one is referred to as *quenched*, that is, to say $\mathbb{P}$-almost sure convergence (to a deterministic or random process) under the law $\mathcal{P}_{\mu_n}$. In [1] another point of view was taken, namely P_{μ_n}-almost sure convergence under the law of the random medium and the exponential random variables $e_{n,i}$. Both imply the weakest form of convergence in law under the joint law of all random variables involved, often misleadingly referred to as *annealed*. The method used in [1] was based on the analysis of the Laplace transform of the clock process and the use of Gaussian comparison theorems. This left no way to deal with a fixed random environment. We will see, however, that we are to use heavily the computations from that paper.

1.1. *Key tools and strategy.* This approach is based on a powerful and illuminating method developed by Durrett and Resnick [13] to prove functional limit theorems for dependent variables. We state their theorem in a specialized form suitable for our applications, which is taken from [15] (see Theorem 2.1).

THEOREM 1.1. *Let Z_i^n be a triangular array of random variables with support in $\mathbb{R}_+$ defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let ν be a sigma-finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, such that $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$. Assume that there exists a sequence a_n, such that for all continuity points x of the distribution function of ν, for all $t > 0$, in $\mathcal{P}$-probability,*

$$\lim_{n\uparrow\infty} \sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{P}(Z_i^n > x | \mathcal{F}_{n,i-1}) = t\nu(x, \infty), \tag{1.7}$$

and

$$\lim_{n\uparrow\infty} \sum_{i=1}^{\lfloor a_n t\rfloor} [\mathcal{P}(Z_i^n > x | \mathcal{F}_{n,i-1})]^2 = 0, \tag{1.8}$$

where $\mathcal{F}_{n,i}$ denotes the σ-algebra generated by the random variables $Z_{n,j}$, $j \le i$. If, moreover,

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\uparrow\infty} \sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}\mathbb{1}_{Z_i^n \le \varepsilon} Z_i^n = 0, \tag{1.9}$$

then

$$\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i} \Rightarrow S_\nu(t), \tag{1.10}$$

where S_ν is the Lévy subordinator with Lévy measure ν and zero drift. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

REMARK. Condition (1.9) ensures that "small" terms in the sum do not contribute to the limit. It is almost a consequence of assumption (1.7) and the hypothesis on the limiting measure ν. However, in the general context of triangular arrays, one can easily construct counterexamples if (1.9) is not imposed.

REMARK. We emphasize that the result holds in the (usual) J_1-topology, since this is crucial for applications to correlation functions. See [26] for an extensive discussion of topologies on càdlàg spaces.

The straightforward idea is to apply this theorem with $Z_{n,i} \equiv c_n^{-1}\lambda_n^{-1}(J_n(i))e_{n,i}$. This was done in [15] (see Theorem 1.3.) and applied to the case of Bouchaud's trap models [15] and in the random energy model [16, 17] where it allowed the author to extend all previously know results in a very elegant way.

In models with strong local correlations, such as the p-spin SK model, one cannot, however, expect that with this choice the conditions of the theorem will be satisfied. In fact, one easily convinces oneself that contributions to the sum in (1.10) cannot only come from singly widely separated points i, but that such contributing terms form clusters due to the correlations.

In this paper we show that a good way to proceed in such a situation is to use a suitable blocking. Introduce a new scale, θ_n, and use Theorem 1.1 with the random variables

$$Z_{n,i} \equiv \sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(i))e_{n,i}, \qquad i \geq 1. \tag{1.11}$$

The purpose of this procedure is that if J_n is rapidly mixing, we can hope to choose $\theta_n \ll a_n$ such that the random variables $J_n(\theta_n i)$, $i \in \mathbb{N}$ are close to independent and distributed according to the invariant distribution π_n. But then, under the law $\mathcal{P}_{\mu_n}$, also the random variables $Z_{n,i}$ are close to independent and identically distributed (although with a complicated distribution, that is, a random variable depending on the random environment). That should put us in a position to verify the conditions of Theorem 1.1.

Let us now look at this in more detail.

For $y \in \mathcal{V}_n$ and $u > 0$, let

$$Q_n^u(y) \equiv \mathcal{P}_y\left(\sum_{j=0}^{\theta_n-1} \lambda_n^{-1}(J_n(j))e_{n,j} > c_n u\right) \tag{1.12}$$

be the tail distribution of the aggregated jumps when X_n starts in y. Note that $Q_n^u(y)$, $y \in \mathcal{V}_n$, is a random function on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and so is the function $F_n^u(y)$, $y \in \mathcal{V}_n$ defined through

$$F_n^u(y) \equiv \sum_{x \in \mathcal{V}_n} p_n(y,x)Q_n^u(x). \tag{1.13}$$

Writing $k_n(t) \equiv \lfloor \lfloor a_n t \rfloor / \theta_n \rfloor$, we further define

$$\nu_n^{J,t}(u,\infty) \equiv \sum_{i=0}^{k_n(t)-1} F_n^u\big(J_n(\theta_n(i))\big), \tag{1.14}$$

$$(\sigma_n^{J,t})^2(u,\infty) \equiv \sum_{i=0}^{k_n(t)-1} \big[F_n^u\big(J_n(\theta_n(i))\big)\big]^2. \tag{1.15}$$

Finally, we set

$$\bar{S}_n(k) \equiv \sum_{i=1}^{k}\left(\sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}\right) + c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0} \tag{1.16}$$

and

$$S_n^b(t) \equiv \bar{S}_n(k_n(t)). \tag{1.17}$$

We now formulate four conditions for the sequence S_n to converge to a subordinator. Note that these conditions refer to given sequences of numbers a_n, c_n and θ_n as well as a given realization of the random environment.

CONDITION (A1). There exists a σ-finite measure ν on $(0,\infty)$ satisfying the hypothesis stated in Theorem 1.1, and such that for all $t > 0$ and all $u > 0$,

$$P_{\mu_n}\big(|\nu_n^{J,t}(u,\infty) - t\nu(u,\infty)| < \varepsilon\big) = 1 - o(1) \qquad \forall \varepsilon > 0. \tag{1.18}$$

CONDITION (A2). For all $u > 0$ and all $t > 0$,

$$P_{\mu_n}\big((\sigma_n^{J,t})^2(u,\infty) < \varepsilon\big) = 1 - o(1) \qquad \forall \varepsilon > 0. \tag{1.19}$$

CONDITION (A3). For all $t > 0$,

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\uparrow\infty} \mathcal{E}_{\mu_n} \sum_{i=1}^{\lfloor a_n t\rfloor} \mathbb{1}_{\{\lambda_n^{-1}(J_n(i))e_i \le c_n\varepsilon\}} c_n^{-1}\lambda_n^{-1}(J_n(i))e_i = 0. \tag{1.20}$$

CONDITION (A0′). For all $v > 0$,

$$\sum_{x\in\mathcal{V}_n} \mu_n(x) e^{-v c_n \lambda_n(x)} = o(1). \tag{1.21}$$

THEOREM 1.2. *For all sequences of initial distributions μ_n and all sequences a_n, c_n and $1 \le \theta_n \ll a_n$, for which Conditions (A0′), (A1), (A2) and (A3) are verified, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability [meaning that the terms $o(1)$ converge to zero either almost surely or in probability, resp.], the following holds w.r.t. the same convergence mode*:

$$S_n^b(\cdot) \Rightarrow S_\nu(\cdot), \tag{1.22}$$

where S_ν is the Lévy subordinator with Lévy measure ν and zero drift. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

REMARK. Note that Condition (A0′) is there to ensure that last term in (1.16) converges to zero in the limit $n \uparrow \infty$.

REMARK. The result of this theorem is stated for the *blocked* process $S_n^b(t)$. It implies immediately that under the same hypothesis, the original process $S_n(t)$ [defined in (1.6)] converges to S_ν in the weaker M_1-topology; see [26] for a detailed discussion of Skorokhod topologies. However, the statement of the theorem is strictly stronger than just convergence in M_1, and it is this form that is useful in applications.

REMARK. To extract detailed information on the process X_n, for example the behavior of correlation functions, from the convergence of the blocked clock process, one needs further information on the typical behavior of the process during the θ_n steps of a single block. This is a model-dependent issue, and we will exemplify how this can be done in the context of the p-psin SK model.

We now come to the key step in our argument. This consists in reducing Conditions (A1) and (A2) of Theorem 1.2 to: (i) a *mixing condition* for the chain J_n and (ii) a *law of large numbers* for the random variables Q_n.

Again we formulate three conditions for given sequences a_n, c_n and a given realization of the random environment.

CONDITION (A1-1). Let J_n be a periodic Markov chain with period q. There exists an integer sequence $\ell_n \in \mathbb{N}$, and a positive decreasing sequence ρ_n, satisfying $\rho_n \downarrow 0$ as $n \uparrow \infty$, such that for all pairs $x, y \in \mathcal{V}_n$, and all $i \geq 0$,

$$\sum_{k=0}^{q-1} P_{\pi_n}\bigl(J_n(i+\ell_n+k)=y, J_n(i)=x\bigr) \leq (1+\rho_n)\pi_n(x)\pi_n(y). \tag{1.23}$$

CONDITION (A2-1). There exists a measure ν, as in condition (A1), such that

$$\nu_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \pi_n(x) Q_n^u(x) \to t\nu(u,\infty), \tag{1.24}$$

and

$$(\sigma_n^t)^2(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \sum_{x'\in\mathcal{V}_n} \pi_n(x) p_n^{(2)}(x,x') Q_n^u(x) Q_n^u(x') \to 0, \tag{1.25}$$

where $p_n^{(2)}(x,x') = \sum_{y\in\mathcal{V}_n} p_n(x,y)p_n(y,x')$ are 2-step transition probabilities.

CONDITION (A3-1). For all $t > 0$,

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\uparrow\infty} \lfloor a_n t \rfloor \mathcal{E}_{\pi_n} \mathbb{1}_{\{\lambda_n^{-1}(J_n(0))e_0 \leq c_n\varepsilon\}} c_n^{-1} \lambda_n^{-1}(J_n(0)) e_0 = 0. \tag{1.26}$$

REMARK. The limiting measure ν may be deterministic or random.

