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LARGE DEVIATIONS OF THE EMPIRICAL CURRENTS FOR A BOUNDARY-DRIVEN REACTION DIFFUSION MODEL[1]

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We derive a large deviation principle for the empirical currents of lattice gas dynamics which combine a fast stirring mechanism (Symmetric Simple Exclusion Process) and creation/annihilation mechanisms (Glauber dynamics). Previous results on the density large deviations can be recovered from this general large deviation principle. The contribution of external driving forces due to reservoirs at the boundary of the system is also taken into account.

1. Introduction. A major challenge in nonequilibrium statistical physics is to provide a counterpart to the notion of free energy and to the Gibbs measure which is the cornerstone of the equilibrium statistical physics. Large deviation principles have been proposed as a good alternative to investigate properties of nonequilibrium systems [4, 11, 16]. In particular, a lot of attention has been devoted to the case of lattice gas dynamics for which explicit large deviation principles can be derived in the hydrodynamic scaling (we refer the reader to [4, 11] for recent surveys and further references). Motivated by this recent progress in physics, the original mathematical works [12, 19] on the hydrodynamic large deviations for conservative dynamics have been generalized to take into account the contribution of reservoirs at the boundary of the system [2, 5, 15] and the current of particles flowing through the system [3].

An interesting class of models has been introduced in [8, 9] to describe reaction diffusion equations by combining the Symmetric Simple Exclusion Process (SSEP) to a Glauber dynamics which models the annihilation and creation of particles. In [8, 9], the hydrodynamic limit as well as the fluctuations of the density have been investigated for these models. The density hydrodynamic large deviations have then been proved in [17]. In this paper, we generalize this result by deriving the joint large deviations of the density and of the (conservative and nonconservative) currents flowing in the system. We also take into account the contribution of reservoirs acting at the boundary of the system. Our results were

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motivated by the recent research in nonequilibrium statistical physics on dissipative dynamics and in particular on granular media [1, 21, 23]. We refer to [7] for a more comprehensive discussion on the physical aspects of the large deviations for dissipative systems.

Contrary to the purely conservative dynamics [3], one has to introduce two types of currents: the conservative integrated current Q_t which records the particle jumps from the diffusive part of the dynamics (SSEP) and the nonconservative integrated current K_t associated to the creation annihilation process (Glauber). Heuristically, if one denotes by $\dot{Q}_t(r)$ and $\dot{K}_t(r)$ the instantaneous currents at time t and location r, then the density obeys the following equation:

$$\partial_t \rho_t(r) = -\partial_r \dot{Q}_t(r) + \dot{K}_t(r).$$

The goal of this paper is to compute the asymptotic cost of observing an atypical trajectory of the currents Q, K and of the density ρ when the number of particles tends to infinity. Even so it is apparently more complicated to consider the joint deviations of the empirical currents and the empirical density, it turns out that the structure of the joint large deviation functional I_0 is more transparent as it splits into two distinct contributions involving either the diffusive part or the Glauber part of the dynamics

$$I_0(\rho, Q, K) = I_1(\rho, Q) + I_2(\rho, K).$$

The precise form of the functional I_0 can be found in (2.12). In Section 6, the density large deviation functional derived in [17] is recovered by a contraction principle. This provides a natural interpretation of the density large deviation functional as the optimal combination between the two macroscopic currents Q, K in order to create the atypical density trajectory ρ at a minimal cost $I_1 + I_2$.

Our proof relies on the standard machinery developed to study hydrodynamic large deviations [18, 19], as well as on more recent tools introduced in [2, 3, 5, 15]. Therefore in this paper, we will not detail the aspects of the proof which can be deduced readily from the existing literature, and we will focus on the new features occurring from the nonconservative part of the dynamics. The paper is organized as follows. In Section 2, we introduce the model and state the main results. A strong form of local equilibrium is stated in Section 3. The upper and lower bound of the joint density/current large deviations are derived in Sections 4 and 5. Finally the density large deviations are recovered in Section 6.

2. Notation and results.

2.1. *The microscopic dynamics and notation.* We consider the one-dimensional Symmetric Simple Exclusion Process (SSEP) in the domain $\{-N, \dots, N\}$ with creation and annihilation of particles in the bulk and reservoirs at the boundaries. More precisely, the particles perform random walks with an exclusion constraint which imposes at most one particle per site, and particles can be removed or created in the bulk according to a rate which depends on the local configurations. At the boundaries $\pm N$, two reservoirs maintain constant densities. The dynamics can be viewed as a toy model for chemical reactions where the chemicals are injected at the boundaries, then diffuse and react in the system [8, 9].

The stochastic dynamics is a Markov process on $\{0,1\}^{2N+1}$ whose generator is obtained by adding the generators of the different dynamics

$$L_N = \frac{N^2}{2} L_{0,N} + \frac{N^2}{2} L_{+,N} + \frac{N^2}{2} L_{-,N} + L_{1,N} \tag{2.1}$$

with the SSEP generator

$$L_{0,N} f(\eta) = \sum_{x=-N}^{N-1} [f(\eta^{x,x+1}) - f(\eta)],$$

and creation and annihilation generators at the boundaries depending on the parameters β^+ and β^-

$$L_{+,N} f(\eta) = \big[\eta(N) + \beta_+\big(1 - \eta(N)\big)\big][f(\eta^N) - f(\eta)],$$
$$L_{-,N} f(\eta) = \big[\eta(-N) + \beta_-\big(1 - \eta(-N)\big)\big][f(\eta^{-N}) - f(\eta)],$$

where

$$\eta^x(z) = \begin{cases} \eta(z), & \text{if } z \neq x, \\ 1 - \eta(z), & \text{if } z = x; \end{cases} \qquad \eta^{x,y}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{else.} \end{cases}$$

Finally the creation and annihilation generator in the bulk is given by

$$L_{1,N} f(\eta) = \sum_{x=-N+M+1}^{N-M-1} c(x,\eta)[f(\eta^x) - f(\eta)],$$

where the rate of creation and annihilation $c(x,\cdot)$ is a nonnegative cylindric function with range M; that is, there exists a fixed integer M (possibly equal to 0) such that $c(x,\eta) = c(\eta_{x-M},\dots,\eta_{x+M})$ depends only of the values of η in $\{x-M,\dots,x+M\}$. We remark that the diffusive part of the process is sped up by N^2 to obtain a nontrivial hydrodynamic evolution.

For a given trajectory $\eta : [0,T] \to \{0,1\}^{2N+1}$, let ρ_t^N be the empirical density of particles in $[-1,1]$ at time $t \in [0,T]$

$$\rho_t^N = \frac{1}{N} \sum_{x=-N}^{N} \eta_t(x) \delta_{x/N}.$$

We denote by $Q_t^N(x)$ the conservative current through the edge $(x, x+1)$, that is, the total number of particles that have jumped from x to $x+1$ minus the total number of particles that have jumped from $x+1$ to x between the times 0 and t.

The empirical measure associated to this current is defined as the signed measure on $[-1, 1]$

$$Q_t^N = \frac{1}{N^2} \sum_{x=-N}^{N-1} Q_t^N(x) \delta_{x/N}. \tag{2.2}$$

The renormalization by N^2 takes into account the space renormalization as well as the diffusive scaling of the SSEP dynamics which leads to an extra factor N. We denote by $K_t^N(x)$ the nonconservative current at site x, that is, the total number of particles created minus the total number of particles annihilated at site x between times 0 and t. The corresponding empirical measure is

$$K_t^N = \frac{1}{N} \sum_{x=-N}^{N} K_t^N(x) \delta_{x/N}.$$

For any continuous function $\varphi \in C([-1, 1])$, we will use the notation

$$\langle \rho_t^N \varphi \rangle = \frac{1}{N} \sum_{x=-N}^{N-1} \eta_t(x) \varphi\left(\frac{x}{N}\right).$$

The same notation will be used for K_t^N. As the conservative current applies to edges, we will write for $\varphi \in C^1([-1, 1])$

$$\langle Q_t^N \nabla \varphi \rangle = \frac{1}{N} \sum_{x=-N}^{N-1} Q_t^N(x) \left(\varphi\left(\frac{x+1}{N}\right) - \varphi\left(\frac{x}{N}\right)\right).$$

Note that $\varphi(\frac{x+1}{N}) - \varphi(\frac{x}{N})$ is of order $1/N$ so that the scaling is coherent with (2.2). Finally, for any functions $f(s, r)$ in $[0, T] \times [-1, 1]$, we use the shorthand notation

$$\forall s \in [0, T] \qquad \langle f_s \rangle = \int_{-1}^{1} dr\, f(s, r).$$

The density profiles bounded away from 0 and 1 will be relevant so that we introduce $C_e([-1, 1])$, the set of continuous functions f on $[-1, 1]$, for which there exists a constant $\epsilon > 0$ such that $\epsilon < f < 1 - \epsilon$. Given a function $\gamma \in C_e([-1, 1])$, let ν_γ^N be the Bernoulli product measure on $\{-N, \dots, N\}$ with marginals

$$\forall k \in \{-N, \dots, N\} \qquad \nu_\gamma^N\big(\eta(k) = 1\big) = \gamma\left(\frac{k}{N}\right). \tag{2.3}$$

For $\alpha \in [0, 1]$, the Bernoulli product measure with uniform density α is denoted by ν_α^N. Let $\mathbb{P}_\gamma^N$ be the probability measure associated to the Markov process (ρ_t^N, Q_t^N, K_t^N) on $[0, T]$ with initial measure ν_γ^N on the particle configurations.

We define $\mathcal{M}$, the set of signed measures on $[-1, 1]$, endowed with the weak topology. We also consider $\mathcal{M}_0$, the subset of $\mathcal{M}$ of all absolutely continuous measures, w.r.t. the Lebesgue measure with positive density bounded by 1,

$$\mathcal{M}_0 = \{\rho(x)\, dx \in \mathcal{M}, 0 \le \rho(x) \le 1 \text{ a.e.}\}.$$

In order to consider the joint large deviations of (ρ_t^N, Q_t^N, K_t^N) during the time interval $[0,T]$, we will work on $\mathcal{E} = D([0,T], \mathcal{M}_0 \times \mathcal{M} \times \mathcal{M})$ the space of cad-lag trajectories with values in $\mathcal{M}_0 \times \mathcal{M} \times \mathcal{M}$ endowed with the Skorohod topology [13].

2.2. *The results.* The hydrodynamic behavior of the microscopic dynamics introduced in Section 2.1, can be described in terms of a few macroscopic parameters

$$\forall \alpha \in [0,1] \qquad \begin{aligned} C(\alpha) &= \nu_\alpha\big(c(0,\eta)\big(1-\eta(0)\big)\big), \\ A(\alpha) &= \nu_\alpha\big(c(0,\eta)\eta(0)\big), \end{aligned} \tag{2.4}$$

where C and A represent the average creation and annihilation rates at density α and

$$\forall \alpha \in [0,1] \qquad \sigma(\alpha) = \nu_\alpha\big(\eta_0(1-\eta_1)\big) = \alpha(1-\alpha), \tag{2.5}$$

which is the conductivity of the SSEP. Finally, we denote by $\bar{\rho}_\pm = \frac{\beta_\pm}{1+\beta_\pm}$ the densities at the boundaries imposed by the reservoirs.

We are now ready to state the hydrodynamic limit:

THEOREM 2.1. *Let $\gamma \in C_e([-1,1])$ with $\gamma(-1) = \bar{\rho}_-$ and $\gamma(1) = \bar{\rho}_+$. For each $T > 0$ and $\varphi \in C([0,T]\times[-1,1])$,*

$$\forall \delta > 0 \qquad \lim_{N\to\infty} \mathbb{P}_\gamma^N\left[\left|\int_0^T dt\, \langle \rho_t^N \varphi_t\rangle - \langle \bar{\rho}_t \varphi_t\rangle\right| > \delta\right] = 0,$$

where $\bar{\rho}(t,x)$ is the unique weak solution of

$$\begin{cases} \partial_t \bar{\rho}(t,x) = \frac{1}{2}\Delta \bar{\rho}(t,x) + C(\bar{\rho}(t,x)) - A(\bar{\rho}(t,x)), \\ \bar{\rho}(t,\pm 1) = \bar{\rho}_\pm, \\ \bar{\rho}(0,x) = \gamma(x). \end{cases} \tag{2.6}$$

The meaning of weak solution of (2.6) is recalled in the Appendix (with $H = G = 0$).

As a consequence of Theorem 2.1 a law of large numbers holds for the currents.

THEOREM 2.2. *For any test function $\varphi \in C^1([-1,1])$ and $t > 0$,*

$$\forall \delta > 0 \qquad \lim_{N\to\infty} \mathbb{P}_\gamma^N\left[\left|\langle Q_t^N \varphi\rangle + \frac{1}{2}\int_0^t ds\, \langle \varphi \nabla \bar{\rho}_s\rangle\right| > \delta\right] = 0,$$

$$\lim_{N\to\infty} \mathbb{P}_\gamma^N\left[\left|\langle K_t^N \varphi\rangle - \int_0^t ds\, \langle \varphi\big(C(\bar{\rho}_s) - A(\bar{\rho}_s)\big)\rangle\right| > \delta\right] = 0,$$

where $\bar{\rho}$ is the unique weak solution of (2.6).

Theorems 2.1, 2.2 can be deduced from the methods used to prove the large deviations so that their derivation is omitted; see also [3, 8, 9].

