The Annals of Applied Probability
2012, Vol. 22, No. 5, 1778–1821
DOI: 10.1214/11-AAP813

SPREADING SPEEDS IN REDUCIBLE MULTITYPE BRANCHING RANDOM WALK

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This paper gives conditions for the rightmost particle in the nth generation of a multitype branching random walk to have a speed, in the sense that its location divided by n converges to a constant as n goes to infinity. Furthermore, a formula for the speed is obtained in terms of the reproduction laws. The case where the collection of types is irreducible was treated long ago. In addition, the asymptotic behavior of the number in the nth generation to the right of na is obtained. The initial motive for considering the reducible case was results for a deterministic spatial population model with several types of individual discussed by Weinberger, Lewis and Li [*J. Math. Biol.* **55** (2007) 207–222]: the speed identified here for the branching random walk corresponds to an upper bound for the speed identified there for the deterministic model.

1. Introduction. The process starts with a single particle located at the origin. This particle produces daughter particles, which are scattered in $\mathbb{R}$, to give the first generation. These first-generation particles produce their own daughter particles to give the second generation, and so on. As usual in branching processes, the nth-generation particles reproduce independently of each other. Particles have types drawn from a finite set, $\mathcal{S}$, and the distribution of a particle's family depends on its type. More precisely, reproduction is defined by a point process (with an intensity measure that is finite on bounded sets) on $\mathcal{S} \times \mathbb{R}$ with a distribution depending on the type of the parent. The first component of the point process determines the distribution of that child's reproduction point process, its type, and the second component gives the child's birth position relative to the parent's. Multiple points are allowed, so that in a family there may be several children of the same type born in the same place.

Let Z be the generic reproduction point process, with points $\{(\sigma_i, z_i)\}$, and Z_σ the point process (on $\mathbb{R}$) of those of type σ. Let $\mathbb{P}_\nu$ and $\mathbb{E}_\nu$ be the probability and expectation associated with reproduction from a parent with type $\nu \in \mathcal{S}$. Thus, $\mathbb{E}_\nu Z_\sigma$ is the intensity measure of the positions of children of type σ born to a parent of type ν at the origin. The usual Markov-chain classification ideas can be used to classify the types: the type-space is divided, using the relationship "can

Received September 2008; revised September 2011.
MSC2010 subject classifications. Primary 60J80; secondary 60J85, 92D25.
Key words and phrases. Branching random walk, multitype, speed, anomalous spreading, reducible.

have a descendant of this type," into self-communicating classes, each of which corresponds to an irreducible multitype branching process. Two types are in the same class exactly when each can have a descendant, in some generation, of the other type. A class will be said to precede another if the first can have descendants in the second, and then the second will be said to stem from the first.

Let $Z^{(n)}$ be the nth-generation point process. Let $Z_\sigma^{(n)}$ be the points of $Z^{(n)}$ with type σ. Later, exponential moment conditions on the intensity measure of Z will be imposed that ensure these are well-defined point processes (because the expected numbers in bounded sets are finite). Let $\mathcal{F}^{(n)}$ be the information on all families with the parent in a generation up to and including $n-1$. Hence $Z^{(n)}$ is known when $\mathcal{F}^{(n)}$ is known. Let $m(-\theta)$ be the nonnegative matrix of the Laplace transforms of the intensity measures $\mathbb{E}_\nu Z_\sigma$:

$$(m(\theta))_{\nu\sigma} = \int e^{\theta z}\mathbb{E}_\nu Z_\sigma(dz) = \mathbb{E}_\nu\left[\int e^{\theta z} Z_\sigma(dz)\right].$$

Then it is well known, and verified by induction, that the powers of the matrix m provide the transforms of the intensity measures $\mathbb{E}_\nu Z_\sigma^{(n)}$:

$$\mathbb{E}_\nu\left[\int e^{\theta z} Z_\sigma^{(n)}(dz)\right] = \int e^{\theta z}\mathbb{E}_\nu Z_\sigma^{(n)}(dz) = (m(\theta)^n)_{\nu\sigma}. \tag{1.1}$$

Let $\mathcal{B}_\sigma^{(n)}$ be the rightmost particle of type σ in the nth generation, so that

$$\mathcal{B}_\sigma^{(n)} = \sup\{z : z \text{ a point of } Z_\sigma^{(n)}\}$$

and let $\mathcal{B}^{(n)}$ be the rightmost of these.

When the collection of types is irreducible, so that any type can occur in the line of descent of any type, and there is a $\phi > 0$ such that

$$\sup_{\nu,\sigma}(m(\phi))_{\nu\sigma} < \infty, \tag{1.2}$$

there is a constant Γ such that

$$\frac{\mathcal{B}^{(n)}}{n} \to \Gamma \qquad \text{a.s.-}\mathbb{P}_\nu, \tag{1.3}$$

when the process survives. When this holds the speed, starting in ν, is Γ. This result is in Biggins [(1976a), Theorem 4] and, in a more general framework where time is not assumed discrete, in Biggins (1997), Section 4.1. Furthermore, with the obvious adjustment for periodicity, the same result holds with $\mathcal{B}_\sigma^{(n)}$ in place of $\mathcal{B}^{(n)}$—when the type set is aperiodic this is in Biggins (1976b), Corollary V.4.1. The theory for the irreducible process also provides various formulas for Γ in terms of the reproduction process. The question addressed here is what happens when the set of types is reducible.

Write the transpose of m in the canonical form of a nonnegative matrix, described in Seneta (1973, 1981), Section 1.2. This amounts to ordering the rows, and the labels on the classes, so that when one class stems from another it is also later in the ordering. Then there are irreducible blocks, one for each class, down the diagonal and all other nonzero entries in m are above this diagonal structure. Having done this, call the first class, $\mathcal{C}_1$, the second $\mathcal{C}_2$ up to the final one $\mathcal{C}_K$. Intermediate classes need not be totally ordered by "descends from," so their ordering need not be unique.

Any irreducible matrix has a "Perron–Frobenius" eigenvalue (which is positive, is largest in modulus and has corresponding left and right eigenvectors that are strictly positive)—see Seneta (1973, 1981) or Lancaster and Tismenetsky (1985). For $\theta \geq 0$, let $\exp(\kappa_i(\theta))$ be the "Perron–Frobenius" eigenvalue of the ith irreducible block, which is infinite when any entry is infinite. Let $\kappa_i(\theta) = \infty$ for $\theta < 0$; this is just a device to simplify the formulation, since the development concerns only the right tails of the measures—left tails and the consideration of the leftmost particle are just the mirror image. Call κ_i the PF^+eigenvalue of the corresponding matrix, which with these definitions is not necessarily its "Perron–Frobenius" eigenvalue for strictly negative arguments. As Laplace transforms, the logarithms of the nonzero entries in m are convex. Then κ_i is convex—see Lemma 4.3 below.

Let $\mathcal{D}(f)$ be the set where the function f is not $+\infty$, so that $\mathcal{D}(f) = \{\theta : f(\theta) < \infty\}$. Thus in the irreducible case (1.2) is equivalent to $\mathcal{D}(\kappa) \cap (0, \infty) \neq \varnothing$. Furthermore, since each κ_i is convex, $\mathcal{D}(\kappa_i)$ must be an interval in $[0, \infty)$. For any two classes $\mathcal{C}_i$ and $\mathcal{C}_j$ let

$$\mathcal{D}_{i,j} = \bigcap \{\mathcal{D}(m_{\nu\upsilon}) : \nu \in \mathcal{C}_i, \upsilon \in \mathcal{C}_j, m_{\nu\upsilon} > 0\},$$

which is the set where all of the entries in m linking $\mathcal{C}_i$ to $\mathcal{C}_j$ are finite. For any set of reals A let A^+ be all values either in A or greater than those in A. Thus, $\mathcal{D}^+(f)$ has the form $[\varphi, \infty)$ or (φ, ∞), depending on whether $f(\varphi)$ is finite or not.

Without loss of generality, assume that the initial type ν is in the first class, $\mathcal{C}_1$, and that the speed is sought for a type σ in the final class, $\mathcal{C}_K$. Write $i \to j$ if some $\nu \in \mathcal{C}_i$ can have a child (i.e., an immediate descendant) with a type in $\mathcal{C}_j$ and write $i \Rightarrow j$ when i precedes j so that types in class $\mathcal{C}_i$ can have descendants in some later generation with types in class $\mathcal{C}_j$. Assume also, again without loss, that every other class stems from the first and precedes the last. It is now possible to give a result that illustrates the nature of the result on speed without the weight of additional notation needed for its proof or for the results which establish rather more.

THEOREM 1.1. *Let $\nu \in \mathcal{C}_1$, $\sigma \in \mathcal{C}_K$. Suppose that the process made up of individuals in $\mathcal{C}_1$ alone is supercritical and aperiodic* (*i.e.*, *the mean matrix is primitive and has "Perron–Frobenius" eigenvalue greater than* 1) *and survives with probability* 1. *Assume that*

$$\text{there are } \phi_i \in \mathcal{D}(\kappa_i) \qquad \text{with } 0 < \phi_1 \leq \phi_2 \leq \cdots \leq \phi_K \tag{1.4}$$

$$\text{and } \mathcal{D}^+(\kappa_i) \cap \mathcal{D}(\kappa_j) \subset \mathcal{D}_{i,j} \qquad \text{whenever } i \to j. \tag{1.5}$$

Then

$$\frac{\mathcal{B}_\sigma^{(n)}}{n} \to \Gamma = \max_{i \Rightarrow j} \inf_{0<\varphi\le\theta} \max\left\{\frac{\kappa_i(\varphi)}{\varphi}, \frac{\kappa_j(\theta)}{\theta}\right\} \qquad a.s.\text{-}\mathbb{P}_\nu.$$

The conditions (1.4) *and* (1.5) *both hold when the domain of finiteness of every nonzero entry in the matrix m has the same nonempty intersection with* $[0, \infty)$.

This result, other than the actual form of the limit, will be derived as a by-product of a result on the size of $Z_\sigma^{(n)}[na, \infty)$ described later, in Theorem 2.4. That approach to deriving the speed was used for the one-type process in Biggins (1977) and for the irreducible process in Biggins (1997), Section 4.1. The comparatively simple formula for the limit here is one of the main achievements of this study. One interpretation of this formula for the speed is the following: look at each pair of classes where one precedes the other, compute the speed as though these were the only classes present, and then maximize over all such pairs.

It is probably worth being explicit about some of the assumptions that are not made in Theorem 1.1 and the other main theorems. First, the point processes Z are not constrained to have only a finite number of points. The conditions do mean that there are only a finite number of points in any finite interval, but they do not prevent intervals of the form $(-\infty, a]$ from having an infinite number of points. Second, classes after the first one do not have to be supercritical. Third, classes after the first one do not have to be primitive. Finally, it is not assumed that the dispersal in a class is "nondegenerate," so κ_i could be linear in θ when finite, which for a one-type class corresponds to a deterministic displacement of the family from the parent.

An initially unexpected phenomenon is contained within Theorem 1.1. Its essence can be indicated even in the reducible two-type case. Suppose type a can give rise to both type a and type b particles but type b give rise only to type b. Type a or b considered alone forms a one-type branching random walk with speed Γ_a or Γ_b, respectively. At first sight, it seems plausible that, when $\Gamma_a > \Gamma_b$, both types spread at speed Γ_a, driven by the type a particles, and that otherwise, when $\Gamma_a \le \Gamma_b$, the two types move at their own speeds. This plausible conjecture can be false; it is possible to find examples where, in the presence of type a, the type b speed can be faster than $\max\{\Gamma_a, \Gamma_b\}$. The fundamental reason for this "super-speed" phenomenon is that the speed of spread is caused by the interplay between the exponential growth of the population size and the exponential decay of the tail of the dispersal distribution. It is possible for the growth in numbers of type a, through the numbers of type b they produce, to increase the speed of type b from that of a population without type a. When the type a dispersal distribution has comparatively light tails, that speed can exceed also that of type a. In this cartoon version, to get "super-speed" we need the population of a's to grow quickly but the b's to have more chance of dispersing a long way. This also indicates a complication. There are two possible sources for a comparatively heavy-tailed distribution of the b's. It could be that the a's, in producing children of type b, disperse them widely, or it could be that type b's, in producing b's, produce more spread than type a's producing a's. Either effect can influence the speed of the b's. In Theorem 1.1, (1.4) concerns the growth and dispersion within each irreducible class while (1.5) controls the dispersion involved in moving between classes. The interpretation given above of the formula for the speed shows that, normally, the two-type illustration of super-speed is archetypal—there is no possibility of additional "cooperation" from three or more classes that cannot be exhibited with just two.

The stimulus for considering this problem was the work of Weinberger, Lewis and Li (2007), where a deterministic version is discussed and the phenomenon of "super-speed," which they call "anomalous spreading speed," is identified—although there the actual speed is not identified. They also explore the relevance of the phenomenon in a biological example. There are close relations between these deterministic models—and also certain continuous-time ones which involve coupled reaction-diffusion equations—and the branching models examined here. A discussion of this connection, which is more than an analogy, and further illustration of the "super-speed" phenomenon based on applying the results here in the two-type case can be found in the second half of Biggins (2010).

It turns out that the results for the general case rest on those for a more restricted class of processes. A multitype branching process will be called sequential when each class has children only in its own class and the next one and there is exactly one pair of types linking successive classes. Thus there is just one route through the classes $\mathcal{C}_1, \dots, \mathcal{C}_K$, corresponding to the order of the indices. Also, for $i = 1, \dots, K-1$, there is exactly one type in $\mathcal{C}_i$ that can produce offspring in $\mathcal{C}_{i+1}$, and just one type of offspring in $\mathcal{C}_{i+1}$ that it can produce. The next section describes most of the main results, which concern sequential processes. The shape of the remainder of the paper will be indicated in the course of that section and the subsequent one.

2. Results for the sequential case. Throughout this section, the process will be assumed sequential. In the following one the main results for the general process are given. Several transformations of functions will be needed to describe the results. The first is a version of the Fenchel dual (F-dual) of the function f, given by the convex function

$$f^*(x) = \sup_\theta\{\theta x - f(\theta)\}. \tag{2.1}$$

The second is sweeping strictly positive values to infinity: let

$$f^\circ(a) = \begin{cases} f(a), & \text{when } f(a) \le 0, \\ \infty, & \text{when } f(a) > 0. \end{cases}$$

Also, for any function f let

$$\Gamma(f) = \inf\{a : f(a) > 0\}. \tag{2.2}$$

Then $\Gamma(f) = \Gamma(f^\circ)$. It will also be convenient to have a notation for taking the F-dual and then sweeping positive values to infinity, so let

$$f^{\circledast} = (f^*)^\circ. \tag{2.3}$$

Various properties of such functions are described in Section 4. In particular, $f^{\circledast}$ is continuous when finite. The next two results, which are for the case with only one class, demonstrate why these functions will be useful. Both results are given, with an indication of their proofs, in Biggins [(1997), Section 4.1], and will be discussed further in Section 5, where various results for the irreducible case that are necessary preliminaries for the main proofs are obtained.

PROPOSITION 2.1. *Suppose that there is just one class of types, that the exponential moment condition* (1.2) *holds and that the matrix m is primitive with PF$^+$eigenvalue κ. Let U be the upper end-point of the interval on which κ^* is finite. Then, for $a \neq U$,*

$$\lim_n \frac{1}{n} \log\bigl(\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty)\bigr) = -\kappa^*(a). \tag{2.4}$$

PROPOSITION 2.2. *Under the conditions of Proposition* 2.1 *and the additional assumption that the process is supercritical* [*i.e.*, $\kappa(0) > 0$] *and survives with probability* 1,

$$\lim_n \frac{1}{n} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) = -\kappa^{\circledast}(a) \qquad \bigl(=(\kappa^*)^\circ(a)\bigr) \qquad a.s.\text{-}\mathbb{P}_\nu \tag{2.5}$$

for $a \neq \Gamma(\kappa^*)$ and

$$\frac{\mathcal{B}_\sigma^{(n)}}{n} \to \Gamma(\kappa^*) = \Gamma(\kappa^{\circledast}) \qquad a.s.\text{-}\mathbb{P}_\nu.$$

In this case, there is a simple relationship between the behavior of $Z_\sigma^{(n)}[na, \infty)$ and its expectation. When the expectation decays (geometrically) in (2.4) the actual numbers, described by (2.5), are ultimately zero, leading to the limit there being infinite (which explains the sweeping to infinity). On the other hand, when expected numbers grow the actual numbers grow in the same way. Thus the "expectation-speed" and the "almost-sure-speed" are the same [and are both $\Gamma(\kappa^*)$]. In the reducible process this need not be so—the "expectation-speed" can overestimate the "almost-sure-speed." The discussion here will concentrate on the "almost-sure-speed," but expected numbers, which are easier to study, will be considered briefly in Section 12, mainly to illustrate the point just made.

The result on the speed in Proposition 2.2 is a consequence of the asymptotic behavior of nth-generation numbers in intervals of the form $(-\infty, na]$. The same basic approach is used to study reducible sequential processes. There are two parts to this: showing that a suitable function forms a lower bound and then showing that it also forms an upper bound. As might be anticipated from the role of the moment condition (1.2) in the irreducible case, conditions on the finiteness of the entries in m are needed. For the simplest lower bound these conditions will only concern the entries in the irreducible blocks of m, as in (1.4). But for the upper bound the "off-diagonal" entries have to be controlled too, leading to conditions like (1.5). The basic idea for obtaining both bounds is to use induction on the number of classes, with the formula for the bounds being given by suitable recursions.

Certain properties of the limit $\kappa^{\circledast}$ in (2.5), which is a rate function in the large deviations' sense, are sufficiently important here to merit a name.

DEFINITION 1. A function will be called an r-function if it is increasing and convex, takes a value in $(-\infty, 0)$, is continuous from the left and is infinite when strictly positive.

Whenever r is an r-function $\Gamma(r) > -\infty$. Lemma 5.6 shows that $\kappa^{\circledast}$ is an r-function.

The next theorem, which is proved in Section 6, gives a lower bound on the numbers, and hence on the speed. A notation for the convex minorant is needed. For any two functions f and g, let $\mathfrak{C}[f, g]$ be the greatest lower semi-continuous convex function beneath both of them. (The restriction to lower semi-continuous functions only affects values at the end-points of the set on which a convex function is finite.)

