The Annals of Applied Probability
2012, Vol. 22, No. 5, 2048–2066
DOI: 10.1214/11-AAP823

NONUNIFORM RANDOM GEOMETRIC GRAPHS WITH LOCATION-DEPENDENT RADII

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We propose a *distribution-free* approach to the study of random geometric graphs. The distribution of vertices follows a Poisson point process with intensity function $nf(\cdot)$, where $n \in \mathbb{N}$, and f is a probability density function on $\mathbb{R}^d$. A vertex located at x connects via directed edges to other vertices that are within a *cut-off* distance $r_n(x)$. We prove strong law results for (i) the critical cut-off function so that almost surely, the graph does not contain any node with out-degree zero for sufficiently large n and (ii) the maximum and minimum vertex degrees. We also provide a characterization of the cut-off function for which the number of nodes with out-degree zero converges in distribution to a Poisson random variable. We illustrate this result for a class of densities with compact support that have at most polynomial rates of decay to zero. Finally, we state a sufficient condition for an enhanced version of the above graph to be almost surely connected eventually.

1. Introduction and main results. In this paper we study the asymptotic properties related to connectivity of random geometric graphs where the underlying distribution of the vertices may not be uniform. A random geometric graph (RGG) consists of a set of vertices that are distributed in space independently, according to some common probability density function. The edge set of the graph consists of the set of all pairs of points that are within a specified cut-off distance. Our point of departure from usual random geometric graphs is the specification of a cut-off function $r(\cdot)$ that determines the edge set. A directed edge exists, from a vertex located at x to another vertex located at y, provided the distance between x and y is less than $r(x)$.

Our motivation for the study of such graphs comes from applications in wireless networks. In models of wireless networks as RGGs, the nodes are assumed to be communicating entities that are distributed randomly in space according to some underlying density. Nodes are assumed to communicate effectively with other nodes that are within a cut-off distance, that is, proportional to the transmission power. Hence, the transmission power has to be sufficiently large for the network

Received January 2010; revised July 2011.
[1]Supported in part by UGC SAP IV and grant from DRDO-IISc program on Mathematical Engineering.
MSC2010 subject classifications. Primary 60D05, 60G70; secondary 05C05, 90C27.
Key words and phrases. Random geometric graphs, location-dependent radii, Poisson point process, vertex degrees, connectivity.

to be connected. However, nodes that are within each other's transmission range interfere and thus cannot transmit simultaneously. In order to maximize spatial reuse, that is, the simultaneous use of the medium by several nodes to communicate, the transmission power should be minimized. Thus the asymptotic behavior of the critical radius of connectivity in a random geometric graph as the number of vertices becomes large is of considerable interest.

Often the nodes are assumed to be distributed in $[0, 1]^d$ according to a Poisson point process of intensity $n \in \mathbb{N}$. In this case, it is known that the critical connectivity radius scales as $O((\log n/n)^{1/d})$. If the underlying density is nonuniform but bounded away from zero, then (see Penrose [12]) the asymptotic behavior of the largest nearest neighbor distance in the graph is $O((f_0^{-1} \log n/n)^{1/d})$, where $f_0 > 0$ is the minimum of the density over its support. Note that the asymptotics of the largest nearest neighbor distance is determined by the reciprocal of the minimum of the density, since it is in the vicinity of the minimum that the nodes are sparsely distributed.

In many applications such as mobile ad-hoc networks and sensor networks, the distribution of the nodes may be far from uniform (see, e.g., Foh et al. [2], Santi [13]). In the case of nonuniform distribution of nodes, it is not efficient from the point of view of maximizing spatial reuse, for all nodes to use the same cut-off radius. Nodes near the mode of the density require a much smaller radius than those at locations where the density is small. Further, the infimum of the density over its support could be zero ($f_0 = 0$). In such cases the asymptotics of the largest nearest neighbor distance or the connectivity threshold will be very different from that given above. One of the major objectives in a wireless sensor network is to maximize battery life, and hence it is important to minimize the energy expended in data transmission. These considerations leads us to RGGs with location-dependent choice of radii. In many applications, it is assumed that the nodes know or can effectively estimate their location (Akkaya and Younis [1], Langendoen and Reijers [8]).

We prescribe a formula for a critical location dependent cut-off radius depending on the intensity, so that almost surely the resulting graphs do not have isolated nodes eventually. A useful property of the graphs we construct is that the distribution of out-degree is independent of the location of the nodes. Vertex degree distributions are important in designing algorithms for distributed computations over wireless networks where the performance worsens with increasing vertex degrees (Giridhar and Kumar [3]). We derive strong law bounds for the maximum and minimum vertex degrees. By considering a finer parametrization of the cut-off function, we show that the number of vertices with zero out-degree converges to a Poisson distribution, under some conditions on the underlying density. We illustrate the result with some examples. A result of this nature for usual random geometric graphs with the uniform and exponentially decaying densities of nodes can be found in Penrose [10], Gupta and Iyer [4], respectively.

The solution to the connectivity problem for random geometric graphs for nonvanishing densities with compact support and dimensions $d \geq 2$ can be found, for example, in Chapter 13 of Penrose [9]. In one dimension, this problem is studied for densities in $[0,1]$ with polynomial rate of decay to zero in Han and Makowski [6], while in Gupta, Iyer and Manjunath [5] the density is assumed to be exponential or truncated exponential. In two dimensions, the asymptotic distribution for the critical connectivity threshold for a large class of densities, including elliptically contoured distributions, distributions with independent Weibull-like marginals and distributions with parallel level curves is derived in Hsing and Rootzen [7]. In dimensions $d \geq 2$, Penrose [11] obtains the asymptotic distribution for the connectivity threshold when the nodes are distributed according to a standard normal distribution. In this paper, we derive a sufficient condition for the RGGs with location-dependent radius to be almost surely connected eventually.

In summary, our primary motivation in proposing the study of graphs with location dependent radii is as follows. It is to enable the design and study of wireless and sensor networks that allow nonstandard distribution of nodes obtained by fitting densities to empirical data obtained from actual deployments. Given such a density, each individual node can be programmed to choose a transmission radius depending on its location so that the network is connected with high probability. Further, any change in the underlying distribution over time due to failures, re-deployments, etc., can be easily accommodated by appropriately changing the transmission radii. As far as analyzing these graphs is concerned, the key features to contend with are that the edges are directed and that the cut-off radius is specified implicitly.

In order to state our results we need some notation.

