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THE CONVEX MINORANT OF A LÉVY PROCESS

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We offer a unified approach to the theory of convex minorants of Lévy processes with continuous distributions. New results include simple explicit constructions of the convex minorant of a Lévy process on both finite and infinite time intervals, and of a Poisson point process of excursions above the convex minorant up to an independent exponential time. The Poisson–Dirichlet distribution of parameter 1 is shown to be the universal law of ranked lengths of excursions of a Lévy process with continuous distributions above its convex minorant on the interval [0, 1].

1. Introduction. We present simple explicit constructions of the convex minorant of a Lévy process with continuous distributions on both finite and infinite time intervals, and of a Poisson point process of excursions of the Lévy process above its convex minorant. These constructions bridge a number of gaps in the literature by relating combinatorial approaches to fluctuation theory of random walks related to the cycle structure of random permutations, dating back to the 1950s [cf. Andersen (1950, 1953a, 1953b, 1954); Spitzer (1956)], some features of which were extended to interval partitions associated with the convex minorant of Brownian motion and Brownian bridge by Suidan (2001a, 2001b) and Balabdaoui and Pitman (2009), and results previously obtained for the convex minorants of Brownian motion by Groeneboom (1983) and Pitman (1983), and for Lévy processes by Nagasawa (2000) and Bertoin (2000). In particular, we gain access to the excursions above the convex minorant, which were previously treated only in the Brownian case by Groeneboom (1983) and Pitman (1983).

Our work is part of a larger initiative to understand the convex minorant of processes with exchangeable increments. The case of discrete time is handled in Abramson and Pitman (2011), while Brownian motion is given a more detailed study in Pitman and Ross (2010). Our joint findings are summarized in Abramson et al. (2011).

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1.1. *Statement of results.* Let X be a Lévy process. The following hypothesis is used throughout the paper:

(CD) For all $t > 0$, X_t has a continuous distribution, meaning that for each $x \in \mathbb{R}$, $\mathbb{P}(X_t = x) = 0$.

It is sufficient to assume that X_t has a continuous distribution for some $t > 0$. Equivalently [Sato (1999), Theorem 27.4, page 175] X is not a compound Poisson process with drift.

The *convex minorant* of a function f on an interval $[0, t]$ or $[0, \infty)$ is the greatest convex function c satisfying $c \le f$. We shall only consider functions f which are càdlàg, meaning that $\lim_{h\to 0+} f(t + h) = f(t)$ and that $\lim_{h\to 0-} f(t - h)$ exists; the latter limit will be denoted $f(t-)$.

First properties of the convex minorant of a Lévy process, established in Section 2 and which partially overlap with the Markovian study of convex minorants in Lachieze-Rey (2009), are:

PROPOSITION 1. *Let X be a Lévy process which satisfies* (CD) *and C the convex minorant of X on* $[0, t]$. *The following conditions hold almost surely*:

1. *The open set $\mathscr{O} = \{s \in (0, t) : C_s < X_s \wedge X_{s-}\}$ has Lebesgue measure t.*
2. *For every component interval (g, d) of $\mathscr{O}$, the jumps that X might have at g and d have the same sign. When X has unbounded variation on finite intervals, both jumps are zero.*
3. *If (g_1, d_1) and (g_2, d_2) are different component intervals of $\mathscr{O}$, then their slopes differ*

$$\frac{C_{d_1} - C_{g_1}}{d_1 - g_1} \ne \frac{C_{d_2} - C_{g_2}}{d_2 - g_2}.$$

Let $\mathscr{I}$ be the set of connected components of $\mathscr{O}$; we shall also call them *excursion intervals*. Associated with each excursion interval (g, d) are the *vertices* g and d, the *length* $d - g$, the *increment* $C_d - C_g$ and the *slope* $(C_d - C_g)/(d - g)$.

Our main result is a simple description of the lengths and increments of the excursion intervals of the convex minorant. Indeed, we will consider a random ordering of them which uncovers a remarkable probabilistic structure.

THEOREM 1. *Let (U_i) be a sequence of uniform random variables on* $(0, t)$ *independent of the Lévy process X which satisfies* (CD). *Let* $(g_1, d_1), (g_2, d_2), \dots$ *be the sequence of distinct excursion intervals which are successively discovered by the sequence (U_i). Consider another i.i.d. sequence (V_i) of uniform random variables on* $(0, 1)$ *independent of X, and construct the associated* uniform stick-breaking process L *by*

$$L_1 = tV_1 \quad \textit{and} \quad \textit{for } i \ge 1 \qquad L_{i+1} = V_{i+1}(t - S_i),$$

where

$$S_0 = 0 \quad \textit{and} \quad \textit{for } i \geq 1 \qquad S_i = L_1 + \cdots + L_i.$$

Under hypothesis (CD), *the following equality in distribution holds*:

$$\big((d_i - g_i, C_{d_i} - C_{g_i}), i \geq 1\big) \stackrel{d}{=} \big((L_i, X_{S_i} - X_{S_{i-1}}), i \geq 1\big).$$

The *Poisson–Dirichlet distribution of parameter one* is the law of the decreasing rearrangement of the sequence L when $t = 1$. Theorem 1 implies that the Poisson–Dirichlet distribution of parameter 1 is the universal distribution of the ranked lengths of excursions intervals of the convex minorant of a Lévy process with continuous distributions on $[0, 1]$. What differs between each Lévy process is the distribution of the order in which these lengths appear, that is, the law of the composition of of $[0, 1]$ induced by the lengths of excursion intervals when they are taken in order of appearance. Using Theorem 1 we can form a composition of $[0, 1]$ with that law in the following way. For each pair $(L_i, X_{S_i} - X_{S_{i-1}})$ we generate a slope by dividing the second coordinate, the increment, by the first, the length, and then create a composition of $[0, 1]$ by arranging the sequence L in order of increasing associated slope.

Note that the second sequence of Theorem 1 can also be constructed as follows: given a uniform stick-breaking process L, create a sequence Y_i of random variables which are conditionally independent given L and such that the law of Y_i given L is that of X_{L_i} (X independent of L). Then

$$\big((L_i, Y_i) : i \geq 1\big) \stackrel{d}{=} \big((L_i, X_{S_i} - X_{S_{i-1}}), i \geq 1\big).$$

Theorem 1 provides a way to perform explicit computations. For example, the intensity measure ν_t of the point process Ξ_t with atoms at

$$\{(d - g, C_d - C_g) : (g, d) \text{ is an excursion interval}\}$$

is given by

$$\begin{aligned}\nu_t(A) &= \mathbb{E}\Big(\sum_i \mathbf{1}_{(d_i - g_i, C_{d_i} - C_{g_i}) \in A}\Big)\\ &= \mathbb{E}\Big(\sum_i \mathbf{1}_{(L_i, X_{d_i} - X_{g_i}) \in A}\Big) = \int_0^t \int \mathbf{1}_A(l, x) \frac{1}{l} \mathbb{P}(X_l \in dx)\, dl.\end{aligned}$$

[This follows conditioning on L and then using the intensity measure of L obtained by size-biased sampling; cf. formula (6) in Pitman and Yor (1997).]

We now apply Theorem 1 to fully describe the convex minorant of the Cauchy process as first done in Bertoin (2000). Let X be a Cauchy process characterized by

$$F(x) := \mathbb{P}(X_1 \leq x) = 1/2 + \arctan(x)/\pi.$$

Let C be the convex minorant of X on [0, 1] and D its right-hand derivative,

$$D_t = \lim_{h\to 0+} \frac{C_{t+h} - C_t}{h}.$$

Consider

$$I_x = \inf\{t \ge 0 : D_t > x\} \qquad \text{for } x \in \mathbb{R}.$$

Note that $\mathbb{P}(X_t < xt) = F(x)$ and that therefore, in the setting of Theorem 1, the slopes $(C_{d_i} - C_{g_i})/(d_i - g_i)$ are independent of the lengths $d_i - g_i$. Also, let T be a Gamma subordinator such that

$$\mathbb{E}(e^{-qT_t}) = \left(\frac{1}{1+q}\right)^t.$$

COROLLARY 1. 1. *The symmetric Cauchy process is characterized by the independence of lengths and slopes of excursions intervals on* [0, 1].

2. $(I_x, x \in \mathbb{R})$ *and* $(T_{F(x)}/T_1, x \in \mathbb{R})$ *have the same law.*

Item 2 is due to Bertoin (2000), who used a technique allowing only the study of the convex minorant of a Cauchy process on [0, 1].

Integrating Theorem 1, we obtain a description of the convex minorant considered on the random interval $[0, T_\theta]$ where T_θ is a exponential random variable of parameter θ independent of X.

COROLLARY 2. *Let T be exponential of parameter θ and independent of the Lévy process X which satisfies* (CD). *Let Ξ_T be the point process with atoms at lengths and increments of excursion intervals of the convex minorant of X on* $[0, T]$. *Then Ξ_T is a Poisson point process with intensity*

$$\mu_\theta(dt, dx) = e^{-\theta t}\frac{dt}{t}\mathbb{P}(X_t \in dx).$$

By conditioning on T (which essentially reduces to inverting Laplace transforms and underlies the analysis of the relationship between the Gamma subordinator and the Poisson–Dirichlet distribution), we see that Theorem 1 can be deduced from Corollary 2. The latter can be deduced from the analysis of the independence of pre- and post-minimum processes of a Lévy process run until an independent exponential time found in Greenwood and Pitman (1980). These relationships are discussed in Section 4, where we also explain the results on fluctuation theory for Lévy processes which are found in the literature and which can be deduced from our analysis of the convex minorant.

From Theorem 1 we can also derive the behavior of the convex minorant of X on $[0, \infty)$ as described for a Brownian motion by Groeneboom (1983) and Pitman (1983) and for a Lévy process by Nagasawa (2000). Let Ξ_∞ be the point process of lengths of excursion interval and increments of the convex minorant on $[0, \infty)$.

COROLLARY 3. *The quantity* $l = \liminf_{t\to\infty} X_t/t$ *belongs to* $(-\infty, \infty]$ *and is almost surely constant if and only if the convex minorant of* X *on* $[0, \infty)$ *is almost surely finite. In this case, under* (CD), Ξ_∞ *is a Poisson point process with intensity*

$$\mu_\infty(dt, dx) = \frac{\mathbf{1}_{x<lt}}{t}\mathbb{P}(X_t \in dx)\, dt.$$

Recall, for example, Kyprianou [(2006), Example 7.2], the strong law of large numbers for Lévy processes, which says that if the expectation of X_1 is defined, then

$$\lim_{t\to\infty} \frac{X_t}{t} = \mathbb{E}(X_1) \qquad \text{almost surely.}$$

Hence, if $\mathbb{E}(X_1^-) < \infty$, we can apply the second part of Corollary 3 with $l = \mathbb{E}(X_1)$. In the remaining case when $\mathbb{E}(X_1^-) = \mathbb{E}(X_1^+) = \infty$, let ν be the Lévy measure of X and ν_+ its right-tail given by

$$\nu_+(y) = \nu((y, \infty)).$$

Erickson (1973) provides the necessary and sufficient for $-\infty < l$, which implies that, actually, $l = \infty$,

$$\int_{(-\infty,0)} \frac{|y|}{\nu_+(|y|)}\nu(dy) < \infty$$

(see also Doney (2007), page 39, for a proof).

While it seems natural to first study the convex minorant of a Lévy process on $[0, \infty)$, as was the approach of previous authors, the description of the convex minorant with infinite horizon is less complete, as it is necessarily restricted to slopes $a < l$.

As another application, we can use the stick-breaking representation of Theorem 1 to study the absolute continuity of the location and the value of the minimum of the Lévy process on [0, 1]. Let

$$\underline{X}_t = \min_{s\le t} X_s \quad \text{and} \quad \rho_t \text{ be such that } X_{\rho_t} \wedge X_{\rho_t-} = \underline{X}_t.$$

(Recall that under (CD), the minimum of a Lévy process on $[0, t]$ is attained at an almost surely unique place ρ_t, as deduced from Theorem 1 since $\mathbb{P}(X_t = 0) = 0$.)

THEOREM 2. *Let* X *be a Lévy process such that* 0 *is regular for both half-lines* $(-\infty, 0)$ *and* $(0, \infty)$. *Then*:

1. *The distribution of* ρ_1 *is equivalent to Lebesgue measure on* [0, 1].
2. *If* X_t *has an absolutely continuous distribution for each* $t > 0$, *then the distribution of* $(\rho_1, \underline{X}_1)$ *is equivalent to Lebesgue measure on* $(0, 1] \times (0, \infty)$.

3. *If X_t has an absolutely continuous distribution for each $t > 0$, then the distribution of $(\underline{X}_1, X_1 - \underline{X}_1)$ is equivalent to Lebesgue measure on $(-\infty, 0) \times (0, \infty)$.*

Chaumont (2010) also analyzes absolute continuity properties for the supremum of a Lévy process on a fixed interval using excursion theory for the reflected Lévy process. The densities provided by Theorem 2 (more importantly, the fact that they are almost surely positive) provide one way to construct bridges of the Lévy process X conditioned to stay positive. With these bridges, we can prove a generalization of Vervaat's theorem relating the Brownian bridge and the normalized Brownian excursion [Vervaat (1979), Theorem 1] to a fairly general class of Lévy processes. Details are provided in Uribe Bravo (2011).