THEOREM 2.3. *Consider a sequential process with K classes, $\mathcal{C}_1, \dots, \mathcal{C}_K$, with corresponding PF$^+$eigenvalues $\kappa_1, \dots, \kappa_K$ and in which $\mathcal{C}_1$, considered alone, is primitive, supercritical and survives with probability* 1. *Assume that* (1.4) *holds. Define r_i recursively*:

$$r_1 = \kappa_1^{\circledast} \qquad (=(\kappa_1^*)^\circ); \qquad r_i = \mathfrak{C}[r_{i-1}, \kappa_i^*]^\circ \qquad \textit{for } i = 2, \dots, K. \tag{2.6}$$

Then for $\nu \in \mathcal{C}_1$, $\sigma \in \mathcal{C}_K$ and $a \neq \Gamma(r_K)$

$$\liminf \frac{1}{n} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) \geq -r_K(a) \qquad a.s.\text{-}\mathbb{P}_\nu, \tag{2.7}$$

$$\liminf_n \frac{\mathcal{B}_\sigma^{(n)}}{n} \geq \Gamma(r_K) \qquad a.s.\text{-}\mathbb{P}_\nu \tag{2.8}$$

and r_K is an r-function.

The first complement to this lower bound is presented next. Once additional ideas have been introduced, Theorem 2.6 will give the same conclusions under weaker conditions.

THEOREM 2.4. *In the setup and conditions of Theorem 2.3, suppose also that, for* $i = 1, 2, \ldots, K-1$,

$$\left(\bigcap_{j\le i}\mathcal{D}^+(\kappa_j)\right)\cap\mathcal{D}(\kappa_{i+1})\subset\mathcal{D}_{i,i+1}. \tag{2.9}$$

Then

$$\frac{1}{n}\log\bigl(Z_\sigma^{(n)}[na,\infty)\bigr)\to -r_K(a)\qquad a.s.\text{-}\mathbb{P}_\nu \tag{Nu}$$

for $a \neq \Gamma(r_K)$, *and*

$$\frac{\mathcal{B}_\sigma^{(n)}}{n}\to\Gamma(r_K)\qquad a.s.\text{-}\mathbb{P}_\nu. \tag{Sp}$$

The condition (1.4) ensures that the set on the left in (2.9) contains ϕ_i, and so is not empty. Note that (1.4) and (2.9) just involve comparing the domains of finiteness of the entries in m. Hence these conditions are easily applied in the general (nonsequential) case. Note too that (1.5) in Theorem 1.1 is a stronger assumption than (2.9) in this theorem.

To describe the remaining results in this section, one further transformation is needed. As can be seen from Proposition 2.2, the critical function when looking at actual numbers in the first class is $\kappa^{\circledast}$ (rather than κ^*). Typically, there will be a $\vartheta \in (0,\infty)$ such that for $a \le \Gamma(\kappa^*)$

$$\kappa^*(a)=\sup_\theta\{\theta a-\kappa(\theta)\}=\sup_{\theta\le\vartheta}\{\theta a-\kappa(\theta)\}.$$

Then, with $\hat\kappa(\theta)=\kappa(\theta)$ for $\theta\le\vartheta$ and $\hat\kappa(\theta)=\theta\Gamma(\kappa^*)$ for $\theta>\vartheta$, it turns out that $\kappa^{\circledast}$ is the F-dual of $\hat\kappa$, that is, $\kappa^{\circledast}=(\hat\kappa)^*$. Thus, in examining how actual numbers in the first class influence numbers in the second, $\hat\kappa$ should replace κ. This means that the shape of κ only matters up to a certain point, after which it is replaced by a suitable linear function. The details of κ beyond this point have become irrelevant because they only influence κ^* at positive values, which are swept to infinity.

Although this motivation is on the right lines, it turns out that the actual definition of the transformation is better framed somewhat differently in order to cover all cases. It will also be useful to have a name for the class of functions the transformation will apply to. Under the conditions of Proposition 2.1, κ satisfies the next definition.

DEFINITION 2. A function is k-convex if it is convex, finite for some $\theta > 0$ and infinite for all $\theta < 0$.

The pointwise supremum of a collection of convex functions is convex, and that of a collection of monotone functions is monotone. Hence, for k-convex f, it makes sense to define $f^\natural$ to be the maximal convex function such that $f^\natural \le f$ and $f^\natural(\theta)/\theta$ is monotone decreasing in $\theta \in (0, \infty)$. This function will be identically minus infinity if there are no functions satisfying the constraints. Now let

$$\vartheta(f) = \sup\{\theta : f(\theta) = f^\natural(\theta)\}, \tag{2.10}$$

where it is possible that $\vartheta(f) = \infty$. Proposition 7.1 will show that, in the typical case, $\kappa^\natural(\theta)$ is just the straight line $\theta\Gamma(\kappa^*)$ for $\theta > \vartheta(\kappa)$, and that line is the tangent to κ at $\vartheta(\kappa)$, which connects this definition with the motivation offered in the previous paragraph.

An alternative recursion for the r-functions defined by (2.6) in Theorem 2.3 turns out to be more useful when considering upper bounds. This alternative recursion is given in the next result. Let $\mathfrak{M}[f, g](\theta) = \max\{f(\theta), g(\theta)\}$.

PROPOSITION 2.5. *Assume that* (1.4) *holds. Define f_i recursively*:

$$f_1 = \kappa_1; \qquad f_i = \mathfrak{M}[f_{i-1}^\natural, \kappa_i] \qquad \text{for } i = 2, \dots, K. \tag{2.11}$$

Then $(f_i^\natural)^* = f_i^{\circledast} = \mathfrak{M}[f_{i-1}^\natural, \kappa_i]^{\circledast} = r_i$.

This is proved in Section 7, along with a variety of convexity results that contribute to deriving formulas for the speed. The issues surrounding convexity are more complicated than might be expected on the basis of the known results for the irreducible case. For example, it is easy to construct (reducible) two-type examples where f_2 and r_2 have properties that cannot arise in the one-type (or irreducible) case. In particular, there are examples where f_2 is linear (only) on a finite or a semi-infinite interval and where r_2 is linear (only) on a finite interval.

The notation has now been established to state a result giving (Nu) and hence (Sp) in Theorem 2.4 under weaker conditions. The aim was to make these conditions as general as is practicable, but that does mean they are also quite complex. In Theorem 2.10, (Sp) will be established under yet weaker conditions. Let $\underline{\psi}_i = \inf \mathcal{D}_{i,i+1}$ and $\overline{\psi}_i = \sup \mathcal{D}_{i,i+1}$.

THEOREM 2.6. *In the setup and conditions of Theorem* 2.3, *suppose that* (1.4) *holds and that for $i = 1, 2, \dots, K-1$*,

$$\text{there are } \phi_{i,i+1} \in \mathcal{D}_{i,i+1} \qquad \text{with } 0 < \phi_i \le \phi_{i,i+1} \le \phi_{i+1}. \tag{2.12}$$

Let f_i be as defined at (2.11). *Suppose that, for $i = 1, 2, \dots, K-1$*,

$$\text{either } \kappa_{i+1}(\theta) \ge \theta\big(f_i^\natural(\overline{\psi}_i)/\overline{\psi}_i\big) \text{ for } \theta \in [\overline{\psi}_i, \infty) \quad \text{or} \quad \vartheta(f_i) \le \overline{\psi}_i \tag{2.13}$$

and

$$\bigcap_{j \le i} \mathcal{D}^+(\kappa_j) \cap \mathcal{D}(\kappa_{i+1}) \subset [\underline{\psi}_i, \infty). \tag{2.14}$$

Then (Nu) *and* (Sp) *hold.*

Complementing the lower bound in Theorem 2.3 is a two-stage process, involving first deriving an upper bound and then giving conditions for it to equal the lower bound. The first stage is covered by the next result; its proof is in Section 8. Let $I(A)$ be the indicator function of A and let

$$\chi_i = -\log I(\mathcal{D}_{i-1,i}) \qquad \text{for } i = 2, \dots, K,$$

so that χ_i is zero on $\mathcal{D}_{i-1,i}$ and infinity otherwise.

THEOREM 2.7. *Make the same assumptions as in Theorem 2.3. Define g_i recursively*:

$$g_1 = \kappa_1; \qquad g_i = \mathfrak{M}[(g_{i-1}^{\natural} + \chi_i)^{\natural}, \kappa_i] \qquad \textit{for } i = 2, \dots, K. \tag{2.15}$$

Then

$$\limsup_n \frac{1}{n} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) \le -g_K^{\circledast}(a) \qquad \textit{a.s.-}\mathbb{P}_\nu \tag{2.16}$$

and

$$\limsup_n \frac{\mathcal{B}_\sigma^{(n)}}{n} \le \Gamma(g_K^*) \qquad \textit{a.s.-}\mathbb{P}_\nu. \tag{2.17}$$

Furthermore, $-g_K^{\circledast}(a) < \infty$ for all a if (1.4) *holds and* (2.12) *holds for $i = 1, 2, \dots, K - 1$.*

A key point from Proposition 2.5, for the formulation of the rest of the results in this section, is that $(f_K^*)^\circ = f_K^{\circledast} = r_K$. Using this, and comparing (2.7) and (2.8) with (2.16) and (2.17), immediately gives the following corollary.

COROLLARY 2.8. *Make the same assumptions as in Theorem 2.3. Then* (Nu) *holds if $g_K^{\circledast} = f_K^{\circledast}$ and* (Sp) *holds if $\Gamma(f_K^*) = \Gamma(g_K^*)$.*

Thus, in the light of this corollary, proving Theorems 2.4 and 2.6 will entail showing that the conditions imposed imply that $g_K^{\circledast} = f_K^{\circledast}$. This is done in Section 9.

It is possible that $\Gamma(g_K^*) = \Gamma(f_K^*)$ even though $g_K^{\circledast}$ and $f_K^{\circledast}$ do not agree everywhere. Then the speed would be given through (Sp) of Theorem 2.4, even though the behavior of the numbers was not described by (Nu). To investigate this possibility, alternative formulas for $g_K^{\circledast}$ and for $f_K^{\circledast}$ and their associated speeds are important. Those formulas are given next. The formula for $\Gamma(f_K^*)$ is critical in establishing the simpler one given in Theorem 1.1. Also, the formula for $\Gamma(f_K^*)$ is the same one that is obtained as the upper bound on the speed in a deterministic model by Weinberger, Lewis and Li [(2007), Proposition 4.1], so their bound can be simplified, too.

The conventions that $\mathcal{D}_{0,1} = (0, \infty)$ and $\overline{\psi}_K = \infty$ are now adopted. It is worth noting that in (2.18) θ_K is fixed, but in (2.19) it is one of the free variables in the optimization.

THEOREM 2.9. *For a sequential process as described in Theorem 2.3, let g_K be given by (2.15). Then, for $0 < \theta_K \in \mathcal{D}^+_{K-1,K}$,*

$$\frac{g_K(\theta_K)}{\theta_K} = \inf\left\{\max_i\left\{\frac{\kappa_i(\theta_i)}{\theta_i}\right\} : \theta_1 \le \theta_2 \le \cdots \le \theta_K, \theta_i \in \mathcal{D}^+_{i-1,i}, \theta_i \le \overline{\psi}_i\right\} \tag{2.18}$$

and $g_K(\theta_K) = \infty$ for $0 < \theta_K \notin \mathcal{D}^+_{K-1,K}$. Furthermore,

$$\Gamma(g_K^*) = \inf\left\{\max_i\left\{\frac{\kappa_i(\theta_i)}{\theta_i}\right\} : \theta_1 \le \theta_2 \le \cdots \le \theta_K, \theta_i \in \mathcal{D}^+_{i-1,i}, \theta_i \le \overline{\psi}_i\right\}. \tag{2.19}$$

Let f_K be given by (2.11). These formulas hold with f_K in place of g_K on replacing $\mathcal{D}_{i,i+1}$ by $(0,\infty)$ (and $\overline{\psi}_i$ by ∞) for $i = 1, 2, \ldots, K-1$.

Now, asking when the formulas for $\Gamma(g_K^*)$ and $\Gamma(f_K^*)$ give the same result—that is, when the extra restrictions in the optimization associated with the formula for $\Gamma(g_K^*)$ make no difference—leads to the following theorem. Both it and the previous theorem are proved in Section 10, where a little more is also said about formulas for $\Gamma(f_K^*)$.

THEOREM 2.10. *In the setup and conditions of Theorem 2.3, suppose (2.12), (2.13) and $\vartheta(\kappa_{i+1}) \ge \underline{\psi}_i$ all hold for $i = 1, 2, \ldots, K-1$. Then $\Gamma(g_K^*) = \Gamma(f_K^*)$ and (Sp) holds.*

Theorem 2.7 also raises the question of whether the upper bound there, when it is actually larger than the lower bound in Theorem 2.3, can be matched by a corresponding lower bound. A full study of this is not attempted, but some key results are given in the final section of the paper.

3. From sequential to general. The main idea here is to explain how in the general case the number of particles of a specified type can be decomposed using a finite collection of sequential branching processes. Consider $\sigma \in \mathcal{C}_K$. Each particle of type σ can be labeled by the classes that arise in its ancestry, tracing back to the initial ancestor in $\mathcal{C}_1$, and then by the particular types that link the successive classes. This label will be called its genealogical type. Thus, for example, the branching process arising from

$$m = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & 0 & m_{24} \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & 0 & m_{44} \end{pmatrix}$$

contains exactly three routes through the classes from the first class to the fourth, arising from

$$\begin{pmatrix} m_{11} & m_{14} \\ 0 & m_{44} \end{pmatrix}, \qquad \begin{pmatrix} m_{11} & m_{12} & 0 \\ 0 & m_{22} & m_{24} \\ 0 & 0 & m_{44} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m_{11} & m_{13} & 0 \\ 0 & m_{33} & m_{34} \\ 0 & 0 & m_{44} \end{pmatrix},$$

and each particle in the final class arises from a line of descent following one of these three. For the second phase of the decomposition, each nonzero entry in m_{14} specifies a different type within the first route. Similarly, a pair of nonzero entries, one drawn from m_{12} and the other from m_{24}, specifies a type within the second route.

Slightly more formally, let ℓ be a label for genealogical type (so ℓ records which classes occur in the ancestry and which pairs of types link classes in that ancestry). Now let (σ, ℓ) be an augmented type that indicates those of type σ with genealogical type ℓ. There are only a finite number of different genealogical types, and, by definition,

$$Z_\sigma^{(n)}[na, \infty) = \sum_\ell Z_{\sigma,\ell}^{(n)}[na, \infty). \tag{3.1}$$

Furthermore, each genealogical type corresponds to a sequential branching process embedded within the original one.

The next two results follow by straightforward argument from the decomposition (3.1) and the continuity of r-functions when finite. Note that the minimum of convex functions need not be convex, and so r in this theorem need not be convex, and hence need not be an r-function, but it will share in the other properties of an r-function.

THEOREM 3.1. *Suppose that, for each ℓ, there is an r-function, r_ℓ such that*

$$n^{-1} \log\bigl(Z_{\sigma,\ell}^{(n)}[na, \infty)\bigr) \to -r_\ell(a) \qquad a.s.\text{-}\mathbb{P}_\nu$$

for all $a \neq \Gamma(r_\ell)$. Then

$$n^{-1} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) \to -r(a) = -\min_\ell\{r_\ell(a)\} \qquad a.s.\text{-}\mathbb{P}_\nu$$

for all $a \neq \Gamma(r)$ and

$$n^{-1}\mathcal{B}_\sigma^{(n)} \to \Gamma(r) \qquad a.s.\text{-}\mathbb{P}_\nu.$$

THEOREM 3.2. *Suppose that for each ℓ*

$$n^{-1}\mathcal{B}_{\sigma,\ell}^{(n)} \to \Gamma_\ell \qquad a.s.\text{-}\mathbb{P}_\nu. \tag{3.2}$$

Then

$$n^{-1}\mathcal{B}_\sigma^{(n)} \to \Gamma = \max_\ell \Gamma_\ell \qquad a.s.\text{-}\mathbb{P}_\nu.$$

Obviously Theorems 3.1 and 3.2 can be applied to get the overall speed when (Nu) and (Sp), respectively, hold for every embedded sequential process. The next result shows that this overall speed is often not as difficult to calculate as at first appears. Its proof will be described in Section 11.

THEOREM 3.3. *Suppose that* (3.2) *holds for each embedded sequential process with* $\Gamma_\ell = \Gamma(r_\ell)$ *and its associated* r_ℓ *given by the recursion* (2.6) *in Theorem* 2.3. *Let* Γ *be the maximum speed obtained as in Theorem* 3.2. *Then*

$$\Gamma = \max_{i \Rightarrow j} \{\Gamma(\mathfrak{C}[\kappa_i^{\circledast}, \kappa_j^*])\} = \max_{i \Rightarrow j} \inf_{0<\varphi\le\theta} \max\left\{\frac{\kappa_i(\varphi)}{\varphi}, \frac{\kappa_j(\theta)}{\theta}\right\}.$$

PROOF OF THEOREM 1.1. The conditions ensure that Theorem 2.4 holds for each embedded sequential process. Then Theorem 3.3 gives the result. □

4. Preliminaries. The section introduces various notation and gives some preliminary results on convexity, drawing heavily on other sources. Further convexity results that are more particular to this study will be obtained in later sections.

A convex function is called proper when it is finite somewhere. A proper convex function is called closed when it is lower semi-continuous—see Rockafellar [(1970), Section 7, page 52] for a full discussion. For a convex function on $\mathbb{R}$ that is finite on a nonempty interval, this is the same as demanding continuity from within at the endpoints of its domain of finiteness. The closure $\underline{f}$ of the proper convex function f on $\mathbb{R}$ is obtained by adjusting the values of f at these endpoints to make it closed. Thus $\underline{f} \le f$. By definition, an r-function is proper and closed and so at first sight the nature of the results might suggest that attention could be restricted throughout to closed convex functions. However, this is not so. By using the off-diagonal entry in m, it is easy to construct (reducible) two-type examples where g_2 [given by the recursion (2.15)] is not closed (by being bounded on an open interval but infinite at one of its endpoints).

LEMMA 4.1. (i) *When* f *is convex,* f^* *is a closed convex function, as is* $f^{\circledast}$ *provided it is finite somewhere, and* $(f^*)^* = \underline{f}$.

(ii) *If* f *and* g *are convex functions, then so is* $\mathfrak{M}[f, g]$ *and, provided* $\mathfrak{M}[f, g]$ *is finite somewhere,* $\mathfrak{M}[f, g]^* = \mathfrak{C}[f^*, g^*]$.

PROOF. The first part is all contained in Rockafellar [(1970), Theorem 12.2], except for the claim about $f^{\circledast}$, which follows easily from its definition at (2.3). The first part of (ii) follows directly from the definitions and the second is in Rockafellar (1970), Theorems 9.4, 16.5. □

LEMMA 4.2. *When* f *is k-convex* (*as introduced in Definition* 2):

(i) $f^*(a) > -\infty$ *for all* a;
(ii) $f^*(a) \to \infty$ *as* $a \uparrow \infty$ *and* $\Gamma(f^*) < \infty$;
(iii) f^* *is increasing*;
(iv) $f^*(a) < \infty$ *for some* a;
(v) $f^*(a) \to -\underline{f}(0)$ *as* $a \downarrow -\infty$;
(vi) $\Gamma(f^*) > -\infty$ *if and only if* $\underline{f}(0) > 0$.