1.1. *Notation.* Let f be a continuous probability density function with support $S \subset \mathbb{R}^d$. For any random variable X with density f, we denote it by $X \overset{d}{\sim} f$. Let the metric on $\mathbb{R}^d$ be given by one of the ℓ_p norms $1 \leq p \leq \infty$, denoted by $\|\cdot\|$. Let θ_d denote the volume of the unit ball in $\mathbb{R}^d$. We denote by $\bar{B}$ the closure of the set B. Let $\mathcal{X} = \{X_1, X_2, \dots\}$ be a sequence of i.i.d. points distributed according to f. Let $\{N_n\}_{n\geq 1}$ be a nondecreasing sequence of Poisson random variables with $E[N_n] = n$ and define the sequence of sets

$$\mathcal{P}_n = \{X_1, X_2, \dots, X_{N_n}\}, \qquad n \geq 1. \tag{1.1}$$

Note that $\mathcal{P}_n$ is a Poisson point process with intensity nf. For any $r > 0$ and $x \in \mathbb{R}^d$ we denote by $B(x, r)$ the open ball of radius r centered at x. For any Borel set $B \subset \mathbb{R}^d$ and any point process $\mathcal{P}$, $\mathcal{P}[B]$ represents the number of points of $\mathcal{P}$ in B, and define

$$F(B) := \int_B f(x)\, dx. \tag{1.2}$$

We now define the random geometric graphs of interest with location-dependent radii.

DEFINITION 1. For any $n \geq 1$, let $\mathcal{P}_n$ be the set given by (1.1). For any function $r : \mathbb{R}^d \to [0, \infty)$, the random geometric graph $G_n(f, r)$ is defined to be the graph with vertex set $\mathcal{P}_n$, and the directed edge set

$$E_n = \{\langle X_i, X_j\rangle : X_i, X_j \in \mathcal{P}_n, \|X_i - X_j\| \leq r(X_i)\}.$$

We will also consider an augmented version of the above random geometric graph which is obtained by making all the edges in $G_n(f, r)$ bi-directional.

DEFINITION 2. The enhanced random geometric graph $\tilde{G}_n(f, r)$ associated with the graph $G_n(f, r)$ is defined to be the graph with vertex set $\mathcal{P}_n$ and (undirected) edge set

$$\tilde{E}_n = \big\{\{X_i, X_j\} : X_i, X_j \in \mathcal{P}_n, \langle X_i, X_j\rangle \in E_n \text{ or } \langle X_j, X_i\rangle \in E_n\big\}.$$

In the communication application described in the Introduction, the following procedure will give a graph whose edge set will *contain* the edges of the enhanced graph $\tilde{G}_n$. Upon deployment, the nodes broadcast their radius. All nodes reset their transmission radius to be the maximum of their original radius and the ones they receive from the broadcast. Note that this is done only once. Clearly all the directed links in the original graph now become bi-directional together with possible creation of some directed edges. Thus if the enhanced graph $\tilde{G}_n$ is connected, then so is the graph obtained by this procedure.

1.2. *Main results*. For any fixed $c > 0$, define the sequence of *cut-off* functions $\{r_n(c, x)\}_{n \geq 1}$, via the equation

$$\int_{B(x, r_n(c,x))} f(y)\, dy = c\frac{\log n}{n}, \qquad x \in S. \tag{1.3}$$

Later we will have occasion to take c to be a function of x and n as well. We will denote $G_n(f, r_n(c, \cdot))$ by G_n and the associated enhanced graph by $\tilde{G}_n$, when c, f are fixed and r_n is as defined in (1.3). By the Palm theory for Poisson point processes (Theorem 1.6, [9]), the expected out-degree of any node in G_n will be

$$E[\deg(X_1)] = n \int_{B(x, r_n(c,x))} f(y)\, dy = c \log n,$$

which is the same as the vertex degree in the usual random geometric graph defined on uniform points in the connectivity regime. Let P^x and E^x denote the Palm distributions of $\mathcal{P}_n$ conditional on a vertex located at x. By the Palm theory for Poisson point processes, the expected out-degree of a node located at $x \in \mathbb{R}^d$ in the graph G_n will be

$$E^x[\deg(x)] = c \log n.$$

Thus the expected vertex degree of a node in G_n does not depend on the location of the node. In fact, the number of points of $\mathcal{P}_n \setminus \{x\}$ that fall in $B(x, r_n(c, x))$ under P^x will follow a Poisson distribution with mean $c \log n$.

Let $W_n = W_n(c)$ be the number of nodes in G_n that have zero out-degree, that is,

$$W_n = \sum_{X_i \in \mathcal{P}_n} 1_{\{\mathcal{P}_n[B(X_i, r_n(c, X_i)) \setminus \{X_i\}] = 0\}}. \tag{1.4}$$

For each $n \geq 1$, define

$$d_n = \inf\{c > 0 : W_n = 0\}. \tag{1.5}$$

In other words, d_n is the critical cut-off parameter, that is, the smallest c, so that the graphs $G_n(f, r_n(c, \cdot))$ do not have any node with zero out-degree. Our first result is a strong law for this critical cut-off parameter d_n.

THEOREM 1.1. *Let d_n be the critical cut-off parameter as defined in* (1.5). *Then almost surely*,

$$\lim_{n \to \infty} d_n = 1. \tag{1.6}$$

Let $G_n = G_n(f, r_n(c, \cdot))$ be the random geometric graphs as defined in Definition 1 with r_n as in (1.3). Consider the enhanced random geometric graph $\tilde{G}_n = \tilde{G}_n(f, r_n(c, \cdot))$ associated with G_n; see Definition 2. We now state a strong law result for the critical cut-off parameter to eleminate isolated nodes in the graph $\tilde{G}_n$. Let $\tilde{W}_n$ be the number of isolated nodes, that is, nodes with degree zero, in the enhanced graph $\tilde{G}_n$. Define

$$\tilde{d}_n := \inf\{c > 0 : \tilde{W}_n = 0\}. \tag{1.7}$$

Clearly $\tilde{d}_n \leq d_n$ by construction. The following theorem shows that the threshold required to eleminate isolated nodes in the enhanced graph $\tilde{G}_n$ is the same as for the graph G_n.

THEOREM 1.2. *Let $\tilde{d}_n$ be as defined in* (1.7). *Then, almost surely*,

$$\lim_{n \to \infty} \tilde{d}_n = 1. \tag{1.8}$$

The exact asymptotics for the connectivity threshold for random geometric graphs requires a lot of elaborate computations; see Chapter 13, [9]. We provide a sufficient condition that requires only a local computation at each node and makes use of the connectivity threshold for the usual uniform random geometric graphs.