Our next results will only consider convex minorants on a fixed interval, which we take to be [0, 1].

Theorem 1 gives a construction of the convex minorant by means of sampling the Lévy process at the random, but independent, times of a uniform stick-breaking process. Our second proof of it, which does not rely on fluctuation theory and gives insight into the excursions of X above its convex minorant, depends on the use of the following path transformation. Let u be an element of the excursion set $\mathscr{O}$, and let (g, d) be the excursion interval which contains u. We then define a new stochastic process $X^u = (X_t^u)_{t \le 1}$ by

$$X_t^u = \begin{cases} X_{u+t} - X_u, & 0 \le t < d - u, \\ C_d - C_g + X_{g+t-(d-u)} - X_u, & d - u \le t \le d - g, \\ C_d - C_g + X_{t-(d-g)}, & d - g \le t < d, \\ X_t, & d \le t \le 1. \end{cases} \tag{1}$$

The idea for such a definition is that the graph of the convex minorant of X^u on $[d - g, 1]$ can be obtained from the graph of C by removing (g, d) and closing up the gap adjusting for continuity, while on $[0, d - g]$, X^u goes from 0 to $C_d - C_g$. (Property 2 of Proposition 1 is essential for the transformation to work like this; see Figure 2.) A schematic picture of the path transformation is found in Figure 1 for a typical Brownian motion path.

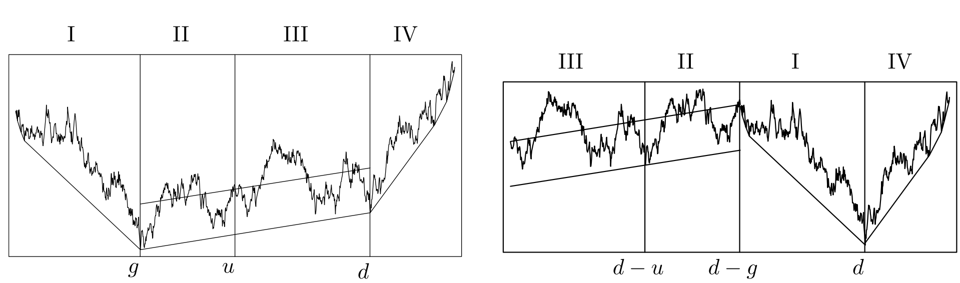

FIG. 1. *Visualization of the path transformation $X \mapsto X^u$ applied to a Brownian motion seen from its convex minorant.*

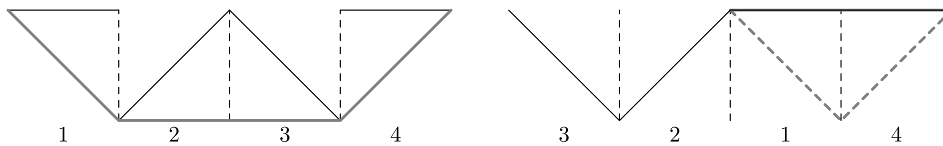

FIG. 2. *Visualization of the path transformation $X \mapsto X^u$ applied to a càdlàg path not satisfying property* 2 *of Proposition* 1.

Theorem 1 then follows from the following invariance result. Indeed, by applying the following path transformation recursively, we can obtain a size-biased sample of the excursion intervals. In particular, the excursion interval containing an independent uniform variable has a uniform length, which begins to explain the stick-breaking process appearing in Theorem 1.

THEOREM 3. *If U is a uniform on* $(0, 1)$ *and independent of X and hypothesis* (CD) *holds*, *the pairs* (U, X) *and* $(d - g, X^U)$ *have the same law.*

Proof of Theorem 3 will be based on the analogous random walk result proved by Abramson and Pitman (2011) as well as analysis on Skorohod space. Abraham and Pitman's discrete time result is an exact invariance property for a similar transformation applied to the polygonal approximation X^n of X given by $X^n_t = X_{[nt]/n}(\lceil nt\rceil/n - t) + X_{\lceil nt\rceil/n}([nt]/n)$; that this approximation does not converge in Skorohod space to X makes the passage to the limit technical, although it simplifies considerably for Lévy processes with unbounded variation, and particularly so for Lévy processes with continuous sample paths. The discrete time result is combinatorial in nature and related to permutations of the increments. Indeed, the discrete time result is based on the fact that for a random walk S with continuous jump distribution, the probability that S lies strictly above the line from $(0, 0)$ to (n, S_n) on $\{0, \ldots, n\}$ is known to be $1/n$, and conditionally on this event, the law of S can be related to a Vervaat-type transform of S. Hence, it is not only possible to verify by combinatorial reasoning that the faces of the convex minorant have the same law as the cycle lengths of a uniform random permutation when both are placed in decreasing order, but also to characterize the path fragments on top of each excursion interval.

Theorem 3 actually gives a much stronger result than Theorem 1 since it grants access to the behavior of X between vertex points of the convex minorant. To see this, consider the Vervaat transformation: for each $t > 0$ and each càdlàg function f, let $\rho_t = \rho_t(f)$ be the location of the last minimum $\underline{f}(t)$ of f on $[0, t]$ and define

$$V_t f(s) = f(\rho_t + s \bmod t) - \underline{f}(t) \qquad \text{for } s \in [0, t].$$

This path transformation was introduced in Vervaat (1979) for the Brownian bridge; its connection to Lévy processes was further studied for stable Lévy processes by Chaumont (1997), for spectrally positive Lévy processes in Miermont (2001), and more general Lévy processes by Fourati (2005).

For each excursion interval (g, d) of $\mathscr{O}$, associate an *excursion* $e^{(g,d)}$ given by

$$e^{(g,d)}(s) = X_{g+s} - C_{g+s} \qquad \text{for } s \in [0, d-g];$$

note that $e^{(g,d)}(0)$ is positive if $X_g > C_g$. Finally, recalling the setting of Theorem 1, let K^i be Knight's bridge,

$$K^i_s = X_{(S_{i-1}+t)} - X_{S_{i-1}} - s\frac{X_{S_i} - X_{S_{i-1}}}{L_i}, \qquad s \in [0, L_i]$$

[the name is proposed because of remarkable universality theorems proven for K^i in Knight (1996)].

THEOREM 4. *The following equality in distribution holds under* (CD):

$$\big((d_i - g_i, C_{d_i} - C_{g_i}, e^{(g_i,d_i)}), i \geq 1\big) \stackrel{d}{=} \big((L_i, X_{S_i} - X_{S_{i-1}}, V_{L_i}(K^i)), i \geq 1\big). \tag{2}$$

Note that the increment $C_d - C_g$ cannot be obtained from the path fragment $e^{(g,d)}$ when X jumps at g or d. This does not happen if X has unbounded variation, thanks to Proposition 1.

The same remark of Theorem 1 holds, namely, the intensity measure of the right-hand side of (2), seen as a point process, admits the expression

$$\mathbb{E}\Big(\sum_i \mathbf{1}_{(L_i, X_{S_i} - X_{S_{i-1}}, V_{L_i}(K^i)) \in A}\Big) = \int_0^1 \int\int \mathbf{1}_A(l, x, e)\frac{1}{l}\kappa_l(dx, de)\, dl$$

in terms of the law of X, where the measure κ_l is the joint law of X_l and the Vervaat transform V_l of $(X_t - tX_l/l, t \in [0, l])$. The measure κ_l is related to Lévy processes conditioned to stay positive [introduced in generality in Chaumont and Doney (2005)] in Uribe Bravo (2011).

This document is organized as follows: we first study the basic properties of the convex minorant of a Lévy process of Proposition 1 in Section 2. Then, examples of the qualitative behaviors of the convex minorants are given in Section 3. Next, we turn to the description of the process of lengths and slopes of excursion intervals up to an independent exponential time in Section 4, where we also discuss how this implies the description of the convex minorant to a deterministic and finite time and on an infinite horizon. Section 4 also explains the relationship between this work and the literature on fluctuation theory for Lévy processes. Section 5 is devoted to the absolute continuity of the location and time of the minimum of a Lévy process with a proof of Theorem 2. Finally, we pass to the invariance of the path transformation (1) for Lévy processes stated as Theorem 3, in Section 6 and to the description of the excursions above the convex minorant implied by Theorem 4 in Section 7.

2. Basic properties of the convex minorant on a finite interval. In this section we will prove Proposition 1. Let $X = (X_t, t \in [0, 1])$ be a Lévy process and consider its convex minorant C on $[0, 1]$ as well as the lower semicontinuous regularization X^l of X given by $X^l = X \wedge X_-$ [with the convention $X_-(0) = X_0 = 0$].

2.1. *Property* 1 *of Proposition* 1. We will first be concerned with the measure of

$$\mathscr{P} = \{t \in [0, 1] : X^l = C\}.$$

A first observation is that $\mathscr{P}$ does not vary under changes in the drift of X. We now prove that $\mathscr{P}$ has Lebesgue measure zero almost surely. Indeed, it suffices to see that for each $t \in (0, 1)$, $t \notin \mathscr{P}$ almost surely. If X has unbounded variation, Rogozin (1968) proves that

$$\liminf_{h \to 0+} \frac{X_h}{h} = -\infty \qquad \text{almost surely (a.s.)}$$

[see, however, the more recent proof at Vigon (2002)], and so by the Markov property at each fixed time t, we get

$$\liminf_{h \to 0+} \frac{X_{t+h} - X_t}{h} = -\infty \quad \text{and} \quad \limsup_{h \to 0+} \frac{X_{t+h} - X_t}{h} = \infty \qquad \text{a.s.}$$

However, at any $\tau \in \mathscr{P}$, we have

$$\liminf_{h \to 0+} \frac{X_{\tau+h} - X_\tau}{h} \geq D(\tau) > -\infty \qquad \text{a.s.,}$$

where D is the right-hand derivative of C. If X has bounded variation, the proof is similar, except that, according to Bertoin [(1996), Proposition 4, page 81], we get

$$\lim_{h \to 0+} \frac{X_{t+h} - X_t}{h} = d$$

almost surely, where d is the drift coefficient. We then see that if $t \in \mathscr{P} \cap (0, 1)$, then $D(t) = d$; the inequality $D(t) \leq d$ follows from the preceding display, and by time reversal we also obtain $d \leq C'_-(t)$. Taking away the drift, we see that t then should be a place where the minimum is achieved. However, t is almost surely not a time when the minimum is reached: defining

$$\tilde{X}_s = \begin{cases} X_{t+s} - X_t, & \text{if } s \leq 1 - t, \\ X_1 - X_t + X_{s-(1-t)}, & \text{if } 1 - t \leq s \leq 1, \end{cases}$$

we know that $\tilde{X}$ has the same law as X. Note that the minimum of X is reached at t if and only if $\tilde{X}$ remains above zero, which happens with positive probability only when 0 is irregular for $(-\infty, 0)$. Hence, t does not belong to $\mathscr{P}$ almost surely whenever 0 is regular for $(-\infty, 0)$. If this is not the case, then 0 is regular for $(0, \infty)$ since X is nonatomic, and applying same argument to the time reversed process $(X_{(1-t)-} - X_1, t \leq 1)$ we see then that $t \notin \mathscr{P}$ almost surely in this remaining case.

2.2. *Property* 2 *of Proposition* 1. We will now show that for an excursion interval (g,d) of X above C, the jumps of X at g and d, denoted ΔX_g and ΔX_d, satisfy $\Delta X_g \Delta X_d \geq 0$. We first prove that, thanks to (CD), X does not have jumps of both signs on the two endpoints of an excursion. The proof depends on different arguments for bounded and unbounded variation: with unbounded variation, actually no jumps occur at the endpoints.

If X has unbounded variation, we again use Rogozin's result:

$$\liminf_{h\to 0+} \frac{X_h}{h} = -\infty \quad \text{and} \quad \limsup_{h\to 0+} \frac{X_h}{h} = \infty,$$

and adapt Millar's proof of his Proposition 2.4 [Millar (1977)] to see that X is continuous on $\{X^l = C\}$. Indeed, for every $\varepsilon > 0$, let $J_1^\varepsilon, J_2^\varepsilon, \dots$ be the jumps of X with size greater than ε in absolute value. Then the strong Markov property applied at J_i^ε implies that

$$\liminf_{h\to 0+} \frac{X_{J_i^\varepsilon+h} - X_{J_i^\varepsilon}}{h} = -\infty \quad \text{and} \quad \limsup_{h\to 0+} \frac{X_{J_i^\varepsilon+h} - X_{J_i^\varepsilon}}{h} = \infty.$$

Hence, at any random time T which is almost surely a jump time of X, we get

$$\liminf_{h\to 0+} \frac{X_{T+h} - X_T}{h} = -\infty;$$

however, if $t \in \{X^l = C\}$, we see that

$$\liminf_{h\to 0+} \frac{X_{t+h} - X_t}{h} \geq D(t) > -\infty.$$

Suppose now that X has bounded variation but infinite Lévy measure. Since our problem (jumping to or from the convex minorant) is invariant under addition of drift, we can assume that the drift coefficient of X is zero, and so

$$\lim_{h\to 0+} \frac{X_h}{h} = 0$$

by Bertoin (1996), Proposition 4, page 81. We will now prove that almost surely, for every component (g,d) of $\{C < X^l\}$, we have

$$\Delta X_g \Delta X_d \geq 0. \tag{3}$$

The argument is similar to the unbounded variation case: at any random time T which is almost surely a jump time of X, we have

$$\lim_{h\to 0+} \frac{X_{T+h} - X_T}{h} = 0.$$

We deduce that if the slope of C on the interval (g,d) is strictly positive, then $\Delta X_g \geq 0$, and so $X_{g-} = C_g$. By time-reversal, we see that if the slope of C is strictly negative on (g,d), then $\Delta X_d \leq 0$ and so $X_d = C_d$. Note that C only has nonzero slopes. Indeed, a zero slope would mean that the infimum of X is attained at least twice, a possibility, that is, ruled out by Proposition 2.2 of Millar (1977) under assumption (CD).

