The Annals of Probability
2012, Vol. 40, No. 6, 2460–2482
DOI: 10.1214/11-AOP688
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RELATIVE COMPLEXITY OF RANDOM WALKS IN RANDOM SCENERIES

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Dedicated to the memory of Dan Rudolph

Relative complexity measures the complexity of a probability preserving transformation relative to a factor being a sequence of random variables whose exponential growth rate is the relative entropy of the extension. We prove distributional limit theorems for the relative complexity of certain zero entropy extensions: RWRSs whose associated random walks satisfy the α-stable CLT ($1 < \alpha \le 2$). The results give invariants for *relative isomorphism* of these.

Introduction. Invariants generalizing entropy and measuring the "*complexity*" of a probability preserving transformation with zero entropy have been introduced in [12, 13] and [17].

Here we consider corresponding "*relative*" notions applied to a transformation over a factor (the classical definitions being retrieved when the factor is trivial).

We give explicit computations of the invariants obtained (distributional limits) for certain *random walks in random sceneries*.

Let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation, and let

$$\mathfrak{P} = \mathfrak{P}(X, \mathcal{B}, m) := \{\text{countable, measurable partitions of } X\}.$$

A T-*generator* is a partition $P \in \mathfrak{P}$ satisfying $\sigma(\bigcup_{n\in\mathbb{Z}} T^n P) = \mathcal{B} \bmod m$.

Given the probability preserving transformation $(X, \mathcal{B}, m, T)$, $P \in \mathfrak{P}$ and $n \ge 1$, the *Hamming metric* on $P_n := \bigvee_{j=0}^{n-1} T^{-j} P$ is given by

$$\overline{d}_n^{(P)}(a^{(1)}, a^{(2)}) := \frac{1}{n}\#\{0 \le k \le n-1 : a_k^{(1)} \ne a_k^{(2)}\},$$

where $a^{(i)} = [a_0^{(i)}, \dots, a_{n-1}^{(i)}] = \bigcap_{j=0}^{n-1} T^{-j} a_j^{(i)}$ $(i = 1, 2)$.

This induces the (T, P, n)-*Hamming pseudometric* on X given by

$$d_n^{(P)}(x, y) := \overline{d}_n^{(P)}(P_n(x), P_n(y)),$$

where $P(z)$ is defined by $z \in P(z) \in P$.

Received January 2010; revised November 2010.

[1]Supported by Israel Science Foundation Grant 1114/08.

MSC2010 subject classifications. Primary 37A35, 60F05; secondary 37A05, 60F17, 37A50.

Key words and phrases. Relative complexity, entropy dimension, random walk in random scenery, $[T, T^{-1}]$ transformation, symmetric stable process, local time.

Relative complexity. The following definitions are relativized versions of those in [12] and [17].

Given a factor $\mathcal{C} \subset \mathcal{B}$ (i.e., a T-invariant sub-σ-algebra) and $n \geq 1$, $\varepsilon > 0$, define $K_{\mathcal{C}}(P, n, \varepsilon) = K_{\mathcal{C}}^{(T)}(P, n, \varepsilon) : X \to \mathbb{R}$ by

$$K_{\mathcal{C}}(P, n, \varepsilon)(x) := \min\left\{\#F : F \subset X, m\left(\bigcup_{z \in F} B(n, P, z, \varepsilon) \| \mathcal{C}\right)(x) > 1 - \varepsilon\right\},$$

where

$$B(n, P, x, \varepsilon) := \{y \in X : d_n^{(P)}(x, y) \leq \varepsilon\},$$

and $m(\cdot \| \mathcal{C})$ denotes conditional measure with respect to $\mathcal{C}$.

Note that

$$B(n, P, x, \varepsilon) = \bigcup_{a \in P_n : \overline{d}_n^{(P)}(a, P_n(y)) \leq \varepsilon} a$$

and is therefore a union of P_n-cylinders.

The random variable $K_{\mathcal{C}}(P, n, \varepsilon)$ is $\mathcal{C}$-measurable, and the family

$$\{K_{\mathcal{C}}(P, n, \varepsilon) : n \geq 1, \varepsilon > 0\}$$

is called the *relative complexity of T with respect to P given $\mathcal{C}$*.

The unwieldiness of this family motivates a search for one sequence which describes its asymptotic properties. For example, one such sequence is given as follows:

It follows from the discussions in [12] and [17] that

$$(\bigstar) \qquad \frac{1}{n} \log K_{\mathcal{C}}(P, n, \varepsilon) \xrightarrow[n \to \infty, \varepsilon \to 0]{m} h(T, P \| \mathcal{C}),$$

where $\xrightarrow{m}$ denotes convergence in measure and $h(T, P \| \mathcal{C})$ denotes the relative entropy of the process (P, T) with respect to $\mathcal{C}$.

We consider a distributional version amplifying this complexity convergence in the case $h(T, P \| \mathcal{C}) = 0$. To "warm up" for this we give a sketch proof of ($\bigstar$) at the end of Section 2.

Complexity sequences. Let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation, let $\mathcal{C} \subset \mathcal{B}$ be a factor, let $\mathcal{K} = \{n_k\}_k$, $n_k \to \infty$ be a subsequence and let $P \in \mathfrak{P}$.

We call the sequence $(d_k)_{k \geq 1}$ $(d_k > 0)$ a *$\mathcal{C}$-complexity sequence along* $\mathcal{K} = \{n_k\}_k$ if $\exists$ a random variable Y on $(0, \infty)$ such that

$$\text{(a)} \qquad \frac{\log K_{\mathcal{C}}(P, n_k, \varepsilon)}{d_k} \xrightarrow[k \to \infty, \varepsilon \to 0]{\mathfrak{d}} Y,$$

where $\xrightarrow{\mathfrak{d}}$ denotes convergence in distribution (or just a *$\mathcal{C}$-complexity sequence* in case $\mathcal{K} = \mathbb{N}$).

For example, if $h(T, P\|\mathcal{C}) > 0$, then according to ($\bigstar$), $(n)_{n\ge 1}$ is a $\mathcal{C}$-complexity sequence for (T, P) with $Y = h(T, P\|\mathcal{C})$.

We'll see (below) that if the distributional convergence (a) holds for some T-generator $P \in \mathfrak{P}$, then it holds $\forall$ T-generators $P \in \mathfrak{P}$ in which case we call the sequence $(d_k)_{k\ge 1}$ a *$\mathcal{C}$-complexity sequence for T along* $\mathcal{K} = \{n_k\}_k, n_k \to \infty$.

The growth rates of these are invariant under *relative isomorphism* (see below).

Relative entropy dimension. This is a relative, subsequence version of the *entropy dimension* in [13].

Let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation, let $\mathcal{C} \subset \mathcal{B}$ be a factor and let $\mathcal{K} = \{n_k\}_k, n_k \to \infty$.

The *upper relative entropy dimension of T with respect to $\mathcal{C}$ along $\mathcal{K}$* is

$$\overline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) := \inf\left\{t \ge 0 : \frac{\log K(P, n_k, \varepsilon)}{n_k^t} \xrightarrow[k\to\infty,\varepsilon\to 0]{m} 0 \;\forall P \in \mathfrak{P}\right\}$$

and

the *lower relative entropy dimension of T with respect to $\mathcal{C}$ along $\mathcal{K}$* is

$$\underline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) := \sup\left\{t \ge 0 : \exists P \in \mathfrak{P}, \frac{\log K(P, n_k, \varepsilon)}{n_k^t} \xrightarrow[k\to\infty,\varepsilon\to 0]{m} \infty\right\}.$$

In case the upper and lower entropy dimensions coincide, we call the mutual value the *relative entropy dimension of T with respect to $\mathcal{C}$ along $\mathcal{K}$* and denote it by $\text{E-dim}_{\mathcal{K}}(T, \mathcal{C})$.

As before, we'll drop reference to $\mathcal{K}$ in case $\mathcal{K} = \mathbb{N}$ writing $\overline{\text{E-dim}}(T, \mathcal{C}) := \overline{\text{E-dim}}_{\mathbb{N}}(T, \mathcal{C})$ and $\underline{\text{E-dim}}(T, \mathcal{C}) := \underline{\text{E-dim}}_{\mathbb{N}}(T, \mathcal{C})$.