THEOREM 1.3. *Assume that for $\mu_n = \pi_n$ and for sequences a_n, c_n, ℓ_n and $\ell_n \leq \theta_n \ll a_n$, Conditions (A1-1), (A2-1), (A3-1) and (A0′) hold $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability. Then the sequence of random stochastic process S_n^b converges to the process S_ν, weakly in the Skorokhod space $D[0,\infty)$ equipped with the J_1-topology, $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability.*

1.2. *Application to the p-spin SK model.* Theorem 1.3 is the central result of this paper. It provides a very nice tool to prove convergence results of clock processes almost surely with respect to the random environment, that is the physically desirable mode. It is capable of dealing with correlations that have an effect, such as are present in the p-spin SK model. In this model, the underlying graphs $\mathcal{V}_n$ are the hypercubes $\Sigma_n = \{-1, 1\}^n$. On Σ_n we consider a Gaussian process, H_n, with zero mean and covariance

$$\mathbb{E} H_n(x) H_n(x') = n R_n(x, x')^p, \tag{1.27}$$

where $R_n(x, x') \equiv \frac{1}{n} \sum_{i=1}^n x_i x_i'$. The random environment, $\tau_n(x)$, is then defined in terms of H_n by

$$\tau_n(x) \equiv \exp(\beta H_n(x)), \tag{1.28}$$

with $\beta \in \mathbb{R}_+$ the inverse temperature. The Markov chain, J_n, is chosen as the simple random walk on Σ_n, that is,

$$p_n(x, x') = \begin{cases} \dfrac{1}{n}, & \text{if } \operatorname{dist}(x, x') = 1, \\ 0, & \text{else;} \end{cases} \tag{1.29}$$

here $\operatorname{dist}(\cdot, \cdot)$ is the graph distance on Σ_n,

$$\operatorname{dist}(x, x') \equiv \frac{1}{2} \sum_{i=1}^n |x_i - x_i'|. \tag{1.30}$$

This chain has for unique invariant measure the measure $\pi_n(x) = 2^{-n}$. Finally, choosing $C = 2^n$ in (1.1), the mean holding times, $\lambda_n^{-1}(x)$, reduce to $\lambda_n^{-1}(x) = \tau_n(x)$.

THEOREM 1.4. *For any $p \geq 3$, there exists a constant $K_p > 0$ that depends on β and γ, and a function $\zeta(p)$, such that for all γ satisfying*

$$0 < \gamma < \min(\beta^2, \zeta(p)\beta), \tag{1.31}$$

the law of the stochastic process

$$S_n^b(t) \equiv e^{-\gamma n} S_n(\theta_n \lfloor t n^{1/2} e^{n\gamma^2/2\beta^2} \theta_n^{-1} \rfloor), \qquad t \geq 0, \tag{1.32}$$

with $\theta_n = \frac{3\ln 2}{2}n^2$, *defined on the space of càdlàg functions equipped with the Skorokhod* J_1*-topology, converges to the law of the stable subordinator* $V_{\gamma/\beta^2}(t), t \geq 0$, *of Lévy measure* $K_p(\gamma/\beta^2)x^{-\gamma/\beta^2-1}\,dx$. *Convergence holds* $\mathbb{P}$*-a.s. if* $p > 4$, *and in* $\mathbb{P}$*-probability, if* $p = 3, 4$.

The function $\zeta(p)$ *is increasing, and it satisfies*

$$\zeta(3) \simeq 1.0291 \quad \textit{and} \quad \lim_{p\to\infty} \zeta(p) = \sqrt{2\log 2}. \tag{1.33}$$

REMARK. This result implies the weaker statement that

$$S_n(t) \equiv e^{-\gamma N} S_n(\lfloor tn^{1/2}e^{n\gamma^2/2\beta^2}\rfloor), \qquad t \geq 0, \tag{1.34}$$

converges in the same way in the M_1-topology.

In [1] an analogous result is proven, with the same constants $\zeta(p)$ and K_p, but convergence there is in law with respect to the random environment (and almost sure with respect to the trajectories J_n). Being able to obtain convergence under the law of the trajectories for fixed environments, as we do here, is a considerable conceptual improvement.

Finally, one must ask whether the convergence of the clock process in the form obtained here is useful for deriving aging information in the sense that we can control the behavior of certain correlation functions. One may be worried that a jump in limit of the coarse-grained clock process refers to a period of time during which the process still may make n^2 steps, and our limit result tells us nothing about how the process moves during that time. We will, however, show that essentially all this time is spent in a single visit to a quite small "trap," within which the process does not make more than $o(n)$ steps.

In this way we prove the almost-sure (or in probability) version of Theorem 1.2 of [1].

THEOREM 1.5. *Let* $A_n^\varepsilon(t,s)$ *be the event defined by*

$$A_n^\varepsilon(t,s) = \{R_n(X_n(te^{\gamma n}), X_n((t+s)e^{\gamma n})) \geq 1-\varepsilon\}. \tag{1.35}$$

Then, under the hypothesis of Theorem 1.4, for all $\varepsilon \in (0,1)$, $t > 0$ *and* $s > 0$,

$$\lim_{N\to\infty} \mathcal{P}_{\pi_n}(A_n^\varepsilon(t,s)) = \frac{\sin\alpha\pi}{\pi}\int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\,du. \tag{1.36}$$

Convergence holds $\mathbb{P}$*-a.s. if* $p > 4$, *and in* $\mathbb{P}$*-probability, if* $p = 3, 4$.

The remainder of the paper is organised as follows. In the next section we prove Theorems 1.2 and 1.3. In Section 3 we apply our main theorem to the p-spin SK model and prove Theorem 1.5.

2. Proof of the main theorems. We now prove our main theorem. The first step is the proof of Theorem 1.2.

2.1. *Proof of Theorem* 1.2.

PROOF. Throughout we fix a realization $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^b(t) \equiv S_n^b(t) - c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}. \tag{2.1}$$

Condition (A0′) ensures that $S_n^b - \widehat{S}_n^b$ converges to zero, uniformly. Thus we must show that under Conditions (A1) and (A2),

$$\widehat{S}_n^b(\cdot) \Rightarrow S_\nu(\cdot). \tag{2.2}$$

This will be a simple corollary of Theorem 1.1. Recall that

$$k_n(t) \equiv \lfloor \lfloor a_n t \rfloor / \theta_n \rfloor, \tag{2.3}$$

and for $i \geq 1$, define

$$Z_{n,i} \equiv \sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}. \tag{2.4}$$

By (1.17) and (2.1), $\widehat{S}_n^b(t) = \sum_{i=1}^{k_n(t)} Z_{n,i}$. We now want to apply Theorem 1.1 to the latter partial sum process. For this let $\{\mathcal{F}_{n,i}, n \geq 1, i \geq 0\}$ be the array of sub-sigma fields of $\mathcal{F}^X$ defined by (with obvious notation) $\mathcal{F}_{n,i} = \sigma(\bigcup_{j \leq \theta_n i}\{J_n(j), e_{n,j}\})$, for $i \geq 0$. Clearly, for each n and $i \geq 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}$ measurable and $\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i}$. Next observe that

$$\begin{aligned} &\mathcal{P}_{\mu_n}(Z_{n,i} > z | \mathcal{F}_{n,i-1}) \\ &\qquad = \sum_{x \in \mathcal{V}_n} \mathcal{P}_{\mu_n}\big(J_n(\theta_n(i-1)+1) = x, Z_{n,i} > z | \mathcal{F}_{n,i-1}\big), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\mathcal{P}_{\mu_n}\big(J_n(\theta_n(i-1)+1) = x, Z_{n,i} > z | \mathcal{F}_{n,i-1}\big) \\ &\qquad = \mathcal{P}_{\mu_n}\big(J_n(\theta_n(i-1)+1) = x, Z_{n,i} > z | J_n(\theta_n(i-1))\big). \end{aligned} \tag{2.6}$$

Using Bayes' theorem and the Markov property, the last line can be written as

$$p_n\big(J_n(\theta_n(i-1)), x\big)\mathcal{P}_{\mu_n}\left(\sum_{j=1}^{\theta_n} c_n^{-1}\lambda_n^{-1}\big(J_n(j-1)\big)e_{n,j-1} > z | J_n(0) = x\right). \tag{2.7}$$

Thus, in view of (1.12), (1.13), (1.14) and (1.15), it follows from (2.5), (2.6) and (2.7) that

$$
\begin{aligned}
\sum_{i=1}^{k_n(t)} \mathcal{P}_{\mu_n}(Z_{n,i} > z|\mathcal{F}_{n,i-1}) &= \sum_{i=1}^{k_n(t)} \sum_{x\in\mathcal{V}_n} p_n\big(J_n\big(\theta_n(i-1)\big), x\big) Q_n^u(x) \\
&= \sum_{i=1}^{k_n(t)} F_n^u\big(J_n\big(\theta_n(i-1)\big)\big) \qquad (2.8)\\
&= \nu_n^{J,t}(u,\infty).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\sum_{i=1}^{k_n(t)} [\mathcal{P}_{\mu_n}(Z_{n,i} > \varepsilon|\mathcal{F}_{n,i-1})]^2 &= \sum_{i=1}^{k_n(t)} \big[F_n^u\big(J_n\big(\theta_n(i-1)\big)\big)\big]^2 \qquad (2.9)\\
&= (\sigma_n^{J,t})^2(u,\infty).
\end{aligned}
$$

From (2.8) and (2.9) it follows that Conditions (A2) and (A1) of Theorem 1.2 are exactly the conditions from Theorem 1.1. Similarly Condition (A3) is Condition 1.9. Therefore the conditions of Theorem 1.1 are verified, and so $\widehat{S}_n^b \Rightarrow S_\nu$ in $D([0,\infty))$ where S_ν is a subordinator with Lévy measure ν and zero drift. $\square$

2.2. *Proof of Theorem* 1.3. The proof of Theorem 1.3 comes in two steps. In the first we use the ergodic properties of the chain J_n to pass from sums along a chain J_n to averages with respect to the invariant measure of J_n.

We assume from now on that the initial distribution μ_n is the invariant measure π_n of the jump chain J_n.

PROPOSITION 2.1. *Let $\mu_n = \pi_n$. Assume that Condition (A1-1) is satisfied. Then, choosing $\theta_n \geq \ell_n$, the following holds: for all $t > 0$ and all $u > 0$, we have that for all $\varepsilon > 0$,*

$$
\begin{aligned}
&P_{\pi_n}\big(|\nu_n^{J,t}(u,\infty) - \nu_n^t(u,\infty)| \geq \varepsilon\big) \qquad (2.10)\\
&\quad \leq \varepsilon^{-2}[\rho_n(\nu_n^t(u,\infty))^2 + (\sigma_n^t)^2(u,\infty)],
\end{aligned}
$$

and

$$
P_{\pi_n}\big((\sigma_n^{J,t})^2(u,\infty) \geq \varepsilon\big) \leq \varepsilon^{-1}(\sigma_n^t)^2(u,\infty). \qquad (2.11)
$$

PROOF. To simplify notation, we only give the proof for the case when the chain J_n is aperiodic, that is, $q = 1$. Details of how to deal with the general periodic case can be found in the proof of Proposition 4.1 of [15].