2.3. *Property* 3 *of Proposition* 1. We now see that, almost surely, all excursion intervals of X above its convex minorant have different slopes. A different argument is given for bounded and unbounded variation processes.

When X has unbounded variation on compact sets, let C^t denote the convex minorant of X on $[0,t]$ so that $C = C^1$. Note that C^t and C agree up to some random time, which we call τ_t; for every fixed $t \in (0,1)$, τ_t cannot equal t as $C_t < X^l_t$ almost surely, as proved in Section 2.1. We will first prove that, almost surely, for every $t \in (0,1) \cap \mathbb{Q}$, whenever the post t process touches a line that extends C^t linearly outwards from one of the excursion intervals of C^t, it crosses it downwards. To see that this is enough, suppose that there were two excursion intervals, (g_1,d_1) and (g_2,d_2), with the same associated slope. Then there would exist $t \in (g_2,d_2) \cap \mathbb{Q}$ such that $g_1 < d_1 \le \tau_t < t$. If the post t process touches the linear extension of the convex minorant over the interval (g_1,d_1) it must cross it downwards. This should occur at d_2, which contradicts $C_{d_2} = X^l_{d_2}$.

To prove the claim that the post t process crosses the extended lines downwards for each fixed $t \in (0,1)$, let $L_i(s) = \alpha_i + \beta_i s$ be the lines extending the segments of C^t (using any ordering which makes the α_i and β_i random variables). Let

$$T_i = \inf\{s \ge 0 : X^l_{t+s} - X_t \le \alpha_i - X_t + \beta_i(t+s)\}.$$

Hence T_i is a stopping time for the filtration $\mathscr{F}_{t+s} = \sigma(X_r : r \le t+s), s \ge 0$, with respect to which $X_{t+s} - X_t, s \ge 0$, is a Lévy process. If X jumps below L_i at time T_i, then the excursion interval of C containing t cannot have slope β_i (and incidentally, β_i is not a slope of C). Since X has infinite variation, Rogozin's result quoted above gives

$$\liminf_{h \to 0+} \frac{X_{T_i+h} - X_{t_i}}{h} = -\infty.$$

Hence, if X is continuous at T_i then X goes below L_i immediately after T_i and β_i cannot be a slope of C. We have seen, however, that in the unbounded variation case, X does not jump at the vertices of excursion intervals.

When X has bounded variation, the argument is similar except in a few places. Suppose the drift of X is zero. We first use

$$\lim_{h \downarrow 0+} \frac{X_{t+h} - X_t}{h} = 0$$

to prove that for every $t \in (0,1)$, whenever the post t process touches a linear extension L_i of C^t on an excursion interval *with positive slope*, by a jump, it crosses it downwards: this is clear if X is continuous at T_i or if it jumps into L_i at T_i. However, X cannot reach L_i from the left and jump away at T_i by quasi-continuity of Lévy processes. By time reversal, we handle the case of negative slopes, and therefore there are no two excursions above the convex minorant with the same slope almost surely by the same arguments as in the unbounded variation case. Again, note that slopes of C are nonzero since under (CD) the minimum of X is attained only once by Proposition 2.2 of Millar (1977).

3. Examples.

3.1. *Lévy processes of bounded variation.* Consider a Lévy process X with paths of bounded variation on compact sets and zero drift such that 0 is regular for $(0, \infty)$ but irregular for $(-\infty, 0)$. Then the cumulative minimum of X is piecewise constant and decreases by jumps; that is, X reaches a new minimum by jumping downwards. It follows that the convex minorant of X on any finite interval has a finite number of segments of negative slopes until it reaches the minimum of X, and all the excursions above the convex minorant end by a jump (and begin continuously). However, since the minimum is attained at a jump time, say at ρ, then $\lim_{t\to 0}(X_{\rho+t} - X_\rho)/t = 0$, and since $X_{\rho+\cdot} - X_\rho$ visits $(0, \infty)$ on any neighborhood of 0, there cannot be a segment of the convex minorant with slope zero, nor a first segment with positive slope. Hence 0 is an accumulation point for positive slopes.

3.2. *The convex minorant of a Cauchy process.*

PROOF OF COROLLARY 1. Let X be a symmetric Cauchy process, such that

$$F(x) := \mathbb{P}(X_1 \le x) = 1/2 + \arctan(x)/\pi.$$

Since

$$\mathbb{E}(e^{iuX_t}) = e^{-t|u|},$$

we see that X is 1-selfsimilar, which means that X_t has the same law as tX_1 for every $t \ge 0$.

If Ξ_1 is the point process of lengths and increments of excursions intervals for the convex minorant on $[0, 1]$, its intensity measure ν_1 has the following form:

$$\nu_1(dl, dx) = \frac{1}{l}\mathbb{P}(X_l \in dx)\,dl.$$

Therefore, the intensity $\tilde{\nu}_1$ of the point process of lengths and slopes of excursions intervals for the convex minorant on $[0, 1]$, say $\tilde{\Xi}_1$, factorizes as

$$\tilde{\nu}_1(dl, ds) = \frac{1}{l}\mathbb{P}(X_1 \in ds)\,dl.$$

Let $Y_1, Y_2, \dots$ be an i.i.d. sequence of Cauchy random variables independent of L; recall that F is their distribution function. From the analysis of the point process Ξ_1 in the forthcoming proof of Lemma 1, the above factorization of the intensity measure $\tilde{\nu}_1$ implies that $\tilde{\Xi}_1$ has the law of the point process with atoms

$$\{(L_i, Y_i) : i \ge 1\}; \tag{4}$$

otherwise said: lengths and slopes are independent for the Cauchy process.

In the converse direction, we see that if lengths and slopes are independent then X is a 1-selfsimilar Lévy process. Indeed, using Theorem 1, we see that X_{L_1}/L_1 and L_1 are independent. Let G be the law of X_{L_1}/L_1. Independence of L_1 and X_{L_1}/L_1 implies that X_t/t has law G for almost all $t \in (0, 1)$, so that $G = F$. As the law of X_t/t is weakly continuous, we see that X_t/t has law F for all $t \in (0, 1)$ and the independence and homogeneity of increments of X implies that X_t/t has law F for all t. However, it is known that a 1-selfsimilar Lévy process is a symmetric Cauchy process, although perhaps seen at a different speed. See Theorem 14.15 and Example 14.17 of Sato (1999).

We finish the proof by identifying the law of $(I_x, x \in \mathbb{R})$. Informally, I_x is the time in which the convex minorant of X on $[0, 1]$ stops using slopes smaller than x. We then see that I has the same law as

$$\tilde{I} = \left(\sum_{i=1}^{\infty} L_i \mathbf{1}_{Y_i \le x}, x \in \mathbb{R}\right).$$

In contrast, if $U_i, i \ge 1$, is an i.i.d. sequence of uniform random variables on $(0, 1)$ independent of L, the process $(T_t/T_1, t \in [0, 1])$ has the representation

$$\left(\sum_{i=1}^{\infty} L_i \mathbf{1}_{U_i \le t}, t \in [0, 1]\right).$$

With the explicit choice $U_i = F(Y_i)$, we obtain the result. □

As a consequence of Corollary 1, we see that the set $\mathscr{C} = \{t \in [0, 1] : C_t = X_t \wedge X_{t-}\}$ is perfect.

3.3. *The convex minorant of stable processes.* Let C be the convex minorant of the Lévy process X on $[0, 1]$. We now point out a dichotomy concerning the set of slopes

$$\mathscr{S} = \left\{\frac{C_d - C_g}{d - g} : (g, d) \text{ is an excursion interval}\right\},$$

when X is a stable Lévy process of index $\alpha \in (0, 2]$ characterized either by the scaling property

$$X_{st} \stackrel{d}{=} s^{1/\alpha} X_t, \qquad s > 0,$$

or the following property of its characteristic function:

$$|\mathbb{E}(e^{iuX_t})| = e^{-tc|u|^\alpha}.$$

COROLLARY 4. *When $\alpha \in (1, 2]$, $\mathscr{S}$ has no accumulation points, and $\mathscr{S} \cap (a, \infty)$ and $\mathscr{S} \cap (-\infty, -a)$ are almost surely infinite for all $a > 0$. If $\alpha \in (0, 1]$, then $\mathscr{S}$ is dense in $\mathbb{R}_+$, $\mathbb{R}_-$, or $\mathbb{R}$ depending on if X is a subordinator, $-X$ is a subordinator or neither condition holds.*

PROOF. When $\alpha \in (1, 2]$, Fourier inversion implies that X_1 admits a continuous and bounded density which is strictly positive. We now make an intensity measure computation for $a < b$:

$$\mathbb{E}(\#\mathscr{S} \cap (a, b)) = \int_0^1 \int_a^b \frac{1}{t} \mathbb{P}(X_t/t \in (a, b))\, dt.$$

Using the scaling properties of X, we see that near $t = 0$, the integrand is asymptotic to $ct^{-1/\alpha}$ where c is the density of X_1 at zero. Since

$$\mathbb{E}(\#\mathscr{S} \cap (a, b)) < \infty$$

for all $a < b$, then $\mathscr{S}$ does not contain accumulation points in $\mathbb{R}$.

If $a > 0$, a similar argument implies that

$$\mathbb{E}(\#\mathscr{S} \cap (a, \infty)) = \infty$$

since $\mathbb{P}(X_1 > 0) > 0$. Unfortunately, this does not imply that $\#\mathscr{S} \cap (a, \infty) = \infty$ almost surely. However, from Theorem 1, we see that $\mathscr{S} \cap [a, \infty)$ has the same law as

$$\sum_{i \geq 1} \mathbf{1}_{Y_i \geq a L_i^{1-1/\alpha}},$$

where L and Y are independent, and Y_i has the same law as X_1. Since $1 - 1/\alpha > 0$ and $L_i \to 0$, we see that $Y_i \geq a L_i^{1-1/\alpha}$ infinitely often, implying that $\#\mathscr{S} \cap (a, \infty) = \infty$ almost surely.

We have already dealt with the Cauchy case, which corresponds to $\alpha = 1$, so consider $\alpha \in (0, 1)$. Arguing as before, we see that

$$\#\mathscr{S} \cap (a, b) \stackrel{d}{=} \sum_{i \geq 1} \mathbf{1}_{Y_i \in L_i^{1-1/\alpha}(a,b)}.$$

Since $1 - 1/\alpha < 0$, we see that $Y_i \in L_i^{1-1/\alpha}(a, b)$ infinitely often as long as $\mathbb{P}(X_1 \in (a, b) > 0)$. Finally, recall that the support of the law of X_1 is $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$ depending on if X is a subordinator, $-X$ is a subordinator or neither condition holds. □

4. Splitting at the minimum and the convex minorant up to an independent exponential time. In this section, we analyze the relationship between Theorem 1 and Corollary 2 and how they link with well-known results of the fluctuation theory of Lévy processes. We also give a proof of Corollary 3.

We will first give a proof of Corollary 2 and show how it leads to a proof of Theorem 1. While the implication is based on very well-known results of fluctuation theory, it is insufficient to prove the more general Theorem 4. Our proof of Theorem 4 is independent of the results of this section.

