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THE STOCHASTIC REFLECTION PROBLEM ON AN INFINITE DIMENSIONAL CONVEX SET AND BV FUNCTIONS IN A GELFAND TRIPLE[1]

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In this paper, we introduce a definition of BV functions in a Gelfand triple which is an extension of the definition of BV functions in [*Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **21** (2010) 405–414] by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Hilbert space H. We prove the existence and uniqueness of a strong solution of this problem when Γ is a regular convex set. The result is also extended to the nonsymmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$.

1. Introduction. A definition of BV functions in abstract Wiener spaces has been given by Fukushima in [9], Fukushima and Hino in [10], based upon Dirichlet form theory. In this paper, we introduce BV functions in a Gelfand triple, which is an extension of BV functions in a Hilbert space defined in [2]. Here, we use a version of the Riesz–Markov representation theorem in infinite dimensions proved by Fukushima using the quasi-regularity of the Dirichlet form (see [13]) to give a characterization of BV functions.

In this paper, we consider the Dirichlet form

$$\mathcal{E}^\rho(u, v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz)$$

(where μ is a Gaussian measure in H and ρ is a BV function) and its associated process. By using BV functions, we obtain a Skorohod-type representation for the associated process, if $\rho = I_\Gamma$ and Γ is a convex set.

As a consequence of these results, we can solve the following stochastic differential inclusion in the Hilbert space H:

$$\begin{cases} dX(t) + \big(AX(t) + N_\Gamma(X(t))\big)\,dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{1.1}$$

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where our solution is strong (in the probabilistic sense), if Γ is regular. Here $A : D(A) \subset H \to H$ is a self-adjoint operator. $N_\Gamma(x)$ is the normal cone to Γ at x and $W(t)$ is a cylindrical Wiener process in H. The precise meaning of the above inclusion will be defined in Section 5.2. The solution to (1.1) is called distorted (if $\rho = I_\Gamma$, reflected) Ornslein–Uhlenbek (OU for short)-process.

Equation (1.1) was first studied (strongly solved) in [15], when $H = L^2(0, 1)$, A is the Laplace operator with Dirichlet or Neumann boundary conditions and Γ is the convex set of all nonnegative functions of $L^2(0, 1)$; see also [23]. In [5], the authors study the situation when Γ is a regular convex set with nonempty interior. They get precise information about the corresponding Kolmogorov operator, but did not construct a strong solution to (1.1).

In this paper, we consider a convex set Γ. If Γ is a regular convex set, we show that I_Γ is a BV-function and thus obtain existence and uniqueness results for (1.1). By a modification of [9] and using [6], we obtain the existence of an (in the probabilistic sense) weak solution to (1.1). Then, we prove pathwise uniqueness. Thus, by a version of the Yamada–Watanabe Theorem (see [12]), we deduce that (1.1) has a unique strong solution. We also consider the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0, 1) | f \geq -\alpha\}, \alpha \geq 0$, and prove our result about Skorohod-type representation and that I_{K_α} is a BV function, if $\alpha > 0$.

The solution of the reflection problem is based on an integration by parts formula. The connection to BV functions is given in Theorem 3.1 below, which is a key result of this paper. It asserts that the integration by parts formula for $\rho \cdot \mu$ gives a characterization of BV functions ρ, in the case where μ is a Gaussian measure. This is an extension of the characterization of BV functions in finite dimension. But an integration by parts formula is in fact enough for the reflection problem. This we show in Section 6, exploiting the beautiful integration by parts formula for $K_\alpha, \alpha \geq 0$, proved in [23], which in case $\alpha = 0$, that is, $K_0 = \{f \in L^2(0, 1) : f \geq 0\}$, is with respect to a non-Gaussian measure, namely a Bessel bridge. Theorem 3.1 applies to prove that I_{K_α} is a BV function, but only if $\alpha > 0$.

This paper is organized as follows. In Section 2, we consider the Dirichlet form and its associated distorted OU-process. We introduce BV functions in Section 3, by which we can get the Skorohod type representation for the OU-process. In Section 4, we analyze the reflected OU-process. In Section 5, we get the existence and uniqueness of the solution for (1.1) if Γ is a regular convex set. We also extend these results to the nonsymmetric case. In Section 6, we consider the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0, 1) | f \geq -\alpha\}, \alpha \geq 0$.

Finally, we would like to mention that apart from contributing to develop the theory of BV functions on infinite dimensional spaces, one main motivation of this paper is to provide the probabilistic counterpart to results in [5] and [6], by exploiting Dirichlet form theory and its associated potential theory.

2. The Dirichlet form and the associated distorted OU-process. Let H be a real separable Hilbert space (with scalar product $\langle\cdot,\cdot\rangle$ and norm denoted by $|\cdot|$). We denote its Borel σ-algebra by $\mathcal{B}(H)$. Assume that:

HYPOTHESIS 2.1. *$A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x\rangle \geq \delta |x|^2$ $\forall x \in D(A)$ for some $\delta > 0$ and A^{-1} is of trace class.*

Since A^{-1} is trace class, there exists an orthonormal basis $\{e_j\}$ in H consisting of eigen-functions for A with corresponding eigenvalues $\alpha_j \in \mathbb{R}$, $j \in \mathbb{N}$, that is,

$$Ae_j = \alpha_j e_j, \qquad j \in \mathbb{N}.$$

Then $\alpha_j \geq \delta$ for all $j \in \mathbb{N}$.

Below $D\varphi : H \to H$ denotes the Fréchet-derivative of a function $\varphi : H \to \mathbb{R}$. By $C_b^1(H)$ we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For $K \subset H$, the space $C_b^1(K)$ is defined as the space of restrictions of all functions in $C_b^1(H)$ to the subset K. μ will denote the Gaussian measure in H with mean 0 and covariance operator

$$Q := \tfrac{1}{2}A^{-1}.$$

Since A is strictly positive, μ is nondegenerate and has full topological support. Let $L^p(H,\mu)$, $p \in [1,\infty]$, denote the corresponding real L^p-spaces equipped with the usual norms $\|\cdot\|_p$. We set

$$\lambda_j := \frac{1}{2\alpha_j} \qquad \forall j \in \mathbb{N}$$

so that

$$Qe_j = \lambda_j e_j \qquad \forall j \in \mathbb{N}.$$

For $\rho \in L^1_+(H,\mu)$ we consider

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz), \qquad u, v \in C_b^1(F),$$

where $F := \mathrm{Supp}[\rho \cdot \mu]$ and $L^1_+(H,\mu)$ denotes the set of all nonnegative elements in $L^1(H,\mu)$. Let $\mathrm{QR}(H)$ be the set of all functions $\rho \in L^1_+(H,\mu)$ such that $(\mathcal{E}^\rho, C_b^1(F))$ is closable on $L^2(F, \rho\cdot\mu)$. Its closure is denoted by $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. We denote by $\mathcal{F}^\rho_e$ the extended Dirichlet space of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$, that is, $u \in \mathcal{F}^\rho_e$ if and only if $|u| < \infty$ $\rho\cdot\mu$-a.e. and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\rho$ such that $\mathcal{E}^\rho(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u$ $\rho\cdot\mu$-a.e. as $n \to \infty$.

THEOREM 2.2. *Let $\rho \in \mathrm{QR}(H)$. Then $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(F;\rho\cdot\mu)$ in the sense of* [13], *Chapter* IV, *Definition* 3.1.

PROOF. The assertion follows from the main result in [21]. $\square$

By virtue of Theorem 2.2 and [13], there exists a diffusion process $M^\rho = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$ on F associated with the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. M^ρ will be called distorted OU-process on F. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1,1) = 0$, M^ρ is recurrent and conservative. We denote by $\mathbf{A}_+^\rho$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^ρ, and define $\mathbf{A}^\rho := \mathbf{A}_+^\rho - \mathbf{A}_+^\rho$. For $A \in \mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}_0^\rho := \{A \in \mathbf{A}^\rho | E_{\rho\cdot\mu}(\{A\}_t) < \infty \ \forall t > 0\}$. Each element in $\mathbf{A}_+^\rho$ has a corresponding positive $\mathcal{E}^\rho$-smooth measure on F by the Revuz correspondence. The set of all such measures will be denoted by S_+^ρ. Accordingly, $A_t \in \mathbf{A}^\rho$ corresponds to a $\nu \in S^\rho := S_+^\rho - S_+^\rho$, the set of all $\mathcal{E}^\rho$-smooth signed measure in the sense that $A_t = A_t^1 - A_t^2$ for $A_t^k \in \mathbf{A}_+^\rho, k = 1, 2$, whose Revuz measures are $\nu^k, k = 1, 2$, and $\nu = \nu^1 - \nu^2$ is the Hahn–Jordan decomposition of ν. The element of $\mathbf{A}^\rho$ corresponding to $\nu \in S^\rho$ will be denoted byA^ν.

Note that for each $l \in H$ the function $u(z) = \langle l, z\rangle$ belongs to the extended Dirichlet space $\mathcal{F}_e^\rho$ and

$$\mathcal{E}^\rho(l(\cdot), v) = \frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z)\, d\mu(z) \qquad \forall v \in C_b^1(F). \tag{2.1}$$

On the other hand, the AF $\langle l, X_t - X_0\rangle$ of M^ρ admits a unique decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l \in H$,

$$\langle l, X_t - X_0\rangle = M_t^l + N_t^l \qquad \forall t \ge 0 \ P_z\text{-a.s.} \tag{2.2}$$

for $\mathcal{E}^\rho$-q.e. $z \in F$.

Now for $\rho \in L^1(H, \mu)$ and $l \in H$, we say that $\rho \in \mathrm{BV}_l(H)$ if there exists a constant $C_l > 0$,

$$\left|\int \langle l, Dv(z)\rangle \rho(z)\, d\mu(z)\right| \le C_l \|v\|_\infty \qquad \forall v \in C_b^1(F). \tag{2.3}$$

By the same argument as in [10], Theorem 2.1, we obtain the following theorem.

THEOREM 2.3. *Let $\rho \in L_+^1$ and $l \in H$.*

(1) *The following two conditions are equivalent*:

(i) $\rho \in \mathrm{BV}_l(H)$.

(ii) *There exists a* (*unique*) *signed measure ν_l on F of finite total variation such that*

$$\frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z)\, d\mu(z) = -\int_F v(z)\nu_l(dz) \qquad \forall v \in C_b^1(F). \tag{2.4}$$

In this case, *ν_l necessarily belongs to $S^{\rho+1}$.*

Suppose further that $\rho \in \mathrm{QR}(H)$. Then the following condition is also equivalent to the above:

(iii) $N^l \in \mathbf{A}_0^\rho$.

In this case, $\nu_l \in S^\rho$ *and* $N^l = A^{\nu_l}$.

(2) M^l *is a martingale AF with quadratic variation process*

$$\langle M^l \rangle_t = t|l|^2, \qquad t \geq 0. \tag{2.5}$$

REMARK 2.4. Recall that the Riesz representation theorem of positive linear functionals on continuous functions by measures is not applicable to obtain Theorem 2.3, (i) ⇒ (ii), because of the lack of local compactness. However, the quasi-regularity of the Dirichlet form provides a means to circumvent this difficulty.