Let X be a random variable with probability density function f with support $S \subset \mathbb{R}^d$, $d \geq 2$. Suppose that f admits a mapping $h : S \to \mathbb{R}^d$ such that $h(X)$ is uniformly distributed on $[0, 1]^d$. For example, if the coordinates of X are independently distributed, then the coordinate mappings of h will be the marginal distributions. Recall that θ_d denotes the volume of the unit ball. For any $\varepsilon > 0$, let $\{m_n\}_{n\geq 1}$ be the sequence defined by

$$m_n(\varepsilon)^d = (1+\varepsilon)\frac{m \log n}{n\theta_d}, \qquad n \geq 1,$$

where

$$m = \max_{0\leq j\leq d-1} \frac{2^j(d-j)}{d}. \tag{1.9}$$

For any set B, we denote by $h(B)$ the image of the set B under h. Suppose that the functions $r_n(c,x)$ are as defined in (1.3). Define the sequence of functions,

$$c_n(\varepsilon, x) := \inf\{c : h(B(x, r_n(c,x))) \supset B(h(x), m_n(\varepsilon))\}, \qquad x \in S, \tag{1.10}$$

and $\{r_n(c_n, \cdot)\}_{n\geq 1}$ to be functions on S that satisfy the equation

$$\int_{B(x, r_n(c_n, x))} f(y)\,dy = c_n(\varepsilon, x)\frac{\log n}{n}, \qquad n \geq 1. \tag{1.11}$$

Let $G_n(f, r_n(c_n))$ be the graphs defined as in Definition 1 with r replaced by $r_n(c_n)$. Let $\tilde{G}_n(f, r_n(c_n))$ be the enhanced graphs associated with $G_n(f, r_n(c_n))$, that is, the graphs obtained by making the edges in $G_n(f, r_n(c_n))$ bi-directional.

THEOREM 1.3. *Let $X \overset{d}{\sim} f$, and suppose that $h(X)$ is uniform on $[0,1]^d$, $d \geq 2$. Then for any $\varepsilon > 0$, almost surely, the sequence of enhanced random geometric graphs $\tilde{G}_n(f, r_n(c_n))$ is connected for all sufficiently large n.*

In particular, the above result implies that

$$P(\tilde{G}_n(f, r_n(c_n)) \text{ is connected }) \to 1 \qquad \text{as } n \to \infty.$$

Our next result is on strong law asymptotics for the maximum and minimum vertex degrees for the sequence of graphs G_n. Let $H : [0,\infty) \longrightarrow [0,\infty)$ be defined by $H(0) = 1$ and

$$H(a) = 1 - a + a\log a, \qquad a > 0. \tag{1.12}$$

The function H has a unique turning point at the minima $a = 1$. Let $H_+^{-1} : [0,\infty) \to [1,\infty)$ be the inverse of H restricted to $[1,\infty)$ and $H_-^{-1} : [0,1] \to [0,1]$ be the inverse of the restriction of H to $[0,1]$.

THEOREM 1.4. *For any $c > 0$, let $\Delta_n = \Delta_n(c)$ be the maximum and $\delta_n = \delta_n(c)$ be the minimum vertex out-degree of the graph $G_n = G_n(f, r_n(c,\cdot))$. Then with probability 1,*

$$\limsup_{n\to\infty} \frac{\Delta_n}{\log n} \leq cH_+^{-1}(c^{-1}). \tag{1.13}$$

If $c < 1$, *then* $\delta_n \to 0$ *almost surely. If* $c > 1$, *then with probability* 1,

$$\liminf_{n\to\infty} \frac{\delta_n}{\log n} \geq c H_-^{-1}(c^{-1}). \tag{1.14}$$

Note that (1.14) does not shed any light on what happens to the minimum vertex degree when $c = 1$. This requires a finer parametrization of the cut-off function. For any $\beta \in \mathbb{R}$, define $\hat{r}_n(\beta, \cdot) : S \to [0, \infty)$ to be functions satisfying

$$F(B(x, \hat{r}_n(\beta, x))) = \int_{B(x,\hat{r}_n(\beta,x))} f(y)\,dy = \frac{\log n + \beta}{n}, \tag{1.15}$$

for all n sufficiently large for which $\log n + \beta > 0$, and arbitrarily otherwise. Since β is fixed, we will write $\hat{r}_n(x)$ for $\hat{r}_n(\beta, x)$. For each $n \geq 1$, define the sets

$$A_n(x) := \{y \in \mathbb{R}^d : \|x - y\| \leq \hat{r}_n(x) + \hat{r}_n(y)\}, \tag{1.16}$$

$$\hat{A}_n(x) := \{y \in \mathbb{R}^d : \max\{\hat{r}_n(x), \hat{r}_n(y)\} \leq \|x - y\| \leq \hat{r}_n(x) + \hat{r}_n(y)\}, \tag{1.17}$$

$$K_n(x, y) := B(y, \hat{r}_n(y)) \setminus B(x, \hat{r}_n(x)), \qquad x, y \in S. \tag{1.18}$$

THEOREM 1.5. *Let* $\hat{W}_n$ *be the number of nodes of out-degree zero in the graph* $\hat{G}_n = G_n(f, \hat{r}_n(\beta, \cdot))$, *where* $\hat{r}_n(\beta, \cdot)$ *is as defined in* (1.15). *Suppose that* f *satisfies the following two conditions for all n sufficiently large*:

(1) *There exists a constant* $\alpha \in (0, 1)$, *such that for all n sufficiently large*

$$\inf_{x\in S} \inf_{y\in\hat{A}_n(x)} F(K_n(x, y)) \geq \alpha\left(\frac{\log n + \beta}{n}\right), \tag{1.19}$$

(2)

$$\sup_{x\in S} F(A_n(x)) = o(n^{\alpha-1}) \qquad \text{as } n \to \infty. \tag{1.20}$$

Then,

$$\hat{W}_n \xrightarrow{d} \mathrm{Po}(e^{-\beta}) \tag{1.21}$$

as $n \to \infty$, *where* $\mathrm{Po}(\lambda)$ *denotes a Poisson random variable with mean* λ.

As noted earlier, Poisson approximation results for the number of isolated nodes are available only for a small class of distributions. Condition (1.20) can be replaced by the following sufficient condition which, as will be shown below, is easier to verify for some classes of densities.

$$\sup_{x\in S} F(B(x, 2\hat{r}_n(x))) = o(n^{\alpha-1}) \qquad \text{as } n \to \infty. \tag{1.22}$$

THEOREM 1.6. *Suppose that the hypothesis* (1.19) *of Theorem* 1.5 *and* (1.22) *are satisfied*, *then* (1.21) *holds true*.