Let us first establish that

$$E_{\pi_n}[\nu_n^{J,t}(y)] = \nu_n^t(u,\infty), \tag{2.12}$$

$$E_{\pi_n}[(\sigma_n^{J,t})^2(u,\infty)] = (\sigma_n^t)^2(u,\infty). \tag{2.13}$$

To this end set

$$\pi_n^{J,t}(x) = k_n^{-1}(t)\sum_{j=1}^{k_n(t)} \mathbb{1}_{\{J_n(\theta_n(j-1))=x\}}, \qquad x \in \mathcal{V}_n. \tag{2.14}$$

Then, equations (1.14) and (1.15) may be rewritten as

$$\nu_n^{J,t}(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n^{J,t}(y)F_n^u(y), \tag{2.15}$$

$$(\sigma_n^{J,t})^2(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n^{J,t}(y)(F_n^u(y))^2. \tag{2.16}$$

Since by assumption the initial distribution is the invariant measure π_n of J_n, the chain variables $(J_n(j), j \geq 1)$ satisfy $P_{\pi_n}(J_n(j) = x) = \pi_n(x)$ for all $x \in \mathcal{V}_n$, and all $j \geq 1$. Hence

$$E_{\pi_n}[\pi_n^{J,t}(y)] = \pi_n(y), \tag{2.17}$$

$$E_{\pi_n}[\nu_n^{J,t}(u,\infty)] = k_n(t)\sum_{x\in\mathcal{V}_n} \pi_n(x)F_n^u(x), \tag{2.18}$$

$$E_{\pi_n}[(\sigma_n^{J,t})^2(u,\infty)] = k_n(t)\sum_{x\in\mathcal{V}_n} \pi_n(x)(F_n^u(x))^2, \tag{2.19}$$

and equations (2.12) and (2.13) now follow readily from these identities. Indeed, inserting (1.13) into (2.18) and using that π_n is the invariant measure of J_n, we get

$$E_{\pi_n}[\nu_n^{J,t}(u,\infty)] = k_n(t)\sum_{y\in\mathcal{V}_n}\sum_{x\in\mathcal{V}_n} \pi_n(x)p_n(x,y)Q_n^u(y), \tag{2.20}$$

$$= k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)Q_n^u(y), \tag{2.21}$$

which proves (2.12). Similarly, inserting (1.13) into (2.19) yields

$$E_{\pi_n}[(\sigma_n^{J,t})^2(u,\infty)] = k_n(t)\sum_{x\in\mathcal{V}_n} \pi_n(x)\biggl(\sum_{y\in\mathcal{V}_n} p_n(x,y)Q_n^u(y)\biggr)^2, \tag{2.22}$$

which gives (2.13), once observed that, by reversibility, $\sum_{x\in\mathcal{V}_n}\pi_n(x)p_n(x,y)\times p_n(x,y') = \pi_n(y)\sum_{x\in\mathcal{V}_n} p_n(y,x)p_n(x,y') = \pi_n(y)p_n^{(2)}(y,y')$.

We are now ready to prove the proposition. In view of (2.13), (2.11) is nothing but a first order Chebyshev inequality. To establish (2.10) set

$$\mathbb{D}_{ij}(x,y) = P_{\pi_n}\bigl(J_n\bigl(\theta_n(i-1)\bigr) = x, J_n\bigl(\theta_n(j-1)\bigr) = y\bigr) - \pi_n(x)\pi_n(y). \tag{2.23}$$

A second-order Chebyshev inequality together with expressions (2.18) of $E_{\pi_n}[\nu_n^{J,t}(u,\infty)]$ yield

$$
\begin{aligned}
&P_{\pi_n}\bigl(|\nu_n^{J,t}(u,\infty) - E_{\pi_n}[\nu_n^{J,t}(u,\infty)]| \geq \varepsilon\bigr) \\
&\quad \leq \varepsilon^{-2} E_{\pi_n}\Biggl[k_n(t) \sum_{y\in\mathcal{V}_n} \bigl(\pi_n^{J,t}(y) - \pi_n(y)\bigr) F_n^u(y)\Biggr]^2 \qquad (2.24)\\
&\quad = \varepsilon^{-2} \sum_{x\in\mathcal{V}_n}\sum_{y\in\mathcal{V}_n} F_n^u(x)F_n^u(y) \sum_{i=1}^{k_n(t)}\sum_{j=1}^{k_n(t)} \mathbb{D}_{ij}(x,y).
\end{aligned}
$$

Now $\sum_{i=1}^{k_n(t)}\sum_{j=1}^{k_n(t)} \mathbb{D}_{ij}(x,y) = (\overline{I}) + (\overline{II})$ where

$$
(\overline{I}) \equiv \sum_{i=1}^{k_n(t)}\sum_{j=1}^{k_n(t)} \mathbb{D}_{ij}(x,y)\mathbb{1}_{\{j\neq i\}} \leq \rho_n k_n^2(t)\pi_n(x)\pi_n(y), \qquad (2.25)
$$

as follows from Condition (A1-1), choosing $\theta_n \geq \ell_n$, and

$$
\begin{aligned}
(\overline{II}) &\equiv \sum_{1\leq i\leq k_n(t)} \mathbb{D}_{ii}(x,x)\mathbb{1}_{\{x=y\}} \\
&= k_n(t)\bigl[P_{\pi_n}\bigl(J_n\bigl(\theta_n(i-1)\bigr)=x\bigr) - \pi_n^2(x)\bigr]\mathbb{1}_{\{x=y\}} \qquad (2.26)\\
&= k_n(t)\pi_n(x)\bigl(1-\pi_n(x)\bigr)\mathbb{1}_{\{x=y\}}.
\end{aligned}
$$

Inserting (2.26) and (2.25) in (2.24) we obtain, using again (2.13) and (2.17), that

$$
\begin{aligned}
&P_{\pi_n}\bigl(|\nu_n^{J,t}(u,\infty) - E_{\pi_n}[\nu_n^{J,t}(u,\infty)]| \geq \varepsilon\bigr) \qquad (2.27)\\
&\quad \leq \varepsilon^{-2}\bigl[\rho_n(\nu_n^t(u,\infty))^2 + (\sigma_n^t)^2(u,\infty)\bigr].
\end{aligned}
$$

Proposition 2.1 is proven. □

PROOF OF THEOREM 1.3. The proof of Theorem 1.3 is now immediate: combine the conclusions of Proposition 2.1 with Condition (A2-1) to get both conditions (A1) and (A2). Finally, Condition (A3) is Condition (A3-1), since we are starting from the invariant measure. □

3. Application to the *p*-spin SK model. In this section we show how Conditions (A1-1) and (A2-1) can be verified in the case of the random hopping time dynamics of the p-spin SK model.

The proof contains four steps, two of which are quite immediate.

Conditions (A1-1) for simple random walk has been established, for example, in [1] and [16]. The following lemma is taken from Proposition 3.12 of [16].

LEMMA 3.1. *Let P_{π_n} be the law of the simple random walk on the hypercube Σ_n started in the uniform distribution. Let $\theta_n = \frac{3\ln 2}{2} n^2$. Then, for any $x, y \in \Sigma_n$ and any $i \geq 0$,*

$$\left| \sum_{k=0}^{1} P_{\pi_n}\big(J_n(\theta_n + i + k) = y, J_n(0) = x\big) - 2\pi_n(x)\pi_n(y) \right| \leq 2^{-3n+1}. \tag{3.1}$$

Clearly this implies that Condition (A1-1) holds.

We now turn to the first part of Condition (A2-1). We will show that

$$\nu_n^t(u, \infty) \to \nu^t(u, \infty) = t K_p u^{-\gamma/\beta^2}, \tag{3.2}$$

almost surely, respectively, in probability, as $n \uparrow \infty$.

3.1. *Laplace transforms*. Instead of proving the convergence of the distribution functions ν_n^t directly, we pass to their Laplace transforms, prove their convergence and then use Feller's continuity lemma to deduce convergence of the original objects.

For $v > 0$, consider the Laplace transforms

$$\begin{aligned} \hat{\nu}_n^t(v) &= \int_0^\infty du e^{-uv} \nu_n^t(u, \infty), \\ \hat{\nu}^t(v) &= \int_0^\infty du e^{-uv} \nu^t(u, \infty). \end{aligned} \tag{3.3}$$

With $Z_n \equiv \sum_{j=0}^{\theta_n - 1} c_n^{-1} \lambda_n^{-1}(J_n(j)) e_{n,j}$, we have, by definition of $\nu_n^t(u, \infty)$,

$$\nu_n^t(u, \infty) = k_n(t) \sum_{x \in \mathcal{V}_n} \pi_n(x) Q_n^u(x) = k_n(t) \mathcal{P}_{\pi_n}(Z_n > u).$$

Hence

$$\begin{aligned} \hat{\nu}_n^t(v) &= \int_0^\infty du e^{-uv} \nu_n^t(u, \infty) \\ &= k_n(t) \int_0^\infty du e^{-uv} \mathcal{P}_{\pi_n}(Z_n > u) \\ &= k_n(t) \frac{1 - \mathcal{E}_{\pi_n}(e^{-vZ_n})}{v}, \end{aligned} \tag{3.4}$$

where the last equality follows by integration by parts.

3.2. *Convergence of* $\mathbb{E}\hat{\nu}_n^t(v)$. The following lemma is an easy consequence of the results of [1]:

LEMMA 3.2. *Let $c_n = e^{\gamma n}$, $a_n = n^{1/2} e^{n\gamma^2/2\beta^2}$. For any $p \geq 3$, and $\beta, \gamma > 0$ such that $\gamma/\beta^2 \in (0,1)$, there exists a finite positive constant, K_p, such that for any $v > 0$,*

$$\lim_{n\uparrow\infty} k_n(t)\mathbb{E}[1 - \mathcal{E}_{\pi_n}(e^{-vZ_n})] = K_p t v^{\gamma/\beta^2}. \tag{3.5}$$

PROOF. We rely essentially on the results of [1]. In that paper the Laplace transforms $\mathbb{E}e^{-vZ_n}$ were computed even for $\theta_n = a_n t$. We just recall the key ideas and the main steps.