Before stating the large deviation principle, we need more notation. Let G and H be smooth functions in $[0, T] \times [-1, 1]$. For a given trajectory (ρ, Q, K) in $\mathcal{E}$, we set

$$J_{G,H}(\rho, Q, K) = J_H^1(\rho, Q) + J_G^2(\rho, K) \tag{2.7}$$

with

$$\begin{aligned} J_H^1(\rho, Q) = \langle Q_T \nabla H_T \rangle - \int_0^T \left\langle Q_s \frac{d}{ds} \nabla H_s \right\rangle ds - \frac{1}{2} \int_0^T \langle \rho_s \Delta H_s \rangle \, ds \\ - \frac{1}{2} \int_0^T \langle \sigma(\rho_s)(\nabla H_s)^2 \rangle \, ds + \frac{1}{2} \bar{\rho}_+ \int_0^T \nabla H(s, 1) \, ds \\ - \frac{1}{2} \bar{\rho}_- \int_0^T \nabla H(s, -1) \, ds, \end{aligned} \tag{2.8}$$

where $\sigma(u) = u(1-u)$ is defined in (2.5) and

$$\begin{aligned} J_G^2(\rho, K) = \langle K_T G_T \rangle - \int_0^T \left\langle K_s \frac{d}{ds} G_s \right\rangle ds \\ - \int_0^T ds \, \langle C(\rho_s)(e^{G_s} - 1) + A(\rho_s)(e^{-G_s} - 1) \rangle, \end{aligned} \tag{2.9}$$

where A and C were introduced in (2.4).

The first functional is related to the contribution of the conservative currents and the second one to the nonconservative currents; see Theorem 2.5. We define

$$J(\rho, Q, K) = \sup_{G,H} J_{G,H}(\rho, Q, K) = \sup_{H \in C^{1,2}} J_H^1(\rho, Q) + \sup_{G \in C^{1,0}} J_G^2(\rho, K),$$

where the supremum is taken on functions $G \in C^{1,0}$ and $H \in C^{1,2}$. Note that the functions G and H can take arbitrary (finite) values at the boundaries.

Define $\mathcal{A}$ as the set of trajectories (ρ, Q, K) satisfying the following two conditions:

- *Conservation law.* For all test function $\varphi \in C^1([-1, 1])$ vanishing at the boundaries

$$\langle \rho_t \varphi \rangle - \langle \rho_0 \varphi \rangle = \langle Q_t \partial_x \varphi \rangle + \langle K_t \varphi \rangle, \qquad Q_0 = 0, K_0 = 0. \tag{2.10}$$

- *Energy condition.* The energy $\mathcal{Q}(\rho)$ of the density trajectory is finite with

$$\begin{aligned} \mathcal{Q}(\rho) = \sup_{\varphi} \bigg\{ \int_0^T dt \int_{-1}^1 dx \, \rho(t, x) \nabla \varphi(t, x) - \frac{1}{2} \int_0^T dt \int_{-1}^1 dx \, \varphi(t, x)^2 \\ - \int_0^T dt \, \big(\bar{\rho}_+ \varphi(t, 1) - \bar{\rho}_- \varphi(t, -1)\big) \bigg\}, \end{aligned} \tag{2.11}$$

  and the supremum is taken over smooth functions φ in $[0, T] \times [-1, 1]$.

DEFINITION 2.3. Trajectories (ρ, Q, K) in $\mathcal{E}$ with densities $(\rho(t,x), Q(t,x), K(t,x))$ belonging to $C^{1,2}$ will be referred to as smooth trajectories.

For smooth trajectories the conservation law reduces to

$$\partial_t \rho + \partial_x \partial_t Q - \partial_t K = 0,$$

where $\partial_t Q, \partial_t K$ are the instantaneous currents, and the energy condition reads

$$\mathcal{Q}(\rho) = \frac{1}{2} \int_0^T dt \int_{-1}^1 dx\, (\nabla \rho(t,x))^2 < \infty$$

with $\rho(t,1) = \bar{\rho}_+$ and $\rho(t,-1) = \bar{\rho}_-$. The energy condition was introduced in [5, 15, 22] to control the approximation procedure in the derivation of the large deviation lower bound; see Theorem 2.7.

Finally we define the dynamical rate function

$$I_0(\rho, Q, K) = \begin{cases} J(\rho, Q, K), & \text{if } (\rho, Q, K) \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.12}$$

To take into account the large deviations of the initial measure ν_γ^N, we introduce for any function $m : [-1,1] \to [0,1]$

$$h_\gamma(m) = \left\langle m \log \frac{m}{\gamma} \right\rangle + \left\langle (1-m) \log \frac{1-m}{1-\gamma} \right\rangle$$

and

$$I_\gamma(\mu) = \begin{cases} h_\gamma(m), & \text{if } \mu(dx) = m(x)\, dx, \\ +\infty, & \text{otherwise.} \end{cases}$$

The rate function is the sum of the dynamical deviation cost from the initial measure and the deviation cost from the hydrodynamic trajectory

$$I(\rho, Q, K) = I_0(\rho, Q, K) + I_\gamma(\rho_0).$$

From now, the initial density profile γ is a given smooth function in $C_e([-1,1])$ equal to $\bar{\rho}_\pm$ at the boundaries. We state the large deviation theorems.

THEOREM 2.4. *For all closed sets $\mathcal{F} \in \mathcal{E}$*

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_\gamma^N [(\rho_t^N, Q_t^N, K_t^N) \in \mathcal{F}] \le - \inf_{(\rho, Q, K) \in \mathcal{F}} I(\rho, Q, K).$$

We first state the lower bound for smooth trajectories.

THEOREM 2.5. *Let (ρ, Q, K) be a smooth trajectory; see Definition 2.3. Then the large deviation functional I_0 has an explicit form*

$$\begin{aligned} I_0(\rho, Q, K) = \int_0^T dt \int_{-1}^1 dx \bigg\{ & \frac{(\partial_t Q(x,t) + \partial_x \rho(x,t))^2}{2\sigma(\rho(x,t))} \\ & + \Phi(\rho(x,t), \partial_t K(x,t)) \bigg\} \end{aligned} \tag{2.13}$$

with

$$\Phi(\rho,\kappa) = C(\rho) + A(\rho) - \sqrt{\kappa^2 + 4A(\rho)C(\rho)} + \kappa \log\left(\frac{\sqrt{\kappa^2 + 4A(\rho)C(\rho)} + \kappa}{2C(\rho)}\right). \tag{2.14}$$

If $C(\rho) = 0$, *then* Φ *becomes*

$$\Phi(\rho,\kappa) = \begin{cases} A(\rho) + \kappa - \kappa \log\left(\dfrac{-\kappa}{A(\rho)}\right), & \text{if } \kappa \le 0, \\ \infty, & \text{if } \kappa > 0. \end{cases}$$

A similar formula holds if $A(\rho) = 0$.

For any open set $\mathcal{O} \in \mathcal{E}$ *containing the smooth trajectory* (ρ, Q, K)

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_\gamma^N[(\rho_t^N, Q_t^N, K_t^N) \in \mathcal{O}] \ge -I(\rho, Q, K).$$

REMARK 2.6. The first contribution to (2.13) comes from the difference between the instantaneous empirical current $\partial_t Q$ and the canonical instantaneous current associated to ρ which is $-\frac{1}{2}\nabla\rho$. This term has already been analyzed in the conservative dynamics [3, 6]. The second term in (2.13) should be interpreted as the large deviation functional associated to Poisson processes with parameters $C(\rho_t)$ and $A(\rho_t)$.

In order to derive the lower bound for general trajectories, we introduce two technical assumptions on the rates (2.4):

ASSUMPTION (L1). The rate A (resp., C) is either concave and positive on $]0, 1[$ or uniformly equal to zero.

ASSUMPTION (L2). The functions A and C are monotonous and

$$\forall z \in [0, 1] \qquad A'(z) \ge 0, \qquad C'(z) \le 0. \tag{2.15}$$

THEOREM 2.7. *Assume Assumptions* (L1) *and* (L2), *then for all open sets* $\mathcal{O} \in \mathcal{E}$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_\gamma^N[(\rho_t^N, Q_t^N, K_t^N) \in \mathcal{O}] \ge -\inf_{(\rho,Q,K)\in\mathcal{O}} I(\rho, Q, K).$$

The concavity Assumption (L1) has been introduced in [17]. As we shall see in Section 5.2, it simplifies the proof of the lower bound; however, it is mainly technical and Theorem 2.7 should be valid without Assumption (L1). We refer to [5, 15, 22] for further results on this generalization in the case of conservative dynamics. Assumption (L2) will be used in the Appendix only to ensure the uniqueness of the weak solutions for singular perturbations of the hydrodynamic equation (2.6).

3. Modified dynamics and local equilibrium. Local equilibrium lies at the heart of the hydrodynamic limit theory, and it ensures that during the time evolution, the local measure remains close to an equilibrium measure with a varying density. In this section, we state, in our framework, a strong form of local equilibrium which will be useful for the derivation of the hydrodynamic large deviations. The proofs are omitted as they follow the scheme introduced in [2, 18, 19].

3.1. *The modified dynamics.* We first define a modification of the original process (2.1) which will be used to derive the large deviations. For smooth functions $G \in C^{1,0}$ and $H \in C^{1,2}$ on $[0,T] \times [-1,1]$, denote $L_N^{G_t,H_t}$, the time-dependent generator given by

$$L_N^{G_t,H_t} = \frac{N^2}{2} L_{0,N}^{H_t} + L_{1,N}^{G_t} + \frac{N^2}{2} L_{+,N} + \frac{N^2}{2} L_{-,N} \tag{3.1}$$

with

$$\begin{aligned} L_{0,N}^{H_t} f(\eta) = \sum_{x=-N}^{N-1} &[f(\eta^{x,x+1}) - f(\eta)] \\ &\times \exp\Big[(\eta(x) - \eta(x+1))\Big(H\Big(t, \frac{x+1}{N}\Big) - H\Big(t, \frac{x}{N}\Big)\Big)\Big], \\ L_{1,N}^{G_t} f(\eta) = \sum_{x=-N}^{N} &c(x,\eta)\big[\eta(x) e^{-G(t,x/N)} + \big(1-\eta(x)\big) e^{G(t,x/N)}\big] \\ &\times [f(\eta^x) - f(\eta)]. \end{aligned}$$

The modified dynamics induces a weak drift in the conservative dynamics. For large N, a particle jumps from x to $x \pm 1$ at rate $\frac{1}{2}(1 \pm \frac{1}{N}\partial_x H(t, \frac{x}{N}))$. $L_{1,N}^{G_t}$ is the generator of a nonconservative dynamics for which the intensity of creation and annihilation varies in time and space according to G. Finally, let $\mathbb{P}_{\gamma,G,H}^N$ be the probability measure associated to the process with initial measure ν_γ^N and generator $L_N^{G_t,H_t}$. We stress the fact that the reservoir dynamics are unchanged. As in Theorem 2.1, one can show that the modified dynamics follows the hydrodynamic limit equation

$$\begin{aligned} \partial_t \rho(t,x) = \tfrac{1}{2}\Delta \rho(t,x) - \nabla(\sigma(\rho(t,x)) \nabla H(t,x)) + C(\rho(t,x)) e^{G(t,x)} \\ - A(\rho(t,x)) e^{-G(t,x)}. \end{aligned} \tag{3.2}$$

3.2. *Local equilibrium.* We set $\Lambda_l^{(x)} = \{y \in \{-N,\dots,N\}$ such that $|y-x| \leq l\}$ and define the local density as

$$\bar{\eta}^l(x) = \frac{1}{|\Lambda_l^{(x)}|} \sum_{y \in \Lambda_l^{(x)}} \eta(y), \tag{3.3}$$

where $|\Lambda_l^{(x)}|$ stands for the number of sites in $\Lambda_l^{(x)}$. Let ψ be a cylindric function with support in $\{-R,\dots,R\}$. Given φ a smooth function on $[0,T]\times[-1,1]$ and $\delta,\varepsilon>0$, we define the set $B_{\delta,\varepsilon,\varphi}(\psi)$ on the trajectories $\{\eta_s\}_{s\le T}$ as

$$B_{\delta,\varepsilon,\varphi}(\psi)=\Biggl\{\{\eta_s\}_{s\le T}, \left|\int_0^T ds\,\frac{1}{2(N-R)}\sum_{x=-N+R}^{N-R}\varphi\Bigl(s,\frac{x}{N}\Bigr)\bigl[\psi(\tau_x\eta_s)-\nu_{\bar\eta_s^{\epsilon N}(x)}(\psi)\bigr]\right|\le\delta\Biggr\}, \tag{3.4}$$

where $\tau_x\eta$ is the configuration η shifted by x.

The local equilibrium property also holds for the modified dynamics (3.1).

THEOREM 3.1. *Given φ, ψ and $\delta>0$, the trajectories concentrate super-exponentially fast on the set $B_{\delta,\varepsilon,\varphi}(\psi)$*

$$\limsup_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\gamma,G,H}\bigl(B_{\delta,\varepsilon,\varphi}(\psi)^c\bigr)=-\infty.$$

Moreover, the reservoirs impose local equilibrium at the boundaries with the densities $\bar\rho_+,\bar\rho_-$. For any continuous function Φ in $[0,T]$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}^N_{\gamma,G,H}\left[\left|\int_0^T ds\,\Phi(s)\bigl(\eta_s(\pm N)-\bar\rho_\pm\bigr)\right|\ge\delta\right]=-\infty.$$

The derivation of Theorem 3.1 follows from the bounds on entropy production [19] which can be adapted to control the boundary terms as in [2].

4. Large deviation upper bound. The derivation of Theorem 2.4 is split into several steps. First an upper bound with the rate function J (2.12) is derived for compact sets, then for closed sets. Finally, we prove that the rate function is infinite for the trajectories which do not belong to the set $\mathcal{A}$ introduced in (2.10), (2.11).