PROOF. When $f(\phi) < \infty$, $f^*(a) \geq \phi a - f(\phi) > -\infty$ giving (i), and, since $\phi > 0$, letting $a \uparrow \infty$ gives (ii). Furthermore, because $f(\theta) = \infty$ for $\theta < 0$,

$$f^*(a) = \sup_{\theta}\{\theta a - f(\theta)\} = \sup_{\theta \geq 0}\{\theta a - f(\theta)\} \leq \sup_{\theta \geq 0}\{\theta a' - f(\theta)\},$$

when $a' \geq a$, so f^* is increasing in a. Since f is finite and convex there must be finite A and B such that $f(\theta) \geq A\theta - B$ for all θ and then $f^*(A) \leq B$, giving (iv). Part (v) follows from Lemma 4.1(i) and Rockafellar (1970), Theorem 27.1(a). Part (vi) follows directly from (iii), (v) and the definition of Γ. $\square$

The next result gives properties of κ arising from irreducible m. It is worth stressing that part (iii) includes claims about one-sided derivatives at the endpoints of $\mathcal{D}(\kappa)$.

LEMMA 4.3. *Suppose κ is the PF$^+$eigenvalue of an irreducible m and that* (1.2) *holds*:

(i) *$\mathcal{D}(\kappa)$ is a (possibly degenerate) interval containing the ϕ in* (1.2);
(ii) *κ is k-convex*;
(iii) *κ is continuous on the closure of $\mathcal{D}(\kappa)$, differentiable on $\mathcal{D}(\kappa)$ and analytic on its interior*;
(iv) *κ is closed.*

PROOF. Clearly (1.2) implies that $\kappa(\phi) < \infty$. For convexity, see Kingman (1961), Miller (1961) and Seneta (1973), Theorem 3.7. Part (ii) follows immediately from this and (1.2). For analyticity on the interior, which is a straightforward application of the implicit function theorem, see Miller [(1961), Theorem 1(a)], Lancaster and Tismenetsky [(1985), Theorem 11.5.1] or Biggins and Rahimzadeh Sani (2005), Theorem 1(i). Each entry in m is continuous on the closure of the set where it is finite and so the same must be true of κ. Hence, when κ is finite at the endpoint of the interval on which it is finite, Rockafellar [(1970), Theorem 24.1] implies that the derivative extends continuously to this endpoint, where the derivative at the endpoint is the one-sided one from within the interval. Part (iv) follows directly from this and part (i). $\square$

5. The irreducible case. The discussion starts with a simple lemma which is easily deduced from Seneta (1973, 1981), Theorems 1.1, 1.5.

LEMMA 5.1. *Let M be an irreducible matrix with all its entries finite and nonnegative. Then M has a "Perron–Frobenius" eigenvalue (which is positive, and of largest modulus) e^ρ, and there is a finite C that is independent of n, ν and σ such that $e^{-n\rho}(M^n)_{\nu\sigma} \leq C$ and, for primitive M, $n^{-1}\log(M^n)_{\nu\sigma} \to \rho$.*

In this section it is assumed that there is just one class of types, so the matrix m is irreducible, that the exponential moment condition (1.2) holds and that m has PF$^+$eigenvalue κ. In fact the matrix m is assumed primitive up to the final result in the section, where periodic m are considered. Though rather simple, that extension to periodic m is important in establishing the main result. Most results in this section are not novel, though several are (I believe) new and their discussion underpins later developments. The first lemma is a simple upper bound on transforms that is an ingredient in the upper bounds on numbers described in the proposition that follows it.

LEMMA 5.2.

$$\limsup_n \frac{1}{n} \log\left(\int e^{\theta x} Z_\sigma^{(n)}(dx)\right) \leq \kappa(\theta) \qquad a.s.\text{-}\mathbb{P}_\nu.$$

PROOF. Using (1.1),

$$\frac{1}{n} \log \int e^{\theta z} \mathbb{E}_\nu Z_\sigma^{(n)}(dz) = \frac{1}{n} \log(m(\theta)^n)_{\nu\sigma}.$$

Lemma 5.1 implies that

$$\limsup_n \frac{1}{n} \log\left(\int e^{\theta x} \mathbb{E}_\nu Z_\sigma^{(n)}(dx)\right) \leq \kappa(\theta) \qquad \text{a.s.-}\mathbb{P}_\nu$$

and so for any $\varepsilon > 0$ and then large enough n

$$\frac{\mathbb{E}_\nu \int e^{\theta x} Z_\sigma^{(n)}(dx)}{\exp(n(\kappa(\theta) + 2\varepsilon))} \leq \exp(-n\varepsilon).$$

This has a finite sum over n, giving the result. $\square$

The next proposition derives three upper bounds; the first concerns expectations, the second the probabilities of certain "extreme" events and the third actual numbers. These upper bounds on numbers are (nearly always) exact: that is the content of Propositions 2.1, 5.5 and 2.5, which are all needed later.

PROPOSITION 5.3. *For all σ, ν, and a,*

$$\limsup_n \frac{1}{n} \log\left(\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty)\right) \leq -\kappa^*(a),$$

$$\limsup_n \frac{1}{n} \log\left(\mathbb{P}_\nu\left(\mathcal{B}_\sigma^{(n)} \geq na\right)\right) \leq \min\{-\kappa^*(a), 0\}$$

and

$$\limsup_n \frac{1}{n} \log\left(Z_\sigma^{(n)}[na, \infty)\right) \leq -\kappa^{\circledast}(a) \qquad a.s.\text{-}\mathbb{P}_\nu.$$

PROOF. For $\theta \geq 0$,

$$e^{\theta na}\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty) \leq \int e^{\theta z}\mathbb{E}_\nu Z_\sigma^{(n)}(dz) = (m(\theta)^n)_{\nu\sigma}$$

so that

$$\log\bigl(\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty)\bigr) \leq -n\theta a + \log((m(\theta)^n)_{\nu\sigma}).$$

Hence, for $\theta \geq 0$, using Lemma 5.1,

$$\limsup_n \frac{1}{n}\log\bigl(\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty)\bigr) \leq -\bigl(\theta a - \kappa(\theta)\bigr).$$

Since κ is defined to be infinite for $\theta < 0$, this holds for all θ and so minimizing the right-hand side over θ gives the first bound. Since

$$\mathbb{P}_\nu\bigl(\mathcal{B}_\sigma^{(n)} \geq na\bigr) = \mathbb{E}_\nu I\bigl(\mathcal{B}_\sigma^{(n)} \geq na\bigr) \leq \mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty),$$

the second follows directly from this. Turning to the third, since

$$e^{\theta na} Z_\sigma^{(n)}[na, \infty) \leq \int e^{\theta z} Z_\sigma^{(n)}(dz),$$

Lemma 5.2, gives

$$\limsup_n \frac{1}{n}\log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) \leq -\bigl(\theta a - \kappa(\theta)\bigr) \qquad \text{a.s.-}\mathbb{P}_\nu$$

and minimizing over θ gives the third bound, with κ^* in place of $\kappa^{⊛}$. However, $Z_\sigma^{(n)}[na, \infty)$ is integer-valued and so can only decay geometrically by being zero for all large n, which implies κ^* can be replaced by $\kappa^{⊛}$. $\square$

PROOF OF PROPOSITION 2.1. This is just an application of suitable large deviation theory based on

$$\frac{1}{n}\log\int e^{\theta z}\mathbb{E}_\nu Z_\sigma^{(n)}(dz) = \frac{1}{n}\log(m(\theta)^n)_{\nu\sigma} \to \kappa(\theta) \qquad \text{for } \theta > 0,$$

which holds by Lemma 5.1. See Biggins [(1995), Section 7] for a little more detail on the method. $\square$

PROPOSITION 5.4.

$$\sup_n \frac{1}{n}\log\bigl(\mathbb{E}_\sigma Z_\sigma^{(n)}[na, \infty)\bigr) = -\kappa^*(a).$$

PROOF. Note that $a_n = \mathbb{E}_\sigma Z_\sigma^{(n)}[na, \infty)$ is supermultiplicative ($a_{n+m} \geq a_n a_m$) and so standard theory of subadditive sequences gives that the supremum agrees with the limit, and the latter has already been identified in Proposition 2.1. $\square$

The next result concerns the decay of the probability of a particle appearing to the right of na. For the one-type process Rouault (1987) gives a result similar to the next one under extra conditions and Rouault [(1993), Theorem 2.1] gives a much sharper one. The multitype case does not seem to have been discussed before.

PROPOSITION 5.5. *For $a \neq U$,*

$$\frac{1}{n}\log\bigl(\mathbb{P}_\nu\bigl(\mathcal{B}_\sigma^{(n)} \geq na\bigr)\bigr) \to \min\{-\kappa^*(a), 0\}.$$

PROOF. Take b with $b \neq U$ and $\kappa^*(b) > 0$. Take $\varepsilon > 0$. Then, using Propositions 2.1 and 5.4, there is an r such that

$$-\kappa^*(b) \geq \frac{1}{r}\log\bigl(\mathbb{E}_\sigma Z_\sigma^{(r)}[rb, \infty)\bigr) \geq -\kappa^*(b) - \varepsilon. \tag{5.1}$$

Starting from an initial ancestor of type σ, regard as its children all its descendants r generations later of type σ and displaced at least rb from the initial particle's position. Identify "children" of these children in the same way, and so on. The resulting process is a (one-type) Galton–Watson process with mean $\mathbb{E}_\sigma Z_\sigma^{(r)}[rb, \infty)$. This process is subcritical, because $\exp(-r\kappa^*(b)) < 1$. Let $N^{(n)}$ be the number in its nth generation. Then, by arrangement, when the initial ancestor is of type σ,

$$N^{(n)} \leq Z_\sigma^{(nr)}[nrb, \infty)$$

so that $N^{(n)} > 0$ implies that $\mathcal{B}_\sigma^{(nr)} \geq nrb$. Hence, using Asmussen and Hering [(1983), Theorem III.1.6] to estimate $\mathbb{P}(N^{(n)} > 0)$,

$$\begin{aligned}\frac{1}{nr}\log\bigl(\mathbb{P}_\sigma\bigl(\mathcal{B}_\sigma^{(nr)} \geq nrb\bigr)\bigr) &\geq \frac{1}{nr}\log\bigl(\mathbb{P}\bigl(N^{(n)} > 0\bigr)\bigr)\\ &\to \frac{1}{r}\log\bigl(\mathbb{E}_\sigma Z_\sigma^{(r)}[rb, \infty)\bigr)\\ &\geq -\kappa^*(b) - \varepsilon.\end{aligned}$$

Now, consider a process started from a type ν. Because m is primitive, there is an s such that m^n has all entries strictly positive for every $n \geq s$. Then, for a suitable T, there is a positive probability of a descendant in generation $s + r'$ of type σ and to the right of T for each of $r' = 0, 1, 2, \ldots, r - 1$. Let p be the minimum of these probabilities. For $b > a$, all sufficiently large n and $r' = 0, 1, 2, \ldots, r - 1$,

$$\begin{aligned}\mathbb{P}_\nu\bigl(\mathcal{B}_\sigma^{(nr+s+r')} \geq (nr + s + r')a\bigr) &\geq \mathbb{P}_\nu\bigl(\mathcal{B}_\sigma^{(nr+s+r')} \geq nrb + T\bigr)\\ &\geq p\mathbb{P}_\sigma\bigl(\mathcal{B}_\sigma^{(nr)} \geq nrb\bigr).\end{aligned}$$

Therefore

$$\begin{aligned}\liminf_n \frac{1}{n}\log\bigl(\mathbb{P}_\nu\bigl(\mathcal{B}_\sigma^{(n)} \geq na\bigr)\bigr) &\geq \liminf_n \frac{1}{nr}\log\bigl(\mathbb{P}_\sigma\bigl(\mathcal{B}_\sigma^{(nr)} \geq nrb\bigr)\bigr)\\ &\geq -\kappa^*(b) - \varepsilon.\end{aligned}$$

This holds for any $\varepsilon > 0$ and $b > a$. Thus, since κ^* is continuous from the right except at U,

$$\liminf_n \frac{1}{n} \log(\mathbb{P}_\nu(\mathcal{B}_\sigma^{(n)} \geq na)) \geq \min\{-\kappa^*(a), 0\}$$

except possibly for $a = U$. The upper bound in Proposition 5.3 completes the proof. □

LEMMA 5.6. *Suppose that the branching process is supercritical* [*i.e.*, $\kappa(0) > 0$]. *Then* $\kappa^{\circledast}$ *is an* r*-function* (*as introduced at Definition* 1).

PROOF. Lemma 4.3 gives that κ is k-convex and closed. Also, $\kappa(0) > 0$ because the process is supercritical. Hence, using Lemma 4.2, κ^* is increasing, less than zero somewhere, and convex. Thus $\kappa^{\circledast}$ is a proper convex function that is strictly negative somewhere, left-continuous and infinite when strictly positive and so is an r-function. □

PROOF OF PROPOSITION 2.5. The argument is very similar to that for Proposition 5.5. It will be convenient to let $\mathcal{S}$ be the survival set of the process, even though $\mathbb{P}_\nu(\mathcal{S}) = 1$. Proposition 5.3 implies that (2.5) holds for $a > \Gamma(\kappa^*)$, with the limit being $-\infty$. Hence, only $a < \Gamma(\kappa^*)$ need to be considered. Take $b > a$ but with $\kappa^*(b) < 0$, which is possible because, by Lemma 5.6, $\kappa^{\circledast}$ is an r-function, and take $\varepsilon \in (0, -\kappa^*(b))$. As in Proposition 5.5, use Propositions 2.1 and 5.4, to choose r such that (5.1) holds. Start from an initial ancestor of type σ, and identify the embedded (one-type) Galton–Watson process as in Proposition 5.5. This now has mean $\mathbb{E}_\sigma Z_\sigma^{(r)}[rb, \infty)$ and is supercritical, because $\exp(-r(\kappa^*(b) + \varepsilon)) > 1$. Let $N^{(n)}$ be the number in its nth generation. Then, using, for example, Asmussen and Hering [(1983), Theorems II.5.1, II.5.6] to get the limit of $n^{-1} \log N^{(n)}$,

$$\begin{aligned}
\frac{1}{nr} \log(Z_\sigma^{(nr)}[nrb, \infty)) &\geq \frac{1}{nr} \log N^{(n)} \\
&\to \frac{1}{r} \log(\mathbb{E}_\sigma Z_\sigma^{(r)}[rb, \infty)) \\
&\geq -\kappa^*(b) - \varepsilon
\end{aligned}$$

on the survival set of $N^{(n)}$, which has positive probability. Three matters remain: allowing initial types different from σ; dealing with generations that are not a multiple of r; and showing the result holds almost surely on the survival set of the whole process and not just that of some embedded one. The argument for dealing with all three is standard, and the idea is not complicated. It is to run the process to some large generation, allow each type σ then present to initiate its own $N^{(n)}$, and then use any that survives to provide a suitable lower bound. Here is a more careful version.

Fix σ. Let $\{z_i^{(s)}:i\}$ be the points of $Z_\sigma^{(s)}$. Recall that $\mathcal{F}^{(s)}$ contains all information on families with the parent in a generation up to and including $s-1$. Let $N_{s,i}^{(n)}$ be the process $N^{(n)}$ initiated by the particle at $z_i^{(s)}$. By arrangement, $N_{s,i}^{(n)}$ contains points in the $(nr+s)$th generation to the right of $nrb+z_i^{(s)}$. Given $\mathcal{F}^{(s)}$, these processes are independent. Let $\mathcal{S}(s)$ be the event that at least one of these processes survives. Fix s and r'. For any i, for all large enough n, $(nr+sr+r')a-z_i^{(sr+r')}\le nrb$ and so

$$Z_\sigma^{(nr+sr+r')}[(nr+sr+r')a,\infty)\ge N_{(sr+r'),i}^{(n)}$$

for all sufficiently large n. Hence

$$\liminf_n \frac{1}{(nr+r')}\log\bigl(Z_\sigma^{(nr+r')}[(nr+r')a,\infty)\bigr)\ge -\kappa^*(b)-\varepsilon \tag{5.2}$$

on $\mathcal{S}(sr+r')$. Furthermore $\mathcal{S}(sr+r')\subset\mathcal{S}((s+1)r+r')\subset\mathcal{S}$ and $\mathbb{P}_\nu(\mathcal{S}(sr+r'))\uparrow\mathbb{P}_\nu(\mathcal{S})$ as $r\uparrow\infty$. Hence (5.2) holds almost surely on $\mathcal{S}$ for each $r'=0,1,2,\dots,r-1$. Also, it holds for any $\varepsilon>0$ and every $b>a$. Since κ^* is continuous from the right at a, this provides the lower bound to complement the upper bound in Proposition 5.3.

Though it does not matter here, it is perhaps worth noting that, because $Z_\sigma^{(n)}[na,\infty)$ is monotone in a, the null set in (2.5) can be taken independent of a. □

Since the proof of Theorem 2.6 will be by induction on K it is worth stating explicitly that the induction starts successfully.

COROLLARY 5.7. *When $K=1$, Theorem 2.6 holds.*

PROOF. For $K=1$, the condition (1.4) is equivalent to (1.2) and the conditions (2.12), (2.13) and (2.14) are vacuous. Proposition 2.2 now gives the required conclusions. □

When m is irreducible with period $d>1$, m^d has d primitive blocks on its diagonal, each with PF$^+$eigenvalue κ^d. These primitive blocks partition the types into d subclasses. The next result deals with the case where ν and σ are in the same subclass. It is possible to say a bit more, dealing with ν and σ in different subclasses, but this is not needed here.

PROPOSITION 5.8. *If "primitive" is replaced by "irreducible with period $d>1$," then Propositions 2.1 and 2.2 and all the results in this section continue to hold, provided "n" is replaced by "nd" and ν and σ come from the same subclass.*

PROOF. Apply the results to the primitive process obtained by only inspecting every dth generation. □

6. Lower bounds on numbers, main results. The objective in this section is to prove Theorem 2.3. The main challenge is to show how in a sequential process the numbers in the penultimate class contribute to numbers in the final class. The first proposition shows two things: that the numbers in the penultimate class drive the numbers of those first in their line of descent to be in the final class and that those numbers drive the first in the line of descent of any other type in the final class. To discuss this, let $F_\sigma^{(n)}$ be the point process of those in generation n of type σ that are first in their line of descent with this type. The subsequent theorem explores how the numbers in $F_\sigma^{(n)}$ combine with the growth of numbers within the class.

PROPOSITION 6.1. *Consider a sequential process. Let $\upsilon \in \mathcal{C}_{K-1}$ and $\tau \in \mathcal{C}_K$ be types for which $m_{\upsilon\tau} > 0$ and let $\nu \in \mathcal{C}_1$. If there is an r-function r such that for all $a < \Gamma(r)$*

$$\liminf \frac{1}{n} \log\bigl(Z_\upsilon^{(n)}[na, \infty)\bigr) \geq -r(a) \qquad a.s.\text{-}\mathbb{P}_\nu,$$

then

$$\liminf_n \frac{1}{n} \log\bigl(F_\sigma^{(n)}[na, \infty)\bigr) \geq -r(a) \qquad a.s.\text{-}\mathbb{P}_\nu \tag{6.1}$$

for all $a \neq \Gamma(r)$ and $\sigma \in \mathcal{C}_K$.