The point in [1] is to first fix a realization of the chain J_n, and to define, for a given realization, the one-dimensional normal Gaussian process

$$U^0(i) \equiv n^{-1/2} H_n(J_n(i)), \tag{3.6}$$

with covariance

$$\Lambda^0_{ij} = n^{-1}\mathbb{E}H_n(J_n(i))H_n(J_n(j)) = R_n(J_n(i), J_n(j))^p. \tag{3.7}$$

Moreover, they define a comparison process, U^1, as follows. Let ν be an integer of order n^ρ, with $\rho \in (1/2, 1)$. Then U^1 has covariance matrix

$$\Lambda^1_{ij} = \begin{cases} 1 - 2pn^{-1}|i-j|, & \text{if } \lfloor i/\nu \rfloor = \lfloor j/\nu \rfloor, \\ 0, & \text{else.} \end{cases} \tag{3.8}$$

Finally they define the interpolating family of processes, for $h \in [0,1]$,

$$U^h(i) \equiv \sqrt{h}U^1(i) + \sqrt{1-h}U^0(i). \tag{3.9}$$

For any normal Gaussian process, U, indexed by $\mathbb{N}$, define the functions

$$F_n(U, v, k) \equiv \exp\left(-vc_n^{-1} \sum_{i=0}^{k-1} e_{n,i} e^{\beta\sqrt{n}U_i}\right) \tag{3.10}$$

and

$$\mathcal{E}_{\pi_n}(F(U,v,k)|\mathcal{F}^J) \equiv G(U,v,k) = \exp\left(-\sum_{i=0}^{k-1} g(vc_n^{-1}e^{\beta\sqrt{n}U_i})\right), \tag{3.11}$$

with $g(x) = \ln(1+x)$.

Then the Laplace transforms we are after can be written as

$$\begin{aligned} \mathbb{E}\mathcal{E}_{\pi_n} e^{-vZ_n} &= \mathbb{E}\mathcal{E}_{\pi_n}(\mathcal{E}_{\pi_n}(e^{-vZ_n}|\mathcal{F}^J)) \\ &= E_{\pi_n}\mathbb{E}G(U^0, v, \theta_n). \end{aligned} \tag{3.12}$$

Here we used that the conditional expectation, given $\mathcal{F}^J$, is just the expectation with respect to the variables $e_{n,i}$, which can be computed explicitly, and gives rise to the function G.

The idea is now that U^1 is a good enough approximation to U^0, for most realizations of the chain J, to allow us to replace U^0 by U^1 in the last line above.

More precisely, we have the following estimate.

LEMMA 3.3. *With the notation above we have that for all* $p \geq 3$,

$$k_n(t) E_{\pi_n} |\mathbb{E}G(U^0, v, \theta_n) - \mathbb{E}G(U^1, v, \theta_n)| \leq tCn^{1/2}/\nu. \tag{3.13}$$

REMARK. In [1] (see Proposition 3.1) it is proven that E_{π_n}-almost surely,

$$\mathbb{E}G(U^0, v, \lfloor a_n t \rfloor) - \mathbb{E}G(U^1, v, \lfloor a_n t \rfloor) \to 0. \tag{3.14}$$

This result would not be expected for our expression, but we do not need this. The proof of Proposition 3.1 of [1], however, directly implies our Lemma 3.3.

The computation of the expression involving the comparison process U^1 is fairly easy. First, note that by independence (and making for simplicity the assumption that θ_n is an integer multiple of ν),

$$\begin{aligned} \mathbb{E}G(U^1, v, \theta_n) &= [\mathbb{E}G(U^1, v, \nu)]^{\theta_n/\nu} \\ &= \bigl[1 - \bigl(1 - \mathbb{E}G(U^1, v, \nu)\bigr)\bigr]^{\theta_n/\nu}. \end{aligned} \tag{3.15}$$

But in [1], Proposition 2.1, it is shown that

$$a_n \nu^{-1}\bigl(1 - \mathbb{E}G(U^1, v, \nu)\bigr) \to K_p v^{\gamma/\beta^2}. \tag{3.16}$$

This implies immediately that

$$k_n(t)\bigl\{1 - \bigl[1 - \bigl(1 - \mathbb{E}G(U^1, v, \nu)\bigr)\bigr]^{\theta_n/\nu}\bigr\} \to K_p v^{\gamma/\beta^2} t, \tag{3.17}$$

as desired. Combining this with Lemma 3.3, the assertion of Lemma 3.2 follows. $\square$

3.3. *Concentration of* ν_n^t. To complete the proof, we need to control the fluctuations of ν_n^t.

LEMMA 3.4. *Under the same hypothesis as in Lemma* 3.2, *there exists an increasing function*, $\zeta(p)$, *such that for all* $p \geq 3$, $\zeta(p) > 1$, *and* $\zeta(p) \uparrow \sqrt{2 \ln 2}$, *such that*, *if* $\gamma/\beta^2 < \min(1, \zeta(p)/\beta)$,

$$\mathbb{E}\bigl(\hat{\nu}_n^t(v) - \mathbb{E}\hat{\nu}_n(v)\bigr)^2 \leq Cn^{1-p/2}. \tag{3.18}$$

PROOF. The proof is again very similar to the proof of Proposition 3.1 in [1]. We have to compute

$$\mathbb{E}(\mathcal{E}_{\pi_n} e^{-vZ_n})^2 = E_{\pi_n} E'_{\pi_n}\bigl(\mathbb{E}\mathcal{E}_{\pi_n}\mathcal{E}'_{\pi_n}\bigl(e^{-v(Z_n+Z'_n)} | \mathcal{F}^J \times \mathcal{F}^{J'}\bigr)\bigr), \tag{3.19}$$

where $Z'_n \equiv \sum_{j=0}^{\theta_n - 1} c_n^{-1} \lambda_n^{-1}(J'_n(j)) e'_{n,j}$, J'_n and $(e'_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$ being, respectively, independent copies of J_n and $(e_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$. To express this as in the previous proof, we introduce the Gaussian process V^0 by

$$V^0(i) \equiv \begin{cases} n^{-1/2} H_n(J_n(i)), & \text{if } 0 \leq i \leq \theta_n - 1, \\ n^{-1/2} H_n(J'_n(i)), & \text{if } \theta_n \leq i \leq 2\theta_n - 1. \end{cases} \tag{3.20}$$

Then, with the notation of (3.11),

$$\mathcal{E}_{\pi_n}\mathcal{E}'_{\pi_n}\big(e^{-v(Z_n+Z'_n)}|\mathcal{F}^J\times\mathcal{F}^{J'}\big)=G(V^0,v,2\theta_n). \tag{3.21}$$

Next we define the comparison process V^1 with covariance matrix

$$\Lambda^2_{ij}\equiv\begin{cases}\Lambda^0_{ij}, & \text{if } \max(i,j)<\theta_n \text{ or } \min(i,j)\ge\theta_n,\\ 0, & \text{else.}\end{cases} \tag{3.22}$$

The point is that

$$E_{\pi_n}E'_{\pi_n}\mathbb{E}G(V^1,v,2\theta_n)=(E_{\pi_n}\mathbb{E}G(V^0,v,\theta_n))^2=(\mathbb{E}\mathcal{E}_{\pi_n}e^{-vZ_n})^2. \tag{3.23}$$

On the other hand, using the standard Gaussian interpolation formula, we obtain the representation

$$\begin{aligned}&\mathbb{E}G(V^1,v,2\theta)-\mathbb{E}G(V^0,v,2\theta)\\&\quad=\frac{1}{2}\int_0^1\sum_{\substack{0\le i<\theta_n\\ \theta_n\le j<2\theta_n}}\Lambda^0_{ij}\mathbb{E}\frac{\partial^2G(V^h,v,2\theta_n)}{\partial v_i\,\partial v_j}\,dh+(i\leftrightarrow j),\end{aligned} \tag{3.24}$$

where the interpolating process V^h is defined analogously to (3.9). The second derivatives of G were computed and bounded in [1] [see equation (3.7) and Lemma 3.2]. We recall the following bounds:

LEMMA 3.5. *With the notation above and the assumptions of Lemma 3.2,*

$$\begin{aligned}&\mathbb{E}\left|\frac{\partial^2G(V^h,v,2\theta_n)}{\partial v_i\,\partial v_j}\right|\\&\quad\le v^2c_n^{-2}\beta^2n\mathbb{E}\big[e^{\beta\sqrt{n}(V^h(i)+V^h(j))}\exp\big(-2g\big(c_n^{-1}ve^{\beta\sqrt{n}V^h(i)}\big)\\&\qquad\qquad-2g\big(c_n^{-1}ve^{\beta\sqrt{n}V^h(j)}\big)\big)\big]\\&\quad\equiv\Xi_n(\Lambda^h_{ij}).\end{aligned} \tag{3.25}$$

Moreover, for $\lambda>0$ small enough,

$$\Xi_n(c)\le\bar{\Xi}_n(c)=\begin{cases}C\big((1-c)^{-1/2}\wedge\sqrt{n}\big)e^{-(\gamma^2n)/(\beta^2(1+c))},\\ \qquad \textit{if } 1>c>\gamma/\beta^2+\lambda-1,\\ Cne^{-n(\beta^2(1+c)-2\gamma)},\\ \qquad \textit{if } c\le(\gamma/\beta^2)+\lambda-1,\end{cases} \tag{3.26}$$

where $C(\gamma,\beta,v,\lambda)$ is a suitably chosen constant independent of n and c.

REMARK. Notice that, since $\gamma/\beta^2<1$ under our hypothesis, we can always choose λ such that the top line in (3.26) covers the case $c\ge0$.

Note that for $c \geq 0$ (see equation (3.25) in [1]),

$$\int_0^1 \Xi_n\big((1-h)c\big)\,dh \leq 2C \exp\biggl(-\frac{\gamma^2 n}{\beta^2(1+c)}\biggr). \tag{3.27}$$

The terms with negative correlation are in principle smaller than those with positive one, but some thought reveals that one cannot really gain substantially over the bound

$$\int_0^1 \Xi_n\big((1-h)c\big)\,dh \leq C \exp\biggl(-\frac{\gamma^2 n}{\beta^2}\biggr), \tag{3.28}$$

that is, used in [1] [see equation (3.24)].