4.1. *The upper bound for compact sets.* In order to compare the original dynamics starting from the initial profile γ to a modified dynamics with regular drifts G,H (3.1) starting from the initial profile ω, we compute the Radon–Nikodym derivative [18],

$$\begin{aligned}\frac{d\mathbb{P}^N_{\omega,G,H}}{d\mathbb{P}^N_\gamma}=\exp\biggl[N\biggl\{&\langle Q_T^N\nabla H_T\rangle-\int_0^T ds\Bigl\langle Q_s^N\frac{d}{ds}\nabla H_s\Bigr\rangle\\&+\langle K_T^N G_T\rangle-\int_0^T ds\Bigl\langle K_s^N\frac{d}{ds}G_s\Bigr\rangle+O_H\Bigl(\frac{1}{N}\Bigr)\biggr\}\\&-\int_0^T R_N(s)\,ds+h_{\gamma,\omega}(\rho_0^N)\biggr],\end{aligned}\tag{4.1}$$

where $h_{\gamma,\omega}(\rho) = \langle \rho \log(\omega/\gamma)\rangle + \langle (1-\rho)\log((1-\omega)/(1-\gamma))\rangle$ and

$$
\begin{aligned}
R_N(s) = \frac{N^2}{2}\Bigg(\sum_{x=-N}^{N-1} & \eta_s(x)\big(1-\eta_s(x+1)\big) \\
& \times \left[\exp\!\left(H\!\left(s,\frac{x+1}{N}\right) - H\!\left(s,\frac{x}{N}\right)\right) - 1\right] \\
& + \eta_s(x+1)\big(1-\eta_s(x)\big)\left[\exp\!\left(H\!\left(s,\frac{x}{N}\right) - H\!\left(s,\frac{x+1}{N}\right)\right) - 1\right]\Bigg) \\
& + \sum_{x=-N}^{N} c(x,\eta_s)\left[\exp\!\left((1-2\eta_s(x))G\!\left(s,\frac{x}{N}\right)\right) - 1\right].
\end{aligned}
\tag{4.2}
$$

Expanding the exponential and summing by parts, we get

$$
\begin{aligned}
R_N(s) = \frac{1}{2}\Bigg\{\sum_{x=-N}^{N-1} & \Bigg[\eta_s(x)\Delta H\!\left(s,\frac{x}{N}\right) \\
& + \frac{1}{4}\left(\nabla H\!\left(s,\frac{x}{N}\right)\right)^2 \big[\eta_s(x)\big(1-\eta_s(x+1)\big) \\
& + \eta_s(x+1)\big(1-\eta_s(x)\big)\big]\Bigg]\Bigg\} \\
& + \frac{1}{2}[\eta_s(N)\nabla H(s,1) - \eta_s(-N)\nabla H(s,-1)] \\
& + \sum_{x=-N}^{N} c(x,\eta_s)\left[\exp\!\left((1-2\eta_s(x))G\!\left(s,\frac{x}{N}\right)\right) - 1\right] + O(1).
\end{aligned}
$$

Thanks to the local equilibrium, the microsopic expressions in $R_N(s)$ can be replaced by their averages. We set

$$
\begin{aligned}
B_{\delta,\varepsilon} = {} & B_{\delta,\varepsilon,(\nabla H)^2}\big(\eta_0(1-\eta_0)\big) \cap B_{\delta,\varepsilon,(\mathrm{e}^{G}-1)}\big(c(0,\eta)\eta_0\big) \\
& \cap B_{\delta,\varepsilon,(\mathrm{e}^{-G}-1)}\big(c(0,\eta)(1-\eta_0)\big) \cap B'_{\delta,\pm},
\end{aligned}
\tag{4.3}
$$

where the sets $B_{\delta,\varepsilon,\varphi}(\cdot)$ were introduced in (3.4) and

$$
B'_{\delta,\pm} = \left\{\left|\int_0^T ds\,\nabla H(s,\pm 1)\big(\eta_s(\pm N) - \bar\rho_\pm\big)\right| \le \delta\right\}.
$$

The super-exponential replacement Theorem 3.1 implies

$$
\forall \delta > 0 \qquad \limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbb{P}_\gamma^N(B^c_{\delta,\varepsilon}) = -\infty.
$$

For trajectories in $B_{\delta,\varepsilon}$, the Radon–Nikodym derivative can be approximated as follows:

$$\frac{d\mathbb{P}^N_{\omega,G,H}}{d\mathbb{P}^N_\gamma} = \exp\biggl[N\biggl(J_{G,H,\epsilon}(\rho^N, Q^N, K^N) + h_{\gamma,\omega}(\rho^N_0) + O_{G,H}\biggl(\frac{1}{N}\biggr) + O(\delta)\biggr)\biggr],$$

where we used the functional

$$\begin{aligned}
&J_{G,H,\epsilon}(\rho, Q, K)\\
&\quad = \langle Q_T \nabla H_T\rangle - \int_0^T ds \biggl\langle Q_s \frac{d}{ds}\nabla H_s\biggr\rangle + \langle K_T G_T\rangle - \int_0^T ds \biggl\langle K_s \frac{d}{ds} G_s\biggr\rangle\\
&\qquad - \frac{1}{2}\int_0^T ds\, \langle \rho_s \Delta H_s + \sigma(\rho_s * i_\epsilon)(\nabla H_s)^2\rangle \qquad (4.4)\\
&\qquad - \int_0^T ds\, \langle C(\rho_s * i_\epsilon)(e^{G_s} - 1) + A(\rho_s * i_\epsilon)(e^{-G_s} - 1)\rangle\\
&\qquad + \frac{1}{2}\bar{\rho}_+ \int_0^T ds \nabla H(s, 1) - \frac{1}{2}\bar{\rho}_- \int_0^T ds \nabla H(s, -1)
\end{aligned}$$

with A, C and σ as in (2.4), (2.5). The function i_ϵ is an approximation of unity

$$\rho^N_s * i_\epsilon = \bar{\eta}^{\epsilon N}_s.$$

For any $\mathcal{O}$ open set in $\mathcal{E}$, we can then deduce the large deviation upper bound

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N}\mathbb{P}^N_\gamma(\mathcal{O})\\
&\quad \le \max\biggl[\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N_\gamma\bigl((\rho^N, Q^N, K^N) \in \mathcal{O} \cap B_{\delta,\varepsilon}\bigr);\\
&\qquad\qquad\qquad\qquad \limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N_\gamma(B^c_{\delta,\varepsilon})\biggr]\\
&\quad \le \limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N_{\omega,G,H}\biggl[\frac{d\mathbb{P}^N_\gamma}{d\mathbb{P}^N_{\omega,G,H}} \mathbf{1}_{\{\mathcal{O}\cap B_{\delta,\varepsilon}\}}(\rho^N, Q^N, K^N)\biggr]\\
&\quad \le \limsup_{N\to\infty} \frac{1}{N}\\
&\qquad \times \log \mathbb{P}^N_{\omega,G,H}\bigl[e^{-N(J_{G,H,\epsilon}((\rho^N, Q^N, K^N)) + h_{\gamma,\omega}(\rho^N_0) + O_{G,H}(1/N) + O(\delta))}\mathbf{1}_{\mathcal{O}}\bigr]\\
&\quad \le \sup_{(\rho,Q,K)\in\mathcal{O}} \{-J_{G,H,\epsilon}(\rho, Q, K) - h_{\gamma,\omega}(\rho_0)\} + O(\delta).
\end{aligned}$$

This is true for any (ϵ, G, H, ω) and any $\delta > 0$, so finally

$$\limsup_{N\to\infty} \frac{1}{N}\mathbb{P}^N_\gamma(\mathcal{O}) \le \inf_{\epsilon,G,H,\omega} \sup_{(\rho,Q,K)\in\mathcal{O}} \{-J_{G,H,\epsilon}(\rho, Q, K) - h_{\gamma,\omega}(\rho_0)\}. \qquad (4.5)$$

The previous bound can then be extended to any compact set $\mathcal{K}$ by using a finite covering with open sets (see [18])

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N(\mathcal{K})\leq-\sup_{(\rho,Q,K)\in\mathcal{K}}\sup_{G,H}\{J_{G,H}(\rho,Q,K)\}=-J(\rho,Q,K).$$

4.2. *The upper bound for closed sets.* We are going to prove the exponential tightness, that is, to exhibit a sequence $\{\mathcal{K}_n\}$ of compact sets in $\mathcal{E}$ such that for any n

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N(\mathcal{K}_n^c)\leq-n. \tag{4.6}$$

The large deviation upper bound will then follow for general closed sets F of $\mathcal{E}$ by noticing

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N(F)&\leq\limsup_{N\to\infty}\frac{1}{N}\log[\mathbb{P}_\gamma^N(F\cap\mathcal{K}_n)+\mathbb{P}_\gamma^N(\mathcal{K}_n^c)]\\&\leq\max\Big[\sup_{(\rho,Q,K)\in F\cap\mathcal{K}_n}J(\rho,Q,K);-n\Big].\end{aligned}$$

Letting n go to ∞ completes the upper bound for closed sets.

In order to build a sequence of compact sets, we need to check first that the measures concentrate on equi-continuous trajectories

LEMMA 4.1. *Let ϕ be a C^1 function on $[-1,1]$. Then we have for all $\epsilon>0$,*

$$\lim_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N\Big[\sup_{|t-s|\leq\delta}|\langle\rho_t^N\phi\rangle-\langle\rho_s^N\phi\rangle|>\epsilon\Big]=-\infty, \tag{4.7}$$

$$\lim_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N\Big[\sup_{|t-s|\leq\delta}|\langle Q_t^N\phi\rangle-\langle Q_s^N\phi\rangle|>\epsilon\Big]=-\infty, \tag{4.8}$$

$$\lim_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N\Big[\sup_{|t-s|\leq\delta}|\langle K_t^N\phi\rangle-\langle K_s^N\phi\rangle|>\epsilon\Big]=-\infty, \tag{4.9}$$

where the supremum is taken over s,t in $[0,T]$.

Second, we need estimates on the total variation norm of the empirical currents.

LEMMA 4.2. *For any time $T>0$, one has*

$$\lim_{a\to\infty}\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N\left(\sup_{0\leq t\leq T}\frac{1}{N}\sum_{x=-N}^{N}|K_t^N(x)|\geq a\right)=-\infty, \tag{4.10}$$

$$\lim_{a\to\infty}\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N\left(\sup_{0\leq t\leq T}\frac{1}{N^2}\sum_{x=-N}^{N}|Q_t^N(x)|\geq a\right)=-\infty. \tag{4.11}$$

Should the currents be bounded (as the density), then Lemma 4.1 would be enough to ensure the exponential tightness [18].

We postpone the derivation of the lemmas and conclude the proof of the exponential tightness (4.6). We focus on the conservative current as the same strategy applies to ρ, K. Fix a sequence ϕ_l of C^2 functions on $[-1,1]$ dense in $C([-1,1])$. We set

$$\mathcal{C}_{l,\delta,m} = \left\{ Q \in D([0,T],\mathcal{M});\ \sup_{|t-s|\le\delta} |\langle Q_t\phi_l\rangle - \langle Q_s\phi_l\rangle| \le \frac{1}{m} \right\}.$$

Using Lemma 4.1, we have

$$\forall l \ge 0, \forall m \ge 1, \forall n \ge 1, \exists \delta(n,m,l) \qquad \mathbb{P}_\gamma^N\big(Q^N \notin \mathcal{C}_{l,\delta(n,m,l),m}\big) \le \exp(-Nnml).$$

We introduce also $\mathcal{C}'_n = \{Q \in D([0,T],\mathcal{M}); \sup_{t\le T}|Q_t| \le k(n)\}$, where $|Q_t|$ stands for the total variation norm of the measure Q_t, and $k(n)$ is chosen, according to Lemma 4.2, such that

$$\mathbb{P}_\gamma^N(Q^N \notin \mathcal{C}'_n) \le \exp(-Nn).$$

Now consider

$$\mathcal{K}_n = \bigcap_{l\ge 0, m\ge 1} \mathcal{C}_{l,\delta(n,m,l),m} \cap \mathcal{C}'_n. \tag{4.12}$$

Ascoli theorem (see [13], Theorem 6.3, page 123) implies that $\mathcal{K}_n$ is a compact set. Combining the previous estimates, we see that $\mathcal{K}_n$ satisfies (4.6).

PROOF OF LEMMA 4.1. We start by proving (4.8) and follow the strategy of [3]. It is enough to show that the expression below goes to $-\infty$ as δ vanishes

$$\max_{0\le k\le T/\delta} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_\gamma^N\Big[\sup_{k\delta\le t\le (k+1)\delta} \langle Q_t^N\phi\rangle - \langle Q_{k\delta}^N\phi\rangle > \epsilon\Big].$$

Thanks to the computation of the Radon–Nikodym derivative of the modified dynamics (4.2) (with $\nabla H = \frac{1}{N}\phi, G = 0$), we know from (4.1) that for all $a > 0$,

$$\begin{aligned}\mathcal{M}_T = \exp\Bigg\{ & N\langle Q_T^N a\phi\rangle \\ & - \frac{1}{2}\int_0^T ds \sum_{x=-N}^{N-1} \bigg(\eta_s(x) a\nabla\phi\Big(\frac{x}{N}\Big) \\ & \qquad + \frac{1}{4}\Big[a\phi\Big(\frac{x}{N}\Big)\Big]^2 [\eta_s(x)\big(1-\eta_s(x+1)\big) \\ & \qquad\qquad + \eta_s(x+1)\big(1-\eta_s(x)\big)]\bigg) \\ & \qquad + N[\eta_s(N)\phi(1) - \eta_s(-N)\phi(-1)] + O(1)\Bigg\}\end{aligned}$$

is a mean one positive martingale. One easily checks that the integral term is bounded above by $C_\phi a^2 NT$ (we will take the limit as a goes to infinity), where C_ϕ is a constant depending only on ϕ. Therefore, multiplying by aN, adding and subtracting the integral term of the logarithm of the martingale and exponentiating,

$$\begin{aligned}
&\mathbb{P}_\gamma^N\Big[\sup_{k\delta \le t \le (k+1)\delta} \langle Q_t^N \phi\rangle - \langle Q_{k\delta}^N \phi\rangle > \epsilon\Big] \\
&\quad \le \mathbb{P}_\gamma^N\Big[\sup_{k\delta \le t \le (k+1)\delta} \frac{\mathcal{M}_t}{\mathcal{M}_{\delta k}} > \exp\{\epsilon a N - C_\phi a^2 N\delta\}\Big].
\end{aligned} \tag{4.13}$$

Taking δ small enough such that $C_\phi a^2 N\delta < \frac{\epsilon a N}{2}$, then Doob's inequality implies that the last expression is bounded above by $\exp[-aN\epsilon/4]$. Letting a go to ∞ completes the proof.