THEOREM 6.2. *Consider any process with final class $\mathcal{C}_K$ having PF$^+$eigenvalue κ and initial type $\nu \notin \mathcal{C}_K$. Suppose that for the r-function r and any $\sigma \in \mathcal{C}_K$, (6.1) holds for all $a < \Gamma(r)$. Then*

$$\liminf_n \frac{1}{n} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) \geq -\mathfrak{C}[r, \kappa^*]^\circ(a) \qquad a.s.\text{-}\mathbb{P}_\nu$$

for all $a < \Gamma(\mathfrak{C}[r, \kappa^*])$.

Before starting the main proofs, three lemmas are proved. The second of these identifies a characterization of $\mathfrak{C}[r, \kappa^*]$ that arises in proving Theorem 6.2.

LEMMA 6.3. *Suppose f is k-convex, r is an r-function and $\mathfrak{M}[r^*, f](\phi) < \infty$ for some $\phi > 0$. Then $\mathfrak{C}[r, f^*]^\circ$ is also an r-function.*

PROOF. By Lemma 4.2, f^* is proper, closed, convex and increasing. Clearly $\mathfrak{C}[r, f^*]^\circ$ is convex. It is increasing, because both r and f^* are, and negative somewhere, because r is. Since $\mathfrak{C}[r, f^*]$ is continuous from the left (by definition) the same must be true of $\mathfrak{C}[r, f^*]^\circ$. Finally, using both parts of Lemma 4.1, $(\mathfrak{M}[r^*, f])^* = \mathfrak{C}[r, f^*]$, and now Lemma 4.2(i) implies that $(\mathfrak{M}[r^*, f])^*$ is not identically $-\infty$. $\square$

LEMMA 6.4. *Under the same conditions as Lemma 6.3, for* $a < \Gamma(\mathfrak{C}[r, f^*])$,

$$\mathfrak{C}[r, f^*](a) = \inf\{\lambda r(b) + (1-\lambda) f^*(c) : (\lambda, b, c) \in A_a, r(b) < 0\},$$

where $A_a = \{(\lambda, b, c) : \lambda \in [0, 1], \lambda b + (1-\lambda)c = a, \lambda r(b) + (1-\lambda) f^*(c) < 0\}$.

PROOF. Let $\mathfrak{c}[f, g]$ be the convex minorant of f and g, so that $\mathfrak{C}[f, g]$ is the closure of $\mathfrak{c}[f, g]$. Since $\mathfrak{C}[r, f^*]$ is increasing and convex, it is continuous and strictly negative on $(-\infty, \Gamma(\mathfrak{C}[r, f^*]))$ and so on that set $\mathfrak{C}[r, f^*](a) = \mathfrak{c}[r, f^*](a)$. Furthermore, using Rockafellar [(1970), Theorem 5.6],

$$\mathfrak{c}[r, f^*](a) = \inf\{\lambda r(b) + (1-\lambda) f^*(c) : \lambda \in [0, 1], \lambda b + (1-\lambda)c = a\},$$

which equals $\inf\{\lambda r(b) + (1-\lambda) f^*(c) : (\lambda, b, c) \in A_a\}$ when $\mathfrak{c}[r, f^*](a) < 0$. It remains to show that the additional constraint $r(b) < 0$ makes no difference, by showing that excluded values of the function can be approximated closely by included ones. The only possibility excluded is $b = \Gamma(r)$, since r is infinity when strictly positive. The corresponding values of the function being minimized can be approximated arbitrarily well when $\lambda < 1$ by taking $b \uparrow \Gamma(r)$ keeping c fixed and adjusting λ. To deal with the $\lambda = 1$ case, where $a = b = \Gamma(r)$, note first that if $f^*(\tilde{a}) = \infty$ for all $\tilde{a} > \Gamma(r)$, then, because $r(\tilde{a}) = \infty$ for all $\tilde{a} > \Gamma(r)$ also, the same will be true of the convex minorant of r and f^*. Then $a = \Gamma(r) = \Gamma(\mathfrak{C}[r, f^*])$, contradicting $a < \Gamma(\mathfrak{C}[r, \kappa^*])$. Hence, there must be a $c > a$ with $f^*(c) < \infty$. Then

$$(1-\varepsilon) r\left(\frac{a - \varepsilon c}{1-\varepsilon}\right) + \varepsilon f^*(c)$$

provides a suitable approximation as $\varepsilon \downarrow 0$. $\square$

LEMMA 6.5. *Let Y_n be Binomial on N_n trials with success probability p_n and* $\sum_n (N_n p_n)^{-1}(1-p_n) < \infty$. *Then* $\log(Y_n) - \log(N_n p_n) \to 0$ *as* $n \to \infty$ *almost surely.*

PROOF. Chebyshev's inequality gives that $P(|Y_n - EY_n| \geq \varepsilon EY_n)$ is bounded above by $(\varepsilon^2 N_n p_n)^{-1}(1-p_n)$, and so Borel–Cantelli gives that $Y_n/(N_n p_n) \to 1$. $\square$

PROOF OF PROPOSITION 6.1. Since $r(a) = \infty$ for $a > \Gamma(r)$, the result holds in these cases. Assume now that $a < \Gamma(r)$. The result is proved first for $\sigma = \tau$. For some T there is a probability $p > 0$ that a particle of type υ has a child of type τ to the right of T, because $m_{\upsilon\tau} > 0$. Then, given $\mathcal{F}^{(n)}$, $F_\tau^{(n+1)}[nb - T, \infty)$ is bounded below by a Binomial variable, Y_n, on $Z_\upsilon^{(n)}[nb, \infty)$ trials with success probability p. Take $b \in (a, \Gamma(r))$ with $r(b) < 0$. Then, by Lemma 6.5, for $\varepsilon > 0$ and then large enough n

$$\log\bigl(F_\tau^{(n+1)}[nb - T, \infty)\bigr) \geq \log(Y_n) \geq \log\bigl(p Z_\upsilon^{(n)}[nb, \infty)\bigr) - \varepsilon.$$

Hence

$$\liminf \frac{1}{n} \log\bigl(F_\tau^{(n+1)}[nb - T, \infty)\bigr) \geq -r(b)$$

and so

$$\liminf \frac{1}{n} \log F_\tau^{(n)}[na, \infty) \geq -r(b) \uparrow -r(a)$$

as $b \downarrow a$, giving (6.1) for $a < \Gamma(r)$ when $\sigma = \tau$.

Suppose now that $\sigma \neq \tau$. Find a sequence of distinct types $\tau = \sigma(0) \neq \sigma(1) \neq \cdots \neq \sigma(c) = \sigma$ such that each type can have children of the type following it in the sequence. For some T, there is a probability $p > 0$ that a particle of type τ has a descendant c generations later to the right of T and of type σ. Let $\widetilde{F}^{(n+c)}$ be the point process of all those in $F_\sigma^{(n+c)}$ with ancestors of type τ in generation n. Then, given $\mathcal{F}^{(n)}$, $\widetilde{F}^{(n+c)}[nb - T, \infty)$ is bounded below by a Binomial variable, Y_n, on $F_\tau^{(n)}[nb, \infty)$ trials with success probability p. Thus

$$\liminf \frac{1}{n} \log \widetilde{F}^{(n)}[na, \infty) \geq -r(a),$$

when $r(a) < 0$. Clearly $F_\sigma^{(n)}[x, \infty) \geq \widetilde{F}^{(n)}[x, \infty)$, giving the result. $\square$

PROOF OF THEOREM 6.2. Let d be the period of $\mathcal{C}_K$. Take $b < \Gamma(r)$ with $r(b) < 0$, $c < \Gamma(\kappa^*)$ with $\kappa^*(c) < 0$, $\varepsilon > 0$ and $\lambda \in [0, 1]$. For each positive integer t, let $n = n(t)$ and $\tilde{n} = \tilde{n}(t)$ be chosen to be increasing in t with $t = n + \tilde{n}d$ and with $n/t \to \lambda$ as $n \to \infty$. Let $N_t = F_\sigma^{(n)}[nb, \infty)$. Then, using the assumption that (6.1) holds, provided $n = n(t) \to \infty$,

$$\begin{aligned}
\liminf_t \frac{1}{t} \log N_t &= \liminf_t \frac{1}{t} \log\bigl(F_\sigma^{(n)}[nb, \infty)\bigr) \\
&= \lambda \liminf_n \frac{1}{n} \log\bigl(F_\sigma^{(n)}[nb, \infty)\bigr) \\
&\geq -\lambda r(b).
\end{aligned}$$

Given $\mathcal{F}^{(n)}$, $Z_\sigma^{(t)}[nb + \tilde{n}dc, \infty)$ is bounded below by N_t independent copies (under $\mathbb{P}_\sigma$) of $Z_\sigma^{(\tilde{n}d)}[\tilde{n}dc, \infty)$. Propositions 2.1, 2.2 and 5.8 imply that most of these copies should have size near $\exp(-\tilde{n}d\kappa^*(c))$. Let Y_t be the number that are not too far below their expectation, that is, the number with

$$\log\bigl(Z_\sigma^{(\tilde{n}d)}[\tilde{n}dc, \infty)\bigr) \geq \tilde{n}d\bigl(-\kappa^*(c) - \varepsilon\bigr).$$

Then, given $\mathcal{F}^{(n)}$, Y_t is a Binomial variable with N_t trials and success probability p_t, where

$$p_t = \mathbb{P}_\sigma\bigl(\log\bigl(Z_\sigma^{(\tilde{n}d)}[\tilde{n}dc, \infty)\bigr) \geq \tilde{n}d\bigl(-\kappa^*(c) - \varepsilon\bigr)\bigr).$$

Propositions 2.2 and 5.8 imply that $p_t \to 1$ provided $\tilde{n}(t) \to \infty$. Now

$$\log\bigl(Z_\sigma^{(t)}[nb + \tilde{n}dc, \infty)\bigr) \geq \log Y_t + \tilde{n}d\bigl(-\kappa^*(c) - \varepsilon\bigr)$$

and, using Lemma 6.5, $Y_t/N_t \to 1$ almost surely when $\sum_t (1/N_t) < \infty$. Let $T(j) = \max\{t : n(t) = j\}$. For suitable small δ and then all sufficiently large n

$$\log N_t = \log\bigl(F_\sigma^{(n)}[nb, \infty)\bigr) \geq n\bigl(-r(b) - \delta\bigr) > 0.$$

Then,

$$\sum_t \frac{1}{N_t} \leq C \sum_j \frac{T(j)}{\exp(j(-r(b) - \delta))}$$

and this is finite provided T does not grow exponentially quickly, for which it suffices that $n(t)^\gamma \geq t$ for some $\gamma > 1$. Putting this together, provided $\tilde{n}(t) \to \infty$ and $n(t)^\gamma \geq t$, which can both be arranged,

$$\liminf_t \frac{1}{t} \log\bigl(Z_\sigma^{(t)}[nb + \tilde{n}dc, \infty)\bigr) \geq \lambda(-r(b)) + (1 - \lambda)\bigl(-\kappa^*(c) - \varepsilon\bigr). \tag{6.2}$$

Note too that

$$\frac{nb + \tilde{n}dc}{t} = \left(\frac{n}{t}b + \frac{\tilde{n}d}{t}c\right) \to \lambda b + (1 - \lambda)c$$

so that (6.2) implies, using continuity of r at b and κ^* at c,

$$\liminf_t \frac{1}{t} \log\bigl(Z_\sigma^{(t)}\bigl(t[\lambda b + (1 - \lambda)c), \infty\bigr)\bigr) \geq -\bigl(\lambda r(b) + (1 - \lambda)\kappa^*(c)\bigr). \tag{6.3}$$

Consider instead the case where $\kappa^*(c) \geq 0$, but still with $t = n(t) + \tilde{n}(t)d$. Let $p_t = \mathbb{P}_\sigma(\mathcal{B}_\sigma^{(\tilde{n}d)} \geq \tilde{n}dc)$. Now, given $\mathcal{F}^{(n)}$, $Z_\sigma^{(t)}[nb + \tilde{n}dc, \infty)$ is bounded below by a Binomial variable, Y_t, on $N_t = F_\sigma^{(n)}[nb, \infty)$ trials with success probability p_t. Much as previously, provided $n(t) \to \infty$, $\tilde{n}(t) \to \infty$ and $n(t)/t \to \lambda$, as $t \to \infty$, Propositions 5.5 and 5.8 give

$$\liminf \frac{1}{t}(\log N_t + \log p_t) \geq -\bigl(\lambda r(b) + (1 - \lambda)\kappa^*(c)\bigr).$$

Therefore, using Lemma 6.5, when $\lambda r(b) + (1 - \lambda)\kappa^*(c) < 0$,

$$\begin{aligned}\liminf \frac{1}{t} \log\bigl(Z_\sigma^{(t)}[nb + \tilde{n}dc, \infty)\bigr) &\geq \liminf \frac{1}{t} \log Y_t \\ &\geq -\bigl(\lambda r(b) + (1 - \lambda)\kappa^*(c)\bigr)\end{aligned}$$

and so, using continuity of r at b, (6.3) holds in this case, too.

Hence (6.3) holds for any $\lambda \in [0, 1]$, any b such that $r(b) < 0$ and any c with $\lambda r(b) + (1 - \lambda)\kappa^*(c) < 0$. Fix a. Maximize the right of (6.3), using Lemma 6.4, over $(\lambda, b, c) \in A_a$ with $r(b) < 0$ to get

$$\liminf_t \frac{1}{t} \log\bigl(Z_\sigma^{(t)}[ta, \infty)\bigr) \geq \mathfrak{C}[r, \kappa^*](a).$$

Now use that $Z_\sigma^{(t)}[ta,\infty)$ is integer-valued to replace $\mathfrak{C}[r,\kappa^*]$ by $\mathfrak{C}[r,\kappa^*]^\circ$. $\square$

PROOF OF THEOREM 2.3. The result holds for $K=1$, by Corollary 5.7. Suppose the result holds for $K-1$. By Lemmas 4.3 and 6.3, r_K has the right properties. Then, by Proposition 6.1 and then Theorem 6.2, (2.7) holds. $\square$

7. Properties of $f^\natural$ and the recursion. The main objectives of this section are to prove Proposition 7.1 giving properties of $f^\natural$ and to establish Proposition 2.5 giving the alternative recursion for r_i.

Recall that $f^\natural$ is the maximal convex function that has $f^\natural(\theta)/\theta$ monotone decreasing in $\theta\in(0,\infty)$ such that $f^\natural\le f$, and that $\vartheta(f)$ is given by (2.10). The next result describes the structure of $f^\natural$ and shows $\vartheta(f)$ is closely connected to $\Gamma(f^*)$. It is worth mentioning that, although this proposition admits other possibilities, in the main results here $\underline{f}(\vartheta)$ and $f(\vartheta)$ will only be different in cases where $f(\vartheta)$ is also infinite. The formula $\Gamma(f^*)=\inf\{f(\theta)/\theta:\theta>0\}$ included in the proposition is the one used for the speed in the irreducible blocks by Weinberger, Lewis and Li (2007) in their model.

PROPOSITION 7.1. *Suppose f is k-convex. Let $\Gamma=\Gamma(f^*)$, $\vartheta=\vartheta(f)$ and $\underline{\psi}=\inf\mathcal{D}(f)$. Then $f^\natural\equiv-\infty$ and $\vartheta=-\infty$ when $\Gamma=-\infty$. Otherwise, $\vartheta\ge 0$ and $f^\natural(\theta)=f(\theta)$ for $0\le\theta<\vartheta$ (by definition). When $0\le\vartheta<\infty$,*

$$f^\natural(\theta)=\theta\Gamma<f(\theta)\qquad\text{for }\theta>\vartheta$$

and

$$f^\natural(\vartheta)=\begin{cases} f(\vartheta)\ge\underline{f}(\vartheta)=\vartheta\Gamma, & \text{when }\vartheta=\underline{\psi},\\ \vartheta\Gamma=\underline{f}(\vartheta)\le f(\vartheta), & \text{when }\vartheta>\underline{\psi}.\end{cases}$$

In all cases,

$$\Gamma=\inf_{\theta>0}\frac{f^\natural(\theta)}{\theta}=\inf_{\theta>0}\frac{f(\theta)}{\theta}. \tag{7.1}$$

When $0\le\vartheta<\infty$, $\Gamma=f(\vartheta)/\vartheta$ provided f is lower semi-continuous at ϑ and, when $\vartheta=\infty$, $\Gamma=\lim_{\theta\uparrow\infty}f(\theta)/\theta$.

Recall that $f^\circledast$ is defined to be $(f^*)^\circ$. Let

$$f^\flat=(f^\circledast)^*=((f^*)^\circ)^*$$

and

$$\vartheta^\flat(f)=\inf\{\theta:f^\flat(\theta)<\underline{f}(\theta)\},$$

which is $+\infty$ when this set is empty. Let $\underline{\psi}=\inf\mathcal{D}(f)$. The next lemma, which will be proved later in the section, says that $f^\natural$ and $f^\flat$ can only be different at $\underline{\psi}$ where the former is $f(\underline{\psi})$ and the latter is $\underline{f}(\underline{\psi})$. This motivates deriving properties of $f^\flat$.

LEMMA 7.2. *Let f be k-convex. Then $\vartheta(f) = \vartheta^\flat(f)$. When $\vartheta(f) = -\infty$, $f^\natural = f^\flat \equiv -\infty$. When $\vartheta(f) \geq 0$, $f^\natural(\theta) = f^\flat(\theta)$ for $\theta > \underline{\psi}$, and $f^\natural(\underline{\psi}) = f(\underline{\psi}) \geq \underline{f}(\underline{\psi}) = f^\flat(\underline{\psi})$.*

The next result establishes some properties of $f^\flat$. In particular, the second part shows that it is a candidate for $f^\natural$, in that it has the right properties. Building on these properties, the result following this one characterizes $f^\flat$.