Next we must compute the probability that Λ_{ij}^0 takes on a specific value. But since Λ_{ij}^0 is a function of $R_n(J_n(i), J_n'(j))$, this turns out to be very easy, namely, since both chains start in the invariant distribution

$$\begin{aligned} &\mathcal{E}_{\pi_n}\mathcal{E}'_{\pi_n}\mathbb{1}_{nR_n(J_n(i),J_n'(j))=m} \\ &\quad = \sum_{x,y\in\mathcal{S}^n} \mathcal{P}_{\pi_n}\big(J_n(i)=x\big)\mathcal{P}'_{\pi_n}\big(J_n'(i)=y\big)\mathbb{1}_{nR_n(x,y)=m} \\ &\quad = 2^{-n}\sum_{x\in\mathcal{S}^n}\mathbb{1}_{nR_n(x,1)=m} = 2^{-n}\binom{n}{(n-m)/2}. \end{aligned} \tag{3.29}$$

Putting all things together, we arrive at the bound

$$\begin{aligned} &k_n(t)^2\big|\mathbb{E}G(V^0,v,2\theta) - (\mathbb{E}G(V^0,v,\theta))^2\big| \\ &\quad \leq \sum_{m=0}^n 2^{-n}\binom{n}{(n-m)/2}\Bigl(\frac{m}{n}\Bigr)^p t^2 n e^{n\gamma^2/\beta^2} 2C\exp\biggl(-\frac{n\gamma^2}{\beta^2(1+(m/n)^p)}\biggr) \\ &\qquad + \sum_{m=0}^n 2^{-n}\binom{n}{(n-m)/2}\Bigl(\frac{m}{n}\Bigr)^p t^2 n e^{n\gamma^2/\beta^2} 2C\exp\biggl(-\frac{n\gamma^2}{\beta^2}\biggr), \end{aligned} \tag{3.30}$$

where we did use that $k_n(t)\theta_n \approx t\sqrt{n}e^{n\gamma^2/\beta^2}$. Clearly the second term is smaller than the first, so we only need to worry about the latter. But this term is exactly the term (3.28) in [1], where it is shown that this is smaller than

$$C' t^2 n^{1-p/2}, \tag{3.31}$$

provided $\gamma < \zeta(p)$. This provides the assertion of our Lemma 3.4 and concludes its proof. $\square$

REMARK. The estimate on the second moment we get here allows to get almost sure convergence only if $p > 4$. It is not quite clear whether this is natural. We were tempted to estimate higher moments to get improved estimates on the convergence speed. However, any straightforward application of the comparison methods used here does produce the same order for all higher moments. We have not been able to think of a tractable way to improve this result.

3.4. *Verification of the second part of Condition (A2-1).* For $u, u' > 0$ define

$$\tilde{\eta}_n^t(u) = \frac{1}{n} k_n(t) \sum_{x \in \mathcal{V}_n} \pi_n(x) (Q_n^u(x))^2, \tag{3.32}$$

$$\eta_n^t(u, u') = k_n(t) \sum_{x \in \mathcal{V}_n} \sum_{x' \in \mathcal{V}_n} \mu_n(x, x') Q_n^u(x) Q_n^{u'}(x'), \tag{3.33}$$

where μ_n is the uniform distribution on pairs of vertices (x, x') that are at distance 2 apart,

$$\mu_n(x, x') = \begin{cases} 2^{-n} \dfrac{2}{n(n-1)}, & \text{if } \operatorname{dist}(x, x') = 2, \\ 0, & \text{else.} \end{cases} \tag{3.34}$$

Equation (1.25) will be verified if we can show that for all $t > 0$ and all $u, u' > 0$, both $\tilde{\eta}_n^t(u)$ and $\eta_n^t(u, u')$ tend to zero, almost surely, respectively, in probability, as $n \uparrow \infty$.

As before we will do this by first passing to the Laplace transform of $\eta_n^t(u, u')$. For $v, v' > 0$, define

$$\begin{aligned} \hat{\eta}_n^t(v, v') &= \int_0^\infty du \int_0^\infty du' e^{-(uv + u'v')} \eta_n^t(u, u'), \\ \hat{\eta}^t(v, v') &= \int_0^\infty du \int_0^\infty du' e^{-(uv + u'v')} \eta^t(u, u'). \end{aligned} \tag{3.35}$$

The reason for considering the two point function $\eta_n^t(u, u')$ is that, integrating by parts as in (3.5), $\hat{\eta}_n^t(v, v')$ takes the convenient form

$$\hat{\eta}_n^t(v, v') = k_n(t) \sum_{x \in \mathcal{V}_n} \sum_{x' \in \mathcal{V}_n} \mu_n(x, x') \frac{1 - \mathcal{E}_x(e^{-vZ_n})}{v} \frac{1 - \mathcal{E}'_{x'}(e^{-v'Z'_n})}{v'}, \tag{3.36}$$

where $\mathcal{E}_x$ (resp., $\mathcal{E}'_{x'}$) denotes the expectation with respect to the law $\mathcal{P}_x$ of the chain X_n started in x (resp., the law $\mathcal{P}'_{x'}$ of an independent copy X'_n started in x').

LEMMA 3.6. *Under the assumptions, and with the notation of Lemma 3.2, for any $v, v' > 0$,*

$$\lim_{n \uparrow \infty} \mathbb{E} \hat{\eta}_n^t(v, v') = 0. \tag{3.37}$$

PROOF. The key idea of the proof is that the first $\bar{\theta}_n = 2n \ln n$ terms in the sums Z_n are irrelevant. With this in mind, we define $W_n \equiv \sum_{j=\bar{\theta}_n}^{\theta_n - 1} c_n^{-1} \times \lambda_n^{-1}(J_n(j)) e_{n,j}$.

Note that

$$
\begin{aligned}
vv'\mathbb{E}\hat{\eta}_n^t(v,v') &= k_n(t)\mathbb{E}[1-\mathcal{E}_{\pi_n}(e^{-vZ_n})] + k_n(t)\mathbb{E}[1-\mathcal{E}'_{\pi_n}(e^{-v'Z'_n})] \\
&\qquad - k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\mathbb{E}\bigl[1-\mathcal{E}_x\mathcal{E}'_{x'}\bigl(e^{-(vZ_n+v'Z'_n)}\bigr)\bigr] \\
&\le k_n(t)\mathbb{E}[1-\mathcal{E}_{\pi_n}(e^{-vZ_n})] + k_n(t)\mathbb{E}[1-\mathcal{E}'_{\pi_n}(e^{-v'Z'_n})] \\
&\qquad - k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\mathbb{E}\bigl[1-\mathcal{E}_x\mathcal{E}'_{x'}\bigl(e^{-(vW_n+v'W'_n)}\bigr)\bigr].
\end{aligned}
\tag{3.38}
$$

Adding and subtracting the term $\mathbb{E}\mathcal{E}_x(e^{-vW_n})\mathbb{E}\mathcal{E}'_{x'}(e^{-v'W'_n})$ to the term $\mathbb{E}\mathcal{E}_x\mathcal{E}'_{x'}(e^{-(vW_n+v'W'_n)})$, the right-hand side of (3.38) is equal to

$$
\begin{aligned}
& k_n(t)\mathbb{E}[1-\mathcal{E}_{\pi_n}(e^{-vZ_n})] + k_n(t)\mathbb{E}[1-\mathcal{E}'_{\pi_n}(e^{-v'Z'_n})] \\
&- k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')[1-\mathbb{E}\mathcal{E}_x(e^{-vW_n})\mathbb{E}\mathcal{E}'_{x'}(e^{-v'W'_n})] \\
&+ k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\bigl(\mathbb{E}\mathcal{E}_x\mathcal{E}'_{x'}\bigl(e^{-(vW_n+v'W'_n)}\bigr) \\
&- \mathbb{E}\mathcal{E}_x(e^{-vW_n})\mathbb{E}\mathcal{E}'_{x'}(e^{-v'W'_n})\bigr).
\end{aligned}
\tag{3.39}
$$

After a little reorganisation, (3.39) is in turn equal to

$$
\begin{aligned}
& k_n(t)\mathbb{E}[\mathcal{E}_{\pi_n}(e^{-vW_n}-e^{-vZ_n}+e^{-v'W_n}-e^{v'Z_n})] \\
&+ vv'k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\mathbb{E}\frac{1-\mathcal{E}_x(e^{-vW_n})}{v}\mathbb{E}\frac{1-\mathcal{E}'_{x'}(e^{-v'W'_n})}{v'} \\
&+ k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\mathcal{E}_x\mathcal{E}_{x'}\bigl(\mathbb{E}\bigl(e^{-(vW_n+v'W'_n)}\bigr)-\mathbb{E}(e^{-vW_n})\mathbb{E}(e^{-v'W'_n})\bigr).
\end{aligned}
\tag{3.40}
$$

Now one deduces readily from Lemma 3.2 that

$$
k_n(t)\mathbb{E}[\mathcal{E}_{\pi_n}(e^{-vW_n}-e^{-vZ_n})] \sim K_p t v^{\gamma/\beta^2}\bar{\theta}_n/\theta_n = O\left(\frac{\ln n}{n}\right)
\tag{3.41}
$$

and tends to zero as $n\uparrow\infty$. Also by Lemma 3.2,

$$
\begin{aligned}
& k_n(t)\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}\mu_n(x,x')\mathbb{E}\frac{1-\mathcal{E}_x(e^{-vW_n})}{v}\mathbb{E}\frac{1-\mathcal{E}'_{x'}(e^{-v'W'_n})}{v'} \\
&= O\bigl(1/k_n(t)\bigr)
\end{aligned}
\tag{3.42}
$$

and tends to zero even much faster. The last term in (3.40) will be controlled by the Gaussian comparison method similar to the proof of Lemma 3.4. Indeed, using the same comparison and interpolation process as in the proof of that lemma, we see that for given trajectories J_n, J_n',

$$\begin{aligned}
&\mathbb{E}\big(e^{-(vW_n+v'W_n')}\big)-\mathbb{E}(e^{-vW_n})\mathbb{E}(e^{-v'W_n'}) \\
&\qquad=\int_0^1 \sum_{\substack{\bar\theta_n\le i<\theta_n\\ \theta_n+\bar\theta_n\le j<2\theta_n}} \Lambda_{ij}^0 \mathbb{E}\frac{\partial^2 G(V^h,v,2\theta_n)}{\partial v_i\,\partial v_j}\,dh.
\end{aligned}\tag{3.43}$$