For the nonconservative current, (4.9) follows in the same way by using the martingale

$$\mathcal{N}_t = \exp\Bigg[\langle a\phi K_T^N\rangle - \int_0^T ds \sum_{x=-N}^{N} c(x,\eta_s)\big[e^{a(1-2\eta_s(x))\phi(x/N)} - 1\big]\Bigg].$$

The proof of (4.7) can be found in [18]. $\square$

PROOF OF LEMMA 4.2. The probability of the event in (4.10) can be estimated from above by the large deviations of $(2N+1)$ independent Poisson processes, and thus

$$\forall a > 0 \qquad \mathbb{P}_\gamma^N\Bigg(\sup_{0\le t\le T} \frac{1}{N}\sum_{x=-N}^{N} |K_t^N(x)| \ge a\Bigg) \le \exp(-NC_a), \tag{4.14}$$

where C_a goes to infinity as a diverges.

We turn now to the bound (4.11) and show that

$$\forall a > 0 \qquad \mathbb{P}_\gamma^N\Bigg(\sup_{0\le t\le T} \frac{1}{N^2}\sum_{x=-N}^{N} |Q_t^N(x)| \ge a\Bigg) \le e^{-NC'_a}, \tag{4.15}$$

where C'_a goes to infinity as a diverges. To prove (4.15), we first use a microscopic identity (which holds at any time),

$$\begin{aligned}
&\forall x \in \{-N,\dots,N\} \\
&\quad Q_t^N(x-1) - Q_t^N(x) = \eta_t^N(x) - \eta_0^N(x) - K_t^N(x).
\end{aligned} \tag{4.16}$$

Therefore (4.14) implies that with probability at least $1 - e^{-NC_a}$, the conservative current through the edge $(x, x+1)$ satisfies

$$\forall t \in [0,T] \qquad |Q_t^N(x)| \le |Q_t^N(-N)| + 2N(1+a).$$

From this, we get

$$
\begin{aligned}
&\mathbb{P}_\gamma^N\left(\sup_{0\le t\le T}\frac{1}{N^2}\sum_{x=-N}^{N-1}|Q_t^N(x)|\ge 5a\right)\\
&\qquad\le e^{-NC_a}+\mathbb{P}^N\left(\sup_{0\le t\le T}\frac{|Q_t^N(-N)|}{N}\ge 3a-2\right)\\
&\qquad\le e^{-NC_at}+\mathbb{P}_\gamma^N\left(\sup_{0\le t\le T}\frac{Q_t^N(-N)}{N}\ge 3a-2\right)\\
&\qquad\quad+\mathbb{P}_\gamma^N\left(\inf_{0\le t\le T}\frac{Q_t^N(-N)}{N}\le -3a+2\right).
\end{aligned}
\tag{4.17}
$$

We bound now the first term on the RHS (the second one can be bounded similarly by symmetry). Using again identity (4.16), we get

$$
\begin{aligned}
&\mathbb{P}_\gamma^N\left(\sup_{0\le t\le T}\frac{Q_t^N(-N)}{N}\ge 3a-2\right)\\
&\qquad\le \mathbb{P}_\gamma^N\left(\sup_{0\le t\le T}\frac{1}{N^2}\sum_{x=-N}^{N-1}Q_t^N(x)\ge a-4\right)+e^{-NC_at}.
\end{aligned}
$$

The terms in the RHS can be estimated as in (4.13) by reducing to a martingale estimate. This completes (4.15). □

4.3. *The set* $\mathcal{A}$. To complete the derivation of the upper bound, we prove that the trajectories concentrate exponentially fast on the set $\mathcal{A}$ introduced in (2.10), (2.11).

The conservation law. Let $C_0^2([-1,1])$ be the set of twice differentiable functions vanishing at the boundary.

LEMMA 4.3. *For any* $\phi\in C_0^2([-1,1])$, *we introduce*

$$
\begin{aligned}
V_T(\rho^N,Q^N,K^N,\phi)&=\frac{1}{N}\sum_{x=-N+1}^{N-1}\phi\left(\frac{x}{N}\right)(\eta_T(x)-\eta_0(x))-\frac{1}{N}\nabla\phi\left(\frac{x}{N}\right)Q_T^N(x)\\
&\qquad+\phi\left(\frac{x}{N}\right)K_T(x).
\end{aligned}
$$

Then for any $\delta>0$,

$$
\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N[|V_T(\rho^N,Q^N,K^N,\phi)|>\delta]=-\infty.
$$

PROOF. For any site x and times $s < t$, the following microscopic relation holds:

$$\eta_t^N(x) - \eta_s^N(x) = [Q_t^N(x-1) - Q_s^N(x-1)] - [Q_t^N(x) - Q_s^N(x)] \\ + K_t^N(x) - K_s^N(x).$$

Summing in x and integrating by parts, the Q-term gives

$$V_T(\rho^N, Q^N, K^N, \phi) = \frac{1}{N} \sum_{x=-N}^{N-1} W_x Q_T^N(x)$$

with

$$W_x = \int_{x/N}^{(x+1)/N} du \left(\frac{x+1}{N} - u\right) \phi''(u),$$

where we used that ϕ vanishes at the boundaries. From this identity we get for any $a > 0$

$$\mathbb{P}_\gamma^N\big(V_T(\rho^N, Q^N, K^N, \phi) > \delta\big) \le e^{-aN\delta} \mathbb{P}_\gamma^N\left(\exp\left(a \sum_{x=-N}^{N-1} W_x Q_T^N(x)\right)\right). \tag{4.18}$$

We first note that W_x is of the order $1/N^2$ and that uniformly in x, $W_{x+1} - W_x$ is of the order ϵ_N/N^2 where ϵ_N vanishes to 0 as N goes to infinity. Thus identity (4.1) implies

$$\frac{d\mathbb{P}_{\gamma,0,H}^N}{d\mathbb{P}_\gamma^N} = \exp\left(a \sum_{x=-N}^{N-1} W_x Q_T^N(x) + a o_\phi(N)\right),$$

where H is such that $H((x+1)/N) - H(x/N) = W_x$. This function is a mean one martingale, and (4.18) leads to

$$\frac{1}{N} \log \mathbb{P}_\gamma^N\big(V_T(\rho^N, Q^N, K^N, \phi) > \delta\big) \le -a\delta + a o_\phi(1).$$

Letting N and then a go to ∞ concludes the proof of the lemma. □

Let (ϕ_k) be a dense sequence of functions in $C_0^2([-1, 1])$, and define

$$\mathcal{A}_n^\delta = \{(\rho, Q, K) \in \mathcal{E} \text{ such that } \forall k \le n, |V_T(\rho, Q, K, \phi_k)| \le \delta\}.$$

The previous lemma gives that for all n and $\delta > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_\gamma^N[(\rho^N, Q^N, K^N) \in F] \le - \inf_{(\rho,Q,K)\in F\cap\mathcal{A}_n^\delta} J(\rho, Q, K).$$

This is true for any n and (ϕ_n) is dense. So letting δ go to 0, we can define $I_0 = +\infty$ for the trajectories which do not satisfy relation (2.10).

Energy condition.

LEMMA 4.4. *For any smooth $\varphi : (0,T) \times [-1,1] \to \mathbb{R}$. There is a constant c_0 such that*

$$\forall k > 0 \qquad \limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_\gamma^N \big(\mathcal{Q}_\varphi(\rho^N) \geq k \big) \leq -k + c_0 T, \tag{4.19}$$

where $\mathcal{Q}_\varphi$ is defined for some suitable constant c

$$\begin{aligned} \mathcal{Q}_\varphi(\rho) = & \int_0^T dt \int_{-1}^1 dx\, \rho(t,x) \nabla\varphi(t,x) - \int_0^T dt\, \big(\bar\rho_+ \varphi(t,1) - \bar\rho_- \varphi(t,-1)\big) \\ & - \frac{1}{2c} \int_0^T dt \int_{-1}^1 dx\, \varphi(t,x)^2. \end{aligned} \tag{4.20}$$

The proof of this lemma follows from [5, 15] and therefore is omitted. By considering a dense sequence of functions $\{\varphi_k\}$, one can deduce from (4.19) [15] that the large deviations are infinite for the trajectories ρ such that $\sup_\varphi \mathcal{Q}_\varphi(\rho) = +\infty$. Note that if ρ is such that $\sup_\varphi \mathcal{Q}_\varphi(\rho) < +\infty$, then ρ is in $\mathbb{L}^2([0,T], \mathbb{H}(]-1,1[))$ and the Riesz representation theorem implies that there exists $\nabla\rho$ such that for any $\varphi : (0,T) \times [-1,1] \to \mathbb{R}$,

$$\begin{aligned} & \int_0^T dt \int_{-1}^1 dx\, \rho(t,x) \nabla\varphi(t,x) - \{\bar\rho_+ \varphi(t,1) - \bar\rho_- \varphi(t,-1)\} \\ & \quad = \int_0^T dt \int_{-1}^1 dx \nabla\rho(t,x) \varphi(t,x). \end{aligned} \tag{4.21}$$

5. Large deviations lower bound. The derivation of the lower bound is split into two parts following the general scheme for hydrodynamic large deviations [18, 19]. First we derive the lower bound for regular trajectories. Then under Assumptions (L1), (L2), we prove that general trajectories can be approximated by regular trajectories. A key feature in this approximation procedure is that the contribution of both currents decouple

$$\begin{aligned} I_0(\rho, Q, K) &= I_1(\rho, Q) + I_2(\rho, K) \\ &= \sup_{H \in C^{1,2}} J_H^1(\rho, Q) + \sup_{G \in C^{1,0}} J_G^2(\rho, K), \end{aligned} \tag{5.1}$$

where J_H^1 and J_G^2 were defined in (2.8) and (2.9). This simplifies some steps in the approximation as both functionals can be analyzed independently, contrary to the study in [17]. Assumption (L1) provides some convexity properties of the large deviation functional which simplify the proof. We follow closely some arguments of [3, 18]. Thus we will sketch the main steps of the proofs and only detail the new aspects related to the nonconservative currents.

5.1. *Lower bound for smooth trajectories.* In this section we derive Theorem 2.5. Suppose ρ, Q, K are smooth in time and space (see Definition 2.3) and that for any time t and x, $\rho(t,x)$ is bounded away from 0 and 1. To the trajectory (ρ, Q, K), one can associate the functions $G(t,x)$ and $H(t,x)$ satisfying

$$\begin{cases} \partial_t Q(t,x) = -\frac{1}{2}\nabla\rho(t,x) + \sigma(\rho(t,x))\nabla H(t,x), \\ \partial_t K(t,x) = C(\rho(t,x))e^{G(t,x)} - A(\rho(t,x))e^{-G(t,x)}. \end{cases} \tag{5.2}$$

The pointwise existence of G comes from the fact that the polynomial $CX^2 - \dot{K}X - A$ has one positive root for positive C and A. Furthermore H is well defined as long as $\sigma(\rho) = \rho(1-\rho) \neq 0$. We choose $H(t,-1) = 0$ and the value of H at $x = 1$ is imposed by (5.2), contrary to the case of density large deviations where H is equal to 0 at both boundaries; see Section 6.

LEMMA 5.1. *For smooth trajectories (ρ, Q, K) the functional I_0 (2.12) is given by*

$$I_0(\rho, Q, K) = J_{G,H}(\rho, Q, K) \tag{5.3}$$

with G and H as in (5.2). *This expression coincides with the explicit form of the functional* (2.13).

PROOF. As (ρ, Q, K) is smooth in time, $J_{\tilde{G},\tilde{H}}$ can be rewritten after integration by parts as

$$\begin{aligned} J_{\tilde{G},\tilde{H}}&(\rho, Q, K) \\ &= \int_0^T \left\langle \left[\partial_s Q_s + \frac{1}{2}\nabla\rho_s - \frac{1}{2}\sigma(\rho_s)\nabla\tilde{H}_s\right]\nabla\tilde{H}_s\right\rangle ds \\ &\quad + \int_0^T \langle \partial_s K_s \tilde{G}_s - C(\rho_s)(e^{\tilde{G}_s} - 1) - A(\rho_s)(e^{-\tilde{G}_s} - 1)\rangle\, ds \end{aligned} \tag{5.4}$$

for any $(\tilde{G}, \tilde{H})$ smooth functions.