In the rest of this section, we shall introduce a special class of $\rho \in \mathrm{QR}(H)$, which will be used in Section 4 below.

A nonnegative measurable function $h(s)$ on $\mathbb{R}^1$ is said to possess the *Hamza property* if $h(s) = 0$ ds-a.e. on the closed set $\mathbb{R}^1 \setminus R(h)$ where

$$R(h) = \left\{ s \in \mathbb{R}^1 : \int_{s-\varepsilon}^{s+\varepsilon} \frac{1}{h(r)}\, dr < \infty \text{ for some } \varepsilon > 0 \right\}.$$

We say that a function $\rho \in L^1_+(H, \mu)$ satisfies *the ray Hamza condition in direction* $l \in H$ ($\rho \in \mathbf{H}_l$ in notation) if there exists a nonnegative function $\tilde{\rho}_l$ such that

$$\tilde{\rho}_l = \rho \ \mu\text{-a.e. and } \tilde{\rho}_l(z + sl) \text{ has the Hamza property in } s \in \mathbb{R}^1 \text{ for each } z \in H.$$

We set $\mathbf{H} := \bigcap_k \mathbf{H}_{e_k}$, where e_k is as in Hypothesis 2.1. A function in the family $\mathbf{H}$ is simply said to satisfy the *ray Hamza condition*. By [1] $\mathbf{H} \subset \mathrm{QR}(H)$, and thus we always have $\rho + 1 \in \mathrm{QR}(H)$, since clearly $\rho + 1 \in \mathbf{H}$.

Next, we will present some explicit description of the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ for $\rho \in \mathbf{H}$.

For $e_j \in H$ as in Hypothesis 2.1, we set $H_{e_j} = \{se_j : s \in \mathbb{R}^1\}$. We then have the direct sum decomposition $H = H_{e_j} \oplus E_{e_j}$ given by

$$z = se_j + x, \qquad s = \langle e_j, z \rangle.$$

Let π_j be the projection onto the space E_{e_j} and μ_{e_j} be the image measure of μ under $\pi_j : H \to E_{e_j}$, that is, $\mu_{e_j} = \mu \circ \pi_j^{-1}$. Then we see that for any $F \in L^1(H, \mu)$

$$\int_H F(z)\mu(dz) = \int_{E_{e_j}} \int_{\mathbb{R}^1} F(se_j + x) p_j(s)\, ds\, \mu_{e_j}(dx), \tag{2.6}$$

where $p_j(s) = (1/\sqrt{2\pi\lambda_j}) e^{-s^2/2\lambda_j}$. Thus by [1], Theorem 3.10, for all $u, v \in D(\mathcal{E}^\rho)$,

$$\mathcal{E}^\rho(u, v) = \sum_{j=1}^\infty \mathcal{E}^{\rho, e_j}(u, v), \tag{2.7}$$

where

$$\mathcal{E}^{\rho,e_j}(u,v) = \frac{1}{2}\int_{E_{e_j}}\int_{R(\rho(\cdot e_j+x))} \frac{d\tilde{u}_j(se_j+x)}{ds} \times \frac{d\tilde{v}_j(se_j+x)}{ds} \times \rho(se_j+x)p_j(s)\,ds\,\mu_{e_j}(dx), \tag{2.8}$$

and $u, \tilde{u}_j$ satisfy $\tilde{u}_j = u$ $\rho\mu$-a.e. and $\tilde{u}_j(se_j+x)$ is absolutely continuous in s on $R(\rho(\cdot e_j + x))$ for each $x \in E_{e_j}$. v and $\tilde{v}_j$ are related in the same way.

3. BV functions and distorted OU-processes in F. As in [10], we introduce some function spaces on H. Let

$$A_{1/2}(x) := \int_0^x \bigl(\log(1+s)\bigr)^{1/2}\,ds, \qquad x \geq 0,$$

and let ψ be its complementary function, namely,

$$\psi(y) := \int_0^y (A'_{1/2})^{-1}(t)\,dt = \int_0^y \bigl(\exp(t^2)-1\bigr)\,dt.$$

Define

$$L(\log L)^{1/2}(H,\mu) := \{f : H \to \mathbb{R} | f \text{ Borel measurable}, A_{1/2}(|f|) \in L^1(H,\mu)\},$$

$$L^\psi(H,\mu) := \{g : H \to \mathbb{R} | g \text{ Borel measurable}, \psi(c|g|) \in L^1(H,\mu) \text{ for some } c > 0\}.$$

From the general theory of Orlicz spaces (cf. [20]), we have the following properties:

(i) $L(\log L)^{1/2}$ and L^ψ are Banach spaces under the norms

$$\|f\|_{L(\log L)^{1/2}} = \inf\left\{\alpha > 0 \Big| \int_H A_{1/2}(|f|/\alpha)\,d\mu \leq 1\right\},$$

$$\|g\|_{L^\psi} = \inf\left\{\alpha > 0 \Big| \int_H \psi(|g|/\alpha)\,d\mu \leq 1\right\}.$$

(ii) For $f \in L(\log L)^{1/2}$ and $g \in L^\psi$, we have

$$\|fg\|_1 \leq 2\|f\|_{L(\log L)^{1/2}}\|g\|_{L^\psi}. \tag{3.1}$$

(iii) Since μ is Gaussian, the function $x \mapsto \langle x, l\rangle$ belongs to L^ψ.

Let c_j, $j \in \mathbb{N}$, be a sequence in $[1,\infty)$. Define

$$H_1 := \left\{x \in H \Big| \sum_{j=1}^\infty \langle x, e_j\rangle^2 c_j^2 < \infty\right\},$$

equipped with the inner product

$$\langle x, y\rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j\rangle\langle y, e_j\rangle.$$

Then clearly $(H_1, \langle\cdot,\cdot\rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H(\equiv H^*) \subset H_1^*.$$

It follows that

$${}_{H_1}\langle z, v\rangle_{H_1^*} = \langle z, v\rangle_H \qquad \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple. Furthermore, $\{\frac{e_j}{c_j}\}$ and $\{c_j e_j\}$ are orthonormal bases of H_1 and H_1^*, respectively.

We also introduce a family of H-valued functions on H by

$$(C_b^1)_{D(A)\cap H_1} := \left\{G : G(z) = \sum_{j=1}^{m} g_j(z) l^j, z \in H, g_j \in C_b^1(H), l^j \in D(A) \cap H_1\right\}.$$

Denote by D^* the adjoint of $D : C_b^1(H) \subset L^2(H,\mu) \to L^2(H,\mu; H)$. That is

$$\operatorname{Dom}(D^*) := \left\{G \in L^2(H,\mu; H)\,\middle|\, C_b^1 \ni u \mapsto \int \langle G, Du\rangle\, d\mu \text{ is continuous with respect to } L^2(H,\mu)\right\}.$$

Obviously, $(C_b^1)_{D(A)\cap H_1} \subset \operatorname{Dom}(D^*)$. Then

$$\int_H D^*G(z) f(z)\mu(dz) = \int_H \langle G(z), Df(z)\rangle \mu(dz) \qquad \forall G \in (C_b^1)_{D(A)\cap H_1}, f \in C_b^1(H). \tag{3.2}$$

For $\rho \in L(\log L)^{1/2}(H,\mu)$, we set

$$V(\rho) := \sup_{G \in (C_b^1)_{D(A)\cap H_1}, \|G\|_{H_1} \le 1} \int_H D^*G(z)\rho(z)\mu(dz).$$

A function ρ on H is called a BV function in the Gelfand triple (H_1, H, H_1^*) [$\rho \in \mathrm{BV}(H, H_1)$ in notation], if $\rho \in L(\log L)^{1/2}(H,\mu)$ and $V(\rho)$ is finite. When $H_1 = H = H_1^*$, this coincides with the definition of BV functions defined in [2] and clearly $\mathrm{BV}(H,H) \subset \mathrm{BV}(H,H_1)$. We can prove the following theorem by a modification of the proof of [9], Theorem 3.1.

REMARK 3.0. The introduction of BV functions in a Gelfand triple is natural and originates from standard ideas when working with infinite dimensional state spaces. The intersection of $\mathrm{BV}_l(H)$, when l runs through $D(A) \cap H_1$, describes functions which are "componentwise of bounded variation" in the sense that their weak partial derivatives are measures. In contrast to finite dimensions, this does not give rise to vector-valued measures representing their total weak derivatives or gradients. Therefore, one introduces an appropriate "tangent space" H_1^* to H, in which these total derivatives can be represented as a H_1^*-valued measure. This approach substantially extends the applicability of the theory of BV functions on Hilbert spaces. We document this by including the well-studied case of linear SPDE with reflection, more precisely, the randomly vibrating Gaussian string, forced to stay above a level $\alpha \geq 0$ (see [15, 23]), which (in the case of $\alpha > 0$) is then just a special case of our general approach.

THEOREM 3.1. (i) $\mathrm{BV}(H, H_1) \subset \bigcap_{l \in D(A) \cap H_1} \mathrm{BV}_l(H)$.

(ii) *Suppose* $\rho \in \mathrm{BV}(H, H_1) \cap L^1_+(H, \mu)$, *then there exist a positive finite measure* $\|d\rho\|$ *on* H *and a Borel-measurable map* $\sigma_\rho : H \to H_1^*$ *such that* $\|\sigma_\rho(z)\|_{H_1^*} = 1 \|d\rho\|$*-a.e,* $\|d\rho\|(H) = V(\rho)$,

$$\int_H D^* G(z) \rho(z) \mu(dz) = \int_H {}_{H_1}\langle G(z), \sigma_\rho(z) \rangle_{H_1^*} \|d\rho\|(dz) \qquad \forall G \in (C_b^1)_{D(A) \cap H_1} \tag{3.3}$$

and $\|d\rho\| \in S^{\rho+1}$.

Furthermore, if $\rho \in \mathrm{QR}(H)$, $\|d\rho\|$ *is* $\mathcal{E}^\rho$*-smooth in the sense that it charges no set of zero* $\mathcal{E}_1^\rho$*-capacity. In particular, the domain of integration* H *on both sides of* (3.3) *can be replaced by* F, *the topological support of* $\rho\mu$.

Also, σ_ρ *and* $\|d\rho\|$ *are uniquely determined, that is, if there are* σ_ρ' *and* $\|d\rho\|'$ *satisfying relation* (3.3), *then* $\|d\rho\| = \|d\rho\|'$ *and* $\sigma_\rho(z) = \sigma_\rho'(z)$ *for* $\|d\rho\|$*-a.e.* z.

(iii) *Conversely, if equation* (3.3) *holds for* $\rho \in L(\log L)^{1/2}(H, \mu)$ *and for some positive finite measure* $\|d\rho\|$ *and a map* σ_ρ *with the stated properties, then* $\rho \in \mathrm{BV}(H, H_1)$ *and* $V(\rho) = \|d\rho\|(H)$.