To control the right-hand side we will exploit the fact that after $\mathcal{O}(n\log n)$ steps, such trajectories are at maximal distance apart with probability close to one. Recalling (1.30), define the distance chain, D_n, on $\{0,1,\dots,n\}$ through

$$D_n(i)=\operatorname{dist}(J_n(i),J_n'(i)),\qquad i\ge 1.\tag{3.44}$$

LEMMA 3.7. *Set $\bar\theta_n=2n\log n$ and $\rho(n)=\sqrt{K\frac{\log n}{n}}$. Then, for K sufficiently large*,

$$P\left(\forall_{\bar\theta_n\le i\le\theta_n} D_n(i)>\frac{n}{2}(1-\rho(n))\,|\,D_n(0)=2\right)\ge 1-n^{-8}.\tag{3.45}$$

Moreover, for any fixed $x,y\in\mathcal{V}_n$,

$$P_x\left(\exists_{\bar\theta_n\le i\le\theta_n}\operatorname{dist}(J_n(i),y)<\frac{n}{2}(1-\rho(n))\right)\le\frac{1}{n^4}.\tag{3.46}$$

PROOF. Observe on the one hand that, denoting by $\mathcal{D}_n$, the transition matrix of the distance chain $\mathcal{D}_n$, one has $\mathcal{D}_n=(\mathcal{Q}_n)^2$, where $\mathcal{Q}_n$ is the transition matrix of the Ehrenfest chain on state space $\{0,\dots,n\}$, namely, the chain with transition probabilities $q_n(i,i+1)=\frac{i}{n}$ and $q_n(i,i-1)=1-\frac{i}{n}$. On the other hand, it is sufficient in order to prove (3.46) to prove it for $y=\mathbf{1}\equiv(1,\dots,1)$, and again, the projection chain $\Pi_n(i)\equiv\operatorname{dist}(J_n(i),\mathbf{1})$, $i\ge1$, is nothing but the Ehrenfest chain on $\{0,\dots,n\}$. Both equations (3.45) and (3.46) then follow from well-known estimates for the Ehrenfest chain; specifically, see [20], page 25, equation below (4.18). $\square$

Let $\mathcal{A}_n\subset\mathcal{F}^J\times\mathcal{F}^{J'}$ be the event $\mathcal{A}_n\equiv\{\forall_{\bar\theta_n\le i\le\theta_n}D_n(i)>\frac{n}{2}(1-\rho(n))\}$. Notice first that on $\mathcal{A}_n$, by the estimates in Lemma 3.5,

$$\begin{aligned}
\int_0^1 \sum_{\substack{\bar\theta_n\le i<\theta_n\\ \theta_n+\bar\theta_n\le j<2\theta_n}} \Lambda_{ij}^0 \mathbb{E}\frac{\partial^2 G(V^h,v,2\theta_n)}{\partial v_i\,\partial v_j}\,dh &\le 2C\theta_n^2\rho(n)\exp(-\gamma^2 n/\beta^2)\\
&=O(k_n(t)^{-2}).
\end{aligned}\tag{3.47}$$

On the other hand, on $\mathcal{A}_n^c$, we still have the bound

$$\begin{aligned}
&\int_0^1 \sum_{\substack{\bar\theta_n \le i < \theta_n \\ \theta_n + \bar\theta_n \le j < 2\theta_n}} \Lambda^0_{ij} \mathbb{E} \frac{\partial^2 G(V^h, v, 2\theta_n)}{\partial v_i \, \partial v_j} \, dh \\
&\qquad \le 2C\theta_n^2 \exp(-\gamma^2 n / 2\beta^2) \\
&\qquad = O(\theta_n / k_n(t)).
\end{aligned} \tag{3.48}$$

Putting all estimates together we arrive at the assertion of the lemma. □

To prove convergence in probability, respectively, almost surely, we just need to use the same concentration estimate as in Lemma 3.4 for the term $k_n(t)\mathcal{E}_{\pi_n}(e^{-vW_n} - e^{-vZ_n})$. Finally, the term $\tilde\eta^t_n(u)$ from (3.32) can be controlled in exactly the same way. This establishes Condition (A2-1).

3.5. *Verification of Condition* (A3-1). To show that Condition (A3-1) holds, we again first prove that the average of the right-hand side vanishes as $\varepsilon \downarrow 0$, and then we prove a concentration result.

LEMMA 3.8. *Under the assumptions of the theorem, there is a constant $K < \infty$, such that*

$$\limsup_{n\uparrow\infty} a_n c_n^{-1} \mathbb{E}\mathcal{E}_{\pi_n} \lambda_n^{-1}(J_n(0)) e_0 \mathbb{1}_{\lambda_n^{-1}(J_n(0)) e_0 \le \varepsilon c_n} \le K\varepsilon^{1-\alpha}. \tag{3.49}$$

PROOF. The proof is through explicit estimates. We must control the integral

$$\begin{aligned}
&\int_0^\infty x e^{-x} \, dx \int_{-\infty}^\infty e^{-z^2/2} \mathbb{1}_{x e^{\beta\sqrt{n} z} \le \varepsilon c_n} e^{\beta\sqrt{n} z} \, dz \\
&\quad = \int_0^\infty x e^{-x} \, dx \left[\int_{-\infty}^{(\ln c_n + \ln(\varepsilon/x))/(\beta\sqrt{n})} e^{-z^2/2 + \beta\sqrt{n} z} \, dz \right] \\
&\quad = \int_0^\infty x e^{-x} \, dx \left[e^{\beta^2 n/2} \int_{-\infty}^{((\ln c_n + \ln(\varepsilon/x))/(\beta\sqrt{n})) - \beta\sqrt{n}} e^{-z^2/2} \, dz \right].
\end{aligned} \tag{3.50}$$

Now for our choice $c_n = \exp(\gamma n)$, the upper integration limit in the z-integral is

$$\begin{aligned}
&\frac{\ln c_n + \ln(\varepsilon/x)}{\beta\sqrt{n}} - \beta\sqrt{n} \\
&\qquad = \sqrt{n}\left(\frac{\gamma}{\beta} - \beta\right) + \frac{\ln\varepsilon - \ln x}{\beta\sqrt{n}}.
\end{aligned} \tag{3.51}$$

Thus, for any $\gamma < \beta^2$, this tends to $-\infty$ uniformly for, say, all $x \le n^2$. We therefore decompose the x-integral in the domain $x \le n^2$ and its complement, and use first that

$$
\begin{aligned}
(3.52)\qquad &\int_{n^2}^{\infty} xe^{-x}\,dx \int e^{-z^2/2} \mathbb{1}_{xe^{\beta\sqrt{n}z} \le \varepsilon c_n} e^{\beta\sqrt{n}z}\,dz \\
&\quad \le \varepsilon n^2 c_n e^{-n^2},
\end{aligned}
$$

which tends to zero, as $n \uparrow \infty$. For the remainder we use the bound

$$
(3.53)\qquad \int_u^{\infty} e^{-z^2/2} \le \frac{1}{u} e^{-u^2/2}.
$$

This yields

$$
\begin{aligned}
&e^{\beta^2 n/2} \int_{-\infty}^{((\ln c_n + \ln(\varepsilon/x))/(\beta\sqrt{n})) - \beta\sqrt{n}} e^{-z^2/2}\,dz \\
&\quad \le e^{\beta^2 n/2} \frac{\exp(-1/2(\sqrt{n}(\beta - \gamma/\beta) - (\ln\varepsilon - \ln x)/\beta\sqrt{n})^2)}{(\beta - \beta^{-1}\gamma)\sqrt{n} - (\ln\varepsilon - \ln x)/\beta\sqrt{n}} \\
(3.54)\qquad &\quad = \frac{\exp(-n(\gamma^2/2\beta^2) + n\gamma)}{\sqrt{n}(\beta - \gamma/\beta) + o(1)} \exp\bigl(-(\gamma/\beta^2 - 1)\ln(\varepsilon/x) + O(n^{-1/2})\bigr) \\
&\quad = c_n a_n^{-1} \frac{1}{\beta - \gamma/\beta + o(1)} \exp\bigl(-(\gamma/\beta^2 - 1)\ln(\varepsilon/x) + O(n^{-1/2})\bigr).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\limsup_{n\uparrow\infty} a_n c_n^{-1} \int_0^{\infty} xe^{-x}\,dx \int_{-\infty}^{\infty} e^{-z^2/2} \mathbb{1}_{xe^{\beta\sqrt{n}z} \le \varepsilon c_n} e^{\beta\sqrt{n}z}\,dz \\
(3.55)\qquad &\quad \le \frac{1}{\beta - \gamma/\beta} \varepsilon^{1-\alpha} \int_0^{\infty} x^{\alpha} e^{-x}\,dx,
\end{aligned}
$$

where $\alpha = \gamma/\beta^2$. This yields the assertion of the lemma. $\square$

To complete the proof, we need a concentration estimate. The first step is a simple Gaussian bound.

LEMMA 3.9. *Let X, Y be centered normal Gaussian random variables with covariance $\mathbb{E}XY = c$. Then, for any $\varepsilon, \varepsilon' > 0$,*

$$
\begin{aligned}
&\frac{\mathbb{E}(e^{\beta\sqrt{n}X} \mathbb{1}_{e^{\beta\sqrt{n}X} \le c_n \varepsilon} e^{\beta\sqrt{n}Y} \mathbb{1}_{e^{\beta\sqrt{n}Y} \le c_n \varepsilon'})}{\mathbb{E}(e^{\beta\sqrt{n}X} \mathbb{1}_{e^{\beta\sqrt{n}X} \le c_n \varepsilon})\mathbb{E}(e^{\beta\sqrt{n}Y} \mathbb{1}_{e^{\beta\sqrt{n}Y} \le c_n \varepsilon'})} \\
(3.56)\qquad &\quad \le \frac{\exp(|c|(2\gamma^2 n + \gamma(\ln\varepsilon + \ln\varepsilon'))/(2\beta^2(1+|c|)))}{\sqrt{1-c^2} - (\gamma/\beta^2)\sqrt{(1-|c|)/(1+|c|)}} \bigl(1 + O(1/n)\bigr).
\end{aligned}
$$

PROOF. The numerator on the left-hand side of (3.56) equals (we assume $c \geq 0$ below, but the same estimate with c replaced by $-c$ can be obtained for $c < 0$)