Let X denote a Lévy process with continuous distributions, C its convex minorant on an interval $[0, T]$ (which can be random), X^l the lower-semicontinuous regularization of X given by $X^l_t = X_t \wedge X_{t-}$ and $\mathcal{O} = \{s \le T : C_s < X^l_s\}$ is the open set of excursions from the convex minorant on $[0, T]$. Thanks to Proposition 1 on the basic properties of the convex minorant, proved in Section 2, we see that the point process of lengths and increments of excursion intervals are equivalently obtained by the following construction, taken from Nagasawa [(2000), Chapter XI]: define

$$X^a_t = X_t - at \quad \text{and} \quad \underline{X}^a_t = \min_{s \le t} X^a_s$$

as well as

$$\rho^a = \sup\{s \le T : X^a_t \wedge X^a_{t-} = \underline{X}^a_t\} \quad \text{and} \quad m^a = X^l_{\rho_a}.$$

The idea behind such definitions is that if $a \mapsto \rho^a$ jumps at a, it is because the convex minorant on $[0, t]$ begins using the slope a at ρ^{a-} and ends using it at ρ^a, while the value of the convex minorant at the beginning of this interval is m^{a-}, and at the end it is m^a. For every fixed a, we know that X^a reaches its minimum only once almost surely. However, at a random a at which ρ^a jumps, the minimum is reached twice, since we know that slopes are used only once on each excursion interval. From this analysis, we see that

$$C_{\rho^a} = X^l_{\rho^a} = m^a$$

and obtain the following important relationship:

$$\Xi_T \text{ is the point process } \{(\rho^a - \rho^{a-}, m^a - m^{a-}) : \rho^{a-} < \rho^a\}.$$

We characterize the two-dimensional process (ρ, m) with the help of the following results. First of all, according to Millar's analysis of the behavior of a Lévy process at its infimum [cf. Millar (1977), Proposition 2.4], if 0 is irregular for $(-\infty, 0)$ then, since 0 is regular for $(0, \infty)$, $X^a_{\rho^a} = \underline{X}^a_{\rho^a}$ almost surely for each fixed a; cf. also the final part of Section 2.2. With this preliminary, Theorem 5 and Lemma 6 from Bertoin [(1996), Chapter VI] can be written as follows:

THEOREM 5. *Let T be exponential with parameter θ and independent of X. For each fixed $a \in \mathbb{R}$, there is independence between the processes*

$$\big(X^a_{(t+\rho^a)\wedge T} - m^a, t \ge 0\big) \quad \textit{and} \quad (X^a_{t\wedge\rho^a}, t \ge 0).$$

Furthermore,

$$\begin{aligned} &\mathbb{E}\big(\exp\big(-\alpha\rho^a + \beta(m^a - a\rho^a)\big)\big) \\ &\quad = \exp\biggl(-\int_0^\infty \int_{-\infty}^0 (1 - e^{-\alpha t + \beta x}) \frac{e^{-\theta t}}{t} \mathbb{P}(X_t - at \in dx)\, dt\biggr). \end{aligned} \tag{5}$$

Formula (5) was proved initially by Pečerskiĭ and Rogozin (1969). Later, Greenwood and Pitman (1980) showed how to deduce it by splitting at the minimum of the trajectory of a Lévy process up to an independent exponential time, a theme which was retaken by Bertoin (1996) to produce the independence assertion of the previous theorem.

PROOF OF COROLLARY 2. The proof follows [Nagasawa (2000)]. We first show that (ρ, m) is a process with independent increments. Let $a < b$. Note that $\rho^b - \rho^a$ is the last time that t such that $X_{\rho^a+t} - m^a - bt$ reaches its minimum, so that Theorem 5 implies the independence of $\rho^{a+b} - \rho^a$ and $\sigma(X_{\cdot\wedge\rho^a})$; denote the latter σ-field as $\mathscr{F}^a$. Also, note that $m^b - m^a$ is the minimum of $X_{(\rho^a+t)\wedge T} - m^a - bt, t \geq 0$. Hence there is also independence between $m^b - m^a$ and $\mathscr{F}^a$. Finally, note that if $a' \leq a$, $(\rho^{a'}, m^{a'})$ are $\mathscr{F}^a$ measurable since $\rho^{a'}$ is the last time that $X_{\cdot\wedge\rho^a} - a'\cdot$ reaches its minimum on $[0, \rho^a]$, and $m^{a'}$ is the value of this minimum.

From the above paragraph, we see that the point process of jumps of (ρ, m), that is, Ξ, is a Poisson random measure: this would follow from (a bidimensional extension of) Theorem 2 and Corollary 2 in Gihman and Skorohod [(1975), Chapter IV.1, pages 263–266] which affirm that the jump process of a stochastically continuous process with independent increments on $\mathbb{R}_+$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$. To show that (ρ, m) is stochastically continuous, we show that it has no fixed discontinuities; this follows because for every fixed $a \in \mathbb{R}$, the minimum of X^a is reached at an unique point almost surely, which implies that, for every fixed a, almost surely, neither ρ nor m can jump at a. To compute the intensity measure ν of Ξ_T, note that the pair (ρ^a, m^a) can be obtained from Ξ_T as

$$(\rho^a, m^a) = \sum_{\substack{(u,v)\in\mathscr{I}\\ C_v - C_u \leq a(v-u)}} (v - u, C_v - C_u). \tag{6}$$

The above equality contains the nontrivial assertion that the additive process (ρ, m) has no deterministic component or, stated differently, that it is the sum of its jumps. For the process ρ, this follows because

$$\sum_{\substack{(u,v)\in\mathscr{I}\\ C_v - C_u \leq a(v-u)}} (v - u) = \mathrm{Leb}(\mathscr{O} \cap \{t \leq T : C'_t \leq a\})$$

which, since $\mathrm{Leb}(\mathscr{O}) = T$ and C' is nondecreasing, gives

$$\sum_{\substack{(u,v)\in\mathscr{I}\\ C_v - C_u \leq a(v-u)}} (v - u) = \sup\{t \leq T : C'_t \leq a\} = \rho^a.$$

To discuss the absence of drift from m, let m^C be the signed measure which assigns each interval (u, v) the quantity $C_v - C_u$. (Because C' is nondecreasing, it is trivial to prove the existence of such a signed measure, to give a Hahn decomposition of it and to see that it is absolutely continuous with respect to Lebesgue measure.) Then

$$\sum_{\substack{(u,v)\in\mathscr{I}\\ C_v-C_u\le a(v-u)}} (C_v - C_u) = m^C(\mathscr{O}\cap\{t\le T : C'_t \le a\})$$

$$= m^C(\{t\le T : C'_t\le a\}) = C_{\rho^a} = X^l_{\rho^a} = m^a.$$

From (6), we get

$$\mathbb{E}\big(\exp(-\alpha\rho^a + \beta m^a)\big) = \exp\biggl(-\int_0^\infty \int_{-\infty}^{at} (1-e^{-\alpha t+\beta x})\nu(dt,dx)\biggr),$$

while from the Pečerskiĭ–Rogozin formula (5), we obtain

$$\begin{aligned}
&\mathbb{E}\big(\exp(-\alpha\rho^a + \beta m^a)\big)\\
&\quad = \exp\biggl(-\int_0^\infty\int_{-\infty}^0 (1-e^{-(\alpha-a\beta)t+\beta x})\frac{e^{-\theta t}}{t}\mathbb{P}(X_t - at\in dx)\,dt\biggr)\\
&\quad = \exp\biggl(-\int_0^\infty\int_{-\infty}^{at} (1-e^{-\alpha t+\beta x})\frac{e^{-\theta t}}{t}\mathbb{P}(X_t\in dx)\,dt\biggr)
\end{aligned}$$

giving

$$\nu(dt,dx) = \frac{e^{-\theta t}}{t}\mathbb{P}(X_t\in dx)\,dt. \qquad \square$$

We now remark on the equivalence between Theorem 1 and Corollary 2 and how either of them implies Corollary 3.

Let L be an uniform stick-breaking sequence and X a Lévy process with continuous distributions which are independent. Let S be the partial sum sequence associated to L, and consider the point process $\tilde{\Xi}_t$ with atoms at

$$\{(tL_i, X_{tS_i} - X_{tS_{i-1}})\}.$$

LEMMA 1. *If T an exponential random variable of parameter θ independent of (X,L), $\tilde{\Xi}_T$ is a Poisson point process with intensity*

$$\mu_\theta(dt,dx) = \frac{e^{-\theta t}}{t}\,dt\,\mathbb{P}(X_t\in dx). \tag{7}$$

PROOF. We recall the relationship between the Gamma subordinator and the stick-breaking process, which was found by McCloskey in his unpublished PhD thesis [McCloskey (1965)] and further examined and extended by Perman, Pitman and Yor (1992). Recall that a Gamma process is a subordinator $(\Gamma_t, t\ge 0)$ characterized by the Laplace exponent

$$\mathbb{E}(e^{-q\Gamma_t}) = \left(\frac{\theta}{\theta+q}\right)^t = \exp\biggl(-t\int_0^\infty (1-e^{-qx})\frac{e^{-\theta x}}{x}\,dx\biggr);$$

the law of Γ_1 is exponential of parameter θ. It is well known that $(\Gamma_t/\Gamma_1, t \le 1)$ is independent of Γ_1. Also, it was proved [McCloskey (1965); Perman, Pitman and Yor (1992)] that the size-biased permutation of the jumps of $(\Gamma_t/\Gamma_1, t \in [0, 1])$ has the same law as the stick-breaking process on $[0, 1]$. Hence if L is a stick-breaking process independent of the exponential T of parameter θ, then the point process with atoms at $\{TL_1, TL_2, \ldots\}$ has the same law as the point process with atoms at the jumps of a Gamma subordinator (of parameter θ) on $[0, 1]$ or, equivalently, a Poisson point process with intensity $e^{-\theta x}/x\,dx$.

If S is the partial sum sequence associated to L, conditionally on $T = t$ and $L = (l_1, l_2, \ldots)$, $(X_{TS_i} - X_{TS_{i-1}}, i \le 1)$ are independent and the law of $X_{TS_i} - X_{TS_{i-1}}$ is that of X_{tl_i}. We deduce that the point process with atoms $\{(TL_i, X_{TS_i} - X_{TS_{i-1}}), i \ge 1\}$ is a Poisson point process with the intensity μ_θ of (7), as shown, for example, in Kallenberg [(2002), Proposition 12.3, page 228] using the notion of randomization of point processes. $\square$

Lemma 1 shows how Theorem 1 implies Corollary 2.

Conversely, if we assume Corollary 2, we know that $\tilde{\Xi}_T$ has the same law as the point process of lengths and increments of excursions intervals on the interval $[0, T]$. However, if Ξ_t is the point process of lengths and increments of excursion intervals on $[0, t]$, then

$$\int_0^\infty \theta e^{-\theta t}\mathbb{E}(e^{-\Xi_t f})\,dt = \mathbb{E}(e^{-\tilde{\Xi}_T f}) = \int_0^\infty \theta e^{-\theta t}\mathbb{E}(e^{-\tilde{\Xi}_t f})\,dt$$

which implies that

$$\mathbb{E}(e^{-\Xi_t f}) = \mathbb{E}(e^{-\tilde{\Xi}_t f})$$

for continuous and nonnegative f. However, this implies the identity in law between Ξ_t and $\tilde{\Xi}_t$, giving Theorem 1.

Let us pass to the proof of Corollary 3. Abramson and Pitman show the discrete time analog using a Poisson thinning procedure.

PROOF OF COROLLARY 3. Suppose $l = \liminf_{t\to\infty} X_t/t \in (-\infty, \infty]$. Then there exists $a \in \mathbb{R}$ and $T > 0$ such that $X_t > at$ for all $t > T$. If C^T is the convex minorant of X on $[0, T]$, and ρ^a is the first instant at which the derivative of C^T is greater than a, then the convex function

$$\tilde{C}_t = \begin{cases} C^T, & \text{if } t < \rho^a, \\ C^T_{T_a} + a(t - T), & \text{if } t \ge \rho^a, \end{cases}$$

lies below the path of X on $[0, \infty)$, implying C^∞, the convex minorant of X on $[0, \infty)$, is finite for every point of $[0, \infty)$.

Conversely, if C^∞ is finite on $[0, \infty)$, for any $t > 0$ we can let $a = \lim_{h\to 0+}(C_{t+h} - C_t)/h \in \mathbb{R}$ and note that $\liminf_{s\to\infty} X_s/s \ge a$.

From Erickson (1973) we see that, actually, $\lim_{t\to\infty} X_t/t$ exists and it is finite if and only if $\mathbb{E}(|X_1|) < \infty$ and $\mathbb{E}(X_1) = l$. Note that the right-hand derivative of C^∞ is never strictly greater than l. This derivative cannot equal l: if $l = \infty$ this is clear while if $l < \infty$, it follows from the fact that the zero mean Lévy process $X_t - lt$ visits $(-\infty, 0)$ [as can be proved, e.g., by embedding a random walk and using, for example, by Chung and Fuchs (1951); Chung and Ornstein (1962)]. However, the derivative also surpasses any level $a < l$. This follows from the definitions of l and C^∞: if the derivative of C^∞ were always less than $l - \varepsilon$, since X_t eventually stays above every line of slope $l - \varepsilon/2$, we would be able to construct a convex function greater than C^∞ and below the path of X.

If $a < l$, let L_a be the last time the derivative of C^∞ is smaller than a. Then for $t > L_a$, we see that

$$C^{L_a} = C^t = C^\infty \qquad \text{on } [0, L_a].$$

We will now work with C^{T_θ}, where T_θ is exponential of parameter θ and independent of X. Then on the set $\{L_a < T_\theta\}$, which has probability tending to 1 as $\theta \to 0$, we have $C^{L_a} = C^{T_\theta} = C^\infty$ on $[0, L_a]$. Recall, however, that if Ξ_θ is a Poisson point process with intensity

$$\mu_\theta(dt, dx) = \frac{e^{-\theta t}}{t}\mathbb{P}(X_t \in dx)\, dt,$$

then Ξ_θ has the law of the lengths and increments of excursions of X above C^{T_θ} by Corollary 2. We deduce that for every $a < l$ the restriction of Ξ_θ to $\{(t, x) : x < at\}$ converges in law as $\theta \to 0$ to the point process with atoms at the lengths and increments of excursions of X above C^∞ with slope less than a. Hence, the excursions of X above C^∞ with slopes $< a$ form a Poisson point process with intensity

$$\frac{\mathbf{1}_{x<at}}{t}\mathbb{P}(X_t \in dx)\, dt.$$

It suffices then to increase a to l to obtain the stated description of Ξ_∞. $\square$

Basic to the analysis of this section has been the independence result for the pre and post minimum processes up to an independent exponential time as well as the Pečerskiĭ and Rogozin formula stated in Theorem 5. Theorem 5 is the building block for the fluctuation theory presented in Bertoin [(1996), Chapter VI] and is obtained there using the local time for the Lévy process reflected at its cumulative minimum process. In the following sections, we will reobtain Theorem 1 and Corollaries 2 and 3 appealing only to the basic results of the convex minorant of Section 2 (and without the use of local time). In particular, this implies the first part of Theorem 5, from which the full theorem follows as shown by Bertoin (1996). Indeed, assuming Theorem 4, if T is exponential with parameter θ and independent of X, if ρ^a is the last time $X^l_t - at$ reaches its minimum on $[0, T]$, and m^a is the value of this minimum, we see that

$$(X^a_{(t+\rho^a)\wedge T} - m^a, t \geq 0)$$

can be obtained from the Poisson point process of excursions of X above its convex minorant with slopes $> a$, while

$$(X^a_{t\wedge\rho^a}, t \geq 0)$$

is obtained from the excursions with slopes $\leq a$. Since the process of excursions (up to an independent time) is a Poisson point process, we obtain the independence of the pre and post minimum processes.