(iv) *Let* $W^{1,1}(H)$ *be the domain of the closure of* $(D, C_b^1(H))$ *with norm*

$$\|f\| := \int_H \big(|f(z)| + |Df(z)|\big) \mu(dz).$$

Then $W^{1,1}(H) \subset \mathrm{BV}(H, H)$ *and equation* (3.3) *is satisfied for each* $\rho \in W^{1,1}(H)$. *Furthermore,*

$$\|d\rho\| = |D\rho| \cdot \mu, \qquad V(\rho) = \int_H |D\rho| \mu(dz), \qquad \sigma_\rho = \frac{1}{|D\rho|} D\rho I_{\{|D\rho|>0\}}.$$

PROOF. (i) Let $\rho \in \mathrm{BV}(H, H_1)$ and $l \in D(A) \cap H_1$. Take $G \in (C_b^1)_{D(A)\cap H_1}$ of the type

$$G(z) = g(z)l, \qquad z \in H, g \in C_b^1(H). \tag{3.4}$$

By (3.2)

$$\begin{aligned}\int_H D^* G(z) f(z)\mu(dz) &= \int_H \langle G(z), Df(z)\rangle \mu(dz) \\ &= -\int_H \langle l, Dg(z)\rangle f(z)\mu(dz) \\ &\quad + 2\int_H \langle Al, z\rangle g(z) f(z)\mu(dz) \qquad \forall f \in C_b^1(H);\end{aligned}$$

consequently,

$$D^* G(z) = -\langle l, Dg(z)\rangle + 2g(z)\langle Al, z\rangle. \tag{3.5}$$

Accordingly,

$$\begin{aligned}\int_H \langle l, Dg(z)\rangle \rho(z)\mu(dz) = &-\int_H D^* G(z)\rho(z)\mu(dz) \\ &+ 2\int_H \langle Al, z\rangle g(z)\rho(z)\mu(dz).\end{aligned} \tag{3.6}$$

For any $g \in C_b^1(H)$, satisfying $\|g\|_\infty \le 1$, by (3.1) the right-hand side is dominated by

$$V(\rho)\|l\|_{H_1} + 4\|\rho\|_{L(\log L)^{1/2}} \|\langle Al, \cdot\rangle\|_{L^\psi} < \infty,$$

hence, $\rho \in \mathrm{BV}_l(H)$.

(ii) Suppose $\rho \in L_+^1(H, \mu) \cap \mathrm{BV}(H, H_1)$. By (i) and Theorem 2.3 for each $l \in D(A) \cap H_1$, there exists a finite signed measure ν_l on H for which equation (2.4) holds. Define

$$D_l^A \rho(dz) := 2\nu_l(dz) + 2\langle Al, z\rangle \rho(z)\mu(dz).$$

In view of (3.6), for any G of type (3.4), we have

$$\int_H D^* G(z)\rho(z)\mu(dz) = \int_H g(z) D_l^A \rho(dz), \tag{3.7}$$

which in turn implies

$$V(D_l^A \rho)(H) = \sup_{g \in C_b^1(H), \|g\|_\infty \le 1} \int_H g(z) D_l^A \rho(dz) \le V(\rho)\|l\|_{H_1}, \tag{3.8}$$

where $V(D_l^A \rho)$ denotes the total variation measure of the signed measure $D_l^A \rho$.

For the orthonormal basis $\{\frac{e_j}{c_j}\}$ of H_1, we set

$$\gamma_\rho^A := \sum_{j=1}^\infty 2^{-j} V(D^A_{e_j/c_j}\rho),$$
$$v_j(z) := \frac{dD^A_{e_j/c_j}\rho(z)}{d\gamma_\rho^A(z)}, \qquad z \in H, j \in \mathbb{N}. \tag{3.9}$$

γ_ρ^A is a positive finite measure with $\gamma_\rho^A(H) \le V(\rho)$ and v_j is Borel-measurable. Since $D^A_{e_j/c_j}\rho$ belongs to $S^{\rho+1}$, so does γ_ρ^A. Then for

$$G_n := \sum_{j=1}^n g_j \frac{e_j}{c_j} \in (C_b^1)_{D(A)\cap H_1}, \qquad n \in \mathbb{N}, \tag{3.10}$$

by (3.7) the following equation holds:

$$\int_H D^* G_n(z)\rho(z)\mu(dz) = \sum_{j=1}^n \int_H g_j(z) v_j(z) \gamma_\rho^A(dz). \tag{3.11}$$

Since $|v_j(z)| \le 2^j$ γ_ρ^A-a.e. and $C_b^1(H)$ is dense in $L^1(H, \gamma_\rho^A)$, we can find $v_{j,m} \in C_b^1(H)$ such that

$$\lim_{m\to\infty} v_{j,m} = v_j \qquad \gamma_\rho^A\text{-a.e.}$$

Substituting

$$g_{j,m}(z) := \frac{v_{j,m}(z)}{\sqrt{\sum_{k=1}^n v_{k,m}(z)^2 + 1/m}} \tag{3.12}$$

for $g_j(z)$ in (3.10) and (3.11) we get a bound

$$\sum_{j=1}^n \int_H g_{j,m}(z) v_j(z) \gamma_\rho^A(dz) \le V(\rho),$$

because $\|G_n(z)\|^2_{H_1} = \sum_{j=1}^n g_{j,m}(z)^2 \le 1$ $\forall z \in H$. By letting $m \to \infty$, we obtain

$$\int_H \sqrt{\sum_{j=1}^n v_j(z)^2} \gamma_\rho^A(dz) \le V(\rho) \qquad \forall n \in \mathbb{N}.$$

Now we define

$$\|d\rho\| := \sqrt{\sum_{j=1}^\infty v_j(z)^2} \gamma_\rho^A(dz) \tag{3.13}$$

and $\sigma_\rho : H \to H_1^*$ by

$$\sigma_\rho(z) = \begin{cases} \sum_{j=1}^{\infty} \frac{v_j(z)}{\sqrt{\sum_{k=1}^{\infty} v_k(z)^2}} \cdot c_j e_j, & \text{if } z \in \left\{ \sum_{k=1}^{\infty} v_k(z)^2 > 0 \right\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Then

$$\|d\rho\|(H) \le V(\rho), \qquad \|\sigma_\rho(z)\|_{H_1^*} = 1 \qquad \|d\rho\|\text{-a.e.,} \tag{3.15}$$

$\|d\rho\|$ is $S^{\rho+1}$-smooth and σ_ρ is Borel-measurable. By (3.11) we see that the desired equation (3.3) holds for $G = G_n$ as in (3.10). It remains to prove (3.3) for any G of type (3.4), that is, $G = g \cdot l$, $g \in C_b^1(H)$, $l \in D(A) \cap H_1$. In view of (3.6), equation (3.3) then reads

$$\begin{aligned} &- \int_H \langle l, Dg(z)\rangle \rho(z)\mu(dz) + 2\int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz) \\ &\qquad = \int_H g(z)\, {}_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz). \end{aligned} \tag{3.16}$$

We set

$$k_n := \sum_{j=1}^n \langle l, e_j\rangle e_j = \sum_{j=1}^n \left\langle l, \frac{e_j}{c_j}\right\rangle_{H_1} \frac{e_j}{c_j}, \qquad G_n(z) := g(z)k_n.$$

Thus, $k_n \to l$ in H_1 and $Ak_n \to Al$ in H as $n \to \infty$. But then also

$$\lim_{n\to\infty} \int_H \langle Dg, k_n\rangle \rho\, d\mu = \int_H \langle Dg, l\rangle \rho\, d\mu$$

and

$$\begin{aligned} &\left| \int_H g(z)\langle Ak_n, z\rangle \rho(z)\mu(dz) - \int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz) \right| \\ &\qquad \le 2\|g\|_\infty \|\rho\|_{L(\log L)^{1/2}} \|\langle Ak_n - Al, \cdot\rangle\|_{L^\psi}. \end{aligned}$$

Furthermore,

$$\lim_{n\to\infty} \int_H g(z)\, {}_{H_1}\langle k_n, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) = \int_H g(z)\, {}_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz).$$

So letting $n \to \infty$ yields (3.16).

If $\rho \in \mathrm{QR}(H)$, we can get the claimed result by the same arguments as above.

Uniqueness follows by the same argument as [10], Theorem 3.9.

(iii) Suppose $\rho \in L(\log)^{1/2}(H, \mu)$ and that equation (3.3) holds for some positive finite measure $\|d\rho\|$ and some map σ_ρ with the properties stated in (ii). Then clearly

$$V(\rho) \le \|d\rho\|(H)$$

and hence $\rho \in \mathrm{BV}(H, H_1)$. To obtain the converse inequality, set

$$\sigma_j(z) := \langle c_j e_j, \sigma_\rho(z)\rangle_{H_1^*} = {}_{H_1}\left\langle \frac{e_j}{c_j}, \sigma_\rho(z)\right\rangle_{H_1^*}, \qquad j \in \mathbb{N}.$$

Fix an arbitrary n. As in the proof of (ii), we can find functions

$$v_{j,m} \in C_b^1(H), \qquad \lim_{m\to\infty} v_{j,m}(z) = \sigma_j(z) \qquad \|d\rho\|\text{-a.e.}$$

Define $g_{j,m}(z)$ by (3.12). Substituting $G_{n,m}(z) := \sum_{j=1}^n g_{j,m}(z)\frac{e_j}{c_j}$ for $G(z)$ in (3.3) then yields

$$\sum_{j=1}^n \int_H g_{j,m}(z)\sigma_j(z)\|d\rho\|(dz) \le V(\rho).$$

By letting $m \to \infty$, we get

$$\int_H \sqrt{\sum_{j=1}^n \sigma_j(z)^2}\,\|d\rho\|(dz) \le V(\rho) \qquad \forall n \in \mathbb{N}.$$

We finally let $n \to \infty$ to obtain $\|d\rho\|(H) \le V(\rho)$.

(iv) Obviously the duality relation (3.2) extends to $\rho \in W^{1,1}(H)$ replacing $f \in C_b^1(H)$. By defining $\|d\rho\|$ and $\sigma_\rho(z)$ in the stated way, the extended relation (3.2) is exactly (3.3). $\square$

THEOREM 3.2. *Let $\rho \in \mathrm{QR}(H)\cap \mathrm{BV}(H, H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 3.1*(ii). *Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$ under P_z there exists an $\mathcal{M}_t$-cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following*: *for $l \in D(A)\cap H_1$*

$$\begin{aligned}\langle l, X_t - X_0\rangle = {}& \int_0^t \langle l, dW_s^z\rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, \sigma_\rho(X_s)\rangle_{H_1^*}\, dL_s^{\|d\rho\|} \\ & - \int_0^t \langle Al, X_s\rangle\, ds \qquad \forall t \ge 0\ P_z\text{-a.s.}\end{aligned} \tag{3.17}$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

In particular, if $\rho \in \mathrm{BV}(H, H)$, then $\forall z \in F \setminus S$, $l \in D(A)\cap H$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle + \frac{1}{2}\int_0^t \langle l, \sigma_\rho(X_s)\rangle\, dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s\rangle\, ds \qquad \forall t \ge 0\ P_z\text{-a.s.}$$

PROOF. Let $\{e_j\}$ be the orthonormal basis of H introduced above. Define for all $k \in \mathbb{N}$

$$W_k^z(t) := \langle e_k, X_t - z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} \, dL_s^{\|d\rho\|} + \int_0^t \langle Ae_k, X_s\rangle \, ds. \tag{3.18}$$