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{(\gamma n+\ln\varepsilon)/(\beta\sqrt{n})}\int_{-\infty}^{(\gamma n+\ln\varepsilon')/(\beta\sqrt{n})}\frac{1}{\sqrt{1-c^2}}e^{-(z_1^2+z_2^2+2cz_1z_2)/(2(1-c^2))}\\
&\qquad\qquad\qquad\qquad\times e^{\sqrt{n}(z_1+z_2)}\,dz_1\,dz_2\\
&\quad=\frac{1}{2\pi\sqrt{1-c^2}}\int_{-\infty}^{(\gamma n+\ln\varepsilon)/\beta\sqrt{n}}\int_{-\infty}^{(\gamma n+\ln\varepsilon')/(\beta\sqrt{n})}e^{\beta\sqrt{n}(z_1+z_2)}e^{-(z_1^2+z_2^2)/2}\\
&\qquad\times e^{-(c(z_1-z_2)^2-c(1-c)(z_1^2+z_2^2))/2(1-c^2)}\,dz_1\,dz_2\\
&\quad\leq\frac{1}{2\pi\sqrt{1-c^2}}\int_{-\infty}^{(\gamma n+\ln\varepsilon)/(\beta\sqrt{n})}\int_{-\infty}^{(\gamma n+\ln\varepsilon')/(\beta\sqrt{n})}e^{\beta\sqrt{n}(z_1+z_2)}e^{-(z_1^2+z_2^2)/2}\\
&\qquad\times e^{+(c(z_1^2+z_2^2))/(2(1+c))}\,dz_1\,dz_2\\
&\quad=\frac{1}{2\pi\sqrt{1-c^2}}\left(\int_{-\infty}^{(\gamma n+\ln\varepsilon)/(\beta\sqrt{n})}e^{\beta\sqrt{n}z}e^{(-z^2/(2(1+c)))}\,dz\right)\\
&\qquad\times\left(\int_{-\infty}^{(\gamma n+\ln\varepsilon')/(\beta\sqrt{n})}e^{\beta\sqrt{n}z}e^{-(z^2/(2(1+c)))}\,dz\right).
\end{aligned}$$

Using standard estimates on the asymptotics of one-dimensional Gaussian integrals the claimed result follows after some straightforward computations. □

We will now use Lemma 3.9 to prove the desired concentration estimate.

LEMMA 3.10. *With the notation above*,

$$\begin{aligned}
&\mathbb{E}\big(\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(0))e_0\mathbb{1}_{\lambda_n^{-1}(J_n(0))e_0\leq\varepsilon c_n}\big)^2\\
&\quad-\big(\mathbb{E}\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(0))e_0\mathbb{1}_{\lambda_n^{-1}(J_n(0))e_0\leq\varepsilon c_n}\big)^2\\
&\qquad\leq Cn^{1-p/2}\big(\mathbb{E}\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(0))e_0\mathbb{1}_{\lambda_n^{-1}(J_n(0))e_0\leq\varepsilon c_n}\big)^2.
\end{aligned}\tag{3.57}$$

PROOF. Writing out everything explicitly, we have

$$\begin{aligned}
&\mathbb{E}\big(\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(0))e_0\mathbb{1}_{\lambda_n^{-1}(J_n(0))e_0\leq\varepsilon c_n}\big)^2-\big(\mathbb{E}\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(0))e_0\mathbb{1}_{\lambda_n^{-1}(J_n(0))e_0\leq\varepsilon c_n}\big)^2\\
&\quad=2^{-2n}\sum_{x,x'\in\Sigma_n}\int dy_1\,dy_2e^{-y_1-y_2}y_1y_2\\
&\qquad\times\big(\mathbb{E}\big(e^{\beta(H_n(x)+H_n(x'))}\mathbb{1}_{e^{\beta H_n(x)}\leq c_n\varepsilon/y_1}\mathbb{1}_{e^{\beta H_n(x')}\leq c_n\varepsilon/y_2}\big)\\
&\qquad\quad-\mathbb{E}\big(e^{\beta H_n(x)}\mathbb{1}_{e^{\beta H_n(x)}\leq c_n\varepsilon/y_1}\big)\mathbb{E}\big(e^{\beta H_n(x')}\mathbb{1}_{e^{\beta H_n(x')}\leq c_n\varepsilon/y_2}\big)\big).
\end{aligned}\tag{3.58}$$

Now the last terms depend only on the covariance of $H_n(x)$ and $H_n(x')$, that is, on $R_n(x,x')$. Using Lemma 3.9, we get, when $R_n(x,x')^p = c$,

$$
\begin{aligned}
&\int dy_1\, dy_2 e^{-y_1-y_2} y_1 y_2 \\
&\qquad \times \bigl(\mathbb{E}\bigl(e^{(\beta H_n(x)+H_n(x'))}\mathbb{1}_{e^{\beta H_n(x)}\le c_n\varepsilon/y_1}\mathbb{1}_{e^{\beta H_n(\sigma')}\le c_n\varepsilon/y_2}\bigr) \\
&\qquad\quad - \mathbb{E}\bigl(e^{\beta H_n(x)}\mathbb{1}_{e^{\beta H_n(\sigma)}\le c_n\varepsilon/y_1}\bigr)\mathbb{E}\bigl(e^{\beta H_n(\sigma')}\mathbb{1}_{e^{\beta H_n(\sigma')}\le c_n\varepsilon/y_2}\bigr)\bigr) \\
&\qquad \le \bigl(e^{cn(\gamma^2/(\beta^2(1+c)))}-1\bigr)\bigl(\mathbb{E}\mathcal{E}_{\pi_n} e^{\beta H_n(\sigma)} e_1 \mathbb{1}_{e^{\beta H_n(\sigma)}e_1\le\varepsilon}\bigr)^2\bigl(1+O(c)\bigr).
\end{aligned}
\tag{3.59}
$$

Thus we have to control

$$
\begin{aligned}
&2^{-2n}\sum_{m\in\{-1,-1+2/n,\dots,1-2/n,1\}}\sum_{x,x'\in\Sigma_n}\mathbb{1}_{R_n(x,x')=m}\bigl(e^{m^p n(\gamma^2/(\beta^2(1+m^p)))}-1\bigr) \\
&\quad = \sum_{m\in\{-1,-1+2/n,\dots,1-2/n,1\}} 2^{-n}\binom{n}{n(m+1)/2}\bigl(e^{m^p n(\gamma^2/(\beta^2(1+m^p)))}-1\bigr).
\end{aligned}
\tag{3.60}
$$

The analysis of the last sum can be carried out in the same way as was done in [1] for a very similar sum. It yields that

$$
\sum_{m=-1}^{1} 2^{-n}\binom{n}{n(m+1)/2}\bigl(e^{m^p n\gamma^2/(\beta^2(1+m^p))}-1\bigr) = Cn^{1-p/2}. \tag{3.61}
$$

□

3.6. *Conclusion of the proof.* Consider first the case $p > 4$. Lemmata 3.2 and 3.4, together with Chebyshev's inequality and the Borel–Cantelli lemma, establish that for each $v > 0$,

$$
\lim_{n\to\infty}\hat{\nu}_n^t(v) = \hat{\nu}^t(v) = K_p v^{\gamma/\beta^2-1}, \qquad \mathbb{P}\text{-a.s.} \tag{3.62}
$$

Together with the monotonicity of $\hat{\nu}_n^t(v)$ and the continuity of the limiting function $\hat{\nu}^t(v)$, this implies that there exists a subset $\Omega_1 \subset \Omega$ of the sample space Ω of the τs with the property that $\mathbb{P}(\Omega_1) = 1$, and such that, on Ω_1,

$$
\lim_{n\to\infty}\hat{\nu}_n^t(v) = \hat{\nu}^t(v) \qquad \forall v > 0. \tag{3.63}
$$

Finally, applying Feller's extended continuity theorem for Laplace transforms of (not necessarily bounded) positive measures (see [14], Theorem 2a, Section XIII.1, page 433) we conclude that, on Ω_1,

$$
\lim_{n\to\infty}\nu_n^t(u,\infty) = \nu^t(u,\infty) = K_p u^{-\gamma/\beta^2} \qquad \forall u > 0. \tag{3.64}
$$

In the cases $p = 3, 4$, where our estimates give only convergence in probability, we obtain convergence of $\nu_n^t(u,\infty)$ in probability, for example, by using the characterization of convergence of probability in terms of almost sure convergence of sub-sequences; see, for example, [23], Section II. 19. This allows us to reduce the proof in this case to that of the case of almost sure convergence.

Thus we have established Conditions (A1-1), (A2-1) and (A3-1) under the stated conditions on the parameters γ, β, p, and Theorem 1.4 follows from Theorem 1.3.

3.7. *Consequences for correlation functions.* We now turn to the proof of Theorem 1.5.

PROOF. The proof of this theorem relies on the following simple estimate. Let us denote by $\mathcal{R}_n$ the range of the coarse grained and rescaled clock process S_n^b. The argument of [1] in the proof of Theorem 1.2 that the event $A_n^\varepsilon(s,t) \cap \{\mathcal{R}_n \cap (s,t) \neq \varnothing\}$ has vanishing probability carries over unaltered. However, while in their case, $A_n^\varepsilon(s,t) \supset \{\mathcal{R}_n \cap (s,t) = \varnothing\}$, was obvious, due to the fact that the coarse graining was done on a scale $o(n)$; this is not immediately clear in our case, where the number of steps within a block is of order n^2. What we have to show is that if the process spends the whole time from s to t within one bloc, then almost all of this time is spent, without interruption, within a small ball of radius εn.

To show that this holds, we will need to establish two facts.

FACT 1. The first fact concerns the random environment. We will show that, if a trajectory within a block of length $\theta_n \sim n^2$ hits a point where the random variables H_n are "big," that is, of order an, then with overwhelming probability, all other sites with "big" H_ns this piece of path meets are within a distance εn from this point. In other words, within one block, the path will never hit two distinct clusters of large values of the random field.

FACT 2. The second fact concerns the properties of the random walk J_n. We will show that the random walk that hits such a cluster of large values will spend there, at most, a time of order εn, and it will not leave that cluster and return to it later within θ_n steps.

These two properties imply our claim.

The proof of the first fact relies on the following elementary estimate for correlated Gaussian variables. Note that the following bound is not optimal but good enough for our purposes.

LEMMA 3.11. *Let X, Y be standard Gaussian variables with covariance* $\mathrm{Cov}(X,Y) = 1-c$, $0 < c < 1/4$. *Then for* $a > 0$,

$$\begin{aligned}
\mathbb{P}(X > a, Y > a(1-c/4)) &\le \frac{1}{a2\pi\sqrt{c}} \exp\left(-\frac{a^2}{2}\left(1+\frac{c}{32}\right)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}a} \exp\left(-\frac{a^2}{2}(1+c)\right).
\end{aligned} \tag{3.65}$$

PROOF. Note that the variables X, Y have the joint density

$$\frac{1}{2\pi\sqrt{2c-c^2}}\exp\biggl(-\frac{x^2}{2}-\frac{(y-(1-c)x)^2}{4c-2c^2}\biggr). \tag{3.66}$$

Next,

$$\begin{aligned}&\mathbb{P}\bigl(X>a, Y>a(1-c/2)\bigr)\\&\quad\le \mathbb{P}\bigl(X>a, |Y-(1-c)X|>ac/4\bigr)+\mathbb{P}\biggl(X>a\frac{1-c/2}{1-c}\biggr).\end{aligned} \tag{3.67}$$

The result is now a trivial application of the standard tail estimates for Gaussian integrals. □

This lemma has the following corollary, which is a precise statement of Fact 1.