Our final result is an illustration of the use of the above theorem to a class of densities with compact support that have at most polynomial rates of decay to zero. Let f be a continuous density with compact support $S \subset \mathbb{R}^d$, $d \geq 2$. Let $B_i = B(x_i, r_i)$, $i = 1, 2, \ldots, k$, be nonintersecting balls such that $f(x) = 0$ on the boundary of these balls. For each $i = 1, 2, \ldots, k$, there exists integers m_i and p_{ij}, $j = 1, 2, \ldots m_i$ and constants $0 \leq \eta_i < r_i < \delta_i$ such that, either

$$f(y) = \sum_{j=1}^{m_i} A_{ij}(\|y - x_i\| - r_i)^{p_{ij}}, \qquad y \in B(x_i, \delta_i) \setminus B_i \quad \text{and}$$

$$f(y) = 0, \qquad y \in B_i \setminus B(x_i, \eta_i),$$

or

$$f(y) = \sum_{j=1}^{m_i} A_{ij}(r_i - \|y - x_i\|)^{p_{ij}}, \qquad y \in B_i \setminus B(x_i, \eta_i) \quad \text{and}$$

$$f(y) = 0, \qquad y \in B(x_i, \delta_i) \setminus B_i,$$

and $f(y) > 0$ elsewhere in S. The density can vanish, for example, over water bodies which may contain islands (which are being approximated by balls), and the density decays to zero polynomially near the boundary of these balls. In particular, S itself could be taken to be a ball with the density decaying polynomially to zero at the boundary of S in a radially symmetric fashion. We will denote by $\mathcal{H}$ the set of all densities of the above form. The class $\mathcal{H}$ contains many of the standard distributions such as the uniform, triangular, beta, etc., truncated versions of standard distributions with unbounded support such as the Gaussian, gamma, etc. In Santi [13] the density is assumed to be continuous and bounded away from 0 over $[0, 1]^2$, which is contained in $\mathcal{H}$. The class $\mathcal{H}$ also contains the higher dimensional extensions of the polynomial densities considered in Foh et al. [2] and Han and Makowski [6]. As remarked earlier, since our motivation was to allow for nonstandard densities, $\mathcal{H}$ contains functions that are the modulus of analytic functions in the interior of S.

COROLLARY 1.7. *The conclusion of Theorem 1.6 holds for any density in class $\mathcal{H}$.*

2. Proofs.

PROOF OF THEOREM 1.1. First we will show that

$$\limsup_{n\to\infty} d_n \leq 1. \tag{2.1}$$

Fix $\varepsilon > 0$ and let $c = 1 + \varepsilon$. Let $n_k = k^b$, $k \geq 1$, where the constant b will be chosen later. Let $W_n(c)$ be as defined in (1.4). For each $n \geq 1$, define the events

$$A_n := \{W_n(c) > 0\}.$$

Recall that $\mathcal{P}_n[B]$ denotes the number of points of the point process $\mathcal{P}_n$ that fall in the set B. Set

$$B_k := \bigcup_{n=n_k}^{n_{k+1}} A_n, \qquad k \geq 1.$$

We will show that

$$\sum_{k=1}^{\infty} P[B_k] < \infty. \tag{2.2}$$

It will then follow by the Borel–Cantelli lemma that almost surely, only finitely many of the events B_k (and hence the A_n) happen. Consequently, with probability 1, $d_n \leq 1 + \varepsilon$, eventually. Since $\varepsilon > 0$ is arbitrary, this will prove (2.1).

For each $k \geq 1$, let H_k be the event that there is a vertex $X \in \mathcal{P}_{n_{k+1}}$ that has out-degree zero in the graph $G(\mathcal{P}_{n_k} \cup \{X\}, r_{n_{k+1}}(c))$, that is,

$$H_k = \bigcup_{X \in \mathcal{P}_{n_{k+1}}} \{\mathcal{P}_{n_k}[B(X, r_{n_{k+1}}(c)) \setminus \{X\}] = 0\}.$$

Since we have assumed the variables N_n to be nondecreasing and the functions $r_n(c, x)$ are nonincreasing in n for each fixed c and x, we have

$$A_n \subset H_k, \qquad n_k \leq n \leq n_{k+1}.$$

Consequently,

$$B_k \subset H_k, \qquad k \geq 1. \tag{2.3}$$

By the Palm theory for Poisson point processes (Theorem 1.6, [9]), (1.2) and (1.3), we have

$$\begin{aligned} P[H_k] &\leq E\Bigg[\sum_{X \in \mathcal{P}_{n_{k+1}}} 1_{\{\mathcal{P}_{n_k}[B(X, r_{n_{k+1}}(c)) \setminus \{X\}] = 0\}}\Bigg] \\ &= n_{k+1} \int_{\mathbb{R}^d} e^{-n_k F(B(x, r_{n_{k+1}}(c)))} f(x)\, dx \\ &= n_{k+1} \exp\left(-c \frac{n_k}{n_{k+1}} \log n_{k+1}\right) \\ &= (k+1)^b \exp\left(-(1+\varepsilon)\left(\frac{k}{k+1}\right)^b \log(k+1)^b\right). \end{aligned} \tag{2.4}$$

Choose $\gamma > 0$ so that $(1-\gamma)(1+\varepsilon) > 1$, and pick b such that

$$\big((1-\gamma)(1+\varepsilon) - 1\big)b > 1.$$

For sufficiently large k, we have

$$\left(\frac{k}{k+1}\right)^b > (1-\gamma).$$

Using this in (2.4) , we get for all k sufficiently large

$$P[H_n] \leq \frac{1}{(k+1)^{((1-\gamma)(1+\varepsilon)-1)b}},$$

which is summable in k. (2.2) now follows from the above inequality and (2.3). This proves (2.1) by the arguments following (2.2). To complete the proof, we need to show that

$$\liminf_{n\to\infty} d_n \geq 1. \tag{2.5}$$

Fix $c < 1$ and pick u such that $c < u < 1$. Choose x_0 such that $f(x_0) > 0$. Since f is continuous, we can and do fix a $R > 0$ satisfying $g_0 u < f_0$, where

$$f_0 = \inf_{x\in B(x_0,R)} f(x), \qquad g_0 = \sup_{x\in B(x_0,R)} f(x).$$

Let $\varepsilon > 0$ be such that

$$\varepsilon^{1/d} + c^{1/d} < u^{1/d}. \tag{2.6}$$

Choose $R_0 > 0$ such that $2R_0 < R$ and let $B_0 := B(x_0, R_0)$. Define the sequence of functions $\{\bar{r}_n(\cdot)\}_{n\geq 1}$ by

$$\bar{r}_n(v)^d = \frac{v \log n}{\theta_d f_0 n}, \qquad 0 \leq v \leq 1.$$

Let σ_n be the maximum number such that there exists σ_n many disjoint balls of radius $\bar{r}_n(u)$ with centers in B_0. Then (see Lemma 2.1, [12]), we can find a constant c_1 such that for all n sufficiently large,

$$\sigma_n \geq \frac{c_1 n}{\log n}. \tag{2.7}$$

Let $\{x_1, x_2, \dots, x_{\sigma_n}\}$ be the deterministic set of points in B_0 such that the balls $B(x_i, \bar{r}_n(u))$, $i = 1, 2, \dots, \sigma_n$, are disjoint. Let $E_n(x)$ be the event that there is exactly one point of $\mathcal{P}_n$ in $B(x, \bar{r}_n(\varepsilon))$ with no other point in $B(x, \bar{r}_n(u))$, that is,

$$E_n(x) = \{\mathcal{P}[B(x, \bar{r}_n(\varepsilon))] = 1, \mathcal{P}_n[B(x, \bar{r}_n(u)) \setminus B(x, \bar{r}_n(\varepsilon))] = 0\}. \tag{2.8}$$