LEMMA 7.3. *Let f be k-convex and $\Gamma = \Gamma(f^*)$.*

(i) *$f^\flat(\theta) = \sup_{a \leq \Gamma}\{\theta a - f^*(a)\}$ when $\Gamma > -\infty$, and $f^\flat \equiv -\infty$ when $\Gamma = -\infty$;*
(ii) *$f^\flat \leq f$ and $f^\flat(\theta)/\theta$ is decreasing as θ increases, so $f^\flat \leq f^\natural$;*
(iii) *When $\theta' \geq \theta$, $f^\flat(\theta') \leq f^\flat(\theta) + (\theta' - \theta)\Gamma$.*

PROOF. Since $f^*(a) > 0$ for $a > \Gamma$ and these are swept to infinity in $f^{\circledast}$, applying the definitions gives (i). Now

$$f^\flat(\theta) = \sup_{a \leq \Gamma}\{\theta a - f^*(a)\} \leq \sup_{a}\{\theta a - f^*(a)\} = \underline{f}(\theta) \leq f(\theta)$$

using Lemma 4.1 for the second equality. Also,

$$\frac{f^\flat(\theta)}{\theta} = \sup_{a \leq \Gamma}\left\{a - \frac{f^*(a)}{\theta}\right\}$$

and $f^*(a) \leq 0$ for these a, so this decreases as θ increases. This proves (ii). Maximizing $\theta' a - f^*(a) = \theta a - f^*(a) + (\theta' - \theta)a$ over $a \leq \Gamma$ completes the proof. $\square$

At this point an additional convexity idea is needed. The subdifferential at ϕ of a convex f, $\partial f(\phi)$, is defined as the set of slopes of possible tangents to f at ϕ. More formally,

$$\partial f(\phi) = \{a : f(\theta) \geq f(\phi) + a(\theta - \phi)\ \forall\theta\}.$$

The set is empty when f is infinite at ϕ or has a one-sided derivative at ϕ that is infinite in modulus, it contains a single value at points where f is differentiable, and it is a nondegenerate closed interval in all other cases; see Rockafellar (1970), Theorems 23.3, 23.4. In the last case the infimum of $\partial f(\phi)$ is the left point of this interval and is the derivative of f from the left there.

LEMMA 7.4. *Suppose f is proper and convex.*

(i) *If f is finite in a neighborhood of ϕ, then $\partial f(\phi) = \partial \underline{f}(\phi)$ and is certainly nonempty.*

(ii) *The following are equivalent*: $\gamma \in \partial f(\phi)$; $\phi\gamma - f(\phi) = f^*(\gamma)$ $(= \sup\{\theta\gamma - f(\theta) : \theta\})$.

(iii) *If* $f(\phi) = \underline{f}(\phi)$, *the statements in* (ii) *are also equivalent to* $\phi \in \partial f^*(\gamma)$ *and to* $\phi\gamma - f^*(\gamma) = \sup\{a\phi - f^*(a) : a\}$ $(= f(\phi))$.

PROOF. The assertion that $\partial f(\phi)$ is nonempty is in Rockafellar (1970), Theorem 23.4. The equivalences are some of the results in Rockafellar (1970), Theorem 23.5. □

LEMMA 7.5. *Let h be k-convex with $h(\phi) < \infty$. Suppose g is convex, $g \geq h$, $g(\phi) = h(\phi)$ and $\gamma \in \partial h(\phi)$. Then*:

(i) $\gamma \in \partial g(\phi)$ *and* $g^*(\gamma) = h^*(\gamma)$;

(ii) *if* $h(\theta) = g(\theta)$ *for all* $\theta \leq \phi$, *then* $g^*(a) = h^*(a)$ *for all* $a \leq \gamma$;

(iii) *if, in addition*, $g(\theta) = \infty$ *for* $\theta > \phi$, *then* $g^*(a) = h^*(\gamma) - \phi(\gamma - a) = \phi a - h(\phi)$ *for* $a > \gamma$.

PROOF. Since $g(\phi) = h(\phi)$ and $g \geq h$,

$$\begin{aligned}\partial h(\phi) &= \{a : h(\theta) \geq h(\phi) + a(\phi - \theta)\ \forall\theta\}\\ &\subset \{a : g(\theta) \geq g(\phi) + a(\phi - \theta)\ \forall\theta\}\\ &= \partial g(\phi).\end{aligned}$$

Thus $\gamma \in \partial h(\phi)$ implies $\gamma \in \partial g(\phi)$, and then Lemma 7.4(ii) gives

$$h^*(\gamma) = \sup_\theta\{\theta\gamma - h(\theta)\} = \phi\gamma - h(\phi) = \phi\gamma - g(\phi) = \sup_\theta\{\theta\gamma - g(\theta)\} = g^*(\gamma).$$

This proves (i). For any θ

$$\begin{aligned}\theta a - h(\theta) &= \theta\gamma - h(\theta) - \theta(\gamma - a)\\ &\leq \phi\gamma - h(\phi) - \theta(\gamma - a)\\ &= \phi a - h(\phi) - (\theta - \phi)(\gamma - a),\end{aligned}$$

and so, when $(\theta - \phi)(\gamma - a) \geq 0$, $\theta a - h(\theta) \leq \phi a - h(\phi)$. Hence, for $a \leq \gamma$

$$h^*(a) = \sup_\theta\{\theta a - h(\theta)\} = \sup_{\theta\leq\phi}\{\theta a - h(\theta)\}$$

and this holds also for g, giving (ii). Also, for $a > \gamma$,

$$\sup_{\theta\leq\phi}\{\theta a - h(\theta)\} = \phi a - h(\phi) = \phi\gamma - h(\phi) - \phi(\gamma - a) = h^*(\gamma) - \phi(\gamma - a)$$

and when $g(\theta) = \infty$ for $\theta > \phi$ the first expression here is $g^*(a)$. □

LEMMA 7.6. *Let f be k-convex, $\Gamma = \Gamma(f^*)$, and $\vartheta = \vartheta^\flat(f)$.*

(i) *If* $\Gamma > -\infty$ *and* $\partial f^*(\Gamma) = \varnothing$ *or* $f^*(\Gamma) < 0$, *then* $f^\flat = \underline{f}$ *and* $\vartheta = \infty$.
(ii) *If* $\Gamma > -\infty$ *and* $\partial f^*(\Gamma) \neq \varnothing$, *then for any* $\phi \in \partial f^*(\Gamma)$

$$f^\flat(\theta) = \begin{cases} \underline{f}(\theta), & \theta \leq \phi, \\ \theta\Gamma - f^*(\Gamma), & \theta \geq \phi. \end{cases}$$

(iii) $f^\flat(\theta) = \underline{f}(\theta)$ *if and only if* $\theta \leq \vartheta$.

PROOF. Assume $\partial f^*(\Gamma) = \varnothing$. Then $f^*(a) = \infty$ for $a > \Gamma$, using Rockafellar (1970), Theorem 23.4. Also, if $f^*(\Gamma) < 0$, then, since f^* is continuous when finite, $f^*(a) = \infty$ for $a > \Gamma$. Hence, in both cases,

$$f^\flat(\theta) = \sup_{a \leq \Gamma}\{\theta a - f^*(a)\} = \sup_a\{\theta a - f^*(a)\} = \underline{f}(\theta),$$

and so $\vartheta^\flat(f) = \inf\{\theta : f^\flat(\theta) < \underline{f}(\theta)\} = \infty$. This give (i). Now assume $\partial f^*(\Gamma) \neq \varnothing$. For any $\phi \in \partial f^*(\Gamma)$, Lemma 7.5 (with $h = f^*$ and $g = f^{\circledast}$) gives (ii) because $(f^*)^* = \underline{f}$.

Turning to the final part, the result is immediate (and without real content) when $\Gamma = -\infty$. It also holds when (i) holds. When (ii) holds $\vartheta^\flat(f) \geq \sup \partial f^*(\Gamma)$, but when $\underline{f}(\phi) = f^\flat(\phi) = \phi\Gamma - f^*(\Gamma)$ Lemma 7.4(ii) gives $\phi \in \partial f^*(\Gamma)$. Hence $\vartheta^\flat(f) = \sup \partial f^*(\Gamma)$ and $f^\flat(\theta) < \underline{f}(\theta)$ for all $\theta > \vartheta^\flat(f)$. $\square$

PROOF OF LEMMA 7.2. Let $\vartheta = \vartheta^\flat(f)$ and $\Gamma = \Gamma(f^*)$. When $\Gamma = -\infty$, $f^*(a) > 0$ for all a, $f^\flat \equiv -\infty$ and $\vartheta = -\infty$. If $f^\natural \not\equiv -\infty$, then, for some finite $A \geq 0$ and B, $A + B\theta \leq f^\natural(\theta) \leq f(\theta)$ and then $f^*(B) \leq -A \leq 0$. Hence when $\Gamma = -\infty$, $f^\natural \equiv -\infty$ and $\vartheta(f) = -\infty$.

Assume now that $\Gamma > -\infty$, so that $f^\natural \not\equiv -\infty$. Then $f^\natural(\underline{\psi}) = f(\underline{\psi})$. By Lemma 7.3(ii), $f^\natural \geq f^\flat$ and using Lemma 7.6 $f^\flat(\underline{\psi}) = \underline{f}(\underline{\psi}) \leq f(\underline{\psi}) = f^\natural(\underline{\psi})$. We need to show that $f^\natural$ and $f^\flat$ agree on $(\underline{\psi}, \infty)$. When $\mathcal{D}(f) = \{\underline{\psi}\}$ the result holds. Hence we may suppose $\mathcal{D}(f)$ has a nonempty interior. Then $f \geq f^\natural \geq f^\flat = \underline{f} = f$ on $(\underline{\psi}, \vartheta)$. Thus the result holds when $\vartheta = \infty$, and so we can assume $\vartheta < \infty$, and hence, by Lemma 7.6(i), that $f^*(\Gamma) = 0$. Then, by Lemma 7.6(ii), $f^\flat(\theta) = f(\theta)$ for $\theta \in (\underline{\psi}, \vartheta)$ and $f^\flat(\theta) = \Gamma\theta$ for $\theta \in [\vartheta, \infty)$. Suppose that for some $\phi > \underline{\psi}$, $f^\natural(\phi) > f^\flat(\phi)$. Hence, $\phi \geq \vartheta$ and $f^\natural(\phi) > \Gamma\phi$. Then

$$\frac{f^\natural(\phi)}{\phi} > \Gamma = \frac{f^\flat(\vartheta)}{\vartheta} = \frac{\underline{f}(\vartheta)}{\vartheta} = \liminf_{\theta \to \vartheta} \frac{f(\theta)}{\theta} \geq \liminf_{\theta \to \vartheta} \frac{f^\natural(\theta)}{\theta},$$

contradicting that $f^\natural(\theta)/\theta$ is decreasing and continuous at ϕ.

It remains to prove $\vartheta(f) = \vartheta$ in this case. Lemma 7.6(iii) gives

$$\vartheta = \inf\{\theta : f^\flat(\theta) < \underline{f}(\theta)\} = \sup\{\theta : f^\flat(\theta) = \underline{f}(\theta)\}$$

and the relationship between $f^\natural$ and $f^\flat$ already established means this equals $\sup\{\theta : f^\natural(\theta) = f(\theta)\}$ which is $\vartheta(f)$. $\square$

PROOF OF PROPOSITION 7.1. This uses Lemmas 7.2 and 7.6. When $\Gamma = -\infty$, Lemma 7.2 contains the result. When $\partial f^*(\Gamma) = \varnothing$ or $f^*(\Gamma) < 0$ the characterization of $f^\natural$ follows from Lemma 7.6(i). In the remaining cases $\vartheta = \vartheta(f) < \infty$ and the characterization follows from Lemma 7.6(ii). The assertion about Γ follows from this characterization. $\square$

The following lemma will be important in later sections. The one after it records various facts needed to prove the alternative recursion in Proposition 2.5.

LEMMA 7.7. *Let f be k-convex and $a \in \partial \underline{f}(\theta)$.*

(i) *If $\theta > \vartheta(f)$, then $f^*(a) > 0$.*
(ii) *If $\theta < \vartheta(f)$, then $f^*(a) \le 0$.*

PROOF. By Lemma 7.2, $\vartheta(f) = \vartheta^\flat(f)$. Lemma 7.4 gives

$$\underline{f}(\theta) = \theta a - f^*(a) = \sup_b\{\theta b - f^*(b)\} \ge \sup_{b \le \Gamma}\{\theta b - f^*(b)\} = f^\flat(\theta).$$

When $\theta > \vartheta(f)$ there is strict inequality, implying that $f^*(a) > 0$.

If $0 = \theta < \vartheta(f)$, then $\Gamma(f^*) > -\infty$ and so $f^*(a) = -\underline{f}(0) < 0$. Otherwise, take $\theta < \theta + \varepsilon < \vartheta(f)$. Note that $f^\flat(\theta)/\theta$ is decreasing on $(0, \infty)$ and equals $\underline{f}(\theta)/\theta$ on $(0, \vartheta(f))$, and that $f^\flat(\theta) = \underline{f}(\theta) = \theta a - f^*(a)$. Therefore

$$\frac{\theta + \varepsilon}{\theta}\big(\theta a - f^*(a)\big) = \frac{\theta + \varepsilon}{\theta}\underline{f}(\theta) \ge \underline{f}(\theta + \varepsilon) \ge (\theta + \varepsilon)a - f^*(a).$$

Thus $-\varepsilon f^*(a)/\theta \ge 0$. $\square$

LEMMA 7.8. *Suppose f and κ are k-convex.*

(i) $f^\circledast = (f^\natural)^\circledast = (f^\natural)^*$ *and* $\underline{f}^\natural = (f^\circledast)^*$;
(ii) $\mathcal{D}(f^\natural) = \mathcal{D}^+(f)$;
(iii) $\mathfrak{M}[f^\natural, \kappa^\natural]^\natural = \mathfrak{M}[f^\natural, \kappa^\natural] \le \mathfrak{M}[f^\natural, \kappa]$.

PROOF. The first part follows easily from Lemmas 4.1 and 7.2, because $f^\flat = (f^\circledast)^*$, and the second from Lemmas 7.2 and 7.3(iii). For the final one, just note that $\mathfrak{M}[f^\natural, \kappa^\natural]$ inherits all the right properties from $f^\natural$ and $\kappa^\natural$. $\square$

PROOF OF PROPOSITION 2.5. By definition (2.6), $f_1^\circledast = \kappa_1^\circledast = r_1$. Suppose the result is true for $i - 1$. By Lemmas 4.1(ii) and 7.8(i)

$$\begin{aligned}(f_i^\natural)^* = f_i^\circledast &= \mathfrak{M}[f_{i-1}^\natural, \kappa_i]^\circledast = (\mathfrak{M}[f_{i-1}^\natural, \kappa_i]^*)^\circ \\ &= \mathfrak{C}[f_{i-1}^\circledast, \kappa_i^*]^\circ = \mathfrak{C}[r_{i-1}, \kappa_i^*]^\circ = r_i\end{aligned}$$

as required. $\square$

LEMMA 7.9. *Let f_i be given by* (2.11). *When* (1.4) *holds*, f_i *is closed and* k-*convex*, $[\phi_i, \infty) \subset \mathcal{D}(f_i^\natural) = \bigcap_{j \le i} \mathcal{D}^+(\kappa_j)$, $-\infty < r_i$ *for each* i, *and if* $f_1(0) > 0$, *then* $f_i(0) > 0$.

PROOF. Using Lemma 4.3, $f_1 = \kappa_1$ is k-convex, and by Lemma 7.8(ii) $\mathcal{D}(f_1^\natural) = \mathcal{D}^+(\kappa_1)$. Hence the result is true for $i = 1$. Suppose the result holds for $i - 1$. By definition,

$$\mathcal{D}(f_i) = \mathcal{D}(\mathfrak{M}[f_{i-1}^\natural, \kappa_i]) = \mathcal{D}(f_{i-1}^\natural) \cap \mathcal{D}(\kappa_i) \supset [\phi_{i-1}, \infty) \cap \mathcal{D}(\kappa_i),$$

which is nonempty, since it contains ϕ_i by (1.4). Thus f_i is k-convex and $\mathcal{D}(f_i^\natural)$ contains $[\phi_i, \infty)$. Furthermore, $f_{i-1}^\natural$ and κ_i are closed, so f_i is, too. Since $\mathcal{D}(f_i)$ is nonempty $\mathcal{D}^+(f_i) = \mathcal{D}(f_{i-1}^\natural) \cap \mathcal{D}^+(\kappa_i)$, and then the induction hypothesis and Lemma 7.8(ii) confirm the formula for $\mathcal{D}(f_i^\natural)$. Now, by Lemma 4.2(i), $-\infty < (f_i^\natural)^* = f_i^{\circledast} = r_i$. Since f_{i-1} is closed, $f_{i-1}(0) > 0$ implies that $f_{i-1}^\natural(0) = f_{i-1}(0)$ and then $f_i(0) \ge f_{i-1}^\natural(0) = f_{i-1}(0) > 0$. $\square$

8. Upper bounds on numbers. Here, Theorem 2.7 will be proved. The first lemma presses the argument deployed at the start of the proof of Proposition 5.3 a little further. It notes that (8.1) implies the apparently stronger (8.3). The minor distinction between $f^\natural$ and $f^\flat$ $(=(f^{\circledast})^*)$, exposed in Lemma 7.2, matters in this result.

LEMMA 8.1. *Suppose that for a* k-*convex* f *with* $\Gamma(f^*) > -\infty$ *and a point processes* $P^{(n)}$

$$\limsup_n \frac{1}{n} \log\left(\int e^{\theta x} P^{(n)}(dx)\right) \le f(\theta) \qquad a.s.\ \forall\theta. \tag{8.1}$$

Then

$$\limsup_n \frac{1}{n} \log\left(P^{(n)}[na, \infty)\right) \le -f^{\circledast}(a) \qquad a.s.\ \forall a \tag{8.2}$$

and

$$\limsup_n \frac{1}{n} \log\left(\int e^{\theta x} P^{(n)}(dx)\right) \le f^\natural(\theta) \qquad a.s.\ \forall\theta. \tag{8.3}$$

PROOF. For $\theta \ge 0$,

$$\theta na + \log P^{(n)}[na, \infty) \le \log \int e^{\theta x} P^{(n)}(dx)$$

and so using (8.1), minimizing over θ, and using that $P^{(n)}[na, \infty)$ is eventually zero when it decays gives (8.2). The assertions (8.1) and (8.3) are the same when

$\vartheta(f) = \infty$. Hence we may assume $\vartheta(f) < \infty$. For $\varepsilon > 0$ and large enough n, $P^{(n)}[n(\Gamma(f^*) + \varepsilon), \infty) = 0$. Then, for $\theta \ge \psi$,

$$\int e^{\theta x} P^{(n)}(dx) \le e^{(\theta-\psi)(\Gamma(f^*)+\varepsilon)n} \int e^{\psi x} P^{(n)}(dx)$$

so that (8.1) gives

$$\limsup \frac{1}{n} \log\biggl(\int e^{\theta x} P^{(n)}(dx)\biggr) \le f(\psi) + (\theta - \psi)\Gamma(f^*) \qquad \text{a.s.}$$

Take $\psi = \theta$ when $\theta < \vartheta(f)$ and when $\theta = \vartheta(f) = \inf \mathcal{D}(f)$, so in these cases the right-hand side is just $f(\theta)$. Otherwise, take $\psi \in \mathcal{D}(f)$ and then let $\psi \to \vartheta(f)$. [If f is lower semi-continuous at $\vartheta(f)$, taking $\psi = \vartheta(f)$ will do.] Then the right-hand side becomes $\underline{f}(\vartheta(f)) + (\theta - \vartheta(f))\Gamma(f^*)$. Proposition 7.1 confirms that the right-hand side is $f^\natural$ in all cases. $\square$

Recall that $-\chi_i$ is the logarithm of the indicator function of the set $\mathcal{D}_{i-1,i}$.