For G and H given by (5.2), we are going to check that

$$J_{G,H}(\rho, Q, K) \geq \sup_{\tilde{G},\tilde{H}} J_{\tilde{G},\tilde{H}}(\rho, Q, K). \tag{5.5}$$

Indeed, take $\tilde{H} = H + F$ and any $\tilde{G}$, then it is easy to see that

$$\begin{aligned} J_{\tilde{G},\tilde{H}}(\rho, Q, K) &= J_{\tilde{G},H}(\rho, Q, K) - \frac{1}{2}\int_0^T ds\, \langle \sigma(\rho_s)|\nabla F_s|^2\rangle \\ &\leq J_{\tilde{G},H}(\rho, Q, K). \end{aligned}$$

Moreover, if $\tilde{G} = G + F$ and $\tilde{H}$ any regular function, then from the expression of J_G^2 (2.9) and identity (5.2), we get

$$
\begin{aligned}
J_{\tilde{G},\tilde{H}}(\rho, Q, K) = J_{G,\tilde{H}}(\rho, Q, K) & \\
&+ \int_0^T ds\, \langle C(\rho_s) e^{G_s}(1 + F_s - e^{F_s})\rangle \\
&+ \int_0^T ds\, \langle A(\rho_s) e^{-G_s}(1 - F_s - e^{-F_s})\rangle \\
\le J_{G,\tilde{H}}(\rho, Q, K),&
\end{aligned}
$$

where we used that $\exp(x) - x - 1 \ge 0$ for any $x \in \mathbb{R}$. This completes the proof of (5.5).

Finally for G and H given by (5.2), one can rewrite (5.4):

$$
\begin{aligned}
J_{G,H}(\rho, Q, K) = \frac{1}{2}\int_0^T &\left\langle \frac{1}{\sigma(\rho_s)}\left(\partial_s Q_s + \frac{1}{2}\nabla\rho_s\right)^2\right\rangle ds \\
&+ \int_0^T \langle C(\rho_s)(1 - e^{G_s} + G_s e^{G_s})\rangle\, ds \\
&+ \int_0^T \langle A(\rho_s)(1 - e^{-G_s} - G_s e^{-G_s})\rangle\, ds.
\end{aligned}
$$

This leads to the explicit form of the functional (2.13). □

REMARK 5.2. For G and H as in (5.2), then the conservation equation $\partial_t \rho = \partial_t K - \nabla\partial_t Q$ implies that ρ obeys the hydrodynamic limit (3.2) of the modified dynamics $\mathbb{P}^N_{G,H}$ (3.1).

We introduce now the set of regular trajectories:

DEFINITION 5.3. Denote by $\mathcal{S}$ the set of trajectories (ρ, Q, K) satisfying $I(\rho, Q, K) < \infty$ and such that:

- ρ is bounded away from 0 and 1: there is $\varepsilon > 0$ such that $\varepsilon < \rho(t, x) < 1 - \varepsilon$ for any (t, x) in $[0, T] \times [-1, 1]$.
- There exist two smooth functions $G \in C^{1,0}$ and $H \in C^{1,2}$ such that (ρ, Q, K) is a weak solution [in the sense of (A.1)] of

$$
\begin{cases}
\partial_t \rho(t, x) = \frac{1}{2}\Delta\rho(t, x) - \nabla(\sigma(\rho(t, x))\nabla H(t, x)) + C(\rho(t, x)) e^{G(t,x)} \\
\qquad\qquad\quad - A(\rho(t, x)) e^{-G(t,x)}, \\
\rho(t, \pm 1) = \bar{\rho}_\pm, \qquad \rho(t = 0, x) = \gamma(x).
\end{cases}
\tag{5.6}
$$

- For any smooth test function φ in $C^{1,1}([0,T]\times[-1,1])$,

$$\begin{aligned}
&\langle Q_T\varphi_T\rangle - \int_0^T \left\langle Q_s \frac{d}{ds}\varphi_s\right\rangle ds \\
&\quad = -\frac{1}{2}\bar{\rho}_+ \int_0^T \varphi(s,1)\,ds + \frac{1}{2}\bar{\rho}_- \int_0^T \varphi(s,-1)\,ds \\
&\qquad + \frac{1}{2}\int_0^T \langle \rho_s \nabla\varphi_s\rangle\,ds + \int_0^T \langle \sigma(\rho_s)\nabla H_s \varphi_s\rangle\,ds,
\end{aligned} \tag{5.7}$$

$$\langle K_T\varphi_T\rangle - \int_0^T \left\langle K_s \frac{d}{ds}\varphi_s\right\rangle ds = \int_0^T ds\,\langle [C(\rho_s)e^{G_s} - A(\rho_s)e^{-G_s}]\varphi_s\rangle. \tag{5.8}$$

The same argument as in Lemma 5.1 implies that for any trajectory (ρ, Q, K) in $\mathcal{S}$

$$\begin{aligned}
I_0(\rho, Q, K) &= J_{G,H}(\rho, Q, K) \\
&= \frac{1}{2}\int_0^T \langle \sigma(\rho_s)|\nabla H_s|^2\rangle\,ds + \int_0^T \langle C(\rho_s)(1 - e^{G_s} + G_s e^{G_s})\rangle\,ds \\
&\quad + \int_0^T \langle A(\rho_s)(1 - e^{-G_s} - G_s e^{-G_s})\rangle\,ds.
\end{aligned} \tag{5.9}$$

We prove now the lower bound for trajectories in $\mathcal{S}$. Theorem 2.5 is a direct consequence of relation (5.9) and of the following proposition.

PROPOSITION 5.4. *For all open sets $\mathcal{O} \in \mathcal{E}$,*

$$\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}_\gamma^N[(\rho^N, Q^N, K^N) \in \mathcal{O}] \ge -\inf_{(\rho,Q,K)\in\mathcal{O}\cap\mathcal{S}} I(\rho, Q, K).$$

PROOF. Let (ρ, Q, K) be in $\mathcal{O}\cap\mathcal{S}$ and satisfying $I(\rho, Q, K) < \infty$. There is smooth (G, H) for which ρ is a weak solution of (5.6). Thanks to Lemma 3.1, there is $\epsilon > 0$ such that the trajectories concentrate in the set $B_{\delta,\varepsilon}$ (4.3)

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}_{\gamma,G,H}^N(B_{\delta,\varepsilon}^c) \le -1. \tag{5.10}$$

The function $J_{G,H,\epsilon}$ defined in (4.4) is continuous on $\mathcal{E}$. Moreover, since ρ_0 is bounded away from 0 and 1, the function $\varphi \to h_{\gamma,\rho_0}(\varphi(0,\cdot))$ is continuous on $D([0,T], \mathcal{M}_0)$. Let $\mathcal{V} \subset \mathcal{O}$ be an open neighborhood of (ρ, Q, K) such that

$$\begin{aligned}
\forall(\varphi, U, L) \in \mathcal{V} \qquad &|J_{G,H,\epsilon}(\varphi, U, L) - J_{G,H,\epsilon}(\rho, Q, K)| < \delta, \\
&|h_{\gamma,\rho_0}(\varphi_0) - h_{\gamma,\rho_0}(\rho_0)| = |h_{\gamma,\rho_0}(\varphi_0) - h_\gamma(\rho_0)| < \delta.
\end{aligned}$$

Using the change of measure (4.1),

$$\begin{aligned}\mathbb{P}_\gamma^N(\mathcal{O}) &\geq \mathbb{P}_\gamma^N(\mathcal{V}) \geq \mathbb{P}_\gamma^N(\mathcal{V}\cap B_{\delta,\epsilon}) \geq \mathbb{P}_{\rho_0,G,H}^N\biggl(\frac{d\mathbb{P}_\gamma^N}{d\mathbb{P}_{\rho_0,G,H}^N}\mathbf{1}_{\mathcal{V}\cap B_{\delta,\epsilon}}\biggr)\\ &\geq \exp\{-N(J_{G,H,\epsilon}(\rho,Q,K)+h_\gamma(\rho_0)+O_{G,H}(N^{-1},\epsilon,\delta))\}\\ &\quad\times \mathbb{P}_{\rho_0,G,H}^N(\mathcal{V}\cap B_{\delta,\epsilon}).\end{aligned}$$

Thanks to (5.10) and the hydrodynamical limit for the perturbed process (3.2)

$$\lim_{N\to\infty}\mathbb{P}_{\rho_0,G,H}^N(\mathcal{V}\cap B_{\delta,\epsilon})=1,$$

where we used the uniqueness of the weak solution of (5.6) (see the Appendix) to conclude that the probability of $\mathcal{V}$ converges to 1. This leads to

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_\gamma^N(\mathcal{O}) \geq -J_{G,H,\epsilon}(\rho,Q,K)-h_{\gamma,\rho_0}(\rho_0)+O_{G,H}(\delta,\epsilon).$$

Letting $\epsilon\downarrow 0$ and $\delta\downarrow 0$, and since (G,H) satisfy (5.6), the proposition is complete. □

5.2. *Approximation for general trajectories*. To complete Theorem 2.7 for general open sets, it remains to prove the following lemma.

LEMMA 5.5. *We assume Assumptions (L1), (L2). For any (ρ,Q,K) such that $I(\rho,Q,K)<\infty$ there is a sequence $(\rho^{(n)},Q^{(n)},K^{(n)})$ in $\mathcal{S}$ converging weakly to (ρ,Q,K) such that*

$$I(\rho,Q,K)=\lim_{n\to\infty}I(\rho^{(n)},Q^{(n)},K^{(n)}). \tag{5.11}$$

Lemma 5.5 implies that

$$\inf_{(\rho,Q,K)\in\mathcal{O}\cap\mathcal{S}}I(\rho,Q,K)=\inf_{(\rho,Q,K)\in\mathcal{O}}I(\rho,Q,K). \tag{5.12}$$

Combining this identity and Proposition 5.4 proves Theorem 2.7.

5.2.1. *Bounding the density away from* 0 *and* 1. We first approximate the density by trajectories bounded away from 0 and 1.

LEMMA 5.6. *Let $P=(\rho,Q,K)$ be a path such that $I(P)<+\infty$. There is $P_\delta=(\rho_\delta,Q_\delta,K_\delta)$ with density ρ_δ uniformly bounded away from* 0 *and* 1 *which converges to P and such that*

$$I(P)=\lim_{\delta\to 0}I(P_\delta).$$

PROOF. Using Assumption (L1), we first establish a property of the functional I_0. We use decomposition (5.1) of I_0. I_1 is a convex functional of (ρ, Q) as it is the supremum of J_H^1 which are convex functionals of (ρ, Q) [we used that $\sigma(\rho) = \rho(1-\rho)$ is concave]. I_2 is not convex, but we use a trick introduced in [17] and decompose I_2 as

$$I_2(\rho, K) = \sup_{G \in C^{1,0}} \tilde{J}_G^2(\rho, K) + \int_0^T dt \, \langle C(\rho_t) + A(\rho_t) \rangle.$$

The large deviation functional can be decomposed into two terms,

$$I_0(\rho, Q, K) = \tilde{I}_0(\rho, Q, K) + \int_0^T dt \, \langle C(\rho_t) + A(\rho_t) \rangle. \tag{5.13}$$

LEMMA 5.7. *The functional $\tilde{I}_0$ is convex and lower semi-continuous for the weak topology.*

PROOF. Since A and C are concave, $\tilde{J}_G^2(\rho, K)$ is a convex functional of (ρ, K). Taking the supremum over G the function remains convex. Thus we deduce that $\tilde{I}_0$ is a convex functional of (ρ, Q, K).

We check now the lower semi-continuity of the functional $\tilde{I}_2(\rho, K) = \sup_{G \in C^{1,0}} \tilde{J}_G^2(\rho, K)$

$$\begin{aligned} \tilde{J}_G^2(\rho, K) = \langle K_T G_T \rangle &- \int_0^T \left\langle K_s \frac{d}{ds} G_s \right\rangle ds \\ &- \int_0^T ds \, \langle C(\rho_s) e^{G_s} + A(\rho_s) e^{-G_s} \rangle. \end{aligned} \tag{5.14}$$

The functional J_H^1 (2.8) can be treated in the same way so that the lemma will be completed.

As $\tilde{I}_2(\rho, K)$ is given as a supremum, it is enough to check that for any G, the functional $\tilde{J}_G^2$ is lower semi-continuous. Let $(\rho^{(n)}, K^{(n)})$ be a sequence converging weakly to (ρ, K). As A, C are concave, one has

$$\begin{aligned} &\int_0^T ds \, \langle C(\rho_s^{(n)}) e^{G_s} + A(\rho_s^{(n)}) e^{-G_s} \rangle \\ &\quad \le \int_0^T ds \, \langle C(\rho_s) e^{G_s} + A(\rho_s) e^{-G_s} \rangle \\ &\qquad + \langle (\rho_s^{(n)} - \rho_s)(C'(\rho_s) e^{G_s} + A'(\rho_s) e^{-G_s}) \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &\liminf_n - \int_0^T ds \, \langle C(\rho_s^{(n)}) e^{G_s} + A(\rho_s^{(n)}) e^{-G_s} \rangle \\ &\quad \ge - \int_0^T ds \, \langle C(\rho_s) e^{G_s} + A(\rho_s) e^{-G_s} \rangle. \end{aligned}$$

As the other part of the functional $\tilde{J}^2_G$ is linear, we conclude that it is lower semi-continuous. □

We turn now to the approximation procedure. Let $\bar{P} = (\varphi, U, M)$ be the solution of the hydrodynamic equation (2.6),

$$\begin{cases} \partial_t \varphi(t,x) = \frac{1}{2}\Delta\varphi(t,x) + C(\varphi(t,x)) - A(\varphi(t,x)), \\ \partial_t U(t,x) = -\frac{1}{2}\nabla\varphi(t,x), \\ \partial_t M(t,x) = C(\varphi(t,x)) - A(\varphi(t,x)), \end{cases}$$

with boundary conditions

$$\varphi_0(0,x) = \gamma(x), \qquad U(0,x) = 0, \qquad M(0,x) = 0,$$
$$\varphi(t,-1) = \bar{\rho}_-, \qquad \varphi(t,1) = \bar{\rho}_+.$$

By construction $I(\bar{P}) = 0$. We set $P_\delta = (1-\delta)P + \delta\bar{P}$ which has a density bounded away from 0 and 1. As $\tilde{I}_0(\bar{P}) < \infty$, the convexity implies that $\tilde{I}_0(P_\delta) \le (1-\delta)\tilde{I}_0(P) + \delta\tilde{I}_0(\bar{P})$ so that

$$\limsup_{\delta\to 0} \tilde{I}_0(P_\delta) \le \tilde{I}_0(P).$$

As P_δ weakly converges to P, the lower semi-continuity of $\tilde{I}_0$ implies

$$\tilde{I}_0(P) \le \liminf_{\delta\to 0} \tilde{I}_0(P_\delta).$$

Thus $\tilde{I}_0(P_\delta)$ converges to $\tilde{I}_0(P)$. Finally h_γ and $\rho \to \int_0^T dt\, \langle C(\rho_t) + A(\rho_t)\rangle$ are continuous for $\|\cdot\|_\infty$. This completes the lemma. □

5.2.2. *Time regularization.* We will prove:

LEMMA 5.8. *For any path $P = (\rho, Q, K)$ such that $I(P) < \infty$, there is a sequence smooth in time $P_\varepsilon = (\rho_\epsilon, Q_\epsilon, K_\epsilon)$ converging weakly to (ρ, Q, K) such that $I(P_\epsilon)$ converges to $I(P)$.*

In the following, only the regularity of K is needed in order to construct a drift G adapted to the nonconservative current (5.16). However, the regularizing sequence has to satisfy the conservation law (2.10) so that ρ, Q, K will be approximated simultaneously.