Simple manipulation of the definitions (using the monotonicity lemma below) shows that:

- if $(d_k)_{k\ge 1}$ is a $\mathcal{C}$-complexity sequence for T along $\mathcal{K} = \{n_k\}_k, n_k \to \infty$, then

$$(☣)\qquad \overline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) = \varlimsup_{k\to\infty} \frac{\log d_k}{\log n_k} \underline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) = \varliminf_{k\to\infty} \frac{\log d_k}{\log n_k}.$$

In fact (☣) also holds under the (more relaxed) assumption of tightness in $(0, \infty)$ of the family $\{Z_{k,\varepsilon} := \frac{\log K_{\mathcal{C}}^{(T)}(P, n_k, \varepsilon)}{d_k} : k \ge 1, \varepsilon > 0\}$ in the sense that for each $\eta > 0$ $\exists K_\eta \in \mathbb{N}, \varepsilon_\eta > 0$ and a compact interval $J \subset (0, \infty)$ such that

$$m([Z_{k,\varepsilon} \notin J]) < \eta \qquad \forall k > K_\eta, 0 < \varepsilon < \varepsilon_\eta.$$

Random walk in random scenery. A *random walk on random scenery* (*RWRS*) is a skew product probability preserving transformation, which we proceed to define in detail:

The *random scenery* is an invertible, probability preserving transformation $(Y, \mathcal{C}, \nu, S)$ and the *random walk on the random scenery* $(Y, \mathcal{C}, \nu, S)$ with *jump* *random variable* ξ (assumed $\mathbb{Z}$-valued) is the skew product $(Z, \mathcal{B}(Z), m, T)$ defined by

$$(\heartsuit) \qquad Z := \Omega \times Y, \qquad m := \mu_\xi \times \nu \quad \text{and} \quad T(x, y) := (Rx, S^{x_0} y),$$

where

$$(\Omega, \mathcal{B}(\Omega), \mu_\xi, R) := \left(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}}), \prod \operatorname{dist} \xi, \text{shift}\right)$$

is the shift of the (independent) jump random variables.

The probability preserving transformation $(\Omega, \mathcal{B}(\Omega), \mu_\xi, R)$ is known as the *base*.

We'll sometimes consider a corresponding RWRS with an *extended base* ξ whose base is an extension of the shift of the jumps

$$\pi : (\Omega', \mathcal{B}', \mu', R') \to (\Omega, \mathcal{B}(\Omega), \mu_\xi, R)$$

and which is defined by

$$(\spadesuit) \qquad Z' := \Omega' \times Y, \qquad m' := \mu' \times \nu \quad \text{and} \quad T'(x, y) := \left(R'x, S^{\pi(x)_0} y\right).$$

The terminology RWRS was coined in [19] where it was attributed to Paul Shields.

There are generalizations of RWRS over more general locally compact topological groups (not considered here) where the RWRS is constructed using a random walk on such a group and whose scenery is a probability preserving action of the group; see [2, 9].

As shown in [23], a RWRS is a K-automorphism if the random walk is aperiodic and the scenery is ergodic.

If the scenery has finite entropy and the random walk is recurrent, then the RWRS has the same entropy as its base.

Possibly the best known RWRS is Kalikow's $[T - T^{-1}]$ transformation, shown in [16] to be not Bernoulli. For a review of this and subsequent work on the Bernoulli properties of RWRSs, see [9].

The *one-sided* RWRS (*defined as above but with* Ω *replaced by the one-sided shift* $\Omega_+ = \mathbb{Z}^{\mathbb{N}}$) is considered, for example, in [15] and [2] where invariants for isomorphism and the induced *cofiltrations* are studied.

A random walk is called α*-stable* ($\alpha \in (0, 2]$) if its jump random variable is α*-stable* in the sense that for some *normalizing constants* $a(n)$ (necessarily $\frac{1}{\alpha}$-regularly varying)

$$\frac{S_n}{a(n)} \xrightarrow{\mathfrak{d}} Y_\alpha,$$

where Y_α has the *standard, symmetric* α*-stable* (SαS) distribution of order α on $\mathbb{R}$ [defined by $\mathbb{E}(e^{itY_\alpha}) = e^{-t^\alpha/\alpha}$]. A RWRS is called α*-stable* if its corresponding random walk is α-stable.

We see that for an extended base RWRS T whose corresponding random walk is aperiodic and α*-stable* ($\alpha \in (1, 2]$):

- the normalizing constants $a(n)$ form a *Base*-complexity sequence for T;
- $\text{E-dim}(T, \text{base}) = \frac{1}{\alpha}$.

Organization of the paper. We state the results more precisely in Section 1. The results on abstract relative complexity are proved in Section 2. In Section 3, we collect some random walk convergence results necessary for the proof of the distributional convergence of relative complexity for RWRS which is done in Section 4.

1. Results.

PROPOSITION 1 (Distributional compactness proposition). *For any* $P \in \mathfrak{P}$, $d_k > 0, n_k \to \infty, \exists k_\ell \to \infty$ *and a random variable* Y *on* $[0, \infty]$ *such that*

$$\text{(a)} \qquad \frac{\log K_{\mathcal{C}}(P, n_{k_\ell}, \varepsilon)}{d_{k_\ell}} \xrightarrow[\ell\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y.$$

THEOREM 2 (Generator theorem). (a) *If there is a countable* T*-generator* $P \in \mathfrak{P}$ *satisfying*

$$\frac{\log K_{\mathcal{C}}(P, n_k, \varepsilon)}{d_k} \xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y,$$

where Y *is a random variable on* $[0, \infty]$, *then*

$$(\sqcup) \qquad \frac{\log K_{\mathcal{C}}(Q, n_k, \varepsilon)}{d_k} \xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y \qquad \forall T\text{-generators } Q \in \mathfrak{P}.$$

(b) *If*

$$\frac{\log K_{\mathcal{C}}(P, n_k, \varepsilon)}{n_k^t} \xrightarrow[n\to\infty,\varepsilon\to 0]{m} 0$$

for some T*-generator* $P \in \mathfrak{P}$, *then* $\overline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) \le t$.

(c) *If*

$$\frac{\log K_{\mathcal{C}}(P, n_k, \varepsilon)}{n_k^t} \xrightarrow[n\to\infty,\varepsilon\to 0]{m} \infty$$

for some $P \in \mathfrak{P}$, *then* $\underline{\text{E-dim}}_{\mathcal{K}}(T, \mathcal{C}) \ge t$.

We'll abuse notation by abbreviating ($\sqcup$) by

$$(\boxtimes) \qquad \frac{1}{d_k} \log K_{\mathcal{C}}^{(T)}(n_k) \approx Y$$

as in, for example,

$$\frac{1}{n} \log K_{\mathcal{C}}^{(T)}(n) \approx h(T \| \mathcal{C}).$$

THEOREM 3 (Distributional convergence theorem). *Let* $(Z, \mathcal{B}(Z), m, T)$ *be an extended base RWRS with* α*-stable, aperiodic jumps* ($\alpha > 1$) *and ergodic scenery* $(Y, \mathcal{C}, \nu, S)$ *satisfying* $0 < h(S) < \infty$, *then*

$$\frac{1}{a(n)} \log K^{(T)}_{\mathcal{B}(\Omega)\times Y}(n) \approx \mathrm{Leb}(B_\alpha([0,1])) \cdot h, \tag{1}$$

where $h := h(S)$, $(a(n))_{n\geq 1}$ *are the normalizing constants of the random walk*, Leb *denotes Lebesgue measure on* $\mathbb{R}$ *and* B_α *is the* $S\alpha S$ *process* (*see below*).

Thus, as advertised at the end of Section 0, $(a(n))_{n\geq 1}$ is a *base*-complexity sequence for T and $\text{E-dim}(T, \text{base}) = \frac{1}{\alpha}$.

Relative isomorphism over a factor. We say that the probability preserving transformations $(X_i, \mathcal{B}_i, m_i, T_i)$ $(i = 1, 2)$ are *relatively isomorphic over the factors* $\mathcal{C}_i \subset \mathcal{B}_i$ $(i = 1, 2)$ if there is an isomorphism $\pi : (X_1, \mathcal{B}_1, m_1, T_1) \to (X_2, \mathcal{B}_2, m_2, T_2)$ satisfying $\pi\mathcal{C}_1 = \mathcal{C}_2$.