Note that the events $\{\mathcal{P}[B(x, \bar{r}_n(\varepsilon))] = 1\}$ and $\{\mathcal{P}[B(x, \bar{r}_n(u)) \setminus B(x, \bar{r}_n(\varepsilon))] = 0\}$ are independent. Hence for any $x \in B_0$, we have

$$\begin{aligned} P[E_n(x)] &= nF(B(x, \bar{r}_n(\varepsilon)))e^{-nF(B(x,\bar{r}_n(\varepsilon)))}e^{-nF(B(x,\bar{r}_n(u)))\setminus B(x,\bar{r}_n(\varepsilon))} \\ &= nF(B(x, \bar{r}_n(\varepsilon)))e^{-nF(B(x,\bar{r}_n(u)))}. \end{aligned}$$

Note that for all $x \in B_0$ and all n sufficiently large, $B(x, \bar{r}_n(u)) \subset B(x_0, R)$. Hence for all n sufficiently large and $x \in B_0$, we have

$$\begin{aligned} P[E_n(x)] &\geq nf_0\theta_d \bar{r}_n(\varepsilon)^d \exp(-ng_0\theta_d \bar{r}_n(u)^d) \\ &= \varepsilon \log n \exp\left(-\frac{g_0 u}{f_0} \log n\right) \\ &= \varepsilon n^{-g_0 u/f_0} \log n. \end{aligned} \tag{2.9}$$

Using the fact that the events $E_n(x_i)$, $i = 1, \dots, \sigma_n$, are independent, the inequality $1 - x \le e^{-x}$, and (2.7), (2.9), we get

$$P\left[\left(\bigcup_{i=1}^{\sigma_n} E_n(x_i)\right)^c\right] \le \exp\bigl(-\varepsilon\sigma_n n^{-(g_0 u)/f_0}\log n\bigr) \le \exp\bigl(-c_1\varepsilon n^{1-(g_0 u)/f_0}\bigr),$$

which is summable in n since $g_0 u < f_0$. Hence, by the Borel–Cantelli lemma, almost surely, for all sufficiently large n the event $E_n(x_i)$ happens for some $i = i(n)$. Hence w.p. 1, for all n sufficiently large, we can find a random sequence $j(n)$ such that $X_{j(n)} \in \mathcal{P}_n$, and there is no other point of $\mathcal{P}_n$ within a distance $\bar{r}_n(u) - \bar{r}_n(\varepsilon)$ of $X_{j(n)}$. By (2.6),

$$\bar{r}_n(u) - \bar{r}_n(\varepsilon) \ge \bar{r}_n(c).$$

Since $X_{j(n)} \in B_0$, from (1.3) and the remark above (2.9), we get for sufficiently large n,

$$\begin{aligned}\int_{B(X_{j(n)}, r_n(c, X_{j(n)}))} f(y)\,dy &= c\frac{\log n}{n} \\ &= \theta_d f_0 \bar{r}_n(c)^d \le \int_{B(X_{j(n)}, \bar{r}_n(c))} f(y)\,dy.\end{aligned} \tag{2.10}$$

Hence $r_n(c, X_{j(n)}) \le \bar{r}_n(c)$. It follows that there is no other point of $\mathcal{P}_n$ in $B(X_{j(n)}, r_n(c, X_{j(n)}))$, that is, $X_{j(n)}$ has out-degree zero in $G_n = G(\mathcal{P}_n, r_n(c))$. Consequently w.p. 1, $d_n \ge c$ for all n sufficiently large. Since $c < 1$, this proves (2.5). □

PROOF OF THEOREM 1.2. Since the graph $\tilde{G}_n$ is obtained by making all the edges in G_n bi-directional, $\tilde{W}_n \le W_n$. Consequently $\tilde{d}_n \le d_n$, and hence by Theorem 1.1 we have

$$\limsup_{n\to\infty} \tilde{d}_n \le 1.$$

Thus it suffices to show that

$$\liminf_{n\to\infty} \tilde{d}_n \ge 1. \tag{2.11}$$

Fix $c < 1$, and let x_0, R, R_0 and $\bar{r}_n$ be as in the second part of the proof of Theorem 1.1. Recall the random sequence $j(n)$, defined on a set of probability one, such that for sufficiently large n, the point $X_{j(n)} \in \mathcal{P}_n$ has no other point of $\mathcal{P}_n$ within a distance $\bar{r}_n(c)$. Since $2R_0 < R$, by the same arguments as in (2.10), we have $r_n(x, c) \le \bar{r}_n(c)$ for all $x \in B(x_0, 2R_0)$ and all n sufficiently large. Thus almost surely, none of the points $X \in \mathcal{P}_n$, that fall in the ball $B(x_0, 2R_0) \setminus B(x_0, R_0)$, have an out-going edge to $X_{j(n)}$ for all sufficiently large n.

On the other hand, for any point $x \notin B(x_0, 2R_0)$, suppose $B(x, r_n(c, x)) \cap B(x_0, R_0) \neq \phi$. Then we can find a point y such that $\|y - x_0\| = 3R_0/2$ such that $B(y, R_0/2) \subset B(x, r_n(c, x))$. However,

$$c\frac{\log n}{n} = \int_{B(x,r_n(c,x))} f(u)\,du \geq \int_{B(y,R_0/2)} f(u)\,du \geq \frac{f_0 \theta R_0^d}{2^d},$$

which clearly is not possible for sufficiently large n. Thus there can be no edge leading from a point of $\mathcal{P}_n$ in $B(x_0, 2R_0)^c$ to $X_{j(n)}$. Hence the points $X_{j(n)} \in \mathcal{P}_n$ have zero in-degree as well for sufficiently large n. Consequently, with probability 1, $\tilde{d}_n \geq c$ for all sufficiently large n for any $c < 1$. This proves (2.11) and thus completes the proof of Theorem 1.2. $\square$

PROOF OF THEOREM 1.3. For any $\varepsilon > 0$, let $c_n = c_n(\varepsilon, \cdot)$, $n \geq 1$, be as defined in (1.10). Consider the random geometric graph $\bar{G}_n$ induced by the mapping h and the enhanced graph $\tilde{G}_n = \tilde{G}_n(f, r_n(c_n, \cdot))$. The vertex set of the graph $\bar{G}_n$ is the set $\{h(X) : X \in \mathcal{P}_n\}$ with edges between any two vertices $Y_i = h(X_i)$ and $Y_j = h(X_j)$ provided there is an edge between X_i and X_j in the graph $\tilde{G}_n$. The vertices of $\bar{G}_n$ are distributed according to a homogenous Poisson point process on $[0, 1]^d$ with intensity n.