COROLLARY 3.12. *Let $H_n(\sigma)$ be the Gaussian process defined in* (1.27). *Let $\mathcal{M}_n \subset \Sigma_n$ be arbitrary. Then, for $\varepsilon > 0$ and all n large enough,*

$$\begin{aligned}&\mathbb{P}\bigl(\exists_{x,x'\in\mathcal{M}_n} : R_n(x,x') < 1-\varepsilon \text{ and}\\&\qquad H_n(x)\ge an \wedge H_n(x')\ge an(1-p\varepsilon/4)\bigr)\\&\qquad\le |\mathcal{M}_n|^2 e^{-na^2/2}e^{-na^2p\varepsilon/64}.\end{aligned} \tag{3.68}$$

A precise version of the second fact is the following lemma.

LEMMA 3.13. *Define the events*

$$\mathcal{W}_\varepsilon(k)\equiv \exists_{\{\theta_n k\le i<j-\varepsilon n\le\theta_n(k+1)\}}\{R_n(J_n(i),J_n(j))\ge 1-\varepsilon\}. \tag{3.69}$$

Then, for any $\varepsilon < 1/4$, there exists a constant $C < \infty$, such that for all n large enough, there exists $c_\varepsilon > 0$, such that

$$P_{\pi_n}(\mathcal{W}_\varepsilon(k)) \le Ce^{-c_\varepsilon n}. \tag{3.70}$$

PROOF. We clearly have to show only that an estimate of the form (3.70) holds, for any $j \ge \varepsilon n$ for the probability $P_{\pi_n}(R_n(J_n(0), J_n(j)) \ge 1-\varepsilon)$. We may also assume that $J_n(0) = \mathbf{1} \equiv (1,\dots,1)$. Observing that $R_n(x,x') = 1-2\,\mathrm{dist}(x,x')$ [see (1.30)], we have $P(R_n((\mathbf{1}, J_n(j))) \ge 1-\varepsilon) = P(\mathrm{dist}((\mathbf{1}, J_n(j))) < \varepsilon/2)$. Now we saw in the proof of Lemma 3.7 that the chain $\Pi_n(j) \equiv \mathrm{dist}(\mathbf{1}, J_n(j))$, $j \ge 1$, is the Ehrenfest chain on $\{0,\dots,n\}$, and again the desired exponential estimate follows from well-known estimates for the latter chain; see, for example, [20]. □

We now continue the proof of Theorem 1.5. As remarked above,

$$\begin{aligned}\mathcal{P}_{\pi_n}(A_n^\varepsilon(s,t)) &= \mathcal{P}_{\pi_n}\bigl(A_n^\varepsilon(s,t)\cap\{\mathcal{R}_n\cap(s,t)=\varnothing\}\bigr)\\&\quad+\mathcal{P}_{\pi_n}\bigl(A_n^\varepsilon(s,t)\cap\{\mathcal{R}_n\cap(s,t)\ne\varnothing\}\bigr),\end{aligned} \tag{3.71}$$

where the second term tends to zero. Next we observe that

$$\begin{aligned}(3.72)\qquad &\mathcal{P}_{\pi_n}\big(A_n^\varepsilon(s,t)\cap\{\mathcal{R}_n\cap(s,t)=\varnothing\}\big)\\ &\quad=\mathcal{P}_{\pi_n}\big(\mathcal{R}_n\cap(s,t)=\varnothing\big)-\mathcal{P}_{\pi_n}\big((A_n^\varepsilon(s,t))^c\cap\{\mathcal{R}_n\cap(s,t)=\varnothing\}\big).\end{aligned}$$

Here the first term is what we want. The event in the second term occurs only if the block-variable, that ensures that the event $\mathcal{R}_n\cap(s,t)=\varnothing$ occurs, contains a very long block or two sub-blocks contributing to its internal "clock-time." Corollary 3.12 and Lemma 3.13 will be used to prove that this tends to zero. To do so, it is convenient to first show that the jump over (s,t) is realized before $k_n(N)$ steps, with high probability.

For any $N<\infty$, we have

$$\begin{aligned}&\mathcal{P}_{\pi_n}\big((A_n^\varepsilon(s,t))^c\cap\{\mathcal{R}_n\cap(s,t)=\varnothing\}\big)\\ (3.73)\qquad &\quad=\sum_{k=0}^{k_n(N)-1}\mathcal{P}_{\pi_n}\big(((A_n^\varepsilon(s,t))^c)\cap\{(s,t)\subset\big(S_n^b(k),S_n^b(k+1)\big)\}\big)\\ &\qquad+\sum_{k=k_n(N)}^{\infty}\mathcal{P}_{\pi_n}\big(((A_n^\varepsilon(s,t))^c)\cap\{(s,t)\subset\big(S_n^b(k),S_n^b(k+1)\big)\}\big).\end{aligned}$$

The second term is bounded by

$$\begin{aligned}&\sum_{k=k_n(N)}^{\infty}\mathcal{P}_{\pi_n}\big(((A_n^\varepsilon(s,t))^c)\cap\{(s,t)\subset\big(S_n^b(k),S_n^b(k+1)\big)\}\big)\\ (3.74)\qquad &\quad\le\mathcal{P}_{\pi_n}\big(S_n^b(N)\le s\big)\to\mathcal{P}\big(V_{\gamma/\beta^2}(N)\le s\big),\end{aligned}$$

where convergence is almost sure (respectively, in probability, if $p=3$ or $p=4$) with respect to the environment, due to the already established convergence of S_n^b. The last probability can be made as small as desired by choosing N sufficiently large. It remains to deal with the first sum on the right-hand side of (3.73).

For a given trajectory J_n, define the event, $\mathcal{G}_\rho(k)\subset\mathcal{F}^\tau$, that in block number k (of size θ_n) two points contribute significantly to the clock that have overlap smaller then $1-\rho$. More precisely,

$$\begin{aligned}(3.75)\qquad \mathcal{G}_\rho(k)\equiv\bigcup_{\substack{k\theta_n\le i<j<(k+1)\theta_n\\ R_n(J_n(i),J_n(j))\le 1-\rho}}&\left\{\lambda_n^{-1}(J_n(i))e_{n,i}\ge\frac{c_n}{\theta_n}(t-s)\right\}\\ &\cap\left\{\lambda_n^{-1}(J_n(j))e_{n,j}\ge\frac{c_n}{\theta_n}n^{-1}\right\}.\end{aligned}$$

Note that Corollary 3.12 implies that the probability of this event, with respect to the law $\mathbb{P}$, is bounded nicely and uniformly in the variables J. Namely,

$$(3.76)\qquad \mathbb{E}\mathcal{P}_{\pi_n}(\mathcal{G}_\rho(k))\le a_n^{-1}e^{-\delta n},$$

for some $\delta > 0$ depending on the choice of ρ. The simplest way to see this is to use that the probability that one of the $e_{n,i}$ is larger than n^2 is smaller than $\exp(-n^2)$, and then use the bound from Corollary 3.12.

On the other hand, on the event $\mathcal{G}_\rho(k)^c$, $(A_n^\varepsilon(s,t))^c \cap \{(s,t) \subset (S_n^b(k), S_n^b(k+1))\}$ can only happen if the following are true: first, there still must exist some i such that $\lambda_n^{-1}(J_n(i))e_{n,i} \geq c_n(t-s)\theta_n^{-1}$, and second, the random walk must realize the event considered in Lemma 3.13.

By these considerations, we have the bound

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{k=0}^{k_n(N)} \mathcal{P}_{\pi_n}\big((A_n^\varepsilon(s,t))^c \cap \{(s,t) \subset (S_n^b(k), S_n^b(k+1))\}\big)\right) \\
&\quad \leq \sum_{k=0}^{k_n(N)} \mathbb{E}\big(\mathcal{P}_{\pi_n}(\mathcal{G}_\rho(k)) + \mathcal{P}_{\pi_n}\big(\{\exists_{k\theta_n \leq i < (k+1)\theta_n} \lambda_n^{-1}(J_n(i))e_{n_i} > c_n\theta_n^{-2}\} \\
&\qquad\qquad \cap \mathcal{W}_\varepsilon(k)\big)\big).
\end{aligned}
\tag{3.77}
$$

Next, we use Lemma 3.13 and similar reasoning as before to see that

$$
\begin{aligned}
&\mathbb{E}\mathcal{P}_{\pi_n}\big(\{\exists_{k\theta_n \leq i < (k+1)\theta_n} \lambda_n^{-1}(J_n(i))e_{n_i} > c_n\theta_n^{-2}\} \cap \mathcal{W}_\varepsilon(k)\big) \\
&\quad = \mathbb{P}\big(\exists_{k\theta_n \leq i < (k+1)\theta_n} \lambda_n^{-1}(J_n(i))e_{n_i} > c_n\theta_n^{-2}\big) P_{\pi_n}(\mathcal{W}_\varepsilon(k)) \\
&\quad \leq \theta_n^2 \mathbb{P}\big(e^{\beta H_n(x)} > c_n n^{-4}\big) C e^{-n\varepsilon c_\varepsilon} + \theta_n e^{-n^2} \\
&\quad \leq \theta_n^2 a_n^{-1} n^{\gamma\sqrt{n}\beta^{-2}} e^{-n\varepsilon c_\varepsilon} + \theta_n e^{-n^2}.
\end{aligned}
\tag{3.78}
$$

Combining all this, we see that

$$
\mathbb{E}\left(\sum_{k=0}^{k_n(N)} \mathcal{P}_{\pi_n}\big((A_n^\varepsilon(s,t))^c \cap \{(s,t) \subset (S_n^b(k), S_n^b(k+1))\}\big)\right) \leq C N e^{-\delta n},
\tag{3.79}
$$

for some positive δ, whatever the choice of ε. But this estimate implies that the term (3.72) converges to zero $\mathbb{P}$-almost surely, for any choice of N. Hence the result is obvious from the J_1 convergence of S_n^b. $\square$

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