PROOF OF LEMMA 5.8. The proof is based on a time convolution and follows similar steps as in Lemma 5.6. We just recall the salient features of the proof (see [18] for further details).

Let ψ_ϵ be a C^∞ approximation of unity such that $\psi_\epsilon = 0$ outside $[0, \epsilon]$ and $\int_0^\epsilon \psi_\epsilon = 1$. To take the convolution product of (ρ, Q, K) with ψ_ϵ, we have to extend the path (ρ, Q, K) beyond the time T. Let r be the solution of the hydrodynamic equation (2.6),

$$\partial_s r_s = \tfrac{1}{2}\Delta r_s + C(r_s) - A(r_s) \qquad \text{with } r_0 = \rho_T.$$

We set for $s > T$,

$$\rho_s = r_{s-T}, \qquad Q_s = Q_T - \frac{1}{2}\int_T^s du \nabla \rho_u,$$
$$K_s = K_T + \int_T^s du\, [C(\rho_u) - A(\rho_u)].$$

Now we define

$$\begin{cases} \rho_\epsilon(t,x) = \rho * \psi_\epsilon(t,x), \\ Q_\epsilon(t,x) = Q * \psi_\epsilon(t,x) - Q * \psi_\epsilon(0,x), \\ K_\epsilon(t,x) = K * \psi_\epsilon(t,x) - K * \psi_\epsilon(0,x). \end{cases} \tag{5.15}$$

The path $(\rho_\epsilon, Q_\epsilon, K_\epsilon)$ satisfies the relation (2.10) and has initial currents equal to 0. We use decomposition (5.1) and approximate independently the functionals I_1 and I_2 by using convexity properties deduced from Assumption (L1). $\square$

5.2.3. *Nonregular drifts.* Thanks to Lemmas 5.6 and 5.8, it is enough to consider a trajectory (ρ, Q, K) smooth in time with a density uniformly bounded away from 0 and 1. We are going to associate to (ρ, Q, K) the drifts H, G as for the trajectories in $\mathcal{S}$ introduced in Definition 5.3. Note that the drifts H, G can be nonsmooth in space.

As $\partial_s K$ exists, the drift G is defined (as for the regular trajectories) as the solution of

$$\forall x \in [-1, 1] \qquad \partial_s K(s,x) = C(\rho(s,x)) e^{G(s,x)} - A(\rho(s,x)) e^{-G(s,x)}. \tag{5.16}$$

Note that K, G solve equation (5.8) and as in the regular case (5.9),

$$\begin{aligned} I_2(\rho, K) = \int_0^T & \langle C(\rho_s)(1 - e^{G_s} + G_s e^{G_s})\rangle \\ & + \langle A(\rho_s)(1 - e^{-G_s} - G_s e^{-G_s})\rangle\, ds. \end{aligned} \tag{5.17}$$

The functional I_1 is the same as the functional of the SSEP (without Glauber rates). Thus the drift of the conservative dynamics can be approximated thanks to the Riesz representation theorem as in [3, 18]. For (ρ, Q) such that $I_1(\rho, Q) < \infty$, there is H in $\mathbb{L}^2([0,T], \mathbb{H}_1(]-1,1[))$ such that

$$I_1(\rho, Q) = \int_0^T \int_{-1}^1 \frac{\sigma(\rho(s,x))}{2} (\nabla H(s,x))^2\, dx\, ds. \tag{5.18}$$

Moreover (5.7) holds. Note that the time regularity is not needed to derive (5.18).

Finally, we check that ρ is a weak solution of (A.1) with the drifts G, H. As the large deviation functional is finite, ρ belongs to $\mathcal{A}$ so that $\mathcal{Q}(\rho)$ is finite and (A.2) is satisfied. Combining the conservation law (2.10) with (5.7) and (5.8) shows that (A.3) holds.

5.2.4. *Approximation paths in $\mathcal{S}$.* We approximate now H, G by smooth drifts H_n, G_n and prove that the associated sequence of paths $(\rho^{(n)}, Q^{(n)}, K^{(n)})$ in $\mathcal{S}$ converges to (ρ, Q, K).

Let H_n be a sequence of C^∞ functions converging to H in $\mathbb{L}^2([0,T], \mathbb{H}_1(]-1, 1[))$. Since C and A satisfy Assumption (L1), and ρ is uniformly bounded away from 0 and 1 (Lemma 5.6), C and A are either uniformly bounded away from 0, or uniformly equal to 0. We will consider only the case $A > 0$, $C > 0$ as the other case follows in the same way. From (5.17), we get that $|G|e^{|G|} \in \mathbb{L}^1([0,T] \times [-1,1])$, so that there is G_n such that

$$\begin{aligned} &\int_0^T \int_{-1}^1 |G_n e^{\pm G_n} - G e^{\pm G}| \xrightarrow[n\to\infty]{} 0 \quad \text{and} \\ &\int_0^T \int_{-1}^1 |e^{\pm G_n} - e^{\pm G}| \xrightarrow[n\to\infty]{} 0. \end{aligned} \tag{5.19}$$

Let $(\rho^{(n)}, Q^{(n)}, K^{(n)})$ be the sequence of paths in $\mathcal{S}$ associated to the regular drifts H_n, G_n. In particular $\rho^{(n)}$ is the weak solution of

$$\begin{cases} \partial_t \rho^{(n)} = \frac{1}{2}\Delta \rho^{(n)} - \nabla(\sigma(\rho^{(n)})\nabla H_n) + C(\rho^{(n)})e^{G_n} - A(\rho^{(n)})e^{-G_n}, \\ \rho^{(n)}(t, \pm 1) = \bar{\rho}_\pm, \qquad \rho^{(n)}(0,\cdot) = \rho(0,\cdot), \end{cases} \tag{5.20}$$

and $Q^{(n)}$ and $K^{(n)}$ are defined as

$$\begin{aligned} Q^{(n)}(t,x) &= \int_0^t \left(-\frac{1}{2}\nabla \rho^{(n)}(s,x) + \sigma(\rho^{(n)}(s,x))\nabla H_n(s,x)\right) ds, \\ K^{(n)}(t,x) &= \int_0^t \left(C(\rho^{(n)}(s,x))e^{G_n(s,x)} - A(\rho^{(n)}(s,x))e^{-G_n(s,x)}\right) ds. \end{aligned} \tag{5.21}$$

From energy estimates [using similar bounds as in the Appendix (A.8)], we get

$$\int_0^T dt \int_{-1}^1 dx \, (\nabla \rho^{(n)})^2 \le C, \tag{5.22}$$

where C is a constant independent of n. As $\rho^{(n)}$ and $\nabla \rho^{(n)}$ are bounded sequences in $\mathbb{L}^2$, they are tight in the weak topology. We want to check the uniqueness of the limiting points. Let R be a limit point of $\rho^{(n)}$. We will prove that R is a weak solution of (A.1). From Section 5.2.3, we know that ρ is also a weak solution of (A.1) so that the uniqueness of the weak solutions (see the Appendix) will imply that $\rho = R$.

We want to take the limit in the weak formulation of (5.20). By construction, for any smooth function φ on $[-1,1]$, we have

$$\int_0^t ds \int_{-1}^1 dx\, \rho^{(n)}(s,x)\nabla\varphi(x) = [\bar\rho_+\varphi(1) - \bar\rho_-\varphi(-1)] - \int_0^t ds \int_{-1}^1 dx \nabla\rho^{(n)}(s,x)\varphi(x).$$

By weak convergence of the subsequence, one has

$$\int_0^t ds \int_{-1}^1 dx\, R(s,x)\nabla\varphi(x) = [\bar\rho_+\varphi(1) - \bar\rho_-\varphi(-1)] - \int_0^t ds \int_{-1}^1 dx \nabla R(s,x)\varphi(x).$$

Furthermore (5.22) implies that the limit ∇R is also in $\mathbb{L}^2$. It remains to take the limit in n in the equation

$$\begin{aligned}\int_{-1}^1 \rho^{(n)}(t)\varphi = {}& \int_{-1}^1 \gamma\varphi - \frac{1}{2}\int_0^t\int_{-1}^1 \nabla\rho^{(n)}\nabla\varphi + \int_0^t\int_{-1}^1 \nabla\varphi\sigma(\rho^{(n)})\nabla H_n \\ &+ \int_0^t\int_{-1}^1 C(\rho^{(n)})e^{G_n}\varphi - \int_0^t\int_{-1}^1 A(\rho^{(n)})e^{-G_n}\varphi.\end{aligned} \tag{5.23}$$

The difficulty is to treat the nonlinear terms. We proceed term by term and start with the nonlinearity C in (5.23) (the term in A can be controlled in the same way). By (5.19), $\exp(G_n)$ converges to $\exp(G)$ in $\mathbb{L}^1$ thus it is enough to check that

$$\lim_{n\to\infty}\int_0^t\int_{-1}^1 C(\rho^{(n)})e^G\varphi = \int_0^t\int_{-1}^1 C(R)e^G\varphi. \tag{5.24}$$

We write

$$\begin{aligned}\int_0^t\int_{-1}^1 &(C(\rho^{(n)}) - C(R))\varphi e^G \\ &= \int_0^t\int_{-1}^1 (C(\rho^{(n)}) - C(\rho^{(n)} * \iota_\delta))\varphi e^G \\ &\quad + \int_0^t\int_{-1}^1 (C(\rho^{(n)} * \iota_\delta) - C(R * \iota_\delta))\varphi e^G \\ &\quad + \int_0^t\int_{-1}^1 (C(R * \iota_\delta) - C(R))\varphi e^G,\end{aligned} \tag{5.25}$$

where ι_δ approximates a Dirac function. We stress that the convolution by ι_δ is a technical step used only to derive the convergence in (5.25). The fact that the boundary conditions of $\rho^{(n)} * \iota_\delta$ are not exactly equal to $\bar\rho_\pm$ has no impact on the convergence.

As δ goes to 0, $R * \iota_\delta$ converges to R. The convergence in the weak topology implies that $\rho^{(n)} * \iota_\delta$ converges a.s. to $R * \iota_\delta$ when n goes to infinity. Suppose that

$$\lim_{\delta \to 0} \sup_n \left| \int_0^t \int_{-1}^1 (C(\rho^{(n)}) - C(\rho^{(n)} * \iota_\delta)) e^G \varphi \right| = 0, \tag{5.26}$$

then choosing δ small and then n large, the convergence in (5.24) follows.

To derive (5.26), we first note that for $x \in]-1+\delta, 1-\delta[$

$$\begin{aligned} \rho^{(n)} * \iota_\delta(x) - \rho^{(n)}(x) & \\ &= \int_{-1}^1 dy\, \iota_\delta(x-y) [\rho^{(n)}(y) - \rho^{(n)}(x)] \\ &= \int_{-1}^1 dy\, \iota_\delta(x-y) \int_x^y du\, \partial_u \rho^{(n)}(u) \\ &\le \sqrt{\int_{-1}^1 du\, (\partial_u \rho^{(n)}(u))^2} \int_{-1}^1 dy\, \iota_\delta(x-y) \sqrt{|x-y|} \\ &\le 2\sqrt{\delta} \sqrt{\int_{-1}^1 du\, (\partial_u \rho^{(n)}(u))^2}. \end{aligned} \tag{5.27}$$

Using the uniform bound (5.22), there is a constant C independent of n such that

$$\int_0^t \int_{-1}^1 (\rho^{(n)} * \iota_\delta - \rho^{(n)})^2 \le C\delta. \tag{5.28}$$

It remains to check (5.26). Let $\mathfrak{m}$ be the uniform measure on $[0,t] \times [-1,1]$ given by $\frac{1}{2t}\, ds\, dx$. Given $\psi(s,x)$ a nonnegative function on $[0,t] \times [-1,1]$ such that $\mathfrak{m}(\psi) = 1$, then the entropy inequality implies that for any function φ

$$\mathfrak{m}(\varphi(s,x)\psi(s,x)) \le \mathfrak{m}(\psi(s,x)\log(\psi(s,x))) + \log \mathfrak{m}(\exp(\varphi(s,x))). \tag{5.29}$$

Since A and C are positive, then $\mathfrak{m}(e^{|G|}|G|)$ is bounded, and (5.29) implies

$$\begin{aligned} &\mathfrak{m}\left(|C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})| |\varphi| \frac{e^{|G|}}{\mathfrak{m}(e^{|G|})} \right) \\ &\quad \le \varepsilon_\delta\, \mathfrak{m}\left(\frac{e^{|G|}}{\mathfrak{m}(e^{|G|})} \log \frac{e^{|G|}}{\mathfrak{m}(e^{|G|})} \right) \\ &\qquad + \varepsilon_\delta \log \mathfrak{m}\left(\exp\left(\frac{1}{\varepsilon_\delta} |C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})| |\varphi| \right) \right), \end{aligned}$$

where ε_δ vanishes (in a suitable way to be determined later) as δ goes to 0. Recall that

$$\forall x \in [0, M] \qquad \exp(x) \le 1 + x + \exp(M) x^2 \quad \text{and} \quad \log(x) \le x - 1.$$

As $|C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})||\varphi| \le 2\|C\|_\infty \|\varphi\|_\infty$, the previous inequalities imply

$$\begin{aligned}\log \mathfrak{m}\Big(\exp\Big(\frac{1}{\varepsilon_\delta}|C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})||\varphi|\Big)\Big) \\ \le \frac{1}{\varepsilon_\delta^2}\exp\Big(\frac{\|C\|_\infty\|\varphi\|_\infty}{\varepsilon_\delta}\Big)\mathfrak{m}\big(|C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})||\varphi| \\ + \big(|C(\rho^{(n)} * \iota_\delta) - C(\rho^{(n)})||\varphi|\big)^2\big).\end{aligned}$$

We choose ε_δ such that

$$\lim_{\delta\to 0} \varepsilon_\delta = 0, \qquad \lim_{\delta\to 0} \frac{\sqrt{\delta}}{\varepsilon_\delta}\exp\Big(\frac{\|C\|_\infty\|\varphi\|_\infty}{\varepsilon_\delta}\Big) = 0.$$

Combining the previous inequality and (5.28), we conclude that (5.26) holds.