Here is another example of how the description of the convex minorant up to an independent exponential time leads to a basic result in fluctuation theory: according to Rogozin's criterion for regularity of half-lines, 0 is irregular for $(0, \infty)$ if and only if

$$\int_0^1 \mathbb{P}(X_t \leq 0)/t\, dt < \infty. \tag{8}$$

To see how this might be obtained from Corollary 2, we note that the probability that X does not visit $(0, \infty)$ on some $(0, \varepsilon)$ is positive if and only if the convex minorant up to T_θ has positive probability of not having negative slopes. By Theorem 2 this happens if and only if

$$\int_0^\infty \mathbb{P}(X_t \leq 0)e^{-\theta t}/t\, dt < \infty,$$

which is of course equivalent to (8).

5. Absolute continuity of the minimum and its location.

PROOF OF THEOREM 2. Since 0 is regular for both half-lines, the Lévy process X satisfies assumption (CD), and we can apply Theorem 1.

Let L be an uniform stick-breaking process independent of X, and define its partial sum and residual processes S and R by

$$S_0 = 0, \qquad S_{i+1} = S_i + L_{i+1} \quad \text{and} \quad R_i = 1 - S_i.$$

Set

$$\Delta_i = X_{S_i} - X_{S_{i-1}}.$$

Then the time of the minimum of the Lévy process X on [0, 1], has the same law as

$$\rho = \sum_{i=1}^\infty L_i \mathbf{1}_{\Delta_i < 0},$$

while the minimum of X on [0, 1] (denoted $\underline{X}_1$) and $X_1 - \underline{X}_1$ have the same laws as

$$\sum_{i=1}^\infty \Delta_i \mathbf{1}_{\Delta_i < 0} \quad \text{and} \quad \sum_{i=1}^\infty \Delta_i \mathbf{1}_{\Delta_i > 0}.$$

The basic idea of the proof is to decompose these sums at a random index J; in the case of ρ, into

$$\Sigma_J = \sum_{i=1}^{J} L_i \mathbf{1}_{\Delta_i < 0} \quad \text{and} \quad \Sigma^J = \sum_{i=J+1}^{\infty} L_i \mathbf{1}_{\Delta_i < 0}. \tag{9}$$

The random index (actually a stopping time for the sequence Δ) is chosen so that Σ_J and Σ^J are both positive, and (R_J, Σ_J) has a joint density, which is used to provide a density for Σ using the conditional independence between Σ_J and Σ^J given R_J.

Let I be any stopping time for the sequence Δ which is finite almost surely. We first assert that the sequence $(\Delta_{I+i-1})_{i\geq 1}$ has both nonnegative and strictly negative terms if 0 is regular for both half-lines. Indeed, if 0 is regular for $(-\infty, 0)$, this implies that the convex minorant of X has a segment of negative slope almost surely, which implies the existence of i such that $\Delta_i < 0$ almost surely. If 0 is regular for $(0, \infty)$, a time-reversal assertion proves also the existence of nonnegative terms in the sequence Δ. On the other hand, conditionally on $I = i$ and $L_1 = l_1, \ldots, L_i = l_i$, the sequence $(\Delta_{i-1+j}, j \geq 1)$ has the same law as the sequence Δ but obtained from the Lévy process $X_{(1-l_1-\cdots-l_i)t, t\geq 0}$ which shares the same regularity as X, which implies the assertion.

1. Let I and J be defined by

$$I = \min\{i \geq 1 : \Delta_i \geq 0\} \quad \text{and} \quad J = \min\{j \geq I : \Delta_j < 0\}.$$

By the preceding paragraph, we see that I and J are both finite almost surely. Hence, the two sums Σ_J and Σ^J of (9) are both in the interval (0, 1) and we have

$$\rho = \Sigma_J + \Sigma^J.$$

We now let

$$f(t) = \mathbb{P}(X_t \leq 0)$$

which will allow us to write the density of (Σ_J, R_J); this follows from the computation

$$\begin{aligned} &\mathbb{P}(J = j, L_1 \in dl_1, \ldots, L_j \in dl_j) \\ &\quad = \sum_{i=1}^{j-1} \prod_{k<i} f(l_k) \prod_{i\leq k<j} (1 - f(l_k)) f(l_j) \mathbb{P}(L_1 \in l_1, \ldots, L_j \in l_j) \end{aligned}$$

valid for $j \geq 2$. For $2 \leq i < j$, let

$$g_{i,j}(l_1, \ldots, l_j) = (l_1, \ldots, l_{i-1}, l_i, \ldots, l_{j-2}, l_1 + \cdots + l_i + l_j, 1 - l_i - \cdots - l_j),$$

and define

$$g_{1,2}(l_1, l_2) = (l_2, 1 - l_2 - l_2)$$

as well as

$$g_{1,j}(l_1, \dots, l_j) = (l_1, \dots, l_{j-2}, l_j, 1 - l_1 - \dots - l_j)$$

for $j \geq 3$. Then $g_{i,j}$ is an invertible linear transformation on $\mathbb{R}^j$, and so if B is a Borel subset of $\mathbb{R}^j$ of Lebesgue measure zero, then $g_{i,j}^{-1}(B)$ also has Lebesgue measure zero. If A is a Borel subset of $\mathbb{R}^2$ with Lebesgue measure zero, we get

$$\mathbb{P}\big((\Sigma_J, R_J) \in A\big) \leq \sum_{j=2}^{\infty} \sum_{i=1}^{j-2} \mathbb{P}\big((L_1, \dots, L_j) \in g_{i,j}^{-1}(\mathbb{R}^{j-2} \times A)\big) = 0.$$

Hence, there exists a function g which serves as a joint density of (Σ_J, R_J). We can then let

$$g_r(l) = \frac{g(l, r)}{\int g(l', r)\, dl'}$$

be a version of the conditional density of Σ_J given $R_J = r$.

Using the construction of the stick breaking process and the independence of increments of X we deduce that

$$\tilde{L} = \left(\frac{L_{J+i}}{R_J}, i \geq 1\right)$$

is independent of $(L_{i \wedge J}, \Delta_{i \wedge J}, i \geq 1)$ and has the same law as L. Furthermore, the sequence $(\Delta_{J+i}, i \geq 1)$ is conditionally independent of $(L_{i \wedge J}, \Delta_{i \wedge J})$ given R_J.

We therefore obtain the decomposition

$$\rho = \Sigma_J + R_J \rho^J,$$

where

$$\rho^J = \sum_{i=1}^{\infty} \frac{L_{i+J}}{R_J} \mathbf{1}_{\Delta_{i+J} < 0} = \frac{\Sigma^J}{R_J}.$$

Since ρ^J is a function of $\tilde{L}$, $(\Delta_{J+i}, i \geq 1)$ and R_J, then ρ^J and Σ_J are conditionally independent given R_J. Hence g_{R_J} is also a version of the conditional density of Σ_J given R_J and ρ^J, and we can then write

$$\mathbb{P}(\rho \in dt) = dt \int g_r(t - ry) \mathbb{P}(R_J \in dr, \rho^J \in dy) \tag{10}$$

on $\{J < \infty\}$.

Finally, it remains to see that the density for ρ displayed in equation (10) is positive on $(0, 1)$. We remark that the density of (R_J, Σ_J) is positive on

$$\{(r, \sigma) : 0 < \sigma < 1 - r < 1\}.$$

Indeed, taking r, σ as in the preceding display, we have the explicit computation

$$\begin{aligned}
&\mathbb{P}(J = 2, \Sigma_J \in d\sigma, R_J \in dr) \\
&\quad = \mathbb{P}(\Delta_1 > 0, \Delta_2 < 0, L_2 \in d\sigma, 1 - L_1 - L_2 \in dr) \\
&\quad = \big(1 - f(1 - \sigma - r)\big) f(\sigma) \mathbf{1}_{0<\sigma<1-r<1} \frac{1}{1 - \sigma - r} \, dr \, d\sigma.
\end{aligned}$$

On the other hand, given $t \in (0, 1)$, $\mathbb{P}(R_J < 1 - t) > 0$. Indeed,

$$\begin{aligned}
\mathbb{P}(R_J < 1 - t) &\geq \mathbb{P}(R_J < 1 - t, J = 2) \\
&= \int\int \mathbb{P}(\Delta_1 \geq 0, \Delta_2 < 0, L_1 \in dl_1, 1 - L_1 - L_2 \in dl_2) \mathbf{1}_{l_2 \leq 1-t} \\
&= \int\int \big(1 - f(l_1)\big) f(1 - l_1 - l_2) \frac{1}{1 - l_1} \mathbf{1}_{0<l_2<1-l_1} \mathbf{1}_{l_2<1-t} \\
&> 0,
\end{aligned}$$

since f and $1 - f$ are strictly positive on $(0, 1)$ since 0 is regular for both half lines and so the support of the law of X_t is $\mathbb{R}$ for all $t > 0$. Going back to equation (10), we see that, given $t \in (0, 1)$, on the set $\{(r, y) : 0 < r < 1 - t\}$ we have $t - ry < t < 1 - r$, and so the density $g_r(t - ry)$ is positive. Hence the integral in equation (10) is positive.

2. The proof of absolute continuity of the time and value of the minimum of X on $[0, 1]$ is similar, except that further hypotheses are needed.

First, the value of the minimum of X on $[0, 1]$ has the same distribution as

$$m := \sum_{i=1}^{\infty} \Delta_i \mathbf{1}_{\Delta_i \leq 0}.$$

Since the law of X_t is absolutely continuous with respect to Lebesgue measure for all $t > 0$, we have

$$(\rho, m) = (\Sigma_J, m_J) + (R_J \rho^J, m^J),$$

where

$$m_J = \sum_{i \leq J} \Delta_i \mathbf{1}_{\Delta_i < 0} \quad \text{and} \quad m^J = \sum_{i=1}^{\infty} \Delta_{J+i} \mathbf{1}_{\Delta_{J+i} > 0}.$$

We now prove that:

(a) (ρ_J, m_J) has a conditional density with respect to R_J;
(b) (ρ_J, m_J) and (ρ^J, m^J) are conditionally independent given R_J.

The second assertion follows from our previous analysis of conditional independence in the sequences L and Δ. The first assertion follows from the fact that

$(\Sigma_J, R_J, \Delta_J)$ admit a density on $\{J = j\}$, by a computation similar to the one for (Σ_J, R_J)

$$\begin{aligned}
&\mathbb{P}(J = j, L_1 \in dl_1, \dots, L_j \in dl_j, \Delta_1 \in dx_1, \dots, \Delta_j \in dx_j) \\
&\quad = \sum_{i=1}^{j-1} \mathbf{1}_{x_1,\dots,x_{i-1}<0, x_i,\dots,x_{j-1}>0, x_j<0} \mathbb{P}(X_{l_1} \in dx_1) \cdots \mathbb{P}(X_{l_j} \in dx_j) \\
&\qquad \times \mathbb{P}(L_1 \in dl_1, \dots, L_i \in dl_j)
\end{aligned}$$

so that on $\{J = j\}$, $(L_1, \dots, L_J, \Delta_1, \dots, \Delta_J)$ admit a density with respect to Lebesgue measure, and since $(\Delta_J, R_J, \Delta_J)$ is the image under a surjective linear map of the former variables, the latter admit a joint density. Let f_r be a version of the conditional density of (Σ_J, Δ_J) given $R_J = r$. We then get

$$\begin{aligned}
&\mathbb{P}(\rho \in dt, m \in dx) \\
&\quad = dt\, dx \int f_r(t - rs, x - y) \mathbb{P}(\rho^J \in ds, m^J \in dy, R_J \in dr).
\end{aligned} \tag{11}$$

Regarding the equivalence of the law of (ρ, m) and Lebesgue measure on $(0, 1) \times (-\infty, 0)$, note that a version of the density of (R_J, ρ_J, m_J) is positive on

$$\{(r, s, x) : 0 \le r + s \le 1, x < 0\}.$$

Indeed, we have, for example,

$$\begin{aligned}
&\mathbb{P}(\Delta_1 < 0, \Delta_2 > 0, R_I \in dr, \Sigma_I \in ds, m_I \in dx) \\
&\quad = \mathbb{P}(X_s \in dx)\big(1 - f(1 - r - s)\big) \frac{1}{1 - s} \mathbf{1}_{0 \le r+s \le 1} \mathbf{1}_{x \le 0}.
\end{aligned}$$

Since the law of (ρ^J, m^J, R_J), by analogy with the case of ρ, is seen to charge the set $\{(s, y, r) : t < 1 - r, x < y\}$, we conclude that the expression for the joint density of (ρ, m) given in equation (11) is strictly positive.