By (2.1) and (3.16), we get for all $k \in \mathbb{N}$

$$\mathcal{E}^\rho(e_k(\cdot), g) = \int_H g(z)\langle Ae_k, z\rangle \rho(z)\mu(dz) - \frac{1}{2}\int_H g(z)_{H_1}\langle e_k, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) \qquad \forall g \in C_b^1(H).$$

By Theorem 2.3, it follows that for all $k \in \mathbb{N}$

$$N_t^{e_k} = \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} \, dL_s^{\|d\rho\|} - \int_0^t \langle Ae_k, X_s\rangle \, ds. \tag{3.19}$$

Here we get from (3.18), (3.19) and the uniqueness of decomposition (2.2) that for $\mathcal{E}^\rho$-q.e. $z \in F$,

$$W_k^z(t) = M_t^{e_k} \qquad \forall t \geq 0 \ P_z\text{-a.s.},$$

where the $\mathcal{E}^\rho$-exceptional set and the zero measure set does not depend on e_k. Indeed, we can choose the capacity zero set $S = \bigcup_{j=1}^\infty S_j$, where S_j is the $\mathcal{E}^\rho$-exceptional set for e_j, and for $z \in F \setminus S$, we can use the same method to get a zero measure set independent of e_k. By Dirichlet form theory, we get $\langle M^{e_i}, M^{e_j}\rangle_t = t\delta_{ij}$. So for $z \in F \setminus S$, W_k^z is an $\mathcal{M}_t$-Wiener process under P_z. Thus, with W^z being an $\mathcal{M}_t$-cylindrical Wiener process given by $W^z(t) = (W_k^z(t)e_k)_{k\in\mathbb{N}}$, (3.17) is satisfied for P_z-a.e., where $z \in F \setminus S$. $\square$

4. Reflected OU-processes. In this section, we consider the situation where $\rho = I_\Gamma \in \mathrm{BV}(H, H_1)$, where $\Gamma \subset H$ and

$$I_\Gamma(x) = \begin{cases} 1, & \text{if } x \in \Gamma, \\ 0, & \text{if } x \in \Gamma^c. \end{cases}$$

Denote the corresponding objects σ_ρ, $\|dI_\Gamma\|$ in Theorem 3.1(ii) by $-\mathbf{n}_\Gamma$, $\|\partial\Gamma\|$, respectively. Then formula (3.3) reads

$$\int_\Gamma D^*G(z)\mu(dz) = -\int_F {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*} \|\partial\Gamma\|(dz) \qquad \forall G \in (C_b^1)_{D(A)\cap H_1},$$

where the domain of integration F on the right-hand side is the topological support of $I_\Gamma \cdot \mu$. F is contained in $\bar{\Gamma}$, but we shall show that the domain of integration on the right-hand side can be restricted to $\partial\Gamma$. We need to use the associated distorted OU-process M^{I_Γ} on F, which will be called reflected OU-process on Γ.

First, we consider a μ-measurable set $\Gamma \subset H$ satisfying

$$I_\Gamma \in \mathrm{BV}(H, H_1) \cap \mathbf{H}. \tag{4.1}$$

REMARK 4.1. We emphasize that if Γ is a convex closed set in H, then obviously $I_\Gamma \in \mathbf{H}$. Indeed, for each $z, l \in H$ the set $\{s \in \mathbb{R} | z + sl \in \Gamma\}$ is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_\Gamma \in \mathrm{QR}(H)$.

By a modification of [9], Theorem 4.2, we can prove the following theorem.

THEOREM 4.2. *Let $\Gamma \subset H$ be μ-measurable satisfying condition* (4.1). *Then the support of $\|\partial\Gamma\|$ is contained in the boundary $\partial\Gamma$ of Γ, and the following generalized Gauss formula holds*:

$$\int_\Gamma D^* G(z)\mu(dz) = -\int_{\partial\Gamma} {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*} \|\partial\Gamma\|(dz) \qquad \forall G \in (C_b^1)_{D(A)\cap H_1}. \tag{4.2}$$

PROOF. For any G of type (3.4), we have from (2.1), (3.5) and (3.7) that

$$\mathcal{E}^{I_\Gamma}(l(\cdot), g) - \int_\Gamma g(z)\langle Al, z\rangle \mu(dz) = -\frac{1}{2}\int_F g(z) D_l^A I_\Gamma(dz). \tag{4.3}$$

Since the finite signed measure $D_l^A I_\Gamma$ charges no set of zero $\mathcal{E}_1^{I_\Gamma}$-capacity, equation (4.3) readily extends to any $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $g \in \mathcal{F}_b^{I_\Gamma} := \mathcal{F}^{I_\Gamma} \cap L^\infty(\Gamma, \mu)$.

Denote by Γ^0 the interior of Γ. Then $\Gamma^0 \subset F \subset \bar{\Gamma}$. In view of the construction of the measure $\|dI_\Gamma\|$ in Theorem 3.1, it suffices to show that for $\frac{e_j}{c_j} \in D(A) \cap H_1$

$$V(D_{e_j/c_j}^A I_\Gamma)(\Gamma^0) = 0.$$

By linearity and since positive constants interchange with sup, it suffices to show that

$$V(D_{e_j}^A I_\Gamma)(\Gamma^0) = 0. \tag{4.4}$$

Take an arbitrary $\varepsilon > 0$ and set

$$U := \{z \in H : d(z, H \setminus \Gamma^0) > \varepsilon\}, \qquad V := \{z \in H : d(z, H \setminus \Gamma^0) \geq \varepsilon\},$$

where d is the metric distance of the Hilbert space H. Then $\bar{U} \subset V$ and V is a closed set contained in the open set Γ^0. We define a function h by

$$h(z) := 1 - E_z(e^{-\tau_V}), \qquad z \in F, \tag{4.5}$$

where τ_V denotes the first exit time of M^{I_Γ} from the set V. The nonnegative function h is in the space $\mathcal{F}_b^{I_\Gamma}$ and furthermore it is $\mathcal{E}^{I_\Gamma}$-quasicontinuous because it is M^{I_Γ} finely continuous.

Moreover,

$$h(z) > 0 \quad \forall z \in U, \qquad h(z) = 0 \quad \forall z \in F \setminus V. \tag{4.6}$$

Set

$$\nu_j(dz) := h(z) D^A_{e_j} I_\Gamma(dz) \tag{4.7}$$

and

$$I^j_g := \mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) - \int_\Gamma g(z)h(z)\langle Ae_j, z\rangle \mu(dz). \tag{4.8}$$

Then equation (4.3) with the $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $gh \in \mathcal{F}^{I_\Gamma}_b$ replacing g implies

$$I^j_g = -\frac{1}{2}\int_F g(z)\nu_j(dz).$$

In order to prove (4.4), it is enough to show that $I^j_g = 0$ for any function $g(z)$ of the type

$$g(z) = f(\langle e_j, z\rangle, \langle l_2, z\rangle, \dots, \langle l_m, z\rangle); \qquad l_2, \dots, l_m \in H, f \in C^1_0(R^m) \tag{4.9}$$

for we have then $\nu_j = 0$.

On account of (2.8), we have the expression

$$\begin{aligned} \mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) &= \mathcal{E}^{I_\Gamma, e_j}(e_j(\cdot), gh) \\ &= \frac{1}{2}\int_{E_{e_j}}\int_{R_x} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s)\, ds \mu_{e_j}(dx), \end{aligned} \tag{4.10}$$

where $R_x = R(I_\Gamma(\cdot e_j + x))$, $F_x := \{s : se_j + x \in F\}$ for $x \in E_{e_j}$ and $\tilde{h}$ is a $I_\Gamma \cdot \mu$-version of h appearing in the description of (2.8). For $x \in E_{e_j}$, set

$$V_x := \{s : se_j + x \in V\}, \qquad \Gamma^0_x := \{s : se_j + x \in \Gamma^0\}.$$

We then have the inclusion $V_x \subset \Gamma^0_x \subset R_x \cap F_x$. By (4.6), $h(se_j + x) = 0$ for any $x \in E_{e_j}$ and for any $s \in R_x \setminus V_x$. On the other hand, there exists a Borel set $N \subset E_{e_j}$ with $\mu_{e_j}(N) = 0$ such that for each $x \in E_{e_j} \setminus N$,

$$h(se_j + x) = \tilde{h}(se_j + x) \qquad ds\text{-a.e.}$$

Here we set $h \equiv 0$ on $H \setminus F$. Since $\tilde{h}(\cdot e_j + x)$ is absolutely continuous in s, we can conclude that

$$\tilde{h}(se_j + x) = 0 \qquad \forall x \in E_{e_j} \setminus N, \forall s \in R_x \setminus V_x.$$

Fix $x \in E_{e_j} \setminus N$ and let I be any connected component of the one dimensional open set R_x. Furthermore, for any function g of type (4.9) we denote the support of $g(\cdot e_j + x)$ by K_x (which is a compact set) and choose a bounded open interval J containing K_x. Then $I \cap V_x \cap K_x$ is a closed set contained in the bounded open interval $I \cap J$ and

$$g\tilde{h}(se_j + x) = 0 \qquad \forall s \in (I \cap J) \setminus (I \cap V_x \cap K_x).$$

Therefore, an integration by part gives

$$\int_{I\cap J} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s)\, ds = \int_{I\cap J} \frac{1}{\lambda_j}(g\tilde{h})(se_j + x) s p_j(s)\, ds.$$

Combining this with (4.8) and (4.10), we arrive at

$$\begin{aligned} I_g^j &= \int_{E_{e_j}} \int_{R_x} \frac{1}{2\lambda_j}(g\tilde{h})(se_j + x) s p_j(s)\, ds\, \mu_{e_j}(dx) \\ &\quad - \int_H g(z)h(z)\langle Ae_j, z\rangle I_\Gamma(z)\mu(dz) = 0. \end{aligned}$$

□

Now we state Theorem 3.2 for $\rho = I_\Gamma$.

THEOREM 4.3. *Suppose $\Gamma \subset H$ is a μ-measurable set satisfying condition (4.1). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$, under P_z there exists an $\mathcal{M}_t$-cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$*

$$\begin{aligned} \langle l, X_t - X_0\rangle &= \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s)\rangle_{H_1^*}\, dL_s^{\|\partial\Gamma\|} \\ &\quad - \int_0^t \langle Al, X_s\rangle\, ds \qquad P_z\text{-a.s.} \end{aligned} \tag{4.11}$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in F \setminus S$

$$I_{\partial\Gamma}(X_s)\, dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \qquad P_z\text{-a.s.} \tag{4.12}$$

In particular, if $\rho \in \mathrm{BV}(H, H)$, then $\forall z \in F \setminus S, l \in D(A) \cap H$

$$\begin{aligned} \langle l, X_t - X_0\rangle &= \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle\, dL_s^{\|\partial\Gamma\|} \\ &\quad - \int_0^t \langle Al, X_s\rangle\, ds \qquad \forall t \geq 0\ P_z\text{-a.s.} \end{aligned}$$

PROOF. All assertions except for (4.12) follow from Theorem 3.2 for $\rho := I_\Gamma$. Equation (4.12) follows by Theorem 4.2 and [11], Theorem 5.1.3. □

5. Stochastic reflection problem on a regular convex set. In this section, we consider Γ satisfying [5], Hypothesis 1.1(ii), with $K := \Gamma$, that is:

HYPOTHESIS 5.1. *There exists a convex C^∞ function $g : H \to \mathbb{R}$ with $g(0) = 0$, $g'(0) = 0$, and $D^2 g$ strictly positive definite, that is, $\langle D^2 g(x)h, h\rangle \geq \gamma |h|^2\ \forall h \in H$ for some $\gamma > 0$, such that*

$$\Gamma = \{x \in H : g(x) \leq 1\}, \qquad \partial\Gamma = \{x \in H : g(x) = 1\}.$$

Moreover, we also suppose that $D^2 g$ *is bounded on* Γ *and* $|Q^{1/2}Dg|^{-1} \in \bigcap_{p>1} L^p(H,\mu)$.