Let m be as defined in (1.9). Now by Theorem 13.2, [9] (with $k_n \equiv 0$), we have

$$\lim_{n\to\infty} \frac{n\theta T_n^d}{\log n} = m,$$

almost surely, where T_n is the threshold for simple connectivity in the usual uniform random geometric graph on $[0, 1]^d$ (nodes being distributed according to a homogenous Poisson point process with intensity n).

By definition of $c_n(\varepsilon)$, in the graph $\bar{G}_n$, each vertex is connected to all its neighbors that are within a distance $m_n(\varepsilon)$ almost surely for all sufficiently large n. It follows that almost surely, the graphs $\bar{G}_n$, and hence the graphs $\tilde{G}_n$ are connected for all sufficiently large n. $\square$

PROOF OF THEOREM 1.4. We first show (1.13). Fix $c > 0$ and $\varepsilon \in (0, 1)$. Define the sequence

$$c_n = (1 + \varepsilon)cH_+^{-1}\left(\frac{1 + \varepsilon}{c}\right)\log n, \qquad n \geq 1.$$

Fix a constant b such that $b\varepsilon > 1$, and let $n_k = k^b$, $k \geq 1$. Define the events $A_n := \{\Delta_n \geq c_n\}$, $n \geq 1$. Let

$$B_k := \bigcup_{n=n_k}^{n_{k+1}} A_n, \qquad k \geq 1. \tag{2.12}$$

Since N_n, c_n are increasing, and $r_n(c,x)$ is decreasing pointwise in n, we have

$$B_k \subset \bigcup_{X \in \mathcal{P}_{n_{k+1}}} \{\mathcal{P}_{n_{k+1}}[B(X, r_{n_k}(c,X))] \geq c_{n_k} + 1\}. \tag{2.13}$$

By the Palm theory for Poisson point processes (Theorem 1.6, [9]),

$$\begin{aligned} P(B_k) &\leq E\biggl[\sum_{X \in \mathcal{P}_{n_{k+1}}} 1_{\{\mathcal{P}_{n_{k+1}}[B(X, r_{n_k}(c,X)) \setminus \{X\}] \geq c_{n_k}\}}\biggr] \\ &= n_{k+1} \int_{\mathbb{R}^d} P\bigl(\mathrm{Po}(n_{k+1} F(B(x, r_{n_k}(c,x)))) \geq c_{n_k}\bigr) f(x)\, dx, \end{aligned} \tag{2.14}$$

where $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ. From (1.3), we have for sufficiently large k,

$$n_{k+1} F(B(x, r_{n_k}(c,x))) = \frac{n_{k+1}}{n_k} c \log n_k \leq (1+\varepsilon) cb \log k.$$

Hence for sufficiently large k, by definition of H_+^{-1}, we get

$$\begin{aligned} \frac{c_{n_k}}{n_{k+1} F(B(x, r_{n_k}(c,x)))} &\geq \frac{(1+\varepsilon) c H_+^{-1}((1+\varepsilon)/c) b \log k}{(1+\varepsilon) cb \log k} \\ &= H_+^{-1}\biggl(\frac{1+\varepsilon}{c}\biggr) > 1. \end{aligned} \tag{2.15}$$

Hence using the Chernoff bound for the Poisson distribution (see Lemma 1.2, [9]), we get

$$\begin{aligned} &P\bigl(\mathrm{Po}(n_{k+1} F(B(x, r_{n_k}(c,x)))) \geq c_{n_k}\bigr) \\ &\quad \leq e^{-n_{k+1} F(B(x, r_{n_k}(c,x))) H(c_{n_k}/(n_{k+1} F(B(x, r_{n_k}(c,x)))))}. \end{aligned}$$

Since H is increasing in $[1,\infty)$ and $n_{k+1} > n_k$, using (2.15) we can bound the probability on the left-hand side in the above equation by

$$\exp\biggl(-\frac{n_{k+1}}{n_k} cb \log k H\biggl(H_+^{-1}\biggl(\frac{1+\varepsilon}{c}\biggr)\biggr)\biggr) \leq \exp\bigl(-(1+\varepsilon) b \log k\bigr).$$

Substituting this bound in (2.14), we get for sufficiently large k,

$$P(B_k) \leq \frac{(k+1)^b}{k^{b(1+\varepsilon)}} \leq (1+\varepsilon) \frac{1}{k^{b\varepsilon}},$$

which is summable in k, since $b\varepsilon > 1$. Hence, by the Borel–Cantelli lemma, almost surely only finitely many of the events B_k and hence A_n happen. Hence, almost surely,

$$\frac{\Delta_n}{\log n} \leq (1+\varepsilon) c H_+^{-1}\biggl(\frac{1+\varepsilon}{c}\biggr),$$

eventually. The result now follows since $\varepsilon > 0$ is arbitrary.

The result for $\delta_n(c)$ in case $c < 1$ follows from Theorem 1.1. For the case $c > 1$, the proof of (1.14) is entirely analogous to that of (1.13), and so we provide the corresponding expressions. Fix $\varepsilon \in (0, 1)$ such that $(1+\varepsilon) < (1-\varepsilon)c$. Let $b > 0$ be such that $b\varepsilon > 1$, and define the sequence $n_k = k^b$, $k \geq 1$. Define the events $A_n := \{\delta_n \leq c_n\}$ where

$$c_n = (1-\varepsilon)cH_-^{-1}\left(\frac{1+\varepsilon}{(1-\varepsilon)c}\right)\log n, \qquad n \geq 1.$$

Let the events B_k be as defined in (2.12). The expression analogous to (2.13) will be

$$B_k \subset \bigcup_{X \in \mathcal{P}_{n_k}} \{\mathcal{P}_{n_k}[B(X, r_{n_{k+1}}(c, X))] \leq c_{n_{k+1}} + 1\}. \tag{2.16}$$

For sufficiently large k,

$$n_k F(B(x, r_{n_{k+1}}(c, x))) = \frac{n_k}{n_{k+1}} c \log n_{k+1} \geq (1-\varepsilon)cb\log(k+1).$$

Hence for sufficiently large k, by our choice of ε, we get

$$\begin{aligned} \frac{c_{n_{k+1}}}{n_k F(B(x, r_{n_{k+1}}(c, x)))} &\leq \frac{(1-\varepsilon)cH_-^{-1}((1+\varepsilon)/((1-\varepsilon)c))b\log(k+1)}{(1-\varepsilon)cb\log(k+1)} \\ &= H_-^{-1}\left(\frac{1+\varepsilon}{(1-\varepsilon)c}\right) < 1. \end{aligned} \tag{2.17}$$