LEMMA 8.2. *In a sequential process with $m_{\upsilon\tau} > 0$ for $\upsilon \in \mathcal{C}_{K-1}$ and $\tau \in \mathcal{C}_K$, suppose that for all $\nu \in \mathcal{C}_1$ and θ*

$$\limsup \frac{1}{n} \log\biggl(\int e^{\theta x} Z_\upsilon^{(n)}(dx)\biggr) \le f(\theta) \qquad \textit{a.s.-}\mathbb{P}_\nu,$$

where f is k-convex with $\Gamma(f^*) > -\infty$. Let $g = f^\natural + \chi_K$ and let κ be the PF^+ eigenvalue of the final block in m, corresponding to $\mathcal{C}_K$. Then, for $\sigma \in \mathcal{C}_K$,

$$\limsup \frac{1}{n} \log\biggl(\int e^{\theta x} Z_\sigma^{(n)}(dx)\biggr) \le \mathfrak{M}[g^\natural, \kappa]^\natural(\theta) \qquad \textit{a.s.-}\mathbb{P}_\nu$$

and $\Gamma(\mathfrak{M}[g^\natural, \kappa]^\natural) > -\infty$.

PROOF. Note first that $f^\natural \le g^\natural \le \mathfrak{M}[g^\natural, \kappa]^\natural$, so that $\Gamma(f^*) > -\infty$ implies that $\Gamma(g^*) > -\infty$ and that $\Gamma(\mathfrak{M}[g^\natural, \kappa]^\natural) > -\infty$.

Recall that $F_\tau^{(n)}$ are those in the nth generation that are the first of type τ in their line of descent. Taking conditional expectations,

$$\mathbb{E}\biggl[\int e^{\theta x} F_\tau^{(n+1)}(dx) \Big| \mathcal{F}^{(n)}\biggr] = \biggl(\int e^{\theta x} Z_\upsilon^{(n)}(dx)\biggr) m_{\upsilon\tau}(\theta)$$

and so, using Lemma 8.1 and the definition of g,

$$\limsup \frac{1}{n} \log \mathbb{E}\biggl[\int e^{\theta x} F_\tau^{(n+1)}(dx) \Big| \mathcal{F}^{(n)}\biggr] \le g(\theta) \qquad \text{a.s.-}\mathbb{P}_\nu.$$

Then conditional Borel–Cantelli [e.g., Chen (1978)] gives that

$$\limsup \frac{1}{n} \log\biggl(\int e^{\theta x} F_\tau^{(n)}(dx)\biggr) \le g(\theta) \qquad \text{a.s.-}\mathbb{P}_\nu$$

and a further application of Lemma 8.1 gives that

$$\limsup \frac{1}{n} \log\left(\int e^{\theta x} F_\tau^{(n)}(dx)\right) \le g^\natural(\theta) \qquad \text{a.s.-}\mathbb{P}_\nu.$$

The set of particles obtained as those first in their lines of descent that are either in $\mathcal{C}_K$ or in generation n forms an optional line, as in Jagers (1989). Let $\mathcal{G}^{(n)}$ contain all information on reproduction down lines of descent to particles in this line. In this sequential process the first in any line of descent with a type in $\mathcal{C}_K$ is necessarily of type τ. For any $\sigma \in \mathcal{C}_K$ and θ,

$$\mathbb{E}\left[\int e^{\theta x} Z_\sigma^{(n)}(dx) \middle| \mathcal{G}^{(n)}\right] = \sum_{r=0}^{n} \int e^{\theta x} F_\tau^{(r)}(dx)(m(\theta)^{n-r})_{\tau\sigma}.$$

Hence, the bound just obtained, Lemma 5.1, and routine estimation give

$$\limsup_n \frac{1}{n} \log \mathbb{E}\left[\int e^{\theta x} Z_\sigma^{(n)}(dx) \middle| \mathcal{G}^{(n)}\right] \le \mathfrak{M}[g^\natural, \kappa](\theta) \qquad \text{a.s.-}\mathbb{P}_\nu.$$

Conditional Borel–Cantelli and Lemma 8.1 complete the proof. $\square$

LEMMA 8.3. *Define g_i by (2.15). Then g_K is finite somewhere on $(0, \infty)$ if and only if (1.4) holds and (2.12) holds for $i = 1, 2, \dots, K-1$. When these hold g_K is k-convex,*

$$[\phi_K, \infty) \subset \mathcal{D}(g_K^\natural) = \left(\bigcap_{j \le K} \mathcal{D}^+(\kappa_j)\right) \cap \left(\bigcap_{j \le K-1} \mathcal{D}^+_{j,j+1}\right),$$

$g_K^\natural$ is continuous on $\mathcal{D}(g_K^\natural)$, and $-g_K^*(a) < \infty$ for some finite a.

PROOF. Assume $g_K(\phi_K)$ is finite. Then $\phi_K \in \mathcal{D}(\kappa_K)$ and there is a $\phi_{K-1,K} \le \phi_K$ such that $(g_{K-1}^\natural + \chi_K)(\phi_{K-1,K}) < \infty$, which implies that $\phi_{K-1,K} \in \mathcal{D}_{K-1,K}$ and that there is a $\phi_{K-1} \le \phi_{K-1,K}$ with $g_{K-1}(\phi_{K-1})$ finite. Hence, by induction on K, $g_K(\phi)$ finite for some positive ϕ implies that (1.4) holds and (2.12) holds for $i = 1, 2, \dots, K-1$.

Now suppose (1.4) holds and (2.12) holds for $i = 1, 2, \dots, K-1$. All the assertions of the lemma then hold with $g_1 = \kappa_1$ in place of g_K. Suppose all the assertions hold for g_{K-1}. Then

$$\mathcal{D}(g_{K-1}^\natural + \chi_{K-1}) = \mathcal{D}^+(g_{K-1}) \cap \mathcal{D}_{K-1,K} \supset [\phi_{K-1}, \infty) \cap \mathcal{D}_{K-1,K} \ni \phi_{K-1,K}.$$

Since this is nonempty,

$$\mathcal{D}(g_K) = \mathcal{D}\big(\mathfrak{M}[(g_{K-1}^\natural + \chi_{K-1})^\natural, \kappa_K]\big) = \mathcal{D}^+(g_{K-1}) \cap \mathcal{D}^+_{K-1,K} \cap \mathcal{D}(\kappa_K)$$

and g_K is continuous there, because $g_{K-1}^\natural$ is by assumption and κ_K is by Lemma 4.3. Furthermore $\mathcal{D}(g_K) \supset [\phi_{K-1}, \infty) \cap \mathcal{D}(\kappa_K) \ni \phi_K$ and so is nonempty. Then, using Lemma 7.8(ii),

$$\mathcal{D}(g_K^\natural) = \mathcal{D}^+(g_K) = \mathcal{D}^+(g_{K-1}) \cap \mathcal{D}^+_{K-1,K} \cap \mathcal{D}^+(\kappa_K) \supset [\phi_K, \infty),$$

and $g^\natural$ is continuous there. Substituting for $\mathcal{D}^+(g_{K-1})$ gives the formula for $\mathcal{D}^+(g_K)$. Lemma 4.2(i) gives the final part and the induction is complete. $\square$

PROOF OF THEOREM 2.7. Note first that the final assertion is contained in Lemma 8.3. Now, by Lemma 8.1, it is enough to show that

$$\limsup \frac{1}{n} \log\left(\int e^{\theta x} Z_\sigma^{(n)}(dx)\right) \le g_K(\theta) \qquad \text{a.s.-}\mathbb{P}_\nu$$

and that $\Gamma(g_K^*) > -\infty$. Both hold when $K = 1$, the first by Lemma 5.2, the second by combining Lemmas 4.2(vi), 4.3(iv) and the assumption that $\kappa_1(0) > 0$. Assume the result holds for $K - 1$. Then it holds also for K, by Lemma 8.2 with $f = g_{K-1}$ and $\kappa = \kappa_K$. $\square$

9. Matching the lower and upper bounds. In this section Theorems 2.4 and 2.6 will be proved, using Theorem 2.7. These are cases where the upper bound on numbers matches the lower bound based on Theorem 2.3. The simpler theorem will be discussed first.

PROOF OF THEOREM 2.4. Let f_i and g_i be as (2.11) and (2.15). Clearly $g_1 = f_1 = \kappa_1$. Assume $g_{i-1} = f_{i-1}$. Note first that $(f_{i-1}^\natural + \chi_i)^\natural \ge f_{i-1}^\natural$ and so

$$g_i = \mathfrak{M}[(g_{i-1}^\natural + \chi_i)^\natural, \kappa_i] = \mathfrak{M}[(f_{i-1}^\natural + \chi_i)^\natural, \kappa_i] \ge \mathfrak{M}[f_{i-1}^\natural, \kappa_i] = f_i.$$

By Lemma 7.9, (2.9) is equivalent to $\mathcal{D}(f_{i-1}^\natural) \cap \mathcal{D}(\kappa_i) \subset \mathcal{D}_{i-1,i}$ $(=\mathcal{D}(\chi_i))$, and when this holds $\mathfrak{M}[f_{i-1}^\natural + \chi_i, \kappa_i] = \mathfrak{M}[f_{i-1}^\natural, \kappa_i]$. Then,

$$g_i = \mathfrak{M}[(f_{i-1}^\natural + \chi_i)^\natural, \kappa_i] \le \mathfrak{M}[f_{i-1}^\natural + \chi_i, \kappa_i] = \mathfrak{M}[f_{i-1}^\natural, \kappa_i] = f_i.$$

Hence $g_i = f_i$. Thus, by induction, $g_K = f_K$. Then $g_K^{\circledast} = f_K^{\circledast}$, which by Corollary 2.8 gives the result. $\square$

The proof just given relies on a simple estimation of $(f_{i-1}^\natural + \chi_i)^\natural$ and then $\mathcal{D}(\kappa_i)$ making χ_i irrelevant. To deal with more cases it is necessary to refine the estimation of $(f_{i-1}^\natural + \chi_i)^\natural$ and make a more careful comparison of the result with κ_i. This is done next.

LEMMA 9.1. *Suppose f and κ are k-convex with $\Gamma(f^*) > -\infty$. Suppose C is a convex set, and let $\chi(\theta) = -\log I(\theta \in C)$, $\underline{\psi} = \inf C$ and $\overline{\psi} = \sup C$. Let $\chi_1(\theta) = -\log I(\theta \in C^+)$ and $\chi_2(\theta) = -\log I(\theta \in (-\infty, \overline{\psi}])$.*

(i) $\Gamma(\mathfrak{M}[(f^\natural + \chi)^\natural, \kappa]^*) > -\infty$.
(ii) *If $\mathcal{D}(f^\natural) \cap C \ne \varnothing$ and $f^\natural$ is continuous from the right at $\overline{\psi}$, then*

$$(f^\natural + \chi)^\natural(\theta) = \begin{cases} (f^\natural + \chi)(\theta), & \theta < \overline{\psi}, \\ \theta\big(f^\natural(\overline{\psi})/\overline{\psi}\big), & \theta \ge \overline{\psi}. \end{cases}$$

(iii) *If, in addition to the conditions in* (ii),

$$\text{(9.1)}\qquad \textit{either } \kappa(\theta) \ge \theta\big(f^{\natural}(\overline{\psi})/\overline{\psi}\big) \textit{ for } \theta \in [\overline{\psi}, \infty) \quad \textit{or} \quad \vartheta(f) \le \overline{\psi},$$

then

$$\mathfrak{M}[(f^{\natural} + \chi)^{\natural}, \kappa] = \mathfrak{M}[f^{\natural} + \chi_1, \kappa].$$

(iv) *If, in addition to the conditions in* (ii), $\mathcal{D}(f^{\natural}) \cap \mathcal{D}(\kappa) \subset [\underline{\psi}, \infty)$, *then*

$$\mathfrak{M}[(f^{\natural} + \chi)^{\natural}, \kappa] = \mathfrak{M}[(f^{\natural} + \chi_2)^{\natural}, \kappa],$$

except possibly at $\underline{\psi}$, *and when they differ there the left-hand side is infinite.*

(v) *When the conditions in both* (iii) *and* (iv) *hold*, $\mathfrak{M}[(f^{\natural} + \chi)^{\natural}, \kappa] = \mathfrak{M}[f^{\natural}, \kappa]$ *except possibly at* $\underline{\psi}$, *and when they differ there the left-hand side is infinite.*

PROOF. The proof of part (i) mimics the first part of the proof of Lemma 8.2. The form of $(f^{\natural} + \chi)^{\natural}$ in (ii) follows from Proposition 7.1. Now, assume (9.1) holds. In the first case, $(f^{\natural} + \chi)^{\natural}$ is dominated by κ in $[\overline{\psi}, \infty)$ and equals $f^{\natural}$ on C. In the second, since $\vartheta(f) \le \overline{\psi} < \infty$ and $f^{\natural}$ is continuous from the right at $\overline{\psi}$, $\Gamma(f^*) = f^{\natural}(\overline{\psi})/\overline{\psi}$ by Proposition 7.1; and so $(f^{\natural} + \chi)^{\natural} = f^{\natural}$ on C^+, and this also holds when $\overline{\psi} = \infty$. Hence in both cases $\mathfrak{M}[(f^{\natural} + \chi)^{\natural}, \kappa] = \mathfrak{M}[f^{\natural} + \chi_1, \kappa]$, proving (iii). By (ii), $(f^{\natural} + \chi)^{\natural}$ and $(f^{\natural} + \chi_2)^{\natural}$ agree for $\theta \ge \overline{\psi}$, and $(f^{\natural} + \chi_2)^{\natural} = f^{\natural}$ for $\theta < \overline{\psi}$. Since $\mathcal{D}(\mathfrak{M}[f^{\natural}, \kappa]) = \mathcal{D}(f^{\natural}) \cap \mathcal{D}(\kappa)$, $\mathfrak{M}[(f^{\natural} + \chi)^{\natural}, \kappa]$ and $\mathfrak{M}[f^{\natural}, \kappa]$ agree (and are both infinite) on $(-\infty, \underline{\psi})$ and by (ii) they agree on $(\underline{\psi}, \overline{\psi})$. They also agree at $\underline{\psi}$ when $\underline{\psi} \in C$ and when it is not $(f^{\natural} + \chi)$ is infinite there. This proves (iv). The final part is an application of (iv) to $f + \chi_1$. $\square$

PROOF OF THEOREM 2.6. Note first that, by Lemma 7.8(ii), $\mathcal{D}^+(g_{K-1}) = \mathcal{D}(g_{K-1}^{\natural})$. Also, Lemmas 7.8(ii) and 7.9 show that the left of (2.14) is just $\mathcal{D}^+(f_i) \cap \mathcal{D}(\kappa_{i+1})$.

The proof is by induction. For it, add in the additional assertion that $g_K^{\natural} = f_K^{\natural}$, except possibly at $\inf \mathcal{D}(f_K)$ when $g_K^{\natural}$ is infinite there. The result, including this additional assertion, is true for $K = 1$. Assume the result and the addition are true for $K - 1$. When (1.4) holds and (2.12) holds for $i = 1, 2, \dots, K - 1$, Lemma 8.3 implies that $g_{K-1}^{\natural}$ is finite at $\overline{\psi}_{K-1}$ and so equals $f_{K-1}^{\natural}$ and is continuous from the right there. Also, by the induction hypothesis $\mathcal{D}(g_{K-1}^{\natural}) \subset \mathcal{D}(f_{K-1}^{\natural})$ [and equals it unless $f_{K-1}^{\natural}$ is finite and $g_{K-1}^{\natural}$ infinite at $\inf \mathcal{D}(f_{K-1}^{\natural}) = \inf \mathcal{D}(f_{K-1})$]. Hence (2.13) and (2.14) with $i = K - 1$ mean Lemma 9.1(v) applies. Together with the induction hypothesis this gives $g_K = \mathfrak{M}[f_{K-1}^{\natural}, \kappa_K] = f_K$ except possibly at $\underline{\psi}_{K-1}$ and $\inf \mathcal{D}(f_{K-1}^{\natural})$, where they can only differ with g_K being infinite. Furthermore, by Lemma 8.3, $f_K(\phi_K) \le g_K(\phi_K) < \infty$. Since both functions are proper and convex, and f_K is closed, they can only differ by g_k being greater, and infinite, at the endpoints of $\mathcal{D}(f_K)$. Hence $g_K^\natural = f_K^\natural$ except possibly at $\inf \mathcal{D}(f_K^\natural)$. Then these two functions have the same F-dual, that is, $g_K^\circledast = f_K^\circledast$. $\square$

10. Formulas for the speed. The main objective here is to establish Theorem 2.9 giving an alternative formula for the speed $\Gamma(g_K^*)$, which plays a critical role in the proof of Theorem 2.10. A few other remarks are also included about computing the speed.

There are several alternative formulas for $\Gamma(f^*)$ from the irreducible case that apply more widely to any k-convex f. One is contained in (7.1) in Proposition 7.1. Another is that $\Gamma = \sup\{a : f^*(a) \le 0\}$, which holds because f^* is convex and increasing. Furthermore, by convexity Γ is the unique solution to $f^*(\Gamma) = 0$, provided only that there are a u and v with $f^*(u) < 0 \le f^*(v) < \infty$.

When f is differentiable throughout $\mathcal{D}(f)$ and there is a θ such that $\theta f'(\theta) - f(\theta) = 0$, then $\Gamma(f^*) = f'(\theta)$—this is straightforward calculus when θ is in the interior of $\mathcal{D}(\kappa)$, and all cases are covered by Rockafellar (1970), Theorem 23.5(b). Then $\Gamma(f^*)$ can be found by solving $f(\theta) = \theta f'(\theta)$ for θ. This is certainly relevant in the irreducible case, since Lemma 4.3(iii) gives that $f = \kappa$ is differentiable, but need not be once there is more than one class.