COROLLARY 4 (Relative isomorphism corollary). *If the probability preserving transformations* $(X_i, \mathcal{B}_i, m_i, T_i)$ $(i = 1, 2)$ *are relatively isomorphic over the factors* $\mathcal{C}_i \subset \mathcal{B}_i$ $(i = 1, 2)$, *then* $\forall \mathcal{K} = \{n_k : k \geq 1\}$, $d_k > 0$,

$$\frac{1}{d_k} \log K^{(T_1)}_{\mathcal{C}_1}(n_k) \approx Y \quad \Leftrightarrow \quad \frac{1}{d_k} \log K^{(T_2)}_{\mathcal{C}_2}(n_k) \approx Y; \tag{2}$$

$$\begin{gathered} \overline{\text{E-dim}}_{\mathcal{K}}(T_1, \mathcal{C}_1) = \overline{\text{E-dim}}_{\mathcal{K}}(T_2, \mathcal{C}_2), \\ \underline{\text{E-dim}}_{\mathcal{K}}(T_1, \mathcal{C}_1) = \underline{\text{E-dim}}_{\mathcal{K}}(T_2, \mathcal{C}_2). \end{gathered} \tag{3}$$

COROLLARY 5 (Relative isomorphism of RWRSs). *Suppose that the aperiodic, stable, extended base RWRSs* $(Z_i, \mathcal{B}_i, m_i, T_i)$ $(i = 1, 2)$ *have sceneries with positive finite entropy and are relatively isomorphic over their bases.*

If $(Z_1, \mathcal{B}_1, m_1, T_1)$ *has* α*-stable jumps, then so does* $(Z_2, \mathcal{B}_2, m_2, T_2)$ *and*

$$a^{(2)}(n)h(S^{(2)}) \underset{n\to\infty}{\sim} a^{(1)}(n)h(S^{(1)}),$$

where $a^{(i)}$ *denotes the sequence of normalizing constants of the random walk associated to* T_i $(i = 1, 2)$.

2. Relative complexity. In this section, we prove Proposition 1, Theorem 2 and Corollary 4 which are relative versions of results appearing in [12, 17] and [13] (see the remark after the proof of Proposition 1).

PROOF OF PROPOSITION 1 (the distributional compactness proposition). Define $F_k : \mathbb{R}_+ \times (0, 1) \to [0, 1]$ by $F_k(q, \varepsilon) := E(\exp[\frac{-q \log K_{\mathcal{C}}(P, n_k, \varepsilon)}{d_k}])$, then $F_k(q, \varepsilon) \leq F_k(q', \varepsilon')$ whenever $q \geq q'$, $\varepsilon \leq \varepsilon'$.

By Helly's theorem and diagonalization, $\exists$:

• a countable set $\Gamma \subset (0,1)$;

• $F : \mathbb{Q}_+ \times (0,1) \to [0,1]$ such that $F(q,\varepsilon) \le F(q',\varepsilon')$ whenever $q \ge q', \varepsilon \le \varepsilon'$; and a subsequence $k_\ell \to \infty$ such that

$$F_{k_\ell}(q,\varepsilon) \xrightarrow[\ell\to\infty]{} F(q,\varepsilon) \qquad \forall \varepsilon \in (0,1) \setminus \Gamma, q \in \mathbb{Q}_+.$$

By the monotonicity of F, $F(q,\varepsilon) \downarrow F(q)$ as $\varepsilon \downarrow 0$, whence

$$F_{k_\ell}(q,\varepsilon) \xrightarrow[\ell\to\infty,\varepsilon\to 0]{} F(q) \qquad \forall q \in \mathbb{Q}_+.$$

Thus $\exists$ a random variable Y on $[0,\infty]$ such that $F(q) = E(e^{-qY})$ and

$$\frac{\log K_{\mathcal{C}}(P, n_{k_\ell}, \varepsilon)}{d_{k_\ell}} \xrightarrow[\ell\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y. \qquad \square$$

REMARK. To see a connection with definition 2 in [12], note that it follows from Proposition 1 (in the deterministic case) that for $B_n > 0$, the set

$$\mathfrak{L} := \left\{ C \in [0,\infty] : \exists n_k \to \infty, \frac{\log K(P, n_k, \varepsilon)}{B_{n_k}} \xrightarrow[k\to\infty,\varepsilon\to 0+]{} C \right\} \ne \varnothing$$

and

$$\lim_{\varepsilon\to 0+} \overline{\lim_{n\to\infty}} \frac{\log K(P,n,\varepsilon)}{B_n} = \sup \mathfrak{L}.$$

We turn next to the proof of the generator Theorem 2.

MONOTONICITY LEMMA. *Suppose that $P, Q \in \mathfrak{P}$ are countable partitions such that $P \prec Q$, and suppose that*

$$\frac{\log K_{\mathcal{C}}(P, n_k, \varepsilon)}{d_k} \xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y \quad \text{and} \quad \frac{\log K_{\mathcal{C}}(Q, n_k, \varepsilon)}{d_k} \xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Z;$$

then $Y \le Z$ [in the sense that $E(e^{-tY}) \ge E(e^{-tZ})$ $\forall t > 0$].

PROOF. $P \prec Q \to K_{\mathcal{C}}(P,n,\varepsilon) \le K_{\mathcal{C}}(Q,n,\varepsilon)$. $\square$

LEMMA 1. *For $k \ge 1, \varepsilon > 0, x \in X$ and large $n \ge 1$,*

$$K_{\mathcal{C}}(P_k, n, 2\varepsilon)(x) \le K_{\mathcal{C}}\left(P, n, \frac{\varepsilon}{k}\right)(x) \le K_{\mathcal{C}}\left(P_k, n, \frac{\varepsilon}{2}\right)(x),$$

where $P_k := \bigvee_{j=0}^{k-1} T^{-j} P$.

PROOF. Calculation shows that $d_n^{(P_k)}(x,y) = k d_n^{(P)}(x,y) \pm \frac{k^2}{n}$ whence for large n, $B(n, P_k, x, \frac{\varepsilon}{2}) \subseteq B(n, P, x, \frac{\varepsilon}{k}) \subseteq B(n, P, x, 2\varepsilon)$. $\square$

LEMMA 2. *Let* $P = \{P_n\}_{n\ge 1}, Q = \{Q_n\}_{n\ge 1} \in \mathfrak{P}$ *be ordered partitions with* $\sum_{n\ge 1} m(P_n \Delta Q_n) < \delta$, *then* $\forall \varepsilon > 0, \exists N$ *such that* $\forall n \ge N$,

$$m\big(\{x \in X : K_{\mathcal{C}}(Q, n, \varepsilon)(x) \ge K_{\mathcal{C}}(P, n, 2\varepsilon + 2\delta)(x)\}\big) > 1 - \varepsilon.$$

PROOF. Define $N_P, N_Q : X \to \mathbb{N}$ by $x \in P_{N_P(x)}, x \in Q_{N_Q(x)}$.
By the ergodic theorem, for a.e. $x \in X, n \ge 1$ large

$$\frac{1}{n}\#\{0 \le k \le n-1 : N_P(T^k x) \ne N_Q(T^k x)\} = \frac{1}{n}\sum_{k=0}^{n-1} 1_{\Delta}(T^k x) < \delta,$$

where $\Delta := \bigcup_{n\ge 1} P_n \Delta Q_n$. It follows that $\forall \varepsilon > 0, \exists N \ge 1$ and sets $A_n \in \mathcal{B}$ $(n \ge N)$ so that for $n \ge N$:

- $m(A_n) > 1 - \varepsilon$;
- $m(A_n \| \mathcal{C})(x) > 1 - \varepsilon \ \forall x \in A_n$;
- $d_n^{(Q)}(x, y) < d_n^{(P)}(x, y) + 2\delta \ \forall x, y \in A_n$.

Thus

$$B(n, Q, x, r) \cap A_n \subseteq B(n, P, x, r + 2\delta) \qquad \forall x \in A_n, r > 0.$$

Now fix $x \in A_n$, and suppose that $F \subset X$, $|F| = K_{\mathcal{C}}(Q, n, \varepsilon)(x)$ and $m(\bigcup_{z\in F} B(n, Q, z, \varepsilon) \| \mathcal{C})(x) > 1 - \varepsilon$.