3. The proof of the absolute continuity of $(\underline{X}_1, X_1 - \underline{X}_1)$ follows the same method of proof, starting with the fact that these random variables have the same joint law as

$$(\Delta^-, \Delta^+) = \sum_{i=1}^{\infty} \Delta_i (\mathbf{1}_{\Delta_i < 0}, \mathbf{1}_{\Delta_i > 0}),$$

which we can again decompose at the random index

$$I = \min\{i \ge 1 : \text{there exist } j, j \le i \text{ such that } \Delta_j < 0, \Delta_{j'} > 0\}$$

into

$$(\Delta^-, \Delta^+) = (\Delta_I^-, \Delta_I^+) + (\Delta^{-,I}, \Delta^{+,I}),$$

where

$$(\Delta_I^-, \Delta_I^-) = \sum_{i \le I} \Delta_i (\mathbf{1}_{\Delta_i < 0}, \mathbf{1}_{\Delta_i > 0}).$$

Since:

(a) $(R_I, \Delta_I^-, \Delta_I^+)$ have a joint density which can be taken positive on $(0, 1) \times (-\infty, 0) \times (0, \infty)$, and

(b) (Δ_I^-, Δ_I^+) and $(\Delta^{-,I}, \Delta^{+,I})$ are conditionally independent given R_I,

we see that (Δ^-, Δ^+) admit a joint density which can be taken positive on $(-\infty, 0) \times (0, \infty)$. □

6. An invariant path transformation for Lévy processes. The aim of this section is to prove Theorem 3. This will be done (almost) by applying the continuous mapping theorem to the embedded random walk $(X_{k/n}, k = 0, \dots, n)$ and a continuous function on Skorohod space. The argument's technicalities are better isolated by focusing first on some special cases in which the main idea stands out. Therefore, we first comment on the case when X has continuous sample paths, then we handle the case when X has paths of unbounded variation on compact intervals, to finally settle the general case.

We rely on a discrete version of Theorem 3, which was discovered by Abramson and Pitman (2011). Let $S^n = (S^n_t, t \in [0, n])$ be the process obtained by interpolating between the values of n steps of a random walks which jumps every $1/n$, and let C^n be its convex minorant. Let $V_0^n, V_1^n, \dots, V_k^n$ be the endpoints of the segments defining the convex minorant C^n. Let U_n be uniform on $\{1/n, \dots, 1\}$. Since there exists an unique j such that

$$U_n \in (V_j^n, V_{j+1}^n],$$

let us define

$$g_n = V_j^n \quad \text{and} \quad d_n = V_{j+1}^n$$

as the excursion interval of S^n above C^n which straddles U_n. Mimicking the definition of the path transformation (1), let us define

$$S_t^{n,U_n} = \begin{cases} S^n_{U_n+t} - S^n_{U_n}, & \text{if } 0 \le t \le d_n - U_n, \\ S^n_{d_n} - S^n_{U_n} + S^n_{g_n+t-(d_n-U_n)} - S^n_{g_n}, & \text{if } d_n - U_n \le t \le d_n - g_n, \\ S^n_{d_n} + S^n_{t-(d_n-g_n)}, & \text{if } d_n - g_n \le t \le d_n, \\ S^n_t, & \text{if } d_n \le t. \end{cases}$$

THEOREM 6 [Abramson and Pitman (2011)]. *If the distribution function of* $S^n_{1/n}$ *is continuous, then the pairs*

$$(U_n, S^n) \quad \textit{and} \quad (d_n - g_n, S^{n,U_n})$$

have the same law.

To prove Theorem 3 we will use Theorem 6 with the random walk obtained by sampling our Lévy process X at points of the form $1/n$ and take the limit as $n \to \infty$. The details are a bit technical in general but simplify considerably when X is continuous or when it reaches its convex minorant continuously.

The main tool for the passage to the limit is a lemma regarding approximation of the endpoints of the interval of the convex minorant that contains a given point. Let $f : [0, 1] \to \mathbb{R}$ be a càdlàg function which starts at zero and is left continuous at 1 and c its convex minorant. Let also $f^l = f \wedge f_-$ be the lower semicontinuous regularization of f, and define with it the excursion set away from the convex minorant $\mathscr{O} = \{c < f^l\}$. For all u belonging to the open set $\mathscr{O}$ we can define the quantities $g < u < d$ as the left and right endpoints of the excursion interval of $\mathscr{O}$ that contains u. We define the *slope* of c at u as the quantity

$$m_u = \frac{c(d) - c(g)}{d - g} = c'(u).$$

The notations $g_u(f), d_u(f)$ and $m_u(f)$ will be preferred when the function f or the point u are not clear from context. We will first be interested in continuity properties of the quantities g_u, d_u and m_u when varying the function f.

Recall that a sequence f_n in the space of càdlàg functions on $[0, 1]$ converges to f in the Skorohod J_1 topology if there exist a sequence of increasing homeomorphisms from $[0, 1]$ into itself such that $f_n - f \circ \lambda_n$ converges uniformly to 0 on $[0, 1]$.

LEMMA 2. *If*:

1. f *is continuous at* u;
2. $u \in \mathscr{O}$;
3. *the function*

$$f^l(t) - \frac{d - t}{d - g} f^l(g) + \frac{t - g}{d - g} f^l(d) \qquad \textit{for } t \in [0, 1]$$

is zero only on $\{g, d\}$;

4. $f_n \to f$ *in the Skorohod* J_1 *topology and* $u_n \to u$, *then*

$$g_{u_n}(f_n) \to g_u(f), \qquad d_{u_n}(f_n) \to d_u(f) \quad \textit{and} \quad m_{u_n}(f_n) \to m_u(f).$$

The proof is presented in Section 6.3. We now pass to the analysis of the particular cases when our Lévy process X has continuous paths, or when it reaches its convex minorant continuously.

6.1. *Brownian motion with drift.* In this subsection, we will prove Theorem 3 when X is a (nondeterministic) Lévy process with continuous paths, that is, a (nonzero multiple of) Brownian motion with drift.

Let f be a continuous function on [0, 1], and consider the continuous function $\varphi_u f$ given by

$$(12)\qquad \varphi_u f(t) = \begin{cases} f(u+t) - f(u), & \text{if } 0 \le d-u, \\ f(d) - f(u) + f\big(g + t - (d-u)\big) - f(g), & \\ & \text{if } d-u \le t \le d-g, \\ f(d) - f(g) + f\big(t - (d-g)\big), & \text{if } d-g \le t \le d, \\ f(t), & \text{if } t \le d. \end{cases}$$

If f, f_n, u and u_n satisfy the hypotheses of Lemma 2 (which implies that $f_n \to f$ uniformly), then $g(f_n) \to g(f)$ and $d(f_n) \to d(f)$. Therefore, it is simple to verify that $(u, f) \mapsto (d - g, \varphi_u f)$ is continuous at (u, f) when the space of continuous functions on [0, 1] is equipped with the uniform norm. When X is a Lévy process with continuous paths and distributions, that is, a Brownian motion with drift, consider its polygonal approximation with step $1/n$ obtained by setting

$$X^n_{k/n} = X_{k/n} \qquad \text{for } k \in \{0, 1, \dots, n\}$$

and extending this definition by linear interpolation on $[0, 1]$. Then $X^n \to X$ uniformly on $[0, 1]$; it is at this point that the continuity of the paths of X is important. Now, if U is uniform on $[0, 1]$ and independent of X, and we set $U_n = n\lceil U/n\rceil$, then $(d_n - g_n, \varphi_{U_n} X^n) \to (d - g, \varphi_U X)$. However, Theorem 6 says that (U_n, X^n) and $(d_n - g_n, \varphi_{U_n} X^n)$ have the same law. We conclude that (U, X) and $(d - g, \varphi_U X)$ have the same law, which is the conclusion of Theorem 3 in this case.

6.2. *Absence of jumps at the convex minorant.* In this subsection, we will prove Theorem 3 when X is a Lévy process of unbounded variation on compact sets [which automatically satisfies (CD)]. We now let f be a càdlàg function on $[0, 1]$ and let c stand for its convex minorant. We will suppose that f is continuous on the set $\{c = f^l\}$, which holds whenever f is the typical trajectory of X, thanks to 2 of Proposition 1.

Again, for all $u \in \{c < f\} = \{c < f \wedge f_-\} = \mathscr{O}$ we define g and d as the left and right endpoints of the excursion interval that contains u. Since f has jumps, its polygonal approximation does not converge to it in Skorohod space, but if we define

$$f_n(t) = f([nt]/n),$$

then f_n converges in the Skorohod J_1 topology to f as $n \to \infty$; cf. Billingsley (1999), Chapter 2, Lemma 3, page 127. This will called the piecewise constant approximation to f with span $1/n$ and is the way we will choose to approximate a Lévy process when it has jumps. The first complication in this case is that the discrete invariant path transformation was defined for the polygonal approximation and not for the piecewise constant approximation to our Lévy process. For this reason, we will have to define a more flexible path transformation than in the continuous case: for $u_1 < u_2 < u_3 \in (0, 1)$, we define $\varphi_{u_1,u_2,u_3} f$ by

$$\varphi_{u_1,u_2,u_3} f(t) = \begin{cases} f(u_2+t) - f(u_2), & 0 \le t < u_3 - u_2, \\ f(u_3) - f(u_2) + f\big(u_1 + t - (u_3 - u_2)\big) - f(u_1), & \\ & u_3 - u_2 \le t \le u_3 - u_1, \\ f(u_3) - f(u_1) + f\big(t - (u_3 - u_1)\big), & \\ & u_3 - u_1 \le t < u_3, \\ f(t), & u_3 \le t. \end{cases} \tag{13}$$

The path transformation φ_u of (12) corresponds to $\varphi_{g,u,d}$. We are interested in $\varphi_{g,U,d} X$, which will be approximated by $\varphi_{\tilde{g}_n, U_n, \tilde{d}_n} X^n$ where $\tilde{g}_n$ and $\tilde{d}_n$ are the left and right endpoints of the excursion of the polygonal approximation to X of span $1/n$ which contains $U_n = \lceil Un \rceil / n$, and X^n is the piecewise constant approximation to X with span $1/n$. We are forced to use both the vertices of the convex minorant of the polygonal approximation and the piecewise constant approximation, since $X^n \to X$ (in the Skorohod J_1 topology), but with $(\tilde{g}_n, \tilde{d}_n)$ we can define a nice invariant transformation: Theorem 6 asserts that

$$(U_n, \varphi_{\tilde{g}_n, U_n, \tilde{d}_n} X^n) \quad \text{and} \quad (\tilde{d}_n - \tilde{g}_n, X^n)$$

have the same law. Indeed, Theorem 6 is an assertion about the increments of a random walk and the polygonal and piecewise approximations to X of span $1/n$ are constructed from the same increments.

Lemma 2 tells us that $(\tilde{g}_n, \tilde{d}_n) \to (d, g)$. It is therefore no surprise that

$$\varphi_{\tilde{g}_n, U_n, \tilde{d}_n} X^n \to \varphi_{g,U,d} X,$$

telling us that (U, X) and $(d - g, \varphi_{g,U,d} X)$ have the same law whenever X satisfies (CD) and has unbounded variation on finite intervals. Convergence follows from the following continuity assertion:

LEMMA 3. *If f is continuous at (u_1, u_2, u_3), $f_n \to f$ in the Skorohod J_1 topology, and $u_i^n \to u_i$ for $i = 1, 2, 3$, then*

$$\varphi_{u_1^n, u_2^n, u_3^n} f_n \to \varphi_{u_1,u_2,u_3} f.$$

Lemma 3 is an immediate consequence of the following convergence criterion found in Ethier and Kurtz (1986), Proposition III.6.5, page 125.

PROPOSITION 2. *A sequence f_n of càdlàg functions on $[0, 1]$ converges to f in the Skorohod J_1 topology if and only if for every sequence $(t_n) \subset [0, 1]$ converging to t*:

1. $|f_n(t_n) - f(t)| \wedge |f_n(t_n) - f(t-)| \to 0$;
2. *if* $|f_n(t_n) - f(t)| \to 0$, $t_n \le s_n \to t$, *then* $|f_n(s_n) - f(t)| \to 0$;

3. *if* $|f_n(t_n) - f(t-)| \to 0$, $s_n \le t_n$ *and* $s_n \to t$, *then* $|f_n(s_n) - f(t)| \to 0$.