LEMMA 10.1. *Suppose that f and κ are k-convex with $\Gamma(f^*) > -\infty$, that $\chi = -\log I(\theta \in C)$ for a convex C, that $g = \mathfrak{M}[(f^\natural + \chi)^\natural, \kappa]$ and that this g is finite somewhere* [*so* $\mathcal{D}(f^\natural) \cap C \cap \mathcal{D}(\kappa) \neq \varnothing$]. *Let $\overline{\psi} = \sup C$. For $0 < \theta \notin C^+$, $g(\theta) = \infty$. For $0 < \theta \in C^+$,*

$$\frac{g(\theta)}{\theta} = \inf\left\{\max\left\{\frac{f(\phi)}{\phi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta, \phi \le \overline{\psi}\right\}, \tag{10.1}$$

where the condition $\phi \le \overline{\psi}$ can be omitted when (9.1) *holds and $f^\natural$ is continuous from the right at $\overline{\psi}$.*

PROOF. It is immediate from its definition that $g(\theta) = \infty$ for $0 < \theta \notin C^+$. By definition $f^\natural(\theta)/\theta$ is decreasing as θ increases for any convex f. For $\theta \in C^+$,

$$\begin{aligned}
\frac{g(\theta)}{\theta} &= \max\left\{\frac{(f^\natural + \chi)^\natural(\theta)}{\theta}, \frac{\kappa(\theta)}{\theta}\right\} \\
&= \inf\left\{\max\left\{\frac{(f^\natural + \chi)^\natural(\phi)}{\phi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta\right\} \\
&= \inf\left\{\max\left\{\frac{f^\natural(\phi)}{\phi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta, \phi \in C\right\}.
\end{aligned} \tag{10.2}$$

Proposition 7.1 relates $f^\natural$ and f: $f^\natural(\theta)/\theta$ and $f(\theta)/\theta$ agree and are decreasing up to $\vartheta(f)$; when $\vartheta(f) < \infty$, the former is constant and the latter is larger for

$\theta > \vartheta(f)$, and either the two agree at $\theta = \vartheta(f)$ or the latter is larger. Hence,

$$\frac{g(\theta)}{\theta} = \inf\left\{\max\left\{\frac{f(\varphi)}{\varphi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta, \varphi \le \phi \in C\right\}.$$

This is (10.1) when $\overline{\psi} \in C$. When it is not, the limit of $f(\varphi)/\varphi$ as $\varphi \uparrow \overline{\psi}$ is no greater than $f(\overline{\psi})/\overline{\psi}$ and so replacing $\varphi \le \phi \in C$ by $\varphi \le \overline{\psi}$ in the formula will not change the output.

Lemma 9.1(iii) shows that if (9.1) holds and $f^{\natural}$ is continuous from the right at $\overline{\psi}$, then the restriction to $\phi \in C$ in (10.2) can be replaced by $\phi \in C^+$. Then f can replace $f^{\natural}$ if this restriction is dropped, too; that is, for $\theta \in C^+$,

$$\begin{aligned} \frac{g(\theta)}{\theta} &= \inf\left\{\max\left\{\frac{f^{\natural}(\phi)}{\phi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta, \phi \in C^+\right\} \\ &= \inf\left\{\max\left\{\frac{f(\phi)}{\phi}, \frac{\kappa(\theta)}{\theta}\right\} : 0 < \phi \le \theta\right\}. \end{aligned}$$ □

PROOF OF THEOREM 2.9. The result is true for $K = 1$ as is the additional condition that $\Gamma(g_1^*) > -\infty$. Assume it is true along with this additional condition for $K-1$. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{K-1})$, $h(\boldsymbol{\theta}) = \max\{\kappa_i(\theta_i)/\theta_i : i \le K-1\}$ and let Δ_ϕ be the set the infimum is taken over in (2.18) for "$K-1$" so that the induction hypothesis is

$$\frac{g_{K-1}(\phi)}{\phi} = \inf\{h(\boldsymbol{\theta}) : \boldsymbol{\theta} \in \Delta_\phi\}.$$

By the previous lemma, for $0 < \theta \in \mathcal{D}^+_{K-1,K}$,

$$\frac{g_K(\theta)}{\theta} = \inf\left\{\max\left\{\frac{g_{K-1}(\phi)}{\phi}, \frac{\kappa_K(\theta)}{\theta}\right\} : 0 < \phi \le \theta, \phi \le \overline{\psi}_{K-1}\right\}.$$

Now

$$\max\left\{\frac{g_{K-1}(\phi)}{\phi}, \frac{\kappa_K(\theta)}{\theta}\right\} = \max\left\{\inf\{h(\boldsymbol{\theta}) : \boldsymbol{\theta} \in \Delta_\phi\}, \frac{\kappa_K(\theta)}{\theta}\right\}$$

and reordering the maximum and infimum on the right makes no difference. This gives g_K in the required form and Lemma 9.1(i) gives that $\Gamma(g_K^*) > -\infty$, completing the induction. Then the formula for $\Gamma(g_K)$ is, by Proposition 7.1, obtained by minimizing also over θ. The result for f_K is just a special case. □

LEMMA 10.2. *Assume (2.12) holds. In (2.18) and (2.19) the conditions "$\theta_i \le \overline{\psi}_i$" can be dropped if (2.13) holds for $i = 1, 2, \dots, K-1$. The conditions "$\theta_i \in \mathcal{D}^+_{i-1,i}$" can be dropped in (2.18) if $\vartheta(\kappa_{i+1}) \ge \underline{\psi}_i$ for $i = 1, \dots, K-2$ and from (2.19) if this holds also for $i = K-1$. When both sets of conditions in (2.19) can be dropped,* $\Gamma(g_K^*) = \Gamma(f_K^*)$.

PROOF. Lemma 8.3 gives that $g_i^\natural$ is continuous at $\overline{\psi}_i$. Then the proof that the conditions $\theta_i \le \overline{\psi}_i$ can be dropped in (2.18) is by induction on i using the last part of Lemma 10.1. When $\vartheta(\kappa_{i+1}) \ge \underline{\psi}_i$ for $i = 1, \dots, K-2$ the extra possibilities included by discarding the conditions $\theta_i \in \mathcal{D}^+_{i-1,i}$ for $i = 2, \dots, K-1$ in (2.18) are larger than those included and so make no difference to the infimum. (Here $\theta_K \in \mathcal{D}^+_{K-1,K}$ cannot be excluded, since the infimum is not over θ_K.) The argument simplifying (2.19) is the same. $\square$

PROOF OF THEOREM 2.10. This is contained in Lemma 10.2. $\square$

11. Simplifying the formula for the speed.

LEMMA 11.1. *Assume f and κ are k-convex, that $\underline{f}(0) > 0$ and that $g = \mathfrak{M}[f^\natural, \kappa]$ is finite somewhere. Let $\vartheta = \vartheta(g)$ [and, for later, $\Gamma = \Gamma(g^*)$]. Then the following hold*:

(i) $g^\natural \ge \mathfrak{M}[f^\natural, \kappa^\natural]$;
(ii) $\vartheta(\kappa) \le \vartheta$;
(iii) $g(\theta) = g^\natural(\theta) = \mathfrak{M}[f^\natural, \kappa^\natural](\theta)$ *for* $\theta < \vartheta$.

PROOF. Let $\varphi = \inf\{\theta : \kappa(\theta) > \mathfrak{M}[f^\natural, \kappa^\natural](\theta)\}$. Observe that

$$\mathfrak{M}[f^\natural, \kappa] = g \ge g^\natural = \mathfrak{M}[f^\natural, \kappa]^\natural \ge \mathfrak{M}[f^\natural, \kappa^\natural]^\natural = \mathfrak{M}[f^\natural, \kappa^\natural],$$

where the final equality is from Lemma 7.8(iii), which gives (i). There is equality throughout when $\theta < \vartheta(\kappa)$, since then $\kappa^\natural(\theta) = \kappa(\theta)$, and also when $\theta < \varphi$. This implies that $\vartheta(\kappa) \le \vartheta$, proving (ii), and that $\varphi \le \vartheta$. Note too, for later in the proof, that $\vartheta(\kappa) \le \varphi$, because $\kappa^\natural$ and κ agree for $\theta < \vartheta(\kappa)$. It remains to show that $\vartheta \le \varphi$. It is certainly true that $\vartheta \le \varphi$ when $\varphi = \infty$. Also if $\kappa(\theta) = \infty$ for all $\theta > \varphi$, then $g(\theta) = \infty$ for $\theta > \varphi$, but, by Proposition 7.1, $g^\natural$ is finite for $\theta > \varphi$ and so $\vartheta \le \varphi$. In the remaining case $\varphi < \infty$, κ is finite on $(\varphi, \varphi + \varepsilon)$ for some $\varepsilon > 0$, and there are $\theta_i \downarrow \varphi$ taken from this interval with $g(\theta_i) = \kappa(\theta_i)$. By Lemma 7.4(i) $\partial\kappa(\theta_i)$ is nonempty. Hence, by Lemma 7.5, $g^*(a) = \kappa^*(a)$ for $a \in \partial\kappa(\theta_i)$. Since $\vartheta(\kappa) \le \varphi$, Lemma 7.7(i) implies that $\kappa^*(a) > 0$. Hence $g^*(a) > 0$ and Lemma 7.7(ii) gives $\vartheta \le \varphi$. $\square$

LEMMA 11.2. *Use the setup of Lemma 11.1.*

(i) *If* $\Gamma = \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$, *then* $g^\natural(\theta) = \mathfrak{M}[f^\natural, \kappa^\natural](\theta)$ *except possibly at* $\theta = \vartheta$.

(ii) *If* $\Gamma > \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$, *then* $\vartheta < \infty$, *and* $(\mathfrak{M}[f^\natural, \kappa^\natural](\theta) - \theta\Gamma)$ *is strictly positive when* $\theta < \vartheta$ *and strictly negative when* $\theta > \vartheta$.

PROOF. Lemma 11.2(iii) gives $g^{\natural}(\theta) = g(\theta) = \mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)$ for $\theta < \vartheta$. Assume that $\Gamma = \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$ and that $\vartheta < \infty$. Then Proposition 7.1 implies that $g^{\natural}(\theta) = \theta\Gamma$ for $\theta > \vartheta$. Similarly, $\kappa^{\natural}(\theta) = \theta\Gamma(\kappa^*)$ for $\theta > \max\{0, \vartheta(\kappa)\}$. If $\Gamma = \Gamma(\kappa^*)$, $g^{\natural}$ and $\kappa^{\natural}$ agree for $\theta > \vartheta$. If instead, $\Gamma = \Gamma(f^*) > \Gamma(\kappa^*)$, then, for $\theta > \vartheta$, $f^{\natural}(\theta) \geq \theta\Gamma(f^*) = g^{\natural}(\theta)$. Hence, in both cases, using also Lemma 11.1(i), $g^{\natural}(\theta) = \mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)$ for $\theta > \vartheta$.

Assume now that $\Gamma > \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$. Take a such that

$$\max\{\Gamma(f^*), \Gamma(\kappa^*)\} < a < \Gamma.$$

Using Lemma 4.1(ii) and the definition of $\Gamma(\cdot)$, $\mathfrak{M}[f^{\natural}, \kappa^{\natural}]^*(a) = \mathfrak{C}[f^{\circledast}, \kappa^{\circledast}](a) = \infty$ and $g^*(a) < 0$. Hence g and $\mathfrak{M}[f^{\natural}, \kappa^{\natural}]$ differ somewhere and so Lemma 11.1(iii) implies that $\vartheta < \infty$.

Since $g(\theta) \geq \Gamma\theta$ for all θ, $\mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta) = g(\theta) \geq \Gamma\theta$ for $\theta < \vartheta$ and $\theta\Gamma = g^{\natural}(\theta) \geq \mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)$ for $\theta > \vartheta$. It remains to show these inequalities are strict. Since $\mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)/\theta$ is decreasing it can only equal Γ on an interval that, if nonempty, includes ϑ. If the interval has a nonempty interior, then, by convexity of $\mathfrak{M}[f^{\natural}, \kappa^{\natural}]$, $\mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta) \geq \Gamma\theta$ for all θ, contradicting that $\mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)/\theta \to \max\{\Gamma(f^*), \Gamma(\kappa^*)\} < \Gamma$ as $\theta \to \infty$. $\square$

LEMMA 11.3. *In the setup of Lemma 11.1 assume also that $f^{\natural}$ and κ are closed.*

(i) *If $\Gamma = \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$, then $g^{\natural} = \mathfrak{M}[f^{\natural}, \kappa^{\natural}]$.*

(ii) *If $\Gamma > \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$, then $g^{\natural}(\theta) = \theta\Gamma$ when $\theta \geq \vartheta$ and $g^{\natural}(\theta) = \mathfrak{M}[f^{\natural}, \kappa^{\natural}](\theta)$ when $\theta < \vartheta$.*

PROOF. When $f^{\natural}$ and κ are closed so are $\kappa^{\natural}$, g, $g^{\natural}$ and $\mathfrak{M}[f^{\natural}, \kappa^{\natural}]$. Part (i) now follows from Lemma 11.2(i) and part (ii) from Proposition 7.1 and Lemma 11.1(iii). $\square$

LEMMA 11.4. *In the setup of Lemma 11.1, assume $\Gamma > \max\{\Gamma(f^*), \Gamma(\kappa^*)\}$. Then $g(\theta) = \kappa(\theta) > f^{\natural}(\theta)$ on (ϑ, ∞).*

(i) *If $\mathcal{D}(\kappa) = \{\phi\}$, then $\vartheta = \phi$, $\kappa(\vartheta) < f^{\natural}(\vartheta) = g(\vartheta) < \infty$ and g is infinite elsewhere.*

(ii) *If $\mathcal{D}(\kappa)$ is not a single point, then, for some $\varepsilon > 0$, $g(\theta) = f^{\natural}(\theta) > \kappa(\theta)$ on $(\vartheta - \varepsilon, \vartheta)$.*

PROOF. Using the definition of g and Lemma 11.2(ii),

$$\mathfrak{M}[f^{\natural}, \kappa] = g(\theta) > g^{\natural}(\theta) = \Gamma\theta > \mathfrak{M}[f^{\natural}, \kappa^{\natural}] \qquad \text{for } \theta \in (\vartheta, \infty).$$

Thus g agrees with κ and strictly exceeds $f^{\natural}$ on (ϑ, ∞).

If $\vartheta = \inf \mathcal{D}(\kappa) < \sup \mathcal{D}(\kappa)$, then the closures of g and κ agree everywhere, giving $\Gamma = \Gamma(\kappa^*)$, which has been ruled out. Hence either $\mathcal{D}(\kappa) = \{\vartheta\}$ and $\kappa(\vartheta) <$

$f^{\natural}(\vartheta)$, giving (i), or $\inf \mathcal{D}(\kappa) < \vartheta \le \sup \mathcal{D}(\kappa)$. Assume the latter, so that there is an $\varepsilon > 0$ such that κ is finite, and continuous, on $(\vartheta - \varepsilon, \vartheta)$ and so $\kappa^{\natural}$ is finite and continuous on $(\vartheta - \varepsilon, \infty)$. When $f^{\natural}$ is infinite on $(-\infty, \vartheta)$ the result holds. Hence by adjusting ε, we can now assume $f^{\natural}$ is also finite on $(\vartheta - \varepsilon, \infty)$. Say $\vartheta(\kappa) = \vartheta$. Using continuity on $(\vartheta - \varepsilon, \infty)$, Proposition 7.1 and Lemma 11.1(iii),

$$\Gamma\vartheta = g^{\natural}(\vartheta) = \max\{f^{\natural}(\vartheta), \kappa^{\natural}(\vartheta)\} > \Gamma(\kappa^*)\vartheta = \kappa^{\natural}(\vartheta).$$

A further use of continuity now gives $f^{\natural}(\theta) > \kappa^{\natural}(\theta) = \kappa(\theta)$ on $(\vartheta - \varepsilon, \vartheta)$ after, if necessary, taking ε smaller. This proves (ii) in this case.

Say now that $\vartheta(\kappa) < \vartheta$, which by Lemma 11.1(ii) is the only other possibility, and adjust ε so that $\vartheta(\kappa) \le \vartheta - \varepsilon$. Suppose, for a contradiction, that there is a $\psi \in (\vartheta - \varepsilon, \vartheta)$ with $\kappa(\psi) = g(\psi)$. Take $a \in \partial\kappa(\psi)$, which is nonempty. By Lemma 7.7(i), $\kappa^*(a) > 0$ because $\psi > \vartheta(\kappa)$, but $g \ge \kappa$ and so Lemma 7.5 gives $\kappa^*(a) = g^*(a)$. However, by Lemma 7.7(ii), $\psi < \vartheta$ implies $g^*(a) \le 0$. Hence there is no such ψ and so $g = f^{\natural} > \kappa$ on $(\vartheta - \varepsilon, \vartheta)$. □

LEMMA 11.5. *In the setup and conditions of Proposition* 2.5, *suppose that* $\kappa_1(0) > 0$ *and that* $\Gamma(f_K^*) > \max\{\Gamma(f_{K-1}^*), \Gamma(\kappa_K^*)\}$. *Then*

$$f_K = \mathfrak{M}\Big[\max_j \kappa_j^{\natural}, \kappa_K\Big].$$

PROOF. For $i = 1, 2, \dots, K$, let

$$h_i = \mathfrak{M}\Big[\max_{j \ge i} \kappa_j^{\natural}, \kappa_K\Big]$$

so that $h_K = \kappa_K$. Now suppose that

$$f_K = \mathfrak{M}[f_i^{\natural}, h_{i+1}], \tag{11.1}$$

which is true, by definition, for $i = K - 1$. Induction will be used to show that this holds also for $i = 1$, which is the required result because $f_1^{\natural} = \kappa_1^{\natural}$.