REMARK 5.2. By [5], Lemma 1.2, Γ is convex and closed and there exists some constant $\delta > 0$ such that $|Dg(x)| \le \delta \ \forall x \in \Gamma$.

5.1. *Reflected OU processes on regular convex sets.* Under Hypothesis 5.1, by [6], Lemma A.1, we can prove that $I_\Gamma \in \mathrm{BV}(H,H) \cap \mathrm{QR}(H)$:

THEOREM 5.3. *Assume that Hypothesis* 5.1 *holds. Then* $I_\Gamma \in \mathrm{BV}(H,H) \cap \mathrm{QR}(H)$.

PROOF. We first note that trivially by Remark 4.1 we have that $I_\Gamma \in \mathrm{QR}(H)$. Let

$$\rho_\varepsilon(x) := \exp\biggl(-\frac{(g(x)-1)^2}{\varepsilon} 1_{\{g\ge 1\}}\biggr), \qquad x \in H.$$

Thus,

$$\lim_{\varepsilon\to 0} \rho_\varepsilon = I_\Gamma.$$

Moreover,

$$D\rho_\varepsilon = -\frac{2}{\varepsilon}\rho_\varepsilon 1_{\{g\ge 1\}} Dg(g-1) \qquad \mu\text{-a.e.}$$

By [6], Lemma A.1, we have for $\varphi \in C_b^1(H)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \varphi(x) 1_{\{g(x)\ge 1\}}\bigl(g(x)-1\bigr)\langle Dg(x), z\rangle \rho_\varepsilon(x)\mu(dx)\\
&\quad = \frac{1}{2}\int_{\partial\Gamma} \varphi(y)\langle n(y), z\rangle \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dy),
\end{aligned}$$

where $n := Dg/|Dg|$ is the exterior normal to $\partial\Gamma$ at y and $\mu_{\partial\Gamma}$ is the surface measure on $\partial\Gamma$ induced by μ (cf. [5, 6, 14]), whereas by (3.2) for any $\varphi \in C_b^1(H)$ and $z \in D(A)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \varphi(x) 1_{\{g(x)\ge 1\}}\bigl(g(x)-1\bigr)\langle Dg(x), z\rangle \rho_\varepsilon(x)\mu(dx)\\
&\quad = -\lim_{\varepsilon\to 0}\frac{1}{2}\int_H \langle D\rho_\varepsilon(x), \varphi(x) z\rangle \mu(dx)\\
&\quad = -\frac{1}{2}\lim_{\varepsilon\to 0}\int_H \rho_\varepsilon(x) D^*(\varphi z)(x)\mu(dx)\\
&\quad = -\frac{1}{2}\int_H 1_\Gamma(x) D^*(\varphi z)(x)\mu(dx).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_H 1_\Gamma(x) D^*(\varphi z)(x)\mu(dx) \\
(5.1)\qquad &= -\int_{\partial\Gamma} \varphi(x)\langle n(x), z\rangle \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dx) \qquad \forall z \in D(A), \varphi \in C_b^1.
\end{aligned}$$

By the proof of [6], Lemma A.1, we get that g is a nondegenerate map. So we can use the co-area formula (see [14], Theorem 6.3.1, Chapter V, or [6], (A.4)):

$$\int_H f\mu(dx) = \int_0^\infty \left[\int_{g=r} f(y)\frac{1}{|Q^{1/2}Dg(y)|}\mu_{\Sigma_r}(dy)\right] dr.$$

By [14], Theorem 6.2, Chapter V, the surface measure is defined for all $r \geq 0$, moreover [14], Theorem 1.1, Corollary 6.3.2, Chapter V, imply that $r \mapsto \mu_{\Sigma_r}$ is continuous in the topology induced by $D_r^p(H)$ for some $p \in (1,\infty), r \in (0,\infty)$ (cf. [14]) on the measures on $(H, \mathcal{B}(H))$. Take $f \equiv 1$ in the co-area formula, then by the continuity property of the surface measure with respect to r we have that $\frac{1}{|Q^{1/2}Dg(y)|}\mu_{\Sigma_r}(dy)$ is a finite measure supported in $\{g = r\}$. By Remark 5.2 and since $\mu_{\partial\Gamma} = \mu_{\Sigma_1}$, we have that $\frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}$ is a finite measure. And hence by Theorem 3.1(iii), we get $I_\Gamma \in \mathrm{BV}(H, H)$. $\square$

Thus by Theorem 4.3, we immediately get the following.

THEOREM 5.4. *Assume Hypothesis 5.1. Then there exists an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$, under P_z there exists an $\mathcal{M}_t$-cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$*

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle\, dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s\rangle\, ds \qquad \forall t \geq 0\ P_z\text{-a.e.},$$

where $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ and

$$\|\partial\Gamma\|(dy) = \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dy),$$

where $\mu_{\partial\Gamma}$ is the surface measure induced by μ (cf. [5, 6, 14]).

REMARK 5.5. It can be shown that for $x \in \partial\Gamma$, $\mathbf{n}_\Gamma(x) = \frac{Dg}{|Dg|}$ is the exterior normal to Γ, that is, the unique element in H of unit length such that

$$\langle \mathbf{n}_\Gamma(x), y - x\rangle \leq 0 \qquad \forall y \in \Gamma.$$

5.2. *Existence and uniqueness of solutions*. Let $\Gamma \subset H$ and our linear operator A satisfy Hypotheses 5.1 and 2.1, respectively. Consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + \big(AX(t) + N_\Gamma(X(t))\big)\,dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{5.2}$$

where $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, that is,

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \le 0\ \forall y \in \Gamma\}.$$

DEFINITION 5.6. A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t))$, $t \in [0, T]$, is called a solution of (5.2) if the following conditions hold:

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ P-a.s.;
(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)\,dL_s = dL_s \qquad P\text{-a.s.}$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle\,dL_s - \int_0^t \langle Al, X_s\rangle\,ds \qquad \forall t \ge 0\ P\text{-a.s.},$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

REMARK 5.7. By Remark 5.5, we know that $\mathbf{n}_\Gamma(x) \in N_\Gamma(x)$ for all $x \in \partial\Gamma$. Hence by Definition 5.6(ii), it follows that Definition 5.6 is appropriate to define a solution for the multi-valued equation (5.2).

We denote the semigroup with the infinitesimal generator $-A$ by $S(t)$, $t \ge 0$.

DEFINITION 5.8. A pair of continuous $H \times \mathbb{R}$ valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t))$, $t \in [0, T]$, is called a mild solution of (5.2) if:

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ P-a.s.;
(ii) L is an increasing process with the property

$$I_{\partial\Gamma}(X_s)\,dL_s = dL_s \qquad P\text{-a.s.}$$

and

$$X_t = S(t)x + \int_0^t S(t-s)\,dW_s - \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\,dL_s \qquad \forall t \in [0, T]\ P\text{-a.s.},$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ. In particular, the appearing integrals have to be well defined.

LEMMA 5.9. *The process given by*

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\,dL_s$$

is P-a.s. continuous and adapted to $\mathcal{F}_t$, $t \in [0,T]$. This especially implies that it is predictable.

PROOF. As $|S(t-s)\mathbf{n}_\Gamma(X_s)| \le M_T|\mathbf{n}_\Gamma(X_s)|$, $s \in [0,T]$, the integrals $\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\,dL_s$, $t \in [0,T]$, are well defined. For $0 \le s \le t \le T$,

$$\begin{aligned}
&\left|\int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)\,dL_u - \int_0^t S(t-u)\mathbf{n}_\Gamma(X_u)\,dL_u\right| \\
&\qquad \le \left|\int_0^s [S(s-u)-S(t-u)]\mathbf{n}_\Gamma(X_u)\,dL_u\right| + \left|\int_s^t S(t-u)\mathbf{n}_\Gamma(X_u)\,dL_u\right| \\
&\qquad \le \int_0^s |[S(s-u)-S(t-u)]\mathbf{n}_\Gamma(X_u)|\,dL_u + \int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|\,dL_u,
\end{aligned}$$

where the first summand converges to zero as $s \uparrow t$ or $t \downarrow s$, because

$$\left|1_{[0,s)}(u)[S(s-u)-S(t-u)]\mathbf{n}_\Gamma(X_u)\right| \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

For the second summand, we have

$$\int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|\,dL_u \le M_T(L_t - L_s) \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

By the same arguments as in [19], Lemma 5.1.9, we conclude that the integral is adapted to $\mathcal{F}_t$, $t \in [0,T]$. $\square$

THEOREM 5.10. $(X(t), L_t)$, $t \in [0,T]$, *is a solution of* (5.2) *if and only if it is a mild solution.*

PROOF. ($\Rightarrow$) First, we prove that for arbitrary $\zeta \in C^1([0,T], D(A))$ the following equation holds:

$$\begin{aligned}
\langle X_t, \zeta_t\rangle &= \langle x, \zeta_0\rangle + \int_0^t \langle \zeta_s, dW_s\rangle - \int_0^t \langle \mathbf{n}_\Gamma(X_s), \zeta_s\rangle\,dL_s \\
&\quad + \int_0^t \langle X_s, -A\zeta_s + \zeta_s'\rangle\,ds \qquad \forall t \ge 0\ P\text{-a.s.}
\end{aligned} \tag{5.3}$$

If $\zeta_s = \eta f_s$ for $f \in C^1([0,T])$ and $\eta \in D(A)$, by Itô's formula we have the above relation for such ζ. Then by [19], Lemma G.0.10, and the same arguments as the proof of Proposition G.0.11 we obtain the above formula for all $\zeta \in C^1([0,T], D(A))$. As in [19], Proposition G.0.11, for the resolvent $R_n :=$

$(n+A)^{-1}: H \to D(A)$ and $t \in [0,T]$ choosing $\zeta_s := S(t-s)nR_n\eta$, $\eta \in H$, we deduce from (5.3) that

$$\begin{aligned}
\langle X_t, nR_n\eta\rangle &= \langle x, S(t)nR_n\eta\rangle + \int_0^t \langle S(t-s)nR_n\eta, dW_s\rangle \\
&\quad - \int_0^t \langle \mathbf{n}_\Gamma(X_s), S(t-s)nR_n\eta\rangle\, dL_s \\
&\quad + \int_0^t \langle X_s, AS(t-s)nR_n\eta\rangle + \langle X_s, -AS(t-s)nR_n\eta\rangle\, ds \\
&= \left\langle S(t)x + \int_0^t S(t-s)\, dW_s + \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\, dL_s, nR_n\eta\right\rangle \\
&\qquad \forall t \in [0,T]\ P\text{-a.s.}
\end{aligned}$$

Letting $n \to \infty$, we conclude that $(X(t), L_t)$, $t \in [0,T]$, is a mild solution.