We turn now to the nonlinearity σ. Using the decomposition

$$\begin{aligned}\int_0^t \int_{-1}^1 &[\sigma(\rho^{(n)}) - \sigma(R)]\nabla\varphi_s \nabla H_s \\ &= \int_0^t \int_{-1}^1 [\sigma(\rho^{(n)}) - \sigma(\rho^{(n)} * \iota_\delta)]\nabla\varphi_s \nabla H_s \\ &\quad + \int_0^t \int_{-1}^1 [\sigma(\rho^{(n)} * \iota_\delta) - \sigma(R * \iota_\delta)]\nabla\varphi_s \nabla H_s \\ &\quad + \int_0^t \int_{-1}^1 [\sigma(R * \iota_\delta) - \sigma(R)]\nabla\varphi_s \nabla H_s,\end{aligned}$$

the last two terms vanish in the limit as in (5.25). The first term can be controlled, thanks to the uniform bound (5.28) and a Cauchy–Schwarz estimate as $\nabla\varphi\nabla H$ belongs to $\mathbb{L}^2$. This concludes the convergence of (5.23).

Following the same proof, (5.21) implies that $(Q^{(n)}, K^{(n)})$ converges weakly to (Q, K).

5.2.5. *I-convergence.* We finally complete the proof of Lemma 5.5. Let $(\rho^{(n)}, Q^{(n)}, K^{(n)})$ be the regularizing sequence defined in (5.20). We start by proving

$$\limsup_{n\to\infty} I_0(\rho^{(n)}, Q^{(n)}, K^{(n)}) \le I_0(\rho, Q, K). \tag{5.30}$$

From expression (5.18) of I_1 and the weak convergence of $\rho^{(n)}$ to ρ, we get

$$\begin{aligned}I_1(\rho, Q) &= \int_0^T \int_{-1}^1 \frac{\sigma(\rho(s,x))}{2}\nabla H(s,x)^2\,dx\,ds \\ &= \lim_{\delta\to 0}\int_0^T \int_{-1}^1 \frac{\sigma([\rho_s * \iota_\delta](x))}{2}\nabla H(s,x)^2\,dx\,ds\end{aligned}$$

$$= \lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{2}\int_0^T\int_{-1}^1 \nabla H(s,x)^2\sigma\big(\big[\rho_s^{(n)}*\iota_\delta\big](x)\big)\,dx\,ds$$
$$\geq \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{2}\int_0^T\int_{-1}^1 (\nabla H(s,x))^2\big[\sigma\big(\rho_s^{(n)}\big)*\iota_\delta(x)\big]\,dx\,ds,$$

where we used the concavity of σ in the last inequality. Notice that

$$\int_0^T\int_{-1}^1 (\nabla H(s,x))^2\big[\sigma\big(\rho_s^{(n)}\big)*\iota_\delta(x)\big]\,dx\,ds$$
$$= \int_0^T\int_{-1}^1 (\nabla H(s,x))^2\sigma\big(\rho_s^{(n)}(x)\big)\,dx\,ds$$
$$+ \int_0^T\int_{-1}^1 [(\nabla H_s)^2*\iota_\delta(x)-(\nabla H(s,x))^2]\sigma\big(\rho_s^{(n)}(x)\big)\,dx\,ds.$$

As σ is bounded, and $(\nabla H)^2$ is integrable, the last term vanishes uniformly in n when δ goes to 0, so that

$$I_1(\rho,Q)\geq \limsup_{n\to\infty}\frac{1}{2}\int_0^T\int_{-1}^1 \nabla H(s,x)^2\sigma\big(\rho_s^{(n)}(x)\big)\,dx\,ds.$$

As $\nabla H^{(n)}$ converges to ∇H in $\mathbb{L}^2$, and σ is bounded, we conclude that

$$I_1(\rho,Q)\geq \limsup_{n\to\infty} I_1\big(\rho^{(n)},Q^{(n)}\big). \tag{5.31}$$

Similarly for I_2, the concavity and boundedness of C and A imply that

$$I_2(\rho,K)\geq \limsup_{n\to\infty}\int_0^T\int_{-1}^1 C\big(\rho^{(n)}\big)(1-e^G+Ge^G)+A\big(\rho^{(n)}\big)(1-e^{-G}-Ge^{-G}).$$

Then we use the convergence properties on $G^{(n)}$ (5.19) and the fact that C and A are bounded to conclude that

$$I_2(\rho,K)\geq \limsup_{n\to\infty} I_2\big(\rho^{(n)},K^{(n)}\big). \tag{5.32}$$

Estimates (5.31) and (5.32) imply (5.30).

The converse inequality

$$\liminf_{n\to\infty} I_0\big(\rho^{(n)},Q^{(n)},K^{(n)}\big)\geq I_0(\rho,Q,K) \tag{5.33}$$

can be deduced from the weak convergence of $(\rho^{(n)},Q^{(n)},K^{(n)})$ to (ρ,Q,K) and from the decomposition (5.13): the limit follows for the term $\tilde{I}_0$ from its lower semi-continuity, and the convergence of the second term $\int_0^T dt\,\langle C(\rho_t^{(n)})+A(\rho_t^{(n)})\rangle$ can be obtained as in (5.24).

6. The density large deviations. In this section, we recover the density large deviation principle (first derived in [17]) by optimizing the functional I over the currents. We assume that the rates A, C satisfy Assumptions (L1), (L2). The contraction principle [10] implies that the density large deviation functional is given by

$$\mathcal{F}(\rho) = \inf_{(Q,K)} I(\rho, Q, K), \tag{6.1}$$

where the infimum is taken over the currents (Q, K). Using the approximation procedure of Section 5.2, we will check that it is enough to consider regular density profiles and the modified functional

$$\hat{\mathcal{F}}(\rho) = \inf_{(Q,K)_{\text{smooth}}} I(\rho, Q, K), \tag{6.2}$$

where the infimum is taken now over the smooth currents (Q, K).

Step 1. In this first step, the explicit solution of the variational problem (6.2) is computed for smooth density profiles. The functional (6.2) can be rewritten as [see (5.9)]

$$\begin{aligned}\hat{\mathcal{F}}(\rho) = \inf_{(G,H)} \biggl\{ & \frac{1}{2} \int_0^T \langle \sigma(\rho_s) |\nabla H_s|^2 \rangle \, ds \\ & + \int_0^T \langle C(\rho_s)(1 - e^{G_s} + G_s e^{G_s}) \rangle \, ds \\ & + \int_0^T \langle A(\rho_s)(1 - e^{-G_s} - G_s e^{-G_s}) \rangle \, ds \biggr\},\end{aligned} \tag{6.3}$$

where G, H are smooth functions such that the conservation relation (2.10) holds

$$\partial_s \rho_s = \tfrac{1}{2} \Delta \rho_s - \partial_x (\sigma(\rho_s) \nabla H_s) + \bigl(C(\rho_s) e^{G_s} - A(\rho_s) e^{-G_s} \bigr). \tag{6.4}$$

We first check that the infimum is reached for functions H with boundary conditions $H(s, -1) = H(s, 1) = 0$ for any time s. Let f be a smooth function in $[0, T]$. Perturbing H into $H(s, x) + f(s) \int_0^x \frac{1}{\sigma(\rho(s,u))} \, du$, we see that the conservation law (6.4) is preserved (it simply amounts to adding a constant conservative current) and

$$\begin{aligned}\int_0^T & \left\langle \sigma(\rho_s) \left| \nabla H_s + \frac{\alpha f(s)}{\sigma(\rho(s,x))} \right|^2 \right\rangle ds \\ &= \int_0^T \langle \sigma(\rho_s) |\nabla H_s|^2 \rangle \, ds \\ &\quad + 2 \int_0^T \bigl(H(s,1) - H(s,-1) \bigr) f(s) + \int_0^T \left\langle \frac{(f(s))^2}{\sigma(\rho(s,x))} \right\rangle ds.\end{aligned}$$

As H minimizes the integral, this implies that H vanishes at the boundaries.

Suppose that an extremum is reached at (H, G), and consider a perturbation with the new drifts $H + h$ and $G + g$. Then constraint (6.4) implies that h, g satisfy the relation

$$\partial_x(\sigma(\rho_s)\nabla h_s) + g_s\big(C(\rho_s)e^{G_s} + A(\rho_s)e^{-G_s}\big) = 0. \tag{6.5}$$

A perturbation of (6.3) around the extremum H, G leads to

$$0 = \int_0^T \langle\sigma(\rho_s)\nabla H_s \nabla h_s\rangle\, ds + \int_0^T \big\langle g_s G_s\big(C(\rho_s)e^{G_s} + A(\rho_s)e^{-G_s}\big)\big\rangle\, ds. \tag{6.6}$$

Since H vanishes at the boundaries, relation (6.5) combined with (6.6) leads to

$$0 = \int_0^T \big\langle g_s[G_s - H_s]\big(C(\rho_s)e^{G_s} + A(\rho_s)e^{-G_s}\big)\big\rangle\, ds.$$

This holds for any g so that the extremum is such that $G = H$ with H determined by

$$\partial_t\rho = \tfrac{1}{2}\Delta\rho - \nabla(\sigma(\rho)\nabla H) + C(\rho)e^{H} - A(\rho)e^{-H}. \tag{6.7}$$

Thus if H satisfies (6.7) and $G = H$, then an extremum is reached for the corresponding currents $(\hat{Q}, \hat{K})$. Since the functional $I(\rho, Q, K)$ is convex w.r.t. (Q, K) [thanks to the representation (2.8), (2.9)], the extremum $(\hat{Q}, \hat{K})$ has to be a global minimum, thus

$$\begin{aligned}\hat{\mathcal{F}}(\rho) = \frac{1}{2}\int_0^T \langle\sigma(\rho_s)|\nabla H_s|^2\rangle\, ds &+ \int_0^T \langle C(\rho_s)(1 - e^{H_s} + H_s e^{H_s})\rangle\, ds \\ &+ \int_0^T \langle A(\rho_s)(1 - e^{-H_s} - H_s e^{-H_s})\rangle\, ds.\end{aligned}$$

Step 2. To approximate $\mathcal{F}$ (6.1) in terms of $\hat{\mathcal{F}}$ (6.2), we first check that the minimum is reached in (6.1). Consider a sequence (Q^n, K^n) which realizes the infimum. By the tightness argument (Section 4.2) the sequence belongs to a compact set and therefore has a weak limit $(Q^\star, K^\star)$. From the lower semi-continuity of the functional $(Q, K) \to I(\rho, Q, K)$, this weak limit is a minimizer

$$\mathcal{F}(\rho) = I(\rho, Q^\star, K^\star). \tag{6.8}$$

Let $(\rho_\varepsilon, Q^\star_\varepsilon, K^\star_\varepsilon)$ be a smooth sequence [as in (5.20)] converging to $(\rho, Q^\star, K^\star)$ such that

$$I(\rho_\varepsilon, Q^\star_\varepsilon, K^\star_\varepsilon) \to I(\rho, Q^\star, K^\star).$$

As $\hat{\mathcal{F}}(\rho_\varepsilon) \le I(\rho_\varepsilon, Q^\star_\varepsilon, K^\star_\varepsilon)$, one has

$$\limsup_{\varepsilon\to 0} \hat{\mathcal{F}}(\rho_\varepsilon) \le \lim_{\varepsilon\to 0} I(\rho_\varepsilon, Q^\star_\varepsilon, K^\star_\varepsilon) = \mathcal{F}(\rho).$$

Using (6.7), there are smooth $(\hat{Q}_\varepsilon, \hat{K}_\varepsilon)$ such that

$$\hat{\mathcal{F}}(\rho_\varepsilon) = I(\rho_\varepsilon, \hat{Q}_\varepsilon, \hat{K}_\varepsilon).$$

The sequence $(\rho_\varepsilon, \hat{Q}_\varepsilon, \hat{K}_\varepsilon)$ has a bounded large deviation cost, and thus it belongs to a compact set. There is a subsequence such that $(\hat{Q}_\varepsilon, \hat{K}_\varepsilon)$ converges weakly to $(\hat{Q}, \hat{K})$. One gets

$$\liminf_{\varepsilon\to 0} \hat{\mathcal{F}}(\rho_\varepsilon) \geq I(\rho, \hat{Q}, \hat{K}) \geq \mathcal{F}(\rho).$$

This limit follows from the decomposition (5.13). The term $\tilde{I}_0$ converges by lower semi-continuity, and the convergence of the second term $\int_0^T dt\, \langle C(\rho_\varepsilon(t)) + A(\rho_\varepsilon(t))\rangle$ can be obtained as in (5.24) (ρ_ε is just a subsequence extracted from $\rho^{(n)}$).