Again using the Chernoff bound and proceeding as in the previous proof, we will get

$$\begin{aligned} P(B_k) &\leq n_k \exp\left(-(1-\varepsilon)cb\log(k+1)\frac{1+\varepsilon}{(1-\varepsilon)c}\right), \\ &\leq (1+\varepsilon)\frac{1}{(k+1)^{b\varepsilon}}, \end{aligned}$$

which is summable in k. Since $\varepsilon > 0$ is arbitrary, (1.14) now follows by the Borel–Cantelli lemma and the arguments used earlier to infer (1.13). $\square$

The proof of Theorem 1.5 uses the following lemma, which is a straightforward extension of Theorem 6.7, [9]. Let d_{TV} denote the total variation distance between two random variables. Let $\hat{r}_n(\beta)$, $\hat{G}_n$ and $\hat{W}_n$, be as in Theorem 1.5. We will use the notation, $B_n(x) = B(x, \hat{r}_n(x))$,

$$\bar{B}_n(x) = \{y : \|y - x\| \leq 3\max\{\hat{r}_n(x), \hat{r}_n(y)\}\},$$

$$\hat{B}_n(x) = \{y : \max\{\hat{r}_n(x), \hat{r}_n(y)\} \leq \|y - x\| \leq 3\max\{\hat{r}_n(x), \hat{r}_n(y)\}\},$$

and $\mathcal{P}_n^x = \mathcal{P}_n \cup \{x\}$. Recall that $\mathcal{P}_n(B)$ denotes the number of points of $\mathcal{P}_n$ that fall in the set B.

LEMMA 2.1. *Let f be a continuous density on $\mathbb{R}^d$. Then, for the graph $\hat{G}_n$, we have*

$$d_{\mathrm{TV}}(\hat{W}_n, \mathrm{Po}(E[\hat{W}_n])) \le \min\left(3, \frac{1}{E[\hat{W}_n]}\right)(I_n^{(1)} + I_n^{(2)}), \tag{2.18}$$

where

$$\begin{aligned} I_n^{(1)} &= n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\bar{B}_n(x)} f(y)\,dy \\ &\quad \times P[\mathcal{P}_n(B_n(x)) = 0] P[\mathcal{P}_n(B_n(y)) = 0], \end{aligned} \tag{2.19}$$

$$\begin{aligned} I_n^{(2)} &= n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\hat{B}_n(x)} f(y)\,dy \\ &\quad \times P[\mathcal{P}_n^y(B_n(x)) = 0, \mathcal{P}_n^x(B_n(y)) = 0]. \end{aligned} \tag{2.20}$$

PROOF. The proof is identical to the proof of Theorem 6.7, [9] with the following obvious change. In the definition of dependency neighborhood, the parameter r is replaced by the function $\sup_{x \in H_{mi} \cup H_{mj}} \hat{r}_n(x)$. $\square$

PROOF OF THEOREM 1.5. By the Palm theory for Poisson point processes,

$$E[\hat{W}_n] = n \int_{\mathbb{R}^d} f(x)\,dx\, e^{-n \int_{B_n(x)} f(y)\,dy},$$

where $B_n(x)$ is as defined above Lemma 2.1. Using (1.15), we get

$$E[\hat{W}_n] = e^{-\beta}.$$

Hence, by Lemma 2.1, it suffices to show that $I_n^{(1)}, I_n^{(2)}$ converge to zero as $n \to \infty$. Again, using the Palm theory and (1.15), we get

$$\begin{aligned} I_n^{(1)} &= n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\bar{B}_n(x)} f(y)\,dy\, e^{-n \int_{B_n(x)} f(u)\,du} e^{-n \int_{B_n(y)} f(v)\,dv} \\ &= e^{-2\beta} \int_{\mathbb{R}^d} f(x)\,dx \int_{\bar{B}_n(x)} f(y)\,dy \to 0 \end{aligned} \tag{2.21}$$

as $n \to \infty$, by the dominated convergence theorem, since $\hat{r}_n(\beta, x) \to 0$ for each $x \in \mathbb{R}^d$. Next we will show that $I_n^{(2)} \to 0$, as $n \to \infty$.

$$\begin{aligned} I_n^{(2)} &= n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\hat{B}_n(x)} f(y)\,dy\, e^{-n \int_{B_n(x) \cup B_n(y)} f(u)\,du} \\ &= I_n^{(21)} + I_n^{(22)}, \end{aligned} \tag{2.22}$$

where

$$I_n^{(21)} := n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\hat{B}_n(x) \cap \{y : \|y - x\| \ge \hat{r}_n(x) + \hat{r}_n(y)\}} f(y)\,dy\, e^{-n \int_{B_n(x) \cup B_n(y)} f(u)\,du},$$

$$I_n^{(22)} := n^2 \int_{\mathbb{R}^d} f(x)\,dx \int_{\hat{A}_n(x)} f(y)\,dy\, e^{-n \int_{B_n(x) \cup B_n(y)} f(u)\,du},$$

where $\hat{A}_n(x)$ is as defined in (1.17). Consider the inner integral in $I_n^{(21)}$. On the set $\hat{B}_n(x) \cap \{y : \hat{r}_n(x) + \hat{r}_n(y) \le \|x - y\|\}$, we have $B_n(x) \cap B_n(y) = \phi$, and hence by (1.15)

$$\int_{B_n(x)\cup B_n(y)} f(u)\,du = \int_{B_n(x)} f(u)\,du + \int_{B_n(y)} f(u)\,du = 2\frac{\log n + \beta}{n}.$$

Thus, $I_n^{(21)}$ converges to zero using the same arguments as in (2.21). It remains to show that $I_n^{(22)} \to 0$ as $n \to \infty$. Since $B_n(x) \cup B_n(y) = B_n(x) \cup K_n(x, y)$, where $K_n(x, y) = B_n(y) \setminus B_n(x)$, we get from (1.15), (1.19), that $I_n^{(22)}$ is bounded by a constant times

$$n^{1-\alpha} \int_{\mathbb{R}^d} f(x) F(A_n(x))\,dx, \tag{2.23}$$

where $A_n(x)$ is as defined in (1.16). The expression in (2.23) converges to zero as $n \to \infty$ by (1.20). □

PROOF OF THEOREM 1.6. Note that in the proof of Theorem 1.5, (1.20) is used only to prove that (2.23) converges to zero. Hence it suffices to show that (2.23) converges to zero under (1.22). Let $A_n(x)$ be as defined in (1.16). Then

$$\begin{aligned} A_n(x) &= \big(A_n(x) \cap \{\hat{r}_n(y) \le \hat{r}_n(x)\}\big) \cup \big(A_n(x) \cap \{\hat{r}_n(x) \le \hat{r}_n(y)\}\big) \\ &\subset B(x, 2\hat{r}_n(x)) \cup \big(A_n(x) \cap \{\hat{r}_n(x) \le \hat{r}_n(y)\}\big). \end{aligned}$$