Let $F_1 := \{z \in F : B(n, Q, z, \varepsilon) \cap A_n \ne \varnothing\}$, and for $z \in F_1$, choose $z' \in B(n, Q, z, \varepsilon) \cap A_n$, then

$$\bigcup_{z\in F_1} B(n, Q, z', 2\varepsilon) \supset \bigcup_{z\in F} B(n, Q, z, \varepsilon) \setminus A_n^c.$$

On the other hand,

$$\bigcup_{z\in F_1} B(n, Q, z, 2\varepsilon) \cap A_n \subset \bigcup_{z\in F_1} B(n, P, z, 2\varepsilon + 2\delta),$$

whence for $x \in A_n$,

$$m\Big(\bigcup_{z\in F_1} B(n, P, z, 2\varepsilon + 2\delta) \| \mathcal{C}\Big)(x) > 1 - 2\varepsilon$$

and

$$K_{\mathcal{C}}(Q, n, \varepsilon)(x) \ge |F_1| \ge K_{\mathcal{C}}(P, n, 2\varepsilon + 2\delta)(x). \qquad \square$$

PROOF OF THEOREM 2 (Generator theorem). We only prove (a), the proofs of (b) and (c) being analogous.

To prove (a), we show that every subsequence of $\{n_k\}$ has a sub-subsequence (also denoted $\{n_k\}$) along which $\frac{\log K_{\mathcal{C}}(Q, n_k, \varepsilon)}{d_k} \xrightarrow[k\to\infty, \varepsilon\to 0]{\mathfrak{d}} Y$.

Fix a subsequence. By Proposition 1, $\exists$ a random variable Z on $[0,\infty]$ and a sub-subsequence along which

$$\frac{\log K_{\mathcal{C}}(P,n_k,\varepsilon)}{d_k}\xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Y \quad\text{and}\quad \frac{\log K_{\mathcal{C}}(Q,n_k,\varepsilon)}{d_k}\xrightarrow[k\to\infty,\varepsilon\to 0]{\mathfrak{d}} Z.$$

It suffices to show that $E(e^{-tY})=E(e^{-tZ})$ $\forall t>0$. We'll show that $E(e^{-tY})\le E(e^{-tZ})$ $\forall t>0$ (the reverse inequality following by symmetry).

To this end, fix $t>0,\varepsilon>0$.

- First choose $\kappa_0\ge 1$ and $\delta>0$ such that $\forall k\ge\kappa_0$, $0<r<\delta$

$$E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(P,n_k,r)}{d_k}\biggr]\biggr)=E(e^{-tY})\pm\varepsilon$$

and

$$E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(Q,n_k,r)}{d_k}\biggr]\biggr)=E(e^{-tZ})\pm\varepsilon.$$

- Next, for $0<r<\delta$, $\exists N=N_r\ge 1$, $Q^{(r)}\prec P_N$ with

$$\sum_{j\ge 1} m\bigl(Q_j^{(r)}\Delta Q_j\bigr)<r.$$

- Using Lemma 2 $\exists\kappa_r>\kappa_0$ such that

$$E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(Q^{(r)},n_k,r)}{d_k}\biggr]\biggr) < E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(Q,n_k,4r)}{d_k}\biggr]\biggr)+\varepsilon \qquad \forall k\ge\kappa_r.$$

Using Lemma 1, $\exists K_r>\kappa_r$ such that for $k>K_r$,

$$\begin{aligned} E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(Q^{(r)},n_k,r)}{d_k}\biggr]\biggr) &\ge E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(P_N,n_k,r)}{d_k}\biggr]\biggr)\\ &\ge E\biggl(\exp\biggl[\frac{-t\log K_{\mathcal{C}}(P,n_k,r/(2N))}{d_k}\biggr]\biggr)\\ &= E(e^{-tY})-\varepsilon. \end{aligned}$$

Thus $E(e^{-tY})\le E(e^{-tZ})+3\varepsilon$ $\forall\varepsilon,t>0$.

As mentioned above, this proves Theorem 2(a). $\square$

We note that Corollary 4 follows immediately from Theorem 2.

PROOF SKETCH OF ($\bigstar$). Set

$$\Pi_n(x):=\{a\in P_0^{n-1}(T): m(a\|\mathcal{C})(x)>0\},$$

$$\Phi_{n,\varepsilon}(x) := \min\Big\{|F| : F \subset \Pi_n(x), m\Big(\bigcup_{a\in F} a\|\mathcal{C}\Big)(x) > 1-\varepsilon\Big\},$$

$$\mathcal{Q}(P,n,\varepsilon)(x) := \max\{\#\{c \in \Pi_n(x) : \overline{d}_n(a,c) \le \varepsilon\} : a \in \Pi_n(x)\},$$

where $\overline{d}_n$ is (T, P, n)-Hamming distance on P_n, then

$$\frac{\Phi_{n,\varepsilon}(x)}{\mathcal{Q}(P,n,\varepsilon)(x)} \le K_{\mathcal{C}}^{(T)}(P,n,\varepsilon)(x) \le \Phi_{n,\varepsilon}(x). \tag{🚲}$$

By the Shannon–MacMillan–Breiman theorem [7], a.s., as $n \to \infty$

$$I(P_n\|\mathcal{C})(x) = \log \frac{1}{m(P_n(x)\|\mathcal{C})(x)} = h(T, P\|\mathcal{C})n\big(1+o(1)\big),$$

whence by a standard counting argument, a.s.,

$$\frac{1}{n}\log_2 \Phi_{n,\varepsilon} \xrightarrow[n\to\infty,\varepsilon\to 0]{} h(T\|\mathcal{C}). \tag{⚽}$$

By direct estimation,

$$\frac{1}{n}\log \mathcal{Q}(P,n,\varepsilon)(x) \le \frac{1}{n}\log\left(|P|^{\varepsilon n}\binom{n}{\varepsilon n}\right) \xrightarrow[n\to\infty,\varepsilon\to 0]{} 0, \tag{☢}$$

whence by (🚲),

$$\frac{1}{n}\log K_{\mathcal{C}}(P,n,\varepsilon) \xrightarrow[n\to\infty,\varepsilon\to 0]{m} h(T, P\|\mathcal{C}). \tag{★}$$

□

The proof of Theorem 3 in Section 4 is also via (🚲) and versions of (⚽) and (☢) where the denominators n are replaced by the a sequence of normalizing constants of the random walk $a(n) = o(n)$.

More information on random walk is needed and developed in Section 3. The version of (⚽) is established (essentially as in Section 7 of [1]) using Skorohod's invariance principle and properties of the range of the random walk. The proof of the (☢) analogue uses the invariance principle for local time as well.

3. Random walks. In this section we consider the random walk limit theorems we need to prove Theorem 3.

These are consequences of the *weak invariance principle* and an *invariance principle for local time* as in Borodin's theorem (below); and the properties of limit processes involved.

• As in [14], the *SαS process* B_α (for $0 < \alpha \le 2$) is a random function in $D([0, 1])$, the *Donsker space* of CADLAG functions (Polish when equipped with the Skorokhod metric, see [3]) with independent, SαS distributed increments (B_2 is aka *Brownian motion*).

• The *weak invariance principle* says that for a α-stable, random walk $S_n = \sum_{k=1}^n \xi_k$,

$$B_{\xi,n} \xrightarrow{\mathfrak{d}} B_\alpha$$

in $D([0, 1])$ where

$$B_{\xi,n}(t) := \frac{1}{a_\xi(n)} S_{[nt]},$$

and $a_\xi(n)$ are the normalizing constants (of the random walk) satisfying

$$\frac{S_n}{a_\xi(n)} \xrightarrow{\mathfrak{d}} Y_\alpha,$$

where $\mathbb{E}(e^{itY_\alpha}) = e^{-|t^\alpha|/\alpha}$.

See [10] for the case $\alpha = 2$ and [14] for $0 < \alpha \le 2$.

Local time. For $1 < \alpha \le 2$, the *local time at* $x \in \mathbb{R}$ of the SαS process B_α is defined by

$$L_\alpha(t, x) := \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon,x+\varepsilon]}(B_\alpha(s))\, ds$$

the limit being known to exists a.s. As shown in [6], a.s., $L_\alpha \in C_0([0, 1] \times \mathbb{R})$, the space of continuous functions on $[0, 1] \times \mathbb{R}$ tending to zero at infinity, which is Polish when equipped with the sup-norm.

We need more information about the unit range $B_\alpha([0, 1])$ of the SαS process.

LEMMA 3 [11]. *With probability* 1, $B_\alpha([0, 1])$ *is Riemann integrable in* $\mathbb{R}$ *and* $L_\alpha(1, x) > 0$ *for* Leb-*a.e.* $x \in B_\alpha([0, 1])$.