($\Leftarrow$) By Lemma 5.9 and [19], Theorem 5.1.3, we have

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\, dL_s \quad \text{and} \quad \int_0^t S(t-s)\, dW_s, \qquad t \in [0,T],$$

have predictable versions. And we use the same notation for the predictable versions of the respective processes. As (X_t, L_t) is a mild solution, for all $\eta \in D(A)$ we get

$$\begin{aligned}
\int_0^t \langle X_s, A\eta\rangle\, ds &= \int_0^t \langle S(s)x, A\eta\rangle\, ds \\
&\quad - \int_0^t \left\langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)\, dL_u, A\eta\right\rangle ds \\
&\quad + \int_0^t \left\langle \int_0^s S(s-u)\, dW_u, A\eta\right\rangle ds \qquad \forall t \in [0,T]\ P\text{-a.s.}
\end{aligned}$$

The assertion that $(X(t), L_t)$, $t \in [0,T]$, is a solution of (5.2) now follows as in the proof of [19], Proposition G.0.9, because

$$\begin{aligned}
&\int_0^t \left\langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)\, dL_u, A\eta\right\rangle ds \\
&\quad = \int_0^t \int_0^s \left\langle \mathbf{n}_\Gamma(X_u), -\frac{d}{ds} S(s-u)\eta\right\rangle dL_u\, ds \\
&\quad = -\left\langle \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)\, dL_s, \eta\right\rangle + \left\langle \int_0^t \mathbf{n}_\Gamma(X_s)\, dL_s, \eta\right\rangle. \qquad \square
\end{aligned}$$

Below, we prove (5.2) has a unique solution in the sense of Definition 5.6.

THEOREM 5.11. *Let $\Gamma \subset H$ satisfy Hypothesis 5.1. Then the stochastic inclusion (5.2) admits at most one solution in the sense of Definition 5.6.*

PROOF. Let (u, L^1) and (v, L^2) be two solutions of (5.2), and let $\{e_k\}_{k\in N}$ be the eigenbasis of A from above. We then have

$$\langle e_k, u(t)-v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s)-v(s)\rangle\, ds + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\, dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\, dL_s^2 = 0.$$

Setting $\phi_k(t) := \langle e_k, u(t)-v(t)\rangle$, we obtain

$$\begin{aligned}
\phi_k^2(t) &= 2\int_0^t \phi_k(s)\, d\phi_k(s) \\
&= -2\biggl(\int_0^t \langle \alpha_k e_k, u(s)-v(s)\rangle \langle e_k, u(s)-v(s)\rangle\, ds \\
&\qquad + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^1 \\
&\qquad - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^2\biggr) \qquad (5.4)\\
&\le -2\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^1 \\
&\quad + 2\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^2.
\end{aligned}$$

By the dominated convergence theorem for all $t \ge 0$, we have P-a.s.

$$\sum_{k\le N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^1 \to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s)-v(s)\rangle\, dL_s^1 \qquad \text{as } N\to\infty$$

and

$$\sum_{k\le N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s)-v(s)\rangle\, dL_s^2 \to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s)-v(s)\rangle\, dL_s^2 \qquad \text{as } N\to\infty.$$

Summing over $k \le N$ in (5.4) and letting $N\to\infty$ yield that for all $t\ge 0$ P-a.s.

$$|u(t)-v(t)|^2 \le 2\int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s)-u(s)\rangle\, dL_s^1 + 2\int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s)-v(s)\rangle\, dL_s^2.$$

By Remark 5.5 it follows that

$$|u(t) - v(t)|^2 \leq 0,$$

which implies

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t). \qquad \square$$

Combining Theorems 5.4 and 5.11 with the Yamada–Watanabe theorem, we now obtain the following theorem.

THEOREM 5.12. *If Γ satisfies Hypothesis 5.1, then there exists a Borel set $M \subset H$ with $I_\Gamma \cdot \mu(M) = \mu(\Gamma)$ such that for every $x \in M$, (5.2) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 5.6 and $P(X_0 = x) = 1$. Moreover, $X(t) \in M$ for all $t \geq 0$ P-a.s.*

PROOF. By Theorems 5.4 and 5.11, one sees that [12], Theorem 3.14(a) is satisfied for the solution (X, L). So, the assertion follows from [12], Theorem 3.14(b). $\square$

REMARK 5.13. Following the same arguments as in the proof of [22], Theorem 2.1, we can give an alternative proof of Theorem 5.12 for a stronger notion of strong solutions (see, e.g., [22]). Also, because of Theorem 5.10, by a modification of [16], Theorem 12.1, we can prove the Yamada Watanabe theorem for the mild solution in Definition 5.8, and then also a corresponding version of Theorem 5.12 for mild solutions for (5.2). This will be contained in forthcoming work.

5.3. *The nonsymmetric case.* In this section, we extend our results to the nonsymmetric case. For $\Gamma \subset H$ satisfying Hypothesis 5.1, we consider the nonsymmetric Dirichlet form,

$$\mathcal{E}^\Gamma(u, v) = \int_\Gamma \left(\frac{1}{2}\langle Du(z), Dv(z)\rangle + \langle B(z), Du(z)\rangle v(z)\right)\mu(dz), \qquad u, v \in C_b^1(\Gamma),$$

where B is a map from Γ to H such that

$$\begin{aligned} &B \in L^\infty(\Gamma \to H, \mu), \\ &\int_\Gamma \langle B, Du\rangle \, d\mu \geq 0 \qquad \text{for all } u \in C_b^1(\Gamma), u \geq 0. \end{aligned} \tag{5.5}$$

Then $(\mathcal{E}, C_b^1(\Gamma))$ is a densely defined bilinear form on $L^2(\Gamma;\mu)$ which is positive definite, since for all $u \in C_b^1(\Gamma)$

$$\mathcal{E}^\Gamma(u,u) = \int_\Gamma \frac{1}{2}\big(\langle Du(z), Du(z)\rangle + \langle B(z), Du^2(z)\rangle(z)\big)\mu(dz) \geq 0.$$

Furthermore, by the same argument as [13], Section II.3.e, we have $(\mathcal{E}, C_b^1(\Gamma))$ is closable on $L^2(\Gamma,\mu)$ and its closure $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a Dirichlet form on $L^2(\Gamma,\mu)$. We denote the extended Dirichlet space of $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ by $\mathcal{F}_e^\Gamma$: Recall that $u \in \mathcal{F}_e^\Gamma$ if and only if $|u| < \infty$ $I_\Gamma \cdot \mu$-a.e. and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\Gamma$ such that $\mathcal{E}^\Gamma(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u$ $I_\Gamma \cdot \mu$-a.e. as $n \to \infty$. This Dirichlet form satisfies the weak sector condition

$$|\mathcal{E}_1^\Gamma(u,v)| \leq K\mathcal{E}_1^\Gamma(u,u)^{1/2}\mathcal{E}_1^\Gamma(v,v)^{1/2}.$$

Furthermore, we have the following theorem.

THEOREM 5.14. *Suppose $\Gamma \subset H$ satisfies Hypothesis 5.1. Then $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a quasi-regular local Dirichlet form on $L^2(\Gamma;\mu)$.*

PROOF. The assertion follows by [13], Section IV. 4b, and [23]. □

By virtue of Theorem 5.14 and [13], there exists a diffusion process $M^\Gamma = (X_t, P_z)$ on Γ associated with the Dirichlet form $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$. Since constant functions are in $\mathcal{F}^\Gamma$ and $\mathcal{E}^\Gamma(1,1) = 0$, M^Γ is recurrent and conservative. We denote by $\mathbf{A}_+^\Gamma$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^Γ, and define $\mathbf{A}^\Gamma = \mathbf{A}_+^\Gamma - \mathbf{A}_+^\Gamma$. For $A \in \mathbf{A}^\Gamma$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}_0^\Gamma = \{A \in \mathbf{A}^\Gamma | E_{I_\Gamma \cdot \mu}(\{A\}_t) < \infty \ \forall t > 0\}$. Each element in $\mathbf{A}_+^\Gamma$ has a corresponding positive $\mathcal{E}^\Gamma$-smooth measure on Γ by the Revuz correspondence. The totality of such measures will be denoted by S_+^Γ. Accordingly, $\mathbf{A}^\Gamma$ corresponds to $S^\Gamma = S_+^\Gamma - S_+^\Gamma$, the set of all $\mathcal{E}^\Gamma$-smooth signed measure in the sense that $A_t = A_t^1 - A_t^2$ for $A_t^k \in \mathbf{A}_+^\rho$, $k = 1, 2$, whose Revuz measures are ν^k, $k = 1, 2$, and $\nu = \nu^1 - \nu^2$ is the Hahn–Jordan decomposition of ν. The element of $\mathbf{A}$ corresponding to $\nu \in S$ will be denoted byA^ν.

Note that for each $l \in H$ the function $u(z) = \langle l, z\rangle$ belongs to the extended Dirichlet space $\mathcal{F}_e^\Gamma$ and

$$\mathcal{E}^\Gamma(l(\cdot), v) = \int_\Gamma \left(\frac{1}{2}\langle l, Dv(z)\rangle + \langle B(z), l\rangle v(z)\right)\mu(dz) \qquad \forall v \in C_b^1(\Gamma). \tag{5.6}$$

On the other hand, the AF $\langle l, X_t - X_0\rangle$ of M^Γ admits a decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l \in H$

$$\langle l, X_t - X_0\rangle = M_t^l + N_t^l \qquad \forall t \geq 0 \ P_z\text{-a.s.} \tag{5.7}$$

for $\mathcal{E}^\rho$-q.e. $z \in \Gamma$.

Then we have the following theorem.

THEOREM 5.15. *Suppose $\Gamma \subset H$ satisfies Hypothesis 5.1.*

(1) *The next three conditions are equivalent*:

(i) $N^l \in A_0$.

(ii) $|\mathcal{E}^\Gamma(l(\cdot), v)| \le C\|v\|_\infty \ \forall v \in C_b^1(\Gamma)$.

(iii) *There exists a finite* (*unique*) *signed measure ν_l on Γ such that*

$$\mathcal{E}^\Gamma(l(\cdot), v) = -\int_\Gamma v(z)\nu_l(dz) \qquad \forall v \in C_b^1(\Gamma). \tag{5.8}$$

In this case, ν_l is automatically smooth and

$$N^l = A^{\nu_l}.$$

(2) *M^l is a martingale AF with quadratic variation process*

$$\langle M^l\rangle_t = t|l|^2, \qquad t \ge 0. \tag{5.9}$$

PROOF. (1) By [17], Theorem 5.2.7, and the same arguments as in [8], we can extend Theorem 6.2 in [8] to our nonsymmetric case to prove the assertions.

(2) Since

$$\mathcal{E}^\Gamma(u, v) = \int_\Gamma \left(\frac{1}{2}\langle Du(z), Dv(z)\rangle + \langle B(z), Du(z)\rangle v(z)\right)\mu(dz), \qquad u, v \in \mathcal{F}^\Gamma,$$

by [17], Theorem 5.1.5, for $u \in C_b^1(\Gamma)$, $f \in \mathcal{F}^\Gamma$ bounded we have

$$\begin{aligned}
&\int \tilde{f}(x)\mu_{\langle M^{[u]}\rangle}(dx)\\
&\quad = 2\mathcal{E}^\Gamma(u, uf) - \mathcal{E}^\Gamma(u^2, f)\\
&\quad = 2\int_\Gamma \left(\frac{1}{2}\langle Du(z), D(u\tilde{f})(z)\rangle + \langle B(z), Du(z)\rangle u(z)\tilde{f}(z)\right)\mu(dz)\\
&\qquad\quad - \int_\Gamma \left(\frac{1}{2}\langle D(u(z)^2), D\tilde{f}(z)\rangle + \langle B(z), D(u^2)(z)\rangle \tilde{f}(z)\right)\mu(dz)\\
&\quad = \int_\Gamma \langle Du(z), Du(z)\rangle \tilde{f}(z)\mu(dz).
\end{aligned}$$

Here $\tilde{f}$ denotes the $\mathcal{E}^\Gamma$-quasi-continuous version of f, $\mu_{\langle M^{[u]}\rangle}$ is the Reuvz measure for $\langle M^{[u]}\rangle$ and $M^{[u]}$ is the martingale additive functional in the Fukushima decomposition for $u(X_t)$. Hence, we have

$$\mu_{\langle M^{[u]}\rangle}(dz) = I_\Gamma\langle Du(z), Du(z)\rangle \cdot \mu(dz).$$

By [17], (5.1.3), we also have

$$e(\langle M^l\rangle) = e(M^l) = \int_\Gamma \frac{1}{2}\langle l, l\rangle \mu(dz),$$

where $e(M^l)$ is the energy of M^l. Then (5.9) easily follows. □

By Theorem 3.1, we can now prove the following theorem.

THEOREM 5.16. *Suppose $\Gamma \subset H$ satisfies Hypothesis 5.1. Then there is an $\mathcal{E}^\Gamma$-exceptional set $S \subset \Gamma$ such that $\forall z \in \Gamma \setminus S$, under P_z there exists an $\mathcal{M}_t$-cylindrical Wiener process W^z, such that the sample paths of the associated OU-process M^Γ on Γ satisfy the following: for $l \in D(A) \cap H_1$*

$$\begin{aligned}\langle l, X_t - X_0\rangle = &\int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s)\rangle_{H_1^*}\, dL_s^{\|\partial\Gamma\|} \\ &- \int_0^t \langle Al, X_s\rangle\, ds - \int_0^t \langle l, B(X_s)\rangle\, ds \qquad P_z\text{-a.s.}\end{aligned} \tag{5.10}$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in \Gamma \setminus S$

$$I_{\partial\Gamma}(X_s)\, dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \qquad P_z\text{-a.s.} \tag{5.11}$$

Here $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ, and

$$\|\partial\Gamma\|(dy) = \frac{|Dg(y)|}{|Q^{1/2}Dg(y)|}\mu_{\partial\Gamma}(dy),$$

where $\mu_{\partial\Gamma}$ the surface measure induced by μ.

PROOF. By (5.6) and (3.16), we have

$$\begin{aligned}\mathcal{E}^\Gamma(l(\cdot), v) &= \int_\Gamma \frac{1}{2}\langle l, Dv(z)\rangle + \langle B(z), l\rangle v(z)\mu(dz) \\ &= \int_\Gamma \langle B(z), l\rangle v(z)\mu(dz) + \int_\Gamma v(z)\langle Al, z\rangle\mu(dz) \\ &\quad + \frac{1}{2}\int_{\partial\Gamma} v(z)\langle l, \mathbf{n}_\Gamma(z)\rangle \|\partial\Gamma\|(dz).\end{aligned}$$

Thus, by Theorem 5.15

$$\begin{aligned}N_t^l = &-\left\langle Al, \int_0^t X_s(\omega)\, ds\right\rangle - \left\langle l, \int_0^t B(X_s(\omega))\, ds\right\rangle \\ &- \frac{1}{2}\left\langle l, \int_0^t \mathbf{n}_\Gamma(X_s(\omega))\, dL_s^{\|\partial\Gamma\|}(\omega)\right\rangle.\end{aligned}$$

By Theorem 5.15 and the same method as in Theorem 3.2 one then proves the first assertion, and the last assertion follows by Theorems 5.3 and 5.4. □

Let $\Gamma \subset H$ and our linear operator A satisfy Hypotheses 5.1 and 2.1, respectively. As in Section 5.2 we shall now prove the existence and uniqueness of a solution of the following stochastic differential inclusion on the Hilbert space H,

$$\begin{cases} dX(t) + \big(AX(t) + B(X(t)) + N_\Gamma(X(t))\big)\, dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{5.12}$$

where B satisfies condition (5.5), $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, that is,

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \leq 0 \ \forall y \in \Gamma\}.$$

DEFINITION 5.17. A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t))$, $t \in [0, T]$, is called a solution of (5.12) if the following conditions hold:

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ P-a.s.;
(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)\, dL_s = dL_s \qquad P\text{-a.s.},$$

and for any $l \in D(A)$ we have

$$\begin{aligned}\langle l, X_t - x\rangle = {}& \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle\, dL_s - \int_0^t \langle l, B(X_s)\rangle\, ds \\ & - \int_0^t \langle Al, X_s\rangle\, ds \qquad \forall t \geq 0 \ P\text{-a.s.},\end{aligned}$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

Below we prove (5.12) has a unique solution in the sense of Definition 5.17.

THEOREM 5.18. *Let $\Gamma \subset H$ satisfy Hypothesis 5.1 and B satisfy the monotonicity condition*

$$\langle B(u) - B(v), u - v\rangle \geq -\alpha |u - v|^2 \tag{5.13}$$

for all $u, v \in \mathrm{dom}(B)$, for some $\alpha \in [0, \infty)$ independent of u, v. The stochastic inclusion (5.12) admits at most one solution in the sense of Definition 5.17.

PROOF. Let (u, L^1) and (v, L^2) be two solutions of (5.12), and let $\{e_k\}_{k \in N}$ be the eigenbasis of A from above. We then have

$$\begin{aligned}& \langle e_k, u(t) - v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle\, ds + \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle\, ds \\ & \quad + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\, dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\, dL_s^2 = 0.\end{aligned}$$

Setting $\phi_k(t) := \langle e_k, u(t) - v(t)\rangle$, and we have

$$\begin{aligned}\phi_k^2(t) &= 2\int_0^t \phi_k(s)\, d\phi_k(s) \\ &= -2\biggl(\int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle \langle e_k, u(s) - v(s)\rangle\, ds \qquad (5.14)\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, ds \\
&\qquad + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^1 \\
&\qquad\qquad - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^2 \Big) \\
&\le -2 \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, ds \\
&\quad - 2 \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^1 \\
&\quad + 2 \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^2.
\end{aligned}
$$

By the same argument as Theorem 5.11, we have the following P-a.s.:

$$
\begin{aligned}
&\sum_{k \le N} \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, ds \\
&\qquad \to \int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle \, ds \qquad \text{as } N \to \infty, \\
&\sum_{k \le N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^1 \\
&\qquad \to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s) - v(s)\rangle \, dL_s^1 \qquad \text{as } N \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k \le N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle \, dL_s^2 \\
&\qquad \to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle \, dL_s^2 \qquad \text{as } N \to \infty.
\end{aligned}
$$

Summing over $k \le N$ in (5.14) and letting $N \to \infty$ yield that for all $t \ge 0$, P-a.s.

$$
\begin{aligned}
&|u(t) - v(t)|^2 + 2 \int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle \, ds \\
&\quad \le 2 \int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s) - u(s)\rangle \, dL_s^1 + 2 \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle \, dL_s^2.
\end{aligned}
$$

By Remark 5.5, it follows that

$$
|u(t) - v(t)|^2 + 2 \int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle \, ds \le 0.
$$

By (5.13) and Gronwall's lemma, it follows that

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t). \qquad \square$$

Combining Theorems 5.16 and 5.18 with the Yamada–Watanabe theorem, we obtain the following.

THEOREM 5.19. *If Γ satisfies Hypothesis 5.1 and B in (5.12) satisfies (5.13), then there exists a Borel set $M \subset H$ with $I_\Gamma \cdot \mu(M) = \mu(\Gamma)$ such that for every $x \in M$, (5.12) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 5.17 and $P(X_0 = x) = 1$. Moreover, $X(t) \in M$ for all $t \geq 0$ P-a.s.*

PROOF. The proof is completely analogous to that of Theorem 5.12. $\square$

6. Reflected OU-processeses on a class of convex sets. Below for a topological space X we denote its Borel σ-algebra by $\mathcal{B}(X)$. In this section, we consider the case where $H := L^2(0,1)$, $\rho = I_{K_\alpha}$, where $K_\alpha := \{f \in H | f \geq -\alpha\}, \alpha \geq 0$, and $A = -\frac{1}{2}\frac{d^2}{dr^2}$ with Dirichlet boundary conditions on $(0,1)$. So in this case $e_j = \sqrt{2}\sin(j\pi r)$, $j \in \mathbb{N}$, is the corresponding eigenbases. We recall that (cf. [23]) we have $\mu(C_0([0,1])) = 1$. In [23], L. Zambotti proved the following integration by parts formulae in this situation:

- for $\alpha > 0$,

$$\int_{K_\alpha} \langle l, D\varphi\rangle \, d\mu$$
$$= -\int_{K_\alpha} \varphi(x)\langle x, l''\rangle \mu(dx) - \int_0^1 dr\, l(r) \int \varphi(x)\sigma_\alpha(r, dx)$$
$$\forall l \in D(A), \varphi \in C_b^1(H),$$

- for $\alpha = 0$,

$$\int_{K_0} \langle l, D\varphi\rangle \, d\nu$$
$$= -\int_{K_0} \varphi(x)\langle x, l''\rangle \nu(dx) - \int_0^1 dr\, l(r) \int \varphi(x)\sigma_0(r, dx) \tag{6.1}$$
$$\forall l \in D(A), \varphi \in C_b^1(H),$$

where ν is the law of the Bessel Bridge of dimension 3 over [0, 1] which is zero at 0 and 1, $\sigma_\alpha(r,dx)=\sigma_\alpha(r)\mu_\alpha(r,dx)$, and for $\alpha>0$, σ_α is a positive bounded function, and for $\alpha=0$, $\sigma_0(r)=\frac{1}{\sqrt{2\pi r^3(1-r)^3}}$, where $\mu_\alpha(r,dx)$, $\alpha\geq 0$, are probability kernels from $(H,\mathcal{B}(H))$ to $([0,1],\mathcal{B}([0,1]))$.

REMARK 6.1. Since each l in $D(A)$ has a second derivative in L^2, its first derivative is bounded, hence l goes faster than linear to zero at any point where l is zero, in particular at the boundary points $r=0$ and $r=1$. Hence, the second integral in the right-hand side of the above equality is well defined.

We know by (3.5) that for all $l\in D(A)$

$$D^*(\varphi(\cdot)l)=-\langle l,D\varphi\rangle-\varphi\langle l'',\cdot\rangle.$$

Hence, for $\alpha>0$,

$$\int_{K_\alpha} D^*(\varphi(\cdot)l)\,d\mu=\int_0^1 l(r)\int\varphi(x)\sigma_\alpha(r,dx)\,dr \qquad \forall l\in D(A),\varphi\in C_b^1(H). \tag{6.2}$$

Now take

$$c_j:=\begin{cases}(j\pi)^{1/2+\varepsilon}, & \text{if } \alpha>0,\\ (j\pi)^{\beta}, & \text{if } \alpha=0,\end{cases} \tag{6.3}$$

where $\varepsilon\in(0,\frac{3}{2}]$ and $\beta\in(\frac{3}{2},2]$, respectively, and define

$$H_1:=\left\{x\in H\,\Big|\,\sum_{j=1}^\infty\langle x,e_j\rangle^2c_j^2<\infty\right\}$$

equipped with the inner product

$$\langle x,y\rangle_{H_1}:=\sum_{j=1}^\infty c_j^2\langle x,e_j\rangle\langle y,e_j\rangle.$$

We note that $D(A)\subset H_1$ continuously for all $\alpha\geq 0$, since $\varepsilon\leq\frac{3}{2}$, $\beta\leq 2$. Furthermore, $(H_1,\langle\cdot,\cdot\rangle_{H_1})$ is a Hilbert space such that $H_1\subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1\subset H(\equiv H^*)\subset H_1^*.$$

It follows that

$${}_{H_1}\langle z,v\rangle_{H_1^*}=\langle z,v\rangle_H \qquad \forall z\in H_1,v\in H,$$

and that (H_1,H,H_1^*) is a Gelfand triple.

The following is the main result of this section.

THEOREM 6.2. *If* $\alpha > 0$, *then* $I_{K_\alpha} \in \mathrm{BV}(H, H_1) \cap \mathbf{H}$.

PROOF. First, for σ_α as in (6.2) we show that for each $B \in \mathcal{B}(H)$ the function $r \mapsto \sigma_\alpha(r, B)$ is in H_1^* and that the map $B \mapsto \sigma_\alpha(\cdot, B)$ is in fact an H_1^*-valued measure of bounded variation, that is,

$$\sup\left\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\bigcup_{n=1}^{\infty}} B_n\right\} < \infty,$$

that is,

$$\sup\left\{\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j^{-2}\left(\int_0^1 \sigma_\alpha(r, B_n)\sin(j\pi r)\,dr\right)^2\right)^{1/2} : \right.$$
$$\left. B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\bigcup_{n=1}^{\infty}} B_n\right\} < \infty,$$

where $\dot{\bigcup}_{n=1}^{\infty} B_n$ means disjoint union.

For $\alpha > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j^{-2}\left(\int_0^1 \sigma_\alpha(r, B_n)\sin(j\pi r)\,dr\right)^2\right)^{1/2} \\
&\quad \le \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j^{-2}\left(\int_0^1 \sigma_\alpha(r, B_n)\,dr\right)^2\right)^{1/2} \\
&\quad \le C\sum_{n=1}^{\infty}\int_0^1 \sigma_\alpha(r, B_n)\,dr \\
&\quad = C\int_0^1 \sigma_\alpha(r)\,dr < \infty.
\end{aligned}$$

Thus, σ_α in (6.2) is of bounded variation as an H_1^*-valued measure. Hence by the theory of vector-valued measures (cf. [3], Section 2.1), there is a unit vector field $n_\alpha : H \to H_1^*$, such that $\sigma_\alpha = n_\alpha\|\sigma_\alpha\|$, where

$$\|\sigma_\alpha\|(B) := \sup\left\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, B = \dot{\bigcup_{n=1}^{\infty}} B_n\right\}$$

is a nonnegative measure, which is finite by the above proof. So (6.2) becomes

$$\int_{K_\alpha} D^*(\varphi(\cdot)l)\,d\mu = \int_H {}_{H_1}\langle \varphi(x)l, n_\alpha(x)\rangle_{H_1^*}\|\sigma_\alpha\|(dx) \qquad \forall l \in D(A), \varphi \in C_b^1(H),$$

which by linearity extends to all $G \in (C_b^1)_{D(A)\cap H_1}$. Thus by Theorem 3.1(iii), we get that $I_{K_\alpha} \in \mathrm{BV}(H, H_1)$.

$I_{K_\alpha} \in \mathbf{H}$ follows by Remark 4.1. $\square$

REMARK 6.3. It has been proved by Guan Qingyang that I_{K_α} is not in $\mathrm{BV}(H, H)$.

THEOREM 6.4. *For $\alpha = 0$, then there exist a positive finite measure $\|\sigma_0\|$ on H and a Borel-measurable map $n_0 : H \to H_1^*$ such that $\|n_0(z)\|_{H_1^*} = 1$ $\|\sigma_0\|$-a.e., and*

$$\begin{aligned}(6.4)\qquad & -\int_{K_0} \langle l, D\varphi\rangle\, d\nu - \int_{K_0} \varphi(x)\langle x, l''\rangle \nu(dx) \\ &= \int {}_{H_1}\langle \varphi(x) l, n_0(x)\rangle_{H_1^*} \|\sigma_0\|(dx) \qquad \forall l \in D(A), \varphi \in C_b^1(H).\end{aligned}$$

PROOF. For $\alpha = 0$ using that $|\sin(j\pi r)| \le 2j\pi r(1-r)$ $\forall r \in [0, 1]$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j^{-2}\left(\int_0^1 \sigma_0(r, B_n)\sin(j\pi r)\, dr\right)^2\right)^{1/2} \\ &\quad \le \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j^{-2}\left(\int_0^1 \sigma_0(r, B_n) 2j\pi r(1-r)\, dr\right)^2\right)^{1/2} \\ &\quad \le C\sum_{n=1}^{\infty}\int_0^1 \sigma_0(r, B_n) r(1-r)\, dr \\ &\quad = C\int_0^1 \sigma_0(r) r(1-r)\, dr < \infty.\end{aligned}$$

Thus, σ_0 in (6.1) is of bounded variation as an H_1^*-valued measure. Hence by the theory of vector-valued measures (cf. [3], Section 2.1), there is a unit vector field $n_0 : H \to H_1^*$, such that $\sigma_0 = n_0\|\sigma_\alpha\|$, where

$$\|\sigma_0\|(B) := \sup\left\{\sum_{n=1}^{\infty} \|\sigma_0(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, B = \dot{\bigcup_{n=1}^{\infty}} B_n\right\}$$

is a nonnegative measure, which is finite by the above proof. So the result follows by (6.1). $\square$

Since here $\mu(K_0) = 0$, we have to change the reference measure of the Dirichlet form. Consider

$$\mathcal{E}^{K_0}(u, v) = \frac{1}{2}\int_{K_0} \langle Du, Dv\rangle\, d\nu, \qquad u, v \in C_b^1(K_0).$$

Since $I_{K_0} \in \mathbf{H}$ by Remark 4.1, the closure of $(\mathcal{E}^{I_{K_0}}, C_b^1(K_0))$ is also a quasi-regular local Dirichlet form on $L^2(F; \rho \cdot \nu)$ in the sense of [13], Chapter IV, Definition 3.1. As before, there exists a diffusion process $M^{I_{K_0}} = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$ on F associated with this Dirichlet form. $M^{I_{K_0}}$ will also be called distorted OU-process on K_0. As before, $M^{I_{K_0}}$ is recurrent and conservative. As before, we also have the associated PCAF and the Revuz correspondence.

Combining these two cases: for $\alpha > 0$ by Theorem 3.2 and for $\alpha = 0$ by the same argument as Theorem 3.2, since we have (6.4), we have the following theorem.

THEOREM 6.5. *Let $\rho := I_{K_\alpha}, \alpha \geq 0$ and consider the measure $\|\sigma_\alpha\|$ and n_α appearing in Theorems 6.2 and 6.4. Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \setminus S$, under P_z there exists an $\mathcal{M}_t$-cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l \in D(A)$*

$$\begin{aligned}\langle l, X_t - X_0\rangle = &\int_0^t \langle l, dW_s\rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, n_\alpha(X_s)\rangle_{H_1^*}\, dL_s^{\|\sigma_\alpha\|} \\ &- \int_0^t \langle Al, X_s\rangle\, ds \qquad P_z\text{-a.e.}\end{aligned} \tag{6.5}$$

Here $L_t^{\|\sigma_\alpha\|}$ is the real valued PCAF associated with $\|\sigma_\alpha\|$ by the Revuz correspondence with respect to M^ρ, satisfying

$$I_{\{X_s+\alpha\neq 0\}}\, dL_s^{\|\sigma_\alpha\|} = 0, \tag{6.6}$$

and for $l \in H_1$ with $l(r) \geq 0$ we have

$$\int_0^t {}_{H_1}\langle l, n_\alpha(X_s)\rangle_{H_1^*}\, dL_s^{\|\sigma_\alpha\|} \geq 0. \tag{6.7}$$

Furthermore, for all $z \in F$

$$P_z\big[X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)\big] = 1. \tag{6.8}$$

PROOF. For $\alpha > 0$, the first part of the assertion follows by Theorem 3.2 and the uniqueness part of Theorem 3.1(ii). For $\alpha = 0$, the assertion follows by the same argument as in Theorem 3.2. (6.6) and (6.7) follow by the property of σ_α in [23]. By [18], p. 135, Theorem 2.4, we have $C_0[0,1]$ is a Borel subset of $L^2[0,1]$. By [11], (5.1.13), we have

$$E_{\rho\mu}\bigg[\int_{k-1}^k 1_{F\setminus C_0[0,1]}(X_s)\, ds\bigg] = \rho\mu(F \setminus C_0[0,1]) = 0 \qquad \forall k \in \mathbb{N},$$

hence

$$E_{\rho\mu}\left[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)\,ds\right]=0.$$

Since $E_x[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)\,ds]$ is a 0-excessive function in $x \in K_\alpha$, it is finely continuous with respect to the process X. Then for $\mathcal{E}^\rho$-q.e. $z \in F$,

$$E_z\left[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)\,ds\right]=0,$$

thus, for $\mathcal{E}^\rho$-q.e. $z \in F$,

$$P_z\left[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)\,ds=0\right]=1.$$

As a consequence, we have that $\Lambda_0 := \{X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)\}$ is measurable and for $\mathcal{E}^\rho$-q.e. $z \in F$

$$P_z(\Lambda_0)=1.$$

As $\Lambda_0 = \bigcap_{t\in\mathbb{Q},t>0}\theta_t^{-1}\Lambda_0$ and since by [4] we have that the semigroup associated with X_t is strong Feller, by the Markov property as in [7], Lemma 7.1, we obtain that for any $z \in F$, $t \in \mathbb{Q}$, $t > 0$,

$$P_z(\theta_t^{-1}\Lambda_0)=1.$$

Hence, for any $z \in F$ we have

$$P_z\big[X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)\big]=1. \qquad \square$$

REMARK 6.6. We emphasize that in the present situation it was proved in [15], Theorem 1.3, that for all initial conditions $x \in H$, there exists a unique strong solution to (1.1). By [23], the solution in [15] is associated to our Dirichlet form, hence satisfies (6.5) by Theorem 6.5. Hence, it follows that the solution in [15], Theorem 1.3, is solution to an infinite-dimensional Skorohod problem.

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