In particular, we see that if f is continuous at t, then $f_n(t_n) \to f(t)$. The above criterion is clearly necessary for convergence since if $f_n \to f$, then there exist a sequence $(\lambda_n, n \in \mathbb{N})$ of increasing homeomorphisms of $[0, 1]$ into itself such that $f_n - f \circ \lambda_n$ converges to zero uniformly. If $t_n \to t$, then $f_n(t_n)$ will be close to either $f(t-)$ or $f(t)$ depending on if $\lambda_n(t_n) < t$ or $\lambda_n(t_n) \ge t$. By using the above criterion, we focus on the real problem for continuity for the transformation φ_{u_1,u_2,u_3}, namely, that nothing wrong happens at $u_3 - u_2$, $u_3 - u_1$ and u_3.

PROOF OF LEMMA 3. Let us prove that for every $t \in [0, 1]$, the conditions of Proposition 2 hold for $\varphi_{u_1^n,u_2^n,u_3^n} f_n$ and $\varphi_{u_1,u_2,u_3} f$.

Let λ_n be increasing homeomorphisms of $[0, 1]$ into itself such that

$$f_n - f \circ \lambda_n \to 0$$

uniformly. We proceed by cases.

$t < u_3$. Eventually $t < u_3^n$, so that $\varphi_{u_1^n,u_2^n,u_3^n} f_n(t) = f_n(t)$ and $\varphi_{u_1,u_2,u_3} f(t) = f(t)$. Since f_n and f satisfy the conditions of Proposition 2 at time t, the same holds for their images under the path transformation.

$t < u_3 - u_2$. Eventually $t < u_3^n - u_2^n$ so that

$$\varphi_{u_1,u_2,u_3} f(t) = f(u_2 + t) - f(u_2) \quad \text{and} \quad \varphi_{u_1^n,u_2^n,u_3^n} f^n(t) = f^n(u_2^n + t) - f^n(u_2^n).$$

Since f is continuous at u_2, Proposition 2 implies that $f^n(u_2^n) \to f(u)$, so that $\varphi_{u_1^n,u_2^n,u_3^n} f^n(t)$ can be made arbitrarily close to either $\varphi_{u_1,u_2,u_3} f(t)$ or $\varphi_{u_1,u_2,u_3} f(t-)$ depending on if

$$u_n + t_n < \lambda_n^{-1}(u + t) \quad \text{or} \quad u_n + t_n \ge \lambda_n^{-1}(u + t).$$

$t \in (u_3 - u_2, u_3) \setminus \{u_3 - u_1\}$ is analogous to the preceding case. For $t \in \{u_3 - u_2, u_3 - u_1, u_3\}$ set

$$v_1 = u_3 - u_2, \qquad v_2 = u_3 - u_1 \quad \text{and} \quad v_3 = u_3.$$

Since f is continuous at u_3, condition 3 gives

$$f_n(u_i^n) \to f(u_i) \qquad \text{for } i = 1, 2, 3,$$

and so

$$\varphi_{u_1,u_2,u_3} f(v_i^n) \to \varphi_{u_1,u_2,u_3} f(v_i) \qquad \text{for } i = 1, 2, 3. \qquad \square$$

6.3. *The general case.* In this subsection, we prove Theorem 3 for a Lévy process X under the sole assumption (CD).

The challenge to overcome in the remaining case, in which X can jump into and out of the convex minorant, is to show how one can handle the jumps; although a result in the vein of Lemma 3 will play a prominent role in our analysis, a more careful inspection of how g_n differs from g is needed in order to sort the following problem: in general, the operation of rearranging pieces of càdlàg paths is not continuous and depends sensitively on the points at which the rearrangement is made. A simple example helps to clarify this: consider $f = \mathbf{1}_{[1/3,1]} + \mathbf{1}_{[2/3,1]}$, so that if $u_1 = 1/3, u_2 = 1/2$ and $u_3 = 2/3$, we have $\varphi_{u_1,u_2,u_3} f = \mathbf{1}_{[1/6,1]} + \mathbf{1}_{[2/3,1]}$. Note that if $u_1^n \to u_1$ and $u_1^n \in (0, 1/2)$, then

$$\varphi_{u_1^n,u_2,u_3} f = \begin{cases} \mathbf{1}_{[1/6,1]} + \mathbf{1}_{[1/6+1/3-u_1^n,1]}, & \text{if } u_1^n \in (0, 1/3], \\ \mathbf{1}_{[1/6,1]} + \mathbf{1}_{[1-u_1^n,1]}, & \text{if } u_1^n \in [1/3, 1/2). \end{cases}$$

We conclude that $\varphi_{u_1^n,u_2,u_3} f \to \varphi_{u_1,u_2,u_3} f$ if and only if $u_1^n \geq 1/3$ eventually.

Let $f : [0, 1] \to \mathbb{R}$ be a càdlàg function which starts at zero and c its convex minorant on $[0, 1]$. Let also $f^l = f \wedge f_-$ be the lower semicontinuous regularization of f. As before, the component intervals of the open set $\mathscr{O} = \{c < f^l\}$ are called the excursion intervals of f, and that for $u \in \mathscr{O}$, (g, d) is the excursion interval that contains u.

We first give the proof of Lemma 2; the proof depends on another lemma with a visual appeal, which is to be complemented with Figure 3.

LEMMA 4. *If for a càdlàg function* $f : [0, 1] \to \mathbb{R}$:

1. *there exist closed intervals* A *and* B *in* $[0, 1]$ *such that* $\inf B - \sup A > 0$ *and*
2. *there exists* $\delta > 0$ *and*

$$h < \delta \frac{\inf B - \sup A}{\inf B \vee (1 - \sup A)}$$

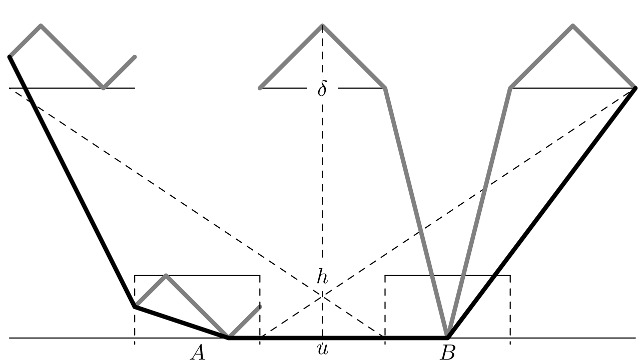

FIG. 3. *Visual content of Lemma 4.*

such that

$$f > \delta \text{ on } [0,1] \setminus A \cup B \quad \text{and} \quad \min_{x \in A \cup B} f^l(x) < h,$$

then for all $u \in (\sup A, \inf B)$,

$$g_u \in A, \qquad d_u \in B \quad \text{and} \quad m_u \le \frac{h}{\inf B - \sup A}.$$

PROOF. This assertion can be checked by cases. We consider 3 possible positions for g_u and three other for $d_u : g_u < \inf A$, $g_u \in A$ and $g_u \in (\sup A, u)$ and similarly $d_u \in (u, \inf B)$, $d_u \in B$ and $d_u > \sup B$. We number each from 1 to 3 and write $C_{i,j}$ for the corresponding case. We trivially discard the cases

$$C_{1,1}, C_{1,3}, C_{3,1}, C_{3,3}$$

for each one would force $c(g)$ to be above the zero slope line through $(0, \delta)$, hence to pass above g on A and B. The case $C_{2,1}$ would force c (hence f) to be above δ on B while $C_{3,2}$ would force f to be above δ on A, hence both are discarded. We finally discard the case $C_{2,3}$ (and by a similar argument $C_{3,2}$) because of our choice of h, since a line from a point of $A \times [0, h]$ to $[\sup B, 1] \times [\delta, \infty)$ passes above h on B. □

PROOF OF LEMMA 2. Set $u \in \{c < f^l\}$, and write g and d for $g_u(f)$ and $d_u(f)$ so that $g < u < d$. Recall that c is linear on $(g_u(f), d_u(f))$. By considering instead

$$t \mapsto f(t) - \frac{d-t}{d-g} f^l(g) + \frac{t-g}{d-g} f^l(d) \quad \text{and}$$
$$t \mapsto f_n(t) - \frac{d-t}{d-g} f^l(g) + \frac{t-g}{d-g} f^l(d),$$

our assumptions allow us to reduce to the case

$$f^l(g) = f^l(d) = 0 \quad \text{and} \quad f^l > 0 \qquad \text{on } [0,1] \setminus \{g, d\}.$$

We will now consider the case $0 < g < d < 1$, the cases $g = 0$ or $d = 1$ being handled similarly.

For every

$$\varepsilon < g \wedge (1-d) \wedge \frac{d-g}{2}$$

we can define

$$\begin{aligned}\delta(\varepsilon) &= \inf\{f(t) : t \in [0, g-\varepsilon] \cup [g+\varepsilon, d-\varepsilon] \cup [d+\varepsilon, 1]\} \\ &= \min\{f^l(t) : t \in [0, g-\varepsilon] \cup [g+\varepsilon, d-\varepsilon] \cup [d+\varepsilon, 1]\}.\end{aligned}$$

Then $\delta(\varepsilon) > 0$ for $\varepsilon > 0$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $g < u < d$, we can choose ε small enough so that

$$u \in (g + \varepsilon, d - \varepsilon).$$

Since $f_n \to f$, there exists a sequence of increasing homeomorphisms λ_n of [0, 1] converging uniformly to the identity such that

$$f_n - f \circ \lambda_n$$

converges uniformly to zero. (If f is continuous, λ_n can be taken equal to the identity function.)

Also, given h_n eventually bounded away from 0,

$$\min\{f^l(t) : t \in (g - \varepsilon, g + \varepsilon)\} < h_n \quad \text{and} \quad \min\{f^l(t) : t \in (d - \varepsilon, d + \varepsilon)\} < h_n$$

for large enough n. Hence

$$\min\{f_n^l(t) : t \in (\lambda_n^{-1}(g - \varepsilon), \lambda_n^{-1}(g + \varepsilon))\} < h_n$$

and

$$\min\{f_n^l(t) : t \in (\lambda_n^{-1}(d - \varepsilon), \lambda_n^{-1}(d + \varepsilon))\} < h_n$$

for large enough n. The particular h_n we will consider is

$$\begin{aligned} h_n &= \delta(\varepsilon) \frac{\lambda_n^{-1}(d - \varepsilon) - \lambda_n^{-1}(g + \varepsilon)}{\lambda_n^{-1}(d - \varepsilon) \vee (1 - \lambda_n^{-1}(g + \varepsilon))} \\ &\to \delta(\varepsilon) \frac{(d - g - 2\varepsilon)}{((d - \varepsilon) \vee (1 - g - \varepsilon))} > 0 \end{aligned}$$

which is eventually positive. Since $f > \delta(\varepsilon)$ on $[0, g - \varepsilon] \cup [g + \varepsilon, d - \varepsilon] \cup [d + \varepsilon, 1]$, then

$$f_n > \delta \qquad \text{on } [0, \lambda_n^{-1}(g - \varepsilon)] \cup [\lambda_n^{-1}(g + \varepsilon), \lambda_n^{-1}(d - \varepsilon)] \cup [\lambda_n^{-1}(d + \varepsilon), 1],$$

and Lemma 4 now tells us that

$$\begin{aligned} g_{u_n}(f_n) &\in (\lambda_n^{-1}(g - \varepsilon), \lambda_n^{-1}(g + \varepsilon)), \\ d_{u_n}(f_n) &\in (\lambda_n^{-1}(d - \varepsilon), \lambda_n^{-1}(d + \varepsilon)) \end{aligned}$$

and

$$m_{u_n}(g_n) \leq h_n / (\lambda_n(d - \varepsilon) - \lambda_n(g + \varepsilon)),$$

so that eventually

$$\begin{aligned} &g_{u_n}(f_n) \in (g - 2\varepsilon, g + 2\varepsilon), \qquad d_{u_n}(f_n) \in (d - 2\varepsilon, d + 2\varepsilon) \quad \text{and} \\ &m_{u_n}(f_n) \leq 2\delta / (d - \varepsilon) \vee (1 - g - \varepsilon). \end{aligned}$$

□

REMARK. In the context of the above proof, if we suppose that $f(g-) = c(g) < f(g)$ and $f(d-) = c(d)$, then for h_n eventually bounded away from zero, we actually have

$$\min\{f^l(t) : t \in [g - \varepsilon, g)\} < h_n$$

for large enough n, and so we get

$$g_{u_n}(f_n) < \lambda_n^{-1}(g).$$

This remark is crucial to the proof of Theorem 3.

REMARK. Let c_n be the convex minorant of f_n. Under the hypotheses of Lemma 2, we can actually deduce that if $t_n \to g$, then $c_n(t_n) \to c(g)$, while if $t_n \to d$, then $c_n(t_n) \to c(d)$. This is because of the following result about convergence of convex functions.