Combining both estimates, we deduce that $\mathcal{F}$ can be approximated by $\hat{\mathcal{F}}$.

APPENDIX: UNIQUENESS OF THE WEAK SOLUTIONS

Let H be in $\mathbb{L}^2([0,T], \mathbb{H}_1(]-1,1[))$ and $|G|\exp(|G|)$ in $\mathbb{L}^1([0,T]\times\,]-1,1[)$. Given an initial data γ, a weak solution of

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho - \partial_x(\sigma(\rho)\partial_x H) + C(\rho)e^G - A(\rho)e^{-G}, \\ \bar{\rho}(t,\pm 1) = \bar{\rho}_\pm, \qquad \bar{\rho}(0,x) = \gamma(x), \end{cases} \tag{A.1}$$

is defined as:

- The density ρ is in $\mathbb{L}^2([0,T], \mathbb{H}^1(]-1,1[))$, that is, there is a function in $\mathbb{L}^2([0,T]\times\,]-1,1[)$ which will be denoted by $\nabla\rho$ such for every $t \in [0,T]$ and every function $\varphi \in C^1([-1,1])$,

$$\begin{aligned} &\int_0^t ds \int_{-1}^1 dx\, \rho(s,x)\nabla\varphi(x) - \{\bar{\rho}_+\varphi(1) - \bar{\rho}_-\varphi(-1)\}t \\ &\qquad = \int_0^t ds \int_{-1}^1 dx \nabla\rho(s,x)\varphi(x), \end{aligned} \tag{A.2}$$

  where $\bar{\rho}_\pm$ are fixed boundary conditions.
- For every $t \in [0,T]$ and every function $\varphi \in C^1([-1,1])$ vanishing at the boundaries,

$$\begin{aligned} &\int_{-1}^1 dx\, \rho(t,x)\varphi(x) - \int_{-1}^1 dx\, \gamma(x)\varphi(x) \\ &\quad = -\int_0^t ds \int_{-1}^1 dx \nabla\rho(s,x)\nabla\varphi(x) \\ &\qquad + \int_0^t ds \int_{-1}^1 dx\, \sigma(\rho(s,x))\nabla H(s,x)\nabla\varphi(x) \\ &\qquad + \int_0^t ds \int_{-1}^1 dx\, \big(C(\rho(s,x))e^{G(s,x)} - A(\rho(s,x))e^{-G(s,x)}\big)\varphi(x). \end{aligned} \tag{A.3}$$

The hydrodynamic limit (2.6) corresponds to $H = G = 0$.

In this Appendix, we derive the uniqueness of the weak solutions. The main technical difficulty comes from the fact that G and $\partial_x H$ are unbounded; see [14] for bounded drifts. Note that at this stage Assumption (L1) is irrelevant. We will rely on Assumption (L2) on A, C which can be interpreted as follows. Equation (A.1) in a strong form reads

$$\partial_t \rho(t,x) = \Delta\rho(t,x) - \partial_x(\sigma(\rho(t,x))\nabla H(t,x)) - V'_{G(t,x)}(\rho(t,x)),$$

where the reaction term is determined by the space–time dependent potential $V_{G(t,x)}$ with $V'_g(\rho) = -e^g C(\rho) + e^{-g} A(\rho)$. Assumption (L2) ensures that the potential $V_{G(t,x)}$ is convex uniformly in $G(t,x)$. Thus the reaction and the diffusion terms are both contractions, and the solution will be unique. We adapt to our framework the argument of [15, 20].

We consider two initial data ρ_0^1, ρ_0^2 and the corresponding weak solutions ρ^1, ρ^2. We are going to prove that the $\mathbb{L}^1$-norm $\|\rho_t^1 - \rho_t^2\|_1$ decreases in time. For a given $\delta > 0$, we introduce the regularized absolute value

$$U_\delta(v) = \frac{v^2}{2\delta} 1_{\{|v| \le \delta\}} + \left(|v| - \frac{\delta}{2}\right) 1_{\{|v| > \delta\}}. \tag{A.4}$$

Define

$$V_\delta = \{(s,x) \in [0,T] \times [-1,1], \text{ such that } |\rho^1(s,x) - \rho^2(s,x)| \le \delta\}. \tag{A.5}$$

Step 1. We are going to check that for times $t < t'$,

$$\begin{aligned} &\int_{-1}^1 dx\, U_\delta\big(\rho^1(t',x) - \rho^2(t',x)\big) - \int_{-1}^1 dx\, U_\delta\big(\rho^1(t,x) - \rho^2(t,x)\big) \\ &\quad = -\frac{1}{\delta} \int_t^{t'} \int_{-1}^1 ds\, dx\, 1_{\{(x,s) \in V_\delta\}} \nabla(\rho^1 - \rho^2)(s,x) \\ &\qquad\qquad \times \{\nabla(\rho^1 - \rho^2)(s,x) + \bar\sigma(s,x) \nabla H(s,x)\} \\ &\qquad + \frac{1}{\delta} \int_t^{t'} \int_{-1}^1 ds\, dx\, U'_\delta\big(\rho^1(s,x) - \rho^2(s,x)\big) \\ &\qquad\qquad \times \big\{\bar C(s,x) e^{G(s,x)} - \bar A(s,x) e^{-G(s,x)}\big\}, \end{aligned} \tag{A.6}$$

where we set

$$\bar A(t,x) = A(\rho^1(t,x)) - A(\rho^2(t,x)), \qquad \bar C(t,x) = C(\rho^1(t,x)) - C(\rho^2(t,x)),$$
$$\bar\sigma(t,x) = \sigma(\rho^1(t,x)) - \sigma(\rho^2(t,x)).$$

To prove (A.6), we follow the regularization scheme introduced in [5]; see the proof of their Theorem 4.6. For $\varepsilon > 0$, denote by $R_\varepsilon^D : [-1,1]^2 \to \mathbb{R}^+$ (resp., R_ε^N) the resolvent of the Dirichlet Laplacian Δ_D (resp., Neumann Δ_N)

$$R_\varepsilon^D = (\mathrm{Id} - \varepsilon\Delta_D)^{-1}, \qquad R_\varepsilon^N = (\mathrm{Id} - \varepsilon\Delta_N)^{-1}.$$

The mollified trajectory is defined by

$$\rho^{\varepsilon}(t,x) = \bar\rho(x) + R^D_{\varepsilon}(\rho_t - \bar\rho)(x),$$

where $\bar\rho$ stands for the linear profile between $\bar\rho_+$ and $\bar\rho_-$. Note that the resolvent of the Dirichlet Laplacian preserves the boundary conditions of the mollified trajectory.

As ρ is a weak solution of (A.3), one has

$$\begin{aligned}
\partial_t \rho^{\varepsilon}(t,x) &= \partial_t \int_{-1}^{1} dy\, R^D_{\varepsilon}(x,y)\big(\rho(t,y) - \bar\rho(y)\big)\\
&= \int_{-1}^{1} dy\, \partial_y R^D_{\varepsilon}(x,y)\big\{-\partial_y\big(\rho(t,y) - \bar\rho(y)\big) + \sigma(\rho(t,y))\nabla H(t,y)\big\}\\
&\quad + \int_{-1}^{1} dy\, R^D_{\varepsilon}(x,y)\big(C(\rho(t,y))e^{G(t,y)} - A(\rho(t,y))e^{-G(t,y)}\big),
\end{aligned}$$

where we used the fact that $R^D_{\varepsilon}(x,1) = R^D_{\varepsilon}(x,-1) = 0$ for any x.

From the relation $\partial_y R^D_{\varepsilon}(x,y) = -\partial_x R^N_{\varepsilon}(x,y)$, one has

$$\begin{aligned}
\partial_t \rho^{\varepsilon}(t,x) &= \partial_x \int_{-1}^{1} dy\, R^N_{\varepsilon}(x,y)\big\{\partial_y\big(\rho(t,y) - \bar\rho(y)\big) - \sigma(\rho(t,y))\nabla H(t,y)\big\}\\
&\quad + \int_{-1}^{1} dy\, R^D_{\varepsilon}(x,y)\big(C(\rho(t,y))e^{G(t,y)} - A(\rho(t,y))e^{-G(t,y)}\big).
\end{aligned} \tag{A.7}$$

Taking the time derivative, we obtain

$$\begin{aligned}
&\partial_t \int_{-1}^{1} dx\, U_\delta\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big)\\
&\qquad = \int_{-1}^{1} dx\, U'_\delta\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big)\partial_t\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big)
\end{aligned}$$

As $\partial_x R^D_{\varepsilon}(x,y) = -\partial_y R^N_{\varepsilon}(x,y)$, and we get

$$\begin{aligned}
\partial_x\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big) &= -\int_{-1}^{1} dy\, \partial_y R^N_{\varepsilon}(x,y)\big(\rho^1(t,y) - \rho^2(t,y)\big)\\
&= \int_{-1}^{1} dy\, R^N_{\varepsilon}(x,y)\big(\partial_y\rho^1(t,y) - \partial_y\rho^2(t,y)\big),
\end{aligned}$$

where we used that ρ^1, ρ^2 are in $\mathbb{L}^1([0,T], \mathbb{H}_1(]-1,1[))$. Thus we can write

$$\partial_x\big[U'_\delta\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big)\big] = 1_{\{x\in V^{\varepsilon}_{\delta,t}\}} R^N_{\varepsilon}\big(\partial_x(\rho^1 - \rho^2)\big)(t,x)$$

with the notation

$$V^{\varepsilon}_{\delta,t} = \{x \in [-1,1], \text{ such that } |\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)| \le \delta\}.$$

Using relation (A.7), one obtains

$$\begin{aligned}
&\partial_t \int_{-1}^{1} dx\, U_\delta\big(\rho^{1,\varepsilon}(t,x) - \rho^{2,\varepsilon}(t,x)\big) \\
&\quad = - \int_{-1}^{1} dx \int_{-1}^{1} dy\, 1_{\{x \in V_{\delta,t}^{\varepsilon}\}} R_\varepsilon^N\big(\partial_x(\rho^1 - \rho^2)\big)(t,x) \\
&\qquad\qquad\qquad \times R_\varepsilon^N(x,y)\big\{\partial_y\big(\rho^1(t,y) - \rho^2(t,y)\big) + \bar\sigma(t,y)\nabla H(t,y)\big\} \\
&\qquad + \iint_{-1}^{1} dx\, dy\, U_\delta'\big((\rho^{1,\varepsilon} - \rho^{2,\varepsilon})(t,x)\big) R_\varepsilon^D(x,y) \\
&\qquad\qquad \times \big(\bar C(t,y)e^{G(t,y)} - \bar A(t,y)e^{-G(t,y)}\big).
\end{aligned}$$

Taking the limit as ε tends to 0, we recover (A.6).

Step 2. First note that from Assumption (L2), one has

$$U_\delta'\big((\rho^1 - \rho^2)(t,x)\big)\big(\bar C(t,y)e^{G(t,y)} - \bar A(t,y)e^{-G(t,y)}\big) \le 0.$$

Thus the reaction term acts as a contraction. As $\sigma(\rho) = \rho(1-\rho)$, then

$$\forall (s,x) \in V_\delta \qquad |\bar\sigma(s,x)| \le \delta.$$

Thus (A.6) implies

$$\begin{aligned}
&\langle U_\delta(\rho_{t'}^1 - \rho_{t'}^2)\rangle - \langle U_\delta(\rho_t^1 - \rho_t^2)\rangle \\
&\quad \le -\frac{1}{\delta}\int_t^{t'} ds\, \big\langle 1_{\{(x,s)\in V_\delta\}}(\nabla\rho_s^1 - \rho_s^2)^2\big\rangle \\
&\qquad + \int_t^{t'} ds\, \big\langle 1_{\{(x,s)\in V_\delta\}}|\nabla(\rho_s^1 - \rho_s^2)||\nabla H_s|\big\rangle
\end{aligned} \tag{A.8}$$

and using the fact that

$$2|\nabla(\rho^1 - \rho^2)(s,x)||\nabla H(s,x)| \le \frac{1}{\delta}|\nabla(\rho^1 - \rho^2)(s,x)|^2 + \delta|\nabla H(s,x)|^2,$$

we obtain

$$\begin{aligned}
&\langle U_\delta(\rho_{t'}^1 - \rho_{t'}^2)\rangle - \langle U_\delta(\rho_t^1 - \rho_t^2)\rangle \\
&\quad \le -\frac{1}{2\delta}\int_t^{t'} ds\, \big\langle 1_{\{(x,s)\in V_\delta\}}(\nabla\rho_s^1 - \rho_s^2)^2\big\rangle + \delta\int_t^{t'} ds\, \langle|\nabla H_s|^2\rangle \\
&\quad \le \delta\int_t^{t'} ds\, \langle|\nabla H_s|^2\rangle.
\end{aligned}$$

Recall that H belongs to $\mathbb{L}^2([0,T],\mathbb{H}_1(]-1,1[)$; thus as δ tends to 0, the LHS converges to 0. Furthermore U_δ converges to the absolute value function. This implies that

$$\forall t \leq t' \qquad \int_{-1}^{1} dx\, |\rho^1(t', x) - \rho^2(t', x)| \leq \int_{-1}^{1} dx\, |\rho^1(t, x) - \rho^2(t, x)|,$$

from which the uniqueness of the weak solutions follows.

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