REMARK. More is true when $\alpha = 2$. Brownian motion B_2 is a.s. continuous whence $B_2([0, 1]) = [\min_{t\in[0,1]} B_2(t), \max_{t\in[0,1]} B_2(t)]$. The Ray–Knight theorem [21, 25], states that a.s., $L_2(1, x) > 0$ iff $x \in B_2([0, 1])^o = (\min_{t\in[0,1]} B_2(t), \max_{t\in[0,1]} B_2(t))$.

Here and throughout, we denote the *interior* (maximal open subset) of $F \subset \mathbb{R}$ by F^o.

It is an interesting question as to whether this version of the Ray–Knight theorem persists for $1 < \alpha < 2$, that is, whether $B_\alpha([0, 1])^o = \{x \in \mathbb{R} : L_\alpha(1, x) > 0\}$ with probability 1.

PROOF OF LEMMA 3. By continuity of $x \mapsto L_\alpha(1, x)$, a.s.

$$B_\alpha([0, 1])^o \supset \{x \in \mathbb{R} : L_\alpha(1, x) > 0\},$$

and it suffices to show that with probability 1,

$$\mathrm{Leb}\big(B_\alpha([0, 1]) \cap \{x \in \mathbb{R} : L_\alpha(1, x) = 0\}\big) = 0.$$

To see this, for $F \in D([0,1])$ and $y \in [0,1]$, define

$$L(F)(y) := \overline{\lim_{n\to\infty}}\, 2n \int_0^1 1_{(y-1/n,y+1/n)}(F(t))\,dt$$
$$= \overline{\lim_{n\to\infty}}\, 2n \int_0^1 1_{(F(t)-1/n,F(t)+1/n)}(y)\,dt$$

and define for $F \in D([0,1]), t \in [0,1]$

$$\Phi(F,t) := (F, F(t), L(F)(F(t))) \in D([0,1]) \times \mathbb{R}^2.$$

We claim that $\Phi : D([0,1]) \times [0,1] \to D([0,1]) \times \mathbb{R}^2$ is Borel measurable.

To see this, note first that $F : [0,1] \to \mathbb{R}$ is bounded, Borel measurable ($F \in D([0,1])$ being a uniform limit of step functions), whence $L(F) : \mathbb{R} \to \mathbb{R}$ is bounded, Borel measurable. Thus Φ is Borel measurable.

Next, we claim that

$$\mathbf{A} := \{(F,y) \in D([0,1]) \times \mathbb{R} : y \in F([0,1]), L(F)(y) = 0\}$$

is an analytic set in $D([0,1]) \times \mathbb{R}$.

This is because

$$\mathbf{A} = \Pi\big(\Phi\big(D([0,1]) \times [0,1]\big) \cap \big(D([0,1]) \times \mathbb{R} \times \{0\}\big)\big),$$

where $\Pi : D([0,1]) \times \mathbb{R}^2 \to D([0,1]) \times \mathbb{R}$ is the projection $\Pi(F,x,y) := (F,x)$.

Thus $\mathbf{A}$ is Prob $\times$ Leb-Lebesgue measurable in $D([0,1]) \times \mathbb{R}$ where Prob := dist $B_\alpha \in \mathcal{P}(D([0,1]))$.

Next, a.s., $L(B_\alpha)(y) = L_\alpha(1,y)$ and (see, e.g., [18] and references therein) $\forall y \in \mathbb{R}$,

$$\mathrm{Prob}\big(\big[B_\alpha([0,1]) \ni y \text{ and } L_\alpha(1,y) = 0\big]\big) = 0.$$

Thus, using Fubini's theorem,

$$\begin{aligned}
&E\big(\mathrm{Leb}\big(B_\alpha([0,1]) \cap [L_\alpha(1,\cdot) = 0]\big)\big)\\
&\quad = \mathrm{Prob} \times \mathrm{Leb}(\mathbf{A})\\
&\quad = \int_{\mathbb{R}} E\big(1_{B_\alpha([0,1])}(y) 1_{[L_\alpha(1,y)=0]}\big)\,dy\\
&\quad = \int_{\mathbb{R}} \mathrm{Prob}\big(\big[B_\alpha([0,1]) \ni y \text{ and } L_\alpha(1,y) = 0\big]\big)\,dy\\
&\quad = 0.
\end{aligned}$$

□

Random walk local time. The *local time of the random walk* is

$$N_{n,k}(x) := \#\{0 \le j \le n-1 : S_j(x) = k\} \qquad (n \ge 1, k \in \mathbb{Z}, x \in \Omega).$$

We define the linear interpolation of N by

$$\widehat{N}(n+s,k+t):=(1-s)(1-t)N_{n,k}+s(1-t)N_{n+1,k}\\ +(1-s)tN_{n,k+1}+stN_{n+1,k+1}$$

for $s,t\in[0,1]$, $n\in\mathbb{N}$, $k\in\mathbb{Z}$, and let

$$L_{\xi,n}(t,x):=\frac{1}{\overline{a}_\xi(n)}\widehat{N}(nt,a_\xi(n)x),$$

where $\overline{a}_\xi(x):=\int_0^x\frac{1}{a_\xi(t)}\wedge 1\,dt$.

REMARKS. (i) Since $a_\xi(x)$ is $\frac{1}{\alpha}$-regularly varying, we have $\overline{a}_\xi(x)\sim\frac{\alpha}{\alpha-1}\frac{x}{a_\xi(x)}$.
(ii) $L_{\xi,n}\in C_0([0,1]\times\mathbb{R})$.

Borodin's theorem [4, 5]. Suppose that $(S_1,S_2,\dots)$ is an aperiodic, α-stable random walk on $\mathbb{Z}$ with $1<\alpha\le 2$, then

$$(B_{\xi,n},L_{\xi,n})\xrightarrow[n\to\infty]{\mathfrak{d}}(B_\alpha,L_\alpha)\qquad\text{in } D([0,1])\times C_0([0,1]\times\mathbb{R}).$$

Borodin's theorem strengthens the invariance principle for local time in Section 2 of [19].

Next we state the main lemma of this section. To this end we first establish some notation.

Convergence in distribution via convergence in measure. For the rest of this section, we'll fix $(S_1,S_2,\dots)$, an aperiodic, α-stable random walk on $\mathbb{Z}$ with $1<\alpha\le 2$ defined on $(\Omega,\mathcal{B}(\Omega),\mu)$ as before and use the following (seemingly stronger but) equivalent "coupling version" of Borodin's theorem.

Let $(\mathbf{\Omega},\mathfrak{F}):=\Omega\times(D([0,1])\times C_0([0,1]\times\mathbb{R}))$ equipped with its Borel sets.

BORODIN'S THEOREM ([4, 5]). *There is a probability* $\mathbf{P}\in\mathcal{P}(\mathbf{\Omega},\mathfrak{F})$ *such that*

$$\mathbf{P}\big(A\times\big(D([0,1])\times C_0([0,1]\times\mathbb{R})\big)\big)\\ =\mu(A)\qquad\forall A\in\mathcal{B}(\Omega);$$

$$\mathbf{P}\big(\Omega\times[(B_\alpha,L_\alpha)\in B]\big)\\ =\mathbf{Q}\big([(B_\alpha,L_\alpha)\in B]\big)\qquad\forall B\in\mathcal{B}\big(D([0,1])\times C_0([0,1]\times\mathbb{R})\big);$$

where $\mathbf{Q}=\operatorname{dist}(B_\alpha,L_\alpha)\in\mathcal{P}(D([0,1])\times C_0([0,1]\times\mathbb{R})$ *and such that*

$$(B_{\xi,n},L_{\xi,n})\xrightarrow[n\to\infty]{m}(B_\alpha,L_\alpha)\qquad\textit{in } D([0,1])\times C_0([0,1]\times\mathbb{R}).$$

This "coupling version" is given in [4] and [5]. Equivalence with the distributional version above follows from a general theorem of Skorokhod; see [3].

We'll use the following proposition.