Assume (11.1) holds for i and consider $f_i^{\natural} = \mathfrak{M}[f_{i-1}^{\natural}, \kappa_i]^{\natural}$. Using Lemmas 4.3, 7.9 and 11.3, there are two possibilities. One is that $f_i^{\natural} = \mathfrak{M}[f_{i-1}^{\natural}, \kappa_i^{\natural}]$ everywhere, in which case,

$$f_K = \max\{f_{i-1}^{\natural}, \kappa_i^{\natural}, h_{i+1}\} = \mathfrak{M}[f_{i-1}^{\natural}, h_i], \tag{11.2}$$

giving (11.1) for $i - 1$. Otherwise, $\vartheta(f_i) < \infty$ and

$$f_i^{\natural}(\theta) = \begin{cases} \theta\Gamma(f_i^*), & \text{for } \theta \ge \vartheta(f_i), \\ \mathfrak{M}[f_{i-1}^{\natural}, \kappa_i^{\natural}](\theta), & \text{for } \theta < \vartheta(f_i). \end{cases}$$

Thus (11.2) holds for $\theta < \vartheta(f_i)$. Also, $\Gamma(f_i^*) \le \Gamma(f_{K-1}^*) < \Gamma(f_K^*)$, which implies that $f_K^*(\Gamma(f_i^*)) < 0$. Hence, for all θ, $\theta\Gamma(f_i^*) < f_K(\theta)$ and so, in particular, when $\theta \ge \vartheta(f_i)$

$$f_K(\theta) = \mathfrak{M}[f_i^{\natural}, h_{i+1}](\theta) = \max\{\theta\Gamma(f_i^*), h_{i+1}(\theta)\} = h_{i+1}(\theta).$$

Thus, using this and Lemma 7.8(iii),

$$h_{i+1}(\theta) > \theta\Gamma(f_i^*) = f_i^\natural(\theta) \geq \mathfrak{M}[f_{i-1}^\natural, \kappa_i^\natural](\theta).$$

Hence, (11.2) also holds when $\theta \geq \vartheta(f_i)$. This shows that (11.2) always holds when (11.1) holds, which completes the inductive step. $\square$

LEMMA 11.6. *In a sequential process satisfying $\kappa_1(0) > 0$ and (1.4), let r_K be given by the recursion (2.6) described in Theorem 2.3. Then*

$$\Gamma(r_K) = \max_{i \Rightarrow j}\{\Gamma(\mathfrak{C}[\kappa_i^{\circledast}, \kappa_j^*])\} = \max_{i \Rightarrow j}\{\Gamma(\mathfrak{M}[\kappa_i^\natural, \kappa_j]^*)\}.$$

PROOF. Note first that for a sequential process $i \Rightarrow j$ is the same as $i < j$. Take f_i as in Proposition 2.5, so that $r_i = f_i^{\circledast} = (f_i^\natural)^*$. Let $\Gamma = \Gamma(r_K)$ $(=\Gamma(f_K^*))$ and $\vartheta = \vartheta(f_K)$. Since $\Gamma(\kappa_K^*) \leq \Gamma(\mathfrak{C}[\kappa_1^{\circledast}, \kappa_K^*])$, it would be enough to establish the result for $\Gamma(r_{K-1})$ in the case where $\Gamma = \max\{\Gamma(r_{K-1}), \Gamma(\kappa_K^*)\}$. Consequently, we can assume that $\Gamma > \max\{\Gamma(r_{K-1}), \Gamma(\kappa_K^*)\}$. Now, Lemma 11.2 gives $\vartheta < \infty$, and $f_K^*(\Gamma) \leq 0$ implies that $\Gamma\theta \leq f_K(\theta)$ everywhere.

Let

$$h = \max\{\kappa_j^\natural : j \leq K-1\}.$$

If h is infinite on $(-\infty, \vartheta)$, then there is a $J < K$ with $\kappa_J^\natural$ infinite on $(-\infty, \vartheta)$. If $\mathcal{D}(\kappa_K) = \{\vartheta\}$, then, by Lemma 11.4(i), there is a $J < K$ with $\kappa_J^\natural(\vartheta) > \kappa_K(\vartheta)$. In both these cases Lemma 11.4 implies that $f_K = \mathfrak{M}[\kappa_J^\natural, \kappa_K]$ and so $\Gamma = \Gamma(\mathfrak{M}[\kappa_J^\natural, \kappa_K]^*)$. Otherwise, using Lemma 11.4(ii), there is an $\varepsilon > 0$ such that h and κ_K are finite and continuous on $(\vartheta - \varepsilon, \vartheta)$. Now, suppose that $h(\vartheta) > \kappa_K(\vartheta)$, and take $J < K$ with $\kappa_J(\vartheta) = h(\vartheta)$. Using the continuity of κ_K when finite, there is an $\varepsilon > 0$ such that $\kappa_K(\theta) < \kappa_J(\theta)$ on $(\vartheta - \varepsilon, \vartheta)$. Also, Lemma 11.4 implies that κ_K is infinite on (ϑ, ∞). Therefore, since $\kappa_J^\natural(\theta)/\theta$ is decreasing in θ,

$$\Gamma = \inf_{\theta > 0} \frac{f_K(\theta)}{\theta} \geq \inf_{\theta > 0} \frac{\mathfrak{M}[\kappa_J^\natural, \kappa_K](\theta)}{\theta} = \frac{\kappa_J^\natural(\vartheta)}{\vartheta} = \frac{f_K(\vartheta)}{\vartheta} = \Gamma \tag{11.3}$$

and so again $\Gamma = \Gamma(\mathfrak{M}[\kappa_J^\natural, \kappa_K]^*)$.

This leaves the case where, for some $\varepsilon > 0$, f_K is finite on $(\vartheta - \varepsilon, \vartheta]$ and $h(\vartheta) \leq \kappa_K(\vartheta)$. Then $\kappa_j^\natural$ is continuous on $(\vartheta - \varepsilon, \infty)$ for every j and thus by Lemma 11.4, $f_K(\theta) = h(\theta) > \kappa_K(\theta)$ on $(\vartheta - \varepsilon, \vartheta)$ and $f_K(\theta) = \kappa_K(\theta) > h(\theta)$ on (ϑ, ∞). By continuity and Lemma 11.3(ii), $h(\vartheta) = \kappa_K(\vartheta) = \Gamma\vartheta$. Let $\mathcal{I}$ be those $j < K$ with $\kappa_j^\natural(\vartheta) = \Gamma\vartheta$ and let $\tilde{h} = \max\{\kappa_j^\natural : j \in \mathcal{I}\}$. By reducing ε if necessary, $f_K = \tilde{h} > \kappa_K$ on $(\vartheta - \varepsilon, \vartheta)$. Let $\gamma_j = \inf \partial\kappa_j^\natural(\vartheta)$ and take J to be an index giving $\min\{\gamma_j : j \in \mathcal{I}\}$. Take $\varepsilon' > 0$. Then, for some $\delta > 0$, for $\theta \in (\vartheta - \delta, \vartheta)$ and $j \in \mathcal{I}$,

$$\kappa_j^\natural(\theta) \leq \kappa_j^\natural(\vartheta) + (\gamma_j - \varepsilon')(\theta - \vartheta) \qquad \big(=\Gamma\vartheta + (\gamma_j - \varepsilon')(\theta - \vartheta)\big)$$

for otherwise, by convexity, $(\gamma_j - \varepsilon') \in \partial\kappa_j^\natural(\vartheta)$. Then, taking the max of these over $j \in \mathcal{I}$ with δ as the minimum of those needed gives

$$f_K(\theta) = \tilde{h}(\theta) \le \Gamma\vartheta + (\gamma_J - \varepsilon')(\theta - \vartheta)$$

for $\theta \in (\vartheta - \delta, \vartheta)$. But $\Gamma\theta \le f_K(\theta)$ everywhere. Hence

$$(\gamma_J - \varepsilon')(\vartheta - \theta) \le \Gamma(\vartheta - \theta)\theta \in (\vartheta - \delta, \vartheta)$$

and so $\gamma_J \le \Gamma$. Therefore, for $\theta \le \vartheta$,

$$f_K(\theta) \ge \kappa_J^\natural(\theta) \ge \Gamma\vartheta + \gamma_J(\theta - \vartheta) \ge \Gamma\theta$$

and for $\theta > \vartheta$, $f_K(\theta) = \kappa_K(\theta)$ and is strictly greater than both $\kappa_J(\theta)$ and $\Gamma\theta$. Thus (11.3) holds in this case, too, giving $\Gamma = \Gamma(\mathfrak{M}[\kappa_J^\natural, \kappa_K]^*)$. $\square$

PROOF OF THEOREM 3.3. Applying Lemma 11.6 to every sequential process gives the first formula for Γ. Fix $i \Rightarrow j$. Let $f = \kappa_i$, $\kappa = \kappa_j$ and $g = \mathfrak{M}[f^\natural, \kappa]$ so that $\Gamma(\mathfrak{C}[\kappa_i^\circledast, \kappa_j^*]) = \Gamma(g^*)$. Now, an application of Lemma 10.1 (with $C = [0, \infty)$) and then of (7.1) in Proposition 7.1 gives the second formula. $\square$

12. Expected numbers.

THEOREM 12.1. *Consider a sequential process with K classes, $\mathcal{C}_1, \dots, \mathcal{C}_K$, with corresponding PF$^+$eigenvalues $\kappa_1, \dots, \kappa_K$ and in which $\mathcal{C}_1$ is primitive. Suppose that*

$$\bigcap_{j \le K} \mathcal{D}(\kappa_j) \ne \varnothing \quad \textit{and} \quad \bigcap_{j \le i+1} \mathcal{D}(\kappa_j) \subset \mathcal{D}_{i,i+1} \qquad \textit{for } i = 1, \dots, K-1. \tag{12.1}$$

Define R_i recursively by $R_1 = \kappa_1^*$ and $R_i = \mathfrak{C}[R_{i-1}, \kappa_i^*]$ for $i = 2, \dots, K$. Then

$$\frac{1}{n}\log\bigl(\mathbb{E}_\nu Z_\sigma^{(n)}[na, \infty)\bigr) \to -R_K(a) \tag{12.2}$$

except possibly at the upper endpoint of the interval on which R_K is finite.

PROOF. Suppose that $m_{\upsilon\tau} > 0$ for $\upsilon \in \mathcal{C}_{K-1}$ and $\tau \in \mathcal{C}_K$. Then

$$\int e^{\theta z}\mathbb{E}_\nu Z_\sigma^{(n)}(dz) = \sum_{r=0}^{n-1}(m(\theta)^r)_{\nu\upsilon} m(\theta)_{\upsilon\tau}(m(\theta)^{n-r-1})_{\tau\sigma}$$

and so, by induction on the number of classes,

$$\frac{1}{n}\log\int e^{\theta z}\mathbb{E}_\nu Z_\sigma^{(n)}(dz) \to \max_i\{\kappa_i(\theta)\} \qquad \text{for } \theta > 0.$$

The second part of (12.1) ensures the off-diagonal terms have no effect; the first part ensures that the limit here is finite for some $\theta > 0$. Induction on the number of classes shows that R_K is the F-dual of $\max_i\{\kappa_i(\theta)\}$. Now, as in Proposition 2.1, large deviation theory gives (12.2). □

Although R_K is defined recursively it can be defined directly as the convex minorant of $\kappa_1^*, \ldots, \kappa_K^*$. It is easy to see, by induction, that $r_i \geq R_i$, so that $\Gamma(r_K) \leq \Gamma(R_K)$. To see that R_i and r_i really can be different, notice that the order of the classes matters in r_i but does not in R_i. It is easy to give a two-type reducible example where $\Gamma(r_K) < \Gamma(R_K)$. More specifically, arrange κ_1^* and κ_2^* so that:

(i) $\kappa_1^*(\Gamma) = \kappa_2^*(\Gamma) = 0$,
(ii) $\kappa_1^*(x) < \kappa_2^*(x)$ for $x > \Gamma$,
(iii) their convex minorant is less than zero at Γ.

Then in computing $\Gamma(r_K)$, these last two conditions do not matter, and $\Gamma(r_K) = \Gamma$. However, they do matter in computing $\Gamma(R_K)$ which will be bigger than Γ. Note too that, if instead of type 1 preceding type 2 here, type 2 preceded type 1, then $\Gamma(r_K) = \Gamma(R_K)$ and this would be an example of super-speed, as described toward the end of the Introduction and in Biggins (2010).

13. Further lower bounds. Consider a sequential process with $m_{\upsilon\tau} > 0$ for $\upsilon \in \mathcal{C}_{K-1}$ and $\tau \in \mathcal{C}_K$. Once either (2.13) or (2.14) fails for $i = K-1$, the behavior of $\mathbb{E}_\upsilon Z_\tau[x, \infty)$ starts to exert an influence: the spatial spread of the children in the final class (of type τ) born to a parent in the penultimate class (of type υ) matters. It seems that some regularity is needed beyond knowledge of the interval of convergence of $m_{\upsilon\tau}$ to derive a result similar to Theorem 2.4 in this case. The conditions (13.1) and (13.2) in the next result are on the tails of the distribution of average numbers of type τ born to a type υ.

THEOREM 13.1. *Make the same assumptions as in Theorem 2.3; define g_i by the recursion (2.15) in Theorem 2.7 and assume (2.12) holds. Let $\upsilon \in \mathcal{C}_{K-1}$ and $\tau \in \mathcal{C}_K$ be the types for which $m_{\upsilon\tau} \neq 0$ and let*

$$\overline{\psi} = \sup\{\psi : m_{\upsilon\tau}(\psi) < \infty\} = \sup \mathcal{D}_{K-1,K},$$
$$\underline{\psi} = \inf\{\psi : m_{\upsilon\tau}(\psi) < \infty\} = \inf \mathcal{D}_{K-1,K}.$$

Assume also that

$$\lim \frac{1}{n} \log\bigl(Z_\upsilon^{(n)}[na, \infty)\bigr) = -g_{K-1}^{\circledast}(a) \qquad a.s.\text{-}\mathbb{P}_\upsilon.$$

Finally, assume both of the following: if (2.13) fails for $i = K-1$, then

$$\lim_{x\to\infty} \frac{\log \mathbb{E}_\upsilon Z_\tau[x, \infty)}{x} = -\overline{\psi}; \tag{13.1}$$

if (2.14) fails for $i = K-1$, then

$$\lim_{x\to-\infty} \frac{\log \mathbb{E}_\upsilon Z_\tau[x, \infty)}{x} = -\underline{\psi}. \tag{13.2}$$

Then

$$\lim \frac{1}{n} \log\bigl(Z_\sigma^{(n)}[na, \infty)\bigr) = -g_K^{\circledast}(a) \qquad a.s.\text{-}\mathbb{P}_\nu.$$

Note that Kawata [(1972), Theorem 7.7.4] shows that the lim sup of the sequences in (13.1) and (13.2) must be $-\overline{\psi}$ and $-\underline{\psi}$, respectively. Thus, the substance of each condition is that the lim inf equals the corresponding lim sup. This theorem improves on the lower bound in Theorem 2.3 in some cases, and matches the upper bound already obtained. It is not too hard to obtain with the machinery already established.

LEMMA 13.2. *In a sequential process, let $\upsilon \in \mathcal{C}_{K-1}$, $\tau \in \mathcal{C}_K$, $\overline{\psi}$ and $\underline{\psi}$ as in Theorem 13.1 and suppose that for $\nu \in \mathcal{C}_1$, and k-convex f with $\Gamma(f^*) > -\infty$,*

$$\lim \frac{1}{n} \log\bigl(Z_\upsilon^{(n)}[na, \infty)\bigr) = -f^{\circledast}(a) \qquad a.s.\text{-}\mathbb{P}_\nu$$

for $a \neq \Gamma(f^*)$. Let $\chi_1(\theta) = -\log I(\theta \in [\underline{\psi}, \infty))$ and $\chi_2(\theta) = -\log I(\theta \in (-\infty, \overline{\psi}])$. Then

$$\liminf \frac{1}{n} \log\bigl(F_\tau^{(n)}[na, \infty)\bigr) \geq -g^{\circledast}(a) \qquad a.s.\text{-}\mathbb{P}_\nu$$

for all $a < \Gamma(g^*)$, where:

(i) $g = f^{\natural}$ *or*
(ii) $g = f^{\natural} + \chi_2$ *when* (13.1) *holds, or*
(iii) $g = f^{\natural} + \chi_1$ *when* (13.2) *holds, or*
(iv) $g = f^{\natural} + \chi_1 + \chi_2$ *when both* (13.1) *and* (13.2) *hold.*

PROOF. Case (i) is given by Proposition 6.1. Let $C = \mathcal{D}_{K-1,K}$. Case (iv) is considered; the other two are similar. Assume $f^{\natural}(\theta) < \infty$ for some $\theta < \underline{\psi}$ and that $\overline{\psi} < \infty$; otherwise this is equivalent to cases (ii) or (iii). Then

$$g^*(a) = \sup_{\theta \in C}\{\theta a - f^{\natural}(a)\} = \sup_{\theta \in C}\{\theta a - f^{\flat}(a)\}.$$

Let

$$\underline{\gamma} = \inf\{\gamma' : \gamma' \in \partial f^{\flat}(\theta), \theta \in C\}$$

and let $\overline{\gamma}$ be the supremum over the same set: both are finite. Calculations like those in Lemma 7.5 show that

$$g^*(a) = \begin{cases} \underline{\psi} a - f^{\flat}(\underline{\psi}), & a \in (-\infty, \underline{\gamma}], \\ f^{\circledast}(a), & a \in (\underline{\gamma}, \overline{\gamma}), \\ \overline{\psi} a - f^{\flat}(\overline{\psi}), & a \in [\overline{\gamma}, \infty). \end{cases}$$

The number to the right of nc in generation n exceeds $N_n = Z_\upsilon^{(n-1)}[na,\infty)$ independent copies of $Z_\tau[n(c-a),\infty)$ under $\mathbb{P}_\upsilon$. Let the expectation of the latter be $\tilde{e}_n$. Here $a<c$, since $n(c-a)$ must go to infinity, but otherwise a may be chosen freely. When $f^{\circledast}(a)<0$, Lemma 6.5 and (13.1) give

$$\begin{aligned}\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\tau^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] &\geq \liminf_n \frac{1}{n}(\log N_n + \log\tilde{e}_n)\\ &\geq -\big(f^{\circledast}(a)+\overline{\psi}(c-a)\big)\end{aligned}$$

and so, maximizing over the available a,

$$\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\tau^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] \geq \sup_{f^{\circledast}(a)<0,a<c}\{\overline{\psi}a - f^{\circledast}(a)\} - \overline{\psi}c.$$

Since $f^{\circledast}$ is closed, increasing and infinite when positive, $\{f^{\circledast}(a)<0, a<c\}$ may be replaced by $\{a\leq c\}$. Then using Lemmas 7.4 and 7.5

$$\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\tau^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] \geq \begin{cases} f^{\flat}(\overline{\psi})-\overline{\psi}c, & \text{for } c\geq\overline{\gamma},\\ -f^{\circledast}(c), & \text{for } c<\overline{\gamma},\end{cases}$$

when this is strictly positive. Similarly, but with $a>c$, so that $n(c-a)$ goes to minus infinity,

$$\begin{aligned}\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\tau^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] &\geq \liminf_n \frac{1}{n}(\log N_n + \log\tilde{e}_n)\\ &\geq -\big(f^{\circledast}(a)+\underline{\psi}(c-a)\big)\end{aligned}$$

provided the latter is strictly positive. Then, maximizing over $a>c$,

$$\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\sigma^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] \geq \begin{cases} f^{\flat}(\underline{\psi})-\underline{\psi}c, & \text{for } c\leq\underline{\gamma},\\ -f^{\circledast}(c), & \text{for } c>\underline{\gamma},\end{cases}$$

again, provided the latter is strictly positive.

Combining these,

$$\liminf_n \frac{1}{n}\log\mathbb{E}\big[Z_\sigma^{(n)}[nc,\infty)|\mathcal{F}^{(n-1)}\big] \geq -g^*(c),$$

when this is strictly positive. Then conditional Borel–Cantelli and continuity of $g^{\circledast}$ complete the proof. □

PROOF OF THEOREM 13.1. First apply Lemma 9.1 to determine which of the four possibilities in Lemma 13.2 is relevant. Now use Lemma 13.2 to show

$$\liminf \frac{1}{n}\log\big(F_\tau^{(n)}[na,\infty)\big) \geq -g_{K-1}^{\circledast}(a) \qquad \text{a.s.-}\mathbb{P}_\nu,$$

and then use Theorem 6.2 to complete the proof. □

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