Using this, the expression in (2.23) is bounded by

$$n^{1-\alpha}\bigg(\int_{\mathbb{R}^d} f(x) F(B(x, 2\hat{r}_n(x)))\,dx + \int_{\mathbb{R}^d} f(x)\,dx \int_{A_n(x)\cap\{\hat{r}_n(x)\le\hat{r}_n(y)\}} f(y)\,dy\bigg).$$

Applying Fubini's theorem to the second term above, we see that the above expression is bounded by

$$2n^{1-\alpha} \int_{\mathbb{R}^d} f(x) F(B(x, 2\hat{r}_n(x)))\,dx,$$

which converges to zero by (1.22). □

PROOF OF COROLLARY 1.7. Let $B = B(0, 1)$. Consider the following two classes of densities:

$$\mathcal{C}_+ := \bigg\{ f : \mathbb{R}^d \to [0, \infty),\ f \text{ continuous },\ \int_{\mathbb{R}^d} f(y)\,dy = 1,\ \inf_{x\in \operatorname{Supp}(f)} f(x) > 0 \bigg\},$$

where $\operatorname{Supp}(f)$ is the support of f. Denote by $\mathcal{C}_E$ the set of functions $f : B \to [0, \infty)$ such that for some $r \in (0, 1)$ and $p \in \mathbb{N}$, $f(x) = 0$, $x \in B(0, r)$, and $f(x) = A(\|x\| - r)^p$, $x \in B \setminus B(0, r)$, with $\int_B f(y)\,dy = 1$. Let $\mathcal{C}_I$ be the set of functions $f : B \to [0, \infty)$ such that for some $p \in \mathbb{N}$, $f(x) = A(1 - \|x\|)^p$, $x \in B$, with

$\int_B f(y)\,dy = 1$. We first prove the result for densities $f \in \mathcal{C}_+ \cup \mathcal{C}_E \cup \mathcal{C}_I$. To do this we need to verify conditions (1.19) and (1.22).

Step 1. Fix $f \in \mathcal{C}_+$. Let $S = \mathrm{Supp}(f)$ and $f_* = \inf_{x\in S} f(x) > 0$ and $f^* = \sup_{x\in S} f(x)$. By (1.15), for any $x \in S$ and for all n large enough, we get

$$f_*\theta_d \hat{r}_n(x)^d \le \frac{\log n + \beta}{n} \le f^*\theta_d \hat{r}_n(x)^d,$$

or

$$\frac{1}{f^*\theta_d}\frac{\log n + \beta}{n} \le \hat{r}_n(x)^d \le \frac{1}{f_*\theta_d}\frac{\log n + \beta}{n}. \tag{2.24}$$

To prove (1.19), note that for any $y \in \hat{A}_n(x)$, we can inscribe a ball of radius $\hat{r}_n(y)/2$ inside $K_n(x, y)$. From this and (2.24), for all n sufficiently large, we get

$$\begin{aligned} F(K_n(x, y)) &\ge f_*\theta_d\left(\frac{\hat{r}_n(y)}{2}\right)^d \\ &\ge \left(\frac{f_*}{2^d f^*}\right)\left(\frac{\log n + \beta}{n}\right). \end{aligned}$$

This proves (1.19) with $\alpha = \frac{f_*}{2^d f^*} < 1$. By (2.24) we have

$$F(B(x, 2\hat{r}_n(x))) \le f^* 2^d \theta_d \hat{r}_n(x)^d = o(n^{1-\alpha}) \qquad \text{as } n \to \infty. \tag{2.25}$$

This proves (1.22). Thus the Poisson convergence result holds for any $f \in \mathcal{C}_+$.

Step 2. Next we prove the result for $f \in \mathcal{C}_E$. Proof for $f \in \mathcal{C}_I$ is similar and so we omit it. Let $B_r = B \setminus B(0, r)$. Recall that $f(x) = 0$ over $B(0, r)$ and is of the form $f(x) = A(\|x\| - r)^p$ over B_r.

For any $x, y \in B_r$, we have by (1.15),

$$F(B(y, \hat{r}_n(y))) = F(B(x, \hat{r}_n(x))) = \frac{\log n + \beta}{n}. \tag{2.26}$$

Note that the density is radially increasing, that is, $f(x) \le f(y)$ if $\|x\| \le \|y\|$. If $y \in \hat{A}_n(x)$, then $y \notin B(x, \hat{r}_n(x))$. If $\|x\| \le \|y\|$, then using (2.26) and the monotonicity of f we get

$$F(K_n(x, y)) = F\big(B(y, \hat{r}_n(y)) \setminus B(x, \hat{r}_n(x))\big) \ge \frac{1}{2}\left(\frac{\log n + \beta}{n}\right).$$

On the other hand if $y \in \hat{A}_n(x)$ and $\|y\| \le \|x\|$, then by (2.26) and the monotonicity of f we have $F(B(y, \hat{r}_n(y)) \cap B(x, \hat{r}_n(x))) \le \frac{1}{2}(\frac{\log n + \beta}{n})$. Hence

$$F(K_n(x, y)) \ge \frac{1}{2}\left(\frac{\log n + \beta}{n}\right).$$

Thus (1.19) holds with $\alpha = \frac{1}{2}$. Next we verify (1.22) over B_r. By (1.15), we have

$$\frac{\log n + \beta}{n} = F(B(x, \hat{r}_n(x))) = \int_{B(x, \hat{r}_n(x)) \cap B_r} A(\|y\| - r)^p\,dy. \tag{2.27}$$

By changing to polar coordinates, the integral on the right-hand side of the above equation has the bounds

$$c_1\hat{r}_n(x)^{p+d} \le \int_{B(x,\hat{r}_n(x))\cap B_r} A(\|y\|-r)^p\,dy \le c_2\hat{r}_n(x)^{p+d}, \tag{2.28}$$

for some positive constants c_1, c_2. From (2.27) and (2.28), we get

$$\hat{r}_n(x)^{p+d} \le c_1^{-1}\frac{\log n+\beta}{n},$$

and hence for some constant c,

$$F(B(x,2\hat{r}_n(x))) \le c_2(2\hat{r}_n(x))^{p+d} \le c\frac{\log n+\beta}{n} = o(n^{-1/2}).$$

This proves (1.22). The result now follows for any $f \in \mathcal{C}_E$ by Theorem 1.6.

Step 3. Let $f \in \mathcal{H}$. Set $S_0 = S \setminus S_1$ where $S_1 = \bigcup_{i=1}^k (B(x_i,\delta_i)\setminus B(x_i,\eta_i))$. The conditions of Theorem 1.6 hold for S_0 by Step 1, and over S_1 by Step 2. This completes the proof of Corollary 1.7. □

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