PROPOSITION. *If M is a metric space, and $\Psi : D([0,1]) \times C_0([0,1] \times \mathbb{R}) \to M$ is continuous, then*

$$\Psi(B_{\xi,n}, L_{\xi,n}) \xrightarrow[n\to\infty]{m} \Psi(B_\alpha, L_\alpha) \qquad \textit{in } M.$$

LOCAL TIME LEMMA. *For $E \subset \mathbb{R}$ a finite union of closed, bounded intervals,*

$$Y_{E,n} := \frac{1}{\overline{a}_\xi(n)} \min_{k \in a_\xi(n)E} N_{n,k} \xrightarrow[n\to\infty]{m} \min_{x\in E} L_\alpha(1,x).$$

PROOF. Since $E \subset \mathbb{R}$ a finite union of closed, bounded intervals, we have (using tightness of $\{L_{\xi,n} : n \in \mathbb{N}\}$ in $C_0([0,1] \times \mathbb{R})$) that

$$Y_{E,n} - \min_{x\in E} L_{\xi,n}(1,x) \xrightarrow[n\to\infty]{m} 0.$$

The function $F : D([0,1]) \times C_0([0,1] \times \mathbb{R}) \to \mathbb{R}$ defined by

$$F(X,Y) := \min_{t\in E} Y(t)$$

is continuous. By the proposition, and Borodin's theorem,

$$F(B_{\xi,n}, L_{\xi,n}) \xrightarrow[n\to\infty]{m} F(B_\alpha, L_\alpha) = \min_{x\in E} L_\alpha(1,x).$$

The lemma follows from this. $\square$

Hyperspace. Let $\mathcal{H}$ be the hyperspace of all nonempty closed, bounded subsets of $\mathbb{R}$. Equip $\mathcal{H}$ with the *Hausdorff metric*,

$$h(A,A') := \inf\{r > 0 : A \subset \mathcal{N}(A',r) \text{ and } A' \subset \mathcal{N}(A,r)\}$$

for $A, B \in \mathcal{H}$ where, for $A \in \mathcal{H}$, $x \in \mathbb{R}$ and $r > 0$,

$$\mathcal{N}(A,r) := \left\{ x \in \mathbb{R} : \inf_{y\in A} |x-y| \le r \right\}.$$

As is well known, $(\mathcal{H}, h)$ is a locally compact, separable metric space.

The *range* of the $\mathbb{Z}$-random walk is $V_n := \{S_j : 0 \le j \le n-1\}$. Note that $\frac{1}{a(n)} V_n = B_{\xi,n}([0,1]) \in \mathcal{H}$.

HYPERSPACE CONVERGENCE LEMMA. *Suppose that $(S_1, S_2, \ldots)$ is an aperiodic, α-stable random walk on $\mathbb{Z}$ with $1 < \alpha \le 2$, then*

$$\frac{1}{a(n)} V_n \xrightarrow{m} \overline{B_\alpha([0,1])} \qquad \textit{in } \mathcal{H}.$$

PROOF. The function $X \mapsto \overline{X([0,1])}$ is continuous $D([0,1]) \to \mathcal{H}$. $\square$

Dyadic partitions and sets. For $\kappa \in \mathbb{N}$, let Δ_κ be the *dyadic partition of order* κ defined by

$$\Delta_\kappa := \left\{\left[\frac{p}{2^\kappa}, \frac{p+1}{2^\kappa}\right], p \in \mathbb{Z}\right\}.$$

A *closed dyadic set* is a finite union of elements of $\bigcup_{\kappa\geq 1} \Delta_\kappa$. An *open dyadic set* is the interior of a closed dyadic set. The *order* of a dyadic set is the minimal $\kappa \in \mathbb{N}$ so that the (closure of the) dyadic set is a union of elements of Δ_κ. Let $\mathcal{D}_\kappa^o$ and $\overline{\mathcal{D}}_\kappa$ denote the collections of open and closed dyadic sets of order κ, respectively.

For $E \subset \mathbb{R}$ bounded, nonempty and $\kappa \geq 1$ let:

• $C_\kappa(E)$ be the largest closed dyadic set of order κ contained in E^o;

• $U_\kappa(E)$ be the smallest open dyadic set of order κ containing E. Note that $C_\kappa(E) \subset U_\kappa(E) \neq \varnothing$ and that it is possible that $C_\kappa(E) = \varnothing$.

For $\kappa \in \mathbb{N}$, $\Upsilon \in \mathcal{D}_\kappa^o$, $\Gamma \in \overline{\mathcal{D}}_\kappa$ satisfying $\Gamma \subset \Upsilon$, define the set

$$\mathcal{U}(\kappa, \Gamma, \Upsilon) := \{E \in \mathcal{H} : C_\kappa(E) = \Gamma, U_\kappa(E) = \Upsilon\}.$$

These sets are not open in $\mathcal{H}$, but are Borel sets in $\mathcal{H}$ with the additional property that

☺ $\forall E \in \mathcal{U}(\kappa, \Gamma, \Upsilon) \exists \delta > 0$ such that $\mathcal{N}(E, \delta) \subset \Upsilon$.

The sets $\mathcal{U}(\kappa, \Gamma, \Upsilon)$ and $\mathcal{U}(\kappa, \Gamma', \Upsilon')$ are disjoint unless $\Gamma = \Gamma'$ and $\Upsilon = \Upsilon'$.

Admissibility. For $\mathcal{E} > 0$, we call a pair $(\Gamma, \Upsilon) \in \bigcup_{\kappa\geq 1} \overline{\mathcal{D}}_\kappa \times \mathcal{D}_\kappa^o$ *$\mathcal{E}$-admissible* if:

(i) $\mu = \mu(\Gamma, \Upsilon) := \mathrm{Leb}(\Upsilon \setminus \Gamma) < \mathcal{E}$;

and for $N \in \mathbb{N}$, we call (Γ, Υ) *$(N, \mathcal{E})$-admissible* if in addition

(ii) $MH(3\mu) + 3\mu \log N < \mathcal{E}$ where $M = M(\Gamma, \Upsilon) := \mathrm{Leb}(\Upsilon)$ and $H(t) := -t \log t - (1-t)\log(1-t)$.

Note that if $A \subset \mathbb{R}$ is Riemann integrable, then $\forall \mathcal{E} > 0, \exists$ a $(N, \mathcal{E})$-admissible pair $(\Gamma, \Upsilon) \in \overline{\mathcal{D}}_\kappa \times \mathcal{D}_\kappa^o$ so that $\Gamma \subset A \subset \Upsilon$ and that in this case $(C_\kappa(A), U_\kappa(A))$ is also $(N, \mathcal{E})$-admissible.

LEMMA 4. *For each $\mathcal{E} > 0, N \in \mathbb{N}, \varepsilon > 0, \exists \kappa \in \mathbb{N}$ and $\theta > 0$ and a finite collection of $(N, \mathcal{E})$-admissible pairs*

$$\{(\Gamma_j, \Upsilon_j)\}_{j\in J} \subset \overline{\mathcal{D}}_\kappa \times \mathcal{D}_\kappa^o$$

satisfying:

(i)

$$\mathbf{P}\Big(\biguplus_{j\in J} G_j\Big) > 1 - \varepsilon \qquad \textit{where } G_j := \big[\overline{B_\alpha([0,1])} \in \mathcal{U}(\kappa, \Gamma_j, \Upsilon_j)\big].$$

For large enough $n \geq 1$,

(ii)

$$\mathbf{P}(G_{j,\theta,n}) > (1-\varepsilon)\cdot\mathbf{P}(G_j) \qquad \forall j\in J,$$

where

$$G_{j,\theta,n} := \left[\frac{1}{\overline{a}(n)}\min_{k\in a(n)\Gamma_j} N_{n,k} > \theta, \frac{1}{a(n)}V_n \subset \Upsilon_j\right]\cap G_j.$$

PROOF. By Lemma 3, $B_\alpha([0,1])$ is a.s. Riemann integrable, so $\exists\kappa\in\mathbb{N}$ and a finite collection of $(N,\mathcal{E})$-admissible pairs $\{(\Gamma_j,\Upsilon_j)\}_{j\in J}\subset\overline{\mathcal{D}}_\kappa\times\mathcal{D}_\kappa^o$ satisfying (i).

Suppose that $\mathbf{P}(G_j)\ge\eta>0\;\forall j\in J$.

By Lemma 3, $\min_{x\in\Gamma_j} L_\alpha(1,x)>0$ a.s. on G_j $\forall j\in J$. This and (☺) ensure that $\exists\theta>0$ such that $\forall j\in J$,

$$\mathbf{P}\Big(\Big[\min_{x\in\Gamma_j} L_\alpha(1,x)>2\theta\Big]\cap\big[\mathcal{N}(\overline{B_\alpha([0,1])},\theta)\subset\Upsilon_j\big]\cap G_j\Big)>\Big(1-\frac{\varepsilon}{2}\Big)\mathbf{P}(G_j).$$

By the local time, and hyperspace convergence lemmas, for $n\ge1$ large

$$\mathbf{P}\Big(\Big[h\Big(\frac{1}{a(n)}V_n,\overline{B_\alpha([0,1])}\Big)\ge\theta\Big]\Big)<\frac{\eta\varepsilon}{4};$$

$$\mathbf{P}\Big(\Big[\Big|\min_{x\in\Gamma_j} L_\alpha(1,x)-\frac{1}{a(n)}\min_{k\in a_\xi(n)\Gamma_j} N_{n,k}\Big|>\theta\Big]\Big)<\frac{\eta\varepsilon}{4}.$$

Statement (ii) follows from this. $\square$

4. Relative complexity of RWRS. We prove Theorem 3(1).

Fix a finite, S-generator $\beta\in\mathfrak{P}(Y,\mathcal{C},\mu)$, and let $P=P_\beta\in\mathfrak{P}(Z,\mathcal{B},m)$ defined by $P(x,y):=\alpha(x)\times\beta(y)$ where $\alpha(x):=[x_0]$, then

$$P_0^{n-1}(T)(x,y)=\alpha_0^{n-1}(R)(x)\times\beta_{V_n(x)}(S)(y),$$

where

$$\beta_{V_n(x)}(S):=\bigvee_{k\in V_n(x)} S^{-k}\beta.$$

Define for $n\in\mathbb{N}$, $\varepsilon>0$ [as in the proof of (★)], $\Pi_n:\Omega\to 2^{P_0^{n-1}(T)}$ by

$$\Pi_n(x):=\big\{a\in P_0^{n-1}(T): m\big(a\|\mathcal{B}(\Omega)\times Y\big)(x)>0\big\}.$$

Note that for fixed $x\in\Omega$, if $z\in\Pi_n(x)$, then z is of form

$$z=(x_0^{n-1},w):=[x_0^{n-1}]\times\bigvee_{j\in V_n(x)} S^{-j}w_j\ (w_j\in\beta).$$

Now define $\Phi_{n,\varepsilon}, \mathcal{Q}(P, n, \varepsilon) : \Omega \to \mathbb{N}$ by

$$\Phi_{n,\varepsilon}(x) := \min\left\{\#F : F \subset \Pi_n(x) : m\left(\bigcup_{a\in F} a \| \mathcal{B}(\Omega) \times Y\right)(x) > 1 - \varepsilon\right\};$$

$$\mathcal{Q}(P, n, \varepsilon)(x) := \max\{\#\{c \in \Pi_n(x) : \overline{d}_n(a, c) \le \varepsilon\} : a \in \Pi_n(x)\},$$

where $\overline{d}_n$ is the (T, P, n)-Hamming metric

$$\overline{d}_n([a_0, \dots, a_{n-1}], [c_0, \dots, c_{n-1}]) = \frac{1}{n}\#\{0 \le k \le n-1 : a_k \ne c_k\}.$$

As before,

(🚲) $$\frac{\Phi_{n,\varepsilon}(x)}{\mathcal{Q}(P, n, \varepsilon)(x)} \le K^{(T)}_{\mathcal{B}(\Omega)\times Y}(P, n, \varepsilon)(x) \le \Phi_{n,\varepsilon}(x).$$

To establish Theorem 3(1), it suffices by (🚲) to show that

(⚽) $$\frac{1}{a(n)} \log_2 \Phi_{n,\varepsilon} \xrightarrow[n\to\infty,\varepsilon\to 0]{\mathfrak{d}} \mathrm{Leb}(B_\alpha([0, 1]))h(S, \beta)$$

and

(☢) $$\frac{1}{a(n)} \log_2 \mathcal{Q}(P, n, \varepsilon) \xrightarrow[n\to\infty,\varepsilon\to 0]{m} 0.$$

Proof of (⚽). In order to use Lemma 4, we consider $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}), \mathbf{m}, \mathbf{T})$ where

$$\begin{aligned}\mathbf{Z} &:= \mathbf{\Omega} \times Y \cong Z \times D([0, 1]) \times C_0([0, 1] \times \mathbb{R}),\\ \mathbf{m} &:= \mathbf{P} \times \nu,\\ \mathbf{T}(x, y, t) &:= (T(x, y), t)\end{aligned}$$

and prove that on $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}), \mathbf{m})$,

(⚽̂) $$\frac{1}{a(n)} \log_2 \Phi_{n,\varepsilon} \xrightarrow[n\to\infty,\varepsilon\to 0]{m} \mathrm{Leb}(B_\alpha([0, 1]))h(S, \beta);$$

[which implies (⚽) on $(Z, \mathcal{B}(Z), m)$], deducing (⚽̂) from

(⚽̃) $$\frac{1}{a(n)} I\left(P_0^{n-1}(T) \| \mathcal{B}(\Omega) \times Y\right) \xrightarrow[n\to\infty]{m} \mathrm{Leb}(B_\alpha([0, 1]))h(S, \beta),$$

where $I(\alpha \| \mathcal{C})$ is *conditional information* defined by

$$I(\alpha\|\mathcal{C})(x) := \log \frac{1}{\mathbf{m}(\alpha(x)\|\mathcal{C})(x)}.$$

Proof of ($\tilde{\circledast}$). By the above

$$P_0^{n-1}(T)(x, y) = \alpha_0^{n-1}(R)(x) \times \beta_{V_n(x)}(S)(y),$$

whence

$$I\big(P_0^{n-1}(T)\|\mathcal{B}(\Omega) \times Y\big)(x) = \log \frac{1}{\nu(\beta_{V_n(x)}(S)(y))} = I\big(\beta_{V_n(x)}(S)\big)(y).$$

The idea of the proof is to approximate $V_n(x)$ with sequences of sets of form $F_{\Lambda,n} := (a(n)\Lambda) \cap \mathbb{Z}$ where $\Lambda \subset \mathbb{R}$ is a finite union of disjoint, bounded, intervals.

Any such sequence $\{F_{\Lambda,n} : n \geq 1\}$ satisfies *Føllner's condition*:

$$\frac{\#(F_{\Lambda,n} \Delta (F_{\Lambda,n} + j))}{\#F_{\Lambda,n}} \xrightarrow[n\to\infty]{} 0 \qquad \forall j \in \mathbb{Z}.$$

Moreover,

$$\#F_{\Lambda,n} = (a(n)\Lambda) \cap \mathbb{Z} = a(n)\mathrm{Leb}(\Lambda) \pm 2M,$$

where Λ is a union of M disjoint intervals.

Thus, by Kieffer's Shannon–MacMillan theorem ([20]—see also [24])

$$\text{(SM)} \qquad \frac{1}{a(n)} I(\beta_{F_{\Lambda,n}}(S)) \xrightarrow[n\to\infty]{m} h(S, \beta)\mathrm{Leb}(\Lambda).$$

Fix $\varepsilon > 0$ and let $\kappa \in \mathbb{N}, \theta > 0$ and the finite collection $\{(\Gamma_j, \Upsilon_j)\}_{j \in J} \subset \overline{\mathcal{D}}_\kappa \times \mathcal{D}_\kappa^o$ of ε-admissible pairs be as in Lemma 4.

By (SM) for large $n \geq 1, \exists H_n \in \mathcal{B}(Y)$ so that $\nu(H_n) > 1 - \varepsilon$ and such that $\forall y \in H_n, \Lambda \in \{\Gamma_j, \Upsilon_j\}_{j \in J}$,

$$\frac{1}{a(n)} I(\beta_{F_{\Lambda,n}}(S))(y) = (1 \pm \varepsilon) h(S, \beta)\mathrm{Leb}(\Lambda).$$

For $\omega \in G_{j,\theta,n}$,

$$\frac{1}{a(n)} V_n(\omega) \subset \mathcal{N}(\overline{B_\alpha([0, 1])}, \theta) \subset \Upsilon_j$$

and

$$(a(n)\Gamma_j) \cap \mathbb{Z} \subset \{k \in \mathbb{Z} : N_{n,k}(\omega) > \theta \overline{a}(n)\} \subset V_n(x).$$

Thus

$$F_{\Gamma_j,n} \subset V_n(\omega) \subset F_{\Upsilon_j,n},$$

whence for $y \in H_n$,

$$\begin{aligned} &h(S,\beta)\mathrm{Leb}(B_\alpha([0,1])) - \varepsilon \\ &\quad< \frac{1}{a(n)} I(\beta_{\Lambda_{n,\Gamma_j}}(S)) \le \frac{1}{a(n)} I(\beta_{V_n(\omega)}(S)) \\ &\quad\le \frac{1}{a(n)} I(\beta_{\Lambda_{n,\Upsilon_j}}(S)) \\ &\quad< h(S,\beta)\mathrm{Leb}(B_\alpha([0,1])) + \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbf{P}\Big(\Big[\frac{1}{a(n)} I\big(\beta_{V_n(\omega)}(S)\big) = h(S,\beta)\mathrm{Leb}(B_\alpha([0,1])) \pm \varepsilon\Big]\Big) \\ &\quad\ge \sum_{j\in J} \mathbf{P}(G_{j,\theta,n})\nu(H_n) \\ &\quad> (1-\varepsilon)^2 \sum_{j\in J} \mathbf{P}(G_j) \\ &\quad> (1-\varepsilon)^3. \end{aligned} \tag{$\tilde{\circledast}$}$$

REMARK. We note that the methods of the proof of ($\tilde{\circledast}$) can be adapted to prove Theorem 7.1 in [22], namely

$$\frac{\#V_n}{a(n)} \xrightarrow{\mathfrak{d}} \mathrm{Leb}(B_\alpha([0,1])).$$

Proof of ($\hat{\circledast}$). By ($\tilde{\circledast}$), $\forall \varepsilon > 0, \exists N_\varepsilon$ such that $\forall n > N_\varepsilon \exists G_n \in \mathcal{B}(\mathbf{\Omega})$ so that for $x \in G_n$,

$$\nu(H_{n,x}) > 1 - \varepsilon,$$

where

$$H_{n,x} := \big\{y \in Y : \nu(\beta_{V_n(x)}(S)(y)) = e^{-a(n)\mathrm{Leb}(B_\alpha([0,1])h(S,\beta)(1\pm\varepsilon))}\big\}.$$

Let $F_{n,\varepsilon,x} := \{\beta_{V_n(x)}(S)(y) : y \in H_{n,x}\}$. It follows that

$$\log \#F_{n,\varepsilon,x} = a(n)\mathrm{Leb}\big(B_\alpha([0,1])h(S,\beta)(1\pm\varepsilon)\big).$$

Thus

$$\log \Phi_{n,\varepsilon}(x) \le a(n)\mathrm{Leb}\big(B_\alpha([0,1])h(S,\beta)(1+\varepsilon)\big).$$

On the other hand, if $F \subset \Pi_n(x), m(\bigcup_{a\in F} a \| \mathcal{B}(\Omega) \times Y)(x) > 1 - \varepsilon$, then $F \supset F_{n,2\varepsilon,x}$, whence

$$\log \Phi_{n,\varepsilon}(x) \ge a(n)\mathrm{Leb}\big(B_\alpha([0,1])h(S,\beta)(1-2\varepsilon)\big). \tag{$\hat{\circledast}$}$$

Proof of (☢). Fix $\varepsilon = \mathcal{E} > 0$. Let $\kappa \in \mathbb{N}$ and $\theta > 0$ and the finite collection of $(\#\beta, \mathcal{E})$-admissible pairs

$$\{(\Gamma_j, \Upsilon_j)\}_{j\in J} \subset \overline{\mathcal{D}}_\kappa \times \mathcal{D}^o_\kappa$$

be as in Lemma 4.

(1) For large n, $x \in G_{j,n,\theta}$, $a = (x_0^{n-1}, w)$, $a' = (x_0^{n-1}, w') \in \Pi_n(x)$,

$$\#\{i \in V_n(x) : w_i \neq w'_i\} \leq a(n)\left(\mu_j + \frac{d_n(a,a')}{\theta}\right),$$

where $\mu_j := \mu(\Gamma_j, \Upsilon_j) = \mathrm{Leb}(\Upsilon_j \setminus \Gamma_j)$.

PROOF. Let $x \in G_{j,n,\theta}$, and let

$$K_n(x) := \{i \in V_n(x) : w_i \neq w'_i\}.$$

Then since

$$\Lambda_{n,\Gamma_j} \subset V_n(x) \subset \Lambda_{n,\Upsilon_j}$$

we have that

$$K_n \subset K_n \cap \Lambda_{n,\Gamma_j} \cup \Lambda_{n,\Upsilon_j} \setminus \Lambda_{n,\Gamma_j},$$

whence, for large n,

$$\#K_n \leq \#(K_n \cap \Lambda_{n,\Gamma_j}) + \#(\Lambda_{n,\Upsilon_j} \setminus \Lambda_{n,\Gamma_j}) \leq \#(K_n \cap \Lambda_{n,\Gamma_j}) + \mu_j a(n).$$

Now,

$$\begin{aligned}
\#(K_n \cap \Lambda_{n,\Gamma_j}) &\leq \frac{1}{\theta \overline{a}(n)} \sum_{k\in\Lambda_{n,\Gamma_j}} N_{n,k} 1_{K_n}(k) \\
&= \frac{1}{\theta \overline{a}(n)} \sum_{k=0}^{n-1} \#\{0 \leq i \leq n-1 : w_{s_i(x)} \neq w'_{s_i(x)}\} \\
&= \frac{n}{\theta \overline{a}(n)} d_n(a,a') \\
&\lesssim \frac{1}{\theta} a(n) d_n(a,a'). \qquad \square
\end{aligned}$$

(2) For n large,

$$\max_{x\in G_{j,n,\theta}} \mathcal{Q}\left(P, n, \frac{\mu_j}{\theta}\right)(x) \leq e^{\mathcal{E}a(n)(1+o(1))}.$$

PROOF. Fix $x \in G_{j,n,\theta}$, $z = (x_0^{n-1}, u) \in \Pi_n(x)$, then

$$\left\{a \in \Pi_n(x) : a \subset B\left(n, P, z, \frac{\mu_j}{\theta}\right)\right\}$$
$$\underset{\sim}{\subseteq} \left\{v \in \beta^{V_n} : d_n((x_0^{n-1}, u), (x_0^{n-1}, v)) < \frac{\mu_j}{\theta}\right\}$$
$$\overset{(1)}{\subseteq} \{v \in \beta^{V_n} : \#\{i \in V_n(x) : v_i \neq u_i\} \leq 2\mu_j a(n)\}.$$

Thus for n large,

$$\begin{aligned}
\#\left\{\Pi_n(x) : a \subset B\left(n, P, z, \frac{\mu_j}{\theta}\right)\right\} &\leq \binom{\#V_n(x)}{2\mu_j a(n)} |\beta|^{2\mu_j a(n)} \\
&\leq \binom{M_j a(n)}{2\mu_j a(n)} |\beta|^{2\mu_j a(n)} \qquad \text{where } M_j := \mathrm{Leb}(\Upsilon_j); \\
&\leq e^{M_j H(2\mu_j) a(n)(1+o(1))} |\beta|^{2\mu_j a(n)} \qquad \text{by Stirling's formula;} \\
&= e^{(M_j H(2\mu_j) + 2\mu_j \log|\beta|) a(n)(1+o(1))} \\
&= e^{\mathcal{E} a(n)(1+o(1))}.
\end{aligned}$$

□

By (2), if $\delta = \delta(\mathcal{E}) := \min_{j \in J} \frac{\mu_j}{\theta}$, then $\delta > 0$ and

$$(☢) \qquad \mathbf{P}([\log_2 \mathcal{Q}(P, n, \delta) < \mathcal{E} a(n)]) > \sum_{j \in J} \mathbf{P}(G_{j,\theta,n}) > (1 - \varepsilon)^2.$$

As mentioned above, this establishes Theorem 3.

Concluding remarks and questions. Recently in [8], Borodin's theorem [4] (coupling version) has been established for strongly aperiodic random walks driven by Markov chains, and Theorem 2 can now be proven with the same methods in this case.

However, Theorem 2 applies neither to a RWRS whose jump random variables are 1-stable nor to a generalized RWRS over $\mathbb{Z}^2$ whose jump random variables are centered and in the domain of attraction of standard normal distribution on $\mathbb{R}^2$. Other methods are needed to treat these cases due to the lack of "smooth local time" of the relevant limit processes.

It is still conceivable that in both cases there are 1-regularly varying relative complexity sequence whence (or otherwise)

$$\text{E-dim}(T, \text{Base}) = 1.$$

Nothing is known about the relative complexity of generalized RWRSs over continuous groups (as in [2]) or of "smooth RWRSs" (as in [26]).

Acknowledgments. The author would like to acknowledge helpful conversations with Omer Adelman, Nathalie Eisenbaum and Haya Kaspi. Lemma 3 is due to Haya Kaspi and Nathalie Eisenbaum and reproduced here with their kind permission.

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