PROPOSITION 3. *If c_n and c are convex functions on $[0, 1]$, for some $a \in (0, 1)$ we have $c_n(a) \to c(a)$, and if the two sequences $(c_n(0))$ and $(c_n(1))$ are bounded, then for every sequence $a_n \to a$ we have $c_n(a_n) \to c(a)$.*

PROOF. If $a_n \le a$, we can use the inequalities

$$c_n(a_n) \le c_n(a)\frac{a_n}{a} + c_n(0)\frac{a - a_n}{a}$$

and

$$c_n(a_n) \ge c_n(a)\frac{1 - a}{1 - a_n} + c_n(1)\frac{a - a_n}{1 - a_n}.$$

We get an analogous pair of inequalities when $a_n \ge a$, which allows us to conclude that the sequence $(c_n(a) - c_n(a_n))$ goes to zero. □

Given $u_1 < u_2 < u_3$, we now define a new càdlàg function $\psi_{u_1,u_2,u_3} f$ as follows:

$$\psi_{u_1,u_2,u_3} f(t) = \begin{cases} f(u_2 + t) - f(u_2), & 0 \le t < u_3 - u_2, \\ c(u_3) - c(u_1) + f\big(u_1 + t - (u_3 - u_2)\big) - f(u_2), & \\ & u_3 - u_2 \le t \le u_3 - u_1, \\ c(u_3) - c(u_1) + f\big(t - (u_3 - u_1)\big), & u_3 - u_1 \le t < u_3, \\ f(t), & u_3 \le t. \end{cases}$$

The difference with the path transformations of (12) and (13) is that we now use the convex minorant c instead of only the function f. This has the effect of choosing where to place the jumps that f might make as it approaches its convex minorant. Note, however, that $\psi_{u_1,u_2,u_3} f = \varphi_{u_1,u_2,u_3} f$ if $f = c$ at u_1 and u_3.

Our next task will be to analyze the continuity of $f \mapsto \psi_{g,u,d} f$ on Skorohod space, with special emphasis on the approximations we will use. For every n, f_n and $\tilde{f}_n$ will be the piecewise constant and polygonal approximations to f with span $1/n$, we set $u_n = \lceil nu \rceil / n$, and

$$g_n = g_{u_n}(f_n), \qquad d_n = d_{u_n}(f_n), \qquad \tilde{g}_n = g_{u_n}(\tilde{f}_n) \quad \text{and} \quad \tilde{d}_n = d_{u_n}(\tilde{f}_n).$$

LEMMA 5. *Under the hypotheses of Lemma 2, if either*

$$f(g) = c(g) \quad \textit{and} \quad f(d) = c(d) \quad \textit{or} \quad f(g-) = c(g) \quad \textit{and} \quad f(d-) = c(d),$$

then

$$\psi_{\tilde{g}_n, u_n, \tilde{d}_n} f_n \to \psi_{g,u,d} f$$

in the Skorohod J_1 topology.

PROOF. Since we have already analyzed what happens when f is continuous at g and d, the essence of the argument will be illustrated when

$$f(g) = c(g) < f(g-) \quad \text{and} \quad f(d) = c(d) < f(d-).$$

As in the proof of Lemma 3, we verify that for every $t \in [0, 1]$, the conditions of Proposition 2 hold for $\psi_{\tilde{g}_n, u_n, \tilde{d}_n} f_n$ and $\psi_{d,u,g} f$ at time t.

Let λ_n be a sequence of increasing homeomorphisms of [0, 1] such that

$$f_n - f \circ \lambda_n \to 0$$

uniformly. The crucial part of the argument is to use the remarks after Lemma 2 from which we deduce that

$$\lambda_n^{-1}(g) \le g_n \quad \text{and} \quad \lambda_n^{-1}(d) \le d_n.$$

Since f_n is the piecewise constant approximation to f, then λ_n^{-1} must eventually take g and d to $[ng]/n$ and $[nd]/n$. But comparing the convex minorants of the piecewise constant and polygonal approximations to f with span $1/n$ leads to

$$g_n - 1/n \le \tilde{g}_n \quad \text{and} \quad d_n - 1/n \le \tilde{d}_n$$

so that

$$\lambda_n^{-1}(g) \le \tilde{g}_n \quad \text{and} \quad \lambda_n^{-1}(d) \le \tilde{d}_n.$$

Again using the remarks after the proof of Proposition 13, we see that

$$c_n(\tilde{g}_n) \to c(g) \quad \text{and} \quad c_n(\tilde{d}_n) \to c(d).$$

The conditions of Proposition 2 can now be verified at times $t \in [0, 1] \setminus d - u, d$ as in the proof of Lemma 13, while for $t \in \{d - u, d\}$, the proof is similar and hence will be illustrated when $t = d - g$. Since $f_n - f \circ \lambda_n \to 0$ uniformly, the jump of f

at g is approximated by the jump of f_n at $\lambda_n^{-1}(g)$. We reduce to cases by taking subsequences: when $t_n > \tilde{d}_n - u_n$ for all n, then $t_n + u_n > \lambda_n^{-1}(d)$ so that

$$\psi_{\tilde{g}_n, u_n, \tilde{d}_n} f(t_n) \to f(d) - f(g) + f(g) - f(u) = f(d) - f(u).$$

On the other hand, when $t_n \leq d_n - u_n$ for all n, we see that

$$\psi_{\tilde{g}_n, u_n, \tilde{d}_n} f(t_n) \text{ is close to } f(d-) - f(u) \text{ or } f(d) - f(u)$$

depending on if

$$t_n + u_n < \lambda_n^{-1}(d) \quad \text{or} \quad t_n + u_n \geq \lambda_n^{-1}(d).$$

Hence, the conditions of Proposition 2 are satisified at $t = d - u$. $\square$

We finally proceed to the proof of Theorem 3.

PROOF OF THEOREM 3. Thanks to Proposition 1, X almost surely satisfies the conditions of Lemma 5 at U. Hence, $(d_n - g_n, \psi_{\tilde{g}_n, U_n, \tilde{d}_n}(X^n))$ converges in law to $(d - g, \psi_{d,U,g} X)$ thanks to Lemmas 2 and 5, as well as the continuous mapping theorem. Since (U_n, X^n) converges in law to (U, X) and the laws of (U_n, X^n) and $(d_n - g_n, \psi_{\tilde{g}_n, U_n, \tilde{d}_n}(X_n))$ are equal by Theorem 6, then (U, X) and $(d - g, X^U)$ have the same law. $\square$

7. Excursions above the convex minorant on a fixed interval. In this section we will prove Theorem 4, which states the equality in law between two sequences. We recall the setting: X is a Lévy process such that X_t has a continuous distribution for every $t > 0$, C is its convex minorant on $[0, 1]$, $X^l = X \wedge X_-$ is the lower semicontinuous regularization of X, $\mathscr{O} = \{C < X^l\}$ is the excursion set, $\mathscr{I}$ is the set of excursion intervals of $\mathscr{O}$, for each $(g, d) \in \mathscr{I}$, and we let $e^{(g,d)}$ be the excursion associated to (g, d) given by

$$e_s^{(g,d)} = X_{(g+s)\wedge d} - C_{(g+s)\wedge d}.$$

We ordered the excursion intervals to state Theorem 1 by sampling them with an independent sequence of uniform random variables on $[0, t]$.

The first sequence of interest is

$$\big((d_i - g_i, C_{d_i} - C_{g_i}, e^{(g_i, d_i)}), i \geq 1\big).$$

The second sequence is obtained with the aid of an independent stick-breaking process and the Vervaat transformation. Recall that $V_t f$ stands for the Vervaat transform of f on $[0, t]$. Let $V_1, V_2, \ldots$ be an i.i.d. sequence of uniform random variables on $(0, 1)$, and construct

$$L_1 = V_1, \qquad L_n = V_n(1 - V_1) \cdots (1 - V_{n-1}) \quad \text{and} \quad S_i = L_1 + \cdots + L_i.$$

This sequence helps us to break up the paths of X into the independent pieces Y^i, $i = 1, 2, \dots$ given by

$$Y_t^i = X_{S_{i-1}+t} - X_{S_{i-1}}, \qquad 0 \le t \le L_i,$$

from which we can define the sequence of Knight bridges,

$$K_t^i = Y_t^i - \frac{t}{L_i} Y_{L_i}^i, \qquad 0 \le t \le L_i.$$

Our second sequence is

$$\bigl((L_i, X_{S_i} - X_{S_{i-1}}, V_{L_i}(K^i)), i \ge 1\bigr).$$

To prove the equality in law, we will use Theorem 3 to obtain a process $\tilde{X}$ which has the same law as X, as well as a stick-breaking sequence $\tilde{L}$ independent of $\tilde{X}$ such that, with analogous notation, the pointwise equality

$$\bigl((d_i - g_i, C_{d_i} - C_{g_i}, e^{(g_i, d_i)}), i \ge 1\bigr) = \bigl((L_i, \tilde{X}_{\tilde{S}_i} - \tilde{X}_{\tilde{S}_{i-1}}, V_{\tilde{L}_i}(\tilde{K}^i)), i \ge 1\bigr)$$

holds. This proves Theorem 4.

Let us start with the construction of $\tilde{X}$ and $\tilde{L}$. Apart from our original Lévy process X, consider an i.i.d. sequence of uniform random variables $U_1, U_2, \dots$ independent of X. Consider first the connected component (g_1, d_1) of $\{C < X \wedge X_-\}$ which contains U_1 and let X^1 be the result of applying the path transformation of Theorem 3 to X at the points g_1, U_1 and d_1. We have then seen that $\tilde{V}_1 = d_1 - g_1$ is uniform on [0, 1] and independent of X^1. Set $\tilde{S}_0 = 0$ and $\tilde{L}_1 = \tilde{V}_1$.

Consider now the convex minorant C^1 of

$$Z^1 = X^1_{\tilde{L}_1 + \cdot} - X^1_{\tilde{L}_1}$$

on $[0, 1 - \tilde{L}_1]$: we assert that it is obtainable from the graph of C by erasing the interval (g_1, d_1) and closing up the gap, arranging for continuity. Formally, we assert the equality

$$C_t^1 = \begin{cases} C_t, & \text{if } t \in [0, g_1), \\ C_{t - g_1 + d_1} - (C_{d_1} - C_{g_1}), & \text{if } t \in [g_1, 1 - \tilde{L}_1]. \end{cases}$$

Note that C^1 is continuous on $[0, 1 - \tilde{L}_1]$ by construction and it is convex by a simple analysis. To see that C^1 is the convex minorant of Z^1, we only need to prove that at g_1 it coincides with $Z^1_{g_1} \wedge Z^1_{g_1-}$; cf. Figure 2 to see how it might go wrong. If $X_{d_1} = C_{d_1}$, then

$$Z^1_{g_1} = X_{d_1} - (C_{d_1} - C_{g_1}) = C_{g_1} = C^1_{d_1},$$

while if $X_{d_1-} = C(d_1) < X_{d_1}$, then property 2 of Proposition 1 implies that $X_{g_1-} = C(g_1)$ and

$$Z^1_{d_1-} = X^1_{d_1-} - X^1_{\tilde{L}_1} = C_{d_1} - C_{g_1} + X_{g_1-} - (C_{d_1} - C_{g_1}) = C_{g_1} = C^1_{g_1}.$$

Let (g_2, d_2) be the connected component of $\{C^1 < Z^1\} \subset [0, 1 - \tilde{L}_1]$ that contains $U_2(1 - \tilde{L}_1)$ and define

$$\tilde{L}_2 = d_2 - g_2, \qquad \tilde{V}_2 = \frac{d_2 - g_2}{1 - \tilde{V}_1}$$

as well as the process X^2 which will be the concatenation of X^1 on $[0, \tilde{V}_1]$ as well as the path transformation of Z^1 on $[0, 1 - \tilde{V}_1]$; that is, Z^1 transformed according to the path transformation of Theorem 3 with parameters $g_2, U_2(1 - \tilde{L}_1), d_2$. From Theorem 3 and the independence of

$$Z^2 = X^1_{\cdot + \tilde{V}_1} - X^1_{\tilde{V}_1} \quad \text{and} \quad X^1_{\cdot \wedge \tilde{V}_1}$$

we see that:

1. X^2 has the same law as X^1;
2. $\tilde{V}_1$ and $\tilde{V}_2$ are independent of X^2, and $\tilde{V}_2$ is independent of $\tilde{V}_1$ and has an uniform distribution on $(0, 1)$;
3. the convex minorant C^2 of Z^2 on $[0, 1 - \tilde{L}_1 - \tilde{L}_2]$ is obtained from C^1 by deleting the interval (g_2, d_2) and closing up the gap arranging for continuity.

Now it is clear how to continue the recursive procedure to obtain, at step n a sequence $\tilde{V}_1, \dots, \tilde{V}_n$ and a process X^n such that if $\tilde{L}_n = \tilde{V}_n(1 - \tilde{V}_{n-1}) \cdots (1 - \tilde{V}_1)$ and $\tilde{S}_n = \tilde{L}_1 + \cdots + \tilde{L}_n$; then:

1. X^n has the same law as X.
2. X^n, $\tilde{V}_1, \dots, \tilde{V}_n$ are independent an the latter n variables are uniform on $(0, 1)$.
3. Let C^n is the convex minorant of

$$Z^n = X^n_{\tilde{S}_n + \cdot} - X^n_{\tilde{S}_n}$$

on $[0, 1 - \tilde{S}_n]$. Then C^n is obtained from C^{n-1} by removing the selected interval (g_n, d_n) and closing up the gap arranging for continuity.
4. X^n coincides with X^{n-1} on $[0, \tilde{S}_{n-1}]$.

From property 4 above, it is clear that X^n converges pointwise on $[0, 1]$ almost surely: it clearly does on $[0, 1)$ and $X^n_1 = X_1$. Also, we see that $\tilde{X}$ has the same law as X and that it is independent of $V_1, V_2, \dots,$ which is an i.i.d. sequence of uniform